The enclosure method for the anisotropic Maxwell system

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Abstract

We develop an enclosure-type reconstruction scheme to identify penetrable and impenetrable obstacles in electromagnetic field with anisotropic medium in $\mathbb{R}^3$. The main difficulty in treating this problem lies in the fact that there are so far no complex geometrical optics solutions available for the Maxwell's equation with anisotropic medium in $\mathbb{R}^3$. Instead, we derive and use another type of special solutions called oscillating-decaying solutions. To justify this scheme, we use Meyers' L^p estimate, for the Maxwell system, to compare the integrals coming from oscillating-decaying solutions and those from the reflected solutions.

Keywords: enclosure method, reconstruction, oscillating-decaying solutions, Runge approximation property, Meyers L^p estimates.

1 Introduction and statement of the results

Let Ω be a bounded C^∞-smooth domain in $\mathbb{R}^3$ with connected complement $\mathbb{R}^3 \setminus \overline{\Omega}$ and D be a subset of Ω with Lipschitz boundary. We are concerned with the electromagnetic wave propagation in an anisotropic medium in $\mathbb{R}^3$ with the electric permittivity $\epsilon = (\epsilon_{ij}(x))$ a 3×3 positive definite matrix and $\epsilon(x) = \epsilon_0(x)$ in $\Omega\backslash\bar{D}$. We also assume that $\epsilon(x) = \epsilon_0(x) - \epsilon_D(x)\chi_D(x)$ with $\epsilon_0 \in C^\infty(\Omega)$ a positive definite 3×3 symmetric matrix and $\epsilon_D(x)$ is a positive 3×3 symmetric matrix and μ a smooth scalar function defined on Ω such that there exist $\mu_c > 0$ and $\epsilon_c > 0$ verifying

$$\mu(x) \geq \mu_c > 0 \text{ and } \sum_{i.j=1}^{3} \epsilon_{ij}(x)\xi_i\xi_j \geq \epsilon_c|\xi|^2 \ \forall\xi \in \mathbb{R}^3,\ \forall x \in \Omega. \tag{1.1}$$

If we denote by E and H the electric and the magnetic fields respectively, then the electromagnetic wave propagation by a penetrable obstacle problem reads as

$$\begin{cases} \nabla\times E - ik\mu H = 0 & \text{in } \Omega, \\ \nabla\times H + ik\epsilon E = 0 & \text{in } \Omega, \\ \nu\times E = f & \text{on } \partial\Omega, \end{cases} \tag{1.2}$$

with $\epsilon = \epsilon_0 - \epsilon_D\chi_D$, and the one by the impenetrable obstacle as

$$\begin{cases} \nabla\times E - ik\mu H = 0 & \text{in } \Omega\backslash\bar{D}, \\ \nabla\times H + ik\epsilon E = 0 & \text{in } \Omega\backslash\bar{D}, \\ \nu\times E = f & \text{on } \partial\Omega, \\ \nu\times H = 0 & \text{on } \partial D, \end{cases} \tag{1.3}$$

where ν is the unit outer normal vector on $\partial\Omega \cup \partial D$ and $k > 0$ is the wave number. In this paper, we assume that k is not an eigenvalue for (1.2) and (1.3).

Impedance Map: We define the impedance map $\Lambda_D : TH^{-\frac{1}{2}}(\partial\Omega) \to TH^{-\frac{1}{2}}(\partial\Omega)$ by

$$\Lambda_D(\nu \times H|_{\partial\Omega}) = (\nu \times E|_{\partial\Omega}),$$

where $TH^{-\frac{1}{2}}(\partial\Omega) := \{f \in H^{-\frac{1}{2}}(\partial\Omega) | \nu \cdot f = 0\}$ and $\times$ is the standard cross product in $\mathbb{R}^3$. We denote by $\Lambda_\emptyset$ the impedance map for the domain without an obstacle.

Consider the anisotropic Maxwell system

$$\begin{cases} \nabla \times E - ik\mu H = 0 & \text{in } \Omega, \\ \nabla \times H + ik\epsilon E = 0 & \text{in } \Omega, \end{cases} \tag{1.4}$$

where μ and ϵ satisfy (1.1). We are interested in the question reconstructing the shape of D using the impedance map Λ_D. This geometrical inverse problem is quite well studied in the literature see [4] and several methods have been proposed to solve it. In this paper, we focus on one of these method, called the enclosure method, which is initiated by Ikehata, see for examples [2, 3], and developed by many researchers [7, 9, 14, 18, 19, 20], [6, 19] for the acoustic model, [5, 9] for the Lamé model and [7, 21] for the Maxwell model. The testing functions used in [7, 21] are complex geometric optics (CGO) solutions of the isotropic Maxwell's equation. The construction of CGO solutions for isotropic inhomogeneous Maxwell's equations is first proposed in [17]. After that, the authors in [8] also constructed CGO solutions for some special anisotropic Maxwell's equations. However, there are not yet of CGO solutions for general anisotropic Maxwell system. Besides, CGO solutions, another kind of special solutions for anisotropic elliptic system was proposed for substitution in [15] and [16]. They are called oscillating-decaying (OD) solutions. Inspired by [17] and [15], our idea is to reduce (1.4) to an elliptic systems and then use the results in [15] to construct oscillating-decaying type solutions to the anisotropic Maxwell system. Precisely, we can decompose the equation (1.4) into two decoupled strongly elliptic systems. The main difference between the construction of the oscillating-decaying solutions in [15] and ours is about the higher derivatives of oscillating-decaying solutions.

One of the main differences between the CGOs and the oscillating-decaying solutions is that, roughly speaking, given a hyperplane, an oscillating decaying solution is oscillating very rapidly along this plane and decaying exponentially in the direction transversely to the same plane. Oscillating-decaying solutions are special solutions with the phase function having nonnegative imaginary part. In addition, these oscillating decaying solutions are only defined on a half plane. To use them as inputs for our detection algorithm, we need to extend them to the whole domain Ω. One way to do the extension is to use the Runge approximation property for the anisotropic Maxwell's equation. The Runge approximation property will help us to find a sequence of approximated solutions which are defined on Ω, satisfy (1.4) and their limit is the oscillating-decaying solution. Note that it was first recognized by Lax [10] that the Runge approximation property is a consequence of the weak unique continuation property. In [11], the authors already proved the unique continuation property and based on it we derive the Runge approximation property for the anisotropic Maxwell's equation.

To be more precise, let ω be a unit vector in $\mathbb{R}^3$, denote $\Omega_t(\omega) = \Omega \cap \{x | x \cdot \omega > t\}$, $\Sigma_t(\omega) = \Omega \cap \{x | x \cdot \omega = t\}$ and set (E_t, H_t) to be the oscillating-decaying solution for the anisotropic Maxwell's equation in $\Omega_t(\omega)$.

Support function: For $\rho \in \mathbb{S}^2$, we define the support function of D by $h_D(\rho) = \inf_{x \in D} x \cdot \rho$.

When $t = h_D(\omega)$, which means $\Sigma_t(\omega)$ touches ∂D, we cannot apply the Runge approximation property to (E_t, H_t) in $\Omega_t(\omega)$. Therefore, we need to enlarge the domain $\Omega_t(\omega)$ such that the OD solutions exist and the Runge approximation property works. Let η be a positive real number, denote $\Omega_{t-\eta}(\omega)$ and $\Sigma_{t-\eta}(\omega)$ and note that $\Omega_{t-\eta}(\omega) \supset \Omega_t(\omega)\ \forall \eta > 0$. We can find $(E_{t-\eta}, H_{t-\eta})$ to be the OD solution in $\Omega_{t-\eta}(\omega)$. By the Runge approximation property, there exists a sequence of functions $\{(E_{\eta,\ell}, H_{\eta,\ell})\}$ satisfying the Maxwell system in Ω such that $(E_{\eta,\ell}, H_{\eta,\ell})$ converges to $(E_{t-\eta}, H_{t-\eta})$ as $\ell \to \infty$ in $L^2(\Omega_{t-\eta}(\omega))$ and in $H(curl, D)$ by interior estimates since $D \Subset \Omega_{t-\eta}(\omega)$. In addition we show that $(E_{t-\eta}, H_{t-\eta})$ converges to (E_t, H_t) in $H(curl, D)$ as $\eta \to 0$. Then we can define the indicator function as follows.

Indicator function: For $\rho \in S^2$, $\tau > 0$ and $t > 0$ we define the indicator function

$$I_\rho(\tau, t) := \lim_{\eta \to 0} \lim_{\ell \to \infty} I_\rho^{\eta,\ell}(\tau, t),$$

where

$$I_\rho^{\eta,\ell}(\tau, t) := ik\tau \int_{\partial\Omega} (\nu \times H_{\eta,\ell}) \cdot (\overline{(\Lambda_D - \Lambda_\emptyset)(\nu \times H_{\eta,\ell})} \times \nu) dS.$$

Goal: We want to characterize the convex hull of the obstacle D from the impedance map Λ_D.

The answer to this goal is the following theorem.

Theorem 1.1. *Let $\rho \in \mathbb{S}^2$. For the penetrable (or impenetrable) obstacle case, we have the following characterization of $h_D(\rho)$.*

$$\begin{cases} \lim_{\tau \to \infty} |I_\rho(\tau, t)| = 0 \text{ when } t < h_D(\rho), \\ \liminf_{\tau \to \infty} |I_\rho(\tau, h_D(\rho))| > 0, \end{cases}$$

To prove Theorem 1.1, for the penetrable obstacle case, we need an appropriate L^p estimate of the corresponding reflected solution. We follow the idea in [7] to prove a global L^p estimate for the curl of the solutions of the anisotropic Maxwell's equation, for p near 2 and $p \leq 2$.

To prove Theorem 1.1, in the impenetrable obstacle case, we use layer potential arguments as in [7] coupled with appropriate L^p estimates. Precisely, first, we use the well-posedness for an exterior isotropic Maxwell's system with the Silver-Müller radiation condition and, in particular, the layer potential theory to find a suitable estimate for the solution of this exterior problem. Second, we decompose the reflected solution into two functions, one satisfies the reflected Maxwell's equation with a zero boundary data, the other satisfies the original anisotropic Maxwell's equation with the same boundary conditions which come from the reflected equation. For the first decomposed function, we use the L^p estimates, and for the second function, we will use the well-posedness, in L^2, for the anisotropic Maxwell's system. Combining these two steps, we derive the full estimate for the reflected solution in the impenetrable obstacle case.

This paper is organized as follows. In the section 2, we give decompose the anisotropic Maxwell system into two strongly elliptic systems. In section 3, we use the elliptic systems derived in the section 2 to build the oscillating-decaying solutions for the Maxwell system. Then, we give the Runge approximation for the anisotropic Maxwell equation in section 4. In section 5, we prove the Theorem 1.1 for both penetrable and impenetrable obstacle case. Finally, in the last section, as an appendix, we provide some technical details which we postponed in the main text and recall some useful estimates for solutions of the Maxwell system. Before closing this introduction, let us mention that in the whole text whenever we use the word smooth it means C^∞-smooth.

2 Reduction to strongly elliptic systems

Our goal is to construct the oscillating-decaying (OD) solution for the following anisotropic time-harmonic Maxwell's system

$$\begin{cases} \nabla \times E = ik\mu H \\ \nabla \times H = -ik\epsilon E \\ \operatorname{div}(\epsilon E) = 0 \\ \operatorname{div}(\mu H) = 0, \end{cases} . \tag{2.1}$$

where E, H denote the electric and magnetic field intensity respectively, and μ denotes the positive scalar permeability, ϵ denotes the permittivity, which is a real, symmetric, positive definite 3×3 matrix.

Inspired by [17], the first step of constructing OD solutions is to reduce (2.1) to a strongly elliptic system. In fact, we reduce the anisotropic Maxwell's system (2.1) to two separate strongly elliptic equations (2.3), while in [17] the isotropic Maxwell's system is reduced to an elliptic (a single Schrödinger) system with coupled zero-th order term. The following theorem is our reduction result.

Theorem 2.1. *We set E and H of the following forms*

$$\begin{cases} E = -\frac{i}{k}\epsilon^{-1}\nabla \times (\mu^{-1}(\nabla \times B)) - \epsilon^{-1}(\nabla \times A) \\ H = \frac{i}{k}\mu^{-1}\nabla \times (\epsilon^{-1}(\nabla \times A)) - \mu^{-1}(\nabla \times B) \end{cases} \tag{2.2}$$

with A, B satisfying the strongly elliptic systems

$$\begin{cases} \mu\nabla tr(M^A\nabla A) - \nabla \times (\epsilon^{-1}(\nabla \times A)) + k^2\mu A = 0 \\ \epsilon\nabla tr(M^B\nabla B) - \nabla \times (\mu^{-1}(\nabla \times B)) + k^2\epsilon B = 0 \end{cases} , \tag{2.3}$$

where M^A, M^B are introduced in Theorem 2.4, then E and H satisfy (2.1).

Remark 2.2. Theorem 2.1 shows that, if we can find solutions of (2.3), then we can find solutions of (2.1).

Proof. In this proof, we will show the process of the reduction. And the proof that the systems (2.3) are strongly elliptic systems will be postponed to Theorem 2.4.

As in [17], we set the following two auxiliary functions which are similar to what they used:

$$\Phi = \frac{i}{k}\mathrm{div}(\epsilon E)$$

and

$$\Psi = \frac{i}{k}\mathrm{div}(\mu H).$$

Note that Φ and Ψ are actually zero by the Maxwell's equation. We consider the following first-order matrix differential operator P

$$P = \begin{pmatrix} 0 & \mathrm{div}(\epsilon(\cdot)) & 0 & 0 \\ \mu^{-1}\nabla & 0 & \nabla\times & 0 \\ 0 & -\nabla\times & 0 & \epsilon^{-1}\nabla \\ 0 & 0 & \mathrm{div}(\mu(\cdot)) & 0 \end{pmatrix}.$$

Note that P is a 8×8 matrix. Let

$$Y = \begin{pmatrix} \Phi \\ E \\ H \\ \Psi \end{pmatrix}$$

Then the problem (2.1) can be rewritten as follows:

$$PY = -ikVY,$$

where

$$V = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \epsilon & 0 & 0 \\ 0 & 0 & \mu & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Thus, the Maxwell's system (2.1) implies

$$(P + ikV)Y = 0 \text{ and } \Phi = \Psi = 0. \tag{2.4}$$

It is easy to see that conversely (2.4) implies the Maxwell's system, and hence they are equivalent.

The first idea of the reducing process is to construct a suitable $\widetilde{Q}$, which can make $(P+ikV)\widetilde{Q}$ a "good" second-order differential operator. Then, a solution $\mathbf{X}$ for the problem

$$(P + ikV)\widetilde{Q}X = 0 \tag{2.5}$$

will give rise to a solution $Y = \widetilde{Q}X$ for

$$(P + ikV)Y = 0.$$

Moreover, if we find the solution X such that the first and the last component of $Y = \widetilde{Q}X$ are zero, then we obtain solutions for the Maxwell's system.

We try the matrix differential operator $\widetilde{Q} = Q - ikI$, where

$$Q = \begin{pmatrix} 0 & \mathrm{div}(\epsilon(\cdot)) & 0 & 0 \\ \nabla & 0 & \epsilon^{-1}(\nabla \times (\cdot)) & 0 \\ 0 & -\mu^{-1}(\nabla \times (\cdot)) & 0 & \nabla \\ 0 & 0 & \mathrm{div}(\mu(\cdot)) & 0 \end{pmatrix}. \tag{2.6}$$

Then

$$\begin{aligned}
&(P + ikV)\widetilde{Q} \\
&= (P + ikV)(Q - ikI) \\
&= PQ - ikP + ikVQ + k^2 V \\
&= \begin{pmatrix} \mathrm{div}(\epsilon\nabla) & 0 & 0 & 0 \\ 0 & L_1 & 0 & 0 \\ 0 & 0 & L_2 & 0 \\ 0 & 0 & 0 & \mathrm{div}(\mu\nabla) \end{pmatrix} \\
&+ \begin{pmatrix} 0 & -ik\mathrm{div}(\epsilon(\cdot)) & 0 & 0 \\ -ik\mu^{-1}\nabla & 0 & -ik\nabla\times & 0 \\ 0 & ik\nabla\times & 0 & -ik\epsilon^{-1}\nabla \\ 0 & 0 & -ik\mathrm{div}(\mu(\cdot)) & 0 \end{pmatrix} \\
&+ \begin{pmatrix} 0 & ik\mathrm{div}(\epsilon(\cdot)) & 0 & 0 \\ ik\epsilon\nabla & 0 & ik\nabla\times & 0 \\ 0 & -ik\nabla\times & 0 & ik\mu\nabla \\ 0 & 0 & ik\mathrm{div}(\mu(\cdot)) & 0 \end{pmatrix} \\
&+ \begin{pmatrix} k^2 & 0 & 0 & 0 \\ 0 & k^2\epsilon & 0 & 0 \\ 0 & 0 & k^2\mu & 0 \\ 0 & 0 & 0 & k^2 \end{pmatrix} \\
&= \begin{pmatrix} \mathrm{div}(\epsilon\nabla) + k^2 & 0 & 0 & 0 \\ ik(\epsilon - \mu^{-1})\nabla & L_1 + k^2\epsilon & 0 & 0 \\ 0 & 0 & L_2 + k^2\mu & ik(\mu - \epsilon^{-1})\nabla \\ 0 & 0 & 0 & \mathrm{div}(\mu\nabla) + k^2 \end{pmatrix},
\end{aligned}$$

where

$$L_1 = \mu^{-1}\nabla(\mathrm{div}(\epsilon(\cdot))) - \nabla \times (\mu^{-1}(\nabla \times (\cdot))) \tag{2.7}$$

$$L_2 = \epsilon^{-1}\nabla(\mathrm{div}(\mu(\cdot))) - \nabla \times (\epsilon^{-1}(\nabla \times (\cdot))). \tag{2.8}$$

A prominent feature of the above operator is that it decomposes the original eight-component system into two four-component systems. Precisely, Set

$$X = \begin{pmatrix} \varphi \\ e \\ h \\ \psi \end{pmatrix},$$

then (2.5) can be separated into two systems:

$$\begin{cases} \mathrm{div}(\epsilon\nabla\varphi) + k^2\varphi = 0 \\ L_1 e + k^2\epsilon e + ik(\epsilon - \mu^{-1})\nabla\varphi = 0. \end{cases}$$

and

$$\begin{cases} \operatorname{div}(\mu\nabla\psi) + k^2\psi = 0 \\ L_2 h + k^2\mu h + ik(\mu - \epsilon^{-1})\nabla\psi = 0. \end{cases}$$

Moreover,

$$\begin{aligned} Y &= \widetilde{Q}X \\ &= \left[\begin{pmatrix} 0 & \operatorname{div}(\epsilon(\cdot)) & 0 & 0 \\ \nabla & 0 & \epsilon^{-1}(\nabla\times(\cdot)) & 0 \\ 0 & -\mu^{-1}(\nabla\times(\cdot)) & 0 & \nabla \\ 0 & 0 & \operatorname{div}(\mu(\cdot)) & 0 \end{pmatrix} - ikI \right] X \\ &= \begin{pmatrix} \operatorname{div}(\epsilon e) - ik\varphi \\ \nabla\varphi + \epsilon^{-1}(\nabla\times h) - ike \\ -\mu^{-1}(\nabla\times \mathbf{e}) + \nabla\psi - ikh \\ \operatorname{div}(\mu h) - ik\psi \end{pmatrix}. \end{aligned}$$

Therefore, the problem of finding the solutions X of

$$(P + ikV)\widetilde{Q}X = 0 \text{ with the first and last component of } \widetilde{Q}X \text{ being } 0 \tag{2.9}$$

is equivalent to the problem of finding solutions of the following two separate systems:

$$\begin{cases} \operatorname{div}(\epsilon e) - ik\varphi = 0, \\ \operatorname{div}(\epsilon\nabla\varphi) + k^2\varphi = 0, \\ \mu^{-1}\nabla(\operatorname{div}(\epsilon e)) - \nabla\times(\mu^{-1}(\nabla\times e)) + k^2\epsilon e + ik(\epsilon - \mu^{-1})\nabla\varphi = 0, \end{cases} \tag{2.10}$$

and

$$\begin{cases} \operatorname{div}(\mu h) - ik\psi = 0, \\ \operatorname{div}(\mu\nabla\psi) + k^2\psi = 0, \\ \epsilon^{-1}\nabla(\operatorname{div}(\mu h)) - \nabla\times(\epsilon^{-1}(\nabla\times h)) + k^2\mu h + ik(\mu - \epsilon^{-1})\nabla\psi = 0. \end{cases} \tag{2.11}$$

Notice that if we set e in the following form

$$e = -\frac{i}{k}(\nabla\varphi + \epsilon^{-1}(\nabla\times A)), \tag{2.12}$$

then the first equation of (2.10) becomes the same as the second one. For the third equation, we have

$$\begin{aligned} &\mu^{-1}\nabla\big(\operatorname{div}(\epsilon e)\big) - \nabla\times\big(\mu^{-1}(\nabla\times e)\big) + k^2\epsilon e + ik(\epsilon - \mu^{-1})\nabla\varphi \\ &= -\frac{i}{k}\mu^{-1}\nabla\left(\operatorname{div}(\epsilon\nabla\varphi)\right) + \frac{i}{k}\nabla\times\Big(\mu^{-1}\big[\nabla\times(\epsilon^{-1}(\nabla\times A))\big]\Big) \\ &\quad - ik\epsilon\nabla\varphi - ik(\nabla\times A) + ik\epsilon\nabla\varphi - \frac{i}{k}\mu^{-1}\nabla\Big(k^2\varphi\Big) \\ &= -\frac{i}{k}\mu^{-1}\nabla\left(\operatorname{div}(\epsilon\nabla\varphi) + k^2\varphi\right) + \frac{i}{k}\nabla\times\Big(\mu^{-1}\big[\nabla\times(\epsilon^{-1}(\nabla\times A))\big]\Big) - ik\nabla\times A \\ &= 0 + \frac{i}{k}\nabla\times\Big(\mu^{-1}\big[\nabla\times(\epsilon^{-1}(\nabla\times A))\big]\Big) - ik\nabla\times A, \end{aligned}$$

by the second equation of (2.10). Thus, by letting e be of the form (2.12), the system (2.10) reduces to

$$\begin{cases} \operatorname{div}\bigl(\gamma\nabla\varphi\bigr) + k^2\varphi = 0, \\ \nabla\times\left(\mu^{-1}\bigl[\nabla\times(\epsilon^{-1}(\nabla\times A))\bigr] - k^2 A\right) = 0. \end{cases} \tag{2.13}$$

Similarly, by letting

$$h = -\frac{i}{k}(\nabla\psi + \mu^{-1}(\nabla\times B))$$

for some vector field B, we can reduce (2.11) to the following system:

$$\begin{cases} \operatorname{div}\bigl(\mu\nabla\psi\bigr) + k^2\psi = 0, \\ \nabla\times\left(\epsilon^{-1}\bigl[\nabla\times(\mu^{-1}(\nabla\times B))\bigr] - k^2 B\right) = 0. \end{cases} \tag{2.14}$$

To resume, if we can find solutions φ, A, ψ and B of (2.13) and (2.14), we can find solutions of the problem (2.9) and therefore the original problem (2.1).

