Manuscript submitted to
AIMS' Journals
Volume **X**, Number **0X**, XX **200X**

doi:10.3934/xx.xx.xx.xx

pp. **X–XX**

RECONSTRUCTION OF PENETRABLE OBSTACLES IN THE ANISOTROPIC ACOUSTIC SCATTERING

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Abstract. We develop an enclosure-type reconstruction scheme to identify penetrable obstacles in acoustic waves with anisotropic medium in $\mathbb{R}^3$. The main difficulty of treating this problem lies in the fact that there are no complex geometrical optics solutions available for the acoustic equation with anisotropic medium in $\mathbb{R}^3$. Instead, we will use another type of special solutions called oscillating-decaying solutions. Even though that oscillating-decaying solutions are defined only on the half space, we are able to give necessary boundary inputs by the Runge approximation property. Moreover, since we are considering a Helmholtz-type equation, we turn to Meyers' L^p estimate to compare the integrals coming from oscillating-decaying solutions and those from reflected solutions.

1. **Introduction.** In the study of inverse problems, we are interested in the special type of solutions for elliptic equations or systems which play an essential role since the pioneer work of Caldéron. Sylvester and Uhlmann [13] introduced complex geometrical optics (CGO) solutions to solve the inverse boundary value problems of the conductivity equation. Based on CGO solutions, Ikehata proposed the so called enclosure method to reconstruct the impenetrable obstacle, for more details, see [2, 3, 4]. There are many results concerning this reconstruction algorithm, such as [9, 15]. The researchers constructed CGO-solutions with polynomial-type phase function of the Helmholtz equation $\Delta u + k^2 u = 0$ or the elliptic system with the Laplacian as the principal part.

When the medium is anisotropic, we need to consider more general elliptic equations, such as anisotropic scalar elliptic equation in a bounded domain $\Omega \subset \mathbb{R}^3$,

$$\nabla \cdot (A^0(x)\nabla u) + k^2 u = 0, \tag{1.1}$$

where $A^0(x) = (a^0_{ij}(x))$, $a^0_{ij}(x) = a^0_{ji}(x)$, and we assume the uniform ellipticity condition, that is, for all $\xi = (\xi_1, \xi_2, \cdots \xi_n) \in \mathbb{R}^n$, $\lambda^0|\xi|^2 \leq \sum_{i,j} a^0_{ij}(x)\xi_i\xi_j \leq \Lambda^0|\xi|^2$ and $x \in \Omega$. In two dimensional case, we can transform (1.1) to an isotropic equation by using isothermal coordinates, then we can apply the CGO-solutions for this case, which can be found in [14]. When $\Omega \subset \mathbb{R}^3$, we cannot directly transform (1.1) to an isotropic equation as we do in $\mathbb{R}^2$, thus we need to use the oscillating-decaying solutions in our reconstruction algorithm. In [10], the author introduced oscillating-decaying solutions for the conductivity equation $\nabla \cdot (\gamma(x)\nabla u) = 0$ with the isotropic

2010 *Mathematics Subject Classification.* Primary: 35R30; Secondary: 35J15.

Key words and phrases. enclosure method, reconstruction, oscillating-decaying solutions, Runge approximation property, Meyers L^p estimates.

conductivity.

We make the following assumptions.

1. Let $\Omega \subset \mathbb{R}^3$ be a bounded C^∞-smooth domain and assume that D is an unknown obstacle with Lipschitz boundary such that $D \Subset \Omega \subset \mathbb{R}^3$ with an inhomogeneous index of refraction subset of a larger domain Ω.
2. Let $A(x) = (a_{ij}(x))$ and $A^0(x) = (a^0_{ij}(x))$ be symmetric matrices with $a_{ij}(x) = a^0_{ij}(x) + \widetilde{a_{ij}}(x)\chi_D$, where each $a^0_{ij}(x)$ is bounded C^∞-smooth, $\widetilde{A}(x) = (\widetilde{a_{ij}}(x)) \in L^\infty(D)$ is regarded as a perturbation in the unknown obstacle D and $\widetilde{A}(x)\xi \cdot \xi \geq \widetilde{\lambda}|\xi|^2$ for any $\xi \in \mathbb{R}^3$ and $x \in D$ with some $\widetilde{\lambda} > 0$. Further $A(x)$ satisfies $\lambda|\xi|^2 \leq A(x)\xi \cdot \xi \leq \Lambda|\xi|^2$ for some constants $0 < \lambda \leq \Lambda$.

Now, let $k > 0$ and consider the steady state anisotropic acoustic wave equation with Dirichlet boundary condition

$$\begin{cases} \nabla \cdot (A(x)\nabla u) + k^2 u = 0 & \text{in } \Omega \\ u = f & \text{on } \partial\Omega. \end{cases} \tag{1.2}$$

For the unperturbed case, we have

$$\begin{cases} \nabla \cdot (A^0(x)\nabla u_0) + k^2 u_0 = 0 & \text{in } \Omega \\ u_0 = f & \text{on } \partial\Omega. \end{cases} \tag{1.3}$$

In this paper, we assume that k^2 is not a Dirichlet eigenvalue of the operator $-\nabla \cdot (A\nabla \bullet)$ and $-\nabla \cdot (A^0\nabla \bullet)$ in Ω. It is known that for any $f \in H^{1/2}(\partial\Omega)$, there exists a unique solution u to (1.2). We define the Dirichlet-to-Neumann map $\Lambda_D : H^{1/2}(\partial\Omega) \to H^{-1/2}(\partial\Omega)$ in the anisotropic case as the following.

Definition 1.1. $\Lambda_D f := A\nabla u \cdot \nu = \sum_{i,j=1}^3 a_{ij}\partial_j u \cdot \nu_i$ and $\Lambda_\emptyset f := A^0 \nabla u_0 \cdot \nu = \sum_{i,j=1}^3 a^0_{ij}\partial_j u_0 \cdot \nu_i$, where $\nu = (\nu_1, \nu_2, \nu_3)$ is a unit outer normal on $\partial\Omega$.

Inverse problem: Identify the location and the convex hull of D from the DN-map Λ_D.

The domain D can also be considered as an inclusion embedded in Ω. The aim of this work is to give a reconstruction algorithm for this problem. Note that the information on the medium parameter $\widetilde{A}(x) = (\widetilde{a_{ij}}(x))$ inside D is not known a priori.

The main tool in our reconstruction method is the oscillating-decaying solutions for the second order anisotropic elliptic differential equations. We use the results from the paper [11] to construct the oscillating-decaying solution. In section 2, we will construct the oscillating-decaying solutions for anisotropic elliptic equations. Note that even if $k = 0$, which means the equation is $\nabla \cdot (A(x)\nabla u) = 0$, we do not know of any CGO-type solutions. Roughly speaking, given a hyperplane, an oscillating-decaying solution is oscillating very rapidly along this plane and decaying exponentially in the direction transverse to the same plane. Oscillating-decaying solutions are special solutions with the imaginary part of the phase function non-negative. Note that the domain of the oscillating-decaying solutions is not over the whole Ω, so we need to extend such solutions to the whole domain. Fortunately, the Runge approximation property provides us a good approach to extend this special solution in Section 3.

In Ikehata's work, the CGO-solutions are used to define the indicator function (see [4] for the definition). In order to use the oscillating-decaying solutions to the inverse problem of identifying an inclusion, we employ the Runge approximation property to redefine the indicator function. It was Lax [5] that first recognized the Runge approximation property is a consequence of the weak unique continuation property. In our case, it is clear that the anisotropic elliptic equation has the weak unique continuation property if the leading part is Lipschitz continuous. Finally, the main theorem and reconstruction algorithm will be presented in Section 4. We remark that the reconstruction algorithm in this paper is weaker than the standard enclosure method for instance, in the sense that our method does not explain what happens to the indicator function after the probing hyperplane has met the obstacle. The results in Section 4 only imply that the indicator function is zero when the hyperplane has not touched the obstacle, and becomes nonzero at the touching point.

