Nearly cloaking for the elasticity system with residual stress

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Abstract

The nearly cloaking via transformation optics approach for the isotropic elastic wave fields is considered. This work extends the study of the nearly cloaking scheme to the elasticity system with residual stress in $\mathbb{R}^N$ for $N = 2, 3$, which is an anisotropic elasticity system. It is worth mentioning that there are no minor symmetric properties for the elastic tensor with residual stress and this system is invariant under a coordinate transformation. Therefore, we think the elasticity system residual stress model is more natural than the isotropic elasticity system in designing the elastic cloaking medium in the physical sense. In addition, the main difficulty of treating this problem lies in the fact that there are no layer potential theory for the residual stress system. Instead, we will derive suitable elliptic estimates for the elasticity system residual stress by comparing with the Lamé system to achieve desired results.

Key words: Elastic cloaking, nearly cloaking, anisotropic elasticity system, residual stress
Mathematics Subject Classification: 74B10, 35R30, 35J25, 35R30, 74J20.

1 Introduction

This work is concerned with the cloaking theory of the elastic waves with residual stress in $\mathbb{R}^N$ for $N = 2, 3$. We formulate the mathematical problem in the following. Let Ω a bounded simply connected C^∞-smooth domain in $\mathbb{R}^N$, for $N = 2, 3$ and $u(x) = (u_i(x))_{i=1}^N$ is the displacement vector field. Consider the boundary value time-harmonic elasticity system

$$\begin{cases} \sum_{j,k,l=1}^N \dfrac{\partial}{\partial x_j}\left(C_{ijkl}(x)\dfrac{\partial u_k}{\partial x_l}\right) + \kappa^2 \rho u_i = 0 & \text{in } \Omega \text{ for } i = 1, 2, \cdots, N, \\ \mathcal{N}_{\mathcal{C}} u = \phi \in H^{-1/2}(\partial\Omega)^N & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\mathcal{C} = (C_{ijkl})$ is a four tensor, $\kappa \in \mathbb{R}$ is the frequency, $\rho = \rho(x)$ denotes the density of the medium and $\mathcal{N}_{\mathcal{C}} u$ is the Neumann data defined as

$$\mathcal{N}_{\mathcal{C}} u := \left(\sum_{j,k,l=1}^N \nu_j C_{1jkl}\frac{\partial u_k}{\partial x_l}, \sum_{j,k,l=1}^N \nu_j C_{2jkl}\frac{\partial u_k}{\partial x_l}, \cdots, \sum_{j,k,l=1}^N \nu_j C_{Njkl}\frac{\partial u_k}{\partial x_l} \right)$$

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is the *boundary traction* on $\partial\Omega$ with $\nu = (\nu_1, \nu_2, \cdots, \nu_N)$ denoting the unit outer normal on $\partial\Omega$. In (1.1), $\rho = \rho_R + \sqrt{-1}\rho_I$ is a complex-valued function with $\rho_R > 0$ and $\rho_I \geq 0$. In order to simplify notations, we denote $\nabla \cdot (\mathcal{C}\nabla u)$ componentwisely by

$$\nabla \cdot (\mathcal{C}\nabla u)_i = \sum_{j,k,l=1}^{N} \frac{\partial}{\partial x_j}\left(C_{ijkl}(x)\frac{\partial u_k}{\partial x_l}\right) \text{ for } i = 1, 2, \cdots, N.$$

In addition, for any $v = (v_1, v_2, \cdots, v_N) \in H^1(\Omega)^N$, we can rewrite (1.1) by the variational formula

$$\mathcal{B}_{\mathcal{C}}(u, v) := \int_{\Omega} \left\{ \sum_{i.j.k.l=1}^{N} C_{ijkl}\frac{\partial u_k}{\partial x_l}\frac{\partial \overline{v_i}}{\partial x_j} - \kappa^2 \rho u_i \overline{v_i} \right\} dx = \int_{\partial\Omega} \phi \cdot \overline{v} dS, \tag{1.2}$$

where $\overline{v_i}$ is the complex conjugate of v_i for $i = 1, 2, \cdots, N$. Recall that the strong convexity condition is given as: there exists $c_0 > 0$ such that for all $N \times N$ symmetric matrix $\varepsilon = (\varepsilon_{ij})_{i,j=1}^{N}$,

$$\sum_{i,j,k,l=1}^{N} C_{ijkl}\varepsilon_{ij}\varepsilon_{kl} \geq c_0 \sum_{i,j=1}^{N} |\varepsilon_{ij}|^2, \text{ for all } x \in \Omega. \tag{1.3}$$

Via the strong convexity condition (1.3) and the Gårding's inequality

$$\mathcal{B}_{\mathcal{C}}(u, u) \geq c_0 \sum_{i,j=1}^{N} \|\nabla u\|^2_{L^2(\Omega)^{N\times N}} - \kappa^2 \|\rho\|_{L^\infty(\Omega)} \|u\|^2_{L^2(\Omega)^N} \text{ for all } u \in H^1(\Omega)^N,$$

then there exists a unique weak solution to (1.2) for all frequency $\kappa \in \mathbb{R}_+$ except for a discrete set $\mathcal{D}$ with the accumulating point at infinity. By using the well-posed property of (1.1), we can define the boundary Neumann-to-Dirichlet (NtD) map as

$$\Lambda_{\mathcal{C},\rho} : H^{-1/2}(\partial\Omega)^N \to H^{1/2}(\partial\Omega)^N \text{ with } \Lambda_{\mathcal{C},\rho}\phi = u|_{\partial\Omega}, \tag{1.4}$$

where $u \in H^1(\Omega)^N$ is the unique solution of (1.1).

In this paper, for the elasticity system with residual stress, we consider the following boundary value problem (1.1) with the elasticity tensor $\mathcal{C}(x) = (C_{ijkl}(x))_{i,j,k,l=1}^{N}$ and

$$C_{ijkl}(x) = \lambda(x)\delta_{ij}\delta_{kl} + \mu(x)(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) + t_{jl}(x)\delta_{ik}, \tag{1.5}$$

where δ_{ij} is the Kronecker delta and $\lambda(x), \mu(x)$ are the Lamé parameters and $\mathcal{C}(x)$ satisfies the strong convexity condition (1.3). The second-rank tensor $T(x) = (t_{jl}(x))_{j,l=1}^{N}$ is the residual stress and satisfies the following conditions:

1. Symmetry:

$$t_{jl}(x) = t_{lj}(x), \text{ for } j, l = 1, 2, \cdots, N, \text{ for all } x \in \Omega. \tag{1.6}$$

2. Divergence free:

$$\nabla \cdot T(x) = \sum_{l} \partial_{x_l} t_{jl}(x) = 0, \text{ for } j = 1, 2, \cdots, N, \text{ for all } x \in \Omega. \tag{1.7}$$

3. Vanishing at the boundary:

$$T(x)\cdot\nu = 0 \text{ or } \sum_{l=1}^{N} t_{jl}(x)\nu_l = 0 \text{ for all } x\in\partial\Omega \tag{1.8}$$

for $j=1,2,\cdots,N$, where $\nu=(\nu_1,\nu_2,\cdots,\nu_N)$ is a unit outer normal on $\partial\Omega$. In fact, (1.7) and (1.8) can be expressed weakly by

$$\int_\Omega T\cdot\nabla v dx = 0$$

for all $v\in H^1(\Omega)^N$.

Moreover, for $N=2,3$, we also assume that the Lamé moduli satisfy the strong convexity condition

$$\mu(x)\geq c_0>0 \text{ and } N\lambda(x)+2\mu(x)\geq c_0>0 \text{ for all } x\in\Omega. \tag{1.9}$$

For more details about the strong convexity property for elasticity systems, we refer readers to [38]. It is easy to see when (1.5) is the elastic four tensor, (1.1) is an *anisotropic* elasticity system. By the symmetric property (1.6), for the elastic tensor with residual stress (1.5), we have the major symmetric property without minor symmetry, which means

$$C_{ijkl}=C_{klij} \text{ but } C_{ijkl}\neq C_{jikl} \text{ for any } i,j,k,l=1,2,\cdots,N. \tag{1.10}$$

In the homogeneous medium, $\lambda(x)\equiv\lambda^{(0)},\mu(x)\equiv\mu^{(0)}$ are the Lamé constants satisfying (1.9) and $T=T(x)=(t_{jl}(x))\in W^{2,\infty}(\Omega)^{N\times N}$ is an arbitrary residual stress coefficient satisfying (1.6)-(1.8) for $i,j=1,2,\cdots,N$. Let $\mathcal{C}(x)\equiv\mathcal{C}^{(0)}$ with

$$C^{(0)}_{ijkl}=\lambda^{(0)}\delta_{ij}\delta_{kl}+\mu^{(0)}(\delta_{ik}\delta_{jl}+\delta_{jk}\delta_{il})+t_{jl}(x)\delta_{ik} \tag{1.11}$$

be a elasticity four tensor with residual stress satisfying (1.3) and $\rho\equiv 1$ be a constant density in a homogeneous medium. In this case, the elasticity system with residual stress (1.1) can be reduced to the following boundary value problem

$$\begin{cases}\mathcal{L}_0^R u_0+\kappa^2 u_0=0 & \text{in } \Omega,\\ \mathcal{N}_{\mathcal{C}^{(0)}}u_0=\phi & \text{on } \partial\Omega,\end{cases} \tag{1.12}$$

where $\mathcal{L}_0^R u:=\nabla\cdot(\mathcal{C}^{(0)}\nabla u_0)$ is the second order elasticity operator with residual stress and the boundary traction $\mathcal{N}_{\mathcal{C}^{(0)}}u_0=\nu\cdot(\mathcal{C}^{(0)}\nabla u_0)$. More precisely, by (1.6) and (1.8), componentwisely, $u_0=(u_{0,1},u_{0,2},\cdots,u_{0,N})$, it is easy to see

$$\sum_{j,k,l=1}^{N}\nu_j\left(t_{jl}(x)\delta_{ik}\right)\frac{\partial u_{0,k}}{\partial x_l}=\sum_{j,k,l=1}^{N}\left(t_{lj}(x)\nu_j\right)\delta_{ik}\frac{\partial u_{0,k}}{\partial x_l}=0 \text{ for } x\in\partial\Omega,$$

for $i=1,2,\cdots,N$ and $N=2,3$, or equivalently, $\nu\cdot(T\nabla u_0)=0$ on $\partial\Omega$, so we can represent the boundary traction for the elasticity system with residual stress by

$$\mathcal{N}_{C^{(0)}}u_0=\begin{cases}2\mu^{(0)}\dfrac{\partial u_0}{\partial\nu}+\lambda^{(0)}\nu\nabla\cdot u_0+\mu^{(0)}\nu^T(\partial_2 u_{0,1}-\partial_1 u_{0,2}) & \text{when } N=2,\\ 2\mu^{(0)}\dfrac{\partial u_0}{\partial\nu}+\lambda^{(0)}\nu\nabla\cdot u_0+\mu^{(0)}\nu\times(\nabla\times u_0) & \text{when } N=3,\end{cases}$$

where ν is a unit outer normal on $\partial\Omega$ and $\nu^T = (-\nu_2, \nu_1) \perp \nu$ and $u_0 = (u_{0,1}, u_{0,2})$ when $N = 2$. In this paper, we can regard the medium $\left\{\Omega; \mathcal{C}^{(0)}, 1\right\}$ as the *free* reference space in our study about the invisible cloaking for the elasticity system with residual stress in this work.