Now let us focus on (2.13) and (2.14). The goal is to find special solutions (e.g. oscillating-decaying solutions) of (2.13) and (2.14). The idea of doing that is to subtract zero terms of the form $\nabla\times\bigl(\nabla tr(M^A\nabla A)\bigr)$ and $\nabla\times\bigl(\nabla tr(M^B\nabla B)\bigr)$ from the second equations of (2.13) and (2.14) for some matrices M^A, M^B, so that they become $\nabla\times(\mathcal{L}^A A) = 0$ and $\nabla\times(\mathcal{L}^B B) = 0$ with $\mathcal{L}^A$ and $\mathcal{L}^B$ being strongly elliptic operators. Precisely, we want to find suitable matrices M^A and M^B such that

$$\mu\nabla tr(M^A\nabla A) - \nabla\times\bigl(\epsilon^{-1}(\nabla\times A)\bigr) + k^2\mu A = 0 \tag{2.15}$$

and

$$\epsilon\nabla tr(M^B\nabla B) - \nabla\times\bigl(\mu^{-1}(\nabla\times B)\bigr) + k^2\epsilon B = 0 \tag{2.16}$$

are strongly elliptic systems. In fact, by letting $M^A = m\mu^{-1}I$ and $M^B = m\mu^{-1}\epsilon$, we can show that (2.15) and (2.16) are strong elliptic systems for arbitrary positive constant m. The proof are given in Theorem 2.4. $\square$

To prove Theorem 2.4, we start with the following computational lemma.

Lemma 2.3. *Let M be a matrix-valued function with smooth entries and $\mathbf{F}$ be a vector field. Then the i-th component of the vector $\nabla\times\bigl(M(\nabla\times\mathbf{F})\bigr)$ is given by*

$$\bigl(\nabla\times\bigl(M(\nabla\times\mathbf{F})\bigr)\bigr)_i = \sum_{j,k,\ell}\widetilde{C}_{ijk\ell}\partial_{j\ell}f_k + \widetilde{R}_i, \tag{2.17}$$

where

$$\widetilde{C}_{ijk\ell} = \delta_{j\ell}M_{ki} + \delta_{ik}M_{\ell j} - \delta_{jk}M_{\ell i} - \delta_{i\ell}M_{kj} + \bigl(\delta_{i\ell}\delta_{jk} - \delta_{ik}\delta_{j\ell}\bigr)tr(M),$$

and $\widetilde{R}_i$ contains the lower order terms. Here, δ_{ij} is the Kronecker delta, M_{ij} is the ij-th entry of M, and $\mathbf{F} = (f_1, f_2, f_3)^T$.

Proof. We prove it by direct computations. For any vectors $\mathbf{a}, \mathbf{b}$, letting $\mathbf{c} = \mathbf{a} \times \mathbf{b}$, we have

$$c_m = \sum_{k\ell} \varepsilon_{m\ell k} a_\ell b_k,$$

where $\mathbf{a} = (a_1, a_2, a_3)^T$, $\mathbf{b} = (b_1, b_2, b_3)^T$, $\mathbf{c} = (c_1, c_2, c_3)^T$ and $\varepsilon_{m\ell k}$ denotes the Levi-Civita symbol. Therefore, we obtain the m-th component of $\nabla \times \mathbf{F}$:

$$\Big(\nabla \times \mathbf{F}\Big)_m = \sum_{k\ell} \varepsilon_{m\ell k} \partial_\ell f_k.$$

Then, the n-th component of $M(\nabla \times \mathbf{F})$ is

$$\Big(M(\nabla \times \mathbf{F})\Big)_n = \sum_{m,k,\ell} M_{nm} \varepsilon_{m\ell k} \partial_\ell f_k.$$

Finally, taking the curl operator on the vector $M(\nabla \times \mathbf{F})$, the i-th component of the resulted vector is

$$\begin{aligned}
\Big(\nabla \times \big(M(\nabla \times \mathbf{F})\big)\Big)_i &= \sum_{j,n,m,k,\ell} \varepsilon_{ijn} \partial_j \big(M_{nm} \varepsilon_{m\ell k} \partial_\ell f_k\big) \\
&= \sum_{j,n,m,k,\ell} \varepsilon_{ijn} \varepsilon_{m\ell k} \big((\partial_j M_{nm}) \partial_\ell f_k + M_{nm} \partial_{j\ell} f_k\big)
\end{aligned}$$

Thus

$$\Big(\nabla \times \big(M(\nabla \times \mathbf{F})\big)\Big)_i = \sum_{j,k,\ell} \widetilde{C}_{ijk\ell} \partial_{j\ell} f_k + \widetilde{R}_i,$$

where

$$\widetilde{C}_{ijk\ell} := \sum_{m,n} \varepsilon_{ijn} \varepsilon_{m\ell k} M_{nm}, \qquad \widetilde{R}_i := \sum_{j,m,n,k,\ell} \varepsilon_{ijn} \varepsilon_{m\ell k} (\partial_j M_{nm}) \partial_\ell f_k.$$

Since

$$\begin{aligned}
\varepsilon_{ijn} \varepsilon_{m\ell k} &= \begin{vmatrix} \delta_{im} & \delta_{i\ell} & \delta_{ik} \\ \delta_{jm} & \delta_{j\ell} & \delta_{jk} \\ \delta_{nm} & \delta_{n\ell} & \delta_{nk} \end{vmatrix} \\
&= \delta_{im}\big(\delta_{j\ell}\delta_{nk} - \delta_{n\ell}\delta_{jk}\big) - \delta_{i\ell}\big(\delta_{jm}\delta_{nk} - \delta_{nm}\delta_{jk}\big) + \delta_{ik}\big(\delta_{jm}\delta_{n\ell} - \delta_{nm}\delta_{j\ell}\big),
\end{aligned}$$

we can obtain

$$\begin{aligned}
\widetilde{C}_{ijk\ell} &= \sum_{mn} \Big(\delta_{im}\big(\delta_{j\ell}\delta_{nk} - \delta_{n\ell}\delta_{jk}\big) - \delta_{i\ell}\big(\delta_{jm}\delta_{nk} - \delta_{nm}\delta_{jk}\big) \\
&\qquad + \delta_{ik}\big(\delta_{jm}\delta_{n\ell} - \delta_{nm}\delta_{j\ell}\big)\Big) M_{nm} \\
&= \big(\delta_{j\ell} M_{ki} - \delta_{jk} M_{\ell i}\big) - \delta_{i\ell} M_{kj} + \delta_{i\ell}\delta_{jk} tr(M) + \delta_{ik} M_{\ell j} - \delta_{ik}\delta_{j\ell} tr(M) \\
&= \delta_{j\ell} M_{ki} + \delta_{ik} M_{\ell j} - \delta_{jk} M_{\ell i} - \delta_{i\ell} M_{kj} + \big(\delta_{i\ell}\delta_{jk} - \delta_{ik}\delta_{j\ell}\big) tr(M).
\end{aligned}$$

□

Theorem 2.4. *Assume that μ is a smooth, positive scalar function and ϵ is a symmetric, positive definite matrix-valued function with smooth entries. The eigenvalues of ϵ are denoted by $\lambda_1(x), \lambda_2(x)$ and $\lambda_3(x)$. Assume there exist positive constants μ_0, Λ, λ such that for all $x \in \Omega$*

$$0 < \mu(x) \le \mu_0$$

$$0 < \lambda \le \lambda_1(x) \le \lambda_2(x) \le \lambda_3(x) \le \Lambda. \tag{2.18}$$

Then (2.15) and (2.16) are uniformly strongly elliptic by letting $M^A = m\mu^{-1}I$ and $M^B = m\mu^{-1}\epsilon$, for arbitrary positive constant m. Here I denotes the 3×3 identity matrix.

Proof. To see whether (2.15) and (2.16) are strongly elliptic, we only have to check the leading order terms of (2.15) and (2.16). We divide this proof into two parts, Part A and Part B, to deal with the equation (2.15) for A and the equation (2.16) for B respectively.

Part A. By Lemma 2.3,

$$\begin{aligned}
&\Big(\mu\nabla tr\big(M^A\nabla A\big) - \nabla\times\big(\epsilon^{-1}(\nabla\times A)\big)\Big)_i \\
&\quad = \sum_{jk\ell}\mu\delta_{ij}\partial_j\big(M^A_{\ell k}\partial_\ell A_k\big) - \sum_{jk\ell}\widetilde{C}^A_{ijk\ell}\partial_{j\ell}A_k - \widetilde{R}^A_i \\
&\quad = \sum_{jk\ell}\big(\mu\delta_{ij}M^A_{\ell k} - \widetilde{C}^A_{ijk\ell}\big)\partial_{j\ell}A_k + \sum_{jk\ell}\mu\delta_{ij}(\partial_j M^A_{\ell k})\partial_\ell A_k - \widetilde{R}^A_i \\
&\quad = \sum_{jk\ell}C^A_{ijk\ell}\partial_{j\ell}A_k + \sum_{jk\ell}\mu\delta_{ij}(\partial_j M^A_{\ell k})\partial_\ell A_k - \widetilde{R}^A_i,
\end{aligned}$$

where $C^A_{ijk\ell} = \mu\delta_{ij}M^A_{\ell k} - \widetilde{C}^A_{ijk\ell}$ are the coefficients of the leading order terms of (2.15) and

$$\widetilde{C}^A_{ijk\ell} = \delta_{j\ell}(\epsilon^{-1})_{ki} + \delta_{ik}(\epsilon^{-1})_{\ell j} - \delta_{jk}(\epsilon^{-1})_{\ell i} - \delta_{i\ell}(\epsilon^{-1})_{kj} + \big(\delta_{i\ell}\delta_{jk} - \delta_{ik}\delta_{j\ell}\big)tr(\epsilon^{-1}).$$

Recall that (2.15) is called uniformly strongly elliptic in some domain Ω if there exists a positive $c_0 > 0$ independent of $x \in \Omega$ such that

$$\sum_{ijk\ell}C^A_{ijk\ell}(x)a_i a_k b_j b_\ell \ge c_0|\mathbf{a}|^2|\mathbf{b}|^2 \tag{2.19}$$

for any $\mathbf{a}, \mathbf{b} \in \mathbb{R}^3$ and for all $x \in \Omega$. Now

$$\begin{aligned}
\sum_{ijk\ell} C^A_{ijk\ell} a_i a_k b_j b_\ell &= \sum_{ijk\ell} \big(\mu \delta_{ij} M^A_{\ell k} - \widetilde{C}^A_{ijk\ell}\big) a_i a_k b_j b_\ell \\
&= \mu(\mathbf{a}\cdot\mathbf{b})(\mathbf{b}^T M^A \mathbf{a}) \\
&\quad - \sum_{ijk\ell} \Bigg(\delta_{j\ell}(\epsilon^{-1})_{ki} + \delta_{ik}(\epsilon^{-1})_{\ell j} - \delta_{jk}(\epsilon^{-1})_{\ell i} \\
&\qquad\qquad - \delta_{i\ell}(\epsilon^{-1})_{kj} + \big(\delta_{i\ell}\delta_{jk} - \delta_{ik}\delta_{j\ell}\big) tr(\epsilon^{-1}) \Bigg) a_i a_k b_j b_\ell \\
&= \mu(\mathbf{a}\cdot\mathbf{b})(\mathbf{b}^T M^A \mathbf{a}) \\
&\quad - \Bigg(|\mathbf{b}|^2(\mathbf{a}^T\epsilon^{-1}\mathbf{a}) + |\mathbf{a}|^2(\mathbf{b}^T\epsilon^{-1}\mathbf{b}) - (\mathbf{a}\cdot\mathbf{b})(\mathbf{b}^T\epsilon^{-1}\mathbf{a}) \\
&\qquad\qquad - (\mathbf{a}\cdot\mathbf{b})(\mathbf{a}^T\epsilon^{-1}\mathbf{b}) + tr(\epsilon^{-1})(\mathbf{a}\cdot\mathbf{b})^2 - tr(\epsilon^{-1})|\mathbf{a}|^2|\mathbf{b}|^2 \Bigg) \\
&= tr(\epsilon^{-1})|\mathbf{a}|^2|\mathbf{b}|^2 - |\mathbf{a}|^2(\mathbf{b}^T\epsilon^{-1}\mathbf{b}) - |\mathbf{b}|^2(\mathbf{a}^T\epsilon^{-1}\mathbf{a}) - tr(\epsilon^{-1})(\mathbf{a}\cdot\mathbf{b})^2 \\
&\quad + 2(\mathbf{a}\cdot\mathbf{b})\big(\mathbf{b}^T\epsilon^{-1}\mathbf{a}\big) + \mu(\mathbf{a}\cdot\mathbf{b})\big(\mathbf{b}^T M^A \mathbf{a}\big)
\end{aligned}$$

since ϵ (and hence ϵ^{-1}) is symmetric. Let S be the orthogonal matrix such that $\epsilon = S^T D S$, where $D = \mathrm{diag}(\lambda_1, \lambda_2, \lambda_3)$. Thus $\epsilon^{-1} = S^T D^{-1} S$. Also let $M^A = S^T N^A S$. By letting $v = S\mathbf{a}/|\mathbf{a}|$ and $w = S\mathbf{b}/|\mathbf{b}|$, it's easy to see that (2.19) holds for all $\mathbf{a}, \mathbf{b} \in \mathbb{R}^3$ iff

$$\begin{aligned}
tr(\epsilon^{-1}) - (\mathbf{w}^T D^{-1}\mathbf{w}) - (v^T D^{-1}\mathbf{v}) - tr(\epsilon^{-1})(\mathbf{v}\cdot\mathbf{w})^2 \\
+ 2(\mathbf{v}\cdot\mathbf{w})\big(\mathbf{w}^T D^{-1}\mathbf{v}\big) + \mu(\mathbf{v}\cdot\mathbf{w})\big(\mathbf{w}^T N^A \mathbf{v}\big) \ge c_0
\end{aligned}$$

for all $\mathbf{v}, \mathbf{w} \in \mathbb{R}^3$ such that $|\mathbf{v}| = |\mathbf{w}| = 1$. Note that $tr(\epsilon^{-1}) = tr(D^{-1}) = \lambda_1^{-1} + \lambda_2^{-1} + \lambda_3^{-1}$. In summary, we find that (2.15) is uniformly strongly elliptic on Ω iff

$$\inf_{\mathbf{x}\in\Omega} \Bigg(\min_{|\mathbf{v}|=|\mathbf{w}|=1} F(\mathbf{v}, \mathbf{w}) \Bigg) > 0, \tag{2.20}$$

where

$$\begin{aligned}
F(\mathbf{v}, \mathbf{w}) &= \Bigg(tr(D^{-1}) - (\mathbf{w}^T D^{-1}\mathbf{w}) - (\mathbf{v}^T D^{-1}\mathbf{v}) - tr(D^{-1})(\mathbf{v}\cdot\mathbf{w})^2 \\
&\qquad + 2(\mathbf{v}\cdot\mathbf{w})\big(\mathbf{w}^T D^{-1}\mathbf{v}\big) \Bigg) + \mu(\mathbf{v}\cdot\mathbf{w})\big(\mathbf{w}^T N^A \mathbf{v}\big) \\
&=: G(\mathbf{v}, \mathbf{w}) + \mu(\mathbf{v}\cdot\mathbf{w})\big(\mathbf{w}^T N^A \mathbf{v}\big).
\end{aligned}$$

We will show that

$$G(\mathbf{v}, \mathbf{w}) \ge \lambda_3^{-1}\big(1 - (\mathbf{v}\cdot\mathbf{w})^2\big) \tag{2.21}$$

under the constraints $|\mathbf{v}| = |\mathbf{w}| = 1$. Then, by choosing $M^A = m\mu^{-1} I$ for some positive constant m, we also have $N^A = m\mu^{-1} I$, and

$$\begin{aligned}
F(\mathbf{v}, \mathbf{w}) &= G(\mathbf{v}, \mathbf{w}) + m(\mathbf{v}\cdot\mathbf{w})^2 \\
&\ge \lambda_3^{-1}\big(1 - (\mathbf{v}\cdot\mathbf{w})^2\big) + m(\mathbf{v}\cdot\mathbf{w})^2 \\
&= \lambda_3^{-1} + (m - \lambda_3^{-1})(\mathbf{v}\cdot\mathbf{w})^2.
\end{aligned}$$

Now since $0 \le (\mathbf{v}\cdot\mathbf{w})^2 \le 1$, if $m \ge \lambda_3^{-1}$, we have $F(\mathbf{v},\mathbf{w}) \ge \lambda_3^{-1}$, while if $m < \lambda_3^{-1}$, we have $F(\mathbf{v},\mathbf{w}) \ge \lambda_3^{-1} + (m - \lambda_3^{-1}) = m$. Remember that $\lambda_3^{-1}(x) \ge \Lambda^{-1}$ on Ω, we conclude that $F(\mathbf{v},\mathbf{w}) \ge \min(\Lambda^{-1}, m)$ for all $|\mathbf{v}| = |\mathbf{w}| = 1$ and all $x \in \Omega$.

It remains to show (2.21). For this, note that

$$G(\mathbf{v},\mathbf{w}) = \sum_{j=1,2,3} \lambda_j^{-1}\Big(1 - w_j^2 - v_j^2 - (\mathbf{v}\cdot\mathbf{w})^2 + 2(\mathbf{v}\cdot\mathbf{w})v_j w_j\Big) =: \sum_j \lambda_j^{-1} K_j.$$

We can prove $K_j \ge 0$ as follows: Since $(\mathbf{v}\cdot\mathbf{w}) - v_1 w_1 = v_2 w_2 + v_3 w_3$, by Schwarz inequality we have

$$|(\mathbf{v}\cdot\mathbf{w}) - v_1 w_1| \le \sqrt{v_2^2 + v_3^2}\sqrt{w_2^2 + w_3^2} = \sqrt{1 - v_1^2}\sqrt{1 - w_1^2}.$$

Taking square, we obtain

$$(\mathbf{v}\cdot\mathbf{w})^2 - 2(\mathbf{v}\cdot\mathbf{w})v_1 w_1 + v_1^2 w_1^2 \le 1 - v_1^2 - w_1^2 + v_1^2 w_1^2,$$

which means $K_1 \ge 0$. Similarly $K_2, K_3 \ge 0$. As a consequence, since $\lambda_1^{-1} \ge \lambda_2^{-1} \ge \lambda_3^{-1}$, we have

$$G(\mathbf{v},\mathbf{w}) \ge \lambda_3^{-1}(K_1 + K_2 + K_3) = \lambda_3^{-1}\big(1 - (\mathbf{v}\cdot\mathbf{w})^2\big),$$

which completes the proof of Part A.

