The Calderón problem for variable coefficients nonlocal elliptic operators

Tuhin Ghosh[*], Yi-Hsuan Lin[†], Jingni Xiao[‡]

Abstract

In this paper, we introduce an inverse problem of a Schrödinger type variable nonlocal elliptic operator $(-\nabla\cdot(A(x)\nabla))^s+q)$, for $0<s<1$. We determine the unknown bounded potential q from the exterior partial measurements associated with the nonlocal Dirichlet-to-Neumann map for any dimension $n\geq 2$. Our results generalize the recent initiative [18] of introducing and solving inverse problem for fractional Schrödinger operator $((-\Delta)^s+q)$ for $0<s<1$. We also prove some regularity results of the direct problem corresponding to the variable coefficients fractional differential operator and the associated degenerate elliptic operator.

Key words. The Calderón problem, nonlocal Schrödinger equation, anisotropic, unique continuation principle, Runge approximation property, degenerate elliptic equations, A_p weight, Almgren's frequency function, doubling inequality
Mathematics Subject Classification: 35R30, 26A33, 35J10, 35J70

1 Introduction

Let $\mathcal{L}$ be an elliptic partial differential operator. We consider an inverse problem associated to the nonlocal fractional operator $\mathcal{L}^s$ with the power $s\in(0,1)$. We introduce the corresponding Calderón problem of determining the unknown bounded potentials $q(x)$ from the exterior measurements on the Dirichlet-to-Neumann (DN) map of the nonlocal Schrödinger equation $(\mathcal{L}^s+q)u=0$. It intends to generalize the recent study on the Calderón problem for the fractional Schrödinger equation [18]. The study of the nonlocal operators is currently an active research area in mathematics and often covers vivid problems coming from different fields including mathematical physics, finance, biology and geology. See the references [4, 37] for subsequent discussions.

The study of inverse problems remains as a popular field in applied mathematics since A.P. Calderón published his pioneering work "On an inverse boundary value problem" [9] in 1980s. The problem proposed by Calderón is: "Is it possible to determine the electrical conductivity of a medium by making voltage and current measurements on its boundary?" For instance, let $\Omega\subset\mathbb{R}^n$ be a smooth domain and $\gamma(x)>0$ on Ω be the conductivity of the medium, then one can determine the conductivity γ from the knowledge of current and voltage

[*]Institute for Advanced Study, Jockey Club, HKUST, Hong Kong, China
[†]Department of Mathematics, University of Washington, Seattle, USA
[‡]Department of Mathematics, Hong Kong Baptist University, Hong Kong, China

on the boundary measurements $\left(u|_{\partial\Omega}, \gamma\frac{\partial u}{\partial\nu}|_{\partial\Omega}\right)$, where u solves the conductivity equation

$$-\nabla\cdot(\gamma\nabla u) = 0 \text{ in } \Omega. \tag{1.1}$$

It gets its momentum with the seminal work of Sylvester and Uhlmann [45], solving the Calderón problem in space dimension $n \geq 3$ by establishing the fact that the conductivity γ gets uniquely determined by the Dirichlet-to-Neumann map $\left(u|_{\partial\Omega} \mapsto \gamma\frac{\partial u}{\partial\nu}\right)$ of the conductivity equation (1.1). Later on the same result has been proved in dimension $n = 2$ also in [5]. However, the conclusion does not true in one dimensional case. We refer readers to a survey article [49] for more information. In a very recent progress the study of Calderón's type inverse problem is being initiated for nonlocal operators, in particular the Calderón problem of the fractional Schrödinger operator $(-\Delta)^s + q(x)$ has been solved in [18].

In this article, we continue the progress by considering more general nonlocal operators $(-\nabla\cdot(A(x)\nabla))^s + q(x)$ where A is possibly variable coefficient *anisotropic* matrix with standard ellipticity and boundedness assumptions on it. This work also offers a comparative study between nonlocal inverse problem of $(-\nabla\cdot(A(x)\nabla))^s + q(x)$ for $0 < s < 1$, and the local inverse problem of $-\nabla\cdot(A(x)\nabla) + q(x)$. The solvability of the local inverse problem is fully known in two dimension, whereas in three and higher dimension it is partially solved for certain class of anisotropic matrix. We will see such difficulties do not arise in our non-local analogue.

In this paper, we consider $\mathcal{L}$ to be a second order linear elliptic operator of the divergence form

$$\mathcal{L} := -\nabla\cdot(A(x)\nabla), \tag{1.2}$$

which is defined in the entire space $\mathbb{R}^n$ for $n \geq 2$, where $A(x) = (a_{ij}(x))$, $x \in \mathbb{R}^n$ is an $n \times n$ symmetric matrix satisfying the ellipticity condition, i.e.,

$$\begin{cases} a_{ij} = a_{ji} \text{ for all } 1 \leq i, j \leq n, \text{ and} \\ \Lambda^{-1}|\xi|^2 \leq \sum_{i,j=1}^n a_{ij}(x)\xi_i\xi_j \leq \Lambda|\xi|^2 \text{ for all } x \in \mathbb{R}^n, \text{ for some } \Lambda > 0. \end{cases} \tag{1.3}$$

Our definition of the fractional power $\mathcal{L}^s$, with its domain $\mathrm{Dom}(\mathcal{L}^s)$, initiated from the spectral theorem. We then extend the operator $\mathcal{L}^s$, by applying the heat kernel and its estimates, as a bounded linear operator

$$\mathcal{L}^s : H^s(\mathbb{R}^n) \longrightarrow H^{-s}(\mathbb{R}^n).$$

The detailed definition of $\mathcal{L}^s$ is included in Section 2.

If Ω is a bounded open set in $\mathbb{R}^n$, let us consider $u \in H^s(\mathbb{R}^n)$ a solution to the Dirichlet problem

$$(\mathcal{L}^s + q)u = 0 \text{ in } \Omega \text{ with } u = g \text{ in } \Omega_e, \tag{1.4}$$

where $q = q(x) \in L^\infty(\Omega)$ and Ω_e is the exterior domain denoted by

$$\Omega_e = \mathbb{R}^n \backslash \overline{\Omega}.$$

We also assume that $\{0\}$ is not an eigenvalue of the operator $(\mathcal{L}^s + q)$, which means

$$\begin{cases} \text{if } w \in H^s(\mathbb{R}^n) \text{ solves } (\mathcal{L}^s + q)w = 0 \text{ in } \Omega \text{ and } w|_{\Omega_e} = 0, \\ \text{then } w \equiv 0. \end{cases} \tag{1.5}$$

For being $q \geq 0$, the condition (1.5) is satisfied. Then for any given $g \in H^s(\Omega_e)$, there exists a unique solution $u \in H^s(\mathbb{R}^n)$ solving the nonlocal problem (1.4) (see Proposition 3.3). Next, we are going to define the associated DN map of the problem (1.4) in an analogues way introduced in [18, Lemma 2.4]. The DN map is given by

$$\Lambda_q : X \to X^*, \tag{1.6}$$

where X is the abstract trace space $X = H^s(\mathbb{R}^n)/\widetilde{H}^s(\Omega)$ such that

$$(\Lambda_q[g], [h]) = \mathcal{B}_q(u, h), \text{ for } g, h \in H^s(\mathbb{R}^n). \tag{1.7}$$

Here $[\cdot]$ stands for the equivalence class in X, i.e., for given $g \in H^s(\mathbb{R}^n)$,

$$[g] = g + \widetilde{g}, \text{ with } \widetilde{g} \in \widetilde{H}^s(\Omega),$$

and $\mathcal{B}_q(\cdot,\cdot)$ in (1.7) is the standard bilinear form associated to the above problem (1.4) explicitly introduced in Subsection 3.2.

The range of the DN map could be interpreted as infinitesimal amount of particles migrating to the exterior domain Ω_e in the steady state free diffusion process in Ω modeled by (1.4) which gets excited due to some source term in Ω_e. Analogue to the diffusion process, similar interpretations might be regarded in the theory of stochastic analysis. For more details, see [1, 11, 35].

Furthermore, if the domain Ω, the potential q in Ω, the source term in Ω_e and the matrix $A(x)$ in (1.2) satisfying (1.3) in $\mathbb{R}^n$ are sufficiently smooth, the DN map is more explicit and is given by (see Remark 3.7)

$$\Lambda_q : H^{s+\beta}(\Omega_e) \to H^{-s+\beta}(\Omega_e) \text{ with } \Lambda_q g = \mathcal{L}^s u|_{\Omega_e},$$

where $g \in H^{s+\beta}(\Omega_e)$ and $\beta \geq 0$ is an arbitrary real number satisfying $\beta \in (s - \frac{1}{2}, \frac{1}{2})$. Heuristically, given an open set $W \subseteq \Omega_e$, we interpret $\Lambda_q g|_W$ as measuring the cost required to maintain the exterior value g in W for the fixed inhomogeneity in the system given by $A(x)$ in the whole space $\mathbb{R}^n$.

The following theorem is the main result in this article. It is a generalization of the fractional Schrödinger inverse problem studied in [18] in any dimension $n \geq 2$. This is also a local data result with exterior Dirichlet and Neumann measurements in arbitrary open (possibly disjoint) sets $\mathcal{O}_1, \mathcal{O}_2 \subseteq \Omega_e$.

Theorem 1.1. *Let $n \geq 2$, and $A(x)$ is a C^∞-smooth matrix-valued function in $\mathbb{R}^n$ satisfying* (1.3)*. Let $\Omega \subseteq \mathbb{R}^n$ be a bounded domain with Lipschitz boundary and let $q_1, q_2 \in L^\infty(\Omega)$ satisfy condition* (1.5)*. Assume that $\mathcal{O}_1, \mathcal{O}_2 \subseteq \Omega_e$ are arbitrary open sets and Λ_{q_j} is the DN map with respect to $(\mathcal{L}^s + q_j)u = 0$ in Ω for $j = 1, 2$. If*

$$\Lambda_{q_1} g|_{\mathcal{O}_2} = \Lambda_{q_2} g|_{\mathcal{O}_2} \text{ for any } g \in C_c^\infty(\mathcal{O}_1), \tag{1.8}$$

then one can conclude that

$$q_1 = q_2 \text{ in } \Omega.$$

Theorem 1.1 can be interpreted as a partial data result for the above nonlocal inverse problem. Analogues resembles can be made with the study of the partial data Calderón's type problem, the richness of such works can be found in [23, 24, 25, 26].

Let us present a comparative study between our non-local inverse problem and the known local inverse problem. We begin with recalling the following local inverse problem as: Determining the uniqueness of the potentials $q_1 = q_2$ in Ω from the information on the associated DN maps $\Lambda_{A,q_1} = \Lambda_{A,q_2}$ on $\partial\Omega$, where the $\Lambda_{A,q_j} : H^{1/2}(\partial\Omega) \to H^{-1/2}(\partial\Omega)$ is the DN map defined by $\Lambda_q(u|_{\partial\Omega}) = (A\nabla u)\cdot\nu|_{\partial\Omega}$ (where ν is the unit outer normal on $\partial\Omega$), and u_j solves

$$(\mathcal{L} + q_j)u_j = -\nabla\cdot(A(x)\nabla u_j) + q_j(x)u_j = 0 \text{ in } \Omega \text{ for } j = 1, 2,$$

with $A \in L^\infty(\Omega)$ satisfying the ellipticity condition (1.3).

It has been answered positively in two dimensional case by using the isothermal coordinate. For $n \geq 3$, the answer is known for a certain class of anisotropic matrices A. This problem has been often addressed via geometry settings which goes as follows: Let (M, g) be a oriented compact Riemannian n-dimensional manifold with C^∞-smooth boundary ∂M and let q be a continuous potential on M. Consider

$$(-\Delta_g + q)u = 0 \text{ in } M, \tag{1.9}$$

where

$$\Delta_g = \sum_{j,k=1}^{n} (\det g)^{-1/2} \frac{\partial}{\partial x^j}\left((\det g)^{1/2} g^{jk} \frac{\partial}{\partial x^k}\right)$$

is the Laplace-Beltrami operator on (M, g) and $g = (g_{jk})$ with $(g_{jk}) = (g^{jk})^{-1}$. If $\{0\}$ is not an eigenvalue of $-\Delta_g + q$, we have the corresponding DN map on ∂M defined by

$$\Lambda_{g,q} : H^{1/2}(\partial M) \to H^{-1/2}(\partial M) \text{ by } \Lambda_{g,q}(u|_{\partial M}) := \left.\sum_{j,k=1}^{n} g^{jk} \frac{\partial u}{\partial x_j}\nu_k\right|_{\partial M},$$

where $\nu = (\nu_1, \nu_2, \cdots, \nu_n)$ is the unit outer normal on ∂M. The connection between the matrix $A = (a_{jk})$ and the metric g^{jk} can be made as

$$g^{jk}(x) = (\det A(x))^{-1/(n-2)} a_{jk}(x) \text{ for } n \geq 3.$$

In the two-dimensional setting, if $\Lambda_{g,q_1} = \Lambda_{g,q_2}$ on ∂M, then $q_1 = q_2$ in M whenever q_1, q_2 are continuous potential on M, see [22]. However in the case of three and higher dimensions, it has been answered only partially. Under special geometries, for instance, when (M, g) is admissible (see [17, Definition 1.5]) and q_1, q_2 are C^∞-smooth, then $\Lambda_{g,q_1} = \Lambda_{g,q_2}$ on ∂M implies $q_1 = q_2$ in M, see [17, Theorem 1.6].

In our paper, we study the inverse problem associated with the nonlocal operator $\mathcal{L}^s + q$, where $\mathcal{L} = -\nabla\cdot(A(x)\nabla)$ and $s \in (0, 1)$. We can determine $q_1 = q_2$ in $\Omega \subseteq \mathbb{R}^n$ for any $n \geq 2$ via the partial information $\Lambda_{q_1} g|_{\mathcal{O}_2} = \Lambda_{q_2} g|_{\mathcal{O}_2}$ for any $g \in C_c^\infty(\mathcal{O}_1)$, with $\mathcal{O}_1$, $\mathcal{O}_2$ being arbitrary open subsets in Ω_e, for any C^∞-smooth matrix-valued function $A(x)$ in $\mathbb{R}^n$ satisfying (1.3).

Note that we do not assume any further special structures on $A(x)$ unlike to the case $s = 1$, for example, the method (see [17]) consists of considering the limiting Carleman weight function for the Laplace-Beltrami operator in M and constructing the corresponding complex geometrical optics (CGO) solutions based on those weights of the problem (1.9). Whereas, our analysis relies on the Runge type approximation result (cf. Theorem 1.2) based on the strong uniqueness property (cf. Theorem 1.3) of the nonlocal operator $\mathcal{L}^s$.

For $A(x)$ being an $n \times n$ identity matrix I_n, then $\mathcal{L}$ becomes the Laplacian operator $(-\Delta)$ and the associated inverse problem for $s = 1$ has been studied extensively. When $n \geq 3$, the global uniqueness result is due to [45] for $q \in L^\infty$ and the authors [10, 32] proved it for the case of $q \in L^p$. When $n = 2$, Bukhgeim [5] proved it for slightly more regular potentials and see [2] for the case of $q \in L^p$. We refer readers to [50] for detailed survey on this inverse problem. For $s \in (0, 1)$, the study of this problem has been recently initiated in [18].

Let us briefly mention the way we prove the uniqueness result $q_1 = q_2$ in Ω as stated in Theorem 1.1. By having the following integral identity

$$\int_\Omega (q_1 - q_2)u_1 u_2\, dx = 0,$$

which we obtain from the assumption on the DN maps (1.8). In particular, by taking $u_j \in H^s(\mathbb{R}^n)$ solving $(\mathcal{L}^s + q_j)u_j = 0$ in Ω with $\mathrm{supp}(u_j) \subset \overline{\Omega} \cup \overline{\mathcal{O}_j}$; finally we derive for any $g \in L^2(\Omega)$

$$\int_\Omega (q_1 - q_2)g\, dx = 0.$$

The proof of the above integral identity will be completed with subsequent requirements of the following *strong uniqueness property* and the *Runge approximation property* for the nonlocal operator $\mathcal{L}^s$, similar to the results known (see [18]) for the fractional Laplacian operator.

Theorem 1.2. *(Strong uniqueness property)Let $n \geq 2$, and $A(x)$ is a C^∞-smooth matrix-valued function in $\mathbb{R}^n$ satisfying* (1.3)*. Let $u \in H^s(\mathbb{R}^n)$ be the function with $u = \mathcal{L}^s u = 0$ in some open set $\mathcal{O}$ of $\mathbb{R}^n$, for $s \in (0, 1)$, then $u \equiv 0$ in $\mathbb{R}^n$.*

Theorem 1.3. *(Runge approximation property)Let $n \geq 2$, and $A(x)$ is a C^∞-smooth matrix-valued function in $\mathbb{R}^n$ satisfying* (1.3)*. Let $\Omega \subseteq \mathbb{R}^n$ be a bounded open set and $D \subseteq \mathbb{R}^n$ be an arbitrary open set containing Ω such that $\mathrm{int}(D \backslash \overline{\Omega}) \neq \emptyset$. Let $q \in L^\infty(\Omega)$ satisfies* (1.5)*, then for any $f \in L^2(\Omega)$, for any $\epsilon > 0$, we can find a function $u_\epsilon \in H^s(\mathbb{R}^n)$ which solves*

$$(\mathcal{L}^s + q)u_\epsilon = 0 \text{ in } \Omega \text{ and } \mathrm{supp}(u_\epsilon) \subseteq \overline{D}$$

and

$$\|u_\epsilon - f\|_{L^2(\Omega)} < \epsilon.$$

Remark 1.4. Runge approximation is the reason why one gets better results for nonlocal equations, and Theorem 1.3 is a further generalization of the recent works of Dipierro-Savin-Valdinoci [15] and Ghosh-Salo-Uhlmann [18] where as a nonlocal operator $(-\Delta)^s$ $(s \in (0, 1))$ has been considered.

