

# Strong Unique Continuation for a Residual Stress System with Gevrey Coefficients

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## Abstract

We consider the problem of the strong unique continuation for an elasticity system with general residual stress. Due to the known counterexamples, we assume the coefficients of the elasticity system are in the Gevrey class of appropriate indices. The main tools are Carleman estimates for product of two second order elliptic operators.

Keywords: strong unique continuation, Gevrey class, Carleman estimates

## 1 Introduction and statement of the results

In this paper, we prove the strong unique continuation property (SUCP) for the isotropic elasticity system with residual stress under appropriate conditions. We formulate the mathematical problem in the following.

Let  $\Omega$  be a connected open domain in  $\mathbb{R}^3$  and consider the time-harmonic elasticity system

$$\nabla \cdot \sigma + \kappa^2 \rho u = 0 \text{ in } \Omega, \quad (1)$$

where  $\sigma = (\sigma_{ij})_{i,j=1}^3$  is the stress tensor field,  $\kappa \in \mathbb{C}$  is the frequency and  $\rho = \rho(x) > 0$  denotes the density of the medium. The vector field  $u(x) = (u_i(x))_{i=1}^3$  is the displacement vector. Suppose that the stress tensor is given by

$$\sigma(x) = T(x) + (\nabla u)T(x) + \lambda(x)(\text{tr}E)I + 2\mu(x)E,$$

where  $E(x) = \frac{\nabla u + \nabla u^t}{2}$  is the infinitesimal strain and  $\lambda(x), \mu(x)$  are the Lamé parameters. The second-rank tensor  $T(x) = (t_{ij}(x))_{i,j=1}^3$  is the residual stress and satisfies

$$t_{ij}(x) = t_{ji}(x), \quad \forall i, j = 1, 2, 3 \text{ and } x \in \Omega$$

and

$$\nabla \cdot T = \sum_j \partial_j t_{ij} = 0 \text{ in } \Omega, \quad \forall i = 1, 2, 3.$$

If we define the elastic tensor  $C = (C_{ijkl})_{i,j,k,l=1}^3$  with

$$C_{ijkl} = \lambda \delta_{ij} \delta_{kl} + \mu (\delta_{jk} \delta_{jl} + \delta_{jk} \delta_{il}) + t_{jl} \delta_{ik},$$

then (1) is equivalent to

$$\nabla \cdot (C \nabla u) + \kappa^2 \rho u = 0 \text{ in } \Omega.$$

We concern the SUCP for (1), i.e., if  $u \in H_{loc}^2(\Omega)$  satisfies (1) and  $u(x)$  vanishes to infinite order at a point  $x_0 \in \Omega$ , then  $u$  must vanish identically in  $\Omega$ . Without loss of generality, we assume  $x_0 = 0$ . A brief history of the results on the (strong) unique continuation for (1) is in the following. In [13], Nakamura and Wang proved the unique continuation property for (1) under the condition  $\max_{i,j} \|t_{ij}\|_\infty$  is small and  $T(x), \lambda(x), \mu(x) \in W^{2,\infty}$  and  $\rho(x) \in W^{1,\infty}$ . In [12], Lin proved the SUCP for (1) under the assumptions that  $T(0) = 0$ ,  $\max_{i,j} \|t_{ij}\|_\infty$  is small,  $\lambda(x), \mu(x)$  and  $\rho(x)$  are in  $C^2$ . In addition, in [15], Uhlmann and Wang proved unique continuation principle for (1) under the conditions  $T(x), \lambda(x), \mu(x) \in W^{2,\infty}$ ,  $\rho(x) \in W^{1,\infty}$  and general residual stress.

Motivated by [15], we want to prove the SUCP for (1) with arbitrary residual stress. In this paper, we will give a reduction algorithm to transform (1) into a special fourth order elliptic system. The main difficulty is that when  $T(0) \neq 0$ , the leading terms of (1) will not be the Laplacian at zero, so we cannot use a perturbation argument to derive suitable Carleman estimates in order to obtain the SUCP. In [2], Alinhac and Baouendi proved the SUCP for any fourth order operator with smooth coefficients verifying  $P = Q_2 Q_1 + R$ , where  $Q_i$ 's are second order elliptic operators with  $Q_i(0, D) = -\Delta$  for  $i = 1, 2$ . Moreover, in [10], Le Borgne proved the SUCP for fourth order differential inequality with  $Q_i$ 's are Lipschitz continuous and  $Q_i(0, D) = -\Delta$  for  $i = 1, 2$ . In [12], Lin introduced  $v = \nabla \cdot u$  and  $w = \nabla \times u$  to transform (1) into a second order differential system, but the system is weakly-coupled, i.e., the principal part of the second order derivatives are not diagonal. Moreover, Lin also introduced a fourth order elliptic system  $P = \Delta Q_i$  with  $Q_i$ 's are second order elliptic operators with  $Q_i(0, D) = \Delta$  for  $i = 1, 2$  and give another approach to derive the SUCP. For more details, we refer readers to [12].

In this note, our transformation will reduce (1) into a fourth order principally diagonal elliptic system with the same leading coefficients. The key observation is that the leading terms of the fourth order elliptic system are the same. Notice that principally diagonal strongly elliptic systems allow the application of Carleman estimates for scalar operators since these estimates are flexible with respect to perturbations by lower order terms. Therefore, it is possible to derive suitable Carleman estimates for the fourth order elliptic system.