2. **Construction of oscillating-decaying solutions.** In this section, we follow the paper [11] to construct the oscillating-decaying solution in the anisotropic elliptic equations. In our case, since we only consider a scalar elliptic equation, its construction is simpler than that in [11]. Consider the anisotropic Helmholtz type equation

$$\nabla \cdot (A(x)\nabla u) + k^2 u = 0 \text{ in } \Omega. \tag{2.1}$$

Note that the oscillating-decaying solutions of

$$\nabla \cdot (A(x)\nabla u) = 0 \text{ in } \Omega$$

will have the same form as the equation (2.1), which means the lower order term $k^2 u$ will not affect the representation of the oscillating-decaying solutions, the following are the construction details. Now, we assume that the domain Ω is an open, bounded smooth domain in $\mathbb{R}^3$ and the coefficients $A(x) = (a_{ij}(x))$ is a symmetric 3×3 matrix satisfying uniformly elliptic condition, which means $\sum_{i,j=1}^3 a_{ij}(x)\xi_i\xi_j \geq c|\xi|^2$, $\forall \xi = (\xi_1, \xi_2, \xi_3) \in \mathbb{R}^3$ for some $c > 0$.

Assume that

$$A(x) = (a_{ij}(x)) \in B^\infty(\mathbb{R}^3) = \{f \in C^\infty(\mathbb{R}^3) : \partial^\alpha f \in L^\infty(\mathbb{R}^3),\ \forall \alpha \in \mathbb{Z}_+^3\}$$

is the anisotropic coefficients. Note that $A(x) \in B^\infty$ already implies that A is Lipschitz continuous and the Lipschitz continuity property of $A(x)$ will apply the weak unique continuation property of (2.1) (see [1] for example).

We give several notations as follows. Assume that $\Omega \subset \mathbb{R}^3$ is an open set with smooth boundary and $\omega \in S^2$ is given. Let $\eta \in S^2$ and $\zeta \in S^2$ be chosen so that $\{\eta, \zeta, \omega\}$ forms an orthonormal system of $\mathbb{R}^3$. We then denote $x' = (x \cdot \eta, x \cdot \zeta)$. Let $t \in \mathbb{R}$, $\Omega_t(\omega) = \Omega \cap \{x \cdot \omega > t\}$ and $\Sigma_t(\omega) = \Omega \cap \{x \cdot \omega = t\}$ be a non-empty open set. We consider a scalar function $u_{\chi_t,b,t,N,\omega}(x,\tau) := u(x,\tau) \in C^\infty(\overline{\Omega_t(\omega)} \backslash \overline{\Sigma_t(\omega)}) \cap C^0(\overline{\Omega_t(\omega)})$ with $\tau \gg 1$ satisfying:

$$\begin{cases} L_A u = \nabla \cdot (A(x)\nabla u) + k^2 u = 0 & \text{in } \Omega_t(\omega) \\ u = e^{i\tau x\cdot\xi}\{\chi_t(x')Q_t(x')b + \beta_{\chi_t,t,b,N,\omega}\} & \text{on } \Sigma_t(\omega), \end{cases} \tag{2.2}$$

where $\xi \in S^2$ laying in the span of η and ζ and fixed $\chi_t(x') \in C_0^\infty(\mathbb{R}^2)$ with $\text{supp}(\chi_t) \subset \Sigma_t(\omega)$, $Q_t(x')$ is a nonzero smooth function and $0 \neq b \in \mathbb{C}^3$. Moreover, $\beta_{\chi_t,b,t,N,\omega}(x',\tau)$ is a smooth function supported in $\text{supp}(\chi_t)$ satisfying:

$$\|\beta_{\chi_t,b,t,N,\omega}(\cdot,\tau)\|_{L^2(\mathbb{R}^2)} \leq c\tau^{-1}$$

for some constant $c > 0$. From now on, we use c, c' and their capitals to denote general positive constants whose values may vary from line to line. As in the paper [11], $u_{\chi_t,b,t,N,\omega}$ can be written as

$$u_{\chi_t,b,t,N,\omega} = w_{\chi_t,b,t,N,\omega} + r_{\chi_t,b,t,N,\omega}$$

with

$$w_{\chi_t,b,t,N,\omega} = \chi_t(x')Q_t e^{i\tau x\cdot\xi} e^{-\tau(x\cdot\omega-t)A_t(x')} b + \gamma_{\chi_t,b,t,N,\omega}(x,\tau) \tag{2.3}$$

and $r_{\chi_t b,t,N,\omega}$ satisfying

$$\|r_{\chi_t,b,t,N,\omega}\|_{H^1(\Omega_t(\omega))} \leq c\tau^{-N-1/2}, \tag{2.4}$$

where $A_t(\cdot) \in B^\infty(\mathbb{R}^2)$ is a complex function with its real part $\mathrm{Re}A_t(x') > 0$, and $\gamma_{\chi_t,b,t,N,\omega}$ is a smooth function supported in $\mathrm{supp}(\chi_t)$ satisfying

$$\|\partial_x^\alpha \gamma_{\chi_t,b,t,N,\omega}\|_{L^2(\Omega_s(\omega))} \leq c\tau^{|\alpha|-3/2} e^{-\tau(s-t)a} \tag{2.5}$$

for $|\alpha| \leq 1$ and $s \geq t$, where $a > 0$ is some constant depending on $A_t(x')$. Without loss of generality, we consider the special case where $t = 0$, $\omega = e_3 = (0, 0, 1)$ and choose $\eta = (1, 0, 0)$, $\zeta = (0, 1, 0)$. The general case can be obtained from this special case by change of coordinates. Define $L = L_A$ and $\widetilde{M}\cdot = e^{-i\tau x'\cdot\xi'} L(e^{i\tau x'\cdot\xi'}\cdot)$, where $x' = (x_1, x_2)$ and $\xi' = (\xi_1, \xi_2)$ with $|\xi'| = 1$, then $\widetilde{M}$ is a differential operator. To be precise, by using $a_{jl} = a_{lj}$, we calculate $\widetilde{M}$ to be given by

$$\begin{aligned}
\widetilde{M} &= -\tau^2 \sum_{jl} a_{jl}\xi_j\xi_l + 2\tau \sum_{jl} a_{jl}(i\xi_l)\partial_j + \sum_{jl} a_{jl}\partial_j\partial_l \\
&\quad + \sum_{jl} (\partial_j a_{jl})(i\tau\xi_l) + \sum_{jl} (\partial_j a_{jl})\partial_l + k^2 \\
&= -\tau^2 \sum_{jl} a_{jl}\xi_j\xi_l + 2\tau \sum_{l} a_{3l}(i\xi_l)\partial_3 + a_{33}\partial_3\partial_3 \\
&\quad + 2\tau \sum_{j\neq 3,l} a_{jl}(i\xi_l)\partial_j + \sum_{j\neq 3,l\neq 3} a_{jl}\partial_j\partial_l \\
&\quad + \sum_{jl} (\partial_j a_{jl})(i\tau\xi_l) + \sum_{jl} (\partial_j a_{jl})\partial_l + k^2
\end{aligned}$$

with $\xi_3 = 0$. Now, we want to solve

$$\widetilde{M}v = 0,$$

which is equivalent to $Mv = 0$, where $M = a_{33}^{-1}\widetilde{M}$. Now, we use the same idea in [11], define $\langle e, f\rangle = \sum_{ij} a_{ij} e_i f_j$, where $e = (e_1, e_2, e_3)$, $f = (f_1, f_2, f_3)$ and denote $\langle e, f\rangle_0 = \langle e, f\rangle |_{x_3=0}$. Let P be a differential operator, and we define the order of P, denoted by $ord(P)$, in the following sense:

$$\|P(e^{-\tau x_3 A(x')}\varphi(x'))\|_{L^2(\mathbb{R}^3_+)} \leq c\tau^{ord(P)-1/2},$$

where $\mathbb{R}^3_+ = \{x_3 > 0\}$, $A(x')$ is a smooth complex function with its real part greater than 0 and $\varphi(x') \in C_0^\infty(\mathbb{R}^2)$. In this sense, similar to [11], we can see that τ, ∂_3 are of order 1, ∂_1, ∂_2 are of order 0 and x_3 is of order -1.