The paper is organized as follows. In Section 2, we will give a brief review of the background knowledge required in our paper, including the definition of the operator $\mathcal{L}^s$. Some results for the Dirichlet problem, including the well-posedness and the definition of the corresponding DN map, associated with the nonlocal operator $\mathcal{L}^s$ will be established in Section 3. In Section 4, we will show that the nonlocal problem in $\mathbb{R}^n$ is related to an extension degenerate local elliptic problem in $\mathbb{R}^n \times (0,\infty)$, which was first characterized by [44]. We also introduce suitable regularity results for the nonlocal operator $\mathcal{L}^s$ in $\mathbb{R}^n$, and its extension operator in $\mathbb{R}^n \times (0,\infty)$. These regularity results play the essential role to achieve our desired results. We hope that these regularity results could be of some independent interests. In Section 5, we will derive the strong unique continuation property (SUCP) for variable fractional operators and we prove Theorems 1.2 and 1.3. In Section 6, we prove the nonlocal type Calderón problem, Theorem 1.1. In Appendix, we offer the proof of the existence, uniqueness, and related properties including the Almgren type frequency function method and the associated doubling inequality for the degenerate elliptic problem.

Acknowledgment

The authors would like to thank Professor Gunther Uhlmann for suggesting the problem, and also to thank Professor Mikko Salo, Professor Gunther Uhlmann and Doctor Hui Yu for helpful discussions. Y.-H. Lin is partially supported by MOST of Taiwan under the project 160-2917-I-564-048 and J. Xiao was partly supported by the Mr. Kwok Yat Wai and Madam Kwok Chung Bo Fun Graduate School Development Fund.

2 Preliminaries

In this section, we will discuss some key properties for the variable coefficients fractional nonlocal operator $\mathcal{L}^s = (-\nabla \cdot (A(x)\nabla))^s$. For $A(x)$ being an identity matrix, the operator $\mathcal{L}^s$ becomes the well-known fractional Laplacian operator $(-\Delta)^s$, and the detailed study about the $(-\Delta)^s$ is available in [3, 6, 7, 8, 38, 39, 41, 42, 43].

2.1 Spectral Theory

We sketch in this section some basis of the spectral theory which will be used in this paper. For details, readers can refer to the references [36, 40, 47], etc.

Let $\mathcal{L}$ be a non-negative definite and self-adjoint operator densely defined in a Hilbert space, say, $L^2(\mathbb{R}^n)$. Let ϕ be a real-valued measurable function defined on the spectrum of $\mathcal{L}$. Then the following defined $\phi(\mathcal{L})$ is also a self-adjoint operator in $L^2(\mathbb{R}^n)$,

$$\phi(\mathcal{L}) := \int_0^\infty \phi(\lambda)\, dE_\lambda,$$

where $\{E_\lambda\}$ is the spectral resolution of $\mathcal{L}$ and each E_λ is a projection in $L^2(\mathbb{R}^n)$ (see for instance, [20]). The domain of $\phi(\mathcal{L})$ is given by

$$\mathrm{Dom}(\phi(\mathcal{L})) = \left\{ f \in L^2(\mathbb{R}^n);\, \int_0^\infty |\phi(\lambda)|^2\, d\|E_\lambda f\|^2 < \infty \right\}.$$

The linear operator $\phi(\mathcal{L}) : \mathrm{Dom}(\phi(\mathcal{L})) \to L^2(\mathbb{R}^n)$ is understood, via Riesz representation theorem, in the following sense,

$$\langle \phi(\mathcal{L})f, g\rangle_{L^2} := \int_0^\infty \phi(\lambda)\, d\langle E_\lambda f, g\rangle_{L^2}, \quad f \in \mathrm{Dom}(\phi(\mathcal{L})),\ g \in L^2(\mathbb{R}^n),$$

where $\langle\cdot,\cdot\rangle$ denotes the (real) inner product in $L^2(\mathbb{R}^n)$.

Now we are in a position to define the fractional operator $\mathcal{L}^s$. Notice that $\lambda^s = \dfrac{1}{\Gamma(-s)}\int_0^\infty (e^{-t\lambda}-1)t^{-1-s}dt$ for $s\in(0,1)$, where $\Gamma(-s) := -\Gamma(1-s)/s$, and Γ is the Gamma function. We define, given $s\in(0,1)$,

$$\mathcal{L}^s := \int_0^\infty \lambda^s \, dE_\lambda = \frac{1}{\Gamma(-s)}\int_0^\infty \left(e^{-t\mathcal{L}} - \mathrm{Id}\right)\frac{dt}{t^{1+s}}, \tag{2.1}$$

where $e^{-t\mathcal{L}}$ given by

$$e^{-t\mathcal{L}} := \int_0^\infty e^{-t\lambda}\, dE_\lambda \tag{2.2}$$

is a bounded self-adjoint operator in $L^2(\mathbb{R}^n)$ for each $t\geq 0$. The operator family $\{e^{-t\mathcal{L}}\}_{t\geq 0}$ is called the heat semigroup associated with $\mathcal{L}$ (cf. [34]). The domain of $\mathcal{L}^s$ is given by

$$\mathrm{Dom}(\mathcal{L}^s) = \left\{ f\in L^2(\mathbb{R}^n); \int_0^\infty \lambda^{2s}\, d\|E_\lambda f\|^2 < \infty \right\}. \tag{2.3}$$

Notice for any $f\in\mathrm{Dom}(\mathcal{L}^s)$ that, $\mathcal{L}^s f\in L^2(\mathbb{R}^n)$ and is given, again in the sense of Riesz representation theorem, by the formula

$$\langle \mathcal{L}^s f, g\rangle_{L^2} = \frac{1}{\Gamma(-s)}\int_0^\infty \left\langle \left(e^{-t\mathcal{L}}f - f\right), g\right\rangle_{L^2} \frac{dt}{t^{1+s}}, \quad g\in L^2(\mathbb{R}^n), \tag{2.4}$$

when $s\in(0,1)$.

Remark 2.1. We remark here that

$$\mathrm{Dom}(\mathcal{L}) \subseteq \mathrm{Dom}(\mathcal{L}^s), \quad s\in(0,1). \tag{2.5}$$

In fact, for any $f\in\mathrm{Dom}(\mathcal{L})\subseteq L^2(\mathbb{R}^n)$, one has

$$\begin{aligned}\int_0^\infty \lambda^{2s}\, d\|E_\lambda f\|^2 &= \int_1^\infty \lambda^{2s}\, d\|E_\lambda f\|^2 + \int_0^1 \lambda^{2s}\, d\|E_\lambda f\|^2 \\ &\leq \int_0^\infty \lambda^2\, d\|E_\lambda f\|^2 + \int_0^\infty d\|E_\lambda f\|^2 \\ &= \|\mathcal{L}f\|^2_{L^2(\mathbb{R}^n)} + \|f\|^2_{L^2(\mathbb{R}^n)} < \infty.\end{aligned}$$

2.2 Sobolev Spaces

For simplicity, we shall always consider real function spaces in this paper. Our notations for Sobolev spaces are mainly followed by [30].

Let $a\in\mathbb{R}$ be a constant. Let $H^a(\mathbb{R}^n) = W^{a,2}(\mathbb{R}^n)$ be the (fractional) Sobolev space endowed with the norm

$$\|u\|_{H^a(\mathbb{R}^n)} := \left\|\mathscr{F}^{-1}\left\{ \langle\xi\rangle^a \mathscr{F}u\right\}\right\|_{L^2(\mathbb{R}^n)},$$

where $\langle\xi\rangle = (1+|\xi|^2)^{\frac{1}{2}}$. It is known that for $s\in(0,1)$, $\|\cdot\|_{H^s(\mathbb{R}^n)}$ has the following equivalent form

$$\|u\|_{H^s(\mathbb{R}^n)} := \|u\|_{L^2(\mathbb{R}^n)} + [u]_{H^s(\mathbb{R}^n)} \tag{2.6}$$

where

$$[u]^2_{H^s(\mathcal{O})} := \int_{\mathcal{O}\times\mathcal{O}} \frac{|u(x)-u(z)|^2}{|x-z|^{n+2s}} dxdz,$$

for any open set $\mathcal{O}$ of $\mathbb{R}^n$.

Given any open set $\mathcal{O}$ of $\mathbb{R}^n$ and $a \in \mathbb{R}$, we denote the following Sobolev spaces,

$$\begin{aligned} H^a(\mathcal{O}) &:= \{u|_{\mathcal{O}};\ u \in H^a(\mathbb{R}^n)\}, \\ \widetilde{H}^a(\mathcal{O}) &:= \text{closure of } C_c^\infty(\mathcal{O}) \text{ in } H^a(\mathbb{R}^n), \\ H_0^a(\mathcal{O}) &:= \text{closure of } C_c^\infty(\mathcal{O}) \text{ in } H^a(\mathcal{O}), \end{aligned}$$

and

$$H^a_{\overline{\mathcal{O}}} := \{u \in H^a(\mathbb{R}^n);\ \mathrm{supp}(u) \subset \overline{\Omega}\}.$$

The Sobolev space $H^a(\mathcal{O})$ is complete under the norm

$$\|u\|_{H^a(\mathcal{O})} := \inf\left\{\|v\|_{H^a(\mathbb{R}^n)}; v \in H^a(\mathbb{R}^n) \text{ and } v|_{\mathcal{O}} = u\right\}.$$

It is known that $\widetilde{H}^a(\mathcal{O}) \subseteq H_0^a(\mathcal{O})$, and that $H^a_{\overline{\mathcal{O}}}$ is a closed subspace of $H^a(\mathbb{R}^n)$.

Lemma 2.2. *([30]) Let Ω be a Lipschitz domain in $\mathbb{R}^n$. Then*

(1) For any $a \in \mathbb{R}$,

$$\begin{aligned} &\widetilde{H}^a(\Omega) = H^a_{\overline{\Omega}} \subseteq H_0^a(\Omega), \\ &(H^a(\Omega))^* = \widetilde{H}^{-a}(\Omega) \text{ and } \left(\widetilde{H}^a(\Omega)\right)^* = H^{-a}(\Omega). \end{aligned}$$

(2) For $a \geq 0$ and $a \notin \{\frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \frac{7}{2}, \ldots\}$,

$$\widetilde{H}^a(\Omega) = H_0^a(\Omega).$$

2.3 The Operator $\mathcal{L}^s$

In this paper, we consider $\mathcal{L}$ to be a linear second order partial differential operator of the divergence form

$$\mathcal{L} := -\nabla \cdot (A(x)\nabla). \tag{2.7}$$

We assume that the $n \times n$ matrix $A(x) = (a_{ij}(x))_{i,j=1}^n$ such that $a_{ij} \in C^\infty(\mathbb{R}^n)$ for $i, j = 1, 2, \cdots, n$ and satisfies (1.3). It is easy to see that the operator $\mathcal{L}$ is well-defined on $C_0^\infty(\mathbb{R}^n)$, which is dense in the Hilbert space $L^2(\mathbb{R}^n)$. However, $\mathcal{L}$ is not self-adjoint in the domain $C_0^\infty(\mathbb{R}^n)$. In fact, one can verify in the case that, the adjoint operator admits the domain $\mathrm{Dom}(\mathcal{L}^*) = \{f \in L^2(\mathbb{R}^n);\ \mathcal{L}f \in L^2(\mathbb{R}^n)\}$, which does not coincide with $C_0^\infty(\mathbb{R}^n)$. In order to define the fractional power $\mathcal{L}^s$ by applying the spectral theory we briefly sketched in Section 2.1, one needs firstly to extend $\mathcal{L}$ as a self-adjoint operator densely defined in $L^2(\mathbb{R}^n)$.

It is known, see for instance [20], that $\mathcal{L}$ with the domain

$$\mathrm{Dom}(\mathcal{L}) = H^2(\mathbb{R}^n) \tag{2.8}$$

is the maximal extension such that $\mathcal{L}$ is self-adjoint and densely defined in $L^2(\mathbb{R}^n)$. Moreover, it is natural to expect that $\mathrm{Dom}(\mathcal{L}^s)$ is close to the Sobolev space $H^{2s}(\mathbb{R}^n)$, which is shown, at least when $s = 1/2$, that $\mathrm{Dom}(\mathcal{L}^s) = H^{2s}(\mathbb{R}^n)$ (cf. [12, 20]). Next, we would like to extend the definition of $\mathcal{L}^s$ from its domain $\mathrm{Dom}(\mathcal{L}^s)$ introduced in (2.3) to $H^s(\mathbb{R})$, using heat kernels and theirs estimates.

It is known that for $\mathcal{L}$ satisfying (1.3) with C^∞-smooth leading coefficient, then the bounded operator $e^{-t\mathcal{L}}$ given in (2.2) admits a symmetric (heat) kernel $p_t(x,z)$ (cf. [20]). In other words, one has for any $t \in \mathbb{R}_+ := (0,\infty)$ and any $f \in L^2(\mathbb{R}^n)$ that

$$\left(e^{-t\mathcal{L}} f\right)(x) = \int_{\mathbb{R}^n} p_t(x,z) f(z)\, dz, \quad x \in \mathbb{R}^n. \tag{2.9}$$

Moreover for any $t \in \mathbb{R}_+$, the kernel $p_t(\cdot,\cdot)$ is symmetric and admits the following estimates (cf. [12]) with some positive constants c_j and b_j, $j = 1, 2$,

$$c_1 e^{-b_1 \frac{|x-z|^2}{t}} t^{-\frac{n}{2}} \le p_t(x,z) \le c_2 e^{-b_2 \frac{|x-z|^2}{t}} t^{-\frac{n}{2}}, \quad x, z \in \mathbb{R}^n. \tag{2.10}$$

By applying similar arguments as in the proof of [8, Theorem 2.4], one has for $f, g \in \mathrm{Dom}(\mathcal{L}^s)$ that

$$\langle \mathcal{L}^s f, g\rangle_{L^2} = \frac{1}{2\Gamma(-s)} \int_0^\infty \int_{\mathbb{R}^n\times\mathbb{R}^n} (f(x) - f(z))(g(x) - g(z)) p_t(x,z) dx dz \frac{dt}{t^{1+s}}, \tag{2.11}$$

Now we define

$$\mathcal{K}_s(x,z) := \frac{1}{\Gamma(-s)} \int_0^\infty p_t(x,z) \frac{dt}{t^{1+s}}. \tag{2.12}$$

It is seen from (2.10) that $\mathcal{K}_s$ enjoys the following pointwise estimate

$$\frac{C_1}{|x-z|^{n+2s}} \le \mathcal{K}_s(x,z) = \mathcal{K}_s(z,x) \le \frac{C_2}{|x-z|^{n+2s}}, \quad x, z \in \mathbb{R}^n, \tag{2.13}$$

with some positive constants C_1, C_2. Hence it is obtain by recalling the norm (2.6) of $H^s(\mathbb{R}^n)$ that for any $f, g \in H^s(\mathbb{R}^n)$, the right hand side (RHS) of (2.11) coincides with

$$\frac{1}{2} \int_{\mathbb{R}^n\times\mathbb{R}^n} (f(x) - f(z))(g(x) - g(z)) \mathcal{K}_s(x,z) dx dz.$$

Therefore, it is natural to extend the definition of $\mathcal{L}^s$ from $\mathrm{Dom}(\mathcal{L}^s)$ to $H^s(\mathbb{R}^n)$ in the following distributional sense

$$\langle \mathcal{L}^s f, g\rangle_{H^{-s}\times H^s} := \frac{1}{2} \int_{\mathbb{R}^n\times\mathbb{R}^n} (f(x) - f(z))(g(x) - g(z)) \mathcal{K}_s(x,z) dx dz. \tag{2.14}$$

Moreover, it is obtained from (2.13) that, there exists a positive constant C such that the operator $\mathcal{L}^s$ defined in (2.14) satisfies

$$|\langle \mathcal{L}^s f, g\rangle_{H^{-s}\times H^s}| \le C \|u\|_{H^s(\mathbb{R}^n)} \|v\|_{H^s(\mathbb{R}^n)}, \quad u, v \in H^s(\mathbb{R}^n). \tag{2.15}$$

Thus, the definition (2.14) gives a bounded linear operator

$$\mathcal{L}^s : H^s(\mathbb{R}^n) \longrightarrow H^{-s}(\mathbb{R}^n).$$

We observe by using the symmetry $\mathcal{K}_s(x,z) = \mathcal{K}_s(z,x)$ that $\mathcal{L}^s$ is also symmetric, namely

$$\langle \mathcal{L}^s f, g \rangle_{H^{-s}\times H^s} = \langle f, \mathcal{L}^s g \rangle_{(H^s, H^{-s})}, \quad f, g \in H^s(\mathbb{R}^n). \tag{2.16}$$

Furthermore, it is obtained that

$$\begin{aligned}
\langle \mathcal{L}^s f, g \rangle_{H^{-s}\times H^s} =& \frac{1}{2} \lim_{\epsilon\to 0^+} \int_{\mathbb{R}^n} \int_{|x-z|>\epsilon} (f(x)-f(z))(g(x)-g(z))\mathcal{K}_s(x,z)dxdz \\
=& \frac{1}{2} \lim_{\epsilon\to 0^+} \int_{\mathbb{R}^n} \int_{|x-z|>\epsilon} (f(x)-f(z))g(x)\mathcal{K}_s(x,z)dxdz \\
&+ \frac{1}{2} \lim_{\epsilon\to 0^+} \int_{\mathbb{R}^n} \int_{|x-z|>\epsilon} (f(x)-f(z))g(x)\mathcal{K}_s(x,z)dzdx \\
=& \int_{\mathbb{R}^n} g(x) \lim_{\epsilon\to 0^+} \int_{|x-z|>\epsilon} (f(x)-f(z))\mathcal{K}_s(x,z)dzdx,
\end{aligned}$$

holds for all $f, g \in H^s(\mathbb{R}^n)$. Hence, one can also write

$$(\mathcal{L}^s f)(x) = \lim_{\epsilon\to 0^+} \int_{|x-z|>\epsilon} (f(x)-f(z))\mathcal{K}_s(x,z)dz, \quad f \in H^s(\mathbb{R}^n). \tag{2.17}$$

3 Dirichlet problems for $\mathcal{L}^s + q$

In a continuation to the general case, we proceed our discussions by introducing the state spaces followed by the Dirichlet problem and associated DN map for for $\mathcal{L}^s + q$.