In general, the SUCP does not hold even the coefficients are smooth, Alinhac gave a counterexample in [1]. Thus, we consider all the coefficients in the Gevrey class and we will use the Carleman estimates proved in [6] for the scalar higher order elliptic equations in order to prove the SUCP for the new fourth order strongly elliptic system.

**Definition 1.1.** We say that  $f \in C^\infty(\Omega)$  belongs to the Gevrey class of order  $s$ , denote it as  $G^s(\Omega)$ , if there exist constants  $c, A$  and multiindices  $\beta$  such that

$$|\partial^\beta f| \leq cA^{|\beta|} |\beta|!^s \text{ in } \Omega.$$

To simplify the notation, from now on, we use  $G^s$  to denote  $G^s(\Omega)$ . In this paper, we assume all the coefficients  $T(x), \lambda(x), \mu(x)$  and  $\rho(x)$  lie in the Gevrey class  $G^s$ . We are interested in the SUCP for (1) with Gevrey coefficients, which means if  $u$  satisfies (1) and  $u$  is flat at the origin in the sense that

$$\sup_{r \leq \delta} r^{-N} \|u\|_{L^2(B(0,r))} < \infty \quad (2)$$

for all  $N$ , then  $u$  vanishes near the origin. If  $u$  is smooth, the condition (2) is equivalent to all partial derivatives of  $u$  vanishing at 0.

The SUCP for the second order elliptic equations in the Gevrey class were studied in many literature [4, 5, 6, 11]. In 1981, Lerner [11] considered a second order elliptic operator  $L$  in  $\mathbb{R}^2$  with simple characteristics and the coefficients in the Gevrey class of order  $s$ . Lerner proved that if  $s$  is smaller than a quantity depending on the principal symbol of  $l_2(0, \mathbb{R}^2)$ , then  $L$  has the SUCP near 0. In [5], the authors extended Lerner's result to  $\mathbb{R}^N$ , which means the SUCP holds for a second order elliptic operator  $L$  in  $\mathbb{R}^N$  with the Gevrey order  $s$  smaller than a quantity depending on the principal symbol of  $l_2(0, \mathbb{R}^N)$ .

Recall that the strongly elliptic condition is given as: there exists  $c_0 > 0$  such that for all vectors  $\xi = (\xi_i)_{i=1}^3$ ,

$$\sum_{ij} a_{ij}(x) \xi_i \xi_j \geq c_0 |\xi|^2, \quad \forall x \in \Omega.$$

In this paper, we assume  $P_1$  and  $P_2$  are two strongly elliptic operators, where

$$P_1(x, D) := \sum_{jk} a_{jk}^1(x) \partial_{x_j x_k}^2 := \sum_{jk} (\mu \delta_{jk} + t_{jk}) \partial_{x_j x_k}^2, \quad (3)$$

$$P_2(x, D) := \sum_{jk} a_{jk}^2(x) \partial_{x_j x_k}^2 := \sum_{jk} ((\lambda + 2\mu) \delta_{jk} + t_{jk}) \partial_{x_j x_k}^2 \quad (4)$$

with  $a_{jk}^1(x) = \mu(x) \delta_{jk} + t_{jk}(x)$  and  $a_{jk}^2(x) = (\lambda(x) + 2\mu(x)) \delta_{jk} + t_{jk}(x)$ . Further, there exists  $c_0 > 0$  such that for any  $\xi = (\xi_i)_{i=1}^3 \in \mathbb{R}^3$

$$\sum_{jk} a_{jk}^1(x) \xi_j \xi_k = \sum_{jk} t_{jk} \xi_j \xi_k + \mu |\xi|^2 \geq c_0 |\xi|^2 \quad (5)$$

$$\sum_{jk} a_{jk}^2(x) \xi_j \xi_k = \sum_{jk} t_{jk} \xi_j \xi_k + (\lambda + 2\mu) |\xi|^2 \geq c_0 |\xi|^2 \quad (6)$$

for all  $x \in \Omega$ , note that  $(a_{jk}^\ell(x))_{j,k=1}^3$  is a symmetric matrix for  $\ell = 1, 2$ .

We also assume that there exists a constant  $\alpha > 0$  such that the eigenvalues  $\lambda_1^\ell \leq \lambda_2^\ell \leq \lambda_3^\ell$  of  $(a_{jk}^\ell(0))$  satisfying

$$\alpha > \frac{\lambda_3^\ell - \lambda_1^\ell}{\lambda_1^\ell} \quad (7)$$

and

$$s < 1 + \frac{1}{\alpha} \quad (8)$$

for  $\ell = 1, 2$ .

The following theorem derives the SUCP for (1) when all the coefficients lie in the Gevrey class  $G^s$ .

**Theorem 1.2.** *Let the residual stress  $(t_{ij}(x))_{i,j=1}^3$ , the Lamé parameters  $\lambda(x)$ ,  $\mu(x)$  and the density of the medium  $\rho(x)$  be in the Gevrey class  $G^s(\Omega)$  with  $s$  satisfying (8). Then for all  $u \in H_{loc}^2(\Omega; \mathbb{R}^3)$  solving (1) and for all  $N > 0$*

$$\int_{R \leq |x| \leq 2R} |u|^2 dx = O(R^N) \text{ as } R \rightarrow 0,$$

then  $u$  is identically zero in  $\Omega$ .