Now according to this order, the principal part M_2 (order 2) of M is:

$$M_2 = -\{D_3^2 + 2\tau \langle e_3, e_3\rangle_0^{-1} \langle e_3, \rho\rangle_0 D_3 + \tau^2 \langle e_3, e_3\rangle_0^{-1} \langle \rho, \rho\rangle_0\}$$

with $D_3 = -i\partial_3$ and $\rho = (\xi_1, \xi_2, 0)$. Note that the principal part M_2 does not involve the lower order term $k^2\cdot$, so we can follow all the constructions in the same procedures as in [11] and we omit details.

3. **Tools and estimates.** In this section, we introduce the Runge approximation property and a very useful elliptic estimate: Meyers L^p-estimates.

3.1. **Runge approximation property.**

Definition 3.1. [5] Let L be a second order elliptic operator, solutions of an equation $Lu = 0$ are said to have the Runge approximation property if, whenever K and Ω are two simply connected domains with $K \subset \Omega$, any solution in K can be approximated uniformly in compact subsets of K by a sequence of solutions in Ω.

There are many applications for Runge approximation property in inverse problems. Similar results for some elliptic operators can be found in [5], [6]. The following theorem is a classical result for Runge approximation property for second order elliptic equations.

Theorem 3.2. *(Runge approximation property) Let $L_0\cdot = \nabla\cdot(A^0(x)\nabla\cdot) + k^2\cdot$ be a second order elliptic differential operator with $A^0(x)$ to be Lipschitz. Assume that k^2 is not a Dirichlet eigenvalue of $-\nabla\cdot(A^0(x)\nabla\cdot)$ in Ω. Let O and Ω be two open bounded domains with smooth boundary in $\mathbb{R}^3$ such that $O \Subset \Omega$ and $\Omega\backslash\overline{O}$ is connected.*

Let $u_0 \in H^1(O)$ satisfy

$$L_0 u_0 = 0 \text{ in } O.$$

Then for any compact subset $K \subset O$ and any $\epsilon > 0$, there exists $U \in H^1(\Omega)$ satisfying

$$L_0 U = 0 \text{ in } \Omega,$$

such that

$$\|u_0 - U\|_{H^1(K)} \le \epsilon.$$

Proof. The proof is standard and it is based on the weak unique continuation property for the anisotropic second order elliptic operator L_0 and the Hahn-Banach theorem. For more details, how to derive the Runge approximation property from the weak unique continuation, we refer readers to [5] $\square$

3.2. **Elliptic estimates and some identities.** We need some estimates for solutions to some Dirichlet problems which will be used in next section. Recall that, for $f \in H^{1/2}(\partial\Omega)$, let u and u_0 be solutions to the Dirichlet problems (1.2) and (1.3), respectively. Note that $a_{ij}(x) = a^0_{ij}(x) + \widetilde{a_{ij}}(x)\chi_D$ and we set $w = u - u_0$, then w satisfies the Dirichlet problem

$$\begin{cases} \nabla\cdot(A(x)\nabla w) + k^2 w = -\nabla\cdot((\widetilde{A}\chi_D)\nabla u_0) & \text{in } \Omega \\ w = 0 & \text{on } \partial\Omega \end{cases} \tag{3.1}$$

where $A(x) = (a_{ij}(x))$, $A^0(x) = (a^0_{ij}(x))$ and $\widetilde{A}(x) = (\widetilde{a_{ij}}(x))$. Then in the following lemmas, we give some estimates for w.

Lemma 3.3. *There exists a positive constant C independent of w such that we have*

$$\|w\|_{L^2(\Omega)} \le C\|\nabla w\|_{L^p(\Omega)}$$

for $\frac{6}{5} \le p \le 2$ if $n = 3$.

Proof. The proof follows from [12] by Freidrich's inequality, see [7] p.258 and use a standard elliptic regularity. $\square$

Lemma 3.4. *There exists $\epsilon \in (0,1)$, depending only on Ω, $A^0(x) = (a^0_{ij}(x))$ and $\widetilde{A}(x) = (\widetilde{a_{ij}}(x))$ such that*

$$\|\nabla w\|_{L^p(\Omega)} \leq C\|u_0\|_{W^{1,p}(D)}$$

for $\max\{2-\epsilon, \frac{6}{5}\} < p \leq 2$ *if* $n = 3$.

Proof. The proof also follows from [12]. Set $f := -(\widetilde{A}\chi_D)\nabla u_0$. Let w_0 be a solution of

$$\begin{cases} \nabla \cdot (A(x)\nabla w_0) = \nabla \cdot f & \text{in } \Omega, \\ w_0 = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.2}$$

The following L^p-estimate of w_0, known as Meyers estimate, follows from [8],

$$\|\nabla w_0\|_{L^p(\Omega)} \leq C\|f\|_{L^p(\Omega)} \tag{3.3}$$

for $p \in (\max\{2-\epsilon, \frac{6}{5}\}, 2]$, where $\epsilon \in (0,1)$ depends on Ω, $A^0(x) = (a^0_{ij}(x))$ and $\widetilde{A}(x) = (\widetilde{a_{ij}}(x))$. We set $W := w - w_0$, then since $w = w_0 + W$, we have

$$\|\nabla w\|_{L^p(\Omega)} \leq \|\nabla w_0\|_{L^p(\Omega)} + \|\nabla W\|_{L^p(\Omega)}. \tag{3.4}$$

Moreover, W satisfies

$$\begin{cases} \nabla \cdot (A(x)\nabla W) + k^2 W = -k^2 w_0 & \text{in } \Omega, \\ W = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.5}$$

By the standard elliptic regularity, we have

$$\|W\|_{H^1(\Omega)} \leq C\|w_0\|_{L^2(\Omega)}.$$

Thus, we get for $p \leq 2$,

$$\|\nabla W\|_{L^p(\Omega)} \leq C\|\nabla W\|_{L^2(\Omega)} \leq C\|W\|_{H^1(\Omega)} \leq C\|w_0\|_{L^2(\Omega)}. \tag{3.6}$$

By Sobolev embedding theorem, we get

$$\|w_0\|_{L^2(\Omega)} \leq C\|w_0\|_{W^{1,p}(\Omega)} \tag{3.7}$$

for $p \geq \frac{6}{5}$ if $n = 3$. Use Poincaré's inequality in L^p spaces ($w_0|_{\partial\Omega} = 0$), we have

$$\|w_0\|_{L^2(\Omega)} \leq C\|\nabla w_0\|_{L^p(\Omega)} \tag{3.8}$$

for $p \geq \frac{6}{5}$ if $n = 3$. Combining (3.3) with (3.4), (3.6) and (3.8), we can obtain

$$\|\nabla w\|_{L^p(\Omega)} \leq C\|f\|_{L^p(\Omega)} \leq C\|u_0\|_{W^{1,p}(D)}$$

for $\max\{2-\epsilon, \frac{6}{5}\} < p \leq 2$ if $n = 3$. $\square$

Recall the Dirichlet-to-Neumann map which we have defined in Section 1: $\Lambda_D f := A\nabla u \cdot \nu$ and $\Lambda_\emptyset f := A^0 \nabla u_0 \cdot \nu$, where $\nu = (\nu_1, \nu_2, \nu_3)$ is a unit outer normal on $\partial\Omega$.

We next prove some useful identities.

Lemma 3.5. $\int_{\partial\Omega}(\Lambda_D - \Lambda_\emptyset) f\bar{f} d\sigma = Re \int_D \widetilde{A}\nabla u_0 \cdot \overline{\nabla u} dx$.