3.1 Well-Posedness

Throughout this section, we shall always let $\Omega \subseteq \mathbb{R}^n$ be a bounded Lipschitz domain, q be a potential in $L^\infty(\Omega)$ and $s \in (0,1)$ be a constant. We consider the following nonlocal Dirichlet problem for the nonlocal operator $\mathcal{L}^s$,

$$\begin{cases} (\mathcal{L}^s + q)u = f & \text{in } \Omega, \\ u = g & \text{in } \Omega_e. \end{cases} \tag{3.1}$$

Define the bilinear form $B_q(\cdot,\cdot)$ by

$$\mathcal{B}_q(v,w) := \langle \mathcal{L}^s v, w \rangle + \int_\Omega q(x)v(x)w(x)\,dx, \quad v, w \in H^s(\mathbb{R}^n) \tag{3.2}$$

with $\mathcal{L}^s$ given by the form (2.14). It is seen from (2.16) that $\mathcal{B}_q$ is symmetric, and from (2.15) that $\mathcal{B}_q$ is a bounded in $H^s(\mathbb{R}^n) \times H^s(\mathbb{R}^n)$, i.e.,

$$|\mathcal{B}_q(v,w)| \le C\|v\|_{H^s(\mathbb{R}^n)}\|w\|_{H^s(\mathbb{R}^n)}, \quad v, w \in H^s(\mathbb{R}^n). \tag{3.3}$$

Note that $\mathcal{B}_q$ can be also regarded as a symmetric bounded bilinear form in the space $\widetilde{H}^s(\Omega)$. In fact, by using (3.3) and the fact that $C_0^\infty(\Omega)$ is dense in $\widetilde{H}^s(\Omega)$, one can define for any $v \in \widetilde{H}^s(\Omega)$ that,

$$\mathcal{B}_q(v,\phi) = \langle \mathcal{L}^s \tilde{v}, \phi \rangle + \int_\Omega q(x)v(x)\phi(x)\,dx, \quad \phi \in C_0^\infty(\Omega), \tag{3.4}$$

where $\tilde{v} \in H^s(\mathbb{R}^n)$ is an extension of v such that $\tilde{v}|_\Omega = v$.

Definition 3.1. (Weak solution) Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$. Given $f \in H^{-s}(\Omega)$ and $g \in H^s(\mathbb{R}^n)$, we say that $u \in H^s(\mathbb{R}^n)$ is a (weak) solution of (3.1) if $\widetilde{u}_g := u - g \in \widetilde{H}^s(\Omega)$ and

$$\mathcal{B}_q(u, \phi) = \langle f, \phi \rangle \quad \text{for any } \phi \in C_0^\infty(\Omega), \tag{3.5}$$

or equivalently

$$\mathcal{B}_q(\widetilde{u}_g, \phi) = \langle f - (\mathcal{L}^s + q)g, \phi \rangle \quad \text{for any } \phi \in C_0^\infty(\Omega). \tag{3.6}$$

Remark 3.2. It is easy to see that the space $C_0^\infty(\Omega)$ of test functions in (3.5) and (3.6) can be replaced by $\widetilde{H}^s(\Omega)$.

The well-posedness of the Dirichlet problem (3.1) is shown by the following more general result.

Proposition 3.3. *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$ and $q \in L^\infty(\Omega)$. The following results hold.*

(1) There is a countable set $\Sigma = \{\lambda_j\}_{j=1}^\infty$ of real numbers $\lambda_1 \leq \lambda_2 \leq \ldots \to \infty$, such that given $\lambda \in \mathbb{R} \setminus \Sigma$, for any $f \in H^{-s}(\Omega)$ and any $g \in H^s(\mathbb{R}^n)$, there is a unique $u \in H^s(\mathbb{R}^n)$ satisfying $u - g \in \widetilde{H}^s(\Omega)$ and

$$\mathcal{B}_q(u, v) - \lambda(u, v)_{L^2} = \langle f, v \rangle \quad \text{for any } v \in \widetilde{H}^s(\Omega). \tag{3.7}$$

Moreover,

$$\|u\|_{H^s(\mathbb{R}^n)} \leq C_0 \left(\|f\|_{H^{-s}(\Omega)} + \|g\|_{H^s(\mathbb{R}^n)} \right), \tag{3.8}$$

for some constant $C_0 > 0$ independent of f and g.

(2) The condition (1.5) *holds if and only if $0 \notin \Sigma$.*

(3) If $q \geq 0$ a.e. in Ω, then $\Sigma \subseteq \mathbb{R}_+$, and hence (1.5) *always holds.*

Proof. It is obtained from (2.13) and (2.14) that

$$\langle \mathcal{L}^s v, v \rangle_{(H^{-s}, H^s)} = \frac{1}{2} \int_{\mathbb{R}^n \times \mathbb{R}^n} |v(x) - v(z)|^2 \mathcal{K}_s(x, z) dx dz \geq c_0 \|v\|_{H^s(\mathbb{R}^n)}, \tag{3.9}$$

for any $v \in H^s(\mathbb{R}^n)$ with some constant $c_0 > 0$ independent of v. As a consequence,

$$\mathcal{B}_q(v, v) + \lambda_0(v, v)_{L^2} \geq c_0 \|v\|^2_{\widetilde{H}^s(\Omega)}, \tag{3.10}$$

for any $v \in \widetilde{H}^s(\Omega)$, where λ_0 is a constant such that $\lambda_0 \leq \|q_-\|_{L^\infty(\Omega)}$ with $q_-(x) := -\min\{0, q(x)\}$. On the other hand, it is easy to see from (3.3) that

$$|\mathcal{B}_q(w, v) + \lambda_0(w, v)_{L^2}| \leq (C + \lambda_0) \|w\|_{\widetilde{H}^s(\Omega)} \|v\|_{\widetilde{H}^s(\Omega)}, \tag{3.11}$$

holds for any $w, v \in \widetilde{H}^s(\Omega)$. Hence, we know that the bilinear form $\mathcal{B}_q(\cdot, \cdot) + \lambda_0(\cdot, \cdot)_{L^2}$ is bounded and coercive. Therefore, given any $f \in H^{-s}(\Omega) = \left(\widetilde{H}^s(\Omega)\right)^*$, there is a unique $u \in \widetilde{H}^s(\Omega)$ such that

$$\mathcal{B}_q(u, v) + \lambda_0(u, v)_{L^2} = \langle f, v \rangle \text{ for any } v \in \widetilde{H}^s(\Omega), \tag{3.12}$$

and that

$$\|u\|_{\widetilde{H}^s(\Omega)} \leq C \|f\|_{H^{-s}(\Omega)} \tag{3.13}$$

with some constant C independent of f. Denote by $\mathcal{G}_0$ the operator mapping f to the solution u of (3.12). Then $\mathcal{G}_0$ is bounded from $H^{-s}(\Omega)$ to $\widetilde{H}^s(\Omega)$ with a bounded inverse.

Now, suppose $\widetilde{u}_g \in \widetilde{H}^s(\Omega)$ satisfying (3.7) with $u = \widetilde{u}_g$. Then one has

$$\widetilde{u}_g = \mathcal{G}_0\left(f + (\lambda + \lambda_0)\tilde{u}_g\right),$$

which implies

$$\left(\frac{1}{\lambda + \lambda_0}\mathrm{Id} - \mathcal{G}_0\right)\widetilde{u}_g = \mathcal{G}_0 f, \tag{3.14}$$

where Id denotes the identity map in $\widetilde{H}^s(\Omega)$. By compact Sobolev embedding, it is observed that $\mathcal{G}_0$ is compact in $\widetilde{H}^s(\Omega)$. Thus by the spectral properties of compact operators, $\mathcal{G}_0^{-1}$ has discrete spectrum $\{\frac{1}{\lambda_j+\lambda_0}\}_{j=1}^{\infty}$ consisting only eigenvalues with $\lambda_j \to \infty$ as j increases. Denote $\Sigma = \{\lambda_j\}_{j=1}^{\infty}$. Then by the Fredholm alternative, one has for any $\lambda \notin \Sigma$, the operator

$$\left(\frac{1}{\lambda + \lambda_0}\mathrm{Id} - \mathcal{G}_0\right) : \widetilde{H}^s(\Omega) \to \widetilde{H}^s(\Omega),$$

is injective and has a bounded inverse. Therefore, the equation (3.14) is uniquely solvable, providing $\lambda \notin \Sigma$, with the following estimate of the solution $\widetilde{u}_g$,

$$\|\widetilde{u}_g\|_{\widetilde{H}^s(\Omega)} \leq C\|f\|_{H^{-s}(\Omega)}, \tag{3.15}$$

for some constant $C > 0$ independent of $\widetilde{u}_g$ and f.

The rest of the proof for the statement (1) is completed by considering $\widetilde{u}_g = u - g$. The result in (2) is a direct consequence of (1). Finally, by taking $\lambda_0 = 0$ in the previous arguments, one already sees (3). $\square$

Next we consider the Dirichlet problem (3.1) with a zero RHS, namely,

$$\begin{cases} (\mathcal{L}^s + q)u = 0 & \text{in } \Omega, \\ u = g & \text{in } \Omega_e. \end{cases} \tag{3.16}$$

In the rest of the paper, we shall always assume that $q \in L^\infty(\Omega)$ satisfies (1.5), or equivalently, $0 \notin \Sigma$ with the set Σ given in Proposition 3.3. Under this assumption, it is shown in Proposition 3.3 that given any $g \in H^s(\mathbb{R}^n)$, the Dirichlet problem (3.16) admits a unique solution $u \in H^s(\mathbb{R}^n)$ such that

$$\|u\|_{H^s(\mathbb{R}^n)} \leq C\|g\|_{H^s(\mathbb{R}^n)}. \tag{3.17}$$

Recall that $u \in H^s(\mathbb{R}^n)$ is called a solution of (3.16) if $u - g \in \widetilde{H}^s(\Omega)$ and $\mathcal{B}_q(u, v) = 0$ for any $v \in \widetilde{H}^s(\Omega)$. We emphasize the following proposition before ending this subsection.

Proposition 3.4. *The solution $u \in H^s(\mathbb{R}^n)$ of* (3.1) *does not depend on the value of $g \in H^s(\mathbb{R}^n)$ in Ω, it depends only on $g|_{\Omega_e}$.*

Proof. Let $g_1, g_2 \in H^s(\mathbb{R}^n)$ be such that $g_1 - g_2 \in \widetilde{H}^s(\Omega) = H^s_{\overline{\Omega}}$. Denote $u_j \in H^s(\mathbb{R}^n)$ as the solution of (3.16) with the Dirichlet data g_j for each $j = 1, 2$. It is observed that

$$\widetilde{u} := u_1 - u_2 = (u_1 - g_1) - (u_2 - g_2) + (g_1 - g_2) \in \widetilde{H}^s(\Omega)$$

and $\mathcal{B}_q(\widetilde{u}, v) = 0$ for any $v \in \widetilde{H}^s(\Omega)$. Thus by the unique solvability of (3.16) with $g = 0$ one has $\widetilde{u} = 0$. □

From Proposition 3.4, one can actually consider the nonlocal problem (3.16) with Dirichlet data in the quotient space

$$X := H^s(\mathbb{R}^n)/H^s_{\overline{\Omega}} \cong H^s(\Omega_e), \tag{3.18}$$

provided that Ω is Lipschitz.

3.2 The DN Map

We define in this section the associated DN map for $\mathcal{L}^s + q$ via the bilinear form $\mathcal{B}_q$ in (3.5).

Proposition 3.5. *(DN map) Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$ for $n \geq 2$, $s \in (0,1)$ and $q \in L^\infty(\Omega)$ satisfy the eigenvalue condition* (1.5). *Let X be the quotient space given in* (3.18). *Define*

$$\langle \Lambda_q[g], [h] \rangle := \mathcal{B}_q(u, h), \quad [g], [h] \in X, \tag{3.19}$$

where $g, h \in H^s(\mathbb{R}^n)$ are representatives of the classes $[g], [h] \in X$ respectively, and $u \in H^s(\mathbb{R}^n)$ is the solution of (3.16) *with the Dirichlet data g. Then,*

$$\Lambda_q : X \to X^*,$$

which is bounded. Moreover, we have the following symmetry property for Λ_q,

$$\langle \Lambda_q[g], [h] \rangle = \langle \Lambda_q[h], [g] \rangle, \quad [g], [h] \in X. \tag{3.20}$$

Proof. We first show that Λ_q given in (3.19) is well-defined. Recall from Remark 3.4 that, the solution to (3.16) with Dirichlet data $\widetilde{g} \in H^s(\mathbb{R}^n)$ is the same as the solution with data g, as long as $\widetilde{g} - g \in \widetilde{H}^s(\Omega)$. Thus the RHS of (3.19) is invariant under the different choices of the representative $g \in H^s(\mathbb{R}^n)$ for $[g] \in X$. In addition, one has

$$\mathcal{B}_q(u, \widetilde{h}) = \mathcal{B}_q(u, h) + \mathcal{B}_q(u, \widetilde{h} - h) = \mathcal{B}_q(u, h),$$

for any $\widetilde{h} \in H^s(\mathbb{R}^n)$ such that $\widetilde{h} - h \in \widetilde{H}^s(\Omega)$. Therefore, the RHS of (3.19) is well determined by $[g], [h] \in X$.

From the boundedness (3.3) of $\mathcal{B}_q$, one has

$$\begin{aligned} |\langle \Lambda_q[g], [h] \rangle| &\leq C \|g_0\|_{H^s(\mathbb{R}^n)} \|h_0\|_{H^s(\mathbb{R}^n)} \\ &\leq C_1 \|[g]\|_X \|[h]\|_X, \end{aligned}$$

by properly choosing representatives $g_0, h_0 \in H^s(\mathbb{R}^n)$ for $[g], [h] \in X$. The symmetry of X is a direct consequence of the symmetry of the bilinear form $\mathcal{B}_q$. The proof is completed. □

Recall that the quotient space X is isometric to $H^s(\Omega_e)$, whenever $\Omega \subseteq \mathbb{R}^n$ is a Lipschitz domain. Hence one can always regard the operator Λ_q defined in Proposition 3.5 as

$$\Lambda_q : H^s(\Omega_e) \to (H^s(\Omega_e))^* = H^{-s}_{\overline{\Omega_e}} = \widetilde{H}^{-s}(\Omega_e).$$

In general, for any $\widetilde{h} \in H^s(\mathbb{R}^n)$ we have

$$
\begin{aligned}
(\Lambda_q[g],[h])_{X^*\times X} &= \mathcal{B}_q(u_g,\widetilde{h}) \\
&= \int_{\mathbb{R}^n} \widetilde{h}\mathcal{L}^s u_g dx + \int_\Omega q u_g \widetilde{h} dx \\
&= \int_{\Omega_e} \widetilde{h}\mathcal{L}^s u_g dx \\
&= \int_{\Omega_e} h\mathcal{L}^s u_g \, dx.
\end{aligned} \tag{3.21}
$$

Then from (3.21) we have

$$
(\Lambda_q[g],[h])_{H^{-s}_{\overline{\Omega}_e}(\mathbb{R}^n)\times H^s(\Omega_e)} = \int_{\Omega_e} h\mathcal{L}^s u_g \, dx, \text{ for any } h \in H^s(\Omega_e).
$$

This implies that

$$
\Lambda_q[g] = \mathcal{L}^s u_g|_{\Omega_e}. \tag{3.22}
$$

Let us continue to give another representation of $\Lambda_q[g]$ involving the Neumann operator $\mathcal{N}_s$. We introduce the anisotropic nonlocal Neumann operator $\mathcal{N}_s$ analogues to the Neumann operator which is initiated in [14] for the fractional Laplacian operator $(-\Delta)^s$. Here we define the anisotropic nonlocal Neumann operator $\mathcal{N}_s$ for $\mathcal{L}^s$ over the exterior domain Ω_e as follows:

$$
\mathcal{N}_s u(x) := \int_\Omega \mathcal{K}_s(x,z)(u(x)-u(z))\, dz, \text{ for } x \in \Omega_e \text{ and } u \in H^s(\mathbb{R}^n) \tag{3.23}
$$

where $\mathcal{K}_s(x,z)$ (cf. (2.12)) is the kernel of $\mathcal{L}$ introduced in (2.17).