This paper is organized as follows. In section 2, we will reduce (1) into a fourth order principally diagonal elliptic system. We use the ideas in [12] and give more detailed transformations. In section 3, we will use the property of the strongly elliptic system in the Gevrey class, then we can get the asymptotic behavior of  $u$  near 0. In section 4, we state the SUCP for the fourth order elliptic system and prove the theorem by using the Carleman estimates.

## 2 Reduction to a fourth order strongly elliptic system

In this section, we want to transform (1) into a principally diagonal fourth order strongly elliptic system. As the calculation in [12]. Let

$$Ru = \nabla \cdot (\nabla u T) \quad (9)$$

with  $Ru = ((Ru)_1, (Ru)_2, (Ru)_3)$ , where  $(Ru)_i = \sum_{jk} t_{jk} \partial_{jk}^2 u_i$ ,  $i = 1, 2, 3$ .

As in Section 2, we set  $U = (u, v, w)^t$ , where  $v = \nabla \cdot u$ ,  $w = \nabla \times u$  and  $u$  satisfies (1). From (1), (9), let  $P_1$  and  $P_2$  be two elliptic operators

$$\begin{aligned} P_1(x, D) &= R + \mu \Delta, \\ P_2(x, D) &= R + (\lambda + 2\mu) \Delta, \end{aligned}$$

then  $(u, v, w)$  satisfies

$$P_1(x, D)u = A_{1,1}(u, v) + A_{1,0}(u, v), \quad (10)$$

$$\begin{aligned} P_2(x, D)v &= - \sum_{jk} \nabla(t_{jk}) \cdot \partial_{jk}^2 u \\ &\quad + A_{2,1}(u, v, w) + A_{2,0}(u, v, w), \end{aligned} \quad (11)$$

$$\begin{aligned} P_1(x, D)w &= - \sum_{jk} \nabla(t_{jk}) \times \partial_{jk}^2 u \\ &\quad + A_{3,1}(u, v, w) + A_{3,2}(u, v, w), \end{aligned} \quad (12)$$

where  $A_{\ell, m}$  are  $m$ -th order differential operators. For more details, we refer reader to [12].

Notice that  $u \in H_{loc}^2(\Omega; \mathbb{R}^3)$  satisfies (10) and  $v = \nabla \cdot u \in H_{loc}^1(\Omega)$  and  $\nabla v \in L_{loc}^2(\Omega)$ , then the right hand side of (10) lies in  $L_{loc}^2(\Omega)$ . Therefore, we use the standard elliptic higher order regularity theory for (10) (see Theorem 2.2 in [7]) and the strongly elliptic property, then we have  $u \in H_{loc}^3(\Omega; \mathbb{R}^3)$ . Iterate the procedures, we obtain  $u \in H_{loc}^k(\Omega; \mathbb{R}^3) \forall k \in \mathbb{N}$  (which implies  $v, w \in H_{loc}^k(\Omega) \forall k \in \mathbb{N}$ ).

Let  $P(x, D)$  be the principal part of the system to get

$$P(x, D)U = (P_1(x, D)u, P_2(x, D)v, P_1(x, D)w)^t,$$

where  $U := (u, v, w)^t : \Omega \rightarrow \mathbb{R}^7$ . Component-wise, we have

$$\begin{aligned} (P(x, D)U)_i &= \mu \Delta u_i + \sum_{jk} t_{jk} \partial_{jk}^2 u_i, \quad i = 1, 2, 3 \\ (P(x, D)U)_i &= (\lambda + 2\mu) \Delta v + \sum_{jk} t_{jk} \partial_{jk}^2 v, \quad i = 4 \\ (P(x, D)U)_i &= \mu \Delta w_{i-4} + \sum_{jk} t_{jk} \partial_{jk}^2 w_{i-4} \quad i = 5, 6, 7. \end{aligned}$$

Now, let us take the second order elliptic operator  $P_2(x, D)$  on (10), we get

$$\begin{aligned} P_2 P_1(x, D)u &= P_2(x, D)[A_{1,1}(u, v) + A_{1,0}(u, v)] \\ &:= \sum_{m=0}^3 B_{1,m}(u, v), \end{aligned} \quad (13)$$

where  $B_{1,m}$  is an  $m$ -th order differential operator. Similarly, we can take  $P_1(x, D)$  on (11) and  $P_2(x, D)$  on (12), then we obtain

$$\begin{aligned} P_1 P_2(x, D)v & \\ &= P_1(x, D)\left(-\sum_{jk} \nabla(t_{jk}) \cdot \partial_{jk}^2 u\right) + P_1(x, D)(A_{2,1}(u, v, w) + A_{2,0}(u, v, w)) \\ &= -P_1(x, D)\left(\sum_{jk} \nabla(t_{jk}) \cdot \partial_{jk}^2 u\right) + \sum_{m=0}^3 B_{2,m}(u, v, w), \end{aligned} \quad (14)$$

and

$$\begin{aligned} P_2 P_1(x, D)w & \\ &= P_2(x, D)\left(-\sum_{jk} \nabla(t_{jk}) \times \partial_{jk}^2 u\right) + P_2(x, D)(A(u, v, w) + A_{3,2}(u, v, w)) \\ &= -P_2(x, D)\left(\sum_{jk} \nabla(t_{jk}) \times \partial_{jk}^2 u\right) + \sum_{m=0}^3 B_{3,m}(u, v, w). \end{aligned} \quad (15)$$

