ACADEMIC YEAR 115

CALCULUS I

Lecture Notes

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Learning Focus 1

Functions, Exponentials, and Inverses

1.1 Functions, domains, and composition

Definition 1.1

A function $f : D \to E$ assigns exactly one value $f(x) \in E$ to each $x \in D$. Its range is $f(D) = \{f(x) : x \in D\}$.

The domain is part of a function: simplifying a formula does not restore excluded inputs. Sums and products use the common domain; a quotient also excludes zeros of its denominator.

Definition 1.2

The composition is $(f \circ g)(x) = f(g(x))$, with domain $\{x \in \operatorname{dom} g : g(x) \in \operatorname{dom} f\}$.

A graph passes the vertical line test when each vertical line meets it at most once.

Notes

Example 1.1

Domains survive simplification

Let $f(u) = \sqrt{u-1}$ and $g(x) = x^2 - 3$. Find $f \circ g$ and $g \circ f$, including their domains. Then find the domain of $h(x) = \ln\big((x-1)/(x+2)\big)$.

Solution

1.2 Exponential functions

Definition 1.3

For $a > 0$, the exponential function is $x \mapsto a^x$. If $a \neq 1$, its domain is $\mathbb{R}$ and its range is $(0, \infty)$. It is strictly increasing for $a > 1$ and strictly decreasing for $0 < a < 1$.

The basic identities are $a^{x+y} = a^x a^y$, $a^{x-y} = a^x / a^y$, and $(a^x)^r = a^{rx}$. The natural base is $e \approx 2.71828$.

Exponential models

For $C > 0$ and real k, in $A(t) = Ce^{kt}$, the ratio $A(t+s)/A(t) = e^{ks}$ depends on the elapsed time s, not on the starting time t.

Notes

Example 1.2

Recovering an exponential function

A positive function has the form $A(t) = Ce^{kt}$ and satisfies $A(1) = 12$, $A(4) = 96$. Find C and k, solve $A(t) = 300$, and show that its doubling time is independent of the starting time.

Solution

1.3 One-to-one functions and their inverses

Definition 1.4

A function is one-to-one if $f(x_1) = f(x_2)$ implies $x_1 = x_2$. For such a function, $f^{-1} : f(D) \to D$ is defined by $f^{-1}(y) = x$ precisely when $f(x) = y$.

Strict increase means $f(x_1) < f(x_2)$ whenever $x_1 < x_2$; strict decrease reverses this inequality.

Theorem 1.1

A strictly monotone function is one-to-one. Its inverse satisfies $f^{-1}(f(x)) = x$ for $x \in D$ and $f(f^{-1}(y)) = y$ for $y \in f(D)$.

The inverse graph is the reflection in $y = x$. The notation f^{-1} does not mean $1/f$.

Proof and notes

Example 1.3

An inverse hidden in a radical

Let $F(x) = x + \sqrt{x^2 + 1}$ for $x \in \mathbb{R}$. Show that $F(x) > 0$, determine its range, and find F^{-1}. Verify that the proposed inverse really works.

Solution

1.4 Logarithms and their laws

Definition 1.5

For $a > 0$, $a \neq 1$, $\log_a x = y$ means $a^y = x$. Its domain is $(0, \infty)$; $\ln x = \log_e x$.

Theorem 1.2

For $x, y > 0$ and $r \in \mathbb{R}$,

$$\log_a(xy) = \log_a x + \log_a y, \quad \log_a(x/y) = \log_a x - \log_a y,$$

$$\log_a(x^r) = r \log_a x, \quad \log_a x = \frac{\ln x}{\ln a}.$$

Supporting review. For $c > 0$, $f(x - c)$ shifts a graph right by c, while $f(x) + c$ shifts it upward Reflections are $-f(x)$ and $f(-x)$.

Proof and notes

Example 1.4

Logarithmic equations and admissible roots

Solve $\ln(x - 1) + \ln(x + 2) = \ln 12$. Explain why solving the resulting polynomial is not by itself a complete solution.

Solution

Exercises 1

Part A Standard Problems

Exercise 1.1

Find the domain of $\sqrt{3-x}/(x^2-4)$.

Exercise 1.2

Let $f(x) = \ln x$ and $g(x) = 1 - x^2$. Find both compositions and their domains.

Exercise 1.3

Solve $4^{2x-1} = 8^{x+2}$.

Exercise 1.4

Find the inverse of $f(x) = (3x-2)/(x+4)$, with its domain and range.

Exercise 1.5

Restrict $f(x) = x^2 - 6x + 5$ to $[3, \infty)$. Find its range and inverse.

Exercises 1

Part A Standard Problems

Exercise 1.6

Solve $\log_2(x-1)+\log_2(x-3)=3$.

Exercise 1.7

For $x>0$, $x\neq 2$, expand $\ln\left(x^3\sqrt{x+1}/(x-2)^4\right)$ without making its domain smaller.

Exercise 1.8

Solve $\ln|x-2|=1$ and state all exclusions.

Exercise 1.9

Find the domain and range of $y=3-2\sqrt{x+1}$. Describe the graph transformations.

Exercise 1.10

A model is $P(t)=P_0e^{kt}$, with $P(0)=800$ and $P(5)=1000$. Find k and the time when $P=1600$

Exercises 1

Part A Standard Problems

Exercise 1.11

Determine whether $f(x) = x^3 - x$ is one-to-one on $\mathbb{R}$, without derivatives.

Exercise 1.12

Solve $e^{2x} - 5e^x + 6 = 0$.

Exercises 1

Part B Challenge Problems

Exercise 1.13

For any $f : \mathbb{R} \to \mathbb{R}$, prove that there is a unique decomposition $f = u + v$ with u even and v odd Give formulas for u and v.

Exercise 1.14

Let $f : A \to B$ and $g : B \to C$. If $g \circ f$ is one-to-one, prove that f is one-to-one. Must g be one-to-one on all of B?

Exercises 1

Part B Challenge Problems

Exercise 1.15

For real $a \neq 0$, let $F_a(x) = (x+a)/(x-1)$ on $\mathbb{R} \setminus \{1\}$. Determine which values of a make F_a a bijection of this domain onto itself with $F_a \circ F_a$ equal to the identity. Treat the exceptional constant case separately.

Exercise 1.16

For $a > 0$, find the range and inverse of $G_a(x) = \ln\left(x + \sqrt{x^2 + a^2}\right)$.

Exercise 1.17

Solve $\log_x 16 = \log_4 x$, with all base restrictions.

Exercises 1

Part B Challenge Problems

Exercise 1.18

Determine the domain of $H(x) = \ln\Big(\ln((x+1)/(x-1))\Big)$.

Exercise 1.19

Let $f(x) = ax + b$, with $a \neq 0$. Classify all such functions for which $f(f(x)) = x$ on $\mathbb{R}$.

Exercise 1.20

For $b > 1$, solve $b^x + b^{-x} = c$ in real x for every real parameter c.

Learning Focus 2

Limits and Continuity

2.1 The precise meaning of a limit

Definition 2.1

Assume f is defined whenever $0 < |x - a| < r$ for some $r > 0$ (a punctured interval about a). We write $\lim_{x \to a} f(x) = L$ if, for every $\varepsilon > 0$, some $\delta > 0$ satisfies

$$0 < |x - a| < \delta \implies |f(x) - L| < \varepsilon.$$

For $x \to a^+$ replace $0 < |x - a| < \delta$ by $0 < x - a < \delta$; for a^- use $0 < a - x < \delta$.

Theorem 2.1

A two-sided limit is L if and only if both one-sided limits are L. A finite limit, when it exists, is unique.

Proof and notes

2.2 Limit laws and algebraic structure

Theorem 2.2

If $f(x) \to L$ and $g(x) \to M$, then $f + g \to L + M$, $cf \to cL$, and $fg \to LM$. If $M \neq 0$, then $f/g \to L/M$.

Polynomials admit direct substitution; rational functions do so where their denominator is nonzero For $0/0$, factor, rationalize, or combine fractions before taking limits.

Theorem 2.3: Squeeze theorem

If $g \leq f \leq h$ near a and $g, h \to L$, then $f \to L$. In radian measure,

$$\lim_{x \to 0} \frac{\sin x}{x} = 1, \quad \lim_{x \to 0} \frac{1 - \cos x}{x} = 0.$$

Proof and notes

Example 2.1

Two limiting mechanisms

Evaluate $\lim_{x \to 0} \dfrac{\sin(5x)}{\sqrt{1 + 4x} - 1}$ without using derivatives. State the standard limits that justify the calculation.

Solution

Example 2.2

A nonlinear epsilon–delta proof

Prove directly from Definition 2.1 that $\lim_{x\to 1}(x^2+x)=2$. Give an explicit choice of δ for each $\varepsilon>0$.

Solution

2.3 Continuity and intermediate values

Definition 2.2

A function is continuous at a if it is defined there and $\lim_{x\to a} f(x)=f(a)$. At an endpoint, continuity is one-sided. Algebraic operations and compositions preserve continuity wherever defined.

Theorem 2.4: Intermediate value theorem

If f is continuous on $[a,b]$ and N lies between $f(a)$ and $f(b)$, then $f(c)=N$ for some $c\in[a,b]$. If N lies strictly between the endpoint values, c can be taken in (a,b).

A jump, a removable hole, an infinite limit, and persistent oscillation are different behaviors.

Proof and notes

Example 2.3

Continuity at two junctions

Determine a, b so that the following function is continuous on $\mathbb{R}$:

$$f(x) = \begin{cases} ax + b, & x < 1, \\ x^2 + 1, & 1 \le x < 2, \\ 3x + a, & x \ge 2. \end{cases}$$

Solution

2.4 Infinite limits and asymptotes

Definition 2.3

The statement $f(x) \to +\infty$ as $x \to a$ means that, for every $M > 0$, some $\delta > 0$ makes $f(x) > M$ whenever $0 < |x - a| < \delta$. For $x \to +\infty$, replace this condition by $x > R$ for some R.

A finite limit $\lim_{x \to +\infty} f(x) = L$ means that, for every $\varepsilon > 0$, there is R such that $x > R$ implies $|f(x) - L| < \varepsilon$. For $x \to -\infty$, use $x < -R$ with $R > 0$. A vertical asymptote $x = a$ requires an infinite one-sided limit. A horizontal asymptote $y = L$ requires a finite limit $f(x) \to L$ at one end of the real line.

Degree comparison

For a rational function, divide by the highest denominator power. Remember $\sqrt{x^2} = |x|$, especially as $x \to -\infty$.

Notes

Example 2.4

Existence and uniqueness before derivatives

Prove that $x^3 + x - 1 = 0$ has exactly one real root and that this root lies in $(0, 1)$. Do not use derivatives.

Solution

Exercises 2

Part A Standard Problems

Exercise 2.1

Evaluate $\lim_{x \to 3} (x^2 - 9)/(x^2 - x - 6)$.

Exercise 2.2

Evaluate $\lim_{h \to 0} (\sqrt{4 + h} - 2)/h$.