Lemma 3.6. *Let $\Omega \subseteq \mathbb{R}^n$ as mentioned above. Then*

$$
\Lambda_q[g] = (\mathcal{N}_s u_g - mg + \mathcal{L}^s(E_0 g))|_{\Omega_e}. \tag{3.24}
$$

where $m \in C^\infty(\Omega_e)$ is given by $m(x) := \int_\Omega \mathcal{K}_s(x,z)\, dz$ and E_0 is extension by zero, i.e. $E_0 g = \chi_{\Omega_e} g$.

Proof. Since Ω is a Lipschitz domain, from (3.22) we have:

$$
\Lambda_q[g] = (\mathcal{L}^s u_g)|_{\Omega_e} = (\mathcal{L}^s(\chi_\Omega u_g) + \mathcal{L}^s(\chi_{\Omega_e} u_g))|_{\Omega_e},
$$

as we know if $g \in H^s(\Omega_e)$, then $g \in H^\alpha(\Omega_e)$ for some $\alpha \in (-1/2, 1/2)$ and hence $E_0 g, u_g \in H^\alpha(\mathbb{R}^n)$. Recall also that χ_Ω and $(1-\chi_\Omega)$ are pointwise multipliers on $H^\alpha(\mathbb{R}^n)$ (see [18]). Now from the pointwise definition of $\mathcal{L}^s$ given in (2.17), and the Neumann operator in (3.23) it simply follows that :

$$
(\mathcal{L}^s(\chi_\Omega u_g))|_{\Omega_e} = (\mathcal{N}_s u_g - mg)|_{\Omega_e}
$$

where $m \in C^\infty(\Omega_e)$ is given by $m(x) = \int_\Omega \mathcal{K}_s(x,z)\, dz$. □

Hence, we have two representation of $\Lambda_q[g]$ are given by (3.22) and (3.24).

Remark 3.7. Let $\Omega \subseteq \mathbb{R}^n$ be a bounded open set with C^∞-smooth boundary. Suppose the matrix $A(x)$ given in (2.7) satisfying (1.3), potential $q(x)$ and the source term $g(x)$ are C^∞-smooth functions in $\mathbb{R}^n$, Ω, and Ω_e , respectively. Then for any $\beta \geq 0$ with $s - \frac{1}{2} < \beta < \frac{1}{2}$, the DN map is given by

$$\Lambda_q : H^{s+\beta}(\Omega_e) \to H^{-s+\beta}(\Omega_e), \text{ with } \Lambda_q g = \mathcal{L}^s u_g|_{\Omega_e},$$

where $u_g \in H^{s+\beta}(\mathbb{R}^n)$ is a solution of $(\mathcal{L}^s + q)u = 0$ in Ω with $u = g$ in Ω_e.

Proof. Since $A(x) \in C^\infty(\mathbb{R}^n)$ and $\mathcal{L}^s$ is the fractional operator with C^∞-smooth coefficients of order $2s$, it bounds to satisfy the s-transmission eigenvalue condition given in [21]. Then the proof becomes analogues to the proof of [18, Lemma 3.1] and we omit here. □

We end this section by deriving couple of results regarding the integral identity in our case.

Lemma 3.8. *(Integral identity) Let $\Omega \subseteq \mathbb{R}^n$ as mentioned above, $s \in (0, 1)$ and $q_1, q_2 \in L^\infty(\Omega)$ satisfy (1.5). For any $g_1, g_2 \in H^s(\Omega_e)$ one has*

$$((\Lambda_{q_1} - \Lambda_{q_2})[g_1], [g_2]) = ((q_1 - q_2)r_\Omega u_1, r_\Omega u_2)_{\mathbb{R}^n} \tag{3.25}$$

where $u_j \in H^s(\mathbb{R}^n)$ solves $(\mathcal{L}^s + q_j)u_j = 0$ in Ω with $u_j|_{\Omega_e} = g_j$ for $j = 1, 2$.

Proof. By (3.20), we have

$$\begin{aligned}((\Lambda_{q_1} - \Lambda_{q_2})[g_1], [g_2]) &= (\Lambda_{q_1}[g_1], [g_2]) - ([g_1], \Lambda_{q_2}[g_2])\\ &= B_{q_1}(u_1, u_2) - B_{q_2}(u_1, u_2)\\ &= ((q_1 - q_2)r_\Omega u_1, r_\Omega u_2)_{\mathbb{R}^n}.\end{aligned}$$

□

4 Extension Problems for $\mathcal{L}^s$

In this section, we introduce an extension problem, which characterize the nonlocal operator $\mathcal{L}^s$. For convenience, we introduce the following notations.

Notations in $\mathbb{R}^{n+1}$

We shall always, unless otherwise specified, refer the notation $(x, y) \in \mathbb{R}^{n+1}$ with $x \in \mathbb{R}^n$ and $y \in \mathbb{R}$. Let $\mathbb{R}^{n+1}_+$ be the (open) upper half space of $\mathbb{R}^{n+1}$, namely, $\mathbb{R}^{n+1}_+ := \{(x, y); x \in \mathbb{R}^n, y > 0\}$ and its boundary $\partial\mathbb{R}^{n+1}_+ := \{(x, 0); x \in \mathbb{R}^n\}$. Given any $x_0 \in \mathbb{R}^n$, $(x_0, y_0) \in \mathbb{R}^{n+1}$ and $R > 0$, we denote the balls

$$\begin{aligned}&B(x_0, R) := \{x \in \mathbb{R}^n : \ |x - x_0| < R\} \subset \mathbb{R}^n,\\ &B^{n+1}((x_0, y_0), R) := \left\{(x, y) \in \mathbb{R}^{n+1} : \ \sqrt{|x - x_0|^2 + |y - y_0|^2} < R\right\},\end{aligned}$$

and as $y_0 = 0$, we set

$$\begin{aligned}&B^{n+1}(x_0, R) := B^{n+1}((x_0, 0), R),\\ &B^{n+1}_+(x_0, R) := B^{n+1}(x_0, R) \cap \{y > 0\}.\end{aligned}$$

Let $\mathcal{D}$ be a Lipschitz domain in $\mathbb{R}^{n+1}$. Let w be an arbitrary A_2 Muckenhoupt weight function (cf. [16, 31]) and we denote $L^2(\mathcal{D}, w)$ to be the weighted Sobolev space containing all functions U which are defined a.e. in $\mathcal{D}$ such that

$$\|U\|_{L^2(\mathcal{D},w)} := \left(\int_{\mathcal{D}} w|U|^2 dxdy\right)^{1/2} < \infty.$$

Define

$$H^1(\mathcal{D}, w) := \{U \in L^2(\mathcal{D}, w);\ \nabla_{x,y} U \in L^2(\mathcal{D}, w)\},$$

where $\nabla_{x,y} := (\nabla, \partial_y) = (\nabla_x, \partial_y)$ is the total derivative in $\mathbb{R}^{n+1}$. It is easy to see that $L^2(\mathcal{D}, w)$ and $H^1(\mathcal{D}, w)$ are Banach spaces with respect to the norms $\|\cdot\|_{L^2(\mathcal{D},w)}$ and

$$\|U\|_{H^1(\mathcal{D},w)} := \left(\|U\|^2_{L^2(\mathcal{D},w)} + \|\nabla_{x,y} U\|^2_{L^2(\mathcal{D},w)}\right)^{1/2},$$

respectively. We shall also make use of the weighted Sobolev space $H^1_0(\mathcal{D}, w)$ which is the closure of $C_0^\infty(\mathcal{D})$ under the $H^1(\mathcal{D}, w)$ norm.

In this work, we will consider the weight function w to be y^{1-2s}, $|y|^{1-2s}$, y^{2s-1} and $|y|^{2s-1}$. It is known (cf. [27]) that $y^{1-2s}, y^{2s-1} \in A_2$ for $s \in (0,1)$ in $\mathbb{R}^{n+1}_+$ and $|y|^{1-2s}$, $|y|^{2s-1} \in A_2$ in $\mathbb{R}^{n+1}$.

Let us consider the following extension problem in $\mathbb{R}^{n+1}_+$

$$\begin{cases} -\mathcal{L}_x U + \frac{1-2s}{y} U_y + U_{yy} = 0 & \text{in } \mathbb{R}^{n+1}_+, \\ U(\cdot, 0) = u(x) & \text{on } \partial\mathbb{R}^{n+1}_+. \end{cases} \tag{4.1}$$

The extension problem is related to the nonlocal equation (1.4), where the nonlocal operator $\mathcal{L}^s$ has been regarded as a Dirichlet-to-Neumann map of the above degenerate local problem (4.1). For convenience, we introduce an auxiliary matrix-valued function $\widetilde{A} : \mathbb{R}^n \to \mathbb{R}^{(n+1)\times(n+1)}$ by

$$\widetilde{A}(x) = \begin{pmatrix} A(x) & 0 \\ 0 & 1 \end{pmatrix}. \tag{4.2}$$

We introduce the following degenerate local operator by

$$\mathscr{L}^{1-2s}_{\widetilde{A}} = \nabla_{x,y} \cdot (y^{1-2s} \widetilde{A}(x) \nabla_{x,y}). \tag{4.3}$$

It can be seen that $y^{-1+2s}\mathscr{L}^{1-2s}_{\widetilde{A}}$ is nothing but the above degenerate local operator introduced in (4.1) as

$$\mathscr{L}^{1-2s}_{\widetilde{A}} = y^{1-2s}\left\{\nabla \cdot (A(x)\nabla) + \frac{1-2s}{y}\partial_y + \partial_y^2\right\}.$$

4.1 Basic properties for the extension problem

Let us begin with the following solvability result of the extension for $\mathcal{L}$, where $\mathcal{L}$ is a second order elliptic operator $\mathcal{L} = -\nabla \cdot (A(x)\nabla)$. Recall that the fractional Sobolev space $H^s(\mathbb{R}^n)$ can be realized as a trace space of the weighted Sobolev space $H^1(\mathbb{R}^{n+1}_+, y^{1-2s})$ for $s \in (0,1)$ (see [48]), i.e., for a given $u \in H^s(\mathbb{R}^n)$, there exists $U_0(x,y) \in H^1(\mathbb{R}^{n+1}_+, y^{1-2s})$ such that $U_0(x,0) = u(x) \in H^s(\mathbb{R}^n)$ with

$$\|U_0\|_{H^1(\mathbb{R}^{n+1}_+, y^{1-2s})} \leq C\|u\|_{H^s(\mathbb{R}^n)}. \tag{4.4}$$

For given $u \in H^s(\mathbb{R}^n)$ and define $H^1_0(\mathbb{R}^{n+1}_+, y^{1-2s}) := \{U \in H^1(\mathbb{R}^{n+1}_+, y^{1-2s}) : U = 0 \text{ on } \partial\mathbb{R}^{n+1}_+\}$, then we say $U(x,y) \in H^1(\mathbb{R}^{n+1}_+, y^{1-2s})$ is a weak solution of the Dirichlet boundary value problem (4.1) whenever $\mathscr{U} := U - U_0 \in H^1_0(\mathbb{R}^{n+1}_+, y^{1-2s})$

$$\begin{aligned} &\int_{\mathbb{R}^{n+1}_+} y^{1-2s}\widetilde{A}(x)\nabla_{x,y}\mathscr{U} \cdot \nabla_{x,y}\phi dxdy \\ = &-\int_{\mathbb{R}^{n+1}_+} y^{1-2s}\widetilde{A}(x)\nabla_{x,y}U_0 \cdot \nabla_{x,y}\phi dxdy, \end{aligned} \tag{4.5}$$

for all $\phi \in C^\infty_c(\mathbb{R}^{n+1}_+)$. The solution $U(x,y) \in H^1(\mathbb{R}^{n+1}_+, y^{1-2s})$ can be also characterized as a unique minimizer of the Dirichlet functional

$$\min_{\Psi \in H^1(\mathbb{R}^{n+1}_+, y^{1-2s})} \left\{ \int_{\mathbb{R}^{n+1}_+} y^{1-2s}\widetilde{A}(x)\nabla_{x,y}\Psi \cdot \nabla_{x,y}\Psi dxdy : \quad \Psi(x,0) = u(x) \right\}.$$

The existence and the uniqueness for the Dirichlet problem with zero exterior data is given in the Appendix. First, we have the following uniqueness result.

Lemma 4.1. *(Unique extension) Let $s \in (0,1)$ and let $\widetilde{A}$ be given by (4.2) with $A(x)$ satisfying (1.3). Given any $u \in H^s(\mathbb{R}^n)$, there exists a unique solution $U \in H^1(\mathbb{R}^{n+1}_+, y^{1-2s})$ of*

$$\begin{cases} \mathscr{L}^{1-2s}_{\widetilde{A}} U = 0 & \text{in } \mathbb{R}^{n+1}_+, \\ U(\cdot, 0) = u & \text{in } \mathbb{R}^n. \end{cases} \tag{4.6}$$

Proof. It is known from [33, 48] that $H^s(\mathbb{R}^n)$ can be regarded as the trace space of $H^1(\mathbb{R}^{n+1}_+, y^{1-2s})$ on $\partial\mathbb{R}^{n+1}_+$. Therefore, one can find a function V in the space $H^1(\mathbb{R}^{n+1}_+, y^{1-2s})$ such that $V(x,0) = u(x)$ for $x \in \mathbb{R}^n$. It is then verified by [16, Theorem 2.2] that there is a unique solution $U \in H^1(\mathbb{R}^{n+1}_+, y^{1-2s})$ of $\mathscr{L}^{1-2s}_{\widetilde{A}} U = 0$ such that $U - V \in H^1_0(\mathbb{R}^{n+1}_+, y^{1-2s})$. Thus, the existence of solution to (4.6) has been proven. The uniqueness is then a simple consequence of [16, Theorem 2.2]. □

Next, we demonstrate the following stability estimate.

Lemma 4.2. *(Stability estimate) Let u and U be the same as in Lemma (4.1), then the stability estimate is given by*

$$\|U\|_{H^1(\mathbb{R}^{n+1}_+, y^{1-2s})} \leq C\|u\|_{H^s(\mathbb{R}^n)}. \tag{4.7}$$

for some $C > 0$ independent of u and U.

Proof. Given $u \in H^s(\mathbb{R}^n)$, there exists $U_0(x,y) \in H^1(\mathbb{R}^{n+1}_+, y^{1-2s})$ such that $U_0(x,0) = u(x)$. Since $U \in H^1(\mathbb{R}^{n+1}_+, y^{1-2s})$ is a weak solution of (4.6), let $V := U - U_0$, then $V \in H^1(\mathbb{R}^{n+1}_+, y^{1-2s})$ is a weak solution of

$$\begin{cases} \nabla \cdot (y^{1-2s}\widetilde{A}\nabla V) = \nabla \cdot G & \text{in } \mathbb{R}^{n+1}_+, \\ V(x,0) = 0 & \text{in } \mathbb{R}^n, \end{cases}$$

where $G := -y^{1-2s}\widetilde{A}(x)\nabla_{x,y}U_0$. It is easy to see that $y^{2s-1}G \in L^2(\mathbb{R}^{n+1}_+, y^{1-2s})$ and

$$\begin{aligned}\int_{\mathbb{R}^{n+1}_+} y^{1-2s}|y^{2s-1}G|^2 dxdy &= \int_{\mathbb{R}^{n+1}_+} y^{1-2s}\left|\widetilde{A}\nabla_{x,y}U_0\right|^2 dxdy \\ &\leq C\int_{\mathbb{R}^{n+1}_+} y^{1-2s}|\nabla_{x,y}U_0|^2 dxdy,\end{aligned}$$

for some universal constant $C > 0$. Thus, from (7.2) in Appendix, we know

$$\begin{aligned}\|V\|_{H^1(\mathbb{R}^{n+1}_+, y^{1-2s})} &\leq C\|y^{-1+2s}G\|_{L^2(\mathbb{R}^{n+1}_+, y^{1-2s})} \\ &\leq C\|\mathscr{U}_0\|_{H^1(\mathbb{R}^{n+1}_+, y^{1-2s})},\end{aligned}$$

for some constant $C > 0$ and the last inequality comes the trace estimate (4.4). Finally, by $U = V + U_0$ and the trace estimate (4.4) again, we conclude that there is a constant $C > 0$ such that

$$\|U\|_{H^1(\mathbb{R}^{n+1}_+, y^{1-2s})} \leq C\|u\|_{H^s(\mathbb{R}^n)},$$

which finished the proof. □

We observe that since from the standard elliptic regularity theory, we get that U is C^∞-smooth in $\mathbb{R}^{n+1}_+$. Consequently, by using the standard weak formulation method, we can obtain that $y^{1-2s}\partial_y U$ converges to some function $h \in H^{-s}(\mathbb{R}^n)$ as $y \to 0$ in $H^{-s}(\mathbb{R}^n)$ as

$$(h, \phi(x,0))_{H^{-s}(\mathbb{R}^n)\times H^s(\mathbb{R}^n)} = \int_{\mathbb{R}^{n+1}_+} y^{1-2s}\widetilde{A}(x)\nabla_{x,y}U \cdot \nabla_{x,y}\phi dxdy, \tag{4.8}$$

for all $\phi \in H^1(\mathbb{R}^{n+1}_+, y^{1-2s})$. In other words, $U \in H^1(\mathbb{R}^{n+1}_+, y^{1-2s})$ is a weak solution of the Neumann boundary value problem

$$\begin{cases}\nabla_{x,y}\cdot(y^{1-2s}\widetilde{A}(x)\nabla_{x,y}U) = 0 & \text{in } \mathbb{R}^{n+1}_+, \\ \lim_{y\to 0} y^{1-2s}\partial_y U = h & \text{in } \mathbb{R}^n \times \{0\}.\end{cases} \tag{4.9}$$

The following proposition characterizes $\lim_{y\to 0} y^{1-2s}\partial_y U = h$ as $d_s h = \mathcal{L}^s u$, for some constant d_s depending on s, which connects the nonlocal problem and the extension problem.