Indeed, we have the following facts about the invariance of coordinates transformation for the elasticity system with residual stress. Let $\widetilde{x} = F(x) : \Omega \to \widetilde{\Omega}$ be an orientation-preserving, bi-Lipschitz mapping, then we have the following push-forward relations of $\mathcal{C}$ and ρ defined by

$$\begin{aligned}
\widetilde{\mathcal{C}} &:= F_*\mathcal{C} = \widetilde{C}_{iqkp}(\widetilde{x}) = \frac{1}{\det M}\left\{\sum_{j,l=1}^{N} C_{ijkl}\frac{\partial \widetilde{x_p}}{\partial x_l}\frac{\partial \widetilde{x_q}}{\partial x_j}\right\}\Bigg|_{x=F^{-1}(\widetilde{x})}, \\
\widetilde{\rho} &:= F_*\rho = \left(\frac{\rho}{\det M}\right)\Bigg|_{x=F^{-1}(\widetilde{x})},
\end{aligned}$$

where $M = \left(\dfrac{\partial \widetilde{x}_i}{\partial x_j}\right)_{i,j=1}^{N}$. To simplify the notation, we set $\widetilde{\nabla} = \nabla_{\widetilde{x}}$ and $\left\{\widetilde{\Omega}; \widetilde{\mathcal{C}}, \widetilde{\rho}\right\} = F_*\left\{\Omega; \mathcal{C}, \rho\right\}$. Besides, for any $N \times N$ symmetric matrix (ε_{ij}), we have

$$\sum_{i,q,k,p=1}^{N} \widetilde{C}_{iqkp}\varepsilon_{iq}\varepsilon_{kp} = \frac{1}{\det M}\sum_{i,j,k,l=1}^{N} C_{ijkl}\widetilde{\varepsilon}_{ij}\widetilde{\varepsilon}_{kl},$$

where

$$\widetilde{\varepsilon}_{ij} = \sum_{q=1}^{N}\frac{\partial \widetilde{x_q}}{\partial x_j}\varepsilon_{iq}, \text{ for } i, j = 1, 2, \cdots, N.$$

Using the strong convexity condition (1.3) and the bi-Lipschitz property of F, we obtain

$$\sum_{i,q,k,p=1}^{N} \widetilde{C}_{iqkp}\varepsilon_{iq}\varepsilon_{kp} \geq c_0 \sum_{i,j=1}^{N} |\widetilde{\varepsilon}_{ij}|^2 \geq \widetilde{c_0}\sum_{i,j=1}^{N} |\varepsilon_{ij}|^2 \tag{1.13}$$

for some $\widetilde{c_0} > 0$. Moreover, the straightforward calculation will give that $\widetilde{C}$ is major symmetric but not minor symmetric, i.e.,

$$\widetilde{C}_{iqkp} = \widetilde{C}_{kpiq}, \text{ but } \widetilde{C}_{iqkp} \neq \widetilde{C}_{qikp} \text{ for any } i, q, k, p = 1, 2, \cdots, N. \tag{1.14}$$

The relations (1.10) and (1.14) make the elastic tensor with residual stress is *invariant* under the change of coordinates. Therefore, the elasticity system with residual stress is invariant under the coordinate transformation. More explicitly, we have the following lemma, which was proved in [23].

Lemma 1.1. *[23] The following relation holds: $u \in H^1(\Omega)^N$ is a solution to*

$$\nabla \cdot (\mathcal{C}\nabla u) + \kappa^2\rho u = 0 \text{ in } \Omega, \tag{1.15}$$

if and only if $\widetilde{u} = u \circ F^{-1} \in H^1(\widetilde{\Omega})^N$ is a solution to

$$\widetilde{\nabla} \cdot (\widetilde{\mathcal{C}}\widetilde{\nabla}\widetilde{u}) + \kappa^2\widetilde{\rho}\widetilde{u} = 0 \text{ in } \widetilde{\Omega}. \tag{1.16}$$

Moreover, if $F =$ Identity on $\partial\Omega$, then

$$\Lambda_{\mathcal{C},\rho} = \Lambda_{\widetilde{\mathcal{C}},\widetilde{\rho}},$$

where $\Lambda_{\mathcal{C},\rho}$ *and* $\Lambda_{\widetilde{\mathcal{C}},\widetilde{\rho}}$ *are the NtD maps associated to (1.15) and (1.16), respectively.*

The study on the invisible cloaking has been attracted the most attention among theoretical and practical point of views. A region is cloaked if its contents together with the cloak are invisible to wave detection. A transformational cloaking using the invariance properties of the conductivity equation was discovered by Greenleaf, Lassas and Uhlmann [20, 21]. In [27, 36], the authors used the electromagnetic (EM) waves to make objects invisible. The basic idea of cloaking is using the invariance of a coordinate transformation for specific systems, such as conductivity, acoustic, electromagnetic, and elasticity systems. We refer readers to the survey articles [10, 18, 19, 35, 39] in physics and mathematics literature for the cloaking theory and their developments. The perfect cloaking can be obtained by the one-point-blowup construction, but it will induce the transformed medium to be singular. The singular structure arises a great challenge for mathematical analysis and practical purposes for the cloaking theory.

In order to handle the singular structure from the perfect cloaking constructions, Greenleaf, Kurylev, Lassas and Uhlmann [15] developed a truncation of singularities methods, which is called the *nearly cloaking* (or approximate cloaking) techniques. More specifically, this nearly cloaking method is via a special singular double coating to defeat the singular structure. In [16, 17, 37], the authors utilized this truncation of singularities methods to approach the nearly cloaking theory. On the other hand, there are many researchers have considered another methods, called the small-inclusion-blowup construction, which regularized the singular medium which was induced by the one-point-blowup construction for the perfect cloaking. The small-inclusion-blowup method was studied by many researchers: [5, 26] for the conductivity model, [1, 4, 11, 25, 28, 29, 30, 31] for the acoustic model, [6, 7, 8, 12] for the Maxwell model and [23] for the Lamé model. In [24], the authors also pointed out that the truncation of singularity construction and the small-inclusion construction are equivalent. Especially, we are interested about the small-inclusion-blowup construction to approach our nearly cloaking theory for the elasticity system with residual stress. In further, there are some physics and mathematics literature for the elastic cloaking theory, such as [9, 13, 14, 34, 35].

However, for the isotropic elastic waves, which is governed by the Lamé system, is not invariance under a coordinate transformation, i.e., the minor symmetric structure will break after a coordinate transformation. This phenomena means that any elastic four tensor with the major and minor symmetric properties, then the transformed elastic tensor will not be minor symmetric anymore. Fortunately, for the elastic tensor with residual stress possesses only the major symmetric property but the minor symmetry breaks, see (1.10), (1.14). These relations imply that the elasticity system with residual stress is invariant via a coordinate transformation and the invariant property makes the elastic waves with residual stress is more natural than the isotropic elastic waves in the physical sense. It is hard to build up the cloaking theory for the elasticity system with residual stress because there are no layer potential theories for such system. Instead, we will derive suitable global estimates for the solutions of the elasticity system with residual stress under appropriate boundary traction conditions.

The paper is organized as follows. In Section 2, we introduce the blowup constructions to achieve the perfect cloaking for our residual stress model. In Section 3, we explain how to avoid the singular structure of the elastic tensor with the residual stress by using the regularization technique, which is the small-inclusion-blowup method. Meanwhile, in order to avoid the non-uniqueness for the NtD map after a coordinate transformation, we state our main results of the nearly cloaking theory for the elasticity system with residual stress by demonstrating the *lossy layer* technique. In Section 4, we provide useful tools and global estimates for this second order anisotropic elasticity systems and complete the proof of our main theorem. Finally, in Appendix, we give a glimpse review of the layer potential theory for the Lamé system, which will be used in the proof of our main theorem.

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2 Perfect cloaking for the elasticity system with residual stress

In this section, we will construct the perfect cloaking for the elasticity system with residual stress by using the one-point-blowup approach to achieve our goal. For $N = 2, 3$, let $\Omega \subset \mathbb{R}^N$ and $D \Subset \Omega$ be bounded and connected C^∞-smooth domains. Moreover, we also assume that $\Omega \backslash \overline{D}$ is connected and D contains the origin in $\mathbb{R}^N$. For any $h > 0$, $D_h = \{hx : x \in D\}$ and let

$$\left\{D_{\frac{1}{2}}; \mathcal{C}^{(a)}, \rho^{(a)}\right\}$$

be the target medium with $D_{\frac{1}{2}}$ denoting the region which we want to cloak.

Definition 2.1. We say the medium $\{\Omega; \mathcal{C}, \rho\}$ to be *regular* if $\mathcal{C}$ satisfies the strong convexity condition (1.3) and the major symmetric condition (1.10).