Part B. For (2.16), we have

$$\begin{aligned}
&\Big(\epsilon\nabla tr\big(M^B\nabla\mathbf{B}\big) - \nabla\times\big(\mu^{-1}(\nabla\times\mathbf{B})\big)\Big)_i \\
&\quad = \sum_{jk\ell} \epsilon_{ij}\partial_j\big(M^B_{\ell k}\partial_\ell B_k\big) - \sum_{jk\ell}\widetilde{C}^B_{ijk\ell}\partial_{j\ell}B_k - \widetilde{R}^B_i \\
&\quad = \sum_{jk\ell}\big(\epsilon_{ij}M^B_{\ell k} - \widetilde{C}^B_{ijk\ell}\big)\partial_{j\ell}B_k + \sum_{jk\ell}\epsilon_{ij}(\partial_j M^B_{\ell k})\partial_\ell B_k - \widetilde{R}^B_i,
\end{aligned} \tag{2.22}$$

where

$$\begin{aligned}
\widetilde{C}^B_{ijk\ell} &= \delta_{j\ell}\mu^{-1}\delta_{ki} + \delta_{ik}\mu^{-1}\delta_{\ell j} - \delta_{jk}\mu^{-1}\delta_{\ell i} \\
&\qquad - \delta_{i\ell}\mu^{-1}\delta_{kj} + \big(\delta_{i\ell}\delta_{jk} - \delta_{ik}\delta_{j\ell}\big)tr(\mu^{-1}I) \\
&= \mu^{-1}\Big(\delta_{i\ell}\delta_{jk} - \delta_{ik}\delta_{j\ell}\Big).
\end{aligned}$$

Denote the coefficients of the leading order terms of (2.22) by $C^B_{ijk\ell}$, we have

$$C^B_{ijk\ell} = \epsilon_{ij}M^B_{\ell k} - \widetilde{C}^B_{ijk\ell} = \epsilon_{ij}M^B_{\ell k} - \mu^{-1}\Big(\delta_{i\ell}\delta_{jk} - \delta_{ik}\delta_{j\ell}\Big).$$

By choosing $M^B = m\mu^{-1}\epsilon$ we obtain

$$\sum_{ijk\ell} C^B_{ijk\ell}a_i a_k b_j b_\ell = \mu^{-1}\Big(m(\mathbf{a}^T\gamma\mathbf{b})^2 - \Big((\mathbf{a}\cdot\mathbf{b})^2 - |\mathbf{a}|^2|\mathbf{b}|^2\Big)\Big)$$

for all $\mathbf{a}, \mathbf{b} \in \mathbb{R}^3$. Remember that $\epsilon = S^T D S$. Since we have assumed $\mu^{-1} \geq \mu_0$ for some positive constant μ_0, by letting $\mathbf{v} = S\mathbf{a}/|\mathbf{a}|$ and $\mathbf{w} = S\mathbf{b}/|\mathbf{b}|$ for $\mathbf{a}, \mathbf{b} \neq 0$, we see to prove $C^B_{ijk\ell} a_i a_k b_j b_\ell \geq c_0 |\mathbf{a}|^2 |\mathbf{b}|^2$ for some constant $c_0 > 0$ is equivalent to prove

$$\inf_{\mathbf{x}\in\Omega} \min_{|\mathbf{v}|=|\mathbf{w}|=1} H(\mathbf{v}, \mathbf{w}) > 0, \tag{2.23}$$

where $H(\mathbf{v}, \mathbf{w}) = m(\mathbf{v}^T D \mathbf{w})^2 + \big(1 - (\mathbf{v} \cdot \mathbf{w})^2\big)$. Although (2.23) looks simpler than (2.20), we fail to find a simple method as before to get a clear lower bound. Nevertheless, it is also easy to see that (2.23) is true by continuity, as follows: If $(\mathbf{v} \cdot \mathbf{w})^2 = 1$, then $\mathbf{v} = \pm \mathbf{w}$, and

$$m(\mathbf{v}^T D \mathbf{w})^2 = m(\lambda_1 v_1^2 + \lambda_2 v_2^2 + \lambda_3 v_3^2)^2 \geq m\lambda_1^2.$$

By continuity, there exists $\varepsilon > 0$ such that for $0 \leq 1 - (\mathbf{v} \cdot \mathbf{w})^2 \leq \varepsilon$ we have $m(\mathbf{v}^T D \mathbf{w})^2 \geq m\lambda_1^2/2$. Thus for $0 \leq 1 - (\mathbf{v} \cdot \mathbf{w})^2 \leq \varepsilon$ we have $H(\mathbf{v}, \mathbf{w}) \geq m\lambda_1^2/2$. While for $1 - (\mathbf{v} \cdot \mathbf{w})^2 > \varepsilon$, $H(\mathbf{v}, \mathbf{w}) > \varepsilon$. Thus under the constraints $|\mathbf{v}| = |\mathbf{w}| = 1$ we obtain

$$H(\mathbf{v}, \mathbf{w}) \geq \min(m\lambda_1^2/2, \varepsilon) \geq \min(m\lambda^2/2, \varepsilon),$$

where recall that λ is the lower bound of $\lambda_1(x)$ on Ω. This completes the proof of Part B.

□

Remark 2.5. One can check that the $\widetilde{C}^A$ and $\widetilde{C}^B$ satisfy $\widetilde{C}^A_{ijk\ell} = \widetilde{C}^A_{k\ell ij}$ and $\widetilde{C}^B_{ijk\ell} = \widetilde{C}^B_{k\ell ij}$. And, by choosing $M^A = m\mu^{-1} I$ and $M^B = m\mu^{-1}\epsilon$ as above, the C^A and C^B also satisfy such symmetry. This additional property is useful in the next section.

3 Construction of oscillating-decaying solutions

In this section, we will use the reduction results in section 2 to construct oscillating-decaying solutions of (2.1). From now on, we suppose that $\mu > 0$ is a C^∞ scalar function and ϵ is a 3×3 real positive definite matrix-valued smooth function (i.e. every entry is a real C^∞ function) and E , H satisfy

$$\begin{cases} \nabla \times E - ik\mu H = 0 & \text{in } \Omega, \\ \nabla \times H + ik\epsilon E = 0 & \text{in } \Omega. \end{cases}$$

In order to obtain the oscillating-decaying solutions of E and H, we have to construct the oscillating-decaying solutions for A and B. We follow the proof in [15] to construct the oscillating-decaying solutions for A and B, but here we need to derive higher derivatives for A and B.

From [15], we borrow notation as follows. Assume that $\Omega \subset \mathbb{R}^3$ is an open set with smooth boundary and $\omega \in S^2$ is given. Let $\eta \in S^2$ and $\zeta \in S^2$ be chosen so that $\{\eta, \zeta, \omega\}$ forms an orthonormal system of $\mathbb{R}^3$. We then denote $x' = (x \cdot \eta, x \cdot \zeta)$. Let $t \in \mathbb{R}$, $\Omega_t(\omega) = \Omega \cap \{x \cdot \omega > t\}$, and $\Sigma_t(\omega) = \Omega \cap \{x \cdot \omega = t\}$ be a nonempty open set.

Theorem 3.1. *Given $\{\eta,\zeta,\omega\}$ an orthonormal system of $\mathbb{R}^3$, $x' = (x\cdot\eta, x\cdot\zeta)$ and $t \in \mathbb{R}$. We set $\Omega_t(\omega) = \Omega \cap \{x\cdot\omega > t\}$ and $\Sigma_t(\omega) = \Omega \cap \{x\cdot\omega = t\}$, then we can construct two types oscillating-decaying solutions for the Maxwell system in $\Omega_t(\omega)$ which can be useful for penetrable and impenetrable obstacles, respectively. There exist two solutions of (3.5) of the forms. The first one is*

$$\begin{cases} E = F_A^1(x)e^{i\tau x\cdot\xi}e^{-\tau(x\cdot\omega-t)A_t^A(x')}b + \Gamma^{A,1}_{\chi_t,b,t,N,\omega}(x,\tau) + r^{A,1}_{\chi_t,b,t,N,\omega}(x,\tau) & \text{in } \Omega_t(\omega), \\ H = F_A^2(x)e^{i\tau x\cdot\xi}e^{-\tau(x\cdot\omega-t)A_t^A(x')}b + \Gamma^{A,2}_{\chi_t,b,t,N,\omega}(x,\tau) + r^{A,2}_{\chi_t,b,t,N,\omega}(x,\tau) & \text{in } \Omega_t(\omega), \end{cases} \tag{3.1}$$

where $F_A^1(x) = O(\tau)$, $F_A^2(x) = O(\tau^2)$ are some smooth functions and for $|\alpha| = j$, $j = 1, 2$, we have

$$\begin{cases} \|\Gamma^{A,j}_{\chi_t,b,t,N,\omega}(x,\tau)\|_{L^2(\Omega_t(\omega))} \le c\tau^{|\alpha|-3/2}e^{-\tau(s-t)a_A}, \\ \|r^{A,j}_{\chi_t,b,t,N,\omega}(x,\tau)\|_{L^2(\Omega_t(\omega))} \le c\tau^{j-N+1/2} \end{cases} \tag{3.2}$$

for some positive constants a_A and c. The second one has the form

$$\begin{cases} E = G_B^2(x)e^{i\tau x\cdot\xi}e^{-\tau(x\cdot\omega-t)A_t^B(x')}b + \Gamma^{B,2}_{\chi_t,b,t,N,\omega}(x,\tau) + r'^{B,2}_{\chi_t,b,t,N,\omega}(x,\tau) & \text{in } \Omega_t(\omega), \\ H = G_B^1(x)e^{i\tau x\cdot\xi}e^{-\tau(x\cdot\omega-t)A_t^B(x')}b + \Gamma^{B,1}_{\chi_t,b,t,N,\omega}(x,\tau) + r^{B,1}_{\chi_t,b,t,N,\omega}(x,\tau) & \text{in } \Omega_t(\omega), \end{cases} \tag{3.3}$$

where $G_B^1(x) = O(\tau)$, $G_B^2(x) = O(\tau^2)$ are some smooth functions and for $|\alpha| = j$, $j = 1, 2$, we have

$$\begin{cases} \|\Gamma^{B,j}_{\chi_t,b,t,N,\omega}(x,\tau)\|_{L^2(\Omega_t(\omega))} \le c\tau^{|\alpha|-3/2}e^{-\tau(s-t)a_B}, \\ \|r^{B,j}_{\chi_t,b,t,N,\omega}(x,\tau)\|_{L^2(\Omega_t(\omega))} \le c\tau^{j-N+1/2} \end{cases} \tag{3.4}$$

for some positive constants a_B and c.

Proof. We want to find special solutions $A, B \in (C^\infty(\overline{\Omega_t(\omega)}\backslash\partial\Sigma_t(\omega)) \cap C^0(\overline{\Omega_t(\omega)}))^3$ with $\tau \gg 1$ satisfying Dirichlet boundary problems

$$\begin{cases} L_A A := \mu\nabla tr(M^A\nabla A) - \nabla\times(\epsilon^{-1}(\nabla\times A)) + k^2\mu A = 0 & \text{in } \Omega_t(\omega), \\ A = e^{i\tau x\cdot\xi}\left\{\chi_t(x')Q_t(x')b + \beta^A_{\chi_t,t,b,N,\omega}\right\} & \text{on } \Sigma_t(\omega), \end{cases} \tag{3.5}$$

and

$$\begin{cases} L_B B := \epsilon\nabla tr(M^B\nabla B) - \nabla\times(\mu^{-1}(\nabla\times B)) + k^2\epsilon B = 0 & \text{in } \Omega_t(\omega), \\ B = e^{i\tau x\cdot\xi}\left\{\chi_t(x')Q_t(x')b + \beta^B_{\chi_t,t,b,N,\omega}\right\} & \text{on } \Sigma_t(\omega), \end{cases} \tag{3.6}$$

where $\xi \in S^2$ lying in the span of $\{\eta,\zeta\}$ is chosen and fixed, $\chi_t(x') \in C_0^\infty(\mathbb{R}^2)$ with $\text{supp}(\chi_t) \subset \Sigma_t(\omega)$, $Q_t(x')$ is a nonzero smooth function, and $0 \ne b \in \mathbb{C}^3$ and N is some large natural number. Moreover, $\beta^A_{\chi_t,b,t,N,\omega}(x',\tau)$, $\beta^B_{\chi_t,b,t,N,\omega}(x',\tau)$ are smooth functions supported in $\text{supp}(\chi_t)$ satisfying

$$\|\beta^A_{\chi_t,b,t,N,\omega}(\cdot,\tau)\|_{L^2(\mathbb{R}^2)} \le c\tau^{-1},\ \|\beta^B_{\chi_t,b,t,N,\omega}(\cdot,\tau)\|_{L^2(\mathbb{R}^2)} \le c\tau^{-1}$$

for some constant $c > 0$. From now on, we use c to denote a general positive constant whose value may vary from line to line. As in [15], A, B satisfy second order strongly elliptic equations; then it can be written as

$$\begin{cases} A = A_{\chi_t,b,t,N,\omega} = w^A_{\chi_t,b,t,N,\omega} + r^A_{\chi_t,b,t,N,\omega}, \\ B = B_{\chi_t,b,t,N,\omega} = w^B_{\chi_t,b,t,N,\omega} + r^B_{\chi_t,b,t,N,\omega} \end{cases}$$

with

$$\begin{cases} w^A_{\chi_t,b,t,N,\omega} = \chi_t(x')Q_t e^{i\tau x\cdot\xi} e^{-\tau(x\cdot\omega-t)A^A_t(x')}b + \Gamma^A_{\chi_t,b,t,N,\omega}(x,\tau), \\ w^B_{\chi_t,b,t,N,\omega} = \chi_t(x')Q_t e^{i\tau x\cdot\xi} e^{-\tau(x\cdot\omega-t)A^B_t(x')}b + \Gamma^B_{\chi_t,b,t,N,\omega}(x,\tau), \end{cases} \tag{3.7}$$

and $r^A_{\chi_t b,t,N,\omega}, r^B_{\chi_t b,t,N,\omega}$ satisfying

$$\|r^A_{\chi_t,b,t,N,\omega}\|_{H^k(\Omega_t(\omega))} \le c\tau^{k-N+1/2}, \ \|r^B_{\chi_t,b,t,N,\omega}\|_{H^k(\Omega_t(\omega))} \le c\tau^{k-N+1/2}, \tag{3.8}$$

where $A^A_t(\cdot)$, $A^B_t(\cdot)$ are smooth matrix functions with its real part $\mathrm{Re}A^A_t(x') > 0$, $\mathrm{Re}A^B_t(x') > 0$, and $\Gamma^A_{\chi_t,b,t,N,\omega}$, $\Gamma^B_{\chi_t,b,t,N,\omega}$ are smooth functions supported in $\mathrm{supp}(\chi_t)$ satisfying

$$\begin{cases} \|\partial^\alpha_x \Gamma^A_{\chi_t,b,t,N,\omega}\|_{L^2(\Omega_s(\omega))} \le c\tau^{|\alpha|-3/2} e^{-\tau(s-t)a_A}, \\ \|\partial^\alpha_x \Gamma^B_{\chi_t,b,t,N,\omega}\|_{L^2(\Omega_s(\omega))} \le c\tau^{|\alpha|-3/2} e^{-\tau(s-t)a_B} \end{cases} \tag{3.9}$$

for $|\alpha| \in \mathbb{N}\cup\{0\}$ and $s \ge t$, where $a_A, a_B > 0$ are some constants depending on $A^A_t(x')$ and $A^B_t(x')$, respectively. We give details of the construction of A and B with the estimates (3.8) and (3.9).

In Appendix 6.1, we derive the explicit representation of A and B. Recall that E and H are represented in terms of A and B as follows:

$$\begin{cases} E = -\dfrac{i}{k}\epsilon^{-1}\nabla\times(\mu^{-1}(\nabla\times B)) - \epsilon^{-1}(\nabla\times A), \\ H = \dfrac{i}{k}\mu^{-1}\nabla\times(\epsilon^{-1}(\nabla\times A)) - \mu^{-1}(\nabla\times B). \end{cases} \tag{3.10}$$

Now, we can show that (E,H) satisfies (3.1), (3.2) and we will use this form to prove Theorem 1.1 for the penetrable case. Similarly, we can show that (E,H) satisfies (3.3), (3.4) in order to prove Theorem 1.1 for the impenetrable case. All we need to do is to differentiate A and B term by term componentwise. For the main terms of A and B, we can differentiate $\chi_t(x')Q_t e^{i\tau x\cdot\xi}e^{-\tau(x\cdot\omega-t)A^A_t(x')}b$ and $\chi_t(x')Q_t e^{i\tau x\cdot\xi}e^{-\tau(x\cdot\omega-t)A^B_t(x')}b$ directly and it is easy to see that

$$\begin{cases} \nabla\times A = \tau\widetilde{F_A}(x)e^{i\tau x\cdot\xi}e^{-\tau(x\cdot\omega-t)A^A_t(x')}b + \nabla\times\Gamma^A_{\chi_t,b,t,N,\omega}(x,\tau) + \nabla\times r^A_{\chi_t,b,t,N,\omega}, \\ \nabla\times B = \tau\widetilde{F_B}(x)e^{i\tau x\cdot\xi}e^{-\tau(x\cdot\omega-t)A^B_t(x')}b + \nabla\times\Gamma^B_{\chi_t,b,t,N,\omega}(x,\tau) + \nabla\times r^B_{\chi_t,b,t,N,\omega}, \end{cases}$$

where $\widetilde{F_A}(x)$ and $\widetilde{F_B}(x)$ are smooth matrix-valued functions and support $\mathrm{supp}(\chi_t(x'))$. For the penetrable obstacle case, we choose $A = w^A_{\chi_t,b,t,N,\omega} + r^A_{\chi_t,b,t,N,\omega}$ to be the oscillating-decaying solution satisfying $L_A A = 0$ and $B \equiv 0$ (also satisfies $L_B 0 = 0$) in $\Omega_t(\omega)$, then (3.10) will become

$$\begin{cases} E = -\epsilon^{-1}(\nabla\times A), \\ H = \dfrac{i}{k}\mu^{-1}\nabla\times(\epsilon^{-1}(\nabla\times A)), \end{cases}$$

which means

$$\begin{cases} E = F^1_A(x)e^{i\tau x\cdot\xi}e^{-\tau(x\cdot\omega-t)A^A_t(x')}b + \Gamma^{A,1}_{\chi_t,b,t,N,\omega}(x,\tau) + r^{A,1}_{\chi_t,b,t,N,\omega}(x,\tau), \\ H = F^2_A(x)e^{i\tau x\cdot\xi}e^{-\tau(x\cdot\omega-t)A^A_t(x')}b + \Gamma^{A,2}_{\chi_t,b,t,N,\omega}(x,\tau) + r^{A,2}_{\chi_t,b,t,N,\omega}(x,\tau), \end{cases}$$

where $F_A^1(x)$, $F_A^2(x)$ are smooth functions consisting of $\mu(x)$, $\epsilon(x)$, $Q_t(x')$, $A_t^A(x')$, and their curls (it can be seen by direct calculation). Moreover, by suitable choice of b (for example, we can choose $b \neq 0$ is not parallel to ξ), we will get $F_A^1(x) = O(\tau)$ and $F_A^2(x) = O(\tau^2)$. Moreover, $\Gamma^{A,1}_{\chi_t,b,t,N,\omega}$ and $\Gamma^{A,2}_{\chi_t,b,t,N,\omega}$ satisfy (3.9) for $|\alpha| = 1$ and $|\alpha| = 2$, respectively, $r^{A,1}_{\chi_t,b,t,N,\omega}$ and $r^{A,1}_{\chi_t,b,t,N,\omega}$ satisfy (3.8) for $k = 1$ and $k = 2$, respectively. Similarly, for the impenetrable obstacle case, we choose $A = 0$ and $B = w^B_{\chi_t,b,t,N,\omega} + r^B_{\chi_t,b,t,N,\omega}$ in $\Omega_t(\omega)$, then

$$\begin{cases} E = G_B^2(x)e^{i\tau x\cdot\xi}e^{-\tau(x\cdot\omega-t)A_t^B(x')}b + \Gamma^{B,2}_{\chi_t,b,t,N,\omega}(x,\tau) + r^{,B,2}_{\chi_t,b,t,N,\omega}(x,\tau), \\ H = G_B^1(x)e^{i\tau x\cdot\xi}e^{-\tau(x\cdot\omega-t)A_t^B(x')}b + \Gamma^{B,1}_{\chi_t,b,t,N,\omega}(x,\tau) + r^{B,1}_{\chi_t,b,t,N,\omega}(x,\tau), \end{cases}$$

where $G_B^1(x) = O(\tau)$ and $G_B^2(x) = O(\tau^2)$ and $\Gamma^{B,j}_{\chi_t,b,t,N,\omega}$ satisfies (3.9) for $|\alpha| = j$ and $r^{B,j}_{\chi_t,b,t,N,\omega}$ satisfies (3.9) for $k = j$. □

4 Runge approximation property

In this section, we derive the Runge approximation property for the following anisotropic Maxwell equation

$$\begin{cases} \nabla\times E - ik\mu H = 0 \\ \nabla\times H + ik\epsilon E = 0 \end{cases} \quad \text{in } \Omega,$$

where μ is a smooth scalar function defined on Ω and ϵ is a 3×3 smooth positive definite matrix. Recall that

$$\mu(x) \geq \mu_0 > 0 \text{ and } \sum_{i.j=1}^{3} \epsilon_{ij}(x)\xi_i\xi_j \geq \epsilon_0|\xi|^2 \ \forall \xi \in \mathbb{R}^3.$$

If we set $u = \begin{pmatrix} H \\ E \end{pmatrix}$ and

$$L := i\begin{pmatrix} \epsilon^{-1} & 0 \\ 0 & \mu^{-1}I_3 \end{pmatrix}\begin{pmatrix} 0 & \nabla\times \\ -\nabla\times & 0 \end{pmatrix} + kI_6, \tag{4.1}$$

then we have

$$Lu = 0, \tag{4.2}$$

where I_j means $j\times j$ identity matrix for $j = 3, 6$.

Theorem 4.1. *Let D and Ω be two open bounded domains with C^∞ boundary in $\mathbb{R}^3$ with $D \Subset \Omega$. If $u \in (H(curl, D))^2$ satisfies*

$$Lu = 0 \text{ in } D.$$

Given any compact subset $K \subset D$ and any $\epsilon > 0$, there exists $U \in (H(curl,\Omega))^2$ such that

$$LU = 0 \text{ in } \Omega,$$

and $\|U - u\|_{H(curl,K)} < \epsilon$, where $\|f\|_{H(curl,\Omega)} = \big(\|f\|_{L^2(\Omega)} + \|curl f\|_{L^2(\Omega)}\big)$.

Proof. The proof is standard and it is based on weak unique continuation property for the anisotropic Maxwell system L in (4.1) and the Hahn-Banach theorem. The unique continuation property of the system L is proved in [11]. For more details, how to derive the Runge approximation property from the weak unique continuation, we refer readers to [10]. □

5 Proof of Theorem 1.1

In this section, we want to use the Runge approximation property and the OD solutions to prove Theorem 1.1. We define B to be an open ball in $\mathbb{R}^3$ such that $\overline{\Omega} \subset B$. Assume that $\widetilde{\Omega} \subset \mathbb{R}^3$ is an open Lipschitz domain with $\overline{B} \subset \widetilde{\Omega}$. Recall we have set $\omega \in S^2$ and $\{\eta, \zeta, \omega\}$ forms an orthonormal basis of $\mathbb{R}^3$ and $t_0 = \inf_{x \in D} x \cdot \omega = x_0 \cdot \omega$, where $x_0 = x_0(\omega) \in \partial D$.

