2022 FALL REAL ANALYSIS (I): MIDTERM (NOVEMBER 4, 2022)

- Please mark your name, student ID, and question numbers clearly on your answer sheet.
- State all reasons, lemmas, theorems clearly, while you are using during answering the questions.
- You can use all results coming from *advanced calculus* without any proofs.
- The notations are the same as in the lectures: $|\cdot|$, $m_*(\cdot)$ and $m(\cdot)$ stand for the volume of rectangles, outer measure, and (Lebesgue) measure, respectively.
- The total score is 120 points. 20 points are bonus for you.

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1. A subset $E \subset \mathbb{R}$ has measure zero if given any $\epsilon > 0$, there exist countable intervals I_n that cover E, such that $A \subset \bigcup_{n=1}^{\infty} I_n$ and $\sum_{n=1}^{\infty} |I_n| < \epsilon$. Please show that
   (a) (10 points) Every countable set in $\mathbb{R}$ has measure zero.
   (b) (10 points) The Cantor set $\mathcal{C}$ in $[0,1]$ has measure zero.
2. (10 points) Let $E \subset \mathbb{R}^n$ be a measurable subset. Let f_n be a sequence of measurable functions defined on E. Show that $g(x) := \limsup_{n\to\infty} f_n(x)$ is a measurable function defined on E.
3. (10 points) Prove or disprove that $|f|$ is measurable $\implies$ f is measurable.
4. (15 points) Let $Z \subset \mathbb{R}^1$ be a subset of measure zero. Show that the set $\{x^2 : x \in Z\}$ also has measure zero in $\mathbb{R}$.
5. (15 points) Suppose $A \subset E \subset B$, where A, B are measurable sets in $\mathbb{R}^n$ of finite measure. If $m(A) = m(B)$, show that E is measurable.
6. Suppose that $f_k \to f$ and $g_k \to g$ in measure on E as $k \to \infty$. Show that
   (a) (5 points) $f_k + g_k \to f + g$ in measure on E as $k \to \infty$.
   (b) (10 points) $f_k g_k \to fg$ in measure on E as $k \to \infty$, provided $m(E) < \infty$.
7. (15 points) Give an example of an f that is not (Lebesgue) integrable, but whose improper Riemann integral exists and is finite.
8. (20 points) Prove the following LDCT: If $\{f_k\}_{k\in\mathbb{N}}$ satisfies $f_k \to f$ in measure on E as $k \to \infty$, and $|f_k| \le g$ for any $k \in \mathbb{N}$, where g is an integrable function. Show that f is integrable and $\int_E f_k \to \int_E f$ as $k \to \infty$[1].

[1]Hint: Show that every subsequence of $\{f_k\}$ has a subsequence $\{f_{k_j}\}$ such that $\int_E f_{k_j} \to \int_E f$.