In this section, we assume the medium $\left\{D_{\frac{1}{2}}; \mathcal{C}^{(a)}, \rho^{(a)}\right\}$ to be arbitrary but regular and consider

$$\left\{\Omega \backslash \overline{D_{\frac{1}{2}}}; \mathcal{C}^{(c)}, \rho^{(c)}\right\}$$

as a suitable layer of elastic medium, which is designed to be the cloaking medium. Let

$$\{\Omega; \mathcal{C}, \rho\} = \begin{cases} \left\{\Omega \backslash \overline{D_{\frac{1}{2}}}; \mathcal{C}^{(c)}, \rho^{(c)}\right\} & \text{in } \Omega \backslash \overline{D_{\frac{1}{2}}}, \\ \left\{D_{\frac{1}{2}}; \mathcal{C}^{(a)}, \rho^{(a)}\right\} & \text{in } D_{\frac{1}{2}}, \end{cases} \tag{2.1}$$

be the medium occupying Ω. We can define the associated NtD map

$$\Lambda_{\mathcal{C},\rho} : H^{-1/2}(\partial\Omega)^N \to H^{1/2}(\partial\Omega)^N \tag{2.2}$$

in the medium (2.1). In addition, the medium $D_{\frac{1}{2}}$ and $\Omega \backslash \overline{D_{\frac{1}{2}}}$ will be specified appropriately in the forthcoming discussions whenever it is necessary. If $-\kappa^2$ be not an eigenvalue of the elliptic operator $\mathcal{L}_0^R$ with the zero boundary traction condition, there exists a unique solution $u_0 \in H^1(\Omega)^N$ to

$$\begin{cases} \mathcal{L}_0^R u_0 + \kappa^2 u_0 = 0 & \text{in } \Omega, \\ \mathcal{N}_{\mathcal{C}^{(0)}} u_0 = \phi & \text{on } \partial\Omega, \end{cases}$$

and the corresponding NtD map $\Lambda_0 : H^{-1/2}(\partial\Omega)^N \to H^{1/2}(\partial\Omega)^N$ is well-defined on the free reference space $\{\Omega; \mathcal{C}^{(0)}, 1\}$ with $\Lambda_0\phi = u_0|_{\partial\Omega}$.

Definition 2.2. The medium $\left\{\Omega\backslash\overline{D_{\frac{1}{2}}}; \mathcal{C}^{(c)}, \rho^{(c)}\right\}$ is said to be *perfect (elastic) cloak* if $\Lambda_{\mathcal{C},\rho}(\phi) = \Lambda_0(\phi)$ for any $\phi \in H^{-1/2}(\partial\Omega)^N$.

On the basis of Definition 2.2, the cloaking layer $\left\{\Omega\backslash\overline{D_{\frac{1}{2}}}; \mathcal{C}^{(c)}, \rho^{(c)}\right\}$ makes itself and the target medium $\left\{D_{\frac{1}{2}}; \mathcal{C}^{(a)}, \rho^{(a)}\right\}$ "invisible" by the exterior elastic wave measurements. Now, we want to analyze the singular structure from the mathematical viewpoint. For $R > 0$, let $B_R = \{x : |x| < R\}$ be a ball centered at the origin and radius $R > 0$. In this section, we take $\Omega = B_2$ and $D_{\frac{1}{2}} = B_1$ to demonstrate the analysis for the singular structures of the perfect cloaking. Consider the singular transformation $F_0 : B_2\backslash\{0\} \to B_2\backslash\overline{B_1}$ by

$$F_0(x) = (1 + \frac{|x|}{2})\frac{x}{|x|}, \quad \text{for } x \in B_2\backslash\{0\}. \tag{2.3}$$

It is easy to see that F_0 blows up at 0 in the reference space to B_1 and F_0 maps $B_2\backslash\{0\}$ to $B_2\backslash\overline{B_1}$ with $F_0|_{\partial B_2}$ =Identity. Denoting $y := F_0(x)$, the transformed medium in $\{B_2\backslash\overline{B_1}; \mathcal{C}^{(c)}, \rho^{(c)}\}$ can be represented as

$$\mathcal{C}^{(c)}(y) = (F_0)_*(\mathcal{C}^{(0)}(x))|_{x=F_0^{-1}(y)} \text{ and } \rho^{(c)}(y) = (F_0)_*(1)(x)|_{x=F_0^{-1}(y)},$$

where $(F_0)_*$ is the push-forward of F_0. Now, we consider the boundary value problem (1.1) in $\Omega = B_2$ with

$$\{B_2; \mathcal{C}, \rho\} = \begin{cases} \{B_2\backslash\overline{B_1}; \mathcal{C}^{(0)}, \rho^{(0)}\} & \text{in } B_2\backslash\overline{B_1}, \\ \{B_1; \mathcal{C}^{(a)}, \rho^{(a)}\} & \text{in } B_1, \end{cases}$$

and the corresponding NtD map is $\Lambda_{\mathcal{C}.\rho}$, which was given by (2.2), then we have

Proposition 2.3. *(Perfect cloaking) We have*

$$\Lambda_0 = \Lambda_{\mathcal{C},\rho} \text{ on } \partial B_2.$$

Proof. Let $\widetilde{\Lambda_0}$ be the NtD map associated with the medium $\{B_2\backslash\{0\}; \mathcal{C}^{(0)}, 1\}$. By Lemma 1.1, we have

$$\widetilde{\Lambda_0} = \Lambda_{\mathcal{C},\rho} \text{ on } \partial B_2.$$

Note that the inhomogeneity of $\{\Omega\backslash\{0\}; \mathcal{C}^{(0)}, 1\}$ is only supported in a singular point $\{0\}$, then we have $\widetilde{\Lambda_0} = \Lambda_0$ and complete the proof. □

The above result for the perfect elastic cloaking is easy and intuitive. Note that the singular transformation F_0 will force the elastic tensor $\mathcal{C}^{(c)}$ not to satisfy the strong convexity condition (1.3). Indeed, Hu and Liu [23] gave explicit analysis for the singular structures of the transformed elastic parameters $\mathcal{C}^{(c)}$ and $\rho^{(c)}$, so we omit the arguments and refer readers to Section 3 in [23] for further and complete discussions.

On the other hand, from practical aspects, we can calculate the elastic tensor with residual stress under the radial case by using the polar coordinates (r,θ) in 2D and the spherical coordinates (r,θ,φ) in 3D. In order to calculate the transformed elastic parameter $\mathcal{C}^{(c)}$ (1.11) explicitly, for convenience, we simplify indexes by $1\to r$, $2\to\theta$, $3\to\varphi$ and use the *Voigt* notation for tensor indices, which are

$$\begin{aligned}&11\to 1,\ 22\to 2\ 33\to 3,\\&23,32\to 4,\ 13,31\to 5,\ 12,21\to 6,\end{aligned}$$

then the residual stress tensor (1.11) can be written as

$$C_{ab}=\Lambda_{ab}+\mathcal{T}_{ab} \text{ for } 1\le a,b\le N \text{ and } N=2,3. \tag{2.4}$$

More explicitly, when $N=2$,

$$\Lambda_{ab}=\begin{pmatrix}\lambda^{(0)}+2\mu^{(0)} & \lambda^{(0)} & 0\\ \lambda^{(0)} & \lambda^{(0)}+2\mu^{(0)} & 0\\ 0 & 0 & \mu^{(0)}\end{pmatrix} \tag{2.5}$$

and

$$\mathcal{T}_{ab}=\begin{pmatrix}t_{11} & 0 & t_{12}\\ 0 & t_{22} & t_{12}\\ t_{12} & t_{12} & t_{11}+t_{22}\end{pmatrix}(x). \tag{2.6}$$

When $N=3$,

$$\Lambda_{ab}=\begin{pmatrix}\lambda^{(0)}+2\mu^{(0)} & \lambda^{(0)} & \lambda^{(0)} & 0 & 0 & 0\\ \lambda^{(0)} & \lambda^{(0)}+2\mu^{(0)} & \lambda^{(0)} & 0 & 0 & 0\\ \lambda^{(0)} & \lambda^{(0)} & \lambda^{(0)}+2\mu^{(0)} & 0 & 0 & 0\\ 0 & 0 & 0 & \mu^{(0)} & 0 & 0\\ 0 & 0 & 0 & 0 & \mu^{(0)} & 0\\ 0 & 0 & 0 & 0 & 0 & \mu^{(0)}\end{pmatrix}$$

and

$$\mathcal{T}_{ab}=\begin{pmatrix}t_{11} & 0 & 0 & 0 & t_{13} & t_{12}\\ 0 & t_{22} & 0 & t_{23} & 0 & t_{12}\\ 0 & 0 & t_{33} & t_{23} & t_{13} & 0\\ 0 & t_{23} & t_{23} & t_{22}+t_{33} & t_{12} & t_{13}\\ t_{13} & 0 & t_{13} & t_{12} & t_{11}+t_{33} & t_{23}\\ t_{12} & t_{12} & 0 & t_{13} & t_{23} & t_{11}+t_{22}\end{pmatrix}(x).$$

Furthermore, let DF_0 be the Jacobian matrix of F_0, for the radial case $r:=|y|$ with $y=F_0(x)$, then we have

$$DF_0=\begin{pmatrix}\frac{1}{2} & 0\\ 0 & \frac{r}{2(r-1)}I_{N-1}\end{pmatrix}, \tag{2.7}$$

where I_{N-1} is an $(N-1)\times(N-1)$ identity matrix for $N=2,3$ and for any $r\in(1,2)$. Now, we can use the following transformation formula

$$\mathcal{C}^{(c)}(y)=\left.\frac{DF_0(x)\boxtimes\mathcal{C}^{(0)}(x)\boxtimes DF_0^t(x)}{\det DF_0(x)}\right|_{x=F_0^{-1}(y)}\quad (t \text{ for transpose}),\tag{2.8}$$

where $\boxtimes$ is the multiplication between a matrix and fourth rank tensor and

$$\det DF_0=\begin{cases}\frac{r}{4(r-1)} & \text{for } N=2,\\ \frac{r^2}{8(r-1)^2} & \text{for } N=3.\end{cases}$$