Exercises 2

Part A Standard Problems

Exercise 2.3

Evaluate $\lim_{x\to 0}(1-\cos 4x)/x^2$, without L'Hôpital's rule.

Exercise 2.4

Evaluate $\lim_{x\to 0} x\sin(1/x^2)$.

Exercise 2.5

Find the one-sided limits at 0 of $|x|/x$. Can any value assigned at 0 make it continuous?

Exercise 2.6

Give an epsilon–delta proof that $\lim_{x\to 3}(5x-2)=13$.

Exercise 2.7

Find a so that $f(x)=(x^2-1)/(x-1)$ for $x\neq 1$ and $f(1)=a$ is continuous.

Exercises 2

Part A Standard Problems

Exercise 2.8

Find all vertical and horizontal asymptotes of $(3x^2+1)/(x^2-4)$.

Exercise 2.9

Evaluate $\lim_{x\to-\infty} \sqrt{x^2+4}/x$.

Exercise 2.10

Show that $\cos x = x$ has a solution in $(0,1)$ using continuity alone.

Exercise 2.11

Evaluate $\lim_{x\to\infty} (\sqrt{x^2+6x} - x)$.

Exercise 2.12

For $x \neq 0$ put $f(x) = \sin(3x)/(2x)$. Find a continuous extension to 0 and justify it.

Exercises 2

Part B Challenge Problems

Exercise 2.13

For $a > 0$, give an explicit epsilon–delta proof of $\lim_{x\to a} 1/x = 1/a$.

Exercise 2.14

Define $f_\alpha(0) = 0$ and $f_\alpha(x) = |x|^\alpha \sin(1/x)$ for $x \neq 0$. For which real α is f_α continuous at 0?

Exercise 2.15

Suppose f is continuous on $[0, 1]$ and maps this interval into itself. Prove that $f(c) = c$ for some $c \in [0, 1]$.

Exercises 2

Part B Challenge Problems

Exercise 2.16

A continuous function on an interval takes only rational values. Prove that it is constant.

Adapted from Courant–John, *Introduction to Calculus and Analysis*, Vol. I, Ch. 1 problems, Sec. 1.2d, printed p. 109.

Exercise 2.17

Suppose $f : \mathbb{R} \to \mathbb{R}$ is continuous at 0 and $f(x + y) = f(x) + f(y)$. Prove that $f(x) = xf(1)$ for all real x.

Adapted from Courant–John, *Introduction to Calculus and Analysis*, Vol. I, Ch. 1 problems, Sec. 1.2d, Problem 5, printed p 109.

Exercise 2.18

Let f be increasing (not necessarily strictly) on $[a, b]$ and have the intermediate value property Prove that f is continuous, including one-sided continuity at the endpoints.

Adapted from Courant–John, *Introduction to Calculus and Analysis*, Vol. I, Ch. 1 problems, Sec. 1.2e, Problem 1, printed p 110.

Exercises 2

Part B Challenge Problems

Exercise 2.19

Suppose f is continuous on $[a, b]$, $a < b$, and $f(a) = f(b)$. Prove that some $c \in [a, (a+b)/2]$ satisfies $f(c) = f(c + (b-a)/2)$.

Exercise 2.20

Define $T(x) = 0$ for irrational x and $T(p/q) = 1/q$ for rational p/q in lowest terms, $q > 0$. Prove that T is continuous at every irrational number and discontinuous at every rational number. Hint in a bounded interval, only finitely many reduced fractions have denominator at most a fixed integer.

Adapted from Courant–John, *Introduction to Calculus and Analysis*, Vol. I, Ch. 1 problems, Sec. 1.2d, Problem 4(b), printed p. 109.

Learning Focus 3

Derivatives and Basic Rules

3.1 The derivative and the tangent line

The average rate of change from a to $a+h$ is $[f(a+h)-f(a)]/h$.

Definition 3.1: Derivative

The derivative at an interior point a is the finite limit

$$f'(a) = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h} = \lim_{x\to a} \frac{f(x)-f(a)}{x-a},$$

when it exists. The tangent line is $y = f(a) + f'(a)(x-a)$.

The function $x \mapsto f'(x)$ is the derivative function. Repeated derivatives are denoted by f'', $f^{(3)}$, and $f^{(n)}$. An infinite difference-quotient limit is not a finite derivative. A function is of class C^1 on an interval if it is continuously differentiable there; on a closed interval the derivative extends continuously to the endpoints.

Notes

Example 3.1

A derivative from first principles

For $a > 0$, use the definition to find the derivative of $f(x) = \sqrt{x}$ at a. Then find its tangent line at $(4, 2)$.

Solution

3.2 Differentiability is stronger than continuity

Theorem 3.1: Differentiability implies continuity

If f is differentiable at a, then it is continuous at a.

For a piecewise function, first match the function values at a junction. Then compute the one-sided difference quotients. Equality of the one-sided derivatives, together with continuity, establishes differentiability for the smooth pieces used here. A corner, a cusp, a vertical tangent, or a discontinuity can prevent differentiability. A continuous function need not be differentiable.

Proof and notes

Example 3.2

A differentiable junction

Find a, b so that

$$f(x) = \begin{cases} ax + b, & x \le 1, \\ x^2 + 1, & x > 1 \end{cases}$$

is differentiable at 1. Verify your answer directly at the junction.

Solution

3.3 Constants, powers, and exponentials

Theorem 3.2: Basic derivative rules

At points where the derivatives exist,

$$(cf+g)' = cf' + g', \quad (x^n)' = nx^{n-1}, \quad (e^x)' = e^x.$$

Here c is constant and n is a positive integer. Also $(c)' = 0$.

The exponential normalization is $\lim_{h\to 0}(e^h - 1)/h = 1$. More generally,

$$(a^x)' = a^x \ln a \quad (a > 0), \quad (x^r)' = rx^{r-1} \quad (x > 0, r \in \mathbb{R}).$$

The general-base and real-power formulas will be justified by the chain rule and logarithmic differentiation in Chapter 4.

Proof and notes

3.4 Product and quotient rules

Theorem 3.3: Product and quotient

If f, g are differentiable at x, then

$$(fg)'(x) = f'(x)g(x) + f(x)g'(x).$$

If also $g(x) \neq 0$, then

$$\left(\frac{f}{g}\right)'(x) = \frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2}.$$

Neither rule is obtained by multiplying or dividing the two derivatives. Factoring a derivative is often more useful than expanding it.

Proof and notes

Example 3.3

A product inside a quotient

Differentiate

$$h(x) = \frac{x^2 e^x}{x+1}, \quad x \neq -1.$$

Give a factored answer and find the tangent line at $x = 1$.

Solution

3.5 Position, velocity, acceleration, and speed

Definition 3.2: Motion on a line

For position $s(t)$, velocity is $v(t) = s'(t)$, acceleration is $a(t) = v'(t)$, and speed is $|v(t)|$.

Positive and negative velocity describe opposite directions. Total distance is the sum of absolute changes in position over intervals on which the direction is fixed; displacement is only the final position minus the initial position. Where $v \neq 0$, speed increases when v and a have the same sign and decreases when they have opposite signs. Split the time interval at every change of direction

Notes

Example 3.4

A complete motion analysis

A particle has position $s(t) = t^3 - 6t^2 + 9t$ meters for $0 \leq t \leq 5$ seconds. Find its velocity and acceleration, intervals of motion in each direction, displacement, total distance, and intervals of increasing speed.

Solution

Exercises 3

Part A Standard Problems

Exercise 3.1

Use the definition to differentiate $f(x) = 1/x$ at any $x \neq 0$.

Exercise 3.2

Use the definition to differentiate $x^2 - 3x + 2$.

Exercise 3.3

Find the tangent line to $y = x^3 - 2x$ at $x = -1$.

Exercise 3.4

Differentiate $7x^{4/3} - 5x^{-1/2} + 3e^x$ for $x > 0$.

Exercise 3.5

Differentiate $(x^2 + 1)(x^3 - 4x + 2)$ without expanding the original product.

Exercises 3

Part A Standard Problems

Exercise 3.6

Differentiate $(x^2+1)/(x^2-1)$ and state its domain.

Exercise 3.7

For $f(x) = x^4 - 3x^2 + e^x$, find $f''(x)$ and $f^{(3)}(0)$.

Exercise 3.8

If $f(2) = 3$, $f'(2) = -1$, $g(2) = 4$, and $g'(2) = 5$, find $(fg)'(2)$ and $(f/g)'(2)$.

Exercise 3.9

Find a, b so that $f(x) = ax + b$ for $x < 2$ and $f(x) = x^2$ for $x \geq 2$ is differentiable at 2.

Exercise 3.10

Determine the derivative of $|x-2|$ wherever it exists. Explain what happens at 2.

Exercises 3

Part A Standard Problems

Exercise 3.11

A particle has $s(t) = t^3 - 9t^2 + 24t$ on $[0, 5]$. Find its rest times and intervals of negative velocity

Exercise 3.12

If $f(1) = 2$ and $f'(1) = 3$, find the derivative at 1 of $x^2 f(x)$.

Exercises 3

Part B Challenge Problems

Exercise 3.13

Define $f(0) = 0$ and $f(x) = x^2 \sin(1/x)$ for $x \neq 0$. Prove directly that $f'(0) = 0$. Does the oscillation prevent differentiability?

Exercise 3.14

For real α, let $f_\alpha(0) = 0$ and $f_\alpha(x) = |x|^\alpha \sin(1/x)$ for $x \neq 0$. Determine exactly when $f'_\alpha(0)$ exists

Exercises 3

Part B Challenge Problems

Exercise 3.15

Using only difference quotients, prove that the derivative of an even differentiable function on $\mathbb{R}$ is odd. What happens for an odd function?

Exercise 3.16

Suppose the symmetric quotient $[f(a+h)-f(a-h)]/(2h)$ has a finite limit. Must $f'(a)$ exist? Give a counterexample and then prove the implication in the other direction.

Exercise 3.17

Suppose f is continuous at a with $f(a)=0$. Prove that $g(x)=(x-a)f(x)$ is differentiable at a, and find $g'(a)$. Is differentiability of f needed?

Exercises 3

Part B Challenge Problems

Exercise 3.18

Let f be differentiable at a and $f(a) = 0$. Prove that $|f|$ is differentiable at a if and only if $f'(a) = 0$.

Exercise 3.19

A polynomial p has a zero at a, so $p(x) = (x - a)q(x)$. Show that $p'(a) = q(a)$, and deduce that $p'(a) = 0$ precisely when $(x - a)^2$ divides p.

Exercise 3.20

A differentiable function satisfies $f(a) = b$ and $f'(a) = m$. Evaluate $\lim_{h \to 0}[f(a + 2h) - f(a - 3h)]/h$

Learning Focus 4

Chain, Implicit, and Inverse Differentiation

4.1 Trigonometric derivatives

Theorem 4.1: Trigonometric rules

Using radian measure,

$$\begin{aligned} (\sin x)' &= \cos x, & (\cos x)' &= -\sin x, \\ (\tan x)' &= \sec^2 x, & (\cot x)' &= -\csc^2 x, \\ (\sec x)' &= \sec x \tan x, & (\csc x)' &= -\csc x \cot x. \end{aligned}$$

Each formula holds on the natural domain of the function. The limits $\sin h/h \to 1$ and $(\cos h - 1)/h \to 0$ are the starting point, not consequences of these derivative formulas.

