2023 FALL REAL ANALYSIS (I): FINAL EXAM (DECEMBER 19, 2023)

- Please mark your name, student ID, and question numbers clearly on your answer sheet.
- State all reasons, lemmas, theorems clearly, while you are using during answering the questions.
- The total score is 120 points. 20 points are your bonus.

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1. (20 points) Prove the following variant **Vitali covering lemma**: If E is covered in the Vitali sense by a family $\mathcal{B}$ of balls, and $0 < m_*(E) < \infty$, then for every $\eta > 0$, there exists a disjoint collection of balls $\{B_j\}_{j=1}^{\infty}$ in $\mathcal{B}$ such that
$$m_*\left(E \setminus \cup_{j=1}^{\infty} B_j\right) = 0 \quad \text{and} \quad \sum_{j=1}^{\infty} m(B_j) \leq (1+\eta) m_*(E).$$

2. Suppose $f \in L^2(\mathbb{R}^n)$ and $k \in L^1(\mathbb{R}^n)$.
   (a) (5 points) Show that $(f * k)(x) = \int f(x-y)k(y)\,dy$ converges for a.e. x.
   (b) (7 points) Prove that $\|f * k\|_{L^2(\mathbb{R}^n)} \leq \|f\|_{L^2(\mathbb{R}^d)} \|k\|_{L^1(\mathbb{R}^d)}$.
   (c) (8 points) Establish $\widehat{(f * k)}(\xi) = \widehat{k}(\xi)\widehat{f}(\xi)$ for a.e. ξ.

3. (15 points) Consider the function
$$f(x) = \sum_{n=0}^{\infty} 2^{-n} e^{2\pi i 2^n x}$$
Prove that f satisfies $|f(x) - f(y)| \leq A_\alpha |x-y|^\alpha$, for each $0 < \alpha < 1$, where A_α is some constant depending on α.

4. (15 points) Let $f, g \in L^1(\mathbb{R}^n)$. Suppose that $\int_E f = \int_E g$ for all measurable subset $E \subset \mathbb{R}^n$. Show that $f = g$ a.e..

5. Suppose $f \in L^1(\mathbb{R}^n)$, and denote f^* to be the Hardy-Littlewood maximal function of f. Show that
   (a) (5 points) f^* is measurable.
   (b) (10 points) f^* satisfies
$$m\left(\{x \in \mathbb{R}^n : f^*(x) > \alpha\}\right) \leq \frac{A}{\alpha} \|f\|_{L^1(\mathbb{R}^n)},$$
   for all $\alpha > 0$ and $A = 3^n$.

6. (20 points) Show that the space $L^2(\mathbb{R}^n)$ is complete in its metric $\|\cdot\|_{L^2(\mathbb{R}^n)}$.

7. (20 points) Suppose that $f \in L^1(\mathbb{R}^n)$ and x is a point in the Lebesgue set of f. Let
$$\mathcal{A}(r) := \frac{1}{r^n} \int_{|y| \leq r} |f(x-y) - f(x)|\,dy, \text{ whenever } r > 0.$$

Then $\mathcal{A}(r)$ is a continuous function in $r > 0$ and

$$\mathcal{A}(r) \to 0 \text{ as } r \to 0.$$