More explicitly, for $\mathcal{C}^{(c)}(y)=(C^{(c)}_{ijkl}(y))_{i,j,k,l=1}^N$, (2.8) is equivalent to

$$\begin{pmatrix} C^{(c)}_{i1k1} & C^{(c)}_{i1k2} & \cdots & C^{(c)}_{i1kN}\\ C^{(c)}_{i2k1} & C^{(c)}_{i2k2} & \cdots & C^{(c)}_{i2kN}\\ \vdots & \vdots & \cdots & \vdots\\ C^{(c)}_{iNk1} & C^{(c)}_{iNk2} & \cdots & C^{(c)}_{iNkN}\end{pmatrix}\tag{2.9}$$

$$=\frac{1}{\det DF_0}DF_0\begin{pmatrix} C^{(0)}_{i1k1} & C^{(0)}_{i1k2} & \cdots & C^{(0)}_{i1kN}\\ C^{(0)}_{i2k1} & C^{(0)}_{i2k2} & \cdots & C^{(0)}_{i2kN}\\ \vdots & \vdots & \cdots & \vdots\\ C^{(0)}_{iNk1} & C^{(0)}_{iNk2} & \cdots & C^{(0)}_{iNkN}\end{pmatrix}DF_0^t$$

for $i,k=1,2,\cdots,N$. Making use of the polar coordinate, (2.5), (2.6), (2.7) and (2.9), we can represent the transformed tensor $\mathcal{C}^{(c)}(y)=\mathcal{C}^{(c)}(r)$ for $N=2$ in the following by straightforward calculation, there are twelve nontrivial entries for the residual stress systems:

$$C^{(c)}_{rrrr}=(\lambda^{(0)}+2\mu^{(0)}+t_{11}(F_0^{-1}(y)))\frac{r-1}{r},\ C^{(c)}_{\theta r\theta r}=(\mu^{(0)}+t_{11}(F_0^{-1}(y)))\frac{r}{r-1},$$
$$C^{(c)}_{rrr\theta}=C^{(c)}_{r\theta rr}=\frac{r-1}{r}t_{12}(F_0^{-1}(y)),\ C^{(c)}_{\theta r\theta\theta}=C^{(c)}_{\theta\theta\theta r}=\frac{r}{r-1}t_{12}(F_0^{-1}(y)),$$
$$C^{(c)}_{rr\theta\theta}=C^{(0)}_{\theta\theta rr}=\lambda^{(0)},\ C^{(c)}_{r\theta\theta r}=C^{(c)}_{\theta rr\theta}=\mu^{(0)},$$
$$C^{(c)}_{r\theta r\theta}=(\mu^{(0)}+t_{22}(F_0^{-1}(y)))\frac{r-1}{r},\ C^{(c)}_{\theta\theta\theta\theta}=(\lambda^{(0)}+2\mu^{(0)}+t_{22}(F_0^{-1}(y)))\frac{r}{r-1}.$$

Note that the strong convexity condition (1.3) will imply the matrix (2.4) to be positive definite (see [38]). Note that for $N=2$, by (1.3) and simple calculation, we can derive

$$\begin{cases}\mu^{(0)}+t_{11}(F_0^{-1}(y))>0\\ \lambda^{(0)}+2\mu^{(0)}+t_{22}(F_0^{-1}(y))>0\end{cases}\quad \text{for any } y\in B_2\backslash\overline{B_1}\tag{2.10}$$

holding. Therefore, (2.10) implies that $C^{(c)}_{\theta r\theta r},C^{(c)}_{\theta\theta\theta\theta}$ have singularities on ∂B_1. On the other hand, by $T\in W^{2,\infty}(B_2)$, it is easy to see that $C^{(c)}_{rrrr},C^{(c)}_{r\theta r\theta}$ vanish on ∂B_1 and. Similar calculation will be valid for the transformed tensor $\mathcal{C}^{(c)}$ for $N=3$ under the spherical coordinates, so we skip details here. For more discussions of the transformed elastic tensors, we refer readers to [9, 14, 23].

Based on the blowup construction, we can obtain the perfect cloaking for the elasticity system with residual stress. However, the elastic materials possess singular structure, so the one-point-blowup construction is not useful for the mathematical analysis and practical applications.

3 Nearly cloaking construction and main result

The perfect cloaking for the elasticity system with residual stress produces singular parameters. In order to avoid singular structures, we introduce the nearly cloaking in this section via the small-inclusion-blowup construction and the *lossy layer* techniques as follows.

3.1 Small-inclusion-blowup and non-uniqueness for NtD maps

In the beginning of this section, we demonstrate our nearly-cloaking algorithm by considering the radial case. For small $h>0$, let $\widehat{F_h}:\overline{B_2}\backslash B_h \to \overline{B_2}\backslash B_1$ and $\widehat{F_h}|_{\partial B_2}$ =Identity be a transformation given by

$$\widehat{F_h}(x) := \left(\frac{2-2h}{2-h}+\frac{|x|}{2-h}\right)\frac{x}{|x|}.$$

Consider the following transformation

$$F_h(x) := \begin{cases} \widehat{F_h}(x) & \text{for } h \le |x| \le 2, \\ \frac{x}{h} & \text{for } |x| < h. \end{cases} \tag{3.1}$$

It is obvious that and $F_h : B_2 \to B_2$ is a bi-Lipschitz, orientation-preserving and $F_h|_{\partial B_2}$ =Identity. By using the same small-inclusion-blowup construction as in [23], we have the following form:

$$\{B_2;\mathcal{C},\rho\} = \begin{cases} \left\{B_2\backslash\overline{B_1};\mathcal{C}_h^{(c)},\rho_h^{(c)}\right\} & \text{in } B_2\backslash\overline{B_1}, \\ \left\{B_1;\mathcal{C}^{(a)},\rho^{(a)}\right\} & \text{in } B_1, \end{cases} \tag{3.2}$$

with the cloaking medium given by

$$\mathcal{C}_h^{(c)} := (F_h)_*(\mathcal{C}^{(0)})(x)|_{x=F_h^{-1}(y)},\ \rho_h^{(c)} := (F_h)_*(1)(x)|_{x=F_h^{-1}(y)},\ y\in B_2\backslash\overline{B_1}.$$

Let Λ_h be the NtD map associated with the configuration (3.2). It is easy to see that $F_h \to F_0$ as $h\to 0$, where F_0 is the singular transformation given by (2.3), so we expect $\Lambda_h \to \Lambda_0$ as $h \to 0$. However, in the medium (3.2), NtD map Λ_h may not be well-defined and we will explain more in the following. For $h>0$ small, we write

$$\left\{B_2;\widetilde{\mathcal{C}},\widetilde{\rho}\right\} := (F_h^{-1})_*\left\{B_2;\mathcal{C},\rho\right\} = \begin{cases} \left\{B_2\backslash\overline{B_h};\mathcal{C}^{(0)},1\right\} & \text{in } B_2\backslash\overline{B_h}, \\ \left\{B_h;\widetilde{\mathcal{C}}^{(a)},\widetilde{\rho}^{(a)}\right\} & \text{in } B_h, \end{cases}$$

where $\left\{B_h;\widetilde{\mathcal{C}}^{(a)},\widetilde{\rho}^{(a)}\right\} = (F_h^{-1})_*\left\{B_1;\mathcal{C}^{(a)},\rho^{(a)}\right\}$.

Since the target medium $\left\{B_1;\mathcal{C}^{(a)},\rho^{(a)}\right\}$ is arbitrary but regular, by (1.13), the small inclusion $\left\{B_h;\widetilde{\mathcal{C}}^{(a)},\widetilde{\rho}^{(a)}\right\}$ is arbitrary but regular, which means we avoid the singular structure by considering the transformation F_h. By Lemma 1.1 again, $\widetilde{\Lambda_h}=\Lambda_h$, where $\widetilde{\Lambda_h}$ is the transformed NtD map associated to the medium $\left\{B_2;\widetilde{\mathcal{C}},\widetilde{\rho}\right\}$ and $\widetilde{\Lambda_h}$ may not be well-defined.

Note that the nearly cloaking construction (3.2) via the transformation (3.1) might fail because of the corresponding NtD map $\widetilde{\Lambda_h}$ may not be well-defined. More specifically, for $x \in \mathbb{R}^N$, $N = 2, 3$ and for any $0 < r_0 < r_1$, we can consider the following elasticity system with residual stress

$$\begin{cases} \nabla \cdot (\mathcal{C}^{(0)} \nabla u_1) + \kappa^2 \rho_1 u_1 = 0 & \text{in } r_0 < |x| < r_1, \\ \nabla \cdot (\mathcal{C}^{(0)} \nabla u_0) + \kappa^2 \rho_0 u_0 = 0 & \text{in } |x| < r_0, \\ \mathcal{N}_{\mathcal{C}^{(0)}} u_1 = 0 & \text{on } |x| = r_1, \\ u_0 = u_1 \text{ and } \mathcal{N}_{\mathcal{C}^{(0)}} u_0 = \mathcal{N}_{\mathcal{C}^{(0)}} u_1 & \text{on } |x| = r_0, \end{cases} \tag{3.3}$$

where $\mathcal{C}^{(0)}$ is the residual stress tensor defined by (1.11), $\kappa > 0$ is a fixed frequency. By varying two positive constants ρ_0, ρ_1, then there are two cases might happen:

1. *Non-resonance effect*: For some constants $\rho_0, \rho_1 > 0$, there exists a unique solution $(u_0, u_1) = (0, 0)$ to (3.3), then the corresponding NtD map on $|x| = r_1$ is well-defined (since $(u_0, u_1) = (0, 0)$ is a trivial solution to (3.3)).
2. *Resonance effect*: For some constants $\rho_0, \rho_1 > 0$, there exists a nontrivial solution $(u_0, u_1) \neq (0, 0)$ to (3.3), which implies the corresponding NtD map on $|x| = r_1$ is not well-defined.