Proof and notes

4.2 The chain rule

Theorem 4.2: Chain rule

If g is differentiable at a and f is differentiable at $g(a)$, then

$$(f \circ g)'(a) = f'(g(a))g'(a).$$

Identify the outer function and then work inward. In particular,

$$\frac{d}{dx}e^{u(x)} = e^{u(x)}u'(x), \quad \frac{d}{dx}a^{u(x)} = a^{u(x)}\ln(a)u'(x).$$

The latter identity follows by writing $a^{u(x)} = e^{u(x)\ln a}$.

Proof and notes

Example 4.1

The chain rule from numerical data

Suppose f, g are differentiable, with $f(2) = 3$, $f'(2) = -1$, $g(1) = 2$, and $g'(1) = 4$. For

$$H(x) = \ln\Big(1 + [f(g(x))]^2\Big),$$

find $H'(1)$ without finding a formula for either f or g.

Solution

4.3 Logarithms, real powers, and variable exponents

Theorem 4.3: Logarithmic derivatives

For differentiable u,

$$(\ln|u|)' = \frac{u'}{u} \quad (u \neq 0), \quad (\log_a u)' = \frac{u'}{u \ln a} \quad (u > 0,\ a > 0,\ a \neq 1).$$

For $x > 0$, writing $x^r = e^{r \ln x}$ gives $(x^r)' = rx^{r-1}$. For positive differentiable u and differentiable v,

$$(u^v)' = u^v \left(v' \ln u + v \frac{u'}{u} \right).$$

Logarithmic differentiation simplifies products, quotients, and variable exponents. State the domain before taking logarithms.

Proof and notes

Example 4.2

A variable exponent and a quotient

For $x > 0$, differentiate

$$y = \frac{(1 + x^2)^{\sin x}}{\sqrt{x}}.$$

Give the derivative in factored form and compute $y'(1)$.

Solution

4.4 Implicit differentiation

When a curve is described by an equation in x and y, regard y as a differentiable function of x Differentiate every term, including the inner factor y' whenever y is differentiated. For a second derivative, differentiate the first differentiated equation again before substituting numerical data Terms such as yy' produce $(y')^2 + yy''$. These calculations apply to a differentiable branch. A zero coefficient of y' is a warning that the branch or tangent needs separate analysis, not permission to divide by zero.

Derivation and notes

Example 4.3

Two implicit derivatives

The curve $x^2 + xy + y^2 = 7$ passes through $(1, 2)$. Find the tangent line there and compute y'' at that point.

Solution

4.5 Inverse derivatives and inverse trigonometric functions

Theorem 4.4: Derivative of an inverse

Let f be continuous and strictly monotone on an open interval I. If f is differentiable at $a \in I$ and $f'(a) \neq 0$, its inverse g is differentiable at $b = f(a)$ and $g'(b) = 1/f'(a)$.

$$(\arcsin x)' = \frac{1}{\sqrt{1-x^2}}, \quad (\arccos x)' = -\frac{1}{\sqrt{1-x^2}} \quad (|x| < 1),$$
$$(\arctan x)' = \frac{1}{1+x^2} \qquad (x \in \mathbb{R}).$$

The branches are $\arcsin x \in [-\pi/2, \pi/2]$, $\arccos x \in [0, \pi]$, and $\arctan x \in (-\pi/2, \pi/2)$.

Proof and notes

Example 4.4

A second derivative of an inverse

Let $f(x) = x^3 + x$ and $g = f^{-1}$. Find $g'(2)$ and $g''(2)$ without solving explicitly for g. You may use that f is a strictly increasing bijection of $\mathbb{R}$.

Solution

Exercises 4

Part A Standard Problems

Exercise 4.1

Differentiate $\sin(2x^3 - 5x)$.

Exercise 4.2

Differentiate $\sec^3(4x)$ on its domain.

Exercise 4.3

Differentiate $e^{x^2 \sin x}$.

Exercises 4

Part A Standard Problems

Exercise 4.4

For $x^3 + y^3 = 6xy$, find y' at points where $y^2 - 2x \neq 0$.

Exercise 4.5

Differentiate $\ln((x^2 + 1)/\sqrt{3x - 2})$ and state the real domain.

Exercise 4.6

Differentiate $\arctan \sqrt{1 + x^2}$.

Exercise 4.7

Differentiate $\arcsin(2x)$ and state where the derivative is finite.

Exercise 4.8

For $x > 0$, differentiate x^x.

Exercises 4

Part A Standard Problems

Exercise 4.9

Differentiate $(\sin x)^{\cos x}$ on an interval where $\sin x > 0$.

Exercise 4.10

Let $f(x) = x^3 + x + 1$. Find $(f^{-1})'(3)$.

Exercise 4.11

Find y'' for the branch of $x^2 + y^2 = 25$ where $y \neq 0$.

Exercise 4.12

Differentiate $2^{x^2} e^{-3x}$ and factor the result.

Exercises 4

Part B Challenge Problems

Exercise 4.13

Suppose f is twice continuously differentiable and strictly monotone near a, with $f'(a) \neq 0$. For $g = f^{-1}$ prove

$$g''(f(a)) = -\frac{f''(a)}{[f'(a)]^3}.$$

Exercise 4.14

For $x > 0$, differentiate x^{x^x}, interpreting exponentiation from the top down: $x^{(x^x)}$.

Exercise 4.15

Use differentiation and the principal branches to establish $\arctan x + \arctan(1/x) = \pi/2$ for $x > 0$ and $-\pi/2$ for $x < 0$. You may use the zero-derivative consequence of the mean value theorem from Chapter 6.

Exercises 4

Part B Challenge Problems

Exercise 4.16

Find all horizontal and vertical tangents to $x^2 + xy + y^2 = 7$. Justify the cases where solving for y' is invalid.

Exercise 4.17

Let f be twice differentiable near a and g twice differentiable near $f(a)$. Derive the second derivative of $g(f(x))$.

Exercise 4.18

Find every point where $h(x) = \arcsin(\sin x)$ is differentiable, and give its derivative.

Exercises 4

Part B Challenge Problems

Exercise 4.19

Suppose f is differentiable on $(0, \infty)$ and $f(xy) = f(x) + f(y)$. Show that $f'(x) = f'(1)/x$. Deduce $f(x) = f'(1) \ln x$, using the zero-derivative theorem from Chapter 6.

Exercise 4.20

Let $f(0) = 0$ and $f(x) = x^2 \sin(1/x)$ for $x \neq 0$. Compute f' everywhere and prove that f' is not continuous at 0.

Learning Focus 5

Linear Approximation and Differentials

5.1 Local linearity

Definition 5.1: Linearization

If f is differentiable at a, its linearization at a is

$$L(x) = f(a) + f'(a)(x - a).$$

For nearby x, use $f(x) \approx L(x)$, not an equality. The precise first-order statement is

$$f(a + h) = f(a) + f'(a)h + h\eta(h), \quad \eta(h) \to 0.$$

Choose a base point where $f(a)$ and $f'(a)$ are known exactly. A small input change alone does not give a numerical error bound unless further information about f is available.

Notes

Example 5.1

Approximation and an exact error identity

Use linearization at 4 to estimate $\sqrt{4.1}$. Then show, without a decimal calculator, whether the estimate is too high or too low.

Solution

5.2 Differentials and relative change

Definition 5.2: Differentials

For $y = f(x)$, write $dx = \Delta x$ and $dy = f'(x)\,dx$. The actual change is $\Delta y = f(x+\Delta x) - f(x)$; for small Δx, $\Delta y \approx dy$.

When $f(x) \neq 0$, the relative differential is

$$\frac{dy}{y} = \frac{f'(x)}{f(x)}\,dx.$$

For $y = Cx^p > 0$, $dy/y = p\,dx/x$. An estimated maximum error from differentials is a first-order estimate, not automatically a rigorous bound on the actual error.

Notes

Example 5.2

Linearizing a logarithmic quotient

For $x > 1$, let $H(x) = \ln((x^2 + 1)/(x - 1))$. Find its linearization at $x = 2$ and use it to estimate $H(2.01)$.

Solution

Example 5.3

A measurement error

A sphere's radius is measured as 10.00 cm, with possible error at most 0.02 cm. Estimate the maximum absolute error and percentage error in its volume. Explain the status of the estimate.

Solution

5.3 Optional applications: related rates and exponential models

Optional topic: Related rates

Write an equation among the time-dependent quantities, differentiate with respect to t, and only then substitute the data for the chosen instant. Keep signs and units.

An exponential model $P(t) = P_0 e^{kt}$ has $P' = kP$. If $k > 0$, its doubling time is $\ln 2/k$; if $k < 0$, its half-life is $\ln 2/|k|$.

Optional topic: Hyperbolic functions

$\sinh x = (e^x - e^{-x})/2$, $\cosh x = (e^x + e^{-x})/2$, and $\tanh x = \sinh x/\cosh x$. Their derivatives are $\cosh x$, $\sinh x$, and $1/\cosh^2 x$, respectively.

Derivation and notes

Example 5.4

A composed function near a convenient point

For $x > 0$, set $F(x) = e^{x^2} \sin(\ln x)$. Find $F'(x)$, its tangent line at $x = 1$, and a linear approximation to $F(1.02)$.

Solution

Exercises 5

Part A Standard Problems

Exercise 5.1

Use linearization to estimate $\sqrt[3]{8.12}$.

Exercise 5.2

Estimate $1/\sqrt{0.98}$ by linearization at 1.

Exercise 5.3

Estimate $\ln(1.03)$ by linearization.

Exercises 5

Part A Standard Problems

Exercise 5.4

Estimate $e^{-0.04}$ using a tangent line.

Exercise 5.5

Find the linearization of $\arctan x$ at 1 and estimate $\arctan(1.02)$.

Exercise 5.6

For $y = x^2$ at $x = 3$, compare dy with Δy when $dx = 0.1$.

Exercise 5.7

A cube has measured side length 20 cm with error at most 0.05 cm. Estimate its volume error and percentage error.

Exercise 5.8

If the radius of a circle increases by approximately 1%, estimate the percentage increase of its area and circumference.

Exercises 5

Part A Standard Problems

Exercise 5.9

Find dy for $y = x^2 \ln x$, $x > 0$, and evaluate it at $x = 1$, $dx = -0.02$.

Exercise 5.10

Optional related rates. A 10 m ladder slides with its foot moving outward at 1 m/s. Find the vertical speed of its top when the foot is 6 m from the wall.

Exercise 5.11

Optional exponential model. A substance has half-life 18 years and initial mass 40 g. Find its mass at time t and at $t = 30$.

Exercise 5.12

Optional hyperbolic functions. Verify $\cosh^2 x - \sinh^2 x = 1$ and differentiate $\ln(\cosh x)$.

