2023 FALL REAL ANALYSIS (I) @ NYCU APPL. MATH.
HOMEWORK 3

- Please answer the following questions in details, which means you need to state all theorems and all reasons you have been using.
- Please mark your name, student ID, and question numbers clearly on your answer sheet. The deadline to hand in the exercise is on **November 23, 2023**.

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(1) Let f be nonnegative and measurable on E and $\omega(\alpha) := \omega_{f,E}(\alpha)$ be finite on $(0,\infty)$. Suppose that $\int_0^\infty \alpha^{p-1}\omega(\alpha)\,d\alpha$ is finite, show that $\lim_{a\to 0^+} a^p\omega(a) = \lim_{b\to\infty} b^p\omega(b) = 0$.

(2) If $f(x)$ is a measurable function on $\mathbb{R}^n$. Show that $F(x,y) := f(x-y)$ is also measurable on $\mathbb{R}^{2n}$.

(3) Let f be measurable and *periodic* with period 1: $f(x+1) = f(x)$ for all x. Suppose that there exists a finite number c such that

$$\int_0^1 |f(a+x) - f(b+x)|\,dx \le c$$

for any a and b. Show that f is integrable in $[0,1]$.

(4) Let f be integrable on $(-\infty,\infty)$, and let $h > 0$ be fixed. Show that

$$\int_{-\infty}^{\infty} \left(\frac{1}{2h} \int_{x-h}^{x+h} f(y)\,dy \right) dx = \int_{-\infty}^{\infty} f(x)\,dx.$$

(5) Let $\mathcal{R}$ denote the set of all rectangles in $\mathbb{R}^2$ that contain the origin, and with sides parallel to the coordinate axis. Consider the maximal operator associated to the family, i.e.,

$$f_{\mathcal{R}}^*(x) = \sup_{R\in\mathcal{R}} \fint_R |f(x-y)|dy.$$

Then show that

(a) $f \mapsto f_{\mathcal{R}}^*$ does not satisfy the weak type $(1,1)$ inequality

$$m\left(\{x : f_{\mathcal{R}}^*(x) > \alpha\}\right) \le \frac{A}{\alpha}\|f\|_{L^1(\mathbb{R}^n)},$$

for all $\alpha > 0$, for all $f \in L^1(\mathbb{R}^2)$ and for some constant $A > 0$.

(b) Using this, one can show that there exists $f \in L^1(\mathbb{R})$ so that for $R \in \mathcal{R}$

$$\limsup_{\operatorname{diam}(R)\to 0} \fint_R f(x-y)\,dy = \infty,$$

for a.e. x.

(6) Suppose that $\varphi \in L^1(\mathbb{R}^n)$ with $\int_{\mathbb{R}^n} \varphi(x)\,dx = 1$. Set $K_\delta(x) := \delta^{-n}\varphi(x/\delta)$, for $\delta > 0$.

(a) Prove that $\{K_\delta\}_{\delta>0}$ is a family of good kernel.

(b) Assume in addition that φ is bounded and supported in a bounded set. Verify that $\{K_\alpha\}_{\alpha>0}$ is an approximation to the identity.

(7) Let $f \in L^1(\mathbb{R}^n)$ and $\{K_\delta\}_{\delta>0}$ be an approximation to the identity. Show that

$$\lim_{\delta\to 0}(f * K_\delta)(x) = f(x),$$

for all $x \in \mathcal{L}(f)$, and the limit holds for a.e. $x \in \mathbb{R}^n$.

(8) Suppose that $\varphi \geq 0$ is an integrable function on $\mathbb{R}^n$, with $\int_{\mathbb{R}^n} \varphi(y)\, dx = 1$. Consider $K_\delta(y) := \delta^{-n}\varphi(\delta^{-1}y)$. Show that
 (a) $\{K_\delta\}_{\delta>0}$ is a family of good kernels.
 (b) Assume in addition that φ is bounded and supported in a bounded set. Prove that $\{K_\delta\}_{\delta>0}$ is an approximation to the identity.

(9) Let $E \subset \mathbb{R}^n$ be a measurable set, and $p \in (0,\infty)$. Let $\{f_k\}_{k=1}^\infty$ be a sequence of functions in $L^p(E)$ such that $f_k(x)$ converges to $f(x)$ for a.e.$x \in E$ as $k \to \infty$. Suppose that there exists a constant $C_0 > 0$ such that $\int_E |f_k|^p \leq C_0$, for all $k \in \mathbb{N}$. Prove that

$$\lim_{k\to\infty} \int_E \left| |f_k|^p - |f_k - f|^p - |f|^p \right| \, dx = 0.$$

(*Hint: This identity gives the missing term in the Fatou's lemma.*)