Proposition 4.3. *Given $u \in H^s(\mathbb{R}^n)$, define*

$$U(x,y) := \int_{\mathbb{R}^n} P^s_y(x,z)u(z)dz, \tag{4.10}$$

where P^s_y is the Poisson kernel given by

$$P^s_y(x,z) = \frac{y^{2s}}{4^s\Gamma(s)}\int_0^\infty e^{-\frac{y^2}{4t}} p_t(x,z)\frac{dt}{t^{1+s}}, \quad x,z \in \mathbb{R}^n,\ y > 0, \tag{4.11}$$

with the heat kernel p_t introduced in Section 2.3. Then $U \in H^1(\mathbb{R}^{n+1}_+, y^{1-2s})$ and is the weak solution of (4.6) *and*

$$\lim_{y\to 0+}\frac{U(\cdot,y) - U(\cdot,0)}{y^{2s}} = \frac{1}{2s}\lim_{y\to 0+} y^{1-2s}\partial_y U(\cdot,y) = \frac{\Gamma(-s)}{4^s\Gamma(s)}\mathcal{L}^s u, \tag{4.12}$$

in $H^{-s}(\mathbb{R}^n)$.

Remark 4.4. Note that Stinga and Torrea [44] proved the equality (4.12) for $u \in \mathrm{Dom}(\mathcal{L}^s)$. Here we extend such results for all $u \in H^s(\mathbb{R}^n)$.

Proof of Proposition 4.3. From [44, Theorem 2.1], we know that

$$U(x,y) := \int_{\mathbb{R}^n} P_y^s(x,z)u(z)dz$$

solves the equation (4.6). From Lemma 4.1, we know that $U \in H^1(\mathbb{R}^{n+1}_+, y^{1-2s})$ due to $u \in H^s(\mathbb{R}^n)$. It remains to demonstrate that (4.12) for $u \in H^s(\mathbb{R}^n)$.

Firstly, we prove $\lim_{y\to 0^+} \dfrac{U(\cdot,y)-U(\cdot,0)}{y^{2s}} = \dfrac{\Gamma(-s)}{4^s\Gamma(s)}\mathcal{L}^s u$, for any $v \in H^s(\mathbb{R}^n)$. By using (7.6) in Appendix, we can deduce

$$\begin{aligned}
&\left\langle \lim_{y\to 0} \frac{U(\cdot,y)-U(\cdot,0)}{y^{2s}}, v \right\rangle_{H^{-s}(\mathbb{R}^n)\times H^s(\mathbb{R}^n)} \\
=&\int_{\mathbb{R}^n} \lim_{y\to 0} \left[\frac{\int_{\mathbb{R}^n} P_y^s(x,z)u(z)dz - u(x)}{y^{2s}} \right] v(x)dx \\
=&\int_{\mathbb{R}^n} \lim_{y\to 0} \frac{\int_{\mathbb{R}^n} P_y^s(x,z)\left(u(z)-u(x)\right)v(x)dz}{y^{2s}} dx \\
=&\frac{1}{4^s\Gamma(s)} \int_{\mathbb{R}^n} \lim_{y\to 0^+} \lim_{\epsilon\to 0^+} \int_{|z-x|>\epsilon} \int_0^\infty e^{-\frac{y^2}{4t}} p_t(x,z)\left(u(z)-u(x)\right)v(x)\frac{dt}{t^{1+s}} dzdx \\
=&\frac{1}{4^s\Gamma(s)} \int_{\mathbb{R}^n} \lim_{\epsilon\to 0^+} \int_{|z-x|>\epsilon} \left(\int_0^\infty p_t(x,z)\frac{dt}{t^{1+s}} \right) \left(u(z)-u(x)\right)v(x)dzdx \\
=&\frac{\Gamma(-s)}{4^s\Gamma(s)} \int_{\mathbb{R}^n} \lim_{\epsilon\to 0^+} \int_{|z-x|>\epsilon} \mathcal{K}_s(x,z)\left(u(z)-u(x)\right)v(x)dzdx \\
=&\frac{\Gamma(-s)}{4^s\Gamma(s)} \left\langle \mathcal{L}^s u, v\right\rangle_{H^{-s}(\mathbb{R}^n)\times H^s(\mathbb{R}^n)}.
\end{aligned}$$

Secondly, we prove $\dfrac{1}{2s}\lim_{y\to 0^+} y^{1-2s}\partial_y U(\cdot,y) = \dfrac{\Gamma(-s)}{4^s\Gamma(s)}\mathcal{L}^s u$ by utilizing the density argument. It is known that for $u \in \mathrm{Dom}(\mathcal{L}^s)$, then (4.12) holds in $L^2(\mathbb{R}^n)$. Consider a sequence $\{u_k\}_{k\in\mathbb{N}} \subseteq \mathrm{Dom}(\mathcal{L}^s)\cap H^s(\mathbb{R}^n)$ such that $u_k \to u$ in $H^s(\mathbb{R}^n)$ as $k \to \infty$. Let $U_k \in H^1(\mathbb{R}^{n+1}_+, y^{1-2s})$ be the solution to (4.1) with the boundary data u_k for each $k \in \mathbb{N}$. Recall from (2.5) and (2.8) that $C_0^\infty(\mathbb{R}^n) \subseteq \mathrm{Dom}(\mathcal{L}^s)$. Thus by [44, Theorem 1.1] and Lemma 4.1, $U_k(x,y)$ can be uniquely represented by

$$U_k(x,y) = \int_{\mathbb{R}^n} P_y^s(x,z)u_k(z)dz, \text{ for } x \in \mathbb{R}^n \text{ and } y \in \mathbb{R}_+. \tag{4.13}$$

Moreover, the following relation

$$\frac{1}{2s}\lim_{y\to 0+} y^{1-2s}\partial_y U_k(\cdot,y) = \frac{\Gamma(-s)}{4^s\Gamma(s)}\mathcal{L}^s u_k \tag{4.14}$$

holds in $L^2(\mathbb{R}^n)$ by [44, Theorem 1.1] again. Following that, from (4.8) or (4.9), we conclude for being $U \in H^1(\mathbb{R}^{n+1}_+, y^{1-2s})$, $\dfrac{1}{2s}\lim_{y\to 0^+} y^{1-2s}\partial_y U(x,y) = h$ exists in $H^{-s}(\mathbb{R}^n)$. For convenience, we set $h_k := \frac{\Gamma(-s)}{4^s\Gamma(s)}\mathcal{L}^s u_k$. Note that $U_k - U$

solves the equation $\mathscr{L}^{1-2s}_{\widetilde{A}}(U_k - U) = 0$ in $\mathbb{R}^{n+1}_+$ with $(U_k - U)(x, 0) = (u_k - u)(x)$, by the stability estimate (4.7), we get

$$\|U_k - U\|_{H^1(\mathbb{R}^{n+1}_+, y^{1-2s})} \leq C\|u_k - u\|_{H^s(\mathbb{R}^n)},$$

for some constant $C > 0$ independent of U_k, U, u_k and u. Hence, $U_k \to U$ in $H^1(\mathbb{R}^{n+1}_+, y^{1-2s})$ due to $u_k \to u$ in $H^s(\mathbb{R}^n)$ as $k \to \infty$. On the other hand, by using the weak formulation (4.8), for any $\phi \in H^1(\mathbb{R}^{n+1}_+, y^{1-2s})$ (recall that $\phi(x, 0) \in H^s(\mathbb{R}^n)$ by the trace characterization), $U_k - U$ satisfies

$$\int_{\mathbb{R}^{n+1}_+} y^{1-2s}\widetilde{A}(x)\nabla_{x,y}(U_k - U) \cdot \nabla_{x,y}\phi dxdy = ((h_k - h), \phi(x, 0))_{H^{-s}(\mathbb{R}^n)\times H^s(\mathbb{R}^n)} \cdot$$

From $U_k \to U$ in $H^1(\mathbb{R}^{n+1}_+, y^{1-2s})$ as $k \to \infty$, we conclude that $h_k \to h$ in $H^{-s}(\mathbb{R}^n)$ as $k \to \infty$. Finally, by the integral representation (2.14) for $\mathcal{L}^s$ and $u_k \to u$ in $H^s(\mathbb{R}^n)$ as $k \to \infty$, we can derive that

$$\begin{aligned}
&(h_k, \phi(x, 0))_{H^{-s}(\mathbb{R}^n)\times H^s(\mathbb{R}^n)} \\
=&\frac{\Gamma(-s)}{4^s\Gamma(s)} (\mathcal{L}^s u_k, \phi(x, 0))_{H^{-s}(\mathbb{R}^n)\times H^s(\mathbb{R}^n)} \\
=&\frac{1}{2}\frac{\Gamma(-s)}{4^s\Gamma(s)} \int_{\mathbb{R}^n\times\mathbb{R}^n} (u_k(x) - u_k(z))(\phi(x, 0) - \phi(z, 0))\mathcal{K}_s(x, z)dxdz \\
\to&\frac{1}{2}\frac{\Gamma(-s)}{4^s\Gamma(s)} \int_{\mathbb{R}^n\times\mathbb{R}^n} (u(x) - u(z))(\phi(x, 0) - \phi(z, 0))\mathcal{K}_s(x, z)dxdz \\
=&\frac{\Gamma(-s)}{4^s\Gamma(s)} (\mathcal{L}^s u, \phi(x, 0))_{H^{-s}(\mathbb{R}^n)\times H^s(\mathbb{R}^n)},
\end{aligned}$$

as $k \to \infty$. By using the uniqueness limit of h_k, then we can conclude that $h = \frac{\Gamma(-s)}{4^s\Gamma(s)}\mathcal{L}^s u$ in $H^{-s}(\mathbb{R}^n)$. Therefore we have verified (4.12) and thus completes the proof. □

Next, we recall the well-known reflection extension for the extension problem.

4.2 Even reflection extension and its related regularity properties

Similar to the fractional Laplacian case, we also have the following reflection property for fractional variable operators. Let us consider $B^{n+1}_+(x_0, R) \subset \mathbb{R}^{n+1}_+$, where $\lim_{y\to 0} y^{1-2s}\partial_y U = 0$ in $B(x_0, R) \subset \Omega$. Now by using the even reflection, we define

$$\widetilde{U}(x, y) := \begin{cases} U(x, y) & \text{if } y \geq 0, \\ U(x, -y) & \text{if } y < 0, \end{cases} \tag{4.15}$$

where $U(x, y)$ solves (4.1). Then $\widetilde{U}(x, y)$ is a solution of the following problem

$$\nabla_{x,y} \cdot (|y|^{1-2s}\widetilde{A}(x)\nabla_{x,y}\widetilde{U}) = 0 \text{ in } B^{n+1}(x_0, R). \tag{4.16}$$

In general, for

$$\nabla_{x,y} \cdot (|y|^{1-2s}\widetilde{A}(x)\nabla_{x,y}V) = \nabla_{x,y} \cdot G \text{ in } B^{n+1}(x_0, R),$$

where G is a vector-valued function satisfies $|y|^{-1+2s}|G| \in L^2(B^{n+1}(x_0,R),|y|^{1-2s})$, then we say V to be a weak solution of the above equation if

$$\int_{B^{n+1}(x_0,R)} |y|^{1-2s}\widetilde{A}(x)\nabla_{x,y}V\cdot\nabla_{x,y}\phi\, dxdy = \int_{B^{n+1}(x_0,R)} G\cdot\nabla_{x,y}\phi\, dxdy,$$

for all $\phi\in C_c^\infty(B^{n+1}(x_0,R))$. We have the following regularity results.

Proposition 4.5. *(a) (Solvability) Let $D\subset\mathbb{R}^{n+1}$ be a bounded domain with C^∞-smooth boundary and G is a vector-valued function satisfies $|y|^{-1+2s}|G|\in L^2(D,|y|^{1-2s})$. Let $g\in H^1(D,|y|^{1-2s})$, then there is a unique solution $V\in H^1(D,|y|^{1-2s})$ of*

$$\nabla_{x,y}\cdot(|y|^{1-2s}\widetilde{A}(x)\nabla_{x,y}V) = \nabla_{x,y}\cdot G \text{ in } D, \tag{4.17}$$

with $V-g\in H_0^1(D,|y|^{1-2s})$.

(b) (Interior Hölder's regularity) Let $D\subset\mathbb{R}^{n+1}$ be a bounded domain with C^∞-smooth boundary and let V be a weak solution to (4.17) where G is a vector-valued function satisfies $|y|^{-1+2s}|G|\in L^{2(n+1)}(D,|y|^{1-2s})$. Then $V\in C^{0,\beta}(D')$ for some $\beta\in(0,1)$ depending on n and s, where $D'\Subset D$ is an arbitrary open set.

(c) (Higher regularity in the x-direction) Let $D\subset\mathbb{R}^{n+1}$ be a bounded domain with C^∞-smooth boundary and let $V\in H^1(D,|y|^{1-2s})$ be a weak solution of

$$\nabla_{x,y}\cdot(|y|^{1-2s}\widetilde{A}(x)\nabla_{x,y}V) = 0 \text{ in } D. \tag{4.18}$$

Then for each fixed $y=y_0$ with $(x,y_0)\in D'$, we have $V(x,y_0)\in C^\infty(D'\cap\{y=y_0\})$ in any open subset $D'\Subset D$, since $A(x)$ is a C^∞-smooth matrix-valued function in $\mathbb{R}^n$ satisfying (1.3).

Proof. The proof of (a) has been established in [16] for $|y|^{1-2s}$ being an A_2 function and the proof of (b) is a direct consequence of [16, Theorem 2.3.12]. We move into showing (c). Set $\Delta_{x_i}^h$ to be the classical difference quotient operator, which means

$$\Delta_{x_i}^h V(x,y) := \frac{V(x+he_i,y)-V(x,y)}{h} \text{ for any } i=1,2,\cdots,n,$$

where we have fixed $y>0$. From straightforward calculation, if $V(x)$ solves (4.18), we get $\Delta_{x_i}^h V(x)$ solves

$$\nabla_{x,y}\cdot(|y|^{1-2s}\widetilde{A}(x+he_i)\nabla_{x,y}(\Delta_{x_i}^h V)) = \nabla_{x,y}\cdot G \text{ in } D_h, \tag{4.19}$$

where the function $G=-|y|^{1-2s}(\Delta_{x_i}^h\widetilde{A})\nabla_{x,y}V(x,y)$ and $D_h\Subset D$ is an arbitrary subset such that the Hausdorff distance between D and D_h greater than h. Note that in the right hand side of (4.19) satisfies the condition $|y|^{-1+2s}G\in L^2(D_h,|y|^{1-2s})$ since $V\in H^1(D,|y|^{1-2s})$. Hence, from (a) and the standard cutoff techniques, we know that $\Delta_{x_i}^h V\in H^1(D_h,|y|^{1-2s})$ and

$$\|\Delta_{x_i}^h V\|_{H^1(D',|y|^{1-2s})}\le K<\infty, \tag{4.20}$$

for any subset $D'\Subset D_h$ and it is easy to see that the constant $K>0$ independent of h since $A(x)$ is a C^∞-smooth matrix-valued function in $\mathbb{R}^n$ satisfying (1.3), such that $\left|\Delta_{x_i}^h \widetilde{A}(x)\right| \leq \|\partial_{x_i}\widetilde{A}\|_{L^\infty(D)} \leq C < \infty$. Recall that $H^1(D, |y|^{1-2s})$ is a reflexive Banach space, by using the same argument as in [19, Lemma 7.24], then $\partial_{x_i} V \in H^1(D', |y|^{1-2s})$ for $i = 1, 2, \cdots, n$ with $\|\nabla_{x,y}(\partial_{x_i} V)\|_{L^2(D', |y|^{1-2s})} \leq K < \infty$, where $K > 0$ is the same constant as in (4.20).

