# 2023 FALL REAL ANALYSIS (I) @ NYCU APPL. MATH.
# HOMEWORK 4

- Please answer the following questions in details, which means you need to state all theorems and all reasons you have been using.
- Please mark your name, student ID, and question numbers clearly on your answer sheet. The deadline to hand in the exercise is on **December 21, 2023**.

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(1) Let $f : E \to \mathbb{R}$ be a measurable function, for either $p = 1$ or $p = \infty$, with $\frac{1}{p} + \frac{1}{q} = 1$. Then show that

$$\|f\|_p = \sup\left\{ \int_E fg \,\middle|\, \|g\|_q \le 1 \text{ and } \int_E fg \text{ exists} \right\}.$$

(2) Show that the Minkowski's inequality $\|f + g\|_p \le \|f\|_p + \|g\|_p$ fails for $p < 1$.

(3) Prove the integral version of Minkowski's inequality for $1 \le p \le \infty$, and measurable function $f(x, y)$:

$$\left[\int \left(\int |f(x,y)|\, dx\right)^p dy\right]^{1/p} \le \int \left(\int |f(x,y)|^p\, dy\right)^{1/p} dx.$$

(4) If $f \in L^2(0, 2\pi)$, show that

$$\lim_{k\to\infty} \int_0^{2\pi} f(x) \cos kx\, dx = \lim_{k\to\infty} \int_0^{2\pi} f(x) \sin kx\, dx = 0.$$

Prove that the same is true if $f \in L^1(0, 2\pi)$.

(5) Prove the parallelogram law holds for $L^2$:

$$\|f + g\|_{L^2}^2 + \|f - g\|_{L^2}^2 = 2\|f\|_{L^2}^2 + 2\|g\|_{L^2}^2.$$

Is this true for $L^p$ when $p \ne 2$? Give you answer in a detailed way.

(6) Suppose that $f_k \to f$ a.e. and that $f_k, f \in L^p$, $1 < p \le \infty$. If $\|f_k\| \le M < \infty$, show that $\int f_k g \to \int fg$ for all $g \in L^q$ with $1/p + 1/q = 1$. Show that the result is false if $p = 1$.

(7) Let $\mathcal{S}$ denote a subspace of a Hilbert space $\mathcal{H}$. Show that $\left(\mathcal{S}^\perp\right)^\perp$ is the smallest closed subspace of $\mathcal{H}$ that contains $\mathcal{S}$.

(8) An operator $T$ is an **isometry** if $\|Tf\| = \|f\|$, for all $f \in \mathcal{H}$ (a Hilbert space).
   (a) Show that if $T$ is an isometry, then $\langle Tf, Tg\rangle = \langle f, g\rangle$, for all $f, g \in \mathcal{H}$.
   (b) If $T$ is an isometry and $T$ is surjective, then $T$ is unitary and $TT^* = \mathrm{Id}$ (the identity map).
   (c) Given an example of an isometry but not unitary.
   (d) Show that if $T^*T$ is unitary then $T$ is isometry.

(9) Let $\{e_k\}_{k=1}^{\infty}$ denote an orthonormal set in a Hilbert space $\mathcal{H}$. If $\{c_k\}_{k=1}^{\infty}$ is a sequence of positive real numbers such taht $\sum_{k=1}^{\infty} c_k^2 < \infty$, then the set

$$A = \left\{ \sum_{k=1}^{\infty} a_k e_k \, : \, |a_k| \leq c_k \right\} \tag{0.1}$$

is compact in $\mathcal{H}$.