Now, if we interchange  $P_1, P_2$  on (14), and use

$$P_2 P_1 = P_1 P_2 - [P_1, P_2],$$

where  $[P_1, P_2]$  is the commutator of two second order elliptic operators, then  $[P_1, P_2]$  is a third order differential operator. Thus, (14) becomes

$$\begin{aligned} P_2 P_1(x, D)v &= -P_1(x, D)\left[\sum_{jk} \nabla(t_{jk}) \cdot \partial_{jk}^2 u\right] \\ &\quad + \sum_{m=0}^3 \widetilde{B}_{2,m}(u, v, w), \end{aligned} \quad (16)$$

where  $\widetilde{B}_{2,m}$  is an  $m$ -th order differential operator and

$$\sum_{m=0}^3 \widetilde{B}_{2,m}(u, v, w) = \sum_{m=0}^3 B_{2,m}(u, v, w) - [P_1, P_2](x, D)v.$$

Now, combine (13), (15) and (16) together, we have

$$P_2 P_1(x, D) \begin{pmatrix} u \\ v \\ w \end{pmatrix} = - \begin{pmatrix} 0 \\ P_1(x, D) [\sum_{jk} \nabla(t_{jk}) \cdot \partial_{jk}^2 u] \\ P_2(x, D) [\sum_{jk} \nabla(t_{jk}) \times \partial_{jk}^2 u] \end{pmatrix} \quad (17)$$

$$+ \sum_{m=0}^3 \begin{pmatrix} B_{1,m}(u, v) \\ \widetilde{B}_{2,m}(u, v, w) \\ B_{3,m}(u, v, w) \end{pmatrix}.$$

Now, for  $P_1 \left( \sum_{jk} \nabla(t_{jk}) \cdot \partial_{jk}^2 u \right)$  in (17), recall that  $P_1(x, D) = R + \mu \Delta$  and  $Ru = \nabla \cdot (\nabla u T)$ , then we have

$$P_1(x, D) \left[ \sum_{jk} \nabla(t_{jk}) \cdot \partial_{jk}^2 u \right] = R \left( \sum_{jk} \nabla(t_{jk}) \cdot \partial_{jk}^2 u \right) + \mu \Delta \left( \sum_{jk} \nabla(t_{jk}) \cdot \partial_{jk}^2 u \right). \quad (18)$$

For the second term of (18), by using the vector identity  $\Delta u = \nabla(\nabla \cdot u) - \nabla \times \nabla \times u = \nabla v - \nabla \times w$ , it is easy to see

$$\begin{aligned} & \Delta \left( \sum_{jk} \nabla(t_{jk}) \cdot \partial_{jk}^2 u \right) \\ &= \sum_{jk} \nabla(t_{jk}) \cdot \partial_{jk}^2 (\Delta u) + \widetilde{A}_{2,3}(u) + \widetilde{A}_{2,2}(u) \\ &= \sum_{jk} \nabla(t_{jk}) \cdot \partial_{jk}^2 (\nabla v - \nabla \times w) + \widetilde{A}_{2,3}(u) + \widetilde{A}_{2,2}(u) \\ &= \widetilde{B}_{2,3}(u, v, w) + \widetilde{A}_{2,2}(u), \end{aligned}$$

where  $\widetilde{A}_{2,m}$  and  $\widetilde{B}_{2,m}$  are  $m$ -th order differential operators and

$$\widetilde{B}_{2,3}(u, v, w) = \sum_{jk} \nabla(t_{jk}) \cdot \partial_{jk}^2 (\nabla v - \nabla \times w) + \widetilde{A}_{2,3}(u).$$

For the first term of (18), we have

$$\begin{aligned} & R \left( \sum_{jk} \nabla(t_{jk}) \cdot \partial_{jk}^2 u \right) \\ &= \sum_{\ell m} t_{\ell m} \partial_{\ell m}^2 \left( \sum_{jk} \nabla(t_{jk}) \cdot \partial_{jk}^2 u \right) \\ &= \sum_{jk} \nabla(t_{jk}) \cdot \left[ \sum_{\ell m} t_{\ell m} \partial_{\ell m}^2 \partial_{jk}^2 u \right] + \widetilde{C}_{2,3}(u) + \widetilde{C}_{2,2}(u) \\ &= \sum_{jk} \nabla(t_{jk}) \cdot \partial_{jk}^2 \left( \sum_{\ell m} t_{\ell m} \partial_{\ell m}^2 u \right) + \widetilde{D}_{2,3}(u) + \widetilde{D}_{2,2}(u) \\ &= \sum_{jk} \nabla(t_{jk}) \cdot Ru + \widetilde{D}_{2,3}(u) + \widetilde{D}_{2,2}(u), \end{aligned}$$

and use (10), we have  $Ru = -\mu \Delta u + A_{1,1}(u, v) + A_{1,0}(u, v)$ , we have