Continue this process, we can apply the difference quotient with respect to the x-direction for any order, then one can derive that $\partial_x^\alpha V \in H^1(D', |y|^{1-2s})$, for any multi-index $\alpha \in (\mathbb{N} \cup \{0\})^n$. Now, since $\partial_x^\alpha V(x, y) \in H^1(D', |y|^{1-2s})$, by using the trace theorem for the weighted Sobolev space again, we have $\partial_x^\alpha V(x, 0) \in H^s(D' \cap \{y = 0\})$ for any $\alpha \in \mathbb{N}^n$, or $V(x, 0) \in H^{m+s}(D' \cap \{y = 0\})$ for any $m \in \mathbb{N} \cup \{0\}$. Now, apply the fractional Sobolev embedding theorem (see [13] for instance), we derive $V(x, 0) \in C^\infty(D' \cap \{y = 0\})$. For each fixed $y = y_0 \neq 0$, the equation (4.18) can be regarded as a standard second order elliptic equation with C^∞-smooth coefficients, by the standard elliptic theory it is easy to see that $V(x, y_0)$ is C^∞-smooth in $D' \cap \{y = y_0\}$ with respect to x. This finishes the proof. □

In the end of this section, we introduce the conjugate equation, which is associated to the degenerate operator $\mathscr{L}_{\widetilde{A}}^{1-2s}$ given by (4.3).

4.3 Conjugate equation and odd reflection

As in [7, Section 2] and [44, Section 2], it is known that if $U \in H^1(\mathbb{R}^{n+1}_+, y^{1-2s})$ is a weak solution to $\mathscr{L}_{\widetilde{A}}^{1-2s} U = 0$ in $\mathbb{R}^{n+1}_+$ then the function

$$W(x, y) := y^{1-2s}\partial_y U(x, y)$$

is a solution to the conjugate equation

$$\mathscr{L}_{\widetilde{A}}^{-1+2s} W = y^{1-2s}\left(-\mathcal{L}_x W + \frac{1-2s}{y} W_y + W_{yy}\right) = 0 \text{ in } \mathbb{R}^{n+1}_+. \tag{4.21}$$

If we assume that $W(x, 0) = 0$ for $x \in B(x_0, R) \subset \Omega$ and use the odd reflection, we define

$$\widetilde{W}(x, y) := \begin{cases} W(x, y), & \text{if } y \geq 0, \\ -W(x, -y), & \text{if } y < 0. \end{cases} \tag{4.22}$$

Then we will prove that that $\widetilde{W} \in H^1(|y|^{-1+2s}, B^{n+1}(x_0, R))$ is a weak solution of

$$\nabla_{x,y} \cdot (|y|^{-1+2s}\widetilde{A}(x)\nabla_{x,y}\widetilde{W}) = 0 \text{ in } B^{n+1}(x_0, R).$$

By using Proposition 4.5, we say $\widetilde{W} \in C^{0,\beta}(B^{n+1}(x_0, R))$ for some $\beta \in (0, 1)$ depending on n and $1 - 2s$.

Lemma 4.6. *(The Conjugate Equation) Let $U \in H^1(\mathbb{R}^{n+1}_+, y^{1-2s})$ be a weak solution of $\mathscr{L}_{\widetilde{A}}^{1-2s} U = 0$ in $\mathbb{R}^{n+1}_+$ and $\lim_{y \to 0} y^{1-2s}\partial_y U = 0$ in $B(x_0, R)$. Then for any $r < R$, the function $W = y^{1-2s}\partial_y U \in H^1(B^{n+1}_+(x_0, r), y^{2s-1})$ solves the conjugate equation $\mathscr{L}_{\widetilde{A}}^{2s-1} W = 0$ weakly in $B^{n+1}_+(x_0, r)$ with $W(x, 0) = 0$ for $x \in B(x_0, r)$.*

Proof. As previous discussions, we define $\widetilde{U}$ and $\widetilde{W}$ to be even and odd extension by (4.15) and (4.22), respectively. Then $\widetilde{U}$ solves

$$\nabla_{x,y}\cdot(|y|^{1-2s}\widetilde{A}(x)\nabla_{x,y}\widetilde{U})=0 \text{ in } B^{n+1}(x_0,R),$$

where we know the fact that $\widetilde{U}\in C^{0,\beta}(B^{n+1}(x_0,R))$ and $\widetilde{U}(\cdot,y)\in C^{\infty}(B(x_0,R))$ for $y>0$. We will show that $\widetilde{W}\in H^1_{loc}(B^{n+1}(x_0,R),|y|^{2s-1})$ as a weak solution of

$$\nabla_{x,y}\cdot(|y|^{2s-1}\widetilde{A}(x)\nabla_{x,y}\widetilde{W})=0 \text{ in } \mathcal{B}\Subset B^{n+1}(x_0,R), \tag{4.23}$$

First, it is easy to see that $\widetilde{W}\in L^2(B^{n+1}(x_0,R),|y|^{2s-1})$. Second, for $0<h\ll 1$, we consider the difference quotient for $\widetilde{U}$, then $\Delta^h_{x_i}\widetilde{U}\in H^1(B^{n+1}(x_0,R),|y|^{1-2s})$ is a weak solution of

$$\nabla_{x,y}\cdot(|y|^{1-2s}\widetilde{A}(x)\nabla_{x,y}(\Delta^h_{x_i}\widetilde{U}))=\nabla_{x,y}\cdot G \text{ in } B^{n+1}(x_0,R), \tag{4.24}$$

for any $h>0$, where the function $G=-|y|^{1-2s}(\Delta^h_{x_i}\widetilde{A})\nabla_{x,y}\widetilde{U}$ and it is easy to see that $|y|^{2s-1}H\in L^2(B^{n+1}(x_0,R),|y|^{1-2s}))$ for $i=1,2,\cdots,n$. Let $\eta\in C^\infty_c(B^{n+1}(x_0,R))$ be a standard cutoff function such that $0\leq\eta\leq 1$ with

$$\eta(x,y)=\begin{cases}1 & \text{for } x\in B^{n+1}(x_0,\frac{3}{4}R),\\ 0 & \text{for } x\notin B^{n+1}(x_0,R),\end{cases}\text{, and } \|\nabla_{x,y}\eta\|_{L^\infty(B^{n+1}(x_0,R))}\leq\frac{C}{R},$$

for some constant $C>0$. Now, consider $\eta^2\Delta^h_{x_i}\widetilde{U}\in H^1(B^{n+1}(x_0,R),|y|^{1-2s})$ as a test function and multiply it on the both sides of (4.24) and do the integration by parts over $B^{n+1}(x_0,R)$, then we have

$$\begin{aligned}&\int_{B^{n+1}(x_0,R)}|y|^{1-2s}\widetilde{A}(x)\nabla_{x,y}(\Delta^h_{x_i}\widetilde{U})\cdot\nabla_{x,y}(\eta^2\Delta^h_{x_i}\widetilde{U})dxdy\\ =&-\int_{B^{n+1}(x_0,R)}|y|^{1-2s}(\Delta^h_{x_i}\widetilde{A})\nabla_{x,y}\widetilde{U}\cdot\nabla_{x,y}(\eta^2\Delta^h_{x_i}\widetilde{U}).\end{aligned} \tag{4.25}$$

From a direct computation, it is not hard to see that

$$\nabla_{x,y}(\eta^2\Delta^h_{x_i}\widetilde{U})=\eta^2\nabla_{x,y}(\Delta^h_{x_i}\widetilde{U})+2(\eta\Delta^h_{x_i}\widetilde{U})\nabla_{x,y}\eta$$

and (4.25) becomes

$$\begin{aligned}&\int_{B^{n+1}(x_0,R)}|y|^{1-2s}\eta^2\left|\nabla_{x,y}(\Delta^h_{x_i}\widetilde{U})\right|^2dxdy\\ \leq& C\int_{B^{n+1}(x_0,R)}|y|^{1-2s}\left|\nabla_{x,y}(\Delta^h_{x_i}\widetilde{U})\right||\eta|\left|\Delta^h_{x_i}\widetilde{U}\right||\nabla_{x,y}\eta|\,dxdy\\ &+C\int_{B^{n+1}(x_0,R)}|y|^{1-2s}|\eta|^2\left|\nabla_{x,y}\widetilde{U}\right|\left|\nabla_{x,y}(\Delta^h_{x_i}\widetilde{U})\right|dxdy\\ &+C\int_{B^{n+1}(x_0,R)}|y|^{1-2s}\left|\nabla_{x,y}\widetilde{U}\right||\eta|\,|\nabla_{x,y}\eta|\left|\Delta^h_{x_i}\widetilde{U}\right|dxdy.\end{aligned} \tag{4.26}$$

Apply the Young's inequality on (4.26) and absorb the highest order term of $\widetilde{U}$ to the left hand side of (4.26), then we can derive

$$\begin{aligned}&\int_{B^{n+1}(x_0,\frac{3}{4}R)}|y|^{1-2s}\left|\nabla_{x,y}(\Delta^h_{x_i}\widetilde{U})\right|^2 dxdy\\ \leq & C\left\{\int_{B^{n+1}(x_0,R)}|y|^{1-2s}\left|\Delta^h_{x_i}\widetilde{U}\right|^2 dxdy+\int_{B^{n+1}(x_0,R)}|y|^{1-2s}\left|\nabla_{x,y}\widetilde{U}\right|^2 dxdy\right\}\\ \leq & C\|\widetilde{U}\|_{H^1(B^{n+1}(x_0,R),|y|^{1-2s})},\end{aligned}$$

where the constant $C>0$ is independent of $\widetilde{U}$ and h. This implies that $\Delta^h_{x_i}(\nabla_{x,y}\widetilde{U})\in L^2(B^{n+1}(x_0,\frac{3}{4}R),|y|^{1-2s})$ and

$$\|\Delta^h_{x_i}(\nabla_{x,y}\widetilde{U})\|_{L^2(B^{n+1}(x_0,\frac{3}{4}R),|y|^{1-2s})}\leq C, \text{ for } i=1,2,\cdots .n,$$

for some constant $C>0$ is independent of $\widetilde{U}$ and h. Then use the same technique as in Proposition (4.5), then one can conclude $\|\partial_{x_i}(\nabla_{x,y}\widetilde{U})\|_{L^2(B^{n+1}(x_0,\frac{3}{4}R),|y|^{1-2s})}$, which means $\partial_{x_i}\widetilde{U}\in H^1(B^{n+1}(x_0,\frac{3}{4}R),|y|^{1-2s})$ for $i=1,2,\cdots,n$.

It remains to show $\partial_y W\in L^2(B^{n+1}_+(x_0,\frac{3}{4}R),y^{2s-1})$. Note that

$$\partial_y W=\partial_y(y^{1-2s}\partial_y U)=y^{1-2s}\left(\frac{1-2s}{y}\partial_y U+\partial_y^2 U\right)=y^{1-2s}\mathcal{L}_x U.$$

Via the fact that $\partial^2_{x_ix_j}\widetilde{U}\in L^2(B^{n+1}(x_0,\frac{3}{4}R),|y|^{1-2s})$, this implies the lemma holds and completes the proof. □

5 Strong unique continuation principle and the Runge approximation property

In order to prove Theorem 1.2, we will be using the strong unique continuation principle (SUCP) for the extension operator $\mathscr{L}^{1-2s}_{\widetilde{A}}$. Our strategy in proving Theorem 1.2 is decomposed into two parts. First, we will prove under the condition of Theorem 1.2, the solution of the extension problem will vanish to infinite order, which is inspired by the proof of [41, Proposition 2.2]. Second, we apply the SUCP for degenerate differential equation, which was introduced by [51, Corollary 3.9]. Combine these two steps, then we can prove the SUCP for the operator $\mathscr{L}^{1-2s}_{\widetilde{A}}$.

5.1 Strong unique continuation principle

We begin with the definition of the vanishing to infinity order for the degenerate case.

Definition 5.1. (Vanishing to infinite order) A function $\Psi\in L^2_{loc}(\mathbb{R}^{n+1}_+,y^{1-2s})$ is vanishing to infinite order at a point $(x_0,0)\in\mathbb{R}^{n+1}_+$ if for every $m\in\mathbb{N}$, we have

$$\lim_{r\to 0}r^{-m}\int_{B^{n+1}(x_0,r)}|y|^{1-2s}\Psi^2(x,y)\,dxdy=0. \tag{5.1}$$

We begin with the first step: Vanishing to infinite order.

Theorem 5.2. *Given $u \in H^s(\mathbb{R}^n)$, let $U \in H^1(\mathbb{R}^{n+1}_+, y^{1-2s})$ be the unique solution of the extension problem (4.1). Suppose that $u = \mathcal{L}^s u = 0$ in $B(x_0, 2R)$. Then U vanishes to infinite order on $B(x_0, R)$.*

Proof. We will follow ideas of proof of [41, Proposition 2.2].

1. We know from Proposition 4.3 that $U = \lim_{y\to 0} y^{1-2s}\partial_y U = 0$ for $x \in B(x_0, 2R)$. Define $W := y^{1-2s}\partial_y U$. Then by Lemma 4.6 we know that $W \in H^1(B^{n+1}_+(x_0, \frac{3}{2}R), y^{1-2s})$ solves $\mathscr{L}^{2s-1}_{\widetilde{A}} W = 0$ in $B^{n+1}_+(x_0, \frac{3}{2}R)$ with $W(x, 0) = 0$. We define $\widetilde{U}$ and $\widetilde{W}$ given by even reflection (4.15) and odd reflection (4.22), respectively. It is straightforwardly verified that $\widetilde{U} \in H^1\left(B^{n+1}(x_0, 2R), |y|^{1-2s}\right)$ satisfies

$$\nabla_{x,y} \cdot (|y|^{1-2s}\widetilde{A}(x)\nabla_{x,y}\widetilde{U}) = 0 \text{ in } B^{n+1}(x_0, 2R), \tag{5.2}$$

and $\widetilde{W} \in H^1\left(B^{n+1}(x_0, \frac{3}{2}R), |y|^{-1+2s}\right)$ is a solution to

$$\nabla_{x,y} \cdot (|y|^{-1+2s}\widetilde{A}(x)\nabla_{x,y}\widetilde{W}) = 0 \text{ in } B^{n+1}(x_0, \frac{3}{2}R). \tag{5.3}$$

Hence recalling by Proposition 4.5, the functions $\widetilde{U}$ and $\widetilde{W}$ are Hölder continuous in $B^{n+1}(x_0, R)$. As a consequence U and W are both Hölder continuous in $\overline{B^{n+1}_+(x_0, R)}$.

2. It can be seen by using the mean value theorem and the fundamental theorem of calculus, for all $h \in C^1((0,1)) \cap C([0,1])$ and any $a \in (-\infty, 1)$ if $h(0) = 0$ and $\lim_{y\to 0} y^a \frac{d}{dy}h(y) = 0$ then

$$\lim_{y\to 0} y^{a-1}h(y) = 0. \tag{5.4}$$

The remaining proof of this theorem follows the proof of [41, Proposition 2.2]. We divided it into the following three steps arguments.

Step 1. One-step improvement

As a solution to (5.2), we know from Lemma 4.5 that $\widetilde{U}$ is $C^{0,\beta}$ in any direction in $\mathbb{R}^{n+1}$ and C^∞ in the x-direction since $\widetilde{A}(x)$ is C^∞-smooth. Thus, we can differentiate (5.2) with respect to all x_i-direction up to an arbitrary order for $i = 1, 2, \cdots, n$, due to the C^∞-smoothness. By using the continuity of U, we know that

$$\lim_{y\to 0^+} y^{1-2s}\partial_y \left(\nabla_x \cdot (A(x)\nabla_x U)\right) = 0. \tag{5.5}$$

Then (5.4) will imply that

$$\lim_{y\to 0^+} y^{-2s}\nabla_x \cdot (A(x)\nabla_x U) = 0. \tag{5.6}$$

Recall that U satisfies the equation

$$\nabla_{x,y} \cdot (y^{1-2s}\widetilde{A}(x)\nabla_{x,y}U) = 0 \text{ in } B^{n+1}_+(x_0, R), \tag{5.7}$$

or equivalently, U fulfills

$$\partial_y(y^{1-2s}\partial_y U) = -y^{1-2s}\nabla_x \cdot (A(x)\nabla_x U).$$

By using (5.5), we have

$$\lim_{y\to 0^+} \partial_y(y^{1-2s}\partial_y U) = 0.$$

Next, recall that $\lim_{y\to 0} y^{1-2s}\partial_y U = d_s\mathcal{L}^s u = 0$ for some constant d_s and use (5.4) again, then we obtain

$$\lim_{y\to 0^+} y^{-2s}\partial_y U = \lim_{y\to 0^+} y^{-2s-1}U = 0.$$

Step 2. Iteration

Let us differentiate (5.7) with respect to y and consider U to be a weak solution of

$$\begin{aligned} \partial_y^2(y^{1-2s}\partial_y U) &= -(1-2s)y^{-2s}\nabla_x \cdot (A(x)\nabla_x U) \\ &\quad -y^{1-2s}\partial_y\nabla_x \cdot (A(x)\nabla_x U) \text{ in } B_+^{n+1}(x_0,R) \end{aligned} \tag{5.8}$$

with

$$\lim_{y\to 0^+} \partial_y(y^{1-2s}\partial_y U) = 0 \text{ for } x \in B(x_0,R).$$

Plug (5.5) and (5.6) into (5.8), we have

$$\lim_{y\to 0^+} \partial_y^2(y^{1-2s}\partial_y U) = \lim_{y\to 0^+} y^{-2s-2}U = 0.$$

As previous arguments, let us take the function $W(x,y) = y^{1-2s}\partial_y U(x,y)$ with $\lim_{y\to 0^+} W(x,y) = 0$, then we can reflect the function W to be $\widetilde{W}(x,y)$ into a whole ball in $\mathbb{R}^{n+1}$(see 5.3). Since $U(x,y)$ is C^∞-smooth in the x-direction, so is $\widetilde{W}(x,y)$. Therefore, we can differentiate $\widetilde{W}(x,y)$ with respect to x-variables with arbitrary order. Then by repeating Step 1, we will obtain the continuity of $\partial_y\left(\nabla_x \cdot (A(x)\nabla W)\right)$ and

$$\lim_{y\to 0} \partial_y\left(\nabla_x \cdot (A(x)\nabla W)\right) = 0.$$

To sum up, after these iterate procedures and use the x-direction derivatives, then we can get

$$\begin{aligned} \lim_{y\to 0^+} \partial_y\left(y^{1-2s}\partial_y(\nabla_x \cdot (A(x)\nabla_x U))\right) &= \lim_{y\to 0^+} y^{-2s}\partial_y(\nabla_x \cdot (A(x)\nabla_x U)) \\ &= \lim_{y\to 0^+} y^{-2s-1}(\nabla_x \cdot (A(x)\nabla_x U)) = 0. \end{aligned}$$

Note that the right hand sides of these terms is obtained by differentiating (5.8) with y direction (in the weak sense) and they may involve higher order derivatives with respect to x-variables, hence, we can use the bootstrap arguments to proceed previous arguments.

