

**2021 FALL CALCULUS 0417: SECOND MIDTERM (NOVEMBER 29,
2021)**

- Please answer the following questions in details, which means you need to state all theorems or results you used. The definitions of terminology were taught in the lectures, so you cannot ask instructor or TA about mathematical definitions while taking the midterm.
 - Please mark your name, student ID, and question numbers clearly on your answer sheet.
 - The exam has a total of 120 points.
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1. (10 points) The function $y = \sec^{-1} x$ is defined via $x = \sec y$, for $0 \leq y < \pi/2$ or $\pi \leq y < 3\pi/2$. Find the value of $\frac{d}{dx} (\sec^{-1} x)$.
2. (10 points) Find the tangent line to the curve $(x^2 + y^2 - 1)^3 = x^3 + y - 1$ at $(1, 0)$.
3. (10 points) Let $f(x) = xe^x$. Find the absolute maximum value of $f(\sin x + 2 \cos x)$, $x \in \mathbb{R}$.
4. (10 points) Let $f(x) = \frac{a}{x^3 + 3x + 4} + \frac{b}{x^3 + x - 2}$, $ab > 0$. Show that $f(x) = 0$ has exactly one real solution.
5. Prove the following statements.
 - (a) (10 points) Consider the logarithmic function $\ln x$ for $x > 0$. Show that $\ln x \leq x - 1$ for any $x > 0$.
 - (b) (10 points) Based on (a), let $p_1, p_2, \dots, p_m > 0$ and $q_1, q_2, \dots, q_m > 0$ be arbitrary numbers such that

$$\sum_{k=1}^m p_k = \sum_{k=1}^m q_k = 1.$$

Show that

$$\sum_{k=1}^m p_k \log q_k \leq \sum_{k=1}^m p_k \log p_k.$$

6. (10 points) Evaluate $\lim_{x \rightarrow \infty} \left(\frac{2^x + 5^x}{2} \right)^{1/x}$.
7. Express the following integrals as limits or Riemann sums, by using the right end points:
 - (a) (5 points) $\int_1^3 \sqrt{4 + x^2} dx$.
 - (b) (5 points) $\int_2^5 \left(x^2 + \frac{1}{x} \right) dx$.

8. (10 points) Evaluate $\lim_{x \rightarrow 0} \int_0^x \frac{\sin t^2}{x^2} dt$.
9. (10 points) Find the derivative of $\int_x^{x^3} \sin t^2 dt$.
10. (Bonus) (20 points) State and prove the L'Hospital's rule (only need to show the case for the indeterminate form $\frac{0}{0}$).