Proof. It is clear that

$$\begin{aligned}\int_{\partial\Omega}(A\nabla u)\cdot\nu\bar{\varphi}d\sigma &= \int_\Omega \nabla\cdot(A\nabla u\bar{\varphi})dx\\ &= \int_\Omega(\nabla\cdot(A\nabla u)\bar{\varphi}+A\nabla u\cdot\overline{\nabla\varphi})dx\\ &= -k^2\int_\Omega u\bar{\varphi}dx+\int_\Omega A\nabla u\cdot\overline{\nabla\varphi}dx\end{aligned}$$

for any $\varphi\in H^1(\Omega)$. Since $u=u_0=f$ on $\partial\Omega$, the left hand side of the identity has the same value whether we take $\varphi=u$ or $\varphi=u_0$, and it is equal to $\int_{\partial\Omega}\Lambda_D f\bar{f}d\sigma$. Hence we have

$$\begin{aligned}\int_{\partial\Omega}\Lambda_D f\bar{f}d\sigma &= -k^2\int_\Omega u\overline{u_0}dx+\int_\Omega A\nabla u\cdot\overline{\nabla u_0}dx\\ &= -k^2\int_\Omega |u|^2dx+\int_\Omega A\nabla u\cdot\overline{\nabla u}dx.\end{aligned}$$

The right hand side of the above identity is real. Hence, by taking the real part, we have

$$\int_{\partial\Omega}\Lambda_D f\bar{f}d\sigma=-k^2\mathrm{Re}\int_\Omega u\overline{u_0}dx+\mathrm{Re}\int_\Omega A\nabla u\cdot\overline{\nabla u_0}dx$$

and

$$\int_{\partial\Omega}\Lambda_\emptyset f\bar{f}d\sigma=-k^2\mathrm{Re}\int_\Omega u\overline{u_0}dx+\mathrm{Re}\int_\Omega A^0\nabla u\cdot\overline{\nabla u_0}dx.$$

Therefore, we have

$$\begin{aligned}\int_{\partial\Omega}(\Lambda_D-\Lambda_\emptyset)f\bar{f}d\sigma &= \mathrm{Re}\int_\Omega(A-A^0)\nabla u\cdot\overline{\nabla u_0}dx \qquad (3.9)\\ &= \mathrm{Re}\int_D\widetilde{A}\nabla u\cdot\overline{\nabla u_0}dx.\end{aligned}$$

□

The estimates in the following lemma play an important role in our reconstruction algorithm.

Lemma 3.6. *We have the following identities:*

$$\begin{aligned}\int_{\partial\Omega}(\Lambda_D-\Lambda_\emptyset)f\bar{f}d\sigma &= -\int_\Omega A\nabla w\cdot\overline{\nabla w}dx+k^2\int_\Omega|w|^2dx \qquad (3.10)\\ &\quad+\int_D\widetilde{A}\nabla u_0\cdot\overline{\nabla u_0}dx,\end{aligned}$$

$$\begin{aligned}\int_{\partial\Omega}(\Lambda_D-\Lambda_\emptyset)f\bar{f}d\sigma &= \int_\Omega A^0\nabla w\cdot\overline{\nabla w}dx-k^2\int_\Omega|w|^2dx \qquad (3.11)\\ &\quad+\int_D\widetilde{A}\nabla u\cdot\overline{\nabla u}dx.\end{aligned}$$

In particular, we have

$$\int_{\partial\Omega}(\Lambda_D-\Lambda_\emptyset)f\bar{f}d\sigma\le k^2\int_\Omega|w|^2dx+C\int_D|\nabla u_0|^2dx, \qquad (3.12)$$

$$\int_{\partial\Omega}(\Lambda_D-\Lambda_\emptyset)f\bar{f}d\sigma\ge c\int_D|\nabla u_0|^2dx-k^2\int_\Omega|w|^2dx, \qquad (3.13)$$

where $C>0$ is a constant depending on $\widetilde{A}(x)$ and c is a constant depending on A, A^0 and $\widetilde{A}$.

Proof. Multiplying the identity

$$\nabla\cdot(A(x)\nabla w)+k^2w+\nabla\cdot(\widetilde{A}\chi_D\nabla u_0)=0$$

by $\bar{w}$ and integrating over Ω, we get

$$\begin{aligned}
0 &= \int_\Omega \nabla\cdot(A\nabla w)\bar{w}dx+\int_\Omega \nabla\cdot(\widetilde{A}\chi_D\nabla u_0)\bar{w}dx+k^2\int_\Omega |w|^2dx\\
&= -\int_\Omega A\nabla w\cdot\overline{\nabla w}dx+\int_{\partial\Omega}(A\nabla w\cdot\nu)\bar{w}d\sigma-\int_\Omega \widetilde{A}\chi_D\nabla u_0\cdot\overline{\nabla w}dx\\
&\quad+\int_{\partial\Omega}(\widetilde{A}\chi_D\nabla u_0\cdot\nu)\bar{w}d\sigma+k^2\int_\Omega|w|^2dx\\
&= -\int_\Omega A\nabla w\cdot\overline{\nabla w}dx-\int_D\widetilde{A}\nabla u_0\cdot\overline{\nabla w}dx+k^2\int_\Omega|w|^2dx\\
&= -\int_\Omega A\nabla w\cdot\overline{\nabla w}dx-\int_D\widetilde{A}\nabla u_0\cdot\overline{\nabla u}dx+k^2\int_\Omega|w|^2dx\\
&\quad+\int_D\widetilde{A}\nabla u_0\cdot\overline{\nabla u_0}dx,
\end{aligned}$$

and use (3.9) we can obtain

$$\int_{\partial\Omega}(\Lambda_D-\Lambda_\emptyset)f\bar{f}d\sigma=-\int_\Omega A\nabla w\cdot\overline{\nabla w}dx+\int_D\widetilde{A}\nabla u_0\cdot\overline{\nabla u_0}dx+k^2\int_\Omega|w|^2dx.$$

Similarly, multiplying the identity

$$\nabla\cdot(\widetilde{A}\chi_D\nabla u)+\nabla\cdot(A^0\nabla w)+k^2w=0$$

by $\bar{w}$ and integrating over Ω, we get

$$\begin{aligned}
0 &= \int_\Omega\nabla\cdot(\widetilde{A}\chi_D\nabla u)\bar{w}dx+\int_\Omega\nabla\cdot(A^0\nabla w)\bar{w}dx+k^2\int_\Omega|w|^2dx\\
&= -\int_D\widetilde{A}\nabla u\cdot\overline{\nabla w}dx-\int_\Omega A^0\nabla w\cdot\overline{\nabla w}dx+k^2\int_\Omega|w|^2dx\\
&= -\int_D\widetilde{A}\nabla u\cdot\overline{\nabla u}dx+\int_D\widetilde{A}\nabla u\cdot\overline{\nabla u_0}dx+k^2\int_\Omega|w|^2dx\\
&\quad-\int_\Omega A^0\nabla w\cdot\overline{\nabla w}dx,
\end{aligned}$$

and use (3.9) again, we can obtain

$$\int_{\partial\Omega}(\Lambda_D-\Lambda_\emptyset)f\bar{f}d\sigma=\int_\Omega A^0\nabla w\cdot\overline{\nabla w}dx-k^2\int_\Omega|w|^2dx+\int_D\widetilde{A}\nabla u\cdot\overline{\nabla u}dx.$$

For the remaining part, (3.12) is an easy consequence of (3.10)

$$\begin{aligned}
\int_{\partial\Omega}(\Lambda_D-\Lambda_\emptyset)f\bar{f}d\sigma &\leq k^2\int_\Omega|w|^2dx+\int_D\widetilde{A}\nabla u_0\cdot\overline{\nabla u_0}dx\\
&= k^2\int_\Omega|w|^2dx+C\int_D|\nabla u_0|^2dx,
\end{aligned}$$

since $\widetilde{A}\in L^\infty(D)$.