Exercises 5

Part B Challenge Problems

Exercise 5.13

For $a > 0$ and $h > -a$, prove the exact identity

$$\sqrt{a} + \frac{h}{2\sqrt{a}} - \sqrt{a+h} = \frac{h^2}{2\sqrt{a}(\sqrt{a+h}+\sqrt{a})^2}.$$

What does it say about the tangent approximation?

Exercise 5.14

Suppose $f(a) = 2$, $f'(a) = 3$, $g(a) = 4$, and $g'(a) = -1$. Find the first-order approximations to $f(a+h)g(a+h)$ and $f(a+h)/g(a+h)$.

Exercise 5.15

Let f have a differentiable inverse near $b = f(a)$, with $f'(a) \neq 0$. Derive a first-order approximation for $f^{-1}(b+k)$. Apply it to $f(x) = x^3 + x$ near $b = 2$.

Exercises 5

Part B Challenge Problems

Exercise 5.16

Assume f is differentiable at a. Prove

$$\lim_{h\to 0} \frac{f(a+h)+f(a-h)-2f(a)}{h} = 0.$$

Does this assert that the numerator divided by h^2 has a limit?

Exercise 5.17

For $x > 0$, let $P(x) = x^\alpha$ with fixed real α. Use differentiability at 1 to evaluate $\lim\limits_{h\to 0}[(1+h)^\alpha - 1]/h$ Then evaluate the limit of $[(1+2h)^\alpha - (1-h)^\alpha]/h$.

Exercise 5.18

For $h > -1$, compare the reciprocal $1/(1+h)$ with its linearization $1-h$. Obtain an exact formula for the error and a bound when $|h| \le 1/2$.

Exercises 5

Part B Challenge Problems

Exercise 5.19

A measured positive length x is used to compute $F(x) = x^p e^{kx}$. Derive its relative differential For $p = 2$, $k = -1$, identify a base point where the first-order relative error vanishes, and explain what that does and does not imply.

Exercise 5.20

Optional related rates. A cone-shaped tank has height 6 m and top radius 3 m. Water enters at 2 m^3/min. Find dh/dt when the water depth is 2 m, and explain why the rate is not constant.

Learning Focus 6

Extrema and the Mean Value Theorem

6.1 Absolute extrema and critical points

Definition 6.1: Extrema

An absolute maximum satisfies $f(c) \geq f(x)$ for all inputs in the domain. A local maximum uses only nearby inputs; minima reverse the inequality. A critical point is an interior domain point where $f' = 0$ or f' fails to exist.

Theorem 6.1: Extreme value theorem

A continuous function on a closed bounded interval attains both its maximum and its minimum.

Theorem 6.2: Fermat's theorem

At an interior local extremum where f is differentiable, $f'(c) = 0$.

For absolute extrema on $[a, b]$, compare all interior critical points and both endpoints.

Proof and notes

Example 6.1

Endpoints matter

Find the absolute maximum and minimum of $f(x) = x + 4/x$ on $[1, 5]$. Explain why testing only $f'(x) = 0$ is insufficient.

Solution

6.2 Rolle's theorem and the mean value theorem

Theorem 6.3: Rolle

If f is continuous on $[a, b]$ with $a < b$, differentiable on (a, b), and $f(a) = f(b)$, then $f'(c) = 0$ for some $a < c < b$.

Theorem 6.4: Mean value theorem

If f is continuous on $[a, b]$ with $a < b$ and differentiable on (a, b), then some $c \in (a, b)$ satisfies

$$f(b) - f(a) = f'(c)(b - a).$$

The conclusion concerns one interior point, not every point. Verify both continuity and differentiability before using it.

Proof and notes

6.3 Derivative bounds give global information

Theorem 6.5: Consequences on an interval

If f is continuous on an interval and differentiable in its interior, then $f' > 0$ implies strict increase, and $f' < 0$ implies strict decrease. If $f' = 0$ throughout the interior, f is constant. If $|f'| \leq M$, then

$$|f(y) - f(x)| \leq M|y - x|.$$

An interval hypothesis is essential: disconnected domains can have unrelated constants. To show a zero is unique, combine existence from continuity with uniqueness from monotonicity or Rolle's theorem.

Proof and notes

Example 6.2

Existence and uniqueness

> Prove that $x^3 + 3x - 1 = 0$ has exactly one real root, and locate it in an interval of length 1.

Solution

Example 6.3

A global estimate from a derivative

Prove for all real a, b that

$$|\arctan b - \arctan a| \leq |b - a|.$$

For $a \neq b$, can equality occur?

Solution

6.4 Monotonicity, concavity, and derivative tests

Theorem 6.6: First derivative test

For continuous f near a critical point c, a change of sign of f' from $+$ to $-$ gives a local maximum; $-$ to $+$ gives a local minimum. The same nonzero sign on both sides gives neither.

Concavity upward means the slopes f' are nondecreasing; concavity downward means they are nonincreasing. A continuous graph point where concavity changes is an inflection point.

Theorem 6.7: Second derivative test

If f' exists near c, $f'(c) = 0$, and $f''(c)$ exists, then $f''(c) > 0$ gives a strict local minimum and $f''(c) < 0$ a strict local maximum. Zero is inconclusive.

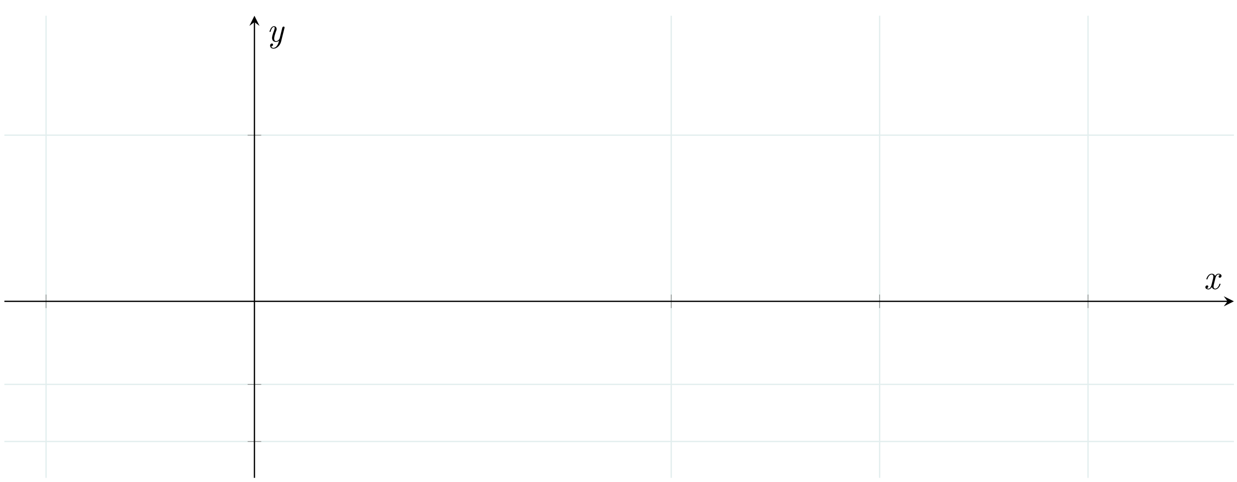
Proof and notes

Example 6.4

A stationary point without an extremum

Analyze $f(x) = x^4 - 4x^3$ on $\mathbb{R}$: critical points, increase and decrease, local and absolute extrema, concavity, and inflection points. Sketch the graph from this information.

Solution



6.5 Optional applications: optimization and Newton iteration

An optimization problem requires a function, a feasible domain, and a check of all candidates including endpoints or limiting behavior. Solving $f' = 0$ alone is not a proof of optimality.

Optional topic: Newton's method

The tangent line to $y = f(x)$ at x_n meets the axis at

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \quad f'(x_n) \neq 0.$$

The method may fail, cycle, or leave the intended domain. A few plausible decimals are not a proof of convergence.

Derivation and notes

Exercises 6

Part A Standard Problems

Exercise 6.1

Find the absolute extrema of $x^3 - 3x + 1$ on $[-2, 2]$.

Exercise 6.2

Verify the mean value theorem for $1/x$ on $[1, 4]$ and find c.

Exercises 6

Part A Standard Problems

Exercise 6.3

Find all c given by Rolle's theorem for $x^3 - x$ on $[-1, 1]$.

Exercise 6.4

If $|f'| \leq 2$ on $(1, 4)$ and f is continuous on $[1, 4]$, prove $|f(4) - f(1)| \leq 6$.

Exercise 6.5

Given $f'(x) = (x-1)^2(x+2)$, determine monotonicity and local extrema without finding f.

Exercise 6.6

Find the local extrema of $x^3 - 6x^2 + 9x + 1$.

Exercise 6.7

Find the concavity and inflection points of $x^3 - 3x^2 + 2$.

Exercises 6

Part A Standard Problems

Exercise 6.8

Classify $x = 0$ for each of x^4, $-x^4$, and x^3. Explain the failure of the second derivative test.

Exercise 6.9

Prove that $e^x = 3 - x$ has exactly one real solution.

Exercise 6.10

Why is $x = 0$ not an inflection point of $f(x) = 1/x$, although the two sides have different concavity?

Exercise 6.11

Optional optimization. Find the largest area of a rectangle with base on the x-axis and upper corners on $y = 12 - x^2$, symmetric about the y-axis.

Exercise 6.12

Optional Newton iteration. Apply two steps to $x^3 - 10 = 0$ starting at $x_0 = 2$.

Exercises 6

Part B Challenge Problems

Exercise 6.13

Let p be a degree-n polynomial, $n \geq 1$, with n distinct real roots. Prove that p' has exactly $n-1$ distinct real roots, one strictly between each consecutive pair of roots of p.

Adapted from David Jerison and MIT, MIT OpenCourseWare 18.01SC, Applications of Differentiation, Problem 2G-4.

Exercise 6.14

Suppose f is three times differentiable on an interval and has four distinct zeros there. Prove that $f^{(3)}$ has a zero between the first and last zeros.

Adapted from David Jerison and MIT, MIT OpenCourseWare 18.01SC, Applications of Differentiation, Problem 2G-5 (one additional derivative).

Exercise 6.15

Prove for $x > 0$ that $e^x > 1 + x + x^2/2$ without using a Taylor series.

Adapted from David Jerison and MIT, MIT OpenCourseWare 18.01SC, Applications of Differentiation, Problem 2G-7(a,b).

Exercises 6

Part B Challenge Problems

Exercise 6.16

For $x > 0$, prove $x/(1+x) < \ln(1+x) < x$.

Exercise 6.17

Determine how many distinct real roots $x^3 - 3x = c$ has, for each real parameter c.

Exercise 6.18

Suppose f is differentiable on $(0, \infty)$ and $f(xy) = f(x) + f(y)$. Prove that $f(x) = C \ln x$ for some constant C.

Exercises 6

Part B Challenge Problems

Exercise 6.19

If f is differentiable and strictly increasing on an interval, must $f'(x) > 0$ everywhere? Prove the best nonnegative conclusion and give a counterexample to strict positivity.