5.1 Penetrable Case

For the anisotropic Maxwell's equation

$$\begin{cases} \nabla \times E = ik\mu H \\ \nabla \times H = -ik\epsilon E \\ \mathrm{div}(\epsilon E) = 0 \\ \mathrm{div}(\mu H) = 0, \end{cases} . \tag{5.1}$$

for any $t \leq t_0$ and $\eta > 0$ small enough, in section 3, we have constructed

$$\begin{cases} E_{t-\eta} = F_A^1(x) e^{i\tau x\cdot\xi} e^{-\tau(x\cdot\omega-(t-\eta))A_t^A(x')} b + \Gamma^{A,1}_{\chi_t,b,t-\eta,N,\omega}(x,\tau) + r^{A,1}_{\chi_t,b,t-\eta,N,\omega}(x,\tau), \\ H_{t-\eta} = F_A^2(x) e^{i\tau x\cdot\xi} e^{-\tau(x\cdot\omega-(t-\eta))A_t^A(x')} b + \Gamma^{A,2}_{\chi_t,b,t-\eta,N,\omega}(x,\tau) + r^{A,2}_{\chi_t,b,t-\eta,N,\omega}(x,\tau), \end{cases}$$

to be the oscillating-decaying solutions satisfying (5.1) in $B_{t-\eta}(\omega) = B \cap \{x | x \cdot \omega > t - \eta\}$, where $F_A^1(x) = O(\tau)$ and $F_A^2(x) = O(\tau^2)$. Moreover, $\Gamma^{A,1}_{\chi_t,b,t-\eta,N,\omega}$ and $\Gamma^{A,2}_{\chi_t,b,t-\eta,N,\omega}$ satisfy (3.9) for $|\alpha| = 1$ and $|\alpha| = 2$, respectively, $r^{A,1}_{\chi_t,b,t-\eta,N,\omega}$ and $r^{A,1}_{\chi_t,b,t-\eta,N,\omega}$ satisfy (3.8) for $k = 1$ and $k = 2$, respectively. Similarly, we have

$$\begin{cases} E_t = F_A^1(x) e^{i\tau x\cdot\xi} e^{-\tau(x\cdot\omega-t)A_t^A(x')} b + \Gamma^{A,1}_{\chi_t,b,t,N,\omega}(x,\tau) + r^{A,1}_{\chi_t,b,t,N,\omega}(x,\tau), \\ H_t = F_A^2(x) e^{i\tau x\cdot\xi} e^{-\tau(x\cdot\omega-t)A_t^A(x')} b + \Gamma^{A,2}_{\chi_t,b,t,N,\omega}(x,\tau) + r^{A,2}_{\chi_t,b,t,N,\omega}(x,\tau), \end{cases}$$

so be the oscillating-decaying solutions satisfying (5.1) in $B_t(\omega) = B \cap \{x | x \cdot \omega > t\}$, where $\Gamma^{A,1}_{\chi_t,b,t,N,\omega}$ and $\Gamma^{A,2}_{\chi_t,b,t,N,\omega}$ satisfy (3.9) for $|\alpha| = 1$ and $|\alpha| = 2$, respectively, $r^{A,1}_{\chi_t,b,t,N,\omega}$ and $r^{A,1}_{\chi_t,b,t,N,\omega}$ satisfy (3.8) for $k = 1$ and $k = 2$, respectively. In fact, from the construction the oscillating-decaying solutions and the property of continuous dependence on parameters in ordinary differential equations in section 3, it is not hard to see that for any τ,

$$\begin{cases} E_{t-\eta} \to E_t \\ H_{t-\eta} \to H_t \end{cases}$$

in $H^2(B_t(\omega))$ as η tends to 0.

Note that $\overline{\Omega_t(\omega)} \subset B_{t-\eta}(\omega)$ for all $t \leq t_0$. By using the Runge approximation property, we can see that there exists a sequence of functions $(E_{\eta,\ell}, H_{\eta,\ell})$, $\ell = 1, 2, \cdots$, such that

$$\begin{cases} E_{\eta,\ell} \to E_{t-\eta} \\ H_{\eta,\ell} \to H_{t-\eta} \end{cases} \quad \text{in } H(curl, B_t(\omega)),$$

as $\ell \to \infty$, where $(E_{\eta,\ell}, H_{\eta,\ell})$ satisfy (5.1) in $\widetilde{\Omega}$ for all $\eta > 0, \ell \in \mathbb{N}$. Recall that the indicator function $I_\rho(\tau, t)$ was defined by the formula:

$$I_\rho(\tau, t) := \lim_{\eta \to 0} \lim_{\ell \to \infty} I_\rho^{\epsilon,\ell}(\tau, t),$$

where

$$I_\rho^{\eta,\ell}(\tau, t) := ik\tau \int_{\partial\Omega} (\nu \times H_{\eta,\ell}) \cdot (\overline{(\Lambda_D - \Lambda_\emptyset)(\nu \times H_{\eta,\ell})} \times \nu) dS.$$

We prove the Theorem 1.1 for the penetrable obstacle case. For the anisotropic penetrable obstacle problem

$$\begin{cases} \nabla \times E - ik\mu H = 0 & \text{in } \Omega, \\ \nabla \times H + ik\epsilon E = 0 & \text{in } \Omega, \\ \nu \times H = f & \text{on } \partial\Omega, \end{cases} \tag{5.2}$$

where k is not an eigenvalue of (5.2). Moreover, we assume μ is a positive smooth scalar function, $\epsilon = \epsilon_0(x) - \chi_D \epsilon_D(x)$, where ϵ_0 is symmetric positive definite smooth matrix, $\epsilon_D(x)$ is a symmetric smooth matrix with $\det \epsilon_D(x) \neq 0 \ \forall x \in D$ and $\chi_D = \begin{cases} 1 & x \in D \\ 0 & \text{otherwise} \end{cases}$. Moreover, we need $\epsilon = \epsilon(x)$ is a positive definite matrix satisfying the uniform elliptic condition. Recall that when $\epsilon(x) = \epsilon_0(x)$, we have constructed E_t and H_t which are oscillating-decaying solutions defined on the half space for the anisotropic Maxwell's equation

$$\begin{cases} \nabla \times E - ik\mu H = 0 & \text{in } \Omega, \\ \nabla \times H + ik\epsilon E = 0 & \text{in } \Omega, \end{cases} \tag{5.3}$$

and $\{(E_{\eta,\ell}, H_{\eta,\ell})\}$ are sequence of functions satisfying (5.3) defined on the whole Ω. Therefore, we can define the boundary data $f_{\eta,\ell} = \nu \times H_{\eta,\ell}$ on $\partial\Omega$ and solve (E, H) satisfies (5.2). Let $\widetilde{H_{\eta,\ell}} = H - H_{\eta,\ell}$ be the reflected solution, then $\widetilde{H_{\eta,\ell}}$ satisfies

$$\begin{cases} \nabla \times (\epsilon^{-1} \nabla \times \widetilde{H_{\eta,\ell}}) - k^2 \mu \widetilde{H_{\eta,\ell}} = -\nabla \times ((\epsilon^{-1}(x) - \epsilon_0^{-1}(x)) \nabla \times H_{\eta,\ell}) \text{ in } \Omega, \\ \nu \times \widetilde{H_{\eta,\ell}} = 0 \text{ on } \partial\Omega. \end{cases} \tag{5.4}$$

Lemma 5.1. *We have the following estimates*

1.

$$-\tau^{-1} I_\rho^{\eta,\ell} \geq \int_D [\epsilon(\epsilon^{-1} - \epsilon_0^{-1})^{-1} \epsilon_0^{-1} \nabla \times H_{\eta,\ell}] \cdot (\nabla \times \overline{H_{\eta,\ell}}) dx - k^2 \int_\Omega \mu |\widetilde{H_{\eta,\ell}}|^2 dx.$$

2.

$$\tau^{-1} I_\rho^{\eta,\ell}(\tau, t) \geq \int_D ((\epsilon_0^{-1} - \epsilon^{-1}) \nabla \times H_{\eta,\ell}) \cdot (\nabla \times \overline{H_{\eta,\ell}}) dx - k^2 \int_\Omega \mu |\widetilde{H_{\eta,\ell}}|^2 dx.$$

Proof. First, we need to prove the following identity

$$\begin{aligned} -\tau^{-1} I_\rho^{\eta,\ell}(\tau, t) \quad = \quad & \int_\Omega \big((\epsilon^{-1} - \epsilon_0^{-1}) \nabla \times H_{\eta,\ell}\big) \cdot (\nabla \times \overline{H_{\eta,\ell}}) dx \\ & - \int_\Omega (\epsilon^{-1} \nabla \times \widetilde{H_{\eta,\ell}}) \cdot (\nabla \times \overline{\widetilde{H_{\eta,\ell}}}) dx - k^2 \int_\Omega \mu |\widetilde{H_{\eta,\ell}}|^2 dx \end{aligned} \tag{5.5}$$

Multiplying $\overline{\widetilde{H_{\eta,l}}}$ in the equation (5.4) and integrating by parts we have

$$\int_\Omega (\epsilon^{-1}\nabla\times\widetilde{H_{\eta,\ell}})\cdot(\nabla\times\overline{\widetilde{H_{\eta,\ell}}})dx - k^2\int_\Omega \mu|\widetilde{H_{\eta,\ell}}|^2dx$$
$$+\int_\Omega((\epsilon^{-1}-\epsilon_0^{-1})\nabla\times H_{\eta,\ell})\cdot(\nabla\times\overline{\widetilde{H_{\eta,\ell}}})dx = 0,$$

$$\int_\Omega (\epsilon^{-1}\nabla\times\widetilde{H_{\eta,\ell}})\cdot(\nabla\times\overline{\widetilde{H_{\eta,\ell}}})dx - k^2\int_\Omega \mu|\widetilde{H_{\eta,\ell}}|^2dx$$
$$-\int_\Omega(\epsilon^{-1}-\epsilon_0^{-1})\nabla\times H_{\eta,\ell})\cdot(\nabla\times\overline{H_{\eta,\ell}})dx \tag{5.6}$$
$$=-\int_\Omega((\epsilon^{-1}-\epsilon_0^{-1})\nabla\times H_{\eta,\ell})\cdot(\nabla\times\overline{H})dx. \tag{5.7}$$

On the other hand, $H(x)$ satisfies

$$\nabla\times(\epsilon^{-1}(x)\nabla\times H(x)) - k^2\mu H(x) = 0, \tag{5.8}$$

then multiply by $H_{\eta,l}(x)$ in the equation (5.8) and integrating by parts we have

$$\int_\Omega((\epsilon^{-1}-\epsilon_0^{-1})\nabla\times H_{\eta,\ell})\cdot(\nabla\times\overline{H})dx \quad = \quad \int_{\partial\Omega}(\epsilon^{-1}\nabla\times\overline{H})\cdot(\nu\times H_{\eta,\ell})ds$$
$$-\int_{\partial\Omega}(\epsilon_0^{-1}\nabla\times H_{\eta,\ell})\cdot(\nu\times\overline{H})ds \tag{5.9}$$

Thus, combine (5.6), (5.9) and $\int_{\partial\Omega}(\nu\times\overline{H_{\eta,\ell}})\cdot(\epsilon_0^{-1}\nabla\times H_{\eta,\ell})ds$ is real, then we have

$$\int_\Omega (\epsilon^{-1}\nabla\times\widetilde{H_{\eta,\ell}})\cdot(\nabla\times\overline{\widetilde{H_{\eta,\ell}}})dx - k^2\int_\Omega \mu|\widetilde{H_{\eta,\ell}}|^2dx$$
$$-\int_\Omega((\epsilon^{-1}-\epsilon_0^{-1})\nabla\times H_{\eta,\ell})\cdot(\nabla\times\overline{H_{\eta,\ell}})dx \tag{5.10}$$
$$=\int_{\partial\Omega}(\nu\times H_{\eta,\ell})\cdot(\epsilon^{-1}\nabla\times\overline{H})ds - \int_{\partial\Omega}(\nu\times\overline{H})\cdot(\epsilon_0^{-1}\nabla\times H_{\eta,\ell})ds$$
$$=\int_{\partial\Omega}(\nu\times H_{\eta,\ell})\cdot(\epsilon^{-1}\nabla\times\overline{H})ds - \int_{\partial\Omega}(\nu\times\overline{H_{\eta,\ell}})\cdot(\epsilon_0^{-1}\nabla\times H_{\eta,\ell})ds$$
$$=\int_{\partial\Omega}(\nu\times H_{\eta,\ell})\cdot(\epsilon^{-1}\overline{\nabla\times H})ds - \int_{\partial\Omega}(\nu\times H_{\eta,\ell})\cdot(\epsilon_0^{-1}\overline{\nabla\times H_{\eta,\ell}})ds$$
$$=\int_{\partial\Omega}(\nu\times H_{\eta,\ell})\cdot[\overline{-ikE+ikE_{\eta,\ell}}]ds$$
$$=ik\int_{\partial\Omega}(\nu\times H_{\eta,\ell})\cdot[\overline{(\Lambda_D-\Lambda_\emptyset)(\nu\times H_{\eta,\ell})}\times\nu]ds$$
$$=\tau^{-1}I_\rho^{\eta,\ell}. \tag{5.11}$$

Second, we show the following identity

$$\int_\Omega (\epsilon_0^{-1}\nabla\times\widetilde{H_{\eta,\ell}})\cdot(\nabla\times\overline{\widetilde{H_{\eta,\ell}}})dx - k^2\int_\Omega \mu|\widetilde{H_{\eta,\ell}}|^2dx \tag{5.12}$$
$$+\int_\Omega((\epsilon^{-1}(x)-\epsilon_0^{-1}(x))\nabla\times H)\cdot(\nabla\times\overline{H})dx$$
$$=-\tau^{-1}I_\rho^{\eta,\ell}.$$

Replacing $H_{\eta,\ell}(x)$ by $H(x) - \widetilde{H_{\eta,\ell}}(x)$ in the equation (5.4), then we have

$$\nabla \times \big((\epsilon^{-1} - \epsilon_0^{-1})\nabla \times H\big) + \nabla \times \left(\epsilon_0^{-1}\nabla \times \widetilde{H_{\eta,\ell}}\right) - k^2\mu\widetilde{H_{\eta,\ell}} = 0 \text{ in } \Omega. \quad (5.13)$$

Multiplying $\overline{\widetilde{H_{\eta,l}}}(x)$ in the equation (5.13) and using integration by parts we have

$$\begin{aligned}
&\int_\Omega \big((\epsilon^{-1} - \epsilon_0^{-1})\nabla \times H\big) \cdot \left(\nabla \times \overline{\widetilde{H_{\eta,\ell}}}\right) dx \\
&+ \int_\Omega \left(\epsilon_0^{-1}\nabla \times \widetilde{H_{\eta,\ell}}\right) \cdot \left(\nabla \times \overline{\widetilde{H_{\eta,\ell}}}\right) dx - k^2 \int_\Omega \mu \left|\widetilde{H_{\eta,\ell}}\right|^2 dx = 0, \qquad (5.14)
\end{aligned}$$

since $\nu \times \widetilde{H_{\eta,l}} = 0$ on $\partial\Omega$. Then we can write equation (5.14) to be

$$\begin{aligned}
&\int_\Omega \left(\epsilon_0^{-1}\nabla \times \widetilde{H_{\eta,\ell}}\right) \cdot \left(\nabla \times \overline{\widetilde{H_{\eta,\ell}}}\right) dx - k^2 \int_\Omega \mu \left|\widetilde{H_{\eta,\ell}}\right|^2 dx \\
&+ \int_\Omega \big((\epsilon^{-1} - \epsilon_0^{-1})\nabla \times H\big) \cdot (\nabla \times \overline{H})dx \\
=& \int_\Omega \big((\epsilon^{-1} - \epsilon_0^{-1})\nabla \times H\big) \cdot (\nabla \times \overline{H_{\eta,\ell}})dx. \qquad (5.15)
\end{aligned}$$

Eliminating $H(x)$ by $\widetilde{H_{\eta,l}}(x) + H_{\eta,l}(x)$ in (5.15) we have

$$\begin{aligned}
&\int_\Omega \left(\epsilon_0^{-1}\nabla \times \widetilde{H_{\eta,\ell}}\right) \cdot \left(\nabla \times \overline{\widetilde{H_{\eta,\ell}}}\right) dx - k^2 \int_\Omega \mu \left|\widetilde{H_{\eta,\ell}}\right|^2 dx \\
&+ \int_\Omega \big((\epsilon^{-1} - \epsilon_0^{-1})\nabla \times H\big) \cdot (\nabla \times \overline{H})dx \\
=& \int_\Omega ((\epsilon^{-1}(x) - \epsilon_0^{-1}(x))\nabla \times H_{\eta,\ell}) \cdot (\nabla \times \overline{H_{\eta,\ell}})dx \\
&+ \int_\Omega ((\epsilon^{-1}(x) - \epsilon_0^{-1}(x))\nabla \times \widetilde{H_{\eta,\ell}}) \cdot (\nabla \times \overline{H_{\eta,\ell}})dx \qquad (5.16)
\end{aligned}$$

Again from (5.4) and by taking the complex conjugate, we can write

$$\nabla \times (\epsilon^{-1}\nabla \times \overline{\widetilde{H_{\eta,\ell}}}) - k^2\mu\overline{\widetilde{H_{\eta,\ell}}} + \nabla \times ((\epsilon^{-1}(x) - \epsilon_0^{-1}(x))\nabla \times \overline{H_{\eta,\ell}}) = 0. \quad (5.17)$$

Multiplying by $\widetilde{H_{\eta,l}}(x)$ in the equation (5.17) and using integration by parts we have

$$\begin{aligned}
&\int_\Omega (\epsilon^{-1}\nabla \times \overline{\widetilde{H_{\eta,\ell}}}) \cdot (\nabla \times \widetilde{H_{\eta,\ell}})dx - k^2 \int_\Omega \mu|\widetilde{H_{\eta,\ell}}|^2 dx \\
+& \int_\Omega ((\epsilon^{-1}(x) - \epsilon_0^{-1}(x))\nabla \times \overline{H_{\eta,\ell}}) \cdot (\nabla \times \widetilde{H_{\eta,\ell}})dx = 0. \qquad (5.18)
\end{aligned}$$

Then from the equations (5.16), (5.18) and the first identity (5.5), we can obtain

$$
\begin{aligned}
&\int_\Omega (\epsilon_0^{-1}\nabla\times\widetilde{H_{\eta,\ell}})\cdot(\nabla\times\overline{\widetilde{H_{\eta,\ell}}})dx - k^2\int_\Omega \mu|\widetilde{H_{\eta,\ell}}|^2dx\\
&+\int_\Omega ((\epsilon^{-1}(x)-\epsilon_0^{-1}(x))\nabla\times H)\cdot(\nabla\times\overline{H})dx\\
=&\int_\Omega ((\epsilon^{-1}(x)-\epsilon_0^{-1}(x))\nabla\times H_{\eta,\ell})\cdot(\nabla\times\overline{H_{\eta,\ell}})dx\\
&-\int_\Omega (\epsilon^{-1}\nabla\times\overline{\widetilde{H_{\eta,\ell}}})\cdot(\nabla\times\widetilde{H_{\eta,\ell}})dx + k^2\int_\Omega \mu|\widetilde{H_{\eta,\ell}}|^2dx\\
=&-\tau^{-1}I_\rho^{\eta,\ell}.
\end{aligned}
\tag{5.19}
$$

Combine (5.19) with the formula

$$
\begin{aligned}
&(\epsilon_0^{-1}\nabla\times\widetilde{H_{\eta,\ell}})\cdot(\nabla\times\overline{\widetilde{H_{\eta,\ell}}}) + ((\epsilon^{-1}-\epsilon_0^{-1})\nabla\times H)\cdot(\nabla\times\overline{H})\\
=&((\epsilon^{-1}-\epsilon_0^{-1})\nabla\times H)\cdot\nabla\times\overline{H} + \epsilon_0^{-1}(\nabla\times H)\cdot(\nabla\times\overline{H})\\
&-2\mathrm{Re}\left\{\epsilon_0^{-1}\nabla\times H\cdot\nabla\times\overline{H_{\eta,\ell}}\right\} + \epsilon_0^{-1}\nabla\times H_{\eta,\ell}\cdot\nabla\times\overline{H_{\eta,\ell}}\\
=&\epsilon^{-1}(\nabla\times H)\cdot(\nabla\times\overline{H}) - 2\mathrm{Re}\left\{\epsilon_0^{-1}\nabla\times H\cdot\nabla\times\overline{H_{\epsilon,l}}\right\} + \epsilon_0^{-1}\nabla\times H_{\epsilon,\ell}\cdot\nabla\times\overline{H_{\epsilon,\ell}}\\
=&\left[\epsilon^{-\frac{1}{2}}\nabla\times H - \epsilon^{\frac{1}{2}}\epsilon_0^{-1}\left(\nabla\times\overline{H_{\eta,\ell}}\right)\right]\cdot\overline{\left[\epsilon^{-\frac{1}{2}}\nabla\times H - \epsilon^{\frac{1}{2}}\epsilon_0^{-1}\left(\nabla\times\overline{H_{\eta,\ell}}\right)\right]}\\
&-\left[\epsilon^{\frac{1}{2}}\epsilon_0^{-1}\left(\nabla\times\overline{H_{\eta,\ell}}\right)\right]\cdot\overline{\left[\epsilon^{\frac{1}{2}}\epsilon_0^{-1}\left(\nabla\times\overline{H_{\eta,\ell}}\right)\right]} + \epsilon_0^{-1}\nabla\times H_{\eta,\ell}\cdot\nabla\times\overline{H_{\eta,\ell}}\\
=&\left[\epsilon^{-\frac{1}{2}}\nabla\times H - \epsilon^{\frac{1}{2}}\epsilon_0^{-1}\left(\nabla\times\overline{H_{\eta,\ell}}\right)\right]\cdot\overline{\left[\epsilon^{-\frac{1}{2}}\nabla\times H - \epsilon^{\frac{1}{2}}\epsilon_0^{-1}\left(\nabla\times\overline{H_{\eta,\ell}}\right)\right]}\\
&+\left(\epsilon_0^{-1}-\epsilon\epsilon_0^{-2}\right)\left(\nabla\times H_{\eta,\ell}\right)\cdot\left(\nabla\times\overline{H_{\eta,\ell}}\right)\\
\geq&[\left(I-\epsilon\epsilon_0^{-1}\right)\epsilon_0^{-1}\nabla\times H_{\eta,\ell}]\cdot(\nabla\times\overline{H_{\eta,\ell}})\\
\geq&[\epsilon(\epsilon^{-1}-\epsilon_0^{-1})^{-1}\epsilon_0^{-1}\nabla\times H_{\eta,\ell}]\cdot(\nabla\times\overline{H_{\eta,\ell}})
\end{aligned}
$$

and note that

$$
\left[\epsilon^{-\frac{1}{2}}\nabla\times H - \epsilon^{\frac{1}{2}}\epsilon_0^{-1}\left(\nabla\times\overline{H_{\eta,\ell}}\right)\right]\cdot\overline{\left[\epsilon^{-\frac{1}{2}}\nabla\times H - \epsilon^{\frac{1}{2}}\epsilon_0^{-1}\left(\nabla\times\overline{H_{\eta,\ell}}\right)\right]}\geq 0.
$$

Therefore, we get

$$
-\tau^{-1}I_\rho^{\eta,\ell}\geq\int_D[\epsilon(\epsilon^{-1}-\epsilon_0^{-1})^{-1}\epsilon_0^{-1}\nabla\times H_{\eta,\ell}]\cdot(\nabla\times\overline{H_{\eta,\ell}})dx - k^2\int_\Omega\mu|\widetilde{H_{\eta,\ell}}|^2dx
$$

which finished the part 1 of lemma 5.1. Finally, again from (5.11), we have

$$
\tau^{-1}I_\rho^{\eta,\ell}\geq\int_\Omega((\epsilon_0^{-1}-\epsilon^{-1})\nabla\times H_{\eta,\ell})\cdot(\nabla\times\overline{H_{\eta,\ell}})dx - k^2\int_\Omega\mu|\widetilde{H_{\eta,\ell}}|^2dx.
$$

□

Remark 5.2. The first inequality will be used when $\left(\epsilon^{-1}-\epsilon_0^{-1}\right)$ is strictly positive definite, i.e.

$$
\xi\cdot(\epsilon^{-1}-\epsilon_0^{-1})\xi\geq\Lambda|\xi|^2 \text{ for all } \xi\in\mathbb{R}^3 \text{ and for some } \Lambda>0;
$$

and the second inequality will be used when $\left(\epsilon_0^{-1}-\epsilon^{-1}\right)$ is strictly positive definite, i.e.

$$
\xi\cdot(\epsilon_0^{-1}-\epsilon^{-1})\xi\geq\lambda|\xi|^2 \text{ for all } \xi\in\mathbb{R}^3 \text{ and for some } \lambda>0.
$$

Now, our work is to estimate the lower order term $\widetilde{H_{\eta,\ell}}$.