The resonance effect will make the corresponding NtD map be not well-defined, which causes the nearly cloaking construction to fail under this situation. However, no matter what cases (resonance or non-resonance effects) occurs, we can make use of the *lossy layer* method to guarantee the corresponding NtD map on the boundary to be well-defined. Thus, we can make the nearly cloaking for the elasticity system with residual stress can be constructed successfully. Therefore, the lossy layer technique is a necessary method to make our near-cloaking construction work.

3.2 Lossy layer techniques

In order to overcome the resonance effects, we introduce a certain damping mechanism, which means that we build a lossy layer between the cloaking region and the cloaked area. Recall that we have assumed $D \Subset \Omega$ be C^∞-smooth domains in $\mathbb{R}^N$ with D containing the origin and $\Omega \backslash \overline{D}$ is connected for $N = 2, 3$. Let $h > 0$ be small, we consider the orientation-preserving and bi-Lipschitz map F_h with

$$\widehat{F_h} : \overline{\Omega} \backslash D_h \to \overline{\Omega} \backslash D \text{ and } \widehat{F_h}|_{\partial\Omega} = \text{Identity},$$

where $D_h = \{hx : x \in D\}$ and define the mapping F by

$$F_h(x) = \begin{cases} \widehat{F_h}(x) & \text{for } x \in \overline{\Omega} \backslash D_h, \\ \frac{x}{h} & \text{for } x \in D_h. \end{cases}$$

It is easy to see that $F_h : \Omega \to \Omega$ is an orientation-preserving and bi-Lipschitz map with $F_h|_{\partial\Omega}$ =Identity.

As in Section 2, $\left\{D_{\frac{1}{2}}; \mathcal{C}^{(a)}, \rho^{(a)}\right\}$ is the medium that we want to cloak. Consider the elastic medium with residual stress as follows:

$$\{\Omega; \mathcal{C}, \rho\} = \begin{cases} \left\{\Omega\backslash\overline{D_{\frac{1}{2}}}; \mathcal{C}^{(c)}, \rho^{(c)}\right\} & \text{in } \Omega\backslash\overline{D_{\frac{1}{2}}}, \\ \left\{D_{\frac{1}{2}}; \mathcal{C}^{(a)}, \rho^{(a)}\right\} & \text{in } D_{\frac{1}{2}}. \end{cases} \tag{3.4}$$

where

$$\{\Omega\backslash\overline{D_{\frac{1}{2}}}; \mathcal{C}^{(c)}, \rho^{(c)}\} = \begin{cases} \left\{\Omega\backslash\overline{D}; \mathcal{C}^{(1)}, \rho^{(1)}\right\} & \text{in } \Omega\backslash\overline{D}, \\ \left\{D\backslash\overline{D_{\frac{1}{2}}}; \mathcal{C}^{(2)}, \rho^{(2)}\right\} & \text{in } D\backslash\overline{D_{\frac{1}{2}}}, \end{cases} \tag{3.5}$$

and

$$\left\{\Omega\backslash\overline{D}; \mathcal{C}^{(1)}, \rho^{(1)}\right\} = (F_h)_* \left\{\Omega\backslash\overline{D_h}; \mathcal{C}^{(0)}, 1\right\}, \tag{3.6}$$

$$\left\{D\backslash\overline{D_{\frac{1}{2}}}; \mathcal{C}^{(2)}, \rho^{(2)}\right\} = (F_h)_* \left\{D_h\backslash\overline{D_{\frac{h}{2}}}; \widetilde{\mathcal{C}}^{(2)}, \widetilde{\rho}^{(2)}\right\}. \tag{3.7}$$

Note that the elastic medium in $D_h\backslash\overline{D_{\frac{h}{2}}}$ is designed by

$$\left\{D_h\backslash\overline{D_{\frac{h}{2}}}; \widetilde{\mathcal{C}}^{(2)}, \widetilde{\rho}^{(2)}\right\}, \ \widetilde{\mathcal{C}}^{(2)} = \gamma h^{2+\delta}\widetilde{\mathcal{C}}^{(0)} \text{ and } \widetilde{\rho}^{(2)} = \alpha + i\beta, \tag{3.8}$$

where α, β, γ and δ are given positive constants and $\left\{D_h\backslash\overline{D_{\frac{h}{2}}}; \widetilde{\mathcal{C}}^{(2)}, \widetilde{\rho}^{(2)}\right\}$ is our critical lossy layer. Note the number $\beta > 0$ is the *damping* parameter of the medium, which guarantees the well-defined property for the NtD map with respect the medium $\{\Omega; \mathcal{C}, \rho\}$ defined by (3.4)-(3.8). Note that the selection of parameters in (3.8) plays the crucial role for the construction of the lossy layer.

Now we consider the boundary value problem

$$\begin{cases} \nabla\cdot(\mathcal{C}\nabla u) + \kappa^2\rho(x)u = 0 & \text{in } \Omega, \\ \mathcal{N}_{\mathcal{C}}u = \phi \in H^{-\frac{1}{2}}(\partial\Omega)^3 & \text{on } \partial\Omega, \end{cases} \tag{3.9}$$

where $\{\Omega; \mathcal{C}, \rho\}$ was introduced by (3.4)-(3.8).

We can state our main result for the nearly cloaking of the elasticity system with residual stress.

Theorem 3.1. *Suppose $-\kappa^2$ is not an eigenvalue of the elliptic operator $\mathcal{L}_0^R$ on $\partial\Omega$ with the zero boundary traction condition $\mathcal{N}_{\mathcal{C}^{(0}}u = 0$ on $\partial\Omega$. Let $\Lambda_{\mathcal{C},\rho}$ and Λ_0 the NtD maps of (3.9) and (1.12), respectively. Then there exists $h^{(0)} > 0$ such that for any $h < h^{(0)}$,*

$$\|\Lambda_{\mathcal{C},\rho} - \Lambda_0\|_{\mathscr{L}(H^{-1/2}(\partial\Omega)^N, H^{1/2}(\partial\Omega)^N)} \le Ch,$$

where $\|\cdot\|_{\mathscr{L}(H^{-1/2}(\partial\Omega)^N, H^{1/2}(\partial\Omega)^N)}$ denotes the operator norm from $H^{-1/2}(\partial\Omega)^N$ to $H^{1/2}(\partial\Omega)^N$ and $C > 0$ is a constant independent of $h, C^{(a)}, \rho^{(a)}$ and δ.

In order to prove Theorem 3.1, notice that by using Lemma 1.1, then we have

$$\Lambda_{\mathcal{C},\rho} = \Lambda_{\widetilde{C},\widetilde{\rho}}, \tag{3.10}$$

where $\widetilde{C} = (F_h^{-1})_*\mathcal{C}$, $\widetilde{\rho} = (F_h^{-1})_*\rho$ and

$$\{\Omega; \widetilde{\mathcal{C}}, \widetilde{\rho}\} = \begin{cases} \mathcal{C}^{(0)}, 1 & \text{in } \Omega\backslash\overline{D_h}, \\ \widetilde{\mathcal{C}}^{(2)}, \widetilde{\rho}^{(2)} & \text{in } D_h\backslash\overline{D_{\frac{h}{2}}}, \\ \widetilde{\mathcal{C}}^{(a)}, \widetilde{\rho}^{(a)} & \text{in } D_{\frac{h}{2}}, \end{cases}$$

with $\widetilde{\mathcal{C}}^{(a)} = (F_h^{-1})_*\mathcal{C}^{(a)}$, $\widetilde{\rho}^{(a)} = (F_h^{-1})_*\rho^{(a)}$ and $\widetilde{\mathcal{C}}^{(2)}, \widetilde{\rho}^{(2)}$ were introduced in (3.8).

Via Lemma 1.2, if $u \in H^1(\Omega)^N$ solves the following elasticity system with residual stress

$$\begin{cases} \nabla \cdot (\mathcal{C}\nabla u) + \kappa^2 \rho u = 0 & \text{in } \Omega, \\ \mathcal{N}_{\mathcal{C}} u = \phi & \text{on } \partial\Omega, \end{cases}$$

then $\widetilde{u} := u \circ F_h^{-1}$ is the solution of

$$\begin{cases} \widetilde{\nabla} \cdot (\widetilde{\mathcal{C}}\widetilde{\nabla}\widetilde{u}) + \kappa^2 \widetilde{\rho}\widetilde{u} = 0 & \text{in } \Omega, \\ \mathcal{N}_{\widetilde{\mathcal{C}}}\widetilde{u} = \phi & \text{on } \partial\Omega. \end{cases} \tag{3.11}$$

Now, let u_0 be a solution of (1.12), by (3.10), then Theorem 3.1 holds by proving the following theorem.

Theorem 3.2. *Suppose $-\kappa^2$ is not an eigenvalue of the elliptic operator $\mathcal{L}_0^R$ on $\partial\Omega$ with the zero boundary traction condition $\mathcal{N}_{\mathcal{C}^{(0)}} u = 0$ on $\partial\Omega$. Then there exists a constant $h_0 > 0$ such that for any $h < h_0$,*

$$\|\widetilde{u} - u_0\|_{H^{1/2}(\partial\Omega)^N} \leq Ch\|\phi\|_{H^{-1/2}(\partial\Omega)^N}, \tag{3.12}$$

where $\widetilde{u}$ and u_0 are solutions of (3.11) and (1.12), respectively and $C > 0$ is a constant independent of $h, \psi, \widetilde{C}, \widetilde{\rho}$ and δ.

4 Elliptic estimates and proof of Theorem 3.2

Before proving Theorem 3.2, we need following estimates, which were introduced in [23]. We will show that the following lemmas holding for the residual stress case. First, we offer an energy for $\widetilde{u}$ in $D_h\backslash\overline{D_{\frac{h}{2}}}$ via a variational method.