$$\begin{aligned}
& R\left(\sum_{jk} \nabla(t_{jk}) \cdot \partial_{jk}^2 u\right) \\
&= \sum_{jk} \nabla(t_{jk}) \cdot \partial_{jk}^2 (-\mu \Delta u + A_{1,1}(u, v) + A_{1,0}(u, v)) + \widetilde{D}_{2,3}(u) + \widetilde{D}_{2,2}(u) \\
&= \sum_{jk} \nabla(t_{jk}) \cdot \partial_{jk}^2 (-\mu(\nabla v - \nabla \times w)) + \sum_{m=0}^3 \widetilde{E}_{2,3}(u, v) \\
&= \sum_{m=0}^3 \widetilde{F}_{2,m}(u, v, w)
\end{aligned}$$

where  $\widetilde{C}_{2,m}, \widetilde{D}_{2,m}, \widetilde{E}_{2,m}$  and  $\widetilde{F}_{2,m}$  are  $m$ -th order differential operators. From the above calculation and (17), we have

$$P_2 P_1(x, D)v = \sum_{m=0}^3 \widetilde{E}_{2,m}(u, v, w), \quad (19)$$

where  $\widetilde{E}_{2,m}$  are  $m$ -th order differential operators. Similarly, for  $P_2\left(\sum_{jk} \nabla(t_{jk}) \times \partial_{jk}^2 u\right)$ , it is easy to see that

$$\Delta\left(\sum_{jk} \nabla(t_{jk}) \times \partial_{jk}^2 u\right) = \widetilde{A}_{3,3}(u, v, w) + \widetilde{A}_{3,2}(u),$$

where  $\widetilde{A}_{3,m}$  is an  $m$ -th order differential operator. Similarly, for  $R\left(\sum_{jk} \nabla(t_{jk}) \times \partial_{jk}^2 u\right)$ , component-wise, we have

$$\begin{aligned}
& \left[ R\left(\sum_{jk} \nabla(t_{jk}) \times \partial_{jk}^2 u\right) \right]_i \\
&= \sum_{\ell m} t_{\ell m} \partial_{\ell m}^2 \left( \sum_{jk} \nabla(t_{jk}) \times \partial_{jk}^2 u \right)_i \\
&= \sum_{\ell m} \sum_{jk} (\nabla(t_{jk}) \times t_{\ell m} \partial_{\ell m}^2 \partial_{jk}^2 u)_i + \widetilde{B}_{3,3}(u) + \widetilde{B}_{3,2}(u) \\
&= \sum_{jk} (\nabla(t_{jk}) \times \partial_{jk}^2 \left( \sum_{\ell m} t_{\ell m} \partial_{\ell m}^2 u \right)_i) + \widetilde{C}_{3,3}(u) + \widetilde{C}_{3,2}(u) \\
&= \sum_{jk} (\nabla(t_{jk}) \times \partial_{jk}^2 R u)_i + \widetilde{D}_{3,3}(u) + \widetilde{D}_{3,2}(u)
\end{aligned}$$

and use (10) again, we obtain

$$\begin{aligned}
& \left[ R \left( \sum_{jk} \nabla(t_{jk}) \times \partial_{jk}^2 u \right) \right]_i \\
&= \sum_{jk} (\nabla(t_{jk}) \times \partial_{jk}^2 [-\mu(\nabla v - \nabla \times w)])_i + \sum_{m=0}^3 \widetilde{E}_{3,m}(u, v) \\
&= \sum_{m=0}^3 \widetilde{F}_{3,m}(u, v, w),
\end{aligned}$$

where  $\widetilde{B}_{3,m}, \widetilde{C}_{3,m}, \widetilde{D}_{3,m}, \widetilde{E}_{3,m}$  and  $\widetilde{F}_{3,m}$  are  $m$ -th order differential operators. Therefore, we transform the equation (15) into

$$P_2 P_1(x, D)w = \sum_{m=0}^3 \widehat{E}_{3,m}(u, v, w), \quad (20)$$

where  $\widehat{E}_{3,m}$  are  $m$ -th order differential operators. From (19), (20) and (17), we can obtain

$$P_2 P_1(x, D) \begin{pmatrix} u \\ v \\ w \end{pmatrix} = \sum_{m=0}^3 \begin{pmatrix} \widehat{E}_{1,m}(u, v, w) \\ \widehat{E}_{2,m}(u, v, w) \\ \widehat{E}_{3,m}(u, v, w) \end{pmatrix},$$

with  $\widehat{E}_{\ell,m}$  are  $m$ -th order differential operators, or equivalently,

$$P_2 P_1 U = \sum_{m=0}^3 \widehat{E}_m(U), \quad (21)$$

with  $\widehat{E}_m = (\widehat{E}_{3,m}, \widehat{E}_{2,m}, \widehat{E}_{1,m})^t$  is an  $m$ -th order differential operator and  $U = (u, v, w)^t$ , which means this fourth-order differential equation has the same leading term  $P_2 P_1$  and all coefficients of (21) lie in  $G^s$ . Moreover, use the elliptic regularity for (21) with Gevrey coefficients, then  $U \in G^s$  by Proposition 2.13 in [3].

### 3 The asymptotic behavior of $u$ near 0

As in Section 2, we set  $U = (u, v, w)^t$ , where  $v = \nabla \cdot u$  and  $w = \nabla \times u$ . If we can prove that  $U$  solves (21) and satisfies the SUCP, then  $u$  solves (1) and fulfills the SUCP. In the following lemma, we describe the asymptotic behavior of  $u$  near 0. Recall that if  $u \in H_{loc}^2(\Omega; \mathbb{R}^3)$ , then  $u \in C^\infty(\Omega)$  by the standard elliptic regularity. Thus,  $\forall k \in \mathbb{N}$ , we can consider  $u \in H_{loc}^k(\Omega; \mathbb{R}^3)$  for arbitrary  $k \in \mathbb{N}$  in the following results.