Exercise 6.20

Assume $a < b$ and $M > 0$. Let f be continuous on $[a, b]$, differentiable on (a, b), with $f(a) = f(b) = 0$ and $|f'| \le M$. Prove

$$|f(x)| \le M \min\{x - a, b - x\} \quad (a \le x \le b),$$

and show that the universal constant $1/2$ in $\max |f| \le M(b - a)/2$ cannot be decreased.

Learning Focus 7

L'Hôpital's Rule, Graphs, and Antiderivatives

7.1 A mean value theorem for two functions

Theorem 7.1: Cauchy mean value theorem

If f, g are continuous on $[a, b]$ with $a < b$ and differentiable on (a, b), then some $c \in (a, b)$ satisfies

$$[f(b) - f(a)]g'(c) = [g(b) - g(a)]f'(c).$$

If g' never vanishes on (a, b), then $g(b) \neq g(a)$ and the equation may be divided to give a ratio of differences.

This theorem compares two changing quantities. It supplies the mechanism behind L'Hôpital's rule.

Proof and notes

7.2 L'Hôpital's rule: hypotheses before calculation

Theorem 7.2: Indeterminate quotients

Suppose f, g are differentiable on a one-sided interval approaching an endpoint a (possibly infinite), $g' \neq 0$, and either $f, g \to 0$ or $|g| \to \infty$. If

$$\lim_{x\to a} \frac{f'(x)}{g'(x)} = L \quad \text{with } L \in \mathbb{R} \cup \{-\infty, +\infty\},$$

then $\lim_{x\to a} f(x)/g(x) = L$. The quotient is taken where $g \neq 0$.

A two-sided application requires matching one-sided conclusions. Forms $0 \cdot \infty$, $\infty - \infty$, and variable powers must first be rewritten. Failure of the derivative-ratio limit does not imply failure of the original limit.

Proof and notes

Example 7.1

A cancellation and a variable power

Evaluate, with justification,

$$\text{(a)}\ \lim_{x\to 0}\frac{e^x-1-x}{x^2}, \qquad \text{(b)}\ \lim_{x\to 0}(1+2x)^{1/x}.$$

Do not use Taylor series.

Solution

7.3 From derivatives to a complete graph

For a graph analysis, identify the domain, intercepts and symmetry; then compute endpoint limits and asymptotes, signs of f', and signs of f''. An oblique asymptote $y = mx + b$ means $f(x) - (mx + b) \to 0$ at the specified end. A sign chart is more informative than a list of derivative zeros. Critical points must belong to the domain, and an inflection point must be a continuous graph point with a genuine change of concavity. Do not join branches through a vertical asymptote

Notes

Example 7.2

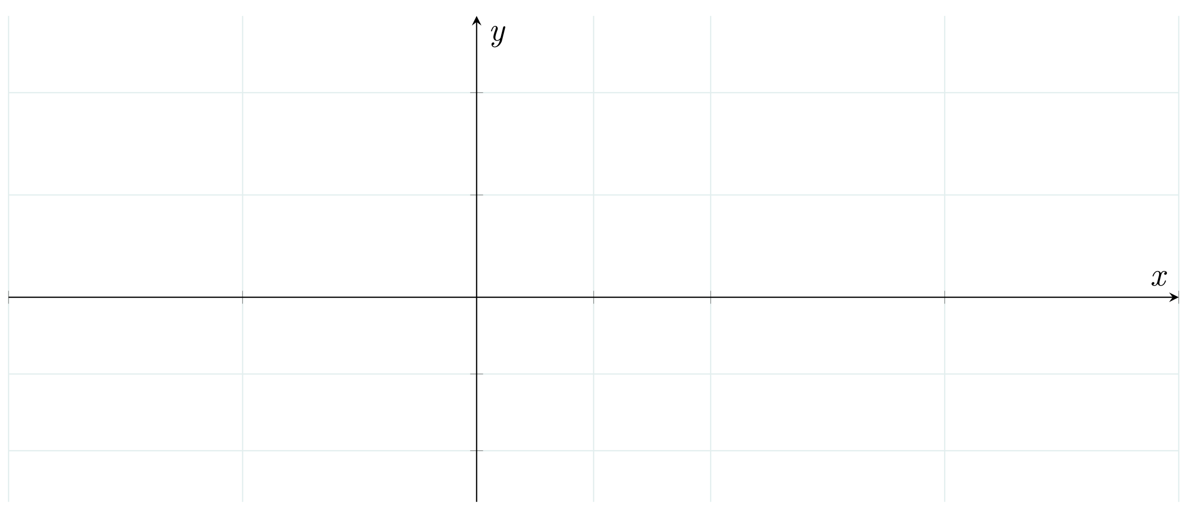
A graph with an oblique asymptote

Analyze and sketch

$$f(x) = \frac{x^2}{x-1}.$$

Find its domain, intercepts, asymptotes, local extrema, concavity, and any inflection points.

Solution



7.4 Antiderivatives and initial conditions

Definition 7.1: Antiderivative

On an interval I, F is an antiderivative of f if $F' = f$ there. The indefinite integral denotes the family $\int f(x)\,dx = F(x) + C$.

Theorem 7.3: Uniqueness up to a constant

Two antiderivatives of the same function on an interval differ by a constant.

$$\begin{aligned} \int x^r\,dx &= \frac{x^{r+1}}{r+1} + C & (r \neq -1), \\ \int \frac{dx}{x} &= \ln|x| + C, & \int e^x\,dx = e^x + C, \\ \int \cos x\,dx &= \sin x + C, & \int \sin x\,dx = -\cos x + C, \\ \int \frac{dx}{1+x^2} &= \arctan x + C. \end{aligned}$$

Use each formula on intervals where it is defined. On a disconnected domain the constants can differ between components.

Proof and notes

Example 7.3

Recovering a function

Find f if $f''(x) = 6x - 4$, $f'(1) = 3$, and $f(0) = 2$.

Solution

Example 7.4

When differentiation of a quotient is unhelpful

Evaluate $\lim_{x\to\infty}(x+\sin x)/x$. Explain why the derivative quotient does not determine this limit.

Solution

Exercises 7

Part A Standard Problems

Exercise 7.1

Evaluate $\lim_{x\to 0}\ln(1+x)/x$.

Exercise 7.2

Evaluate $\lim_{x\to\infty}\ln x/\sqrt{x}$.

Exercise 7.3

Evaluate $\lim_{x\to 0}(\tan x - x)/x^3$.

Exercises 7

Part A Standard Problems

Exercise 7.4

Evaluate $\lim_{x \to 0^+} x \ln x$.

Exercise 7.5

Evaluate $\lim_{x \to 0^+} x^x$.

Exercise 7.6

Evaluate $\lim_{x \to \infty} x(e^{1/x} - 1)$.

Exercise 7.7

Find all antiderivatives of $3x^2 - 4/x$ on $(0, \infty)$.

Exercise 7.8

Solve $F'(x) = \cos x + e^x$ with $F(0) = 2$.

Exercises 7

Part A Standard Problems

Exercise 7.9

Find an antiderivative of $|x|$ on $\mathbb{R}$ that vanishes at 0.

Exercise 7.10

Find the local extrema and concavity of xe^{-x} on $\mathbb{R}$.

Exercise 7.11

Find the oblique asymptote of $(2x^2 + 3x + 4)/(x + 1)$ and the side of the line occupied by each branch.

Exercise 7.12

Find f from $f'' = 2$, $f'(0) = -3$, and $f(2) = 1$.

Exercises 7

Part B Challenge Problems

Exercise 7.13

Determine a, b so that $\lim_{x\to 0}(e^{ax} - 1 - bx)/x^2$ is finite, and find the resulting limit.

Exercise 7.14

Evaluate $\lim_{x\to 0}[\ln(1+x) - x]/x^2$ without a series expansion.

Exercise 7.15

For fixed $a, b > 0$, evaluate $\lim_{x\to 0}(a^x - b^x)/x$ and $\lim_{x\to 0}(a^x - b^x)/(e^x - 1)$.

Exercises 7

Part B Challenge Problems

Exercise 7.16

Evaluate $\lim_{x\to\infty}(1+1/x)^{x+2}$ without invoking a precomputed exponential limit.

Exercise 7.17

Find all C^1 functions on $\mathbb{R}\setminus\{0\}$ whose derivative is $1/x$. Can a single constant always describe the entire family?

Exercise 7.18

For $f_a(x) = x^3/(x^2+a)$ with $a > 0$, find its oblique asymptote and all stationary points. Does the stationary point force an extremum?

Exercises 7

Part B Challenge Problems

Exercise 7.19

Construct differentiable f, g with $f, g \to \infty$ and $f/g \to 1$ as $x \to \infty$, but with f'/g' unbounded Verify every claim.

Exercise 7.20

For fixed real $p > 0$, prove $\ln x/x^p \to 0$ and $x^p/e^x \to 0$ as $x \to \infty$. Do not differentiate a noninteger power an indefinite number of times.

Learning Focus 8

Definite Integrals and the Fundamental Theorem

8.1 Riemann sums and the definite integral

Definition 8.1: Riemann integral

For a partition $a = x_0 < \cdots < x_n = b$, choose $\xi_i \in [x_{i-1}, x_i]$. A bounded function f is Riemann integrable when the sums

$$\sum_{i=1}^{n} f(\xi_i)(x_i - x_{i-1})$$

approach the same finite value as the mesh $\max_i(x_i - x_{i-1}) \to 0$, independently of the tags. This limit is $\int_a^b f(x)\,dx$.

Theorem 8.1: Existence

Every continuous function on $[a, b]$ is Riemann integrable.

The integral is signed accumulation, not always a geometric area. Equal subdivisions are useful but the definition permits unequal subdivisions.

Proof and notes

Example 8.1

Recognizing a limit of sums

Evaluate

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \left(1 + \frac{k}{n}\right)^2 .$$

Identify the interval, subinterval width, and sample points before computing.

Solution

8.2 Algebra and order of integrals

Theorem 8.2: Integral properties

For integrable functions, the integral is linear and additive over adjacent intervals. We set $\int_a^a f = 0$ and $\int_b^a f = -\int_a^b f$. For $a < b$,

$$f \le g \implies \int_a^b f \le \int_a^b g, \quad \left|\int_a^b f\right| \le \int_a^b |f|.$$

In particular, $m \le f \le M$ gives $m(b-a) \le \int_a^b f \le M(b-a)$. For continuous even f, $\int_{-a}^a f = 2\int_0^a f$; for continuous odd f, this integral is zero.

Proof and notes

8.3 The fundamental theorem, first part

Theorem 8.3: FTC I

If f is continuous on $[a, b]$ and

$$F(x) = \int_a^x f(t)\,dt,$$

then F is continuous on $[a, b]$ and differentiable on (a, b), with $F'(x) = f(x)$.

For differentiable endpoints u, v whose values stay in an interval where f is continuous,

$$\frac{d}{dx}\int_{u(x)}^{v(x)} f(t)\,dt = f(v(x))v'(x) - f(u(x))u'(x).$$

No elementary formula for an antiderivative is needed.