Lemma 4.1. *[23] Let $\widetilde{u}$ and u_0 be solutions of (3.11) and (1.12), respectively. Then there exists a constant $C > 0$ depending only on Ω such that*

$$\beta\kappa^2\|\widetilde{u}\|^2_{L^2(D_h\backslash\overline{D_{\frac{h}{2}}})} \leq C\|\phi\|_{H^{-1/2}(\partial\Omega)^N}\|\widetilde{u} - u_0\|_{H^{1/2}(\partial\Omega)^N}.$$

Proof. Note that the residual stress T satisfies (1.7) and (1.8), then $\widetilde{T}$ also satisfies (1.7) and (1.8) after transformation by bi-Lipschitz maps. Multiply $\overline{\widetilde{u}}$ (the complex conjugate of $\widetilde{u}$) to (3.11) and integrate by parts, then

$$-\int_\Omega (\widetilde{\mathcal{C}}\nabla\widetilde{u}) : (\nabla\overline{\widetilde{u}})dx + \kappa^2 \int_\Omega \widetilde{\rho}|\widetilde{u}|^2 dx = -\int_{\partial\Omega} \phi \cdot \overline{\widetilde{u}} dS,$$

where the notation : denotes the standard Frobenius inner product. The remaining proof is the same as Lemma 5.1 in [23] due to (1.8), so we omit here. $\square$

We define the notations

$$\mathcal{N}^{\pm}_{\mathcal{C}^{(0)}}\widetilde{u}(x)=\lim_{\eta\to 0}\mathcal{N}_{\mathcal{C}^{(0)}}\widetilde{u}(x\pm\eta\nu) \text{ for } x\in\partial D_h,$$

where ν is a unit outer normal of ∂D_h. For simplicity, we denote $\Phi^{\pm}(x):=\mathcal{N}^{\pm}_{\mathcal{C}^{(0)}}\widetilde{u}(hx)$ for $x\in\partial D$.

Lemma 4.2. *[23] Let $\widetilde{u}$ and u_0 be solutions of (3.11) and (1.12), respectively. Then there exist constant $C>0$ depending on D,Ω but independent of h,ϕ such that*

$$\begin{aligned}\|\Phi^+\|^2_{H^{-3/2}(\partial D)^N} &\le C\frac{(\gamma+\sqrt{\alpha^2+\beta^2}h^{-\delta}\omega^2)^2}{\beta\gamma^2\omega^2}h^{-2-N}\|\widetilde{u}-u_0\|_{H^{1/2}(\partial\Omega)^N}\\ &\quad\times\|\phi\|_{H^{-1/2}(\partial\Omega)^N}, \qquad (4.1)\\ \|\Phi^-\|^2_{H^{-3/2}(\partial D)^N} &\le C\frac{(\gamma+\sqrt{\alpha^2+\beta^2}h^{-\delta}\omega^2)^2}{\beta\omega^2}h^{2(1+\delta)-N}\|\widetilde{u}-u_0\|_{H^{1/2}(\partial\Omega)^N}\\ &\quad\times\|\phi\|_{H^{-1/2}(\partial\Omega)^N}. \qquad (4.2)\end{aligned}$$

Proof. Using (1.7) and (1.8) of the residual stress $T=(t_{jl})^N_{j,l=1}$ again, the proof can be reduced to the isotropic elasticity case, which was proved in [23]. We refer readers to [23] for the detailed proof. □

To our best knowledge, there are no layer potential theories for the elasticity system with residual stress. Therefore, we want to give appropriate elliptic estimates for the residual stress system by comparing with the Lamé system as follows. Let $\mathscr{C}=(\mathscr{C}_{ijkl})$ and $\mathscr{C}^{(0)}=(\mathscr{C}^{(0)}_{ijkl})$ with

$$\begin{aligned}\mathscr{C}_{ijkl}(x) &:= \lambda(x)\delta_{ij}\delta_{kl}+\mu(x)(\delta_{ik}\delta_{jl}+\delta_{il}\delta_{jk}),\\ \mathscr{C}^{(0)}_{ijkl} &:= \lambda^{(0)}\delta_{ij}\delta_{kl}+\mu^{(0)}(\delta_{ik}\delta_{jl}+\delta_{il}\delta_{jk}),\end{aligned}$$

be isotropic elastic four tensors, where $\lambda(x),\mu(x)$ are Lamé parameters and $\lambda^{(0)},\mu^{(0)}$ are Lamé constants which were introduced in (1.5) and (1.11), respectively. Let Δ be a C^2 bounded domain in $\mathbb{R}^N$ and w_0 be a solution of the following Navier's equation (or homogeneous Lamé system)

$$\begin{cases}\mathcal{L}_0(w_0)+\eta^2 w_0=0 & \text{in } \Delta,\\ \mathcal{N}_{\mathscr{C}^{(0)}}w_0=\phi & \text{on } \partial\Delta,\end{cases} \qquad (4.3)$$

where $\mathcal{L}_0(w_0):=\nabla\cdot(\mathscr{C}^{(0)}\nabla w_0)$ and the boundary traction is defined by

$$\mathcal{N}_{\mathscr{C}^{(0)}}w_0=\begin{cases}2\mu^{(0)}\frac{\partial w_0}{\partial\nu}+\lambda^{(0)}\nu\nabla\cdot u+\mu\nu^T(\partial_2 w_{0,1}-\partial_1 w_{0,2}) & \text{for } N=2,\\ 2\mu^{(0)}\frac{\partial w_0}{\partial\nu}+\lambda^{(0)}\nu\nabla\cdot w_0+\mu^{(0)}\nu\times(\nabla\times w_0) & \text{for } N=3,\end{cases}$$

where ν is a unit outer normal on $\partial\Delta$, $w_0=(w_{0,1},w_{0,2})$ and $\nu^T=(-\nu_2,\nu_1)\perp\nu$ when $N=2$. Suppose $-\eta^2$ is not an eigenvalue of the elliptic operator $\mathcal{L}_0$ with the zero boundary traction condition on $\partial\Delta$, by the strong convexity (1.9), which are

$$\mu^{(0)}\ge c_0>0 \text{ and } N\lambda^{(0)}+2\mu^{(0)}\ge c_0>0,$$

then we know that (4.3) is well-posed. Moreover, the relation

$$\mathcal{N}_{\mathcal{C}^{(0)}} = \mathcal{N}_{\mathscr{C}^{(0)}} \text{ on } \partial\Delta$$

is guaranteed by (1.8). We utilize the following perturbation arguments to construct the near cloaking theory for the elasticity with residual stress.

Now, let $U_0 := u_0 - w_0$ be a reflected solution, where u_0 and w_0 are solutions of (1.12) and (4.3), respectively. It is easy to see that U_0 satisfies the following zero boundary traction problem

$$\begin{cases} \mathcal{L}_0^R U_0 + \kappa^2 U_0 = (\eta^2 - \kappa^2) w_0 - R w_0 & \text{in } \Delta, \\ \mathcal{N}_{\mathcal{C}^{(0)}} U_0 = 0 & \text{on } \partial\Delta, \end{cases} \tag{4.4}$$

where

$$R w_0 := \nabla \cdot (T(x) \nabla w_0) \tag{4.5}$$

and $\mathcal{L}_0^R$ is the second order elliptic operator appeared in (1.12).

The main estimate of this section is the following one.

Proposition 4.3. *(Key estimate) Suppose that Δ is a C^2 bounded domain in $\mathbb{R}^N$ for $N = 2, 3$. Let $U_0 \in H^1(\Delta)^N$ be solutions of (4.4), then there exist constants $C > 0$ independent of U_0 and w_0 such that*

$$\|U_0\|_{H^1(\Delta)^N} \le C \|w_0\|_{H^1(\Delta)^N}. \tag{4.6}$$

Proof. Let $\widehat{U_0}$ be a solution of

$$\begin{cases} \mathcal{L}_0^R \widehat{U_0} = (\eta^2 - \kappa^2) w_0 - R w_0 & \text{in } \Delta, \\ \mathcal{N}_{\mathcal{C}^{(0)}} \widehat{U_0} = 0 & \text{on } \partial\Delta. \end{cases} \tag{4.7}$$

Note that if $\widehat{U}_0$ is a solution of (4.7), then $\widehat{U_0} - \fint_\Delta \widehat{U_0}$ is also a solution of (4.7), where $\fint_\Delta \widehat{U_0} = \frac{1}{|\Delta|} \int_\Delta \widehat{U_0}$. Without loss of generality, we may assume $\int_\Delta \widehat{U_0} dx = 0$. Hence, the Poincaré's inequality will be valid for $\widehat{U_0}$, which means that there exists a constant $C > 0$ independent of $\widehat{U_0}$ such that

$$\|\widehat{U_0}\|_{L^2(\Delta)^N} \le C \|\nabla \widehat{U_0}\|_{L^2(\Delta)^N}. \tag{4.8}$$

Multiply $\overline{\widehat{U_0}}$ (the complex conjugate of $\widehat{U_0}$) on both sides of (4.7) and (1.3), then we can get

$$\|\nabla \widehat{U_0}\|_{L^2(\Delta)^N} \le \left| \int_\Delta \left\{ (R w_0) \cdot \overline{\widehat{U_0}} + (\lambda^2 - \kappa^2) w_0 \cdot \overline{\widehat{U_0}} \right\} dx \right|. \tag{4.9}$$

Use the integration by parts, (1.8) and $T(x) \in W^{2,\infty}(\Omega)$, then the Young's inequality yields that for any $\epsilon > 0$,

$$\left| \int_\Delta (R w_0) \cdot \overline{\widehat{U_0}} dx \right| \le \epsilon \|\nabla \widehat{U_0}\|_{L^2(\Delta)^N} + C(\epsilon) \|\nabla w_0\|_{L^2(\Delta)^N}, \tag{4.10}$$

$$\left| \int_\Delta w_0 \cdot \overline{\widehat{U_0}} dx \right| \le \epsilon \|\widehat{U_0}\|_{L^2(\Delta)^N} + C(\epsilon) \|w_0\|_{L^2(\Delta)^N}. \tag{4.11}$$

Via (4.9), (4.10) and (4.11), we obtain

$$\|\nabla\widehat{U_0}\|_{L^2(\Delta)^N} \le C\|w_0\|_{H^1(\Delta)^N}, \tag{4.12}$$

where $C > 0$ is independent of $\widehat{U_0}$ and w_0. Combine (4.12) and (4.8), we have

$$\|\widehat{U_0}\|_{H^1(\Delta)^N} \le C\|w_0\|_{H^1(\Delta)^N}, \tag{4.13}$$

where $C > 0$ is independent of $\widehat{U_0}$ and w_0.