Step 3. Conclusion

By using the bootstrap arguments, we can get

$$\lim_{y\to 0^+} y^{-m}U(x,y) = 0 \text{ for all } m \in \mathbb{N} \text{ and } x \in B(x_0,R), \tag{5.9}$$

which implies that U vanishes to infinite order in the y-direction on the plane $\partial\mathbb{R}^{n+1}_+$, and in the tangential x-direction it is zero on the plane $\partial\mathbb{R}^{n+1}_+$ and this proves the theorem. □

Corollary 5.3. *Let U be the same function as in Theorem 5.2 and $\widetilde{U}$ be the even reflection of U given by (4.15), then the function $\widetilde{U}$ vanishes to infinite order on $B(x_0,R)$.*

Proof. Since $\widetilde{U}$ is an even reflection of U, then we can repeat the same proof as in Theorem 5.2 in the lower half space $\mathbb{R}^{n+1}\cap\{y<0\}$, then we have

$$\lim_{y\to 0^-} (-y)^{-m}U(x,-y) = 0 \text{ for all } m \in \mathbb{N} \text{ and } x \in B(x_0,R). \tag{5.10}$$

Combining (5.9) and (5.10), we obtain that

$$\lim_{y\to 0} |y|^{-m}\widetilde{U}(x,y) = 0 \text{ for all } m \in \mathbb{N} \text{ and } x \in B(x_0,R), \tag{5.11}$$

which completes the proof. □

Proposition 5.4. *[51, Corollary 3.9] Let $\widetilde{U} \in H^1(B^{n+1}(x_0,1),|y|^{1-2s})$ be a solution to*

$$\nabla_{x,y}\cdot(|y|^{1-2s}\widetilde{A}(x)\nabla_{x,y}\widetilde{U}) = 0 \text{ in } B^{n+1}(0,1). \tag{5.12}$$

Then the equation (5.12) *possesses the SUCP, for a C^∞-smooth matrix-valued function $A(x)$ in $\mathbb{R}^n$ satisfying* (1.3).

Recall that the equation (5.12) has the SUCP if $\widetilde{U} \in H^1(B^{n+1}(0,1),|y|^{1-2s})$ is a weak solution of (5.12) and $\widetilde{U}$ vanishes to infinite order, then $\widetilde{U}\equiv 0$ in $B^{n+1}(0,1)$.

Proof of Proposition 5.4. Firstly, the condition of vanishing to infinite order (5.11) shows that $U \in L^2(B(x_0,R)\times(-r_0,r_0),|y|^{1-2s})$ for $R, r_0 \ll 1$, since

$$\int_{B(x_0,R)\times(-r_0,r_0)} |y|^{1-2s}|\widetilde{U}|^2\,dxdy \le \int_{B(x_0,R)\times(-r_0,r_0)} |y|^{-m}|\widetilde{U}|^2\,dxdy < 1 \tag{5.13}$$

for $y \le r_0 \ll 1$ being sufficiently small enough and for any $m\in\mathbb{N}$ with $m\ge 2$.

On the other hand, by using the doubling inequality (7.23) in Appendix, we have

$$\int_{B^{n+1}(x_0,1)} |y|^{1-2s}|\widetilde{U}|^2 dxdy \le C\int_{B^{n+1}(x_0,\frac{1}{2})} |y|^{1-2s}|\widetilde{U}|^2 dxdy \tag{5.14}$$

where the constant C same as in (7.23). Now, by iterating (5.14), then we have

$$\begin{aligned}
&\int_{B^{n+1}(x_0,1)} |y|^{1-2s}|\widetilde{U}|^2dxdy \\
\leq & C^N \int_{B^{n+1}(x_0,\frac{1}{2^N})} |y|^{1-2s}|\widetilde{U}|^2dxdy \\
\leq & C^N(\frac{1}{2})^{N(m-1)} \int_{B^{n+1}(x_0,\frac{1}{2^N})} |y|^{2-2s-m}|\widetilde{U}|^2dxdy \\
\leq & C^N(\frac{1}{2})^{N(m-1)} \int_{B(x_0,R)\times(-r_0,r_0)} |y|^{-m}|\widetilde{U}|^2dxdy,
\end{aligned}$$

for N large such that $\frac{1}{2^N} < \min\{R, r_0\}$ and for any $m \in \mathbb{N}$ with $m \geq 2$. Now, since $\widetilde{U}$ has vanishing order at x_0, by using (5.13) $\int_{B(x_0,R)\times(-r_0,r_0)} |y|^{-m}|\widetilde{U}|^2dxdy$ remains bounded and $C^N(\frac{1}{2})^{Nm} \to 0$ as $m \to \infty$. This implies $\widetilde{U} = 0$ in $B^{n+1}(x_0, 1)$, which completes the proof. □

Lemma 5.5. *Let $u \in H^s(\mathbb{R}^n)$, if $u = \mathcal{L}^s u = 0$ in any ball $B(x_0, R) \subseteq \mathbb{R}^n$, then $U = 0$ in $\overline{B_+^{n+1}(x_0, R)}$, where U is the function in Theorem 5.2.*

Proof. As $u \in H^s(\mathbb{R}^n)$ satisfying $u|_{B(x_0,R)} = \mathcal{L}^s u|_{B(x_0,R)} = 0$, so from Theorem 5.2, we have U vanishes infinite order on $\partial\mathbb{R}_+^{n+1}$ so does $\widetilde{U}$, where $\widetilde{U}$ is the even reflection of U defined by (4.15). Therefore, by using Proposition 5.4, we have SUCP for (5.12). Consequently it follows $\widetilde{U} = 0$ in $B^{n+1}(x_0, R)$ so $U = 0$ in $\overline{B_+^{n+1}(x_0, R)}$. □

5.2 Proof of Theorem 1.2

Let us begin to prove Theorem 1.2 by using Lemma 5.5.

Proof of Theorem 1.2. We have already shown that $U = 0$ in $\overline{B_+^{n+1}(x_0, R)}$. Now, we will show $U = 0$ in $\mathbb{R}_+^{n+1} \setminus \overline{B_+^{n+1}(x_0, R)}$ also. Let us consider the region $D_\epsilon = \{(x, y) : x \in \mathbb{R}^n \text{ and } \epsilon < y < 1/\epsilon\}$ for any $\epsilon > 0$. Since the weight y^{1-2s} is smooth and positive in $\overline{D_\epsilon}$ for any $s \in (0, 1)$, thus U can be realized as a solution of a uniformly elliptic equation

$$\nabla_{x,y} \cdot (y^{1-2s}\widetilde{A}(x)\nabla_{x,y}U) = 0 \text{ in } \mathbb{R}_+^{n+1} \tag{5.15}$$

in $H^1(D_\epsilon)$ for all $s \in (0, 1)$. Since U also vanishes in $\overline{B_+^{n+1}(x_0, R)} \cap D_\epsilon$, where $\epsilon > 0$ is chosen so small that this set is nonempty, it follows by standard weak unique continuation property for the uniform elliptic equation in a strip domain that U has to vanish in entire D_ϵ. Since this is true for any $\epsilon > 0$ small, one has $U = 0$ in $\mathbb{R}_+^{n+1}$ as required. Hence as a trace of $U \in H^1(\mathbb{R}_+^{n+1}, y^{1-2s})$, $U(x, 0) = u(x) = 0$ in $\mathbb{R}^n$. □

Remark 5.6. We would like to present a different proof using simply integration by parts techniques only, which works for the case $s \leq \frac{1}{2}$ only. In order to establish our claim, $U=0$ in $\mathbb{R}^{n+1}_+$ in this case $s\leq \frac{1}{2}$, we write

$$\begin{aligned}
&\int_{\mathbb{R}^{n+1}_+} y^{1-2s}\widetilde{A}\nabla_{x,y}U\cdot\nabla_{x,y}U\,dx\,dy\\
=&\lim_{R\to\infty}\int_{B^{n+1}_+(0,R)} y^{1-2s}\widetilde{A}\nabla_{x,y}U\cdot\nabla_{x,y}U\,dx\,dy\\
=&\lim_{R\to\infty}\int_{\partial B^{n+1}_+(0;1,R)} y^{1-2s}(\widetilde{A}\nabla_{x,y}U\cdot\nu)U\,dS(x,y), \qquad (5.16)
\end{aligned}$$

where $\partial B^{n+1}_+(0;1,R)=\partial B^{n+1}_+(0,R)\cup\partial B^{n+1}_+(0,1)\cup B^0(0;1,R)$ and $B^0(0;1,R)=\big\{(x,0)\in\mathbb{R}^{n+1};1\leq|x|\leq R\big\}$. Then using the fact $w=0$ on $\partial B^{n+1}_+(0,1)$ and $s\leq\frac{1}{2}$ gives the integrand in (5.16) to be 0 on $B^0(0;1,R)$. Hence,

$$\begin{aligned}
&\int_{\mathbb{R}^{n+1}_+} y^{1-2s}\widetilde{A}\nabla_{x,y}U\cdot\nabla_{x,y}U\,dx\,dy\\
=&\lim_{R\to\infty}\int_{\partial B^{n+1}_+(0,R)} y^{1-2s}(\widetilde{A}\nabla_{x,y}U\cdot\nu)U\,dS(x,y)\\
=&\lim_{R\to\infty}\int_{\mathbb{R}^{n+1}_+\setminus B^{n+1}_+(0,R)} y^{1-2s}\widetilde{A}\nabla_{x,y}U\cdot\nabla_{x,y}U\,dx\,dy=0 \qquad (5.17)
\end{aligned}$$

since $U\in H^1(\mathbb{R}^{n+1}_+,y^{1-2s})$. Thus, it follows from (5.15) and (5.17), $U\equiv 0$ in $\mathbb{R}^{n+1}_+$ and consequently, as a trace of $U\in H^1(\mathbb{R}^{n+1}_+,y^{1-2s})$, $U(x,0)=u(x)=0$ in $\mathbb{R}^n$. This completes the proof of Theorem 1.2 for $H^s(\mathbb{R}^n)$ class of functions whenever $s\leq\frac{1}{2}$.

5.3 Runge approximation property

Let $s\in(0,1)$ and assume that $q\in L^\infty(\Omega)$ satisfy the eigenvalue condition (1.5). We denote $u=u_f\in H^s(\mathbb{R}^n)$ as the unique solution of

$$(\mathcal{L}^s+q)u_f=0 \text{ in } \Omega, \quad \text{with } u_f-f\in\widetilde{H}^s(\Omega).$$

Lemma 5.7. *Let $n\geq 2$, and $A(x)$ is a C^∞-smooth matrix-valued function in $\mathbb{R}^n$ satisfying* (1.3)*. Let $\Omega\subseteq\mathbb{R}^n$ be bounded open set with Lipschitz boundary and $q\in L^\infty(\Omega)$ satisfy the eigenvalue condition* (1.5)*. Let $\mathcal{O}$ be any open subset of Ω_e and consider the set*

$$\mathbb{D}=\{u_f|_\Omega\,;\ f\in C^\infty_c(\mathcal{O})\}.$$

Then $\mathbb{D}$ is dense in $L^2(\Omega)$.

Proof. By the Hahn-Banach theorem, it is only needed to show that for any $v\in L^2(\Omega)$ satisfying $(v,w)_\Omega=0$ for any $w\in\mathbb{D}$, then $v\equiv 0$. Let v be a such function, which means v satisfies

$$(v,r_\Omega u_f)=0, \text{ for any } f\in C^\infty_c(\mathcal{O}). \qquad (5.18)$$

Now, let $\phi \in \widetilde{H}^s(\Omega)$ be the solution of $(\mathcal{L}^s + q)\phi = v$ in Ω. We want to show that for any $f \in C_c^\infty(\mathcal{O})$, the following relation

$$\mathcal{B}_q(\phi, f) = -(v, r_\Omega u_f)_\Omega \tag{5.19}$$

holds. In other words, $\mathcal{B}_q(\phi, w) = (v, r_\Omega w)$ for any $w \in \widetilde{H}^s(\Omega)$. The (5.19) follows due to

$$\mathcal{B}_q(\phi, f) = \mathcal{B}_q(\phi, f - u_f) = (v, r_\Omega(f - u_f))_\Omega = -(v, r_\Omega u_f)_\Omega,$$

where we have used the facts that u_f is a solution and $\phi \in \widetilde{H}^s(\Omega)$. Note that (5.18) and (5.19) imply that

$$\mathcal{B}_q(\phi, f) = 0 \text{ for any } f \in C_c^\infty(\mathcal{O}).$$

Moreover, we know that $r_\Omega f = 0$ because $f \in C_c^\infty(\mathcal{O})$ and we can derive

$$(\mathcal{L}^s\phi, f)_{\mathbb{R}^n} = 0 \text{ for any } f \in C_c^\infty(\mathcal{O}).$$

In the end, we know that $\phi \in H^s(\mathbb{R}^n)$ which satisfies

$$\phi|_{\mathcal{O}} = \mathcal{L}^s\phi|_{\mathcal{O}} = 0.$$

By Theorem 1.2, we obtain $\phi \equiv 0$ and then $v \equiv 0$. □

Remark 5.8. We also refer readers to [28] for more details of the Runge approximation property for the (local) differential equations.

6 Proof of Theorem 1.1

Now, we are ready to prove the global uniqueness result for variable coefficients fractional operators. Even though the proof is similar as the proof in [18], we still give a proof for the completeness.

Proof of Theorem (1.1). If $\Lambda_{q_1} g|_{\mathcal{O}_2} = \Lambda_{q_2} g|_{\mathcal{O}_2}$ for any $g \in C_c^\infty(\mathcal{O}_1)$, where $\mathcal{O}_1$ and $\mathcal{O}_2$ are open subsets of Ω_e, by the integral identity in Lemma 3.8, we have

$$\int_\Omega (q_1 - q_2) u_1 u_2 dx = 0$$

where $u_1, u_2 \in H^s(\mathbb{R}^n)$ solve $(\mathcal{L}^s + q_1)u_1 = 0$ and $(\mathcal{L}^s + q_2)u_2 = 0$ in Ω with u_1, u_2 having exterior values $g_j \in C_c^\infty(\mathcal{O}_j)$, for $j = 1, 2$.

Let $f \in L^2(\Omega)$, and use the approximation lemma 5.7, then there exist two sequences (u_j^1), (u_j^2) of functions in $H^s(\mathbb{R}^n)$ that satisfy

$$\begin{aligned}
&(\mathcal{L}^s + q_1)u_j^1 = (\mathcal{L}^s + q_2)u_j^2 = 0 \text{ in } \Omega, \\
&\operatorname{supp}(u_j^1) \subseteq \overline{\Omega_1} \text{ and } \operatorname{supp}(u_j^2) \subseteq \overline{\Omega_2}, \\
&r_\Omega u_j^1 = f + r_j^1, \quad r_\Omega u_j^2 = 1 + r_j^2,
\end{aligned}$$

where Ω_1, Ω_2 are two open subsets of $\mathbb{R}^n$ containing Ω, and $r_j^1, r_j^2 \to 0$ in $L^2(\Omega)$ as $j \to \infty$. Plug these solutions into the integral identity and pass the limit as $j \to \infty$, then we infer that

$$\int_\Omega (q_1 - q_2) f dx = 0.$$

Since $f \in L^2(\Omega)$ was arbitrary, we conclude that $q_1 = q_2$. □

7 Appendix

At the end of this paper, we present some required materials to complete our paper.

7.1 Stability result for the degenerate problem

In general, we have the following result.