Proof and notes

An integrand depending on the endpoint

When the integrand also contains x

For continuous f on an interval containing 0 and x,

$$\int_0^x f(x-t)\,dt = \int_0^x f(u)\,du.$$

A one-variable primitive proves this identity using the fundamental theorem. Do not merely substitute $t = x$ when differentiating an integral whose integrand also depends on x.

Proof and notes

Example 8.2

Two moving endpoints

For $x > 0$ define

$$H(x) = \int_{\ln x}^{x^2} \sqrt{1+t^4}\,dt.$$

Find $H'(x)$ and the slope of its tangent at $x = 1$.

Solution

8.4 The fundamental theorem, second part, and net change

Theorem 8.4: FTC II

If f is continuous on $[a, b]$ and $A' = f$, then

$$\int_a^b f(x)\,dx = A(b) - A(a).$$

Thus for a continuously differentiable quantity Q, $\int_a^b Q'(t)dt = Q(b) - Q(a)$. In particular, continuous velocity gives displacement $\int_a^b v$ and total distance $\int_a^b |v|$.

Optional topic: Integral mean value theorem

If f is continuous on $[a, b]$, $a < b$, some $c \in [a, b]$ satisfies $f(c) = \frac{1}{b-a} \int_a^b f$.

Proof and notes

Example 8.3

Displacement is not distance

A particle has velocity $v(t) = t^2 - 4t + 3$ on $[0, 4]$. Find its displacement and total distance.

Solution

Example 8.4

Repeated accumulation and a small-interval limit

Let $A(x) = \int_0^x e^{t^2}\,dt$ and $B(x) = \int_0^{x^2} A(u)\,du$. Find $B'(x)$ and evaluate

$$\lim_{x\to 0} \frac{B(x)}{x^4}.$$

Justify the limiting step without using a power series.

Solution

Exercises 8

Part A Standard Problems

Exercise 8.1

Express $\lim_{n\to\infty} \sum_{k=1}^{n} (2 + 3k/n)^2 (3/n)$ as an integral and evaluate it.

Exercise 8.2

Evaluate $\int_{-2}^{2} (x^3 + 2x^2)\,dx$ using symmetry.

Exercises 8

Part A Standard Problems

Exercise 8.3

Evaluate $\int_0^3 |x-1|\,dx$.

Exercise 8.4

If $\int_0^2 f = 3$ and $\int_2^5 f = -1$, find $\int_5^0 f$ and $\int_0^5 (2f+1)$.

Exercise 8.5

Let $F(x) = \int_0^x \cos((x-t)^2)\,dt$ for $x \in \mathbb{R}$. Find $F'(x)$ without quoting differentiation under the integral sign.

Adapted from NYCU, AY 113 Calculus I common examination, A, Question 6; integrand changed.

Exercise 8.6

Find the derivative of $G(x) = \int_x^{2x} e^{-t^2}\,dt$.

Exercises 8

Part A Standard Problems

Exercise 8.7

For $A(x) = \int_0^x (t^2 - 1)dt$, find the intervals of increase and all local extrema.

Exercise 8.8

Choose the single correct value and justify it:

$$\lim_{x \to 0} \frac{1}{x} \int_x^{3x} e^{t^2}\, dt.$$

(A) 1 **(B)** 2 **(C)** 3 **(D)** 6.

Exercise 8.9

Without evaluating the integral, bound $\int_0^2 \sqrt{1 + x^2} dx$ above and below by constants using order.

Exercise 8.10

Find the average value of x^2 on $[-1, 2]$, and locate points where the average is attained.

Exercises 8

Part A Standard Problems

Exercise 8.11

Let $f : \mathbb{R} \to \mathbb{R}$ be continuous and $F(x) = \int_0^x f(t)\,dt$. Select all statements that must be true, and justify every choice or rejection.
(A) $F'(x) = f(x)$ for every real x.
(B) $F''(0)$ must exist.
(C) If f is even, then F is odd.
(D) If $\int_0^1 f(t)\,dt = 0$, then $f \equiv 0$ on $[0, 1]$.

Exercise 8.12

Let $F(x) = \int_0^{x^2} \sqrt{1 + t^3}\,dt$. Find $F''(0)$ and evaluate

$$\lim_{x\to 0} \frac{F(x) - x^2}{x^8}$$

without a series expansion.

Exercises 8

Part B Challenge Problems

Exercise 8.13

Suppose f is continuous and nonnegative on $[a, b]$, with $a < b$ and $\int_a^b f = 0$. Prove $f \equiv 0$.

Exercises 8

Part B Challenge Problems

Exercise 8.14

Suppose continuous f, g on $[a, b]$, $a < b$ satisfy $\int_a^x f = \int_a^x g$ for every x. Prove $f = g$. Explain why equality only at $x = b$ is insufficient.

Exercise 8.15

For continuous f, g on $[a, b]$, $a < b$, prove the integral Cauchy–Schwarz inequality

$$\left(\int_a^b fg\right)^2 \leq \left(\int_a^b f^2\right)\left(\int_a^b g^2\right).$$

Identify equality when $\int g^2 > 0$.

Exercise 8.16

Let f, w be continuous on $[a, b]$, with $w \geq 0$ and $\int_a^b w > 0$. Show that some $c \in [a, b]$ satisfies $\int_a^b fw = f(c) \int_a^b w$.

Exercises 8

Part B Challenge Problems

Exercise 8.17

For continuous f on $[0,1]$, define $K(x) = \int_0^x (x-t)f(t)\,dt$. Prove $K''(x) = f(x)$ on $(0,1)$ without quoting differentiation under the integral sign. Then find $\lim_{x\downarrow 0} K(x)/x^2$ in terms of $f(0)$.

Exercise 8.18

Suppose f is continuous on $[0,1]$ and $f(x) = 1 + \int_0^x f(t)dt$. Determine f, and compute $\int_0^1 f$ without solving any general differential equation.

Exercise 8.19

If f is continuous on $[0,1]$ and $\int_0^1 f = 0$, prove that some $c \in [0,1]$ has $f(c) = 0$. Must there be a zero in $(0,1)$?

Exercises 8

Part B Challenge Problems

Exercise 8.20

Let f be continuously differentiable on $[0,1]$ and $|f'| \leq M$. For $R_n = \frac{1}{n}\sum_{k=1}^{n} f(k/n)$, prove

$$\left| R_n - \int_0^1 f(x)dx \right| \leq \frac{M}{2n}.$$

Learning Focus 9

Substitution, Areas, and Volumes

9.1 Substitution reverses the chain rule

Theorem 9.1: Substitution

If f is continuous on an interval containing the range of a continuously differentiable $g : [a, b] \to \mathbb{R}$, then

$$\int_a^b f(g(x))g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du.$$

For indefinite integrals, write $u = g(x)$ and $du = g'(x)dx$, then return to the original variable For definite integrals, either change the endpoints or substitute back before using the original endpoints. Monotonicity of g is not required for this chain-rule form of substitution.

Proof and notes

Example 9.1

Substitution after algebraic preparation

Evaluate

$$\int_0^1 \frac{x^3}{(1+x^2)^2}\,dx.$$

Explain how the numerator is rewritten after the substitution.

Solution

9.2 Area between curves

Theorem 9.2: Area by slices

For continuous f, g on $[a, b]$, the area between their graphs is

$$A = \int_a^b |f(x) - g(x)|\, dx.$$

If a region is described horizontally by $x = L(y)$ and $x = R(y)$, then $A = \int_c^d [R(y) - L(y)]dy$ provided $R \geq L$.

Find all intersections before deciding which curve is above the other. Split whenever the upper curve or the horizontal boundary changes. A signed difference without an absolute value can cancel area.

Proof and notes

Example 9.2

Area with an absolute value

Find the area enclosed by $y = |x|$ and $y = x^2 - 2$. Identify the intersections and justify any use of symmetry.

Solution

9.3 Volumes by cross sections and washers

Theorem 9.3: Volume by cross sections

If the cross-sectional area perpendicular to an axis is a continuous function $A(x)$ on $[a, b]$, then $V = \int_a^b A(x)dx$. A disk has $A = \pi R^2$ and a washer has $A = \pi(R^2 - r^2)$, where $R \geq r \geq 0$.

Radii are distances from the axis of rotation. For a shifted horizontal axis $y = c$, use vertical distances from c, not simply the heights of the graphs. If a slice crosses the axis, the rotated cross section is a disk rather than a washer with a negative inner radius.

Proof and notes

9.4 Volumes by cylindrical shells

Theorem 9.4: Shells

A thin strip parallel to a rotation axis sweeps out a shell with approximate volume

$$2\pi \cdot \text{radius} \cdot \text{height} \cdot \text{thickness}.$$

Integrate over the strip parameter, provided different strips do not count the same shell volume more than once.

For rotation around $x = c$, use $2\pi \int |x - c|h(x)dx$ on a region lying on one side of the axis. For rotation around $y = c$, horizontal shells use $2\pi \int |y - c|\ell(y)dy$. Choose slices by their direction relative to the rotation axis, not by memorizing an x-formula.

Proof and notes

Example 9.3

One solid, two methods

> The region between $y = x^2$ and $y = x$ for $0 \leq x \leq 1$ is rotated about $y = 2$. Set up and evaluate the volume by washers and by shells.

Solution

Example 9.4

A shifted vertical axis

> Rotate the region under $y = \sqrt{x}$ for $0 \leq x \leq 1$ about $x = 2$. Find the volume using shells. Explain why the radius is not x.

Solution

Exercises 9

Part A Standard Problems

Exercise 9.1

Evaluate $\int x^3\sqrt{x^4+5}\,dx$.

Exercise 9.2

Evaluate $\int_0^{\pi/4} \tan x \sec^2 x\,dx$.

Exercise 9.3

Evaluate $\int (\ln x)/x\,dx$ for $x > 0$.

Exercise 9.4

Find the area between $y = x + 2$ and $y = x^2$ from their left to their right intersection.

Exercise 9.5

Find the area enclosed by $x = y^2 - 1$ and $x = 1 - y$.

Exercises 9

Part A Standard Problems

Exercise 9.6

Rotate the region under $y = 2 - x$ on $[0, 2]$ about the x-axis. Find its volume.

Exercise 9.7

Rotate the region between $y = x$ and $y = x^2$ on $[0, 1]$ about the x-axis. Find the volume.

Exercise 9.8

Rotate the region under $y = 4 - x^2$ on $[0, 2]$ about the y-axis. Use shells. Find the volume.

Exercise 9.9

Set up the volume of the region under $y = \sqrt{x}$ on $[0, 4]$ rotated about $x = 5$, using shells. Then evaluate it.

Exercise 9.10

A solid has cross-sectional area $A(x) = \pi(9 - x^2)$ for $-3 \le x \le 3$. Find its volume.

Exercises 9

Part A Standard Problems

Exercise 9.11

Evaluate $\int_0^1 2xe^{x^2}\,dx$.

Exercise 9.12

Evaluate $\int_0^\pi \sin x \cos x\,dx$ by substitution, even though $\sin x$ is not monotone on this interval.