**Theorem 3.1.** [12] *Let  $u$  be a solution to (1) and for all  $N > 0$*

$$\int_{R \leq |x| \leq 2R} |u|^2 dx = O(R^N) \text{ as } R \rightarrow 0.$$

*Then for  $|\beta| \leq 2$ , we have*

$$\int_{R \leq |x| \leq 2R} |R^{|\beta|} D^\beta u|^2 dx = O(R^N) \text{ as } R \rightarrow 0.$$



*Proof.* The lemma was proved by the Corollary 17.1.4 in Hörmander [9]. By using the Theorem 3.1, we will get the following Corollary.  $\square$

**Corollary 3.2.** *Let  $U = (u, v, w)^t$  with  $v = \nabla \cdot u$  and  $w = \nabla \times u$ . Then for  $|\beta| \leq 1, \forall N > 0$ , we have*

$$\int_{R \leq |x| \leq 2R} |D^\beta U|^2 dx = O(R^N) \text{ as } R \rightarrow 0. \quad (22)$$

In fact, we can get higher derivatives for  $|\beta| \geq 2$  in the Corollary 3.2.

**Lemma 3.3.** [9] *If  $U$  satisfies a fourth order strongly elliptic system (21)*

$$PU = \sum_{m=0}^3 \widehat{E}_m(U),$$

and  $U$  satisfies  $\forall N > 0$ ,

$$\int_{R \leq |x| \leq 2R} |U|^2 dx = O(R^N) \text{ as } R \rightarrow 0.$$

Then it follows that if  $|\beta| \leq 4$  that

$$\int_{R \leq |x| \leq 2R} |R^{|\beta|} D^\beta U|^2 dx = O(R^N) \text{ as } R \rightarrow 0. \quad (23)$$

*Proof.* Since  $U$  satisfies (21), a fourth order strongly elliptic system, by using the Corollary 17.1.4 in [9], we can obtain (23).  $\square$

*Remark 3.4.* In the section 3 of [12], the author proved (23) holding for  $|\beta| \leq 2$ . From Lemma 3.3 and the coefficients of  $P$  are in the Gevrey class  $G^s$ , we have  $U \in G^s$  and

$$\int_{|x| \leq R} |D^\beta U|^2 dx = O(R^N) \text{ as } R \rightarrow 0,$$

for  $|\beta| \leq 4$  and  $\forall N > 0$ .

## 4 Proof of the main theorem

In this section, we want to prove Theorem 1.1. If  $U = (u, v, w)^t$  satisfies (21) and the SUCP, then the SUCP holds for  $u$ , where  $u$  fulfills (1).

### 4.1 SUCP for $U$

In the following theorem, we will prove the SUCP for  $U$ .

**Theorem 4.1.** *Suppose that the second order elliptic operators  $P_\ell$  satisfies (3), (4), (5) and (6) for  $\ell = 1, 2$ .  $\alpha > 0$  satisfies (7) at  $x = 0$  and  $s$  satisfies (8). Let  $P = P_2 P_1$  be a fourth order elliptic operator. Then the SUCP holds for the elliptic system*

$$PU = \sum_{|\beta| \leq 3} a_\beta \partial^\beta U \quad (24)$$

provided the coefficients of  $P_\ell$  are in the Gevrey class  $G^s$ .

*Proof.* The proof follows from [6] and section 1. To prove Theorem 4.1, there are two steps. First, Gevrey regularity of the elliptic system implies the solution  $U$  of (24) is in the Gevrey class  $G^s$  (see Proposition 2.13 in [3]). Use the vanishing order assumption and  $U \in G^s$ , we have

$$|U| \lesssim e^{-|x|^{-\gamma}}, \quad (25)$$

near  $x = 0$  and for some constant  $\gamma > 0$ . Second, we can show that (25) implies  $U$  vanishes near 0 by using appropriate Carleman estimates. In addition, since  $U$  vanishes near 0, by the results in [15], we have  $U \equiv 0$  in  $\Omega$ .  $\square$

*Remark 4.2.* We give the Carleman estimates which were used in the proof of Theorem 4.1 in the following section.

## 4.2 Carleman Estimates

We are going to derive the Carleman estimates with the weight  $e^{\tau|x|^{-\alpha}}$  for the fourth order elliptic operator  $P = P_2P_1$  in this section. The following Carleman estimates for the scalar case has been proven in [5] and [6]. Similar to the scalar elliptic equation, we can derive the following Carleman estimate for the special elliptic system.