5.1.1 Estimate of the lower order term $\widetilde{H_{\eta,\ell}}$

Proposition 5.3. *Assume Ω is a smooth domain and $D \Subset \Omega$. Then there exist a positive constant C and $\delta > 0$ such that*

$$\|\widetilde{H_{\eta,\ell}}\|_{L^2(\Omega)} \leq C\|\nabla \times H_{\eta,\ell}\|_{L^p(D)}$$

for every $p \in (\max\{\frac{4}{3}, \frac{2+\delta}{1+\delta}\}, 2]$.

Proof. We follow the proof of the proposition 3.2 in [7]. Fix $l \in \mathbb{N}$ and we set $f := -(\epsilon^{-1} - \epsilon_0^{-1})(\nabla \times H_{\eta,\ell})$, $g = 0$. Note that, $\epsilon^{-1} - \epsilon_0^{-1} = \epsilon^{-1}(\epsilon_D \chi_D)\epsilon_0^{-1}$ is supported in D. Then the reflected solution $\widetilde{H_{\eta,\ell}}$ satisfies

$$\begin{cases} \nabla \times (\epsilon^{-1} \nabla \times \widetilde{H_{\eta,\ell}}) - k^2 \mu \widetilde{H_{\eta,\ell}} = -\nabla \times ((\epsilon^{-1}(x) - \epsilon_0^{-1}(x)) \nabla \times H_{\eta,\ell}) \text{ in } \Omega, \\ \nu \times \widetilde{H_{\eta,\ell}} = 0 \text{ on } \partial\Omega. \end{cases} \tag{5.20}$$

From the L^p estimate (Theorem 6.4), if we consider the following problem

$$\begin{cases} \nabla \times (\epsilon^{-1} \nabla \times U) + \epsilon_{\max}^{-1} U = \nabla \times f & \text{in } \Omega, \\ \nu \times U = 0 & \text{on } \partial\Omega, \end{cases}$$

has a unique solution in $H_0^{1,q}(curl, \Omega)$, where $\epsilon_{\max}^{-1}$ is the maximum value among all eigenvalues of the matrix $\epsilon^{-1}(x)$ in the region $\overline{\Omega}$. Moreover, we have the estimate

$$\|U\|_{L^p(\Omega)} + \|\nabla \times U\|_{L^p(\Omega)} \leq C\|f\|_{L^p(\Omega)} \tag{5.21}$$

for $p \in (\frac{2+\delta}{1+\delta}, 2]$ for some $\delta > 0$ which depends only on Ω. Now, we set $\Pi_{\eta,\ell} = \widetilde{H_{\eta,\ell}} - U$, then $\Pi_{\eta,\ell}$ satisfies

$$\begin{cases} \nabla \times (\epsilon^{-1} \nabla \times \Pi_{\eta,\ell}) - k^2 \mu \Pi_{\eta,\ell} = (k^2 \mu + \epsilon_{\max}^{-1}) U \text{ in } \Omega, \\ \nu \times \Pi_{\eta,\ell} = 0 \text{ on } \partial\Omega. \end{cases} \tag{5.22}$$

By the well-posedness of (5.22) in $H(curl, \Omega)$ for the anisotropic Maxwell's equation (see Appendix), we have

$$\|\Pi_{\eta,\ell}\|_{L^2(\Omega)} + \|\nabla \times \Pi_{\eta,\ell}\|_{L^2(\Omega)} \leq C\|U\|_{L^2(\Omega)} \tag{5.23}$$

if k is not an eigenvalue. Moreover, for $p \leq 2$, it is to see that

$$\|\Pi_{\eta,\ell}\|_{L^p(\Omega)} + \|\nabla \times \Pi_{\eta,\ell}\|_{L^p(\Omega)} \leq C\|U\|_{L^2(\Omega)}.$$

Following the proof in the proposition 3.2 in [7] again, we denote $B^{p,2}_{\frac{1}{p}}(\Omega)$ to be the Sobolev-Besov space, then we have $U \in B^{p,2}_{\frac{1}{p}}(\Omega)$ and the inclusion map $B^{p,2}_{\frac{1}{p}}(\Omega) \to L^2(\Omega)$ is continuous for $p \in (\frac{4}{3}, 2]$. Moreover, since $\nabla \cdot U = 0$ and $\nu \times U = 0$ on $\partial\Omega$ and use Lemma 6.5 (property 5 in the appendix of [7]), we have the estimate

$$\|U\|_{L^2(\Omega)} \leq C\|U\|_{B^{p,2}_{\frac{1}{p}}(\Omega)} \leq C\{\|U\|_{L^p(\Omega)} + \|\nabla \times U\|_{L^p(\Omega)}\} \tag{5.24}$$

for $p \in (\frac{4}{3}, 2]$. Combining (5.21), (5.23) and (5.24), we obtain

$$\|\Pi_{\eta,\ell}\|_{L^p(\Omega)} + \|\nabla \times \Pi_{\eta,\ell}\|_{L^p(\Omega)} \le C\|f\|_{L^p(\Omega)} \tag{5.25}$$

for $p \in (\max\{\frac{4}{3}, \frac{2+\delta}{1+\delta}\}, 2]$. Since $\widetilde{H_{\eta,\ell}} = \Pi_{\eta,\ell} + U$, by using (5.21) and (5.25), we have

$$\|\widetilde{H_{\eta,\ell}}\|_{L^p(\Omega)} + \|\nabla \times \widetilde{H_{\eta,\ell}}\|_{L^p(\Omega)} \le C\|f\|_{L^p(\Omega)}. \tag{5.26}$$

Since $\nu \times \widetilde{H_{\eta,l}} = 0$ on $\partial\Omega$, we use the Lemma 6.5 again, then we can obtain

$$\begin{aligned} \|\widetilde{H_{\eta,\ell}}\|_{L^2(\Omega)} &\le C\|\widetilde{H_{\eta,\ell}}\|_{B^{p,2}_{\frac{1}{p}}(\Omega)} \\ &\le C\{\|\widetilde{H_{\eta,\ell}}\|_{L^p(\Omega)} + \|\nabla \times \widetilde{H_{\eta,\ell}}\|_{L^p(\Omega)} + \|\nabla \cdot \widetilde{H_{\eta,\ell}}\|_{L^p(\Omega)}\}. \end{aligned} \tag{5.27}$$

In addition, from (5.20), it is easy to see $0 = \nabla\cdot(\mu\widetilde{H_{\eta,\ell}}) = \nabla\mu\cdot\widetilde{H_{\eta,\ell}} + \mu(\nabla\cdot\widetilde{H_{\eta,\ell}})$, then we have

$$\|\nabla \cdot \widetilde{H_{\eta,\ell}}\|_{L^p(\Omega)} \le \frac{\|\nabla\mu\|_{L^\infty(\Omega)}}{\|\mu\|_{L^\infty(\Omega)}}\|\widetilde{H_{\eta,\ell}}\|_{L^p(\Omega)}. \tag{5.28}$$

Finally, use (5.26), (5.27) and (5.28), we will get

$$\begin{aligned} \|\widetilde{H_{\eta,\ell}}\|_{L^2(\Omega)} &\le C\{\|\widetilde{H_{\eta,\ell}}\|_{L^p(\Omega)} + \|\nabla \times \widetilde{H_{\eta,\ell}}\|_{L^p(\Omega)}\} \\ &\le C\|f\|_{L^p(\Omega)} \\ &\le C\|\nabla \times H_{\eta,\ell}\|_{L^p(D)}. \end{aligned} \tag{5.29}$$

□

Remark 5.4. In the reconstruction scheme, we need to take $\limsup_{\ell\to\infty}$ for (5.29) on both sides and $H_{t-\eta} \to H_t$ in $H(curl, \Omega_t(\omega))$ as $\eta \to 0$, then we have

$$\lim_{\eta\to 0}\limsup_{\ell\to\infty} \|\widetilde{H_{\eta,\ell}}\|_{L^2(\Omega)} \le C\|\nabla \times H_t\|_{L^p(D)},$$

for $p \in (\dfrac{4}{3}, 2]$.

In view of the lower bound, we need to introduce the sets $D_{j,\delta} \subset D$, $D_\delta \subset D$ in the following. Recall that $h_D(\rho) = \inf_{x\in D} x \cdot \rho$ and $t_0 = h_D(\rho) = x_0 \cdot \rho$ for some $x_0 \in \partial D$. $\forall\alpha \in \partial D \cap \{x \cdot \rho = h_D(\rho)\} := K$, define $B(\alpha, \delta) = \{x \in \mathbb{R}^3; |x - \alpha| < \delta\}$ $(\delta > 0)$. Note $K \subset \cup_{\alpha\in K} B(\alpha, \delta)$ and K is compact, so there exists $\alpha_1, \cdots, \alpha_m \in K$ such that $K \subset \cup_{j=1}^m B(\alpha_j, \delta)$. Thus, we define

$$D_{j,\delta} := D \cap B(\alpha_j, \delta) \text{ and } D_\delta := \cup_{j=1}^m D_{j,\delta}.$$

It is easy to see that

$$\begin{cases} \int_{D\setminus D_\delta} e^{-p\tau(x\cdot\omega - t_0)A^A_{t_0}(x')} b dx = O(e^{-pa\tau}) \\ \int_{D\setminus D_\delta} e^{-p\tau(x\cdot\omega - t_0)A^B_{t_0}(x')} b dx = O(e^{-pa\tau}) \end{cases}$$

where $A^A_{t_0}(x')$, $A^B_{t_0}(x')$ are smooth matrix-valued functions with bounded entries and their real part strictly greater than 0. so $\exists a > 0$ such that $\mathrm{Re}A^A_{t_0}(x') \ge a > 0$ and $\mathrm{Re}A^B_{t_0}(x') \ge a > 0$. Let $\alpha_j \in K$, by rotation and translation, we may assume $\alpha_j = 0$ and the vector $\alpha_j - x_0 = -x_0$ is parallel to $e_3 = (0,0,1)$. Therefore, we consider the change of coordinates near each α_j as follows:

$$\begin{cases} y' = x' \\ y_3 = x \cdot \rho - t_0, \end{cases}$$

where $x = (x_1, x_2, x_3) = (x', x_3)$ and $y = (y_1, y_2, y_3) = (y', y_3)$. Denote the parametrization of ∂D near α_j by $l_j(y')$, then we have the following estimates. Note that the oscillating-decaying solutions are well-defined in D.

Lemma 5.5. *For $q \le 2$, $\tau \gg 1$, we have the following estimates.*
1.

$$\begin{aligned} \int_D |H_t(x)|^q dx &\le \tau^{2q-1} \sum_{j=1}^m \iint_{|y'|<\delta} e^{-aq\tau l_j(y')} dy' + O(\tau^{2q-1} e^{-qa\delta\tau}) \\ &\quad + O(\tau^{2q} e^{-qa\tau}) + O(\tau e^{-c\tau}) + O(\tau^{-2N+5}) \end{aligned}$$

2.

$$\begin{aligned} \int_D |H_t|^2 dx &\ge C\tau^3 \sum_{j=1}^m \iint_{|y'|<\delta} e^{-2a\tau l_j(y')} dy' - C\tau^3 e^{-2a\delta\tau} \\ &\quad - C\tau e^{-2c\tau} - C\tau^{-2N+5} \end{aligned}$$

3.

$$\begin{aligned} \int_D |E_t(x)|^q dx &\le \tau^{q-1} \sum_{j=1}^m \iint_{|y'|<\delta} e^{-aq\tau l_j(y')} dy' + O(\tau^{q-1} e^{-qa\delta\tau}) \\ &\quad + O(\tau^q e^{-qa\tau}) + O(\tau^{-1}) + O(\tau^{-2N+3}) \end{aligned}$$

4.

$$\begin{aligned} \int_D |E_t|^2 dx &\ge C\tau \sum_{j=1}^m \iint_{|y'|<\delta} e^{-2a\tau l_j(y')} dy' - C\tau e^{-2a\delta\tau} \\ &\quad - C\tau^{-1} - C\tau^{-2N+3}, \end{aligned}$$

where E_t and H_t are oscillating-decaying solutions for the penetrable case defined in $\Omega_t(\omega)$.

Proof. The proof is via the representation of the oscillating-decaying solutions of (E_t, H_t). For $\tau \gg 1 (\tau \ll \tau^2)$, we have

$$\begin{aligned}
\int_D |H_t|^q dx &\leq C\tau^{2q} \int_D e^{-qa\tau(x\cdot\omega - t_0)} dx + C_q \int_D |\Gamma^{A,2}_{\chi_t,b,t,N,\omega}|^q dx \\
&\quad + C_q \int_D |r^{A,2}_{\chi_t,b,t,N,\omega}|^q dx \\
&\leq C\tau^{2q} \int_{D_\delta} e^{-qa\tau(x\cdot\omega - t_0)} dx + C\tau^{2q} \int_{D\setminus D_\delta} e^{-qa\tau(x\cdot\omega - t_0)} dx \\
&\quad + C_q \int_D |\Gamma^1_{A,B,\gamma,\mu}|^q dx + C_q \int_D |r^1_{A,B,\gamma,\mu}|^q dx \\
&\leq C\tau^{2q} \sum_{j=1}^m \iint_{|y'|<\delta} dy' \int_{l_j(y')}^{\delta} e^{-qa\tau y_3} dy_3 + C\tau^{2q} e^{-qa\tau} \\
&\quad + C\|\Gamma^{A,2}_{\chi_t,b,t,N,\omega}\|^2_{L^2(D)} + C\|r^{A,2}_{\chi_t,b,t,N,\omega}\|^2_{L^2(D)} \\
&\leq C\tau^{2q-1} \sum_{j=1}^m \iint_{|y'|<\delta} e^{-aq\tau l_j(y')} dy' - \frac{C}{q}\tau^{2q-1} e^{-qa\delta\tau} \\
&\quad + C\tau^{2q} e^{-qa\tau} + C\tau e^{-ca\tau} + C\tau^{-2N+5},
\end{aligned}$$

where c is a positive constant and a depending only on a_A, a_B. For the lower bound of $\int_D |H_t|^2 dx$, we have

$$\begin{aligned}
\int_D |H_t|^2 dx &\geq C\tau^4 \int_D e^{-2a\tau(x\cdot\omega - t_0)} dx - C\|\Gamma^{A,2}_{\chi_t,b,t,N,\omega}\|^2_{L^2(\Omega_{t_0}(\omega))} \\
&\quad - C\|r^{A,2}_{\chi_t,b,t,N,\omega}\|^2_{L^2(\Omega_{t_0}(\omega))} \\
&\geq C\tau^4 \int_{D_\delta} e^{-2a\tau(x\cdot\omega - t_0)} dx - C\tau e^{-c\tau} - C\tau^{-2N+5}. \\
&\geq C\tau^3 \sum_{j=1}^m \iint_{|y'|<\delta} e^{-2a\tau l_j(y')} dy' - C\tau^3 e^{-2a\delta\tau} \\
&\quad - C\tau e^{-ca\tau} - C\tau^{-2N+5}.
\end{aligned}$$

It is similar to prove the remaining case, so we omit the proof. □

Lemma 5.6. *We have the following estimate*

$$\frac{\|H_t\|^2_{L^2(D)}}{\|E_t\|^2_{L^2(D)}} \geq O(\tau^2), \ \tau \gg 1.$$

Proof. Since ∂D is Lipschitz, we have $l_j(y') \leq C|y'|$. Therefore we have the following estimate

$$\begin{aligned}
C\tau^3 \sum_{j=1}^m \iint_{|y'|<\delta} e^{-2a\tau l_j(y')} dy' &\geq C\tau^3 \sum_{j=1}^m \iint_{|y'|<\delta} e^{-2a\tau|y'|} \\
&\geq C\tau \sum_{j=1}^m \iint_{|y'|<\tau\delta} e^{-2a|y'|} dy' \\
&= O(\tau).
\end{aligned}$$

Then we use Lemma 5.5 to get

$$\begin{aligned}\frac{\|H_t\|^2_{L^2(D)}}{\|E_t\|^2_{L^2(D)}} &\geq C\tau^2\frac{1-\frac{Ce^{-2a\delta\tau}+C\tau^{-2}e^{-2c\tau}+C\tau^{-2N+2}}{\sum_{j=1}^m\iint_{|y'|<\delta}e^{-2a\tau l_j(y')}dy'}}{1-\frac{O(e^{-2\delta a\tau})+O(\tau e^{-ca\tau})+O(\tau^{-2N+2})}{\sum_{j=1}^m\iint_{|y'|<\delta}e^{-2a\tau l_j(y')}dy'}}\\ &= O(\tau^2)\ (\text{if } \tau\gg 1).\end{aligned}$$

□

Lemma 5.7. *If $t = h_D(\rho)$, then for some positive constant C, we have*

$$\liminf_{\tau\to\infty}\int_D \tau|\nabla\times H_t|^2dx \geq C.$$

Proof. Since $l_j(y')\leq C|y'|$, we have

$$\begin{aligned}\int_D|\nabla\times H_t(x)|^2dx &\geq C\int_D|E_t(x)|^2dx\\ &\geq C\tau\sum_{j=1}^m\iint_{|y'|<\delta}e^{-2a\tau l_j(y')}dy' - C\tau e^{-2a\delta\tau}\\ &\quad -C\tau^{-1}-C\tau^{-2N+3}\\ &\geq C\tau\sum_{j=1}^m\iint_{|y'|<\delta}e^{-2a\tau|y'|}dy' - C\tau e^{-2a\delta\tau}\\ &\quad -C\tau^{-1}-C\tau^{-2N+3}\\ &\geq C\tau[\tau^{-2}\sum_{j=1}^m\iint_{|y'|<\tau\delta}e^{-2a|y'|}dy'] - C\tau e^{-2a\delta\tau}\\ &\quad -C\tau^{-1}-C\tau^{-2N+3}\ (\text{as } \tau\gg 1).\end{aligned}$$

Therefore, we have

$$\liminf_{\tau\to\infty}\int_D \tau|\nabla\times H_t|^2dx \geq C.$$

□

Lemma 5.8. *For $p\in(\max\{\frac{4}{3},\frac{2+\delta}{1+\delta}\},2]$. we have the following*

$$\lim_{\eta\to 0}\limsup_{\ell\to\infty}\frac{\|\widetilde{H_{\eta,\ell}}\|^2_{L^2(\Omega)}}{\|\nabla\times H_t\|^2_{L^2(D)}}\leq C\tau^{1-\frac{2}{p}}\ (\tau\gg 1).$$

Proof. From the proposition 5.3, we have

$$\lim_{\eta\to 0}\limsup_{\ell\to\infty}\|\widetilde{H_{\eta,\ell}}\|_{L^2(\Omega)}\leq C\|\nabla\times H_t\|_{L^p(D)}.$$

Then it is easy to see the conclusion. □

Remark 5.9. Recall that the sequence $\{H_{\eta,\ell}\}$ converges to $H_{t+\eta}$ in $H(curl,K)$ as $\ell\to\infty$ for all compact subset $D\Subset K\Subset\Omega$ and $H_{t+\eta}\to H_t$ in $H^2(\Omega_t(\omega))$ as $\eta\to 0$, so we have

$$\|\nabla\times H_{\eta,\ell}\|_{L^p(D)}\to\|\nabla\times H_t\|_{L^p(D)} \text{ and } \|H_{\eta,\ell}\|_{L^2(D)}\to\|H_t\|_{L^2(D)}$$

as $\ell\to\infty$, $\eta\to 0$.