By setting $\mathcal{U}_0 := U_0 - \widehat{U_0}$, we have $U_0 = \widehat{U_0} + \mathcal{U}_0$ and

$$\|U_0\|_{H^1(\Delta)^N} \le C(\|\widehat{U_0}\|_{H^1(\Delta)^N} + \|\mathcal{U}_0\|_{H^1(\Delta)^N}). \tag{4.14}$$

Besides, $\mathcal{U}_0$ satisfies

$$\begin{cases} \mathcal{L}_0^R\nabla\mathcal{U}_0 + \kappa^2\mathcal{U} = -\kappa^2\widehat{U_0} & \text{in } \Delta, \\ \mathcal{N}_{\mathcal{C}^{(0)}}\mathcal{U}_0 = 0 & \text{on } \partial\Delta. \end{cases} \tag{4.15}$$

Recall that the variational formula is

$$\mathcal{B}_{\mathcal{C}^{(0)}}(u,v) := \int_\Delta \left\{ \sum_{i.j.k.l=1}^{3} C_{ijkl}^{(0)} \frac{\partial u_k}{\partial x_l}\frac{\partial \overline{v_i}}{\partial x_j} - \kappa^2\rho u_i\overline{v_i} \right\} dx,$$

then we have

$$\mathcal{B}_{\mathcal{C}^{(0)}}(u,u) \ge c_0\|\nabla u\|^2_{L^2(\Delta)^N} - \kappa^2\|u\|^2_{L^2(\Delta)^N} \text{ for all } u \in H^1(\Delta)^N. \tag{4.16}$$

Moreover, by using (4.16), one can show the well-posedness of (4.15) provided that $-\kappa^2$ is not an eigenvalue of $\mathcal{L}_0^R$ with vanishing boundary traction. We refer readers to [33, Chapter 4] and [32, Chapter 6] for more details. When $-\kappa^2$ is not an eigenvalue of $\mathcal{L}_0^R$ with zero boundary traction, the well-posedness of (4.15) in the $L^2(\Omega)$ Sobolev space implies that

$$\|\mathcal{U}_0\|_{H^1(\Delta)^N} \le C\|\widehat{U_0}\|_{L^2(\Delta)^N}, \tag{4.17}$$

where $C > 0$ is a constant independent of $\mathcal{U}_0$ and $\widehat{U_0}$. Finally, plug (4.13) and (4.17) into (4.14), we can get

$$\|U_0\|_{H^1(\Delta)^N} \le C\|w_0\|_{H^1(\Delta)^N}$$

as desired. This completes the proof. □

The following trace theorem and inverse trace theorem will be used in our proof of Theorem 3.2.

Lemma 4.4. *(Trace and inverse trace theorem) Let $\mathit{\Delta}$ be a C^2 bounded domain in $\mathbb{R}^N$, for any $V \in H^1(\mathit{\Delta})^N$, there exist $c_1, c_2 > 0$ independent of V such that*

$$c_1\|V\|_{H^{1/2}(\partial\mathit{\Delta})^N} \le \|V\|_{H^1(\mathit{\Delta})^N} \le c_2\|V\|_{H^{1/2}(\partial\mathit{\Delta})^N}. \tag{4.18}$$

Proof. This lemma is the standard trace theorem and readers can find the proof in many literature. For example, see [22, Chapter 4] for the detailed proof. □

Now, we prove the following two lemmas by comparing the elasticity system with residual stress with Lamé system in the domain $\Delta := \Omega\backslash\overline{D_h}$ for $h > 0$ small. First, we give a boundary estimate outside small cavities.

Lemma 4.5. *Suppose $-\kappa^2$ is not an eigenvalue of the elliptic operator $\mathcal{L}_0^R$ with zero boundary traction condition in Ω. Let v_0 be a solution of*

$$\begin{cases} \mathcal{L}_0^R v_0 + \kappa^2 v_0 = 0 & \text{in } \Omega\backslash\overline{D_h}, \\ \mathcal{N}_{\mathcal{C}^{(0)}} v_0 = \mathcal{N}_{\mathcal{C}^{(0)}} u_0 & \text{on } \partial D_h, \\ \mathcal{N}_{\mathcal{C}^{(0)}} v = 0 & \text{on } \partial\Omega, \end{cases} \tag{4.19}$$

where u_0 is the solution of (1.12). Then there exists $h_0 > 0$ such that for any $h \in (0, h_0)$,

$$\|v_0\|_{H^{1/2}(\partial\Omega)^N} \leq Ch\|\phi\|_{H^{-1/2}(\partial\Omega)^N}, \tag{4.20}$$

where $C > 0$ independent of h and ϕ.

Proof. Let $\widehat{v_0}$ be a solution of the Lamé system

$$\begin{cases} \mathcal{L}_0 \widehat{v_0} + \eta^2 \widehat{v_0} = 0 & \text{in } \Omega\backslash\overline{D_h}, \\ \mathcal{N}_{\mathcal{C}^{(0)}} \widehat{v_0} = \mathcal{N}_{\mathcal{C}^{(0)}} u_0 & \text{on } \partial D_h, \\ \mathcal{N}_{\mathcal{C}^{(0)}} \widehat{v_0} = 0 & \text{on } \partial\Omega, \end{cases} \tag{4.21}$$

where $-\eta^2$ is not an eigenvalue of $\mathcal{L}_0$ with the zero boundary traction on Ω. By the layer potential techniques for the Lamé system (for example, see [2, 3]), we have the following estimates

$$\|\widehat{v_0}\|_{H^{1/2}(\partial D_h)^N} \leq Ch\|\phi\|_{H^{-1/2}(\partial\Omega)^N}, \tag{4.22}$$

$$\|\widehat{v_0}\|_{H^{1/2}(\partial\Omega)^N} \leq Ch^N\|\phi\|_{H^{-1/2}(\partial\Omega)^N}, \tag{4.23}$$

where (4.23) was proved in [23]. For (4.22), we will offer the proof in our Appendix. Let $\widehat{v} := v_0 - \widehat{v_0}$, where v_0 and $\widehat{v_0}$ are solutions of (4.19) and (4.21), respectively, then $\widehat{v}$ satisfies the following boundary value problem

$$\begin{cases} \mathcal{L}_0^R \widehat{v} + \kappa^2 \widehat{v} = (\eta^2 - \kappa^2)\widehat{v_0} - R\widehat{v_0} & \text{in } \Omega\backslash\overline{D_h}, \\ \mathcal{N}_{\mathcal{C}^{(0)}} \widehat{v} = 0 & \text{on } \partial D_h, \\ \mathcal{N}_{\mathcal{C}^{(0)}} \widehat{v} = 0 & \text{on } \partial\Omega, \end{cases}$$

where R was defined by (4.5). Applying the key estimate (4.6) on the domain $\Delta = \Omega\backslash\overline{D_h}$, there exists $C > 0$ independent of $\widehat{v}$ and $\widehat{v_0}$ such that

$$\|\widehat{v}\|_{H^1(\Omega\backslash\overline{D_h})^N} \leq C\|\widehat{v_0}\|_{H^1(\Omega\backslash\overline{D_h})^N}.$$

By the trace inequality (4.18) and (4.22) on $\Omega\backslash\overline{D_h}$, then it deduces that

$$\begin{aligned} \|v_0\|_{H^{1/2}(\partial(\Omega\backslash\overline{D_h}))^N} &\leq C\|v_0\|_{H^1(\Omega\backslash\overline{D_h})^N} \\ &\leq C\left(\|\widehat{v}\|_{H^1(\Omega\backslash\overline{D_h})^N} + \|\widehat{v_0}\|_{H^1(\Omega\backslash\overline{D_h})^N}\right) \\ &\leq C\|\widehat{v_0}\|_{H^1(\Omega\backslash\overline{D_h})^N} \\ &\leq C\|\widehat{v_0}\|_{H^{1/2}(\partial(\Omega\backslash\overline{D_h}))^N} \\ &\leq Ch\|\phi\|_{H^{-1/2}(\partial\Omega)^N}. \end{aligned}$$

This completes the proof. □

Lemma 4.6. *Suppose $-\kappa^2$ is not an eigenvalue of $\mathcal{L}_0^R$ with the zero boundary traction in Ω. Consider the following boundary value problem*

$$\begin{cases}\mathcal{L}_0^R v+\kappa^2 v=0 & in\ \Omega\backslash\overline{D_h},\\ \mathcal{N}_{\mathcal{C}^{(0)}}v=\psi & on\ \partial D_h,\\ \mathcal{N}_{\mathcal{C}^{(0)}}v=\phi & on\ \partial\Omega.\end{cases}$$

Then there exists $h_0>0$ such that for any $h\in(0,h_0)$,

$$\|v-u_0\|_{H^{1/2}(\partial\Omega)^N}\le C\left(h\|\phi\|_{H^{-1/2}(\partial\Omega)^N}+h^{N-1}\|\psi(h\cdot)\|_{H^{-3/2}(\partial D)^N}\right), \quad (4.24)$$

where u_0 is a solution of (1.12) and $C>0$ is independent of h, φ and ϕ.