**Proposition 4.3.** *Let  $P_\ell(x, D) = \sum_{jk} a_{jk}^\ell(x) \partial_{jk}^2$  be a principally diagonal second order elliptic operator where  $a_{jk}^\ell(x) \in G^s$  satisfies (3), (4), (5) and (6) for  $\ell = 1, 2$ .  $\alpha > 0$  satisfies (7) at  $x = 0$  and  $s$  satisfies (8). Then there exist  $\tau_0 > 0$  and  $r_0 > 0$  such that for  $\tau > \tau_0$  and for all  $V \in C^\infty((B_{r_0} \setminus \{0\}); \mathbb{R}^7)$ ,  $\ell = 1, 2$ , the following inequality holds:*

$$\begin{aligned} & \tau \int |D^2(|x|^{\alpha/2} e^{\tau|x|^{-\alpha}} V)^2 dx + \tau^3 \int |x|^{-4-3\alpha} e^{2\tau|x|^{-\alpha}} V^2 dx \\ & \lesssim \int |e^{2\tau|x|^{-\alpha}} (P_\ell V)|^2 dx. \end{aligned}$$

*Proof.* Since  $P_\ell$  is the principally diagonal second order elliptic operator for  $\ell = 1, 2$ , we can directly follow the consequences in [6] and use the proof in [5]. For more details and classical results, we refer readers to [8, 14].  $\square$

By using the integration by parts, we can get a stronger inequality in the following. For more details, we refer readers to [6] and section 3, then we have

$$\begin{aligned} & \sum_{j=0}^2 \tau^{3-2j} \int e^{2\tau|x|^{-\alpha}} |x|^\alpha |x|^{2(j-2)(1+\alpha)} |D^j V|^2 dx \\ & \lesssim \int |e^{2\tau|x|^{-\alpha}} |P_\ell V|^2 dx, \end{aligned}$$

with  $P_\ell$  satisfying all the assumptions in Proposition 4.2 for  $\ell = 1, 2$ . Note that the right hand side of (11) and (12) involve second order derivatives of  $u$ , we cannot apply the Carleman estimates for the second order differential systems directly to get the SUCP for  $U$ . Since we have transformed (1) into a special fourth order elliptic system with the same leading operator, see (21), then we can derive the Carleman estimates for the operator  $P = P_2P_1$ .

**Corollary 4.4.** [6] Let

$$A = \sum_{jk} a_{jk}(x) \partial_{x_j x_k}^2$$

be a second order strongly elliptic operator with  $a_{ij}$  in the Gevrey class  $G^s$ . Suppose  $\alpha > 0$  satisfying (7) at  $x = 0$ . Then there exists  $\tau_0$  such that for all  $|s|$ ,  $k \leq \nu$  and  $\tau \geq \tau_0$

$$\begin{aligned} & \sum_{j=0}^{k+2} \tau^{3-2j} \int |x|^{\alpha+2j(1+\alpha)} |x|^{2s} e^{2\tau|x|^{-\alpha}} |D^j V|^2 dx \\ & \lesssim \sum_{j=0}^k \tau^{-2j} \int |x|^{2(2+j)(1+\alpha)} |x|^{2s} e^{2\tau|x|^{-\alpha}} |D^j(AV)|^2 dx. \end{aligned} \quad (26)$$

*Proof.* See [6] and section 3. We can use the induction hypothesis to prove the Corollary 4.3.  $\square$

For the fourth order elliptic operator  $PU = P_2 P_1 U$  is the product of two second order elliptic operators which satisfies (21), where  $U = (u, v, w)^t$  and  $P_\ell(x, D)U = \sum_{jk} a_{jk}^\ell(x) \partial_{jk}^2 U$ . Recall that  $a_{jk}^\ell \in G^s$  and  $\alpha > 0$  satisfying (7) uniformly in  $x$  and for  $\ell = 1, 2$ , then we have the following key estimates.

**Proposition 4.5.** We have the following Carleman estimates

$$\sum_{j=0}^4 \tau^{6-2j} \int |x|^{-8-6\alpha} |x|^{2j(1+\alpha)} e^{2\tau|x|^{-\alpha}} |D^j V|^2 dx \leq C \int e^{2\tau|x|^{-\alpha}} |PV|^2 dx.$$

*Proof.* Apply the Corollary 4.2 iteratively, then we have

$$\begin{aligned} & \sum_{j=0}^4 \tau^{6-2j} \int |x|^{-8-6\alpha} |x|^{2j(1+\alpha)} e^{2\tau|x|^{-\alpha}} |D^j V|^2 dx \\ & \lesssim \sum_{j=9}^2 \tau^{3-2j} \int |x|^{-4-3\alpha} |x|^{j(1+\alpha)} e^{2\tau|x|^{-\alpha}} |D^j(P_1 V)|^2 dx \\ & \lesssim \int e^{2\tau|x|^{-\alpha}} |(P_2 P_1 V)|^2 dx = \int e^{2\tau|x|^{-\alpha}} |PV|^2 dx, \end{aligned} \quad (27)$$

where the first inequality is obtained by (26) with  $k = 2$ ,  $s = -4 - \frac{7}{2}\alpha$  and the second inequality is obtained by (26) with  $k = 0$ ,  $s = -2(1 + \alpha)$ . For more details, we refer reader to see [6].  $\square$

Now, we want to prove the SUCP for (1). Here we prove the theorem 2.2.