Finally, for the lower bound, we use

$$\begin{aligned} A^0\nabla w\cdot\overline{\nabla w}+\widetilde{A}\chi_D\nabla u\cdot\overline{\nabla u} &= A\nabla u\cdot\overline{\nabla u}-2\mathrm{Re}A^0\nabla u\cdot\overline{\nabla u_0}+A^0\nabla u_0\cdot\overline{\nabla u_0}\\ &= A(\nabla u-(A)^{-1}A^0\nabla u_0)\cdot(\overline{\nabla u-(A)^{-1}A^0\nabla u_0})\\ &\quad +(A^0-(A^0)(A)^{-1}(A^0))\nabla u_0\cdot\overline{\nabla u_0}\\ &\geq (A^0-(A^0)(A)^{-1}(A^0))\nabla u_0\cdot\overline{\nabla u_0}\\ &\geq c|\nabla u_0|^2, \end{aligned}$$

since $A(\nabla u-(A)^{-1}A^0\nabla u_0)\cdot(\overline{\nabla u-(A)^{-1}A^0\nabla u_0})\geq 0$ and note that $A^0-(A^0)(A)^{-1}(A^0)=A^0(A)^{-1}(A-A^0)=A^0(A)^{-1}\widetilde{A}\chi_D$ is a positive definite matrix by our previous assumptions in section 1. □

Applying Lemma 3.3 to (3.12),

$$\int_{\partial\Omega}(\Lambda_D-\Lambda_\emptyset)f\bar{f}d\sigma\leq C\|u_0\|^2_{W^{1,2}(D)}. \tag{3.14}$$

By (3.13) and the Meyers L^p estimate $\|w\|_{L^2(\Omega)}\leq C\|u_0\|_{W^{1,p}(D)}$, we have

$$\int_{\partial\Omega}(\Lambda_D-\Lambda_\emptyset)f\bar{f}d\sigma\geq c\int_D|\nabla u_0|^2dx-c\|u_0\|^2_{W^{1,p}(D)}. \tag{3.15}$$

4. **Detecting the convex hull of the unknown obstacle.**

4.1. **Main theorem.** Recall that we have constructed the oscillating-decaying solutions in section 2, and note that this solution can not be defined on the whole domain, that is, the oscillating-decaying solutions $u_{\chi_t,b,t,N,\omega}(x,\tau)$ only defined on $\Omega_t(\omega)\subsetneq\Omega$. Nevertheless, with the help of the Runge approximation property, we can only determine the convex hull of the unknown obstacle D by$\Lambda_D f$ for infinitely many f.

We define B to be an open ball in $\mathbb{R}^3$ such that $\overline{\Omega}\subset B$. Assume that $\widetilde{\Omega}\subset\mathbb{R}^3$ is an open smooth domain with $\overline{B}\subset\widetilde{\Omega}$. As in the section 2, set $\omega\in S^2$ and $\{\eta,\zeta,\omega\}$ forms an orthonormal basis of $\mathbb{R}^3$. Suppose $t_0=\inf_{x\in D}x\cdot\omega=x_0\cdot\omega$, where $x_0=x_0(\omega)\in\partial D$. For any $t\leq t_0$ and $\epsilon>0$ small enough, we can construct

$$\begin{aligned} u_{\chi_{t-\epsilon},b,t-\epsilon,N,\omega} &= \chi_{t-\epsilon}(x')Q_{t-\epsilon}(x')e^{i\tau x\cdot\xi}e^{-\tau(x\cdot\omega-(t-\epsilon))A_{t-\epsilon}(x')}b+\gamma_{\chi_{t-\epsilon},b,t-\epsilon,N,\omega}\\ &\quad +r_{\chi_{t-\epsilon},b,t-\epsilon,N,\omega} \end{aligned}$$

to be the oscillating-decaying solution for $\nabla\cdot(A^0(x)\nabla\cdot)+k^2\cdot$ in $B_{t-\epsilon}(\omega)=B\cap\{x\cdot\omega>t-\epsilon\}$, where $\chi_{t-\epsilon}(x')\in C_0^\infty(\mathbb{R}^2)$ and $b\in\mathbb{C}$. Note that in section 2, we have assumed the leading coefficient $A^0(x)\in B^\infty(\mathbb{R}^3)$. Similarly, we have the oscillating-decaying solution

$$u_{\chi_t,b,t,N,\omega}(x,\tau)=\chi_t(x')Q_te^{i\tau x\cdot\xi}e^{-\tau(x\cdot\omega-t)A_t(x')}b+\gamma_{\chi_t,b,t,N,\omega}(x,\tau)+r_{\chi_t,b,t,N,\omega}$$

for L_{A^0} in $B_t(\omega)$. In fact, for any τ, $u_{\chi_{t-\epsilon},b,t-\epsilon,N,\omega}(x,\tau)\to u_{\chi_t,b,t,N,\omega}(x,\tau)$ in an appropriate sense as $\epsilon\to 0$. For details, we refer readers to consult all the details and results in [11], and we list consequences in the following.

$$\chi_{t-\epsilon}(x')Q_{t-\epsilon}(x')e^{i\tau x\cdot\xi}e^{-\tau(x\cdot\omega-(t-\epsilon))A_{t-\epsilon}(x')}b\to\chi_t(x')Q_te^{i\tau x\cdot\xi}e^{-\tau(x\cdot\omega-t)A_t(x')}b$$

in $H^2(B_t(\omega))$ as ϵ tends to 0,

$$\gamma_{\chi_{t-\epsilon},b,t-\epsilon,N,\omega}\to\gamma_{\chi_t,b,t,N,\omega}$$

in $H^2(B_t(\omega))$ as ϵ tends to 0, and finally,

$$r_{\chi_{t-\epsilon},b,t-\epsilon,N,\omega}\to r_{\chi_t,b,t,N,\omega}$$

in $H^1(B_t(\omega))$ as ϵ tends to 0.

Obviously, $B_{t-\epsilon}(\omega)$ is a convex set and $\overline{\Omega_t(\omega)} \subset B_{t-\epsilon}(\omega)$ for all $t \leq t_0$. By using the Runge approximation property, we can see that there exists a sequence of functions $\tilde{u}_{\epsilon,j}$, $j = 1, 2, \cdots$, such that

$$\tilde{u}_{\epsilon,j} \to u_{\chi_{t-\epsilon},b,t-\epsilon,N,\omega} \text{ in } H^1(\overline{\Omega_t(\omega)}),$$

where $\tilde{u}_{\epsilon,j} \in H^1(\widetilde{\Omega})$ satisfy $L_{A^0}\tilde{u}_{\epsilon,j} = 0$ in $\widetilde{\Omega}$ for all ϵ, j. Define the indicator function $I(\tau, \chi_t, b, t, \omega)$ by the formula:

$$I(\tau, \chi_t, b, t, \omega) = \lim_{\epsilon \to 0} \lim_{j \to \infty} \int_{\partial\Omega} (\Lambda_D - \Lambda_\emptyset) f_{\epsilon,j} \overline{f_{\epsilon,j}} d\sigma,$$

where $f_{\epsilon,j} = \tilde{u}_{\epsilon,j}|_{\partial\Omega}$.

Now the characterization of the convex hull of D is based on the following theorem:

Theorem 4.1. *(1) If $t < t_0$, then for any $\chi_t \in C_0^\infty(\mathbb{R}^2)$ and $b \in \mathbb{C}^3$, we have*

$$\limsup_{\tau \to \infty} |I(\tau, \chi_t, b, t, \omega)| = 0.$$

(2) If $t = t_0$, then for any $\chi_{t_0} \in C_0^\infty(\mathbb{R}^2)$ with $x_0' = (x_0 \cdot \eta, x_0 \cdot \zeta)$ being an interior point of supp(χ_{t_0}) and $0 \neq b \in \mathbb{C}$, we have

$$\liminf_{\tau \to \infty} |I(\tau, \chi_{t_0}, b, t_0, \omega)| > 0.$$

Proof. First of all, note that we have a sequence of functions $\{\tilde{u}_{\epsilon,j}\}$ satisfies the equation $\nabla \cdot (A^0 \nabla u) + k^2 u = 0$ in Ω, as in the beginning of the section 3, let $w_{\epsilon,j} = u - \tilde{u}_{\epsilon,j}$, then $w_{\epsilon,j}$ satisfies the Dirichlet problem

$$\begin{cases} \nabla \cdot (A(x) \nabla w_{\epsilon,j}) + k^2 w_{\epsilon,j} = -\nabla \cdot (\widetilde{A} \chi_D \nabla \tilde{u}_{\epsilon,j}) & \text{in } \Omega, \\ w_{\epsilon,j} = 0 & \text{on } \partial\Omega. \end{cases}$$

So we can apply (3.14) directly, which means

$$\int_{\partial\Omega} (\Lambda_D - \Lambda_\emptyset) f_{\epsilon,j} \overline{f_{\epsilon,j}} d\sigma \leq C \|\tilde{u}_{\epsilon,j}\|^2_{H^1(D)} \text{ with } f_{\epsilon,j} = \widetilde{u}_{\epsilon,j}|_{\partial\Omega},$$

where the last inequality obtained by the Hölder's inequality.