Exercises 9

Part B Challenge Problems

Exercise 9.13

For $a > 0$, find the area enclosed by $y = x^2$ and $y = ax$. Express its dependence on a.

Exercise 9.14

For the same region in Exercise 9.13, find the volumes obtained by rotation about the x-axis and about the y-axis.

Exercises 9

Part B Challenge Problems

Exercise 9.15

The region between $y = x^2$ and $y = 1$ for $-1 \leq x \leq 1$ is rotated about the y-axis. Explain why $2\pi \int_{-1}^{1} |x|(1 - x^2)dx$ double counts, and find the correct volume.

Exercise 9.16

For $p > 0$, rotate the region under $y = x^p$ on $[0, 1]$ about $x = 1$. Derive its volume as a function of p and determine whether it increases or decreases with p.

Exercise 9.17

Evaluate $I = \int_0^{\pi/2} x \sin x \cos x \, dx$ using the substitution $x \mapsto \pi/2 - x$ and symmetry, without integration by parts.

Exercises 9

Part B Challenge Problems

Exercise 9.18

For $a > 0$ and continuous f on $[0, a]$, prove $\int_0^a f(x)dx = \int_0^a f(a-x)dx$. Deduce that, if $f(a-x) = f(x)$, then $\int_0^a xf(x)dx = (a/2)\int_0^a f(x)dx$.

Exercise 9.19

The integral $\pi\int_0^1 [(3-x^2)^2 - 4]dx$ describes a washer volume. Give one generating region and axis, then describe the same solid with horizontal shells.

Exercise 9.20

Rotate the region $0 \le x \le 1$, $0 \le y \le x$ about $y = 1/2$. Set up and evaluate its volume by perpendicular slices, taking account of slices that cross the axis.

Learning Focus 10

Integration Techniques

10.1 Integration by parts

Theorem 10.1: Integration by parts

For continuously differentiable u, v,

$$\int u\,v'\,dx = uv - \int u'v\,dx,$$

$$\int_a^b u\,v'\,dx = [uv]_a^b - \int_a^b u'v\,dx.$$

Choose u so differentiation helps, and $dv = v'dx$ so an antiderivative is available. Repeated integration by parts may terminate or return to the original integral, producing an algebraic equation. An inverse trigonometric function can be treated as a product with 1.

Proof and notes

Example 10.1

An inverse function inside an integral

Evaluate $\int_0^1 x \arctan x\,dx$.

Solution

Example 10.2

Returning to the original integral

Evaluate $\int e^{2x} \cos 3x \, dx$ by integration by parts. Keep track of the original integral when it reappears.

Solution

10.2 Trigonometric integrals

For powers of sine and cosine, save one factor of an odd power and convert the rest with $\sin^2 x + \cos^2 x = 1$. If both powers are even, use

$$\sin^2 x = \frac{1 - \cos 2x}{2}, \quad \cos^2 x = \frac{1 + \cos 2x}{2}.$$

For tangent and secant, the identities $1 + \tan^2 x = \sec^2 x$ and the differentials $d(\tan x) = \sec^2 x \, dx$, $d(\sec x) = \sec x \tan x \, dx$ guide the choice. These are strategies, not a single universal algorithm.

Notes

Secant and tangent antiderivatives

Logarithmic trigonometric antiderivatives

On any interval on which $\cos x \neq 0$,

$$\int \sec x\,dx = \ln|\sec x + \tan x| + C,$$

$$\int \tan x\,dx = -\ln|\cos x| + C.$$

The interval restriction and the absolute values are part of these formulas.

Proof and notes

Example 10.3

Parity and logarithmic trigonometric integrals

Evaluate both integrals, choosing a method appropriate to each integrand:

$$\text{(a)} \int \sin^3 x \cos^2 x\,dx, \quad \text{(b)} \int_0^{\pi/4} (\sec x + \tan x)\,dx.$$

Solution

10.3 Trigonometric substitution and domains

For $a > 0$, common substitutions are

radical	substitution	convenient range
$\sqrt{a^2 - x^2}$	$x = a\sin\theta$	$-\pi/2 \le \theta \le \pi/2$
$\sqrt{a^2 + x^2}$	$x = a\tan\theta$	$-\pi/2 < \theta < \pi/2$
$\sqrt{x^2 - a^2}$	$x = a\sec\theta$	$0 \le \theta < \pi/2$ $(x \ge a)$

The range controls signs: $\sqrt{a^2\cos^2\theta} = a|\cos\theta|$. The negative branch $x \le -a$ needs its own parametrization or sign analysis. Complete squares and remove easy factors before choosing a substitution.

Derivation and notes

Example 10.4

A definite trigonometric substitution

Evaluate $\displaystyle\int_0^{3/2} \frac{x^2}{\sqrt{9 - x^2}}\,dx$.

Solution

10.4 Optional numerical integration

For n equal subdivisions of width $h = (b-a)/n$, let $x_i = a + ih$. The midpoint and trapezoidal approximations are

$$M_n = h \sum_{i=1}^{n} f\left(\frac{x_{i-1} + x_i}{2}\right),$$

$$T_n = h \left(\frac{f(a) + f(b)}{2} + \sum_{i=1}^{n-1} f(x_i)\right).$$

For even n, Simpson's approximation is

$$S_n = \frac{h}{3}\Bigg[f(x_0) + f(x_n) + 4 \sum_{\substack{1 \le i \le n-1 \\ i \text{ odd}}} f(x_i) + 2 \sum_{\substack{2 \le i \le n-2 \\ i \text{ even}}} f(x_i)\Bigg].$$

Numerical approximation does not replace a convergence check for an improper integral.

Derivation and notes

Exercises 10

Part A Standard Problems

Exercise 10.1

Evaluate $\int x \ln x \, dx$, $x > 0$.

Exercises 10

Part A Standard Problems

Exercise 10.2

Evaluate $\int x^2 e^x \, dx$.

Exercise 10.3

Evaluate $\int_0^1 \arctan x \, dx$.

Exercise 10.4

Evaluate $\int_0^{\pi/3} \sec x \, dx$. Explain why the absolute value in a general antiderivative can be removed on this interval.

Exercise 10.5

Evaluate $\int \sin^2 x \, dx$.

Exercise 10.6

Evaluate $\int \sin^2 x \cos^3 x \, dx$.

Exercises 10

Part A Standard Problems

Exercise 10.7

Evaluate $\int \tan^3 x \sec^2 x \, dx$.

Exercise 10.8

Evaluate $\int \sec^4 x \, dx$.

Exercise 10.9

Evaluate $\int_0^1 x^2/\sqrt{4-x^2} \, dx$, with an exact answer.

Exercise 10.10

Evaluate $\int dx/(x^2+4)$.

Exercise 10.11

Evaluate $\int_0^1 x^3/\sqrt{1+x^2} \, dx$.

Exercises 10

Part A Standard Problems

Exercise 10.12

Optional numerical integration. Compute M_2 and T_2 for $\int_0^1 x^2 dx$ and compare them to the exact value.

Exercises 10

Part B Challenge Problems

Exercise 10.13

For real a, b with $a^2 + b^2 > 0$, derive

$$\int e^{ax} \cos bx \, dx = \frac{e^{ax}(a \cos bx + b \sin bx)}{a^2 + b^2} + C.$$

Do not divide by a in a way that loses the case $a = 0$.

Exercise 10.14

Let $I_n = \int_0^{\pi/2} \sin^n x \, dx$. For integers $n \geq 2$, prove $I_n = (n-1)I_{n-2}/n$ and compute I_6.

Exercises 10

Part B Challenge Problems

Exercise 10.15

Evaluate $\int_0^1 x(\ln(1+x))^2 dx$.

Exercise 10.16

Evaluate $\int_0^{\pi/4} \sec^3 x\, dx$. Derive any antiderivative beyond the basic formulas already established in this chapter.

Exercise 10.17

Evaluate $\int_0^1 x^2/\sqrt{1+x^2}\, dx$. Do not assume that the first substitution immediately gives a basic antiderivative.

Exercises 10

Part B Challenge Problems

Exercise 10.18

Derive an antiderivative of $\sqrt{x^2 + a^2}$ for $a > 0$, keeping the domain and logarithm argument explicit.

Exercise 10.19

For $n \geq 1$, derive a reduction formula for an antiderivative of $x^n e^x$, and give a closed finite expression.

Exercise 10.20

Let $I = \int_0^{\pi/2} x \sin^2 x \, dx$ and $J = \int_0^{\pi/2} x \cos^2 x \, dx$. Use reflection symmetry to determine $I + J$ Then use a suitable integration technique to determine $I - J$ and evaluate both integrals. Explain why reflection does not make I and J equal.

Learning Focus 11

Partial Fractions and Improper Integrals

11.1 Partial fractions

First perform polynomial division if the numerator degree is at least the denominator degree Factor the remaining denominator over the real numbers. For a repeated linear factor $(x-a)^m$, include every power:

$$\frac{A_1}{x-a}+\frac{A_2}{(x-a)^2}+\cdots+\frac{A_m}{(x-a)^m}.$$

For a repeated irreducible quadratic $q(x)^m$, include

$$\frac{B_1x+C_1}{q(x)}+\cdots+\frac{B_mx+C_m}{q(x)^m}.$$

Clear denominators and compare polynomial coefficients. The numerator over an irreducible quadratic is generally linear, not constant.

Notes

Example 11.1

A repeated factor

Evaluate

$$\int \frac{2x+3}{x(x-1)^2}\,dx.$$

Do not omit the first power of the repeated factor.

Solution

Example 11.2

A radical becomes a rational integral

Evaluate

$$I = \int_0^{1/\sqrt{2}} \frac{x^2}{(1+2x^2)\sqrt{1-x^2}}\,dx.$$

Keep track of the endpoints at every change of variable.

Adapted from NYCU, AY 114 Calculus I common examination, A, calculation 1(c); coefficient and endpoint changed.

Solution

11.2 Improper integrals are separate limits

Definition 11.1: Improper integrals

An integral over $[a, \infty)$ is the limit $\lim_{R\to\infty} \int_a^R f$, when this limit is finite. At a singular endpoint a, use $\lim_{\varepsilon\downarrow 0} \int_{a+\varepsilon}^b f$.

If there is an interior singularity c, both $\int_a^c f$ and $\int_c^b f$ must converge separately. Likewise $\int_{-\infty}^{\infty} f$ requires independent convergence on both sides of a fixed finite point. A Cauchy principal value ties the cutoffs together and is a different notion. Infinity minus infinity is not a convergent improper integral. An improper integral is *absolutely convergent* if the corresponding integral of $|f|$ is finite. Absolute convergence implies convergence; the converse need not hold.

Notes

11.3 Power tests and comparison

Theorem 11.1: Power tests

The integral $\int_1^\infty x^{-p}dx$ converges exactly for $p > 1$, with value $1/(p-1)$. The integral $\int_0^1 x^{-p}dx$ converges exactly for $p < 1$, with value $1/(1-p)$.

