2023 FALL REAL ANALYSIS (I): MIDTERM (OCTOBER 31, 2023)

- Please mark your name, student ID, and question numbers clearly on your answer sheet.
- State all reasons, lemmas, theorems clearly, while you are using during answering the questions.
- You can use all results coming from *advanced calculus* without any proofs.
- The notations are the same as in the lectures: $|\cdot|$, $m_*(\cdot)$ and $m(\cdot)$ stand for the volume of rectangles, outer measure, and (Lebesgue) measure, respectively.
- The total score is 120 points. 20 points are bonus for you.

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1. A subset $E \subset \mathbb{R}$ has measure zero if given any $\epsilon > 0$, there exist countable intervals I_n that cover E, such that $A \subset \bigcup_{n=1}^{\infty} I_n$ and $\sum_{n=1}^{\infty} |I_n| < \epsilon$. Please show that
   (a) (10 points) Every countable set in $\mathbb{R}$ has measure zero.
   (b) (10 points) The Cantor set $\mathcal{C}$ in $[0, 1]$ has measure zero.
2. (10 points) Prove or disprove that $|f|$ is measurable $\implies$ f is measurable.
3. (15 points) Suppose $A \subset E \subset B$, where A, B are measurable sets in $\mathbb{R}^n$ of finite measure. If $m(A) = m(B)$, show that E is measurable.
4. (15 points) Give an example of an f that is not (Lebesgue) integrable, but whose improper Riemann integral exists and is finite.
5. (15 points) Let $\{E_k\}_{k=1}^{\infty}$ be a sequence of sets of $\mathbb{R}^n$ such that $\sum_{k=1}^{\infty} m_*(E_k) < \infty$. Show that $\limsup_{k\to\infty} E_k$ has measure zero.
7. (15 points) If $\int_A f = 0$ for every measurable subset A of a measurable set E, show that $f = 0$ a.e.in E.
6. (15 points) Use Egorov's theorem to prove the bounded convergence theorem.
7. (15 points) Construct a Lebesgue measurable set that is not Borel measurable.