2023 FALL REAL ANALYSIS (I) @ NYCU APPL. MATH.
HOMEWORK 1

- Please answer the following questions in details, which means you need to state all theorems and all reasons you have been using.
- Please mark your name, student ID, and question numbers clearly on your answer sheet. The deadline to hand in the exercise is on **October 5, 2023** in class.

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(1) Let $R \subset \mathbb{R}^n$ be a *rectangle*, prove that $m_*(R) = |R|$.

(2) Let $E \subset \mathbb{R}$ be a set, the outer Jordan content is defined by

$$J_*(E) := \inf \left\{ \sum_{j=1}^{J} |Q_j| : \ E \subset \cup_{j=1}^{J} Q_j \right\}$$

where Q_j's are intervals in $\mathbb{R}$ for $j = 1, 2, \ldots, J$.

(a) Prove that $J_*(E) = J_*(\overline{E})$, where $\overline{E}$ denotes the closure of E.

(b) Construct an example such that $J_*(E) = 1$ but $m_*(E) = 0$, where $m_*(E)$ is the outer measure of E.

(3) Let $E_1, E_2 \subset \mathbb{R}$ be measurable sets. Show that $E_1 \times E_2$ is a measurable set in $\mathbb{R}^2$ and $m(E_1 \times E_2) = m(E_1) \cdot m(E_2)$[1].

(4) Show that the Borel σ-algebra $\mathcal{B}$ in $\mathbb{R}^n$ is the smallest σ-algebra containing the closed sets in $\mathbb{R}^n$.

(5) Let $E \subset \mathbb{R}^n$ be any subset, recalling that the outer measure is defined via

$$m_*(E) = \inf \{ m(\mathcal{O}) : \ E \subset \mathcal{O}, \text{ and } \mathcal{O} \text{ is an open set in } \mathbb{R}^n \}.$$

One can also define an inner measure $m^*(E)$ by

$$m^*(E) = \sup \{ m(F) : \ F \subset E, \text{ and } F \text{ is a closed set in } \mathbb{R}^n \}.$$

Show that

(a) $m^*(E) \le m_*(E)$.

(b) E is measurable if and only if that $m^*(E) = m_*(E)$, provided that $m_*(E) < \infty$.

(6) We have introduced F_σ and G_δ sets in $\mathbb{R}^n$.

(a) Show that a closed set is G_δ but an open set is F_σ.

(b) Construct a set, which is F_σ but not G_δ.

(c) Construct a Borel set, which is neither G_δ nor F_σ.

(7) Construct an example which is a Lebesgue measurable set but not a Borel measurable set.

(8) Try to understand what is a non-measurable set (in the Lebesgue sense)[2].

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[1] In this case we *define* $0 \cdot \infty = 0$

[2] Please do not hand in this problem to your TA, and the construction of non-measurable sets can be found in any references.