Theorem 11.2: Comparison

For nonnegative f, g that are integrable on every closed subinterval away from the same improper endpoint, $f \leq g$ and convergence of $\int g$ imply convergence of $\int f$. If $f \geq g$ and $\int g$ diverges, then $\int f$ diverges. If $f/g \to L \in (0, \infty)$, the two integrals have the same convergence behavior.

Proof and notes

Example 11.3

A parameter in a logarithmic tail

For which real p does

$$\int_e^\infty \frac{dx}{x(\ln x)^p}$$

converge? Evaluate it in the convergent cases.

Solution

Example 11.4

An apparent cancellation at a singularity

Determine whether $\int_{-1}^{1} dx/x$ exists as an improper integral. Compare it with

$$\lim_{\varepsilon \downarrow 0} \left(\int_{-1}^{-\varepsilon} \frac{dx}{x} + \int_{\varepsilon}^{1} \frac{dx}{x} \right).$$

Solution

Exercises 11

Part A Standard Problems

Exercise 11.1

Evaluate $\int dx/[x(x+1)]$.

Exercise 11.2

Evaluate $\int (3x+5)/(x^2+3x+2)\, dx$.

Exercise 11.3

Evaluate $\int (2x^2+x+1)/[x(x^2+1)]\, dx$. State the intervals on which the result is an antiderivative.

Exercises 11

Part A Standard Problems

Exercise 11.4

Evaluate $\int_0^1 2x/[(x^2+1)(x^2+2)]\,dx$.

Exercise 11.5

Determine convergence and value of $\int_1^\infty x^{-3/2}dx$.

Exercise 11.6

Determine convergence and value of $\int_0^1 x^{-1/2}dx$.

Exercise 11.7

Determine convergence of $\int_0^1 dx/x$.

Exercise 11.8

Determine convergence and evaluate $\int_0^\infty e^{-x}/(1+e^{-x})\,dx$.

Exercises 11

Part A Standard Problems

Exercise 11.9

Evaluate $\int_1^\infty dx/[x(x+1)]$.

Exercise 11.10

Evaluate $\int_0^1 \ln x\, dx$ as an improper integral.

Exercise 11.11

Evaluate $\int_1^\infty (\ln x)/x^2\, dx$.

Exercise 11.12

Select all convergent improper integrals and justify each decision:

$$\textbf{(A)}\ \int_0^1 \frac{1-\cos x}{x^{5/2}}\, dx, \quad \textbf{(B)}\ \int_0^1 \frac{1-\cos x}{x^3}\, dx,$$
$$\textbf{(C)}\ \int_1^\infty \frac{\ln x}{x^2}\, dx, \quad \textbf{(D)}\ \int_{-1}^1 \frac{dx}{x^2}.$$

Does symmetric truncation make **(D)** converge?

Exercises 11

Part B Challenge Problems

Exercise 11.13

For real α, β, determine exactly when $\int_0^\infty x^\alpha/(1+x)^\beta\,dx$ converges.

Exercise 11.14

Classify $\int_0^{1/2} x^{-p}(-\ln x)^{-q}dx$ for all real p, q.

Exercise 11.15

Classify and evaluate, when convergent,

$$\int_{e^e}^\infty \frac{dx}{x \ln x (\ln \ln x)^p}.$$

Exercises 11

Part B Challenge Problems

Exercise 11.16

Determine convergence and evaluate

$$\int_0^1 \frac{x^2}{(1+x^2)\sqrt{1-x^2}}\,dx.$$

Justify the improper endpoint before evaluating the integral.

Adapted from NYCU, AY 114 Calculus I common examination, A, calculation 1(c); the endpoint is now singular.

Exercise 11.17

For $p > 0$, compare convergence of $\int_1^\infty \sin x/x^p\,dx$ with absolute convergence when $p > 1$. Prove convergence for every $p > 0$ by integration by parts; no classification for $0 < p \le 1$ of the absolute integral is required.

Exercise 11.18

Show that $\int_1^\infty \sin x/x\,dx$ converges, but $\int_1^\infty |\sin x|/x\,dx$ diverges.

Exercises 11

Part B Challenge Problems

Exercise 11.19

For fixed $c > 0$, evaluate

$$\lim_{\varepsilon \downarrow 0} \left(\int_{-1}^{-\varepsilon} \frac{dx}{x} + \int_{c\varepsilon}^{1} \frac{dx}{x} \right).$$

What does dependence on c show?

Exercise 11.20

For $a, b > 0$, evaluate $\int_0^\infty dx/[(x+a)(x+b)]$, including the case $a = b$, and check continuity of the answer as $b \to a$.

Learning Focus 12

Arc Length and Surface Area

12.1 Arc length of a graph

Theorem 12.1: Arc length

For a continuously differentiable graph $y = f(x)$ on $[a, b]$, its length is

$$L = \int_a^b \sqrt{1 + [f'(x)]^2}\, dx.$$

For a graph $x = g(y)$, interchange the variables: $L = \int_c^d \sqrt{1 + [g'(y)]^2}\, dy$.

The length element is $ds = \sqrt{dx^2 + dy^2}$. If an endpoint derivative is unbounded, first truncate the curve and check the resulting improper integral. A perfect square under the radical may simplify an otherwise difficult-looking problem.

Proof and notes

Example 12.1

A hidden square in an arc-length integrand

Find the length of

$$y = \frac{x^2}{4} - \frac{1}{2}\ln x, \quad 1 \le x \le 2.$$

Solution

12.2 Area of a surface of revolution

Theorem 12.2: Surface area

For a nonnegative C^1 function f on $[a,b]$, revolution about the x-axis gives

$$S = 2\pi \int_a^b f(x)\sqrt{1+[f'(x)]^2}\,dx.$$

In general the surface element is $dS = 2\pi r\,ds$, where r is distance to the axis. For an axis $y = c$, the radius is $|f(x) - c|$; for an axis $x = c$, it is $|x - c|$. Use a parametrization that does not trace the same surface patch more than once. Surface area and volume have different integrands and can have different convergence behavior.

Proof and notes

Example 12.2

A conical surface

Find the area of the surface generated by $y = x$, $0 \le x \le 1$, rotated about the x-axis. Does this formula include the base disk?

Solution

Example 12.3

An endpoint with a vertical tangent

Rotate $y = \sqrt{x}$, $0 \le x \le 1$, about the x-axis. Find the lateral surface area, interpreting the endpoint correctly.

Solution

12.3 Optional parametrized curves

For $x = x(t)$ and $y = y(t)$ of class C^1 on $[\alpha, \beta]$, the length counted along the parametrization is

$$L = \int_\alpha^\beta \sqrt{[x'(t)]^2 + [y'(t)]^2}\, dt.$$

Where $x'(t) \neq 0$, $dy/dx = y'(t)/x'(t)$. For revolution about the x-axis,

$$S = 2\pi \int_\alpha^\beta |y(t)|\sqrt{[x'(t)]^2 + [y'(t)]^2}\, dt,$$

provided the surface is traced once except at negligible boundary points. Changing the speed or reversing direction does not change the geometric length of a singly traced curve.

Derivation and notes

Example 12.4

Finite volume, infinite surface area

For each $R > 1$, rotate the region $1 \leq x \leq R$, $0 \leq y \leq 1/x$ about the x-axis. Determine the limit of the volumes and whether the lateral surface areas remain bounded as $R \to \infty$.

Solution

Exercises 12

Part A Standard Problems

Exercise 12.1

Find the length of $y = 3x + 1$ for $0 \leq x \leq 2$.

Exercise 12.2

Find the length of $y = (2/3)x^{3/2}$ on $[0, 3]$.

Exercises 12

Part A Standard Problems

Exercise 12.3

Set up, but do not evaluate, the length of $y = e^{-x^2}$ on $[0, 1]$.

Exercise 12.4

Find the length of $y = \ln(\cos x)$ on $[0, \pi/4]$.

Exercise 12.5

Find the lateral area generated by the segment $y = 2$, $0 \le x \le 3$, rotated about the x-axis.

Exercise 12.6

Find the lateral surface area generated by $y = 2x$, $0 \le x \le 1$, rotated about the x-axis.

Exercise 12.7

Set up the area generated by $y = x^2$, $0 \le x \le 1$, rotated about the x-axis and about the y-axis.

Exercises 12

Part A Standard Problems

Exercise 12.8

Evaluate the second integral in Exercise 12.7.

Exercise 12.9

Find the length of $x = y^2/4 - (1/2)\ln y$, $1 \le y \le 2$.

Exercise 12.10

Optional parametric curve. Find the length of $x = 3\cos t$, $y = 3\sin t$, $0 \le t \le \pi$.

Exercise 12.11

Set up the surface area from $y = x^2$, $0 \le x \le 1$, rotated about $y = 2$.

Exercise 12.12

Find the lateral area from $y = x$, $0 \le x \le 1$, rotated about $x = 2$.

Exercises 12

Part B Challenge Problems

Exercise 12.13

For a C^1 graph on $[a, b]$, $a < b$, prove that its arc length is at least the distance between its endpoints. Characterize equality.

Exercise 12.14

For $p > 0$, rotate the region $x \geq 1$, $0 \leq y \leq x^{-p}$ about the x-axis. Classify finiteness of both volume and lateral surface area.

Exercise 12.15

Derive the area of a sphere of radius $R > 0$ by revolving $y = \sqrt{R^2 - x^2}$, $-R \leq x \leq R$, about the x-axis. Handle the endpoint derivatives.

Exercises 12

Part B Challenge Problems

Exercise 12.16

Let $a < b$, and let $f : [a, b] \to [c, d]$ be a C^1 strictly increasing bijection with $f' > 0$. Put $g = f^{-1}$ Prove that the arc lengths computed as $y = f(x)$ and $x = g(y)$ agree.

Exercise 12.17

Use dimensionless coordinates and let $f(x) = e^{x^2}$ on $[0, 1]$. Let V be the volume obtained by revolving $0 \le y \le f(x)$ about $x = -7$, let S be the lateral area obtained by revolving the graph about the x-axis, and let L be its arc length. Without evaluating any of these integrals, select the correct order and prove it. You may use $e < 3$.
(A) $V > S > L$ **(B)** $S > V > L$
(C) $V > L > S$ **(D)** $L > S > V$.

Adapted from NYCU, AY 114 Calculus I common examination, A, Question 9; the vertical axis is changed.

Exercise 12.18

Optional parametric curve. The unit circle is parametrized by $x = \cos t$, $y = \sin t$, $0 \le t \le 4\pi$ Compute the length integral and explain the difference from the geometric circumference.

Exercises 12

Part B Challenge Problems

Exercise 12.19

For the graph $y = \sqrt{x}$ on $[0, 1]$, prove that its length is finite and evaluate it by interchanging the coordinates.

Exercise 12.20

Suppose a C^1 plane curve of length L is rotated around an axis in its plane, tracing its surface once, and its distance r to the axis satisfies $0 < r_0 \leq r \leq r_1$. Prove $2\pi r_0 L \leq S \leq 2\pi r_1 L$. Deduce a necessary condition for an infinite-length generating curve to produce finite surface area.