By the Runge approximation property we have

$$\tilde{u}_{\epsilon,j} \to u_{\chi_{t-\epsilon},b,t-\epsilon,N,\omega} \text{ in } H^1(\overline{B_t(\omega)})$$

as $j \to \infty$ and we know that the obstacle $D \subset B_t(\omega)$, so we have

$$\|\tilde{u}_{\epsilon,j} - u_{\chi_{t-\epsilon},b,t-\epsilon,N,\omega}\|_{H^1(D)} \to 0$$

as $j \to \infty$ for all $\epsilon > 0$. Moreover, we know that $u_{\chi_{t-\epsilon},b,t-\epsilon,N,\omega} \to u_{\chi_t,b,t,N,\omega}$ as $\epsilon \to 0$ in $H^1(B_t(\omega))$, which implies

$$\|\tilde{u}_{\epsilon,j} - u_{\chi_t,b,t,N,\omega}\|_{H^1(D)} \to 0$$

as $\epsilon \to 0$, $j \to \infty$. Now by the definition of $I(\tau, \chi_t, b, t, \omega)$, we have

$$I(\tau, \chi_t, b, t, \omega) \leq C \|u_{\chi_t,b,t,N,\omega}\|^2_{H^1(D)}.$$

Now if $t < t_0$, we substitute $u_{\chi_t,b,t,N,\omega} = w_{\chi_t,b,t,N,\omega} + r_{\chi_t,b,t,N,\omega}$ with $w_{\chi_t,b,t,N,\omega}$ being described by (2.3) into

$$I(\tau, \chi_t, b, t, \omega) \leq C(\int_D |u_{\chi_t,b,t,N,\omega}|^2 dx + \int_D |\nabla u_{\chi_t,b,t,N,\omega}|^2 dx)$$

and use estimates (2.4), (2.5) to obtain that

$$|I(\tau, \chi_t, b, t, \omega)| \leq C\tau^{-2N-1}$$

which finishes

$$\limsup_{\tau\to\infty} |I(\tau, \chi_t, b, t, \omega)| = 0.$$

For the second part, as inequality (4.1), we use (3.15), then the similar argument follows. It is easy to get

$$I(\tau, \chi_t, b, t, \omega) \geq c\int_D |\nabla u_{\chi_t,b,t,N,\omega}|^2 dx - c\|u_{\chi_t,b,t,N,\omega}\|^2_{W^{1,p}(D)}, \tag{4.1}$$

For $p \in (\max\{2-\epsilon, \frac{6}{5}\}, 2]$. For the remaining part, we need some extra estimates in the following section. □

4.2. **End of the proof of Theorem 4.1.** For further estimate of the lower bound, we need to introduce the sets $D_{j,\delta} \subset D$, $D_\delta \subset D$ as follows. Recall that $h_D(\omega) = \inf_{x\in D} x\cdot\omega$ and $t_0 = h_D(\omega) = x_0\cdot\omega$ for some $x_0 \in \partial D$. For any $\alpha \in \partial D \cap \{x\cdot\omega = h_D(\omega)\} := K$, define $B(\alpha,\delta) = \{x \in \mathbb{R}^3; |x-\alpha| < \delta\}$ $(\delta > 0)$. Note $K \subset \cup_{\alpha\in K} B(\alpha,\delta)$ and K is compact, so there exists $\alpha_1, \cdots, \alpha_m \in K$ such that $K \subset \cup_{j=1}^m B(\alpha_j,\delta)$. Thus, we define

$$D_{j,\delta} := D\cap B(\alpha_j,\delta) \text{ and } D_\delta := \cup_{j=1}^m D_{j,\delta}.$$

It is easy to see that

$$\int_{D\setminus D_\delta} e^{-p\tau(x\cdot\omega - t_0)A_{t_0}(x')} b dx = O(e^{-pa\delta\tau}),$$

because $A_{t_0}(x') \in B^\infty(\mathbb{R}^2)$ is bounded and its real part strictly greater than 0, so there exists $a > 0$ such that $\mathrm{Re}A_{t_0}(x') \geq a > 0$ on $D\setminus D_\delta$. Let $\alpha_j \in K$, by rotation and translation, we may assume $\alpha_j = 0$ and the vector $\alpha_j - x_0 = -x_0$ is parallel to $e_3 = (0,0,1)$. Therefore, we consider the change of coordinates near each α_j as follows:

$$\begin{cases} y' = x' \\ y_3 = x\cdot\omega - t_0, \end{cases}$$

where $x = (x_1,x_2,x_3) = (x',x_3)$ and $y = (y_1,y_2,y_3) = (y',y_3)$. Denote the parametrization of ∂D near α_j by $l_j(y')$, then we have the following estimates.

Lemma 4.2. *For $q \leq 2$, we have*

$$\begin{aligned}\int_D |u_{\chi_{t_0},b,t_0,N,\omega}|^q dx \quad &\leq \quad c\tau^{-1}\sum_{j=1}^m \iint_{|y'|<\delta} e^{-aq\tau l_j(y')} dy' + O(\tau^{-1}e^{-qa\delta\tau}) \\ &\quad + O(e^{-qa\tau}) + O(\tau^{-3}) + O(\tau^{-2N-1}),\end{aligned} \tag{4.2}$$

$$\begin{aligned}\int_D |u_{\chi_{t_0},b,t_0,N,\omega}|^2 dx \quad &\geq \quad C\tau^{-1}\sum_{j=1}^m \iint_{|y'|<\delta} e^{-2a\tau l_j(y')} dy' + O(\tau^{-1}e^{-2a\delta\tau}) \\ &\quad + O(\tau^{-3}) + O(\tau^{-2N-1}),\end{aligned} \tag{4.3}$$

$$\begin{aligned}\int_D |\nabla u_{\chi_{t_0},b,t_0,N,\omega}|^q dx &\leq C\tau^{q-1}\sum_{j=1}^m \iint_{|y'|<\delta} e^{-qa\tau l_j(y')}dy' + O(\tau^{-1}e^{-aq\delta\tau}) \\ &\quad + O(e^{-qa\tau}) + O(\tau^{-1}) + O(\tau^{-2N-1}), \qquad (4.4)\end{aligned}$$

and

$$\begin{aligned}\int_D |\nabla u_{\chi_{t_0},b,t_0,N,\omega}|^2 dx &\geq C\tau\sum_{j=1}^m \iint_{|y'|<\delta} e^{-2a\tau l_j(y')}dy' + O(\tau^{-1}e^{-2\delta a\tau}) \\ &\quad + O(\tau^{-1}) + O(\tau^{-2N-1}). \qquad (4.5)\end{aligned}$$

Proof. We follow the argument in [12]. We only prove (4.2) and (4.3) and the proof of (4.4) and (4.5) are similar arguments.

For (4.2):

$$\begin{aligned}\int_D |u_{\chi_{t_0},b,t_0,N,\omega}|^q dx &\leq C\int_D e^{-qa\tau(x\cdot\omega-t_0)}dx + C_q\int_D |\gamma_{\chi_{t_0},b,t_0,N,\omega}|^q dx \\ &\quad + C_q\int_D |r_{\chi_{t_0},b,t_0,N,\omega}|^q dx \\ &\leq C\int_{D_\delta} e^{-qa\tau(x\cdot\omega-t_0)}dx + C\int_{D\setminus D_\delta} e^{-qa\tau(x\cdot\omega-t_0)}dx \\ &\quad + C\int_D |\gamma_{\chi_{t_0},b,t_0,N,\omega}|^2 dx + C\int_D |r_{\chi_{t_0},b,t_0,N,\omega}|^2 dx \\ &\leq C\sum_{j=1}^m \iint_{|y'|<\delta} dy' \int_{l_j(y')}^{\delta} e^{-qa\tau y_3}dy_3 + Ce^{-qa\delta\tau} \\ &\quad + C\|\gamma_{\chi_{t_0},b,t_0,N,\omega}\|^2_{L^2(D)} + C\|r_{\chi_{t_0},b,t_0,N,\omega}\|^2_{H^1(D)} \\ &\leq C\tau^{-1}\sum_{j=1}^m \iint_{|y'|<\delta} e^{-aq\tau l_j(y')}dy' - \frac{C}{q}\tau^{-1}e^{-qa\delta\tau} \\ &\quad + Ce^{-qa\delta\tau} + C\tau^{-3} + C\tau^{-2N-1}\end{aligned}$$

note that $D \subset \Omega_{t_0}(\omega)$, which proves (4.1).

