2021 FALL CALCULUS 0417: THIRD MIDTERM (DECEMBER 30, 2021)

- Please answer the following questions in details, which means you need to state all theorems or results you used. The definitions of terminology were taught in the lectures, so you cannot ask instructor or TA about mathematical definitions while taking the midterm.
- Please mark your name, student ID, and question numbers clearly on your answer sheet.
- The exam has a total of 120 points.

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(1) (10 points) Evaluate $\int x \sec^2 x \, dx$.

(2) (10 points) Calculate $\int \dfrac{x^4 + 2x^3 + 1}{(x-1)(x^2+1)^2} \, dx$.

(3) (10 points) Let f be a continuous function on $[0, 1]$. Show that

$$\int_0^1 f(x) \, dx = \int_0^1 f(1-x) \, dx.$$

(4) Consider the curve given by the polar equation $r = a + \cos\theta$, $0 < a < 1$.

   (a) (5 points) Find the constant $a = a_0$ such that the tangent lines at the origin are $y = \frac{\sqrt{3}}{2}x$ and $y = -\frac{\sqrt{3}}{2}x$.

   (b) (5 points) Draw the curve $r = a_0 + \cos\theta$.

   (c) (5 points) Compute the area of the region bounded by the inner loop of $r = a_0 + \cos\theta$.

   (d) (5 points) Compute the surface area given by revolving the inner loop about the x-axis of $r = a_0 + \cos\theta$.

(5) (10 points) Evaluate $\lim\limits_{x \to 0} \dfrac{\int_x^{\tan x} \sqrt{1+t^3} \, dt}{x^3}$.

(6) (10 points) Find the value of α such that

$$F(\alpha) = \int_0^\infty e^{-\alpha x} \sin x \, dx$$

converges and evaluate the integral.

(7) (10 points) Evaluate $\int \cos\theta \cos^5(\sin\theta) \, d\theta$.

(8) For $n \in \mathbb{N}$, let $\Gamma(n) = \int_0^\infty x^{n-1} e^{-x} \, dx$. Show that

   (a) (5 points) $\Gamma(1) = 0! = 1$.

   (b) (10 points) $\Gamma(n+1) = n\Gamma(n)$.

   (b) (5 points) $\Gamma(n+1) = n!$.

(9) (Bonus) Let $I(p) = \int_0^\infty x^p e^{-x^2}\, dx$, where $p \in \mathbb{R}$.

(a) (10 points) Find p so that it converges.

(b) (10 points) Find $I(3)$.