2023 FALL REAL ANALYSIS (I) @ NYCU APPL. MATH.
HOMEWORK 2

- Please answer the following questions in details, which means you need to state all theorems and all reasons you have been using.
- Please mark your name, student ID, and question numbers clearly on your answer sheet. The deadline to hand in the exercise is on **October 26, 2023**.

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(1) Let $a_k \in \mathbb{R}$ be constants and $E_k \subset \mathbb{R}^n$ be subsets, for $k = 1, 2, \ldots, N$. A simple function $f(x) = \sum_{j=1}^{N} a_k \chi_{E_k}(x)$ is measurable if and only if E_k are measurable sets for all $k = 1, 2, \ldots, N$.

(2) (**The Borel-Cantelli lemma**) Suppose $\{E_k\}_{k=1}^{\infty}$ is a collection of countably many measurable subsets of $\mathbb{R}^n$, and

$$\sum_{k=1}^{\infty} m(E_k) < \infty.$$

Let

$$\begin{aligned} E &= \{x \in \mathbb{R}^n : x \in E_k, \text{ for infinitely many } k\} \\ &= \limsup_{k\to\infty}(E_k). \end{aligned}$$

Prove that
(a) E is measurable,
(b) $m(E) = 0$.

(3) If f is integrable in $(0, 1)$, show that $x^k f(x)$ is also integrable in $(0, 1)$, for all $k \in \mathbb{N}$. Moreover, $\int_0^1 x^k f(x)\,dx \to 0$ as $k \to \infty$.

(4) Let $f(x, y)$, $0 \le x, y \le 1$ satisfy the following conditions: For each x, $f(x, y)$ is an integrable function of y, and $\frac{\partial f(x,y)}{\partial x}$ is a bounded function of (x, y). Show that $\frac{\partial f(x,y)}{\partial x}$ is a measurable function of y for each x and

$$\frac{d}{dx}\int_0^1 f(x, y)\,dy = \int_0^1 \frac{\partial}{\partial x} f(x, y)\,dy.$$

(5) Let $E \subset \mathbb{R}^n$ be any measurable subset, and f be a nonnegative measurable function defined on E. Let $(f_m)(E) := \int_E f(x)\,dx$. Show that
(a) f_m is a (Lebesgue) measure.[1]

[1]For example, when $f = 1$, the integration stands for the usual Lebesgue measure.

(b) Let $y = Tx$ be a non-singular linear transformation of $\mathbb{R}^n$. Suppose that $\int_E f(y)\,dy$ exists. Show that

$$\int_E f(y)\,dy = |\det T| \int_{T^{-1}E} f(Tx)\,dx.$$

(6) Let $\{f_k\}$ be a sequence of measurable functions on E. Show that $\sum_{k=1}^{\infty} f_k$ converges absolutely a.e. in E if $\sum_{k=1}^{\infty} \int_E |f_k| < \infty$.

(7) Show that simple functions, step functions, and continuous functions with compact supports are dense in $L^1(\mathbb{R}^n)$, under the graph norm $\|\cdot\|_{L^1(\mathbb{R}^n)}$.

(8) Suppose that f is integrable, and define $f_h(x) = f(x-h)$, for $h \in \mathbb{R}^n$. Show that $\|f_h - f\|_{L^1(\mathbb{R}^n)} \to 0$ as $h \to 0$.

(9) For any $p > 0$, $\int_E |f - f_k|^p \to 0$ as $k \to \infty$, and $\int_E |f_k|^p \le M$ for all $k \in \mathbb{N}$. Show that $\int_E |f|^p \le M$.