For (4.3):

$$\begin{aligned}\int_D |u_{\chi_{t_0},b,t_0,N,\omega}|^2 dx &\geq C\int_D e^{-2a\tau(x\cdot\omega-t_0)}dx - C\|\gamma_{\chi_{t_0},b,t_0,N,\omega}\|^2_{L^2(\Omega_{t_0}(\omega))} \\ &\quad - C\|r_{\chi_{t_0},b,t_0,N,\omega}\|^2_{H^1(\Omega_{t_0}(\omega))} \\ &\geq C\int_{D_\delta} e^{-2a\tau(x\cdot\omega-t_0)}dx - C\tau^{-3} - C\tau^{-2N-1} \\ &= C\tau^{-1}\sum_{j=1}^m \iint_{|y'|<\delta} e^{-2a\tau l_j(y')}dy' - \frac{C}{2}\tau^{-1}e^{-2a\delta\tau} \\ &\quad - C\tau^{-3} - C\tau^{-2N-1}.\end{aligned}$$

□

Recall that we have (4.1), the lower bound of $I(\tau, \chi_{t_0}, b, t_0, \omega)$, so we want to compare the order (in τ) of $\|u_{\chi_{t_0},b,t_0,N,\omega}\|_{L^2(D)}$, $\|\nabla u_{\chi_{t_0},b,t_0,N,\omega}\|_{L^2(D)}$, $\|u_{\chi_{t_0},b,t_0,N,\omega}\|_{L^p(D)}$ and $\|\nabla u_{\chi_{t_0},b,t_0,N,\omega}\|_{L^p(D)}$.

Lemma 4.3. *For* $\max\{2-\epsilon, \frac{6}{5}\} < p \le 2$*, we have the estimates as follows:*

$$\frac{\|\nabla u_{\chi_{t_0},b,t_0,N,\omega}\|^2_{L^2(D)}}{\|u_{\chi_{t_0},b,t_0,N,\omega}\|^2_{L^2(D)}} \ge C\tau^2, \quad \frac{\|u_{\chi_{t_0},b,t_0,N,\omega}\|^2_{L^p(D)}}{\|u_{\chi_{t_0},b,t_0,N,\omega}\|^2_{L^2(D)}} \ge C\tau^{1-\frac{2}{p}}$$

and

$$\frac{\|\nabla u_{\chi_{t_0},b,t_0,N,\omega}\|^2_{L^p(D)}}{\|u_{\chi_{t_0},b,t_0,N,\omega}\|^2_{L^2(D)}} \ge C\tau^{3-\frac{2}{p}}$$

for $\tau \gg 1$.

Proof. The idea of the proof comes from [12], but here we still need to deal with the $\gamma_{\chi_{t_0},b,t_0,N,\omega}$ and $r_{\chi_{t_0},b,t_0,N,\omega}$ in $D \subset \Omega_{t_0}(\omega)$. Note that if ∂D is Lipschitz, in our parametrization $l_j(y')$, we have $l_j(y') \le C|y'|$. Hence,

$$\begin{aligned} \sum_{j=1}^m \iint_{|y'|<\delta} e^{-2a\tau l_j(y')}dy' &\ge C\sum_{j=1}^m \iint_{|y'|<\delta} e^{-2ac\tau|y'|}dy' \\ &\ge C\tau^{-1}\sum_{j=1}^m \iint_{|y'|<\tau\delta} e^{-2|y'|}dy' \\ &= O(\tau^{-1}). \end{aligned}$$

For simplicity, we denote $u_0 := u_{\chi_{t_0},b,t_0,N,\omega}$ in the following calculations. Using Lemma 4.2, we obtain

$$\begin{aligned} &\frac{\int_D |\nabla u_0|^2 dx}{\int_D |u_0|^2 dx} \\ \ge\ & C\frac{\tau\sum_{j=1}^m \iint_{|y'|<\delta} e^{-2a\tau l_j(y')}dy' + O(\tau^{-1}e^{-2a\delta\tau}) + O(\tau^{-1}) + O(\tau^{-2N-1})}{\tau^{-1}\sum_{j=1}^m \iint_{|y'|<\delta} e^{-2a\tau l_j(y')}dy' + O(\tau^{-1}e^{-2a\delta\tau}) + O(\tau^{-3}) + O(\tau^{-2N-1})} \\ \ge\ & C\tau^2 \frac{1 + \frac{O(\tau^{-2}e^{-2a\delta\tau}) + O(\tau^{-2}) + O(\tau^{-2N-2})}{\sum_{j=1}^m \iint_{|y'|<\delta} e^{-2a\tau l_j(y')}dy'}}{1 + \frac{O(e^{-2a\delta\tau}) + O(\tau^{-2}) + O(\tau^{-2N})}{\sum_{j=1}^m \iint_{|y'|<\delta} e^{-2a\tau l_j(y')}dy'}} \\ =\ & O(\tau^2) \end{aligned}$$

as $\tau \gg 1$, where

$$\lim_{\tau\to\infty} \frac{O(\tau^{-2}e^{-2a\delta\tau}) + O(\tau^{-2}) + O(\tau^{-2N-2})}{\sum_{j=1}^m \iint_{|y'|<\delta} e^{-2a\tau l_j(y')}dy'} = 0$$

and

$$\lim_{\tau\to\infty} \frac{O(e^{-2a\delta\tau}) + O(\tau^{-2}) + O(\tau^{-2N})}{\sum_{j=1}^m \iint_{|y'|<\delta} e^{-2a\tau l_j(y')}dy'} = 0.$$

Now, by using the Hölder's inequality with the exponent $q = \dfrac{2}{p} \ge 1$, we have

$$\sum_{j=1}^m \iint_{|y'|<\delta} e^{-pa\tau l_j(y')}dy' \le C\Big(\sum_{j=1}^m \iint_{|y'|<\delta} e^{-2a\tau l_j(y')}dy'\Big)^{\frac{p}{2}}.$$

Hence we use Lemma 4.2 again, we have

$$\frac{(\int_D |u_0|^p dx)^{\frac{2}{p}}}{\int_D |u_0|^2 dx}$$

$$
\begin{aligned}
&\leq C\frac{\tau^{-\frac{2}{p}}(\sum_{j=1}^m\iint_{|y'|<\delta}e^{-pa\tau l_j(y')}dy')^{\frac{2}{p}}+O(\tau^{-\frac{2}{p}}e^{-2a\delta\tau})+O(e^{-2a\tau})}{\tau^{-1}\sum_{j=1}^m\iint_{|y'|<\delta}e^{-2a\tau l_j(y')}dy'+O(\tau^{-1}e^{-2a\delta\tau})+O(\tau^{-3})+O(\tau^{-2N-1})}\\
&\quad+\frac{O(\tau^{-\frac{6}{p}})+O(\tau^{\frac{-4N-2}{p}})}{\tau^{-1}\sum_{j=1}^m\iint_{|y'|<\delta}e^{-2a\tau l_j(y')}dy'+O(\tau^{-1}e^{-2a\delta\tau})+O(\tau^{-3})+O(\tau^{-2N-1})}\\
&\leq C\tau^{-\frac{2}{p}+1}\frac{\sum_{j=1}^m\iint_{|y'|<\delta}e^{-2a\tau l_j(y')}dy'+O(e^{-2a\delta\tau})+O(e^{-2a\tau}\tau^{\frac{2}{p}})}{\sum_{j=1}^m\iint_{|y'|<\delta}e^{-2a\tau l_j(y')}dy'+O(e^{-2a\delta\tau})+O(\tau^{-2})+O(\tau^{-2N})}\\
&\quad+\frac{O(\tau^{-\frac{4}{p}})+O(\tau^{\frac{-4N}{p}})}{\tau^{-1}\sum_{j=1}^m\iint_{|y'|<\delta}e^{-2a\tau l_j(y')}dy'+O(\tau^{-1}e^{-2a\delta\tau})+O(\tau^{-3})+O(\tau^{-2N-1})}\\
&= \tau^{-\frac{2}{p}+1}\frac{1+\frac{O(e^{-2a\delta\tau})+O(e^{-2a\tau}\tau^{\frac{2}{p}})+O(\tau^{-\frac{4}{p}})+O(\tau^{\frac{-4N}{p}})}{\sum_{j=1}^m\iint_{|y'|<\delta}e^{-2a\tau l_j(y')}dy'}}{1+\frac{O(e^{-2a\delta\tau})+O(\tau^{-2})+O(\tau^{-2N})}{\sum_{j=1}^m\iint_{|y'|<\delta}e^{-2a\tau l_j(y')}dy'}}\\
&= O(\tau^{-\frac{2}{p}+1})
\end{aligned}
$$