5.1.2 End of the proof of Theorem 1.1 for the penetrable case

First, we prove the case $t < h_D(\rho)$. From (5.5), we have

$$\begin{aligned} -\tau^{-1} I_\rho^{\eta,\ell}(\tau,t) &= \int_\Omega \big((\epsilon^{-1}-\epsilon_0^{-1})\nabla\times H_{\eta,\ell}\big)\cdot(\nabla\times\overline{H_{\eta,\ell}})dx \\ &\quad -\int_\Omega (\epsilon^{-1}\nabla\times\widetilde{H_{\eta,\ell}})\cdot(\nabla\times\overline{\widetilde{H_{\eta,\ell}}})dx - k^2\int_\Omega \mu|\widetilde{H_{\eta,\ell}}|^2 dx \end{aligned} \tag{5.30}$$

Note that $(\widetilde{E_{\epsilon,\ell}},\widetilde{H_{\epsilon.\ell}})$ satisfies

$$\begin{cases} \nabla\times\widetilde{E_{\eta,\ell}} - ik\mu\widetilde{H_{\eta,\ell}} = 0 & \text{in } \Omega, \\ \nabla\times\widetilde{H_{\eta,\ell}} + ik\gamma\widetilde{E_{\eta,\ell}} = ik(\epsilon_0-\epsilon)E_{\eta,\ell} & \text{in } \Omega, \end{cases}$$

and rewrite it as

$$\nabla\times(\epsilon^{-1}\nabla\times\widetilde{E_{\eta,\ell}}) - k^2\gamma\widetilde{E_{\eta,\ell}} = k^2(\epsilon-\epsilon_0)E_{\eta,\ell}. \tag{5.31}$$

Thus, we can use the same argument from the Remark 5.4 again to (5.31), it is easy to see

$$\|\widetilde{E_{\eta,\ell}}\|_{L^2(\Omega)} \le C\|E_{\eta,\ell}\|_{L^2(D)}.$$

In addition, we use the Maxwell's equation and $\epsilon-\epsilon_0 = -\epsilon_D\chi_D$, then we have

$$\begin{aligned} \int_\Omega (\epsilon^{-1}\nabla\times\widetilde{H_{\eta,\ell}})\cdot(\nabla\times\overline{\widetilde{H_{\eta,\ell}}})dx &= \int_\Omega (-ik\epsilon\widetilde{E_{\eta,\ell}} + ik(\epsilon_0-\epsilon)E_{\eta,\ell}))\cdot(\nabla\times\overline{\widetilde{H_{\eta,\ell}}})dx \\ &\le C\int_\Omega |\widetilde{E_{\eta,\ell}}|^2 dx + C\int_D |E_{\eta,\ell}|^2 dx \\ &\le C\int_D |E_{\eta,\ell}|^2 dx. \end{aligned} \tag{5.32}$$

Thus, from (5.30), Proposition 5.3, Lemma 5.5 and (5.32), we can obtain

$$|\frac{1}{\tau} I_\rho^{\eta,\ell}(\tau,t)| \le \|E_{\eta,\ell}\|^2_{H(curl,D)} + \|H_{\eta,\ell}\|^2_{H(curl,D)}.$$

From taking $\ell\to\infty$ and $\eta\to 0$, we have

$$\begin{aligned} |\frac{1}{\tau} I_\rho(\tau,t)| &\le |\tau\sum_{j=1}^m \iint_{|y'|<\delta} e^{-2a\tau l_j(y')}dy' + O(\tau^2 e^{-2a\delta\tau}) \\ &\quad + O(\tau^2 e^{-2a\tau}) + O(\tau^{-3}) + O(\tau^{-2N+3}) \\ &\le O(\tau^{-1}) + O(\tau^2 e^{-2a\delta\tau}) \\ &\quad + O(\tau^2 e^{-2a\tau}) + O(\tau^{-3}) + O(\tau^{-2N+3}). \end{aligned}$$

In particular, we get

$$\limsup_{\tau\to\infty} |\frac{1}{\tau} I_\rho(\tau,t)| = 0.$$

Second, we prove the case $t = h_D(\rho)$.

Case 1. $\xi\cdot(\epsilon^{-1}-\epsilon_0^{-1})\xi \ge \Lambda|\xi|^2$ for all $\xi\in\mathbb{R}^3$ for some $\Lambda > 0$.

From the inequality in Lemma 5.1, we have

$$\begin{aligned} -\tau^{-1}I_\rho^{\eta,\ell} &\geq \int_D [\epsilon(\epsilon-\epsilon_0^{-1})^{-1}\epsilon_0^{-1}\nabla\times H_{\eta,\ell}]\cdot(\nabla\times\overline{H_{\eta,\ell}})dx - k^2\int_\Omega \mu|\widetilde{H_{\eta,\ell}}|^2dx \\ &\quad -k^2\int_\Omega \mu|\widetilde{H_{\eta,\ell}}|^2dx \\ &\geq C\int_D |\nabla\times H_{\eta,\ell}|^2dx - c\|\widetilde{H_{\eta,\ell}}\|^2_{L^2(\Omega)}. \end{aligned}$$

By using the definition $I_\rho(\tau,t) := \lim_{\eta\to0}\lim_{\ell\to\infty} I_\rho^{\epsilon,\ell}(\tau,t)$, $\{H_{\eta,\ell}\}$ converges to H_t in $H(curl,K)$ for all compact subset $D \Subset K \Subset \Omega$ as $\ell\to\infty$, $\eta\to0$, we have

$$\begin{aligned} \frac{-I_\rho(\tau,t)}{\|\nabla\times H_t\|^2_{L^2(D)}} &\geq C\tau\left[1 - C\lim_{\epsilon\to0}\limsup_{\ell\to\infty}\frac{\|\widetilde{H_\ell}\|^2_{L^2(\Omega)}}{\|\nabla\times H_t\|^2_{L^2(D)}}\right] \\ &\geq C\tau(1-C\tau^{1-\frac{2}{p}}). \end{aligned}$$

Hence, using Lemma 5.7 we deduce that for $\tau \gg 1$,

$$|I_\rho(\tau,h_D(\rho))| \geq C > 0$$

which finishes the proof.

Case 2. $\xi\cdot(\gamma_0^{-1}-\gamma^{-1})\xi \geq \lambda|\xi|^2$ for all $\xi\in\mathbb{R}^3$ for some $\lambda>0$.

Similarly, using the inequality in Lemma 5.1, we have

$$\tau^{-1}I_\rho^{\eta,\ell}(\tau,t) \geq \int_D((\epsilon_0^{-1}-\epsilon^{-1})\nabla\times H_{\eta,\ell})\cdot(\nabla\times\overline{H_{\eta,\ell}})dx - k^2\int_\Omega \mu|H_{\eta,\ell}|^2dx.$$

Then use the same argument as in **Case 1** we can finish the proof.

5.2 Impenetrable Case

We give the proof of the second part of Theorem 1.1, since it is the hardest part. The other cases are easy since we have proved it in the penetrable case. In addition, the upper bound is easy because of the well-posedness and the L^p estimate for the indicator function, but the lower bound is not easy to see. In the following proof, we will use the layer potential properties for the exterior isotropic Maxwell's equation (with the Silver-Müller radiation condition) and the perturbation argument from the anisotropic Maxwell's equation compared with the isotropic case. In the impenetrable case, we have chosen the oscillating-decaying solution as the following form

$$\begin{cases} E_t = G_B^2(x)e^{i\tau x\cdot\xi}e^{-\tau(x\cdot\omega-t)A_t^B(x')}b + \Gamma^{B,2}_{\chi_t,b,t,N,\omega}(x,\tau) + r^{B,2}_{\chi_t,b,t,N,\omega}(x,\tau), \\ H_t = G_B^1(x)e^{i\tau x\cdot\xi}e^{-\tau(x\cdot\omega-t)A_t^B(x')}b + \Gamma^{B,1}_{\chi_t,b,t,N,\omega}(x,\tau) + r^{B,1}_{\chi_t,b,t,N,\omega}(x,\tau), \end{cases}$$

where $G_B^1(x) = O(\tau)$ and $G_B^2(x) = O(\tau^2)$ and $\Gamma^{B,j}_{\chi_t,b,t,N,\omega}$ satisfies (3.9) for $|\alpha| = j$ and $r^{B,j}_{\chi_t,b,t,N,\omega}$ satisfies (3.9) for $k = j$.

We start by the following lemma.

Lemma 5.10. *Assume that μ is a smooth scalar function and γ is a matrix-valued function. Let $(E,H)\in H(curl;\Omega\backslash\bar{D})\times H(curl;\Omega\backslash\bar{D})$ be a solution of* *the problem*

$$\begin{cases} \nabla \times E - ik\mu H = 0 & \text{in } \Omega\backslash\bar{D}, \\ \nabla \times H + i\epsilon E = 0 & \text{in } \Omega\backslash\bar{D}, \\ \nu \times E = f & \text{on } \partial\Omega, \\ \nu \times H = 0 & \text{on } \partial D, \end{cases} \tag{5.33}$$

with $f \in TH^{-1/2}(\partial\Omega)$. *If we put* $f_{\eta,\ell} = \nu \times E_{\eta,\ell}$ *with* $\{E_{\eta,\ell}\}$ *is obtained by the Runge approximation property. Then we have the identity*

$$\begin{aligned} -\frac{1}{\tau} I_\rho^{\eta,\ell}(\tau,t) &= -\int_D \{|\nabla \times E_{\eta,\ell}(x)|^2 - k^2|E_{\eta,\ell}(x)|^2\}dx \\ &\quad - \int_{\Omega\backslash\bar{D}} \{|\nabla \times \widetilde{E_{\eta,\ell}}(x)|^2 - k^2|\widetilde{E_{\eta,\ell}}(x)|^2\}dx \\ &= \int_D \{|\nabla \times H_{\eta,\ell}(x)|^2 - k^2|H_{\eta,\ell}(x)|^2\}dx \\ &\quad + \int_{\Omega\backslash\bar{D}} \{|\nabla \times \widetilde{H_{\eta,\ell}}(x)|^2 - k^2|\widetilde{H_{\eta,\ell}}(x)|^2\}dx \end{aligned}$$

and the inequality

$$-\frac{1}{\tau} I_\rho^{\eta,\ell}(\tau,t) \geq \int_D \{|\nabla \times H_{\eta,\ell}(x)|^2 - k^2|H_{\eta,\ell}(x)|^2\}dx - k^2 \int_{\Omega\backslash\bar{D}} |\widetilde{H_{\eta,\ell}}(x)|^2\}dx,$$

where $\widetilde{E_{\eta,\ell}} = E - E_{\eta,\ell}$ *and* $\widetilde{H_{\eta,\ell}} = H - H_{\eta,\ell}$ *are described in section 5.*

Proof. Use the integration by parts and the boundary condition, we have

$$\int_{\Omega\backslash\bar{D}} \epsilon^{-1}(\nabla\times E)\cdot\overline{(\nabla \times \widetilde{E_{\eta,\ell}})} - k^2\epsilon E\cdot\overline{\widetilde{E_{\eta,\ell}}}dx = -(\int_{\partial\Omega} - \int_{\partial D})ik(\nu\times H)\cdot\overline{\widetilde{E_{\eta,\ell}}}dS = 0.$$

Adding this to

$$\begin{aligned} I_\rho^{\eta,\ell} &= \int_{\partial\Omega} (\nu \times E_{\eta,\ell}) \cdot \overline{(-ikH + ikH_{\eta,\ell})}dS \\ &= \int_{\Omega\backslash\bar{D}} -(\mu^{-1}\nabla \times E_{\eta,\ell}) \cdot (\overline{\nabla \times E}) + k^2(\mu E_{\eta,\ell}) \cdot \bar{E}dx \\ &\quad + \int_\Omega \mu^{-1}|\nabla \times E_{\eta,\ell}|^2 - k^2(\mu E_{\eta,\ell}) \cdot \overline{E_{\eta,\ell}}dx + \int_{\partial D} (\nu \times E_{\eta,\ell}) \cdot (\overline{-ikH})dS \end{aligned}$$

due to the zero boundary condition on ∂D we have the last term is vanishing. □

From the above estimate, it only need to control the lower order term $\int_{\Omega\backslash\bar{D}} |\widetilde{H_{\eta,\ell}}(x)|^2dx$.

5.2.1 Estimate of the lower order term $\widetilde{H_{\eta,\ell}}$

Proposition 5.11. *Let* Ω *be a* C^1 *domain,* $D \Subset \Omega$ *be Lipschitz. Then there exists a positive constant* C *independent of* $(\widetilde{E_{\eta,\ell}}, \widetilde{H_{\eta,\ell}})$ *and* $(E_{\eta,\ell}, H_{\eta,\ell})$ *such that*

$$\int_{\Omega\backslash\bar{D}} |\widetilde{H_{\eta,\ell}}(x)|^2dx \leq C\{\|\nabla \times H_{\eta,\ell}\|^2_{L^p(D)} + \|H_{\eta,\ell}\|^2_{H^{s+1/2}(D)}\},$$

for all p *and* s *such that* $\max\{2-\delta, 4/3\} < p \leq 2$ *and* $0 < s \leq 1$ *with* $\delta > 0$.

Proof. **Step 1.** Before proving the Proposition 5.11, we consider the anisotropic Maxwell's equation in Ω as follows:

$$\begin{cases} \nabla \times E_{\eta,\ell} - ik\mu H_{\eta,\ell} = 0 & \text{in } \Omega, \\ \nabla \times H_{\eta,\ell} + ik\epsilon E_{\eta,\ell} = 0 & \text{in } \Omega, \\ \nu \times E_{\eta,\ell} := f_{\eta,\ell} \in TH^{-1/2}(\partial\Omega) & \text{on } \partial\Omega, \end{cases} \tag{5.34}$$

where $E_{\eta,\ell}$ and $H_{\eta,\ell}$ are solutions of the anisotropic Maxwell's equation. Since $\widetilde{E_{\eta,\ell}} = E - E_{\eta,\ell}$, $\widetilde{H_{\eta,\ell}} = H - H_{\eta,\ell}$, we have

$$\begin{cases} \nabla \times \widetilde{E_{\eta,\ell}} - ik\mu \widetilde{H_{\eta,\ell}} = 0 & \text{in } \Omega \backslash \bar{D}, \\ \nabla \times \widetilde{H_{\eta,\ell}} + ik\gamma \widetilde{E_{\eta,\ell}} = 0 & \text{in } \Omega \backslash \bar{D}, \\ \nu \times \widetilde{E_{\eta,\ell}} = 0 & \text{on } \partial\Omega, \\ \nu \times \widetilde{H_{\eta,\ell}} = -\nu \times H_{\eta,\ell} & \text{on } \partial D. \end{cases} \tag{5.35}$$

Step 2. Let $(E^{ex}_{\eta,\ell}, H^{ex}_{\eta,\ell})$ be the solution of the following well posed exterior Maxwell's problem

$$\begin{cases} \nabla \times E^{ex}_{\eta,\ell} - ikH^{ex}_{\eta,\ell} = 0 & \text{in } \mathbb{R}^3 \backslash \bar{D}, \\ \nabla \times H^{ex}_{\eta,\ell} + ikE^{ex}_{\eta,\ell} = 0 & \text{in } \mathbb{R}^3 \backslash \bar{D}, \\ \nu \times H^{ex}_{\eta,\ell} = -\nu \times H_{\eta,\ell} & \text{on } \partial D, \\ E^{ex}_{\eta,\ell}, H^{ex}_{\eta,\ell} \text{ satisfiy the Silver-Müller radiation condition.} \end{cases} \tag{5.36}$$

We can represent these solutions $E^{ex}_{\eta,\ell}$ and $H^{ex}_{\eta,\ell}$ by the following layer potentials

$$\begin{aligned} H^{ex}_{\eta,\ell}(x) &:= \nabla \times \int_{\partial D} \Phi_k(x,y) f(y) ds(y), \\ E^{ex}_{\eta,\ell}(x) &:= -\frac{1}{ik} \nabla \times H^{ex}_{\eta,\ell}(x), \; x \in \mathbb{R}^3 \backslash \partial D, \end{aligned}$$

where $\Phi_k(x,y) = -\dfrac{e^{ik|x-y|}}{4\pi|x-y|}$, $x, y \in \mathbb{R}^3$, $x \neq y$, is the fundamental solution of the Helmholtz equation and f is the density. Now, we follow the arguments in section 2.1 of [7] and use the same argument for the isotropic Maxwell's equation (5.36), then we have

$$\begin{cases} \|E^{ex}_{\eta,\ell}\|_{L^p(\Omega\backslash\bar{D})} \leq C\{\|\nu \times H_{\eta,\ell}\|_{L^p(\partial D)} + \|\nabla \times H_{\eta,\ell}\|_{L^p(D)}\}, \\ \|H^{ex}_{\eta,\ell}\|_{L^2(\Omega\backslash\bar{D})} \leq C\{\|\nu \times H_{\eta,\ell}\|_{L^p(\partial D)} + \|\nabla \times H_{\eta,\ell}\|_{L^p(D)}\}, \end{cases} \tag{5.37}$$

for $p \in (\frac{4}{3}, 2]$. Moreover, if we define $\mathcal{E}_{\eta,\ell} = \widetilde{E_{\eta,\ell}} - E^{ex}_{\eta,\ell}$, $\mathcal{H}_{\eta,\ell} = \widetilde{H_{\eta,\ell}} - H^{ex}_{\eta,\ell}$, then $\mathcal{E}_{\eta,\ell}$ and $\mathcal{H}_{\eta,\ell}$ satisfy the following Maxwell's equation

$$\begin{cases} \nabla \times \mathcal{E}_{\eta,\ell} - ik\mu \mathcal{H}_{\eta,\ell} = ik(1-\mu) H^{ex}_{\eta,\ell} & \text{in } \Omega \backslash \bar{D}, \\ \nabla \times \mathcal{H}_{\eta,\ell} + ik\epsilon \mathcal{E}_{\eta,\ell} = ik(\gamma - I_3) E^{ex}_{\eta,\ell} & \text{in } \Omega \backslash \bar{D}, \\ \nu \times \mathcal{H}_{\eta,\ell} = 0 & \text{on } \partial\Omega, \\ \nu \times \mathcal{E}_{\eta,\ell} = -\nu \times E^{ex}_{\eta,\ell} & \text{on } \partial D. \end{cases} \tag{5.38}$$

Step 3. Now we decompose $\mathcal{E}_{\eta,\ell} = \mathcal{E}^1_{\eta,\ell} + \mathcal{E}^2_{\eta,\ell}$ and $\mathcal{H}_{\eta,\ell} = \mathcal{H}^1_{\eta,\ell} + \mathcal{H}^2_{\eta,\ell}$, where $(\mathcal{E}^1_{\eta,\ell}, \mathcal{H}^1_{\eta,\ell})$ satisfies the following zero boundary Maxwell's equation

$$\begin{cases} \nabla \times \mathcal{E}^1_{\eta,\ell} - ik\mu\mathcal{H}^1_{\eta,\ell} = ik(1-\mu)H^{ex}_{\eta,\ell} & \text{in } \Omega\backslash\bar{D}, \\ \nabla \times \mathcal{H}^1_{\eta,\ell} + ik\epsilon\mathcal{E}^1_{\eta,\ell} = ik(\epsilon - I_3)E^{ex}_{\eta,\ell} & \text{in } \Omega\backslash\bar{D}, \\ \nu \times \mathcal{E}^1_{\eta,\ell} = \nu \times \mathcal{H}^1_{\eta,\ell} = 0 & \text{on } \partial(\Omega\backslash\bar{D}), \end{cases} \tag{5.39}$$

and $(\mathcal{E}^2_{\eta,\ell}, \mathcal{H}^2_{\eta,\ell})$ satisfies

$$\begin{cases} \nabla \times \mathcal{E}^2_{\eta,\ell} - ik\mu\mathcal{H}^2_{\eta,\ell} = 0 & \text{in } \Omega\backslash\bar{D}, \\ \nabla \times \mathcal{H}^2_{\eta,\ell} + ik\gamma\mathcal{E}^2_{\eta,\ell} = 0 & \text{in } \Omega\backslash\bar{D}, \\ \nu \times \mathcal{H}^2_{\eta,\ell} = 0 & \text{on } \partial\Omega, \\ \nu \times \mathcal{E}^2_{\eta,\ell} = -\nu \times E^{ex}_{\eta,\ell} & \text{on } \partial D. \end{cases} \tag{5.40}$$

First, we deal with the equation (5.39) by using the L^p estimate in $\Omega\backslash\bar{D}$. Note that $(\mathcal{E}^1_{\eta,\ell}, \mathcal{H}^1_{\eta,\ell})$ satisfies (5.39), then we have

$$\begin{cases} \nabla \times (\epsilon^{-1}\nabla \times \mathcal{E}^1_{\eta,\ell}) - k^2\gamma\mathcal{E}^1_{\eta,\ell} = ik\nabla \times [(\mu^{-1} - 1)H^{ex}_{\eta,\ell}] + ik(\gamma - I_3)E^{ex}_{\eta,\ell} & \text{in } \Omega\backslash\bar{D}, \\ \nu \times \mathcal{E}^1_{\eta,\ell} = 0 & \text{on } \partial(\Omega\backslash\bar{D}), \end{cases}$$

and

$$\begin{cases} \nabla \times (\epsilon^{-1}\nabla \times \mathcal{H}^1_{\eta,\ell}) - k^2\mu\mathcal{H}^1_{\eta,\ell} = ik\nabla \times [(I_3 - \epsilon^{-1})E^{ex}_{\eta,\ell}] + ik(1 - \mu)H^{ex}_{\eta,\ell} & \text{in } \Omega\backslash\bar{D}, \\ \nu \times \mathcal{H}^1_{\eta,\ell} = 0 & \text{on } \partial(\Omega\backslash\bar{D}). \end{cases}$$

