2022 FALL REAL ANALYSIS (I): FINAL EXAM (DECEMBER 23, 2022)

- Please mark your name, student ID, and question numbers clearly on your answer sheet.
- State all reasons, lemmas, theorems clearly, while you are using during answering the questions.
- You can use all results coming from *advanced calculus* without any proofs.
- The total score is 140 points. 40 points are your bonus.
- Good luck.

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1. (15 points) Prove the integral version of Minkowski's inequality for $1 \le p < \infty$, and measurable function $f(x,y)$:
$$\left[\int \left(\int |f(x,y)|\,dx\right)^p dy\right]^{1/p} \le \int \left(\int |f(x,y)|^p\,dy\right)^{1/p} dx.$$

2. (15 points) Let $f(x,y)$, $0 \le x, y \le 1$ satisfy the following conditions: For each x, $f(x,y)$ is an integrable function of y, and $\frac{\partial f(x,y)}{\partial x}$ is a bounded function of (x,y). Show that $\frac{\partial f(x,y)}{\partial x}$ is a measurable function of y for each x and
$$\frac{d}{dx}\int_0^1 f(x,y)\,dy = \int_0^1 \frac{\partial}{\partial x} f(x,y)\,dy.$$

3. (15 points) For $1 \le p, q \le \infty$ and $\frac{1}{p} + \frac{1}{q} \ge 1$. Suppose r satisfies $\frac{1}{r} = \frac{1}{p} + \frac{1}{q} - 1$. Prove that
$$\|f * g\|_{L^r} \le \|f\|_{L^p}\|g\|_{L^q},$$
where $*$ denotes the convolution of two functions.

4. (15 points) Let $f \in L^1(\mathbb{R}^n)$ and $\{K_\delta\}_{\delta>0}$ be an approximation to the identity. Show that
$$\lim_{\delta\to 0}(f * K_\delta)(x) = f(x),$$
for all x in the Lebesgue set of f, and the limit holds for a.e. $x \in \mathbb{R}^n$.

5. (20 points) Prove the following variant **Vitali covering lemma**: If E is covered in the Vitali sense by a family $\mathcal{B}$ of balls, and $0 < m_*(E) < \infty$, then for every $\eta > 0$, there exists a disjoint collection of balls $\{B_j\}_{j=1}^\infty$ in $\mathcal{B}$ such that
$$m_*\left(E \setminus \cup_{j=1}^\infty B_j\right) = 0 \quad \text{and} \quad \sum_{j=1}^\infty m(B_j) \le (1+\eta)m_*(E).$$

6. (20 points) Prove that if $\{K_\delta\}_{\delta>0}$ is a family of approximations to the identity, then
$$\sup_{\delta>0} |(f * K_\delta)(x)| \leq c f^*(x),$$
for some constant $c > 0$ and all integrable f. Here f^* denotes the Hardy-Littlewood maximal function fo f.

7. (20 points) Suppose 0 is a point of (Lebesgue) density of the set $E \subset \mathbb{R}$. Show that for each of the individual conditions below there is an infinite sequence of points $x_n \in E$, with $x_n \neq 0$, and $x_n \to 0$ as $n \to \infty$.
   (a) The sequence also satisfies $-x_n \in E$ for all n.
   (b) In addition, $2x_n$ belongs to E for all n.

8. Prove that
   (a) (10 points) If f is integrable on $\mathbb{R}^n$, and f is not identically zero, then
   $$f^*(x) \geq \frac{c}{x^n}, \qquad \text{for some } c > 0 \text{ and for all } |x| \geq 1,$$
   where f^* denotes the Hardy-Littlewood maximal function fo f. [Hint: Use the fact that $\int_B |f| > 0$, for some ball B.]
   (b) (10 points) Conclude that f^* is not integrable on $\mathbb{R}^n$. Then show that the weak type estimate
   $$m(\{x : f^*(x) > \alpha\}) \leq c/\alpha,$$
   for all $\alpha > 0$, whenever $\int |f| = 1$, is best possible in the following sense: if f is supported in the unit ball with $\int |f| = 1$, then
   $$m(\{x : f^*(x) > \alpha\}) \geq c'/\alpha,$$
   for some $c' > 0$ and all sufficiently small $\alpha > 0$.