as $\tau\gg 1$ and

$$
\frac{(\int_D|\nabla u_0|^pdx)^{\frac{2}{p}}}{\int_D|u_0|^2dx}
$$

$$
\begin{aligned}
&\leq C\frac{\tau^{(p-1)\frac{2}{p}}(\sum_{j=1}^m\iint_{|y'|<\delta}e^{-pa\tau l_j(y')}dy')^{\frac{2}{p}}+O(\tau^{-\frac{2}{p}}e^{-2a\delta\tau})+O(e^{-2a\tau})}{\tau^{-1}\sum_{j=1}^m\iint_{|y'|<\delta}e^{-2a\tau l_j(y')}dy'+O(\tau^{-1}e^{-2a\delta\tau})+O(\tau^{-3})+O(\tau^{-2N-1})}\\
&\quad+C\frac{O(\tau^{-\frac{2}{p}})+O(\tau^{\frac{-4N-2}{p}})}{\tau^{-1}\sum_{j=1}^m\iint_{|y'|<\delta}e^{-2a\tau l_j(y')}dy'+O(\tau^{-1}e^{-2a\delta\tau})+O(\tau^{-3})+O(\tau^{-2N-1})}\\
&\leq C\tau^{3-\frac{2}{p}}\frac{\sum_{j=1}^m\iint_{|y'|<\delta}e^{-2a\tau l_j(y')}dy'+O(\tau^{-1}e^{-2a\delta\tau})+O(e^{-2a\tau}\tau^{\frac{2}{p}-1})}{\sum_{j=1}^m\iint_{|y'|<\delta}e^{-2a\tau l_j(y')}dy'+O(e^{-2a\delta\tau})+O(\tau^{-2})+O(\tau^{-2N})}\\
&\quad+C\frac{O(\tau^{-1})+O(\tau^{\frac{-4N}{p}-1})}{+O(\tau^{-\frac{2}{p}})+O(\tau^{\frac{-4N-2}{p}})}\\
&\leq C\tau^{3-\frac{2}{p}}\frac{1+\frac{O(\tau^{-1}e^{-2a\delta\tau})+O(e^{-2a\tau}\tau^{\frac{2}{p}-1})+O(\tau^{-1})+O(\tau^{\frac{-4N}{p}-1})}{\sum_{j=1}^m\iint_{|y'|<\delta}e^{-2a\tau l_j(y')}dy'}}{1+\frac{O(e^{-2a\delta\tau})+O(\tau^{-2})+O(\tau^{-2N})}{\sum_{j=1}^m\iint_{|y'|<\delta}e^{-2a\tau l_j(y')}dy'}}\\
&= O(\tau^{3-\frac{2}{p}})
\end{aligned}
$$

as $\tau\gg 1$. By (4.1) and above estimates, we have

$$
\begin{aligned}
\frac{I(\tau,\chi_t,b,t,\omega)}{\|u_{\chi_t,b,t,N,\omega}\|^2_{L^2(D)}} &\geq C\tau^2-C\tau^{1-\frac{2}{p}}-C\tau^{3-\frac{2}{p}}\\
&\geq C\tau^2
\end{aligned}
$$

for $\tau \gg 1$. On the other hand, for $\|u_{\chi_t,b,t,N,\omega}\|_{L^2(D)}$, we have

$$\begin{aligned}
\int_D |u_{\chi_t,b,t,N,\omega}|^2 dx &\geq C\tau^{-1}\sum_{j=1}^m \iint_{|y'|<\delta} e^{-2a\tau l_j(y')}dy' + O(\tau^{-1}e^{-qa\delta\tau}) \\
&\quad +O(\tau^{-3}) + O(\tau^{-2N-1}) \\
&\geq C\tau^{-1}\sum_{j=1}^m \iint_{|y'|<\delta} e^{-2a\tau|y'|}dy' + O(\tau^{-1}e^{-qa\delta\tau}) \\
&\quad +O(\tau^{-3}) + O(\tau^{-2N-1}) \\
&\geq C\tau^{-2}\sum_{j=1}^m \iint_{|y'|<\tau\delta} e^{-2a|y'|}dy' + O(\tau^{-1}e^{-qa\delta\tau}) \\
&\quad +O(\tau^{-3}) + O(\tau^{-2N-1}) \\
&= O(\tau^{-2}).
\end{aligned}$$

Therefore, we have

$$I(\tau,\chi_t,b,h_D(\omega),\omega) \geq C\tau^2\|u_{\chi_t,b,t,N,\omega}\|^2_{L^2(D)} \geq C > 0$$

for $\tau \gg 1$. □

In view of Theorem 4.1 and Lemma 4.2, we can give an algorithm for reconstructing the convex hull of an inclusion D by the Dirichlet-to-Neumann map Λ_D as long as $A(x)$ and D satisfy the described conditions.

Reconstruction algorithm.

1. Give $\omega \in S^2$ and choose $\eta,\zeta,\xi \in S^2$ so that $\{\eta,\zeta,\xi\}$ forms a basis of $\mathbb{R}^3$ and ξ lies in the span of η and ζ;
2. Choose a starting t such that $\Omega \subset \{x\cdot\omega \geq t\}$;
3. Choose a ball B such that the center of B lies on $\{x\cdot\omega = s\}$ for some $s<t$ and $\Omega \subset \overline{B_t(\omega)}$ and take $0 \neq b \in \mathbb{C}$;
4. Choose $\chi_t \in C_0^\infty(\mathbb{R}^2)$ such that $\chi_t > 0$ in $\Sigma_t(\omega)$ and $\chi_t = 0$ on $\partial\Sigma_t(\omega)$;
5. Construct the oscillating-decaying solution $u_{\chi_{t-\epsilon},b,t-\epsilon,N,\omega}$ in $B_{t-\epsilon}(\omega)$ with $\chi_{t-\epsilon} = \chi_t$ and the approximation sequence $\tilde{u}_{\epsilon,j}$ in $\widetilde{\Omega}$;
6. Compute the indicator function $I(\tau,\chi_t,b,t,\omega)$ which is determined by boundary measurements;
7. If $I(\tau,\chi_t,b,t,\omega) \to 0$ as $\tau\to\infty$, then choose $t'>t$ and repeat steps 4, 5, 6;
8. If $I(\tau,\chi_t,b,t,\omega) \nrightarrow 0$ for some $\chi_{t'}$, then $t' = \inf\{t : I(\tau,\chi_t,b,t,\omega) \nrightarrow 0\}$. Note that finding this t' requires an uncountable number of measurements, so the algorithm can not be realized on a computer;
9. Varying $\omega \in S^2$ and repeat 1 to 8, we can determine the convex hull of D.

Acknowledgement. Y.-H. Lin would like to thank Ms. Irene Cheng to improve the English write up of this article.

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Received xxxx 20xx; revised xxxx 20xx.

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