Now, if we use the same method in the proof of the Proposition 5.3, we will obtain

$$\begin{cases} \|\mathcal{E}^1_{\eta,\ell}\|_{L^p(\Omega\backslash\bar{D})} + \|\nabla \times \mathcal{E}^1_{\eta,\ell}\|_{L^p(\Omega\backslash\bar{D})} \le C\{\|H^{ex}_{\eta,\ell}\|_{L^p(\Omega\backslash\bar{D})} + \|E^{ex}_{\epsilon,\ell}\|_{L^2(\Omega\backslash\bar{D})}\}, \\ \|\mathcal{H}^1_{\eta,\ell}\|_{L^p(\Omega\backslash\bar{D})} + \|\nabla \times \mathcal{H}^1_{\eta,\ell}\|_{L^P(\Omega\backslash\bar{D})} \le C\{\|E^{ex}_{\eta,\ell}\|_{L^p(\Omega\backslash\bar{D})} + \|H^{ex}_{\eta,\ell}\|_{L^2(\Omega\backslash\bar{D})}, \end{cases} \tag{5.41}$$

for any $\frac{4}{3} < p \le 2$. If we combine (5.37) and (5.41) together, we have

$$\|H^1_{\eta,\ell}\|_{L^p(\Omega\backslash\bar{D})} \le C\{\|\nu \times H_{\eta,\ell}\|_{L^p(\partial D)} + \|\nabla \times H_{\eta,\ell}\|_{L^p(D)}\}. \tag{5.42}$$

For $(\mathcal{E}^2_{\eta,\ell}, \mathcal{H}^2_{\eta,\ell})$, we apply the L^2-theory for the anisotropic Maxwell's equation, we get

$$\|\mathcal{H}^2_{\eta,\ell}\|_{L^2(\Omega\backslash\bar{D})} \le \|\mathcal{E}^2_{\eta,\ell}\|_{H(curl,\Omega\backslash\bar{D})} \le C\|\nu\times\mathcal{E}^2_{\eta,\ell}\|_{H^{-1/2}(\partial\Omega)} \le C\|\nu\times E^{ex}_{\eta,\ell}\|_{H^{-1/2}(\partial\Omega)}.$$

Moreover, following the proof in the Lemma 2.3 of [7], we have

$$\|\nu \times E^{ex}_{\eta,\ell}\|_{H^{-1/2}(\partial\Omega)} \le C\|f\|_{L^p(\partial D)},\ \forall p \ge 1,$$

and

$$\|\mathcal{H}^2_{\eta,\ell}\|_{L^2(\Omega\backslash\bar{D})} \le C\{\|\nu \times H_{\eta,\ell}\|^2_{L^p(\partial D)} + \|\nabla \times H_{\eta,\ell}\|^2_{L^p(D)}\}, \tag{5.43}$$

for all $p \in (\frac{4}{3}, 2]$. Recall that $\mathcal{H}_{\eta,\ell} = \mathcal{H}^1_{\eta,\ell} + \mathcal{H}^2_{\eta,\ell}$, by using (5.42) and (5.43), then we have

$$\|\mathcal{H}_{\eta,\ell}\|_{L^2(\Omega\backslash\bar{D})} \le C\{\|\nu \times H_{\eta,\ell}\|_{L^p(\partial D)} + \|\nabla \times H_{\eta,\ell}\|_{L^p(D)}\} \tag{5.44}$$

for all $p \in (\frac{4}{3}, 2]$. Combining (5.37), (5.44) and $\widetilde{H_{\eta,\ell}} = \mathcal{H}_{\eta,\ell} + H^{ex}_{\eta,\ell}$, we get

$$\begin{aligned}\int_{\Omega\setminus\bar{D}} |\widetilde{H_{\eta,\ell}}(x)|^2 dx &\leq \|\mathcal{H}_{\eta,\ell}\|_{L^2(\Omega\setminus\bar{D})} + \|H^{ex}_{\eta,\ell}\|_{L^2(\Omega\setminus\bar{D})} \\ &\leq C\{\|\nu \times H_{\eta,\ell}\|^2_{L^p(\partial D)} + \|\nabla \times H_{\eta,\ell}\|^2_{L^p(D)}\} \quad (5.45)\end{aligned}$$

for all $p \in (\frac{4}{3}, 2]$. Finally, for $s > 0$ and $p \leq 2$ we have $H^s(\partial D) \subset L^2(\partial D) \subset L^p(\partial D)$, then we reduce that

$$\|\nu \times H_{\eta,\ell}\|_{L^p(\partial D)} \leq C\|H_{\eta,\ell}\|_{L^p(\partial D)} \leq C\|H_{\eta,\ell}\|_{H^s(\partial D)}.$$

Note that the trace map from $H^{s+1/2}(D) \to H^s(\partial D)$ is bounded for all $0 < s \leq 1$. So the estimate (5.45) will become

$$\int_{\Omega\setminus\bar{D}} |\widetilde{H_{\eta,\ell}}(x)|^2 dx \leq C\{\|H_{\eta,\ell}\|^2_{H^{s+1/2}(D)} + \|\nabla \times H_{\eta,\ell}\|^2_{L^p(D)}\},$$

for all $p \in (\frac{4}{3}, 2]$ and $0 < s \leq 1$. □

Remark 5.12. Now, if we take $\ell \to \infty$ and $\epsilon \to 0$, we will get

$$\lim_{\eta\to 0} \limsup_{\ell\to\infty} \int_{\Omega\setminus\bar{D}} |\widetilde{H_{\eta,\ell}}(x)|^2 dx \leq C\{\|H_t\|^2_{H^{s+1/2}(D)} + \|\nabla \times H_t\|^2_{L^p(D)}\},$$

where H_t is the oscillating-decaying solution defined on $\Omega_t(\omega)$.

We have the following lemmas for the oscillating-decaying solutions in the same way as we did in section 5, so we omit the proofs.

Lemma 5.13. *For $1 \leq q < \infty$, $\tau \gg 1$, we have the following estimates.*

1.

$$\begin{aligned}\int_D |H_t(x)|^q dx &\leq \tau^{q-1} \sum_{j=1}^m \iint_{|y'|<\delta} e^{-aq\tau l_j(y')} dy' + O(\tau^{q-1} e^{-qa\delta\tau}) \\ &\quad + O(\tau^q e^{-qa\tau}) + O(\tau^{-1}) + O(\tau^{-2N+3})\end{aligned}$$

2.

$$\begin{aligned}\int_D |H_t|^2 dx &\geq C\tau \sum_{j=1}^m \iint_{|y'|<\delta} e^{-2a\tau l_j(y')} dy' - C\tau e^{-2a\delta\tau} \\ &\quad - C\tau^{-1} - C\tau^{-2N+3}\end{aligned}$$

3.

$$\begin{aligned}\int_D |\nabla \times H_t(x)|^q dx &\leq \tau^{2q-1} \sum_{j=1}^m \iint_{|y'|<\delta} e^{-aq\tau l_j(y')} dy' + O(\tau^{2q-1} e^{-qa\delta\tau}) \\ &\quad + O(\tau^{2q} e^{-qa\tau}) + O(\tau e^{-c\tau}) + O(\tau^{-2N+5})\end{aligned}$$

4.

$$\begin{aligned}\int_D |\nabla \times H_t(x)|^2 dx &\geq C\tau^3 \sum_{j=1}^m \iint_{|y'|<\delta} e^{-2a\tau l_j(y')} dy' - C\tau^3 e^{-2a\delta\tau} \\ &\quad - C\tau e^{-c\tau} - C\tau^{-2N+5}\end{aligned}$$

Lemma 5.14. *We have the following estimate*

$$\frac{\|H_t\|^2_{L^2(D)}}{\|\nabla\times H_t\|^2_{L^2(D)}} \le O(\tau^{-2}),\ \tau\gg 1.$$

For $p<2$, we have the following estimate

$$\frac{\|\nabla\times H_t\|^2_{L^p(D)}}{\|\nabla\times H_t\|^2_{L^p(D)}} \le C\tau^{1-\frac{2}{p}},\ \tau\gg 1.$$

Lemma 5.15. *If $t=h_D(\rho)$,then for some positive constant C, we have*

$$\liminf_{\tau\to\infty}\int_D \tau|\nabla\times H_t|^2dx \ge C.$$

5.2.2 End of the proof of Theorem 1.1 for the impenetrable case

By using the same argument in the penetrable case, it is easy to see that

$$\limsup_{\tau\to\infty}|\frac{1}{\tau}I_\rho(\tau,t)|=0$$

for $t>h_D(\rho)$. Recall that from Lemma 5.10, we have

$$-\frac{1}{\tau}I^{\eta,\ell}_\rho(\tau,t)\ge \int_D\{|\nabla\times H_{\eta,\ell}(x)|^2-k^2|H_{\eta,\ell}(x)|^2\}dx-k^2\int_{\Omega\setminus\bar D}|\widetilde{H_{\eta,\ell}}(x)|^2\}dx. \tag{5.46}$$

By using Proposition 6.2, we deduce

$$-\frac{1}{\tau}I^{\eta,\ell}_\rho(\tau,t)\ge \int_D\{|\nabla\times H_{\eta,\ell}(x)|^2-k^2|H_{\eta,\ell}(x)|^2\}dx-C\{\|H_t\|^2_{H^{s+1/2}(D)}+\|\nabla\times H_t\|^2_{L^p(D)}\},$$

where $0<s\le 1$ and $\frac{4}{3}<p\le 2$. We want to estimate $\frac{\|H_t\|^2_{H^{s+1/2}(D)}}{\|\nabla\times H_t\|^2_{L^2(D)}}$, for $0<s\le 1$. Set $r=s+1/2$, then we need to estimate

$$\frac{\|H_t\|^2_{H^r(D)}}{\|\nabla\times H_t\|^2_{L^2(D)}}$$

for $r\in(\frac{1}{2},\frac{3}{2}]$. Using the interpolation inequality, we have

$$\|H_t\|_{H^r(D)}\le C\|H_t\|^{1-r}_{L^2(D)}\|H_t\|^r_{H^1(D)},\ 0\le r\le 1.$$

By the Young's inequality $ab\le \delta^{-\alpha}\frac{a^\alpha}{\alpha}+\delta^\beta\frac{b^\beta}{\beta}$, $\frac{1}{\alpha}+\frac{1}{\beta}=1$, we obtain

$$\begin{aligned}\|H_t\|^2_{H^r(D)} &\le C\left[\frac{\delta^{-\alpha}}{\alpha}\|H_t\|^2_{L^2(D)}+\frac{\delta^\beta}{\beta}\|H_t\|^2_{H^1(D)}\right]\\ &\le C\left[\{(1-r)\delta^{-(1-r)^{-1}}+r\delta^{r^{-1}}\}\|H_0\|^2_{L^2(D)}+r\delta^{r^{-1}}\|\nabla H_t\|^2_{L^2(D)}\right]\end{aligned} \tag{5.47}$$

Recall that $H_t=G^1_B(x)e^{i\tau x\cdot\xi}e^{-\tau(x\cdot\omega-t)A^B_t(x')}b+\Gamma^{B,1}_{\chi_t,b,t,N,\omega}(x,\tau)+r^{B,1}_{\chi_t,b,t,N,\omega}(x,\tau)$ is a smooth function with $G^1_B(x)=O(\tau)$ and $\Gamma^{B,1}_{\chi_t,b,t,N,\omega}$ satisfies (3.9) for $|\alpha|=1$

and $r^{B,1}_{\chi_t,b,t,N,\omega}$ satisfies (3.9) for $k=1$. If we can differentiate H_t componentwisely, we will get $\dfrac{\partial H_t}{\partial x_j} = \dfrac{\partial G^1_B e^{i\tau x\cdot\xi} e^{-\tau(x\cdot\omega - t)A^B_t} b}{\partial x_j} + \dfrac{\partial \Gamma^{B,1}_{\chi_t,b,t,N,\omega}}{\partial x_j} + \dfrac{\partial r^{B,1}_{\chi_t,b,t,N,\omega}}{\partial x_j}$ and

$$\begin{cases} \| \dfrac{\partial N^t_{A,B,\gamma,\mu}}{\partial x_j} \|^2_{L^2(D)} \le C\tau^4 \int_D e^{-2a\tau(x\cdot\rho - t)} dx, \\ \| \dfrac{\partial \Gamma^{2,t}_{A,B,\gamma,\mu}}{\partial x_j} \|_{L^2(D)} \le c\tau^{-1/2} e^{-c\tau}. \\ \| \dfrac{\partial r^{2,t}_{A,B,\gamma,\mu}}{\partial x_j} \|_{L^2(D)} \le c\tau^{-N+3/2}. \end{cases}$$

Then by using the same method as before, it is easy to see that

$$\begin{aligned} \|\nabla H_t\|^2_{L^2(D)} &= \sum_{j=1}^3 \| \frac{\partial H_t}{\partial x_j} \|^2_{L^2(D)} \\ &\le C\tau^4 \int_D e^{-2(x\cdot\rho - t)} dx + c\tau^{-1} e^{-2c\tau} + c\tau^{-2N+3}. \end{aligned}$$

For $t = h_D(\rho)$, we have

$$\begin{aligned} \|\nabla H_t\|^2_{L^2(D)} &\le C\tau^4 \int_D e^{-2a(x\cdot\rho - h_D(\rho))} dx + c\tau^{-1} e^{-2c\tau} + c\tau^{-2N+3} \\ &\le C\tau^4 (\int_{D_\delta} + \int_{D\setminus D_\delta}) e^{-2a(x\cdot\rho - h_D(\rho))} dx + c\tau^{-1} e^{-2\tau(s-t)a} \\ &\quad + c\tau^{-2N+3} \\ &\le C\tau^4 \sum_{j=1}^m \iint_{|y'|<\delta} dy' \int_{l_j(y')}^{\delta} e^{-2a\tau y_3} dy_3 + C\tau^4 e^{-2ac\tau} \\ &\quad + c\tau^{-1} e^{-2c\tau} + c\tau^{-2N+3} \\ &\le C\tau^3 \sum_{j=1}^m \iint_{|y'|<\delta} e^{-2a\tau l_j(y')} dy' - C\tau^3 e^{-2a\delta\tau} \\ &\quad + C\tau^3 e^{-2ac\tau} + c\tau^{-1} e^{-2c\tau} + c\tau^{-2N+3}. \end{aligned} \tag{5.48}$$

From Lemma 5.13 and (5.48), we have

$$\frac{\|\nabla H_t\|^2_{L^2(D)}}{\|\nabla \times H_t\|^2_{L^2(D)}} \le C. \tag{5.49}$$

Combining Lemma 5.13, (5.47) and (5.49) we obtain

$$\begin{aligned} \frac{\|H_t\|^2_{H^r(D)}}{\|\nabla \times H_t\|^2_{L^2(D)}} &\le C\{(1-r)\delta^{-(1-r)^{-1}} + r\delta^{r^{-1}}\} \frac{\|H_t\|^2_{L^2(D)}}{\|\nabla \times H_t\|^2_{L^2(D)}} \\ &\quad + Cr\delta^{r^{-1}} \frac{\|\nabla H_t\|^2_{L^2(D)}}{\|\nabla \times H_t\|^2_{L^2(D)}} \\ &\le C\{(1-r)\delta^{-(1-r)^{-1}} + r\delta^{r^{-1}}\} O(\tau^{-2}) + Cr\delta^{r^{-1}}. \end{aligned}$$

We now choose $p \in (\frac{4}{3}, 2)$, combining (5.46), (5.47) and (5.49) we have

$$\begin{aligned}
\frac{-\frac{1}{\tau} I_\rho^{\epsilon,\ell}(\tau, t)}{\|\nabla \times H_t\|^2_{L^2(D)}} &\geq C - c_1 \frac{\|H_t\|^2_{L^2(D)}}{\|\nabla \times H_t\|^2_{L^2(D)}} - c_2 \frac{\|H_t\|^2_{H^r(D)}}{\|\nabla \times H_t\|^2_{L^2(D)}} - c_3 \frac{\|\nabla \times H_t\|^2_{L^p(D)}}{\|\nabla \times H_t\|^2_{L^2(D)}} \\
&\geq C - c_1\{(1-r)\delta^{-(1-r)^{-1}} + r\delta^{r^{-1}}\} O(\tau^{-2}) - Cr\delta^{r^{-1}} - c_3 \tau^{1-\frac{2}{p}} \\
&\geq C - c_2 r \delta^{r^{-1}}, \ \frac{1}{2} < r < 1, \ \tau \gg 1.
\end{aligned}$$

Hence from Lemma 5.15, we have

$$\liminf_{\tau \to \infty} |I_\rho(\tau, h_D(\rho))| \geq c > 0.$$

6 Appendix

6.1 Construction of the oscillating-decaying solutions A and B

In this subsection, we show how the scheme in [15] can be used to derive the oscillating-decaying solutions A and B. Recall that E and H satisfy equation (2.2), therefore we need to derive estimates of the higher derivatives for A and B. Note that the main term of $w^A_{\chi_t,b,t,N,\omega}$ (resp. $w^B_{\chi_t,b,t,N,\omega}$) is $\chi_t(x')Q_t e^{i\tau x\cdot\xi} e^{-\tau(x\cdot\omega - t)A^A_t(x')} b$ (resp. $\chi_t(x')Q_t e^{i\tau x\cdot\xi} e^{-\tau(x\cdot\omega - t)A^B_t(x')} b$), which can be directly differentiated term by term since it is a multiplication of smooth functions. So we can calculate E and H directly. For convenience, we denote $w = w_{\chi_t,b,t,N,\omega}$ $\gamma = \gamma_{\chi_t,b,t,N,\omega}(x,\tau)$. Without loss of generality, we can use the change of coordinates to assume $t = 0$, $\omega = (0,0,1)$ and $\eta = (1,0,0)$, $\zeta = (0,1,0)$. Define

$$\widetilde{Q_A} := e^{-i\tau x'\cdot\xi'} L_A(e^{i\tau x'\cdot\xi'} \cdot), \ \widetilde{Q_B} := e^{-i\tau x'\cdot\xi'} L_B(e^{i\tau x'\cdot\xi'} \cdot)$$

where $x' = (x_1, x_2)$, $\xi' = (\xi_1, \xi_2)$ with $|\xi'| = 1$ and L_A, L_B have been defined by (3.5) and (3.6). In the following, we will give all the details for the higher derivatives of E and H.

In [15], the authors used the phase plane method to get a first order ODE system and we want to decouple the equation in order to solve it by direct calculations. The method of construction the oscillating-decaying solution is decomposed into several steps:

Step 1. As mentioned before, we set $\widetilde{Q_A} = e^{-i\tau x'\cdot\xi'} L_A(e^{i\tau x'\cdot\xi'} \cdot)$, $\widetilde{Q_B} := e^{-i\tau x'\cdot\xi'} L_B(e^{i\tau x'\cdot\xi'} \cdot)$ and solve $\widetilde{Q_A} v_A = 0$, $\widetilde{Q_B} v_B = 0$. In the following calculations, we only need to consider $\widetilde{Q_A} v_A = 0$ since $\widetilde{Q_B} v_B = 0$ will follow the same calculations. Let $Q_A = C_A \widetilde{Q_A}$ be the operator which satisfies the leading coefficient of ∂_3^2 is 1 and the existence of C_A is given by the strong ellipticity of L_A and we need to solve $Q_A v_A = 0$ (the same reason for the operator $\widetilde{Q_B}$ and Q_B). Now, We introduce the concept of the order in the following manner. We consider τ, ∂_3 are of order 1, ∂_1, ∂_2 are of order 0 and x_3 is of order -1.

