# 2021 FALL MATHEMATICAL LOGIC AND REASONING 1372 1372: MIDTERM (NOVEMBER 8, 2021)

- Please answer the following questions in details, which means you need to state all theorems or results you used.
- Please mark your name, student ID, and question numbers clearly on your answer sheet.
- The exam has a total of 160 points, and we will not normalize your grade.

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1. Let $x, y$ and $z$ be integers. Prove that
   (a) (5 points) if $x$ is odd, then $x^2 + 1$ is even.
   (b) (5 points) If $x$ divides $y$ and $x$ divides $z$, then $x$ divides $y + z$.
   (c) (5 points) if $x$ is even, $y$ is odd, then $x + y$ is odd.
   (d) (5 points) if exactly one of $x, y$ and $z$ is even, then the sum of $x, y$ and $z$ is even.
2. (10 points) Write a proof by contraposition to show that for any real number $x$, if $x^2 + 5x + 6 < 0$, then $2 < x < 3$.
3. (10 points) Suppose $a$ and $b$ are positive integers. Write a proof by contradiction to show that if $ab$ is odd, then both $a$ and $b$ are odd.
4. (20 points) Show that $\sqrt{5}$ is not a rational number[1]
5. Find the power set $2^X$ for the following sets.
   (a) (5 points) $X = \{0, \triangle, \square\}$.
   (b) (5 points) $X = \{\emptyset, \{a\}, \{b\}, \{a, b\}\}$[2].
6. Show that
   (a) (5 points) if $X = \{x \in \mathbb{N} : x^2 < 14\}$ and $Y = \{1, 2, 3\}$, then $X = Y$.
   (b) (5 points) if $A$ is a set, then $A \subseteq 2^X$.
7. Let $A$ and $B$ be two sets. Define the symmetry difference of $A$ and $B$ by
$$A \Delta B = (A - B) \cup (B - A).$$
   Prove that
   (a) (5 points) $A \Delta A = \emptyset$.
   (b) (5 points) $A \Delta B = (A \cup B) - (A \cap B)$.
8. State the following principles:
   (a) (4 points) The principle of mathematical induction.
   (b) (3 points) The principle complete induction.
   (c) (3 points) The well-ordering principle.

*There are more questions in the next page.*

[1] You can use some facts from Chapter 2 I have taught in lectures.
[2] $\emptyset$ denotes the empty set.

## BONUS POINTS

Prove the following statements.

9. (20 points) The principle of mathematical induction implies the well-ordering principle.
10. (20 points) The well-ordering principle implies the principle complete induction.
11. (20 points) The principle complete induction implies the principle of mathematical induction.