*Proof of Theorem 4.1:* The operator  $P = P_2 P_1$  is strongly elliptic in the Gevrey class  $G^s$ , then  $U$  is also in the Gevrey class  $G^s$ . Therefore, we have the vanishing of infinite order implies that

$$|u| \lesssim e^{-|x|^{-\gamma}}$$

for some  $\gamma > \alpha$ . Let  $\chi \in C_0^\infty(\mathbb{R}^3)$  be such that  $\chi \equiv 1$  for  $|x| \leq R$  and  $\chi \equiv 0$  for  $|x| \geq 2R$  ( $R > 0$  is small enough). Then we can apply (27) to the function  $\chi U$ ,

which means

$$\begin{aligned}
& C \sum_{|\beta|=0}^4 \tau^{6-2|\beta|} \int_{|x|<R} |x|^{(2|\beta|-6)(1+\alpha)-2} e^{2\tau|x|^{-\alpha}} |D^\beta U|^2 dx \quad (28) \\
& \leq \int e^{2\tau|x|^{-\alpha}} |PU|^2 dx \\
& \leq \int_{|x|<R} e^{2\tau|x|^{-\alpha}} |PU|^2 dx + \int_{|x|>R} e^{2\tau|x|^{-\alpha}} |P(\chi U)|^2 \\
& \leq \int_{|x|<R} e^{2\tau|x|^{-\alpha}} \left| \sum_{m=0}^3 \widehat{E}_m(U) \right|^2 dx + \int_{|x|>R} e^{2\tau|x|^{-\alpha}} |P(\chi U)|^2,
\end{aligned}$$

by using the reduction elliptic system (21).

If  $\tau$  is large and  $R$  is sufficiently small, then (28) implies

$$\begin{aligned}
& C \sum_{|\beta|=0}^4 \tau^{6-2|\beta|} \int_{|x|<R} |x|^{(2|\beta|-6)(1+\alpha)-2} e^{2\tau|x|^{-\alpha}} |D^\beta U|^2 dx \quad (29) \\
& \leq \int_{|x|>R} e^{2\tau|x|^{-\alpha}} |P(\chi U)|^2,
\end{aligned}$$

for some constant  $C > 0$ . Notice that  $e^{\tau|x|^{-\alpha}} \geq e^{\tau R^{-\alpha}}$  for  $|x| < R$  and  $e^{\tau|x|^{-\alpha}} \leq e^{\tau R^{-\alpha}}$  for  $|x| > R$ . Therefore, we can use (29) to obtain

$$\begin{aligned}
& C \sum_{|\beta|=0}^4 \tau^{6-2|\beta|} \int_{|x|<R} |x|^{(2|\beta|-6)(1+\alpha)-2} |D^\beta U|^2 dx \\
& \leq \int_{|x|>R} |P(\chi U)|^2.
\end{aligned}$$

Let  $\tau \rightarrow \infty$ , we get  $U = 0$  in  $\{|x| < R\}$  for  $R$  small, which implies  $u = 0$  in  $\{|x| < R\}$ . Furthermore, by using the unique continuation principal in [15], we can obtain  $u \equiv 0$  in  $\Omega$ , then we are done.

## References

- [1] Serge Alinhac. Non-unicité pour des opérateurs différentiels à caractéristiques complexes simples. In *Annales scientifiques de l'École Normale Supérieure*, volume 13, pages 385–393. Société mathématique de France, 1980.
- [2] Serge Alinhac and Salah Baouendi. Uniqueness for the characteristic cauchy problem and strong unique continuation for higher order partial differential inequalities. *American Journal of Mathematics*, pages 179–217, 1980.
- [3] Louis Boutet de Monvel and Paul Krée. Pseudo-differential operators and gevrey classes. In *Annales de l'institut Fourier*, volume 17, pages 295–323, 1967.

- [4] F Colombini and C Grammatico. A result on strong uniqueness in gevrey spaces for some elliptic operators. *Communications in Partial Difference Equations*, 30(1-2):39–57, 2005.
- [5] Ferruccio Colombini, Cataldo Grammatico, and Daniel Tataru. Strong uniqueness for second order elliptic operators with gevrey coefficients. *Mathematical Research Letters*, 13(1):15, 2006.
- [6] Ferruccio Colombini and Herbert Koch. Strong unique continuation for products of elliptic operators of second order. *Transactions of the American Mathematical Society*, 362(1):345–355, 2010.
- [7] Mariano Giaquinta. *Introduction to regularity theory for nonlinear elliptic systems*. Birkhauser, 1993.
- [8] Lars Hörmander. *Linear partial differential operators*. Springer, 1963.
- [9] Lars Hörmander. *The analysis of linear partial differential operators III: Pseudo-differential operators*, volume 274. Springer Science & Business Media, 2007.
- [10] Philippe Le Borgne. Strong uniqueness for fourth order elliptic differential operators. In *Carleman Estimates and Applications to Uniqueness and Control Theory*, pages 85–108. Springer, 2001.
- [11] Nicolas Lerner. Resultats d’unicite forte pour des operateurs elliptiques a coefficients gevrey. *Communications in Partial Differential Equations*, 6(10):1163–1177, 1981.
- [12] Ching-Lung Lin. Strong unique continuation for an elasticity system with residual stress. *Indiana University mathematics journal*, 53(2):557–582, 2004.
- [13] Gen Nakamura and Jenn-Nan Wang. Unique continuation for an elasticity system with residual stress and its applications. *SIAM journal on mathematical analysis*, 35(2):304–317, 2003.
- [14] Daniel Tataru. Carleman estimates, unique continuation and applications. *Notes downloadable from <http://math.berkeley.edu/tataru/ucp.html>*, 1999.
- [15] Gunther Uhlmann and Jenn-Nan Wang. Unique continuation property for the elasticity with general residual stress. *Inverse Probl. Imaging*, 3(2):309–317, 2009.