Proof. Let $\mathbb{V}=u_0-v$ in $\Omega\backslash\overline{D_h}$, then $\mathbb{V}$ is a solution of

$$\begin{cases}\mathcal{L}_0^R\mathbb{V}+\kappa^2\mathbb{V}=0 & \text{in } \Omega\backslash\overline{D_h},\\ \mathcal{N}_{\mathcal{C}^{(0)}}\mathbb{V}=\mathcal{N}_{\mathcal{C}^{(0)}}u_0-\psi & \text{on } \partial D_h,\\ \mathcal{N}_{\mathcal{C}^{(0)}}\mathbb{V}=0 & \text{on } \partial\Omega.\end{cases}$$

Decompose $\mathbb{V}:=\mathbb{V}_1-\mathbb{V}_2$ such that $\mathbb{V}_1$ is a solution of

$$\begin{cases}\mathcal{L}_0^R\mathbb{V}_1+\kappa^2\mathbb{V}_1=0 & \text{in } \Omega\backslash\overline{D_h},\\ \mathcal{N}_{\mathcal{C}^{(0)}}\mathbb{V}_1=\mathcal{N}_{\mathcal{C}^{(0)}}u_0 & \text{on } \partial D_h,\\ \mathcal{N}_{\mathcal{C}^{(0)}}\mathbb{V}_1=0 & \text{on } \partial\Omega,\end{cases}$$

and $\mathbb{V}_2$ is a solution of

$$\begin{cases}\mathcal{L}_0^R\mathbb{V}_2+\kappa^2\mathbb{V}_2=0 & \text{in } \Omega\backslash\overline{D_h},\\ \mathcal{N}_{\mathcal{C}^{(0)}}\mathbb{V}_2=\psi & \text{on } \partial D_h,\\ \mathcal{N}_{\mathcal{C}^{(0)}}\mathbb{V}_2=0 & \text{on } \partial\Omega.\end{cases}$$

Making use of (4.23), it is easy to see

$$\|\mathbb{V}_1\|_{H^{1/2}(\partial\Omega)^N}\le Ch^N\|\phi\|_{H^{-1/2}(\partial\Omega)^N}, \quad (4.25)$$

where $C>0$ is independent of $\mathbb{V}_1$ and ϕ. It remains to estimate $\mathbb{V}_2$.

Let $\mathbb{W}$ be a solution of

$$\begin{cases}\mathcal{L}_0\mathbb{W}+\eta^2\mathbb{W}=0 & \text{in } \Omega\backslash\overline{D_h},\\ \mathcal{N}_{\mathscr{C}^{(0)}}\mathbb{W}=\psi & \text{on } \partial D_h,\\ \mathcal{N}_{\mathscr{C}^{(0)}}\mathbb{W}=0 & \text{on } \partial\Omega,\end{cases}$$

and in Section 5 of [23], the authors proved that

$$\|\mathbb{W}\|_{H^{1/2}(\partial\Omega)^N}\le Ch^{N-1}\|\psi(h\cdot)\|_{H^{-3/2}(\partial\Omega)^N},$$

whenever $-\eta^2$ is not an eigenvalue of $\mathcal{L}_0$ with zero boundary traction. Set $\mathbb{V}_3=\mathbb{V}_2-\mathbb{W}$, then $\mathbb{V}_3$ is a solution of

$$\begin{cases}\mathcal{L}_0^R\mathbb{V}_3+\kappa^2\mathbb{V}_3=(\eta^2-\kappa^2)\mathbb{W}-R\mathbb{W} & \text{in } \Omega\backslash\overline{D_h},\\ \mathcal{N}_{\mathcal{C}^{(0)}}\mathbb{V}_3=0 & \text{on } \partial(\Omega\backslash\overline{D_h}),\end{cases}$$

where we have used the same boundary traction, which means $\mathcal{N}_{\mathcal{C}^{(0)}} = \mathcal{N}_{\mathscr{C}^{(0)}}$ on $\partial(\Omega\backslash\overline{D_h})$. Hence, by (4.6) and (4.18), we can derive

$$\begin{aligned} \|\mathbb{V}_3\|_{H^{1/2}(\partial(\Omega\backslash\overline{D_h}))^N} &\leq C\|\mathbb{V}_3\|_{H^1(\Omega\backslash\overline{D_h})^N} \\ &\leq C\|\mathbb{W}\|_{H^1(\Omega\backslash\overline{D_h})^N} \\ &\leq C\left(h\|\phi\|_{H^{1/2}(\partial\Omega)^N} + h^{N-1}\|\psi(h\cdot)\|_{H^{-3/2}(\partial D)^N}\right). \end{aligned} \tag{4.26}$$

Finally, combining (4.25), and (4.26) yields (4.24) to be valid, which means we complete the proof of this lemma. □

Now, we can prove our main result.
Proof of Theorem 3.2. By Lemma 4.2, if we set $\varphi = \mathcal{N}^+_{\mathcal{C}^{(0)}}\widetilde{u}|_{\partial D_h}$, we have $v = \widetilde{u}$ and $\Phi^+ = \varphi(h\cdot)$. By (4.24), we obtain

$$\|\widetilde{u} - u_0\|_{H^{1/2}(\partial\Omega)^N} \leq C\left(h\|\phi\|_{H^{-1/2}(\partial\Omega)^N} + h^{N-1}\|\Phi^+\|_{H^{-3/2}(\partial D)^N}\right). \tag{4.27}$$

Moreover, combine the estimate (4.1) and (4.27), and the Young's inequality will lead the desired estimate (3.12) to hold. We complete the proof. In summary, in this work, we have built up the nearly cloaking theory for the elasticity system with residual stress.

5 Appendix

In Appendix, we utilize layer potential methods for the Lamé system to derive (4.22). For $x \neq y \in \mathbb{R}^N$, let

$$G_\eta(x,y) = \begin{cases} \dfrac{\exp(\sqrt{-1}\eta|x-y|)}{4\pi|x-y|} & \text{when } N=3, \\ \dfrac{i}{4}H_0^{(1)}(\eta|x-y|) & \text{when } N=2, \end{cases}$$

where $H_0^{(1)}$ is the *Hankel function* of the first kind of order 0. The Green's tensor $\Pi(x,y)$ for the Lamé system can be written as

$$\Pi(x,y) = \frac{1}{\mu}G_{k_s}(x,y)I_N + \frac{1}{\eta^2}\mathrm{grad}_x\mathrm{grad}_x^T[G_{k_s}(x,y) - G_{k_p}(x,y)],$$

for $x \neq y \in \mathbb{R}^N$, $N = 2, 3$, where

$$k_p = \frac{\eta}{\sqrt{\lambda+2\mu}},\ k_s = \frac{\eta}{\sqrt{\mu}}$$

are compressional and shear constants and I_N is the $N \times N$ identity matrix.

Let $\mathcal{O}$ be a bounded simply connected domain in $\mathbb{R}^N$ and $\psi(x)$ be a surface density for $x \in \partial\mathcal{O}$, then we can define the single and double layer potentials in the following

$$\begin{aligned} (\mathscr{S}_{\mathcal{O}}\psi)(x) &= \int_{\partial\mathcal{O}} \Pi(x,y)\psi(y)dS(y),\ x \in \mathbb{R}^N\backslash\partial\mathcal{O}, \\ (\mathscr{D}_{\mathcal{O}}\psi)(x) &= \int_{\partial\mathcal{O}} \Xi(x,y)\psi(y)dS(y),\ x \in \mathbb{R}^N\backslash\partial\mathcal{O}, \end{aligned}$$

where $\Xi(x,y)$ is a matrix-valued function with the i-th column vector is

$$[\Xi(x,y)]^T e_i = \mathcal{N}_{\mathscr{C}(0)}[\Xi(x,y)e_i] \text{ on } \partial\mathcal{O}.$$

In addition, we set

$$\begin{aligned}(\mathcal{S}_{\partial\mathcal{O}}\psi)(x) &= \int_{\partial\mathcal{O}} \Pi(x,y)\psi(y)dS(y),\ x\in\partial\mathcal{O},\\ (\mathcal{K}_{\partial\mathcal{O}}\psi)(x) &= \int_{\partial\mathcal{O}} \Xi(x,y)\psi(y)dS(y),\ x\in\partial\mathcal{O}.\end{aligned}$$

Now, we can prove (4.22). Let u_0, v_0, ϕ be the same functions defined in previous sections and let

$$V(x) := \int_{\partial D_h} \Pi(x,y)\mathcal{N}_{\mathscr{C}(0)}u_0 dS(y),\ x\in\Omega\backslash D_h,$$

the authors [23] proved the following estimates

$$\|V\|_{C(\partial D_h)} \le Ch\|\phi\|_{H^{-1/2}(\partial\Omega)^N},\ \|V\|_{C(\partial\Omega)} \le Ch^N\|\phi\|_{H^{-1/2}(\partial\Omega)^N} \tag{5.1}$$

and

$$\|\zeta_1\|_{L^2(\partial D_h)^N} \le Ch^{(N+1)/2}\|\phi\|_{H^{-1/2}(\partial\Omega)^N},\ \|\zeta_2\|_{L^2(\partial\Omega)^N} \le Ch^N\|\phi\|_{H^{-1/2}(\partial\Omega)^N}, \tag{5.2}$$

where $\zeta_1 := w|_{\partial D_h}$, $\zeta_2 := w|_{\partial\Omega}$. By the jump relations for double layer potentials, we get

$$\zeta_1(x) = 2\left[(\mathcal{K}_{\partial D_h}\zeta_1)(x) - (\mathscr{D}_{\partial\Omega}\zeta_2)(x) + V(x)\right] \text{ when } x\in\partial D_h.$$

Use the similar arguments in [23], from (5.1), then we can derive

$$\|V\|_{H^{1/2}(\partial D_h)^N} \le C\|V\|_{C(\partial D_h)^N} \le Ch\|\phi\|_{H^{-1/.2}(\partial\Omega)}.$$

Finally, from (5.2) and the boundedness of $\mathcal{K}_{\partial D_h} : L^2(D_h)^N \to H^1(D_h)^N$, we have

$$\|\mathcal{K}_{\partial D_h}\zeta_1\|_{H^{1/2}(\partial D_h)^N} \le Ch^{(N+1)/2}\|\phi\|_{H^{-1/2}(\partial\Omega)^N}.$$

Finally, from (5.2) again, we obtain

$$\|\mathscr{D}_{\partial\Omega}\zeta_2\|_{H^{1/2}(\partial D_h)^N} \le C\|\mathscr{D}_{\partial\Omega}\zeta_2\|_{C(\partial D_h)^N} \le Ch^N\|\phi\|_{H^{-1/2}(\partial\Omega)^N}$$

as desired, which complete the proof of (4.22).

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