Step 2. Use the Taylor expansion with respect to x_3, we have

$$\begin{aligned}
Q_A(x', x_3) &= Q_A(x', 0) + \cdots + \frac{x_3^{N-1}}{(N-1)!} \partial_3^{N-1} Q_A(x', 0) + R \\
&= Q_A^2 + Q_A^1 + \cdots + Q_A^{-N+1} + R
\end{aligned}$$

where $\mathrm{ord}(Q^j_A) = j$ and $\mathrm{ord}(R) = -N$. Since we hope that $Q_A v_A = 0$, we have

$$Q^2_A v_A = -(Q^2_A + Q^1_A + \cdots + Q^{-N+1}_A + R)v_A := f.$$

Step 3. Following the paper [15], we denote $D_3 = -i\partial_3$, $\rho = (\xi_1, \xi_2, 0)$ and $\langle a, b\rangle = (\langle a, b\rangle_{ik})$ for $a = (a_1, a_2, a_3)$ and $b = (b_1, b_2, b_3)$, where $\langle a, b\rangle_{ik} = \sum_{jl} C^A_{ijkl} a_j b_l$ with C^A_{ijkl} being the leading coefficient of the second order strongly elliptic operator L_A. If we set $W = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$, where

$$\begin{cases} w_1 = v_A \\ w_2 = -\tau^{-1} \langle e_3, e_3\rangle_{x_3=0} D_3 v_A - \langle e_3, \rho\rangle_{x_3=0} v_A \end{cases},$$

and use $f = -(Q^2_A + Q^1_A + \cdots + Q^{-N+1}_A + R)v_A$, then W will satisfy

$$\begin{aligned} D_3 W &= \tau K^A W + \begin{bmatrix} 0 \\ \tau^{-1} \langle e_3, e_3\rangle_{x_3=0} f \end{bmatrix} \\ &= (\tau K^A + K^A_0 + \cdots + K^A_{-N} + S)W \end{aligned}$$

where K^A is a matrix in depending of x_3 which can be diagonlizable by the property of the strong ellipticity of L_A. Note that each K^A_j's only involves the x' derivatives with $\mathrm{ord}(K^A_j) = j$, $\mathrm{ord}(S) = -N - 1$. It is worth to mention that with the help of such special W, then we can solve the ODE system explicitly.
Step 4. Decompose K^A such that

$$\widetilde{K^A} = \tilde{Q}^{-1} K^A \tilde{Q} = \begin{bmatrix} \widetilde{K^A_+} & 0 \\ 0 & \widetilde{K^A_-} \end{bmatrix},$$

where $\mathrm{spec}(\widetilde{K^A_\pm}) \subset \mathbb{C}_\pm := \{\pm Im\lambda > 0\}$ (the existence of $\widetilde{K^A}$ and $\tilde{Q}$ were showed in [15]). If we set $\widehat{W} = \tilde{Q}^{-1} W$, then

$$D_3 \widehat{W} = (\tau \widetilde{K^A} + \widehat{K}_0 + \cdots + \widehat{K}_{-N} + \widehat{S})\widehat{W},$$

Step 5. If we write $\widehat{W} = (I + x_3 A^{(0)} + B^{(0)})\widetilde{W}^{(0)}$ with $A^{(0)}, B^{(0)}$ being differential operators in $\partial_{x'}$ (their coefficients independent of x_3), then

$$\begin{aligned} D_3 \widetilde{W}^{(0)} &= \{\tau \widetilde{K^A} + (\widehat{K}_0 - \tau x_3 A^{(0)} \widetilde{K^A} + \tau x_3 \widetilde{K^A} A^{(0)} - B^{(0)} \widetilde{K^A} \\ &\quad + \widetilde{K^A} B^{(0)} + iA^{(0)}) + \widehat{K'}_{-1} + \cdots\}\widetilde{W}^{(0)} \\ &:= (\tau \widetilde{K^A} + \widetilde{K}_0 + \widehat{K'}_{-1} + \cdots)\widetilde{W}^{(0)} \end{aligned}$$

where $\mathrm{ord}(\widehat{K'}_{-1}) = -1$ and the remainders are at most -2. We choose $A^{(0)}, B^{(0)}$ to be suitable operators and use the same calculations in [15], then we will get

$$\widetilde{K}_0 = \begin{bmatrix} \widetilde{K}_0(1,1) & 0 \\ 0 & \widetilde{K}_0(2,2) \end{bmatrix}$$

to be a diagonal form (here we omit all the details).
Step 6. Finally, following step 5, we can write

$$\begin{aligned}\widehat{W} &= (I+x_3A^{(0)}+\tau^{-1}B^{(0)})(I+x_3^2A^{(1)}+\tau^{-1}x_3B^{(1)}+\tau^{-2}C^{(1)})\cdots\\ &\quad\times(I+x_3^{N+1}A^{(N)}+\tau^{-1}x_3^NB^{(N)}+\tau^{-2}x_3^{N-1}C^{(N)})\widetilde{W}^{(N)}\end{aligned}$$

with suitable $A^{(j)}, B^{(j)}$ and $C^{(j)}$ for $j=0,1,2,\cdots,N$ ($C^{(0)}=0$), then $\widetilde{W}^{(N)}$ satisfies

$$D_3\widetilde{W}^{(N)}=\{\tau\widetilde{K^A}+\widetilde{K}_0+\cdots+\widetilde{K}_{-N}+\widetilde{S}\}\widetilde{W}^{(N)},$$

with all $\widetilde{K}_{-j}$ are decoupled for $0\le j\le N$ and $\mathrm{ord}(\widetilde{S})=-N-1$. If we omit the term $\widetilde{S}$, we can find an approximated solution of the form

$$\hat{v}_A^{(N)}=\sum_{j=0}^{N+1}\hat{v}_{-j,A}^{(N)}$$

satisfying

$$D_3\hat{v}_A^{(N)}=\{\tau\widetilde{K_+^A}+\widetilde{K}_0(1,1)+\cdots+\widetilde{K}_{-N}(1,1)\}\hat{v}_A^{(N)}$$

and each $\hat{v}_{-j,A}^{(N)}$ has to satisfy

$$\begin{cases} D_3\hat{v}_{0,A}^{(N)}=\tau\widetilde{K_+^A}\hat{v}_{0,A}^{(N)}, & \hat{v}_{0,A}^{(N)}|_{x_3=0}=\chi_t(x')b,\\ D_3\hat{v}_{-1,A}^{(N)}=\tau\widetilde{K_+^A}\hat{v}_{-1,A}^{(N)}+\widetilde{K}_0(1,1)\hat{v}_{0,A}^{(N)}, & \hat{v}_{-1,A}^{(N)}|_{x_3=0}=0,\\ \vdots & \\ D_3\hat{v}_{-N-1,A}^{(N)}=\tau\widetilde{K_+^A}\hat{v}_{-N-1,A}^{(N)}+\sum_{j=0}^N\widetilde{K}_{-j}(1,1)\hat{v}_{-j,A}^{(N)}, & \hat{v}_{-N-1,A}^{(N)}|_{x_3=0}=0,\end{cases}$$

where $\chi_t(x')\in C_0^\infty(\mathbb{R}^2)$ and $b\in\mathbb{C}^3$. Thus, by solving this ODE system we can get the following estimates:

$$\|x_3^\beta\partial_{x'}^\alpha(\hat{v}_{-j,A}^{(N)})\|_{L^2(\mathbb{R}_+^3)}\le c\tau^{-\beta-j-1/2} \tag{6.1}$$

for $0\le j\le N+1$. Moreover, if we set $\hat{V}_A^{(N)}=\begin{bmatrix}\hat{v}_A^{(N)}\\ 0\end{bmatrix}$, then it satisfies

$$\begin{cases}\hat{V}_A^{(N)}-\{\tau\widetilde{K^A}+\widetilde{K}_0+\cdots+\widetilde{K}_{-N}\}\hat{V}_A^{(N)}=\tilde{R},\\ \hat{V}_A^{(N)}|_{x_3=0}=\begin{bmatrix}\chi_t(x')b\\ 0\end{bmatrix},\end{cases}$$

where

$$\|\tilde{R}\|_{L^2(\mathbb{R}_+^3)}\le c\tau^{-N-3/2}.$$

Step 7. Finally, if we define the function $\tilde{v}_A=\begin{bmatrix}\tilde{v}_1\\ \tilde{v}_2\\ \tilde{v}_3\end{bmatrix}$, with $\tilde{v}_j$ being the jth component of the vector $\tilde{Q}(I+x_3A^{(0)}+\tau^{-1}B^{(0)})(I+x_3^2A^{(1)}+\tau^{-1}x_3B^{(1)}+$

$\tau^{-2}C^{(1)}) \cdots (I + x_3^{N+1}A^{(N)} + \tau^{-1}x_3^N B^{(N)} + \tau^{-2}x_3^{N-1}C^{(N)})\hat{V}_A^{(N)}$ and set $w_A = \exp(i\tau x' \cdot \xi')\tilde{v}_A$, we will get that

$$
\begin{aligned}
w_A &= Q\exp(i\tau x' \cdot \xi')\exp(i\tau x_3 \widetilde{K_+^A}(x'))\chi_t(x')b + \exp(i\tau x' \cdot \xi')\tilde{\Gamma}(x,\tau) \\
&= Q\exp(i\tau x' \cdot \xi')\exp(-i\tau x_3(-\widetilde{K_+^A}(x')))\chi_t(x')b + \Gamma(x,\tau)
\end{aligned}
$$

and

$$
w_A|_{x_3=0} = \exp(i\tau x' \cdot \xi')(\chi_t(x')Qb + \beta_0(x',\tau),
$$

where $\beta_0(x',\tau) = \tilde{\Gamma}(x',0,\tau)$ is supported in $\mathrm{supp}(\chi_t)$. Note that the function $\tilde{\gamma}$ comes from the combination of $\hat{v}_{-j,A}^{(N)}$'s, for $j = 1, 2, \cdots, N+1$. Now, we derive higher derivative estimates for the oscillating-decaying solutions, back to see all the $\hat{v}_{-j,A}^{(N)}$'s separately. In fact, only need to see $\hat{v}_{-1,A}^{(N)}$. From the estimate (6.1), we know that the estimate is independent of the derivative of x' variables, all we need to concern is the ∂_3 derivative. From the equation

$$
D_3\hat{v}_{-1,A}^{(N)} = \tau\widetilde{K_+^A}\hat{v}_{-1,A}^{(N)} + \widetilde{K}_0(1,1)\hat{v}_{0,A}^{(N)} \tag{6.2}
$$

and the standard regularity theory of ODEs(ordinary differential equations), we know that $\hat{v}_{-1,A}^{(N)} \in C^\infty$ if all the coefficients are smooth. Moreover, note that $\widetilde{K}_+$ independent of x_3, then we can differentiate (6.2) directly, to get

$$
\begin{aligned}
D_3^2\hat{v}_{-1,A}^{(N)} &= D_3[\tau\widetilde{K_+^A}\hat{v}_{-1,A}^{(N)} + \widetilde{K}_0(1,1)\hat{v}_{0,A}^{(N)}] \\
&= \tau\widetilde{K_+^A}(D_3\hat{v}_{-1,A}^{(N)}) + (D_3\widetilde{K}_0(1,1))\hat{v}_{0,A}^{(N)} + \widetilde{K}_0(1,1)D_3\hat{v}_{0,A}^{(N)} \\
&= \tau^2(\widetilde{K_+^A})^2\hat{v}_{-1,A}^{(N)} + \tau\widetilde{K_+^A}\widetilde{K}_0(1,1)\hat{v}_{0,A}^{(N)} + (D_3\widetilde{K}_0(1,1))\hat{v}_{0,A}^{(N)} \\
&\quad + \tau\widetilde{K}_0(1,1)\widetilde{K_+^A}\hat{v}_{0,A}^{(N)}.
\end{aligned}
$$

Thus, we can obtain that

$$
\|x_3^\beta \partial_{x'}^\alpha \partial_3^\eta (\hat{v}_{-1,A}^{(N)})\|_{L^2(\mathbb{R}_+^3)} \leq c\tau^{-\beta+\eta-3/2},
$$

for all $\eta \leq 2$. Inductively, we have

$$
\|x_3^\beta \partial_{x'}^\alpha \partial_3^\eta (\hat{v}_{-1,A}^{(N)})\|_{L^2(\mathbb{R}_+^3)} \leq c\tau^{-\beta+\eta-3/2},
$$

for all $\eta \in \mathbb{N}$. Similarly, for other $\hat{v}_{-j,A}^{(N)}$ with $2 \leq j \leq N+1$, we can get similar estimate in the following:

$$
\|x_3^\beta \partial_{x'}^\alpha \partial_3^\eta (\hat{v}_{-j,A}^{(N)})\|_{L^2(\mathbb{R}_+^3)} \leq c\tau^{\eta-\beta-j-1/2}
$$

$\forall \eta \in \mathbb{N} \cup \{0\}$. Therefore, Γ satisfies

$$
\|\partial_x^\alpha \Gamma\|_{L^2(\Omega_s)} \leq c\tau^{|\alpha|-3/2}e^{-\tau(s-t)\lambda}
$$

on $\Omega_s := \{x_3 > s\} \cap \Omega$ for $s \geq 0$ and $\forall |\alpha| \in \mathbb{N} \cup \{0\}$. Note that since each $\hat{v}_{-j,A}^{(N)}$'s are smooth, we can get the smoothness of $\tilde{R}$ and

$$
\|\partial_x^\alpha \tilde{R}\|_{L^2(\mathbb{R}_+^3)} \leq c\tau^{|\alpha|-N-3/2}
$$

for all $|\alpha| \in \mathbb{N} \cup \{0\}$. Furthermore, we have that

$$\|\partial_x^\alpha (Q_A \widetilde{v_A})\|_{L^2(\Omega_0)} \le c\tau^{|\alpha|-N-1/2}.$$

Step 8. Now let $u = w + r = e^{i\tau x'\cdot\xi'}\tilde{v} + r$ and r be the solution to the boundary value problem

$$\begin{cases} L_A r = -e^{i\tau x'\cdot\xi'}\widetilde{Q_A\widetilde{v_A}} & \text{in } \Omega_0 \\ r = 0 & \text{on } \partial\Omega_0 \end{cases}.$$

However, note that $\Omega_0 = \{x_3 > 0\} \cap \Omega$ is not a smooth domain since $\partial\Omega_0 = (\{x_3 = 0\} \cap \Omega) \cup (\{x_3 > 0\} \cap \partial\Omega)$. Note that the oscillating-decaying solution exists in the half space, from the construction, we know that the solution is independent of the domain Ω. Let $\widetilde{\Omega} \subset \mathbb{R}^3_+$ be a open bounded smooth domain containing Ω with $\{x_3 = 0\} \cap \Omega \subset \partial\widetilde{\Omega}$, from the construction, it is easy to see the form of oscillating-decaying solution does not depend on the domain Ω, then we can extend r to be defined on $\widetilde{\Omega}$ and call it $\tilde{r}(x)$. Here we can also extend $\widetilde{v_A}$ to be defined on $\widetilde{\Omega}$, still denote $\widetilde{v_A}$ and all the decaying estimates will hold since our estimates were considered in $\mathbb{R}^3_+$, then we have

$$\begin{cases} L_A \tilde{r} = -e^{i\tau x'\cdot\xi'}\widetilde{Q_A\widetilde{v_A}} & \text{in } \widetilde{\Omega}, \\ \tilde{r} = 0 & \text{on } \partial\widetilde{\Omega}. \end{cases}$$

Note that all the coefficients are smooth, we apply a well-known elliptic regularity theorem (Theorem2.3, [1]), then we will get $\tilde{r} \in C^k(\Omega)$ $\forall k$ (recall that $\partial\Omega \in C^\infty$) and

$$\|\tilde{r}\|_{H^{k+1}(\Omega;\mathbb{R}^3)} \le c\|\widetilde{Q_A\widetilde{v_A}}\|_{H^k(\Omega;\mathbb{R}^3)}.$$

Hence $\|\partial_x^\alpha r\|_{L^2(\Omega_0)} \le \|\partial_x^\alpha \tilde{r}\|_{L^2(\widetilde{\Omega})} \le c\tau^{|\alpha|-N+1/2}$ for all $|\alpha| \le k$, $\forall k \in \mathbb{N}$. Similarly, we can construct the oscillating decaying solution for $L_B B = 0$. Then we represent A and B to be two oscillating-decaying solution in the following form:

$$\begin{cases} A = & w^A_{\chi_t,b,t,N,\omega} + r^A_{\chi_t,b,t,N,\omega} \text{ in } \Omega_t(\omega), \\ A = & e^{i\tau x\cdot\xi}\{\chi_t(x')Q_t(x')b + \beta^A_{\chi_t,t,b,N,\omega}\} \text{ on } \Sigma_t(\omega), \\ B = & w^B_{\chi_t,b,t,N,\omega} + r^B_{\chi_t,b,t,N,\omega} \text{ in } \Omega_t(\omega), \\ B = & e^{i\tau x\cdot\xi}\{\chi_t(x')Q_t(x')b + \beta^B_{\chi_t,t,b,N,\omega}\} \text{ on } \Sigma_t(\omega), \end{cases}$$

where

$$\begin{cases} w^A_{\chi_t,b,t,N,\omega} = \chi_t(x')Q_t e^{i\tau x\cdot\xi} e^{-\tau(x\cdot\omega - t)A^A_t(x')} b + \gamma^A_{\chi_t,b,t,N,\omega}(x,\tau), \\ w^B_{\chi_t,b,t,N,\omega} = \chi_t(x')Q_t e^{i\tau x\cdot\xi} e^{-\tau(x\cdot\omega - t)A^B_t(x')} b + \gamma^B_{\chi_t,b,t,N,\omega}(x,\tau), \end{cases}$$

$\gamma^A_{\chi_t,b,t,N,\omega}$ and $\gamma^B_{\chi_t,b,t,N,\omega}$ satisfy (3.8) and (3.9).

6.2 Well-posedness and L^p estimate for the anisotropic Maxwell system

In the following, we would list the eigenvalue property and well-posedness results of the following problem: let $\Omega \subset \mathbb{R}^3$ and $K \Subset \Omega$,

$$\begin{cases} \nabla \times E = ik\mu H & \text{in } \Omega \setminus K \\ \nabla \times H = -ik\epsilon E + J & \text{in } \Omega \setminus K \\ \nu \times E = f & \text{on } \partial\Omega \\ \nu \times H = g & \text{on } \partial K, \end{cases} \tag{6.3}$$

where μ, ϵ are symmetric and positive definite matrix-valued functions. More precisely, we assume there exist constants $\mu_0, \mu_1, \lambda_0, \Lambda_0 > 0$ such that

$$\begin{cases} \mu_0 I \leq \mu(x) \leq \mu_1 I, \\ \lambda_0 I \leq \epsilon(x) \leq \Lambda_0 I. \end{cases} \tag{6.4}$$

These well-posedness for the isotropic Maxwell systems can be found in Theorem 4.18 and 4.19 of [13]. However, we have the same result under our assumption (6.4) following the arguments in [13]. Let

$$X = \left\{ u \in H(curl; \Omega \setminus K) | \nu \times u = 0 \text{ on } \partial\Omega \text{ and } u_T \in L^2(\partial K)^3 \text{ on } \partial K \right\}.$$

Definition 6.1. We say (E, H) or E is a weak solution of (6.3) if $E \in X$ and satisfies

$$\left\langle \mu^{-1} \nabla \times E, \nabla \times \phi \right\rangle_{\Omega \setminus K} - k^2 \left\langle \gamma E, \phi \right\rangle_{\Omega \setminus K} = \left\langle ikJ, \phi \right\rangle_{\Omega \setminus K} - \left\langle \mu^{-1} g, \phi_T \right\rangle_{\partial K}, \quad \forall \phi \in X, \tag{6.5}$$

and $\boldsymbol{\nu} \times E = f$ on $\partial\Omega$, where $\phi_T = (\nu \times \phi) \times \nu$ and $\langle \cdot, \cdot \rangle$ denotes the standard Hermitian inner product of L^2 space. Moreover, if (6.5) fails to have a unique solution, then k is called an eigenvalue or a resonance of (6.3).

Lemma 6.2. *There is an infinite discrete set Σ of eigenvalue $k_j > 0$, $j = 1, 2, \ldots$ and corresponding eigenfunctions $E_j \in H_0(curl; \Omega)$, $E_j \neq 0$, such that (6.5) holds with $J = 0$ and $f = g = 0$ is satisfied.*

From the above lemma, we have the following theorem.

Theorem 6.3. *For $k \notin \Sigma$, there exists a unique weak solution $(E, H) \in H(curl; \Omega \backslash \overline{K}) \times H(curl; \Omega \backslash \overline{K})$ of (6.3) given any $f \in H^{-1/2}(Div; \partial\Omega)$, $g \in H^{-1/2}(Div; \partial K)$ and $J \in H^{-1}(\Omega \backslash \overline{K})$. The solution satisfies*

$$\|E\|_{L^2(\Omega \backslash \overline{K})} + \|H\|_{L^2(\Omega \backslash K)} \leq C(\|f\|_{H^{-1/2}(Div; \partial\Omega)} + \|g\|_{H^{-1/2}(Div; \partial K)} + \|J\|_{H^{-1}(\Omega \backslash \overline{K})})$$

for some constant $C > 0$, where

$$H^{-1/2}(Div; \Gamma) := \left\{ f \in H^{-1/2}(\Gamma)^3 \mid \nu \cdot f = 0, \ \nabla_{\partial\Omega} \cdot f \in H^{-1/2}(\Gamma) \right\},$$

$\Gamma = \partial\Omega$ or ∂K.

In the following, we state the L^p theory for the anisotropic Maxwell's system. For this purpose, we define a bilinear form

$$B_A(E, F) := \int_\Omega (A(x) \nabla \times E(x)) \cdot (\nabla \times \overline{F}(x)) dx + M \int_\Omega E(x) \cdot \overline{F}(x) dx$$

for all $E \in H_0^{1,q}(curl, \Omega)$ and $F \in H_0^{1,q'}(curl, \Omega)$ with $\frac{1}{q} + \frac{1}{q'} = 1$. We only state L^p estimate in the following theorem, but we do not prove the theorem. For more details, we refer readers to read [7].

Theorem 6.4. *[7] Let Ω be a a smooth domain. Suppose that $A = A(x)$ is a real symmetric matrix with smooth entries and satisfies the uniform elliptic condition*

$$\lambda|\xi|^2 \leq A(x)\xi \cdot \xi \leq \Lambda|\xi|^2, \text{ for all } \xi \in \mathbb{R}^3,$$

for some constants $0 < \lambda \leq \Lambda < \infty$. Assume q is some number satisfying $2 \leq q < \infty$. Under the condition

$$\inf_{\|F\|_{1,q'}=1} \sup_{\|E\|_{1,q}=1} |B_A E, F)| \geq \frac{1}{K} > 0$$

the Maxwell's systems of the equations

$$\nabla \times (A\nabla \times E) + E = \nabla \times f + g$$

is uniquely solvable in $H_0^{1,q'}(curl, \Omega)$ for each $g \in L^{q'}(\Omega)$ and $f \in L^{q'}(\Omega)$ and the weak solution satisfies

$$\|E\|_{L^{q'}(\Omega)} + \|\nabla \times E\|_{L^{q'}(\Omega)} \leq K\{\|f\|_{L^{q'}(\Omega)} + \|g\|_{L^{q'}(\Omega)}\},$$

where K is a positive constant depending on p.

We end up this appendix with the following lemma on the embedding related to the Sobolev-Besov spaces, for more details, see [12].

Lemma 6.5. *Let $u \in L^p(D)$ such that $\nabla \cdot u \in L^p(D)$ and $\nabla \times u \in L^p(D)$. If $\nu \times u \in L^p(\partial D)$, then also $\nu \cdot u \in L^p(\partial D)$ for $p \in (1, \infty)$. If in addition $1 < p \leq 2$, then $u \in B^{p,2}_{\frac{1}{p}}(D)$ and we have the estimate*

$$\|u\|_{B^{p,2}_{\frac{1}{p}}(D)} \leq C\{\|u\|_{L^p(D)} + \|curlu\|_{L^p(D)} + \|\nabla \cdot u\|_{L^p(D)} + \|\nu \times u\|_{L^p(\partial D)}\}$$

where the Sobolev-Besov space $B^{p,q}_\alpha(D) := [L^p(D), W^{1,p}(D)]_{\alpha,q}$ is obtained by real interpolation for $1 < p, q < \infty$ and $0 < \alpha < 1$.

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