Lemma 7.1. *Let h be a vector-valued function satisfying* $\dfrac{G}{y^{1-2s}} \in L^2(\mathbb{R}^{n+1}_+, y^{1-2s})$, *then the following Dirichlet boundary value problem*

$$\begin{cases} \nabla_{x,y} \cdot (y^{1-2s}\widetilde{A}(x)\nabla_{x,y}V) = \nabla_{x,y} \cdot G \text{ in } \mathbb{R}^{n+1}_+, \\ V(x,0) = 0 \text{ on } \mathbb{R}^n \end{cases} \tag{7.1}$$

has a unique weak solution in $H^1(\mathbb{R}^{n+1}_+, y^{1-2s})$ *satisfying*

$$\|V\|_{H^1(\mathbb{R}^{n+1}_+, y^{1-2s})} \leq C\|y^{-1+2s}G\|_{L^2(\mathbb{R}^{n+1}_+, y^{1-2s})}, \tag{7.2}$$

where the constant $C > 0$ *is independent of* G *and* V*.*

By the weak solution of (7.1) we mean $V \in H^1(\mathbb{R}^{n+1}_+, y^{1-2s})$ solves

$$\int_{\mathbb{R}^{n+1}_+} y^{1-2s}\widetilde{A}(x)\nabla_{x,y}V \cdot \nabla_{x,y}\Psi \, dxdy = \int_{\mathbb{R}^{n+1}_+} y^{-1+2s}G \cdot y^{1-2s}\nabla\Psi \, dxdy, \tag{7.3}$$

for all $\Psi \in H^1_0(\mathbb{R}^{n+1}_+, y^{1-2s})$.

Proof of Lemma 7.1. Let us consider the Dirichlet functional $J : H^1_0(\mathbb{R}^{n+1}_+, y^{1-2s}) \to \mathbb{R}_+$ as

$$J(\Psi) := \int_{\mathbb{R}^{n+1}_+} y^{1-2s}\widetilde{A}(x)\nabla_{x,y}\Psi \cdot \nabla_{x,y}\Psi \, dxdy - \int_{\mathbb{R}^{n+1}_+} y^{-1+2s}G \cdot y^{1-2s}\nabla\Psi dxdy, \tag{7.4}$$

If $V \in H^1_0(\mathbb{R}^{n+1}_+, y^{1-2s})$ is an extremum of $J(\Psi)$ in $H^1_0(\mathbb{R}^{n+1}_+, y^{1-2s})$, then for any $\Psi \in C^\infty_c(\mathbb{R}^{n+1}_+)$, as a function of η,

$$F(\eta) := J(V + \eta\Psi)$$

attains its extremum at $\eta = 0$ and hence $F'(0) = 0$ as

$$\begin{aligned} F'(0) &= \lim_{\eta\to 0} \frac{J(V+\eta\Psi) - J(V)}{\eta} \\ &= 2\int_{\mathbb{R}^{n+1}_+} y^{1-2s}\widetilde{A}(x)\nabla_{x,y}V \cdot \nabla_{x,y}\Psi dxdy \\ &\quad -2\int_{\mathbb{R}^{n+1}_+} y^{-1+2s}h \cdot y^{1-2s}\nabla\Psi dxdy \\ &= 0, \end{aligned}$$

which gives the definition of the weak solution. As we can see from the definition (7.4)

$$J(\Psi) \geq \frac{1}{2}\int_{\mathbb{R}^{n+1}_+} y^{1-2s}|\nabla\Psi|^2 dxdy - \frac{1}{2}\int_{\mathbb{R}^{n+1}_+} y^{-1+2s}|h|^2 dxdy$$
$$\geq -\frac{1}{2}\int_{\mathbb{R}^{n+1}_+} y^{-1+2s}|h|^2 dxdy,$$

that means $J(\Psi)$ is bounded from below in $H^1_0(\mathbb{R}^{n+1}_+, y^{1-2s})$.

Therefore, $\inf_{H^1_0(\mathbb{R}^{n+1}_+, y^{1-2s})} J(\Psi)$ is a finite number. Hence, there exists a minimizing sequence $\{\Psi_k\}_{k=1}^\infty \subset H^1_u(\mathbb{R}^{n+1}_+, y^{1-2s})$ such that

$$\lim_{k\to\infty} J(\Psi_k) = \inf_{H^1_0(\mathbb{R}^{n+1}_+, y^{1-2s})} J(\Psi).$$

Next we observe that, the functional turns out to be weakly lower semi-continuous over its domain of definition, i.e.

$$J(\Psi) \leq \liminf_{k\to\infty} J(\Psi_k), \quad \text{if } \Psi_k \rightharpoonup \Psi \text{ weakly in } H^1_0(\mathbb{R}^{n+1}_+, y^{1-2s}).$$

This simply follows as if $\Psi_k \rightharpoonup \Psi$ weakly in $H^1_0(\mathbb{R}^{n+1}_+, y^{1-2s})$ then

$$\int_{\mathbb{R}^{n+1}_+} y^{1-2s}\widetilde{A}(x)\nabla_{x,y}\Psi \cdot \nabla_{x,y}\Psi \, dxdy$$
$$\leq \liminf_{k\to\infty} \int_{\mathbb{R}^{n+1}_+} y^{1-2s}\widetilde{A}(x)\nabla_{x,y}\Psi_k \cdot \nabla_{x,y}\Psi_k \, dxdy.$$

Thus, if $\{\Psi_k\}_{k=1}^\infty$ is a minimizing sequence, i.e. , if

$$J(\Psi_k) \to \inf_{H^1_0(\mathbb{R}^{n+1}_+, y^{1-2s})} J(V)$$

then there exists a subsequence $\{\Psi_{k_j}\}_{j=1}^\infty$ such that $\Psi_{k_j} \rightharpoonup V$ weakly in $H^1_0(\mathbb{R}^{n+1}_+, y^{1-2s})$ and hence

$$\inf_{H^1_0(\mathbb{R}^{n+1}_+, y^{1-2s})} J(\Psi) \leq J(V) \leq \liminf_{k\to\infty} J(\Psi_k) = \inf_{H^1_0(\mathbb{R}^{n+1}_+, y^{1-2s})} J(\Psi),$$

Therefore, $J(V) = \inf_{H^1_0(\mathbb{R}^{n+1}_+, y^{1-2s})} J(\Psi)$ and we achieve our goal.

Next, we claim that $V \in H^1_0(\mathbb{R}^{n+1}_+, y^{1-2s})$ is the unique minimizer of $J(\Psi)$. Assume that V_1 and $V_2 \in H^1_0(\mathbb{R}^{n+1}_+, y^{1-2s})$ are weak solutions of (7.1), then $V_1 - V_2 \in H^1_0(\mathbb{R}^{n+1}_+, y^{1-2s})$ satisfies the following integral identity

$$\int_{\mathbb{R}^{n+1}_+} y^{1-2s}\widetilde{A}(x)\nabla_{x,y}(V_1 - V_2) \cdot \nabla_{x,y}(V_1 - V_2) \, dxdy = 0,$$

which implies that $V_1 = V_2$. This shows that (7.1) has a unique weak solution in $H^1_0(\mathbb{R}^{n+1}_+, y^{1-2s})$. The remaining stability estimate (7.1) simply follows from (7.3) by taking $\Psi = V$ there, to have

$$\|V\|_{H^1(\mathbb{R}^{n+1}_+, y^{1-2s})} \leq C\|y^{-1+2s}G\|_{L^2(\mathbb{R}^{n+1}_+, y^{1-2s})}$$

for some constant $C > 0$. □

Lemma 7.2. *Let P_y^s be the Poisson kernel given by* (4.11)*. Then*

$$\lim_{y\to 0^+}\int_{\mathbb{R}^n} P_y^s(x,z)dz = 1, \quad x\in\mathbb{R}^n, \tag{7.5}$$

and

$$\int_{\mathbb{R}^n} P_y^s(x,z)dz = 1, \quad x\in\mathbb{R}^n,\, y>0. \tag{7.6}$$

Proof. The limit (7.5) is verified in [44, Theorem 2.1]. We only need to show (7.6). The following identity holds

$$\int_0^\infty e^{-\frac{y^2}{4t}}\frac{e^{-b|x-z|^2/t}}{t^{n/2}}\frac{dt}{t^{1+s}} = \frac{c(n,s,b)}{(b|x-z|^2+y^2)^{\frac{n+2s}{2}}},$$

such that

$$\frac{c_1 y^{2s}}{(b_1|x-z|^2+y^2)^{\frac{n+2s}{2}}} \le P_y^s(x,z) \le \frac{c_2 y^{2s}}{(b_2|x-z|^2+y^2)^{\frac{n+2s}{2}}}, \tag{7.7}$$

with some positive constant b_j, c_j, for $j=1,2$. It is obtained by applying the estimate (7.7) that

$$\begin{aligned}
\lim_{\epsilon\to 0^+}\int_{|z-x|>\epsilon} P_y^s(x,z)dz &= \lim_{\epsilon\to 0^+}\int_{|z-x|>\epsilon}\frac{y^{2s}}{4^s\Gamma(s)}\int_0^\infty e^{-\frac{y^2}{4t}}p_t(x,z)\frac{dt}{t^{1+s}}dz \\
&= \lim_{\epsilon\to 0^+}\int_0^\infty\int_{|z-x|>\epsilon}\frac{y^{2s}}{4^s\Gamma(s)}e^{-\frac{y^2}{4t}}p_t(x,z)\,dz\frac{dt}{t^{1+s}} \\
&= \int_0^\infty \frac{y^{2s}}{4^s\Gamma(s)}e^{-\frac{y^2}{4t}}\frac{dt}{t^{1+s}} \qquad (7.8)\\
&\quad -\lim_{\epsilon\to 0^+}\int_0^\infty\int_{|z-x|\le\epsilon}\frac{y^{2s}}{4^s\Gamma(s)}e^{-\frac{y^2}{4t}}p_t(x,z)\,dz\frac{dt}{t^{1+s}} \\
&= I_1 - \frac{1}{4^s\Gamma(s)}\lim_{\epsilon\to 0^+} I_\epsilon(y),
\end{aligned}$$

where we have used the fact that the heat kernel $p_t(x,z)$ satisfies $\int_{\mathbb{R}^n} p_t(x,z)dz = 1$. We have from the Gamma function that the integral $I_1 = 1$, providing $s\in(0,1)$. We claim that $\lim_{\epsilon\to 0^+} I_\epsilon(y) = 0$ for any $y>0$. In fact, by (2.10) one has

$$\begin{aligned}
I_\epsilon(y) &\le c\int_0^\infty y^{2s}e^{-\frac{y^2}{4t}}\int_{|z-x|\le\epsilon} e^{-b\frac{|x-z|^2}{t}}dz\,\frac{dt}{t^{1+s+n/2}} \\
&= c\int_0^\infty y^{2s}e^{-\frac{y^2}{4t}}\int_{B_\epsilon} e^{-b\frac{|z|^2}{t}}dz\,\frac{dt}{t^{1+s+n/2}} \\
&= 4\pi c\int_0^\infty y^{2s}e^{-\frac{y^2}{4t}}\int_0^\epsilon e^{-b\frac{r^2}{t}}dr\,\frac{dt}{t^{1+s+n/2}} \\
&= 4\pi c_1\int_0^\epsilon \frac{y^{2s}}{(br^2+y^2)^{s+n/2}}dr.
\end{aligned}$$

Therefore, one can pass the limit $\epsilon\to 0^+$ in (7.8) and thus obtain (7.6). □

7.2 Almgren's type frequency function and the doubling inequality for the degenerate problem

Here we mention the strong unique continuation property for the degenerate problem $\nabla_{x,y}\cdot(|y|^{1-2s}\widetilde{A}(x)\nabla_{x,y}\widetilde{U})=0$ in $B^{n+1}(0,1)$. The proof relies on the technique in using the Almgren's frequency function method, which was introduced by Yu [51].

To simplify the notation, let us denote $B_r^{n+1}:=B^{n+1}(0,r)$ and $z=(x,y)\in\mathbb{R}^{n+1}$. For $z\neq 0$, we define

$$\mu(z):=\frac{\left(\widetilde{A}(z)z\right)\cdot z}{|z|^2}\in\mathbb{R}\text{ and }\overrightarrow{\beta}(z):=\frac{\widetilde{A}(z)z}{\mu(z)}\in\mathbb{R}^{n+1},$$

then from the ellipticity condition (1.3), it is easy to see that

$$\widetilde{\Lambda}^{-1}\leq\mu(z)\leq\widetilde{\Lambda}\text{ and }|\overrightarrow{\beta}(z)|\leq\widetilde{\Lambda}|z|\text{ for all }z\in\mathbb{R}^{n+1}$$

for some universal constant $\widetilde{\Lambda}>0$. In addition, by the standard coordinates transformation technique, we may assume that $\widetilde{A}(0)=I_{n+1}$, which is an $(n+1)\times(n+1)$ identity matrix, then we have the following estimates hold for $\mu(z)$ and $\overrightarrow{\beta}(z)=(\beta_1(z),\beta_2(z),\cdots,\beta_{n+1}(z))$:

$$\left|\frac{\partial}{\partial r}\mu(rz)\right|\leq C\text{ for }r>0\text{ and }\frac{\partial\beta_i}{\partial z_j}(z)=\delta_{ij}+O(|z|),\tag{7.9}$$

where δ_{ij} is the Kronecker delta and the constant $C>0$ depends on $\widetilde{A}(z)=(\widetilde{a}_{jk}(z))_{j,k=1}^{n+1}$. The estimates (7.9) were proved in [46, 51], so we skip the details.

Let $\widetilde{U}\in H^1(\mathbb{R}^{n+1}_+,|y|^{1-2s})$ and consider

$$H(r) := \int_{\partial B_r^{n+1}}|y|^{1-2s}\mu(z)|\widetilde{U}(z)|^2dS(z),\tag{7.10}$$

$$D(r) := \int_{B_r^{n+1}}|y|^{1-2s}\left(\widetilde{A}(z)\nabla\widetilde{U}\right)\cdot\nabla\widetilde{U}dz,\tag{7.11}$$

where $\nabla:=\nabla_z=\nabla_{x,y}$ in $\mathbb{R}^{n+1}$ and it is easy to see that $H(r)$ exists for almost every $r>0$ as a surface integral, since the volume integral $(\int_0^R H(r)dr<\infty)$ exists due to $\widetilde{U}\in H^1(\mathbb{R}^{n+1}_+,|y|^{1-2s})$). Next, similar to [29, 46, 51], we define the corresponding *Almgren's frequency function* by

$$N(r):=\frac{rD(r)}{H(r)},$$

and we have the following lemmas.

Lemma 7.3. *For any $r\in(0,1)$, $H(r)=0$ whenever $\widetilde{U}\equiv 0$ in B_r^{n+1}.*

Proof. If $H(r)=0$, it implies that $\widetilde{U}=0$ on ∂B_r^{n+1}. Hence, by the uniqueness of the solution of the degenerate problem (for example, see[16]), we conclude $\widetilde{U}\equiv 0$ in B_r^{n+1}. □

Lemma 7.4. *The function $H(r)$ is differentiable and*

$$H'(r) = \left(\frac{(n+1-2s)}{r} + O(1)\right) H(r) + 2D(r). \tag{7.12}$$

Proof. By change of variables, we have

$$\begin{aligned} H(r) &= \int_{\partial B_r^{n+1}} |y|^{1-2s}\mu(z)|\widetilde{U}(z)|^2 dS \\ &= r^{n+1-2s}\int_{\partial B_1^{n+1}} |y|^{1-2s}\mu(rz)|\widetilde{U}(rz)|^2 dS, \end{aligned}$$

then

$$\begin{aligned} H'(r) &= \frac{d}{dr}H(r) \\ &= (n+1-2s)r^{n-2s}\int_{\partial B_1^{n+1}} |y|^{1-2s}\mu(rz)|\widetilde{U}(rz)|^2 dS \\ &\quad + r^{n+1-2s}\int_{\partial B_1^{n+1}} |y|^{1-2s}\frac{\partial}{\partial r}\mu(rz)|\widetilde{U}(rz)|^2 dS \\ &\quad + 2r^{n+1-2s}\int_{\partial B_1^{n+1}} |y|^{1-2s}\mu(rz)\widetilde{U}(rz)\frac{\partial}{\partial r}\widetilde{U}(rz)dS, \end{aligned}$$

Note that $H'(r)$ exists for a.e. $r>0$ due to $\widetilde{U} \in H^1(\mathbb{R}^{n+1}_+, |y|^{1-2s})$ and $\dfrac{\partial}{\partial r}\mu(rz)$ is bounded by constant $C>0$ (see (7.9)) and after change of variables back, we obtain

$$\begin{aligned} H'(r) &\le \frac{(n+1-2s)}{r}\int_{\partial B_r^{n+1}} |y|^{1-2s}\mu(z)|\widetilde{U}(z)|^2 dS \\ &\quad + C\int_{\partial B_r^{n+1}} |y|^{1-2s}|\widetilde{U}(z)|^2 dS \\ &\quad + 2\int_{\partial B_r^{n+1}} |y|^{1-2s}\mu(z)\widetilde{U}(z)\frac{\partial \widetilde{U}}{\partial \nu}(z)dS, \end{aligned}$$

where ν is a unit outer normal on ∂B_1^{n+1}. By using the regularity assumption for $A(x)$ and $\widetilde{U} \in H^1(\mathbb{R}^{n+1}_+, |y|^{1-2s})$, we have $C\int_{|z|=r} |y|^{1-2s}|\widetilde{U}(z)|^2 dS$ bounded for a.e. $r>0$. Therefore, we have

$$\begin{aligned} H'(r) &= \left(\frac{(n+1-2s)}{r} + O(1)\right) H(r) \\ &\quad + 2\int_{\partial B_r^{n+1}} |y|^{1-2s}\mu(z)\widetilde{U}(z)\frac{\partial \widetilde{U}}{\partial \nu}(z)dS. \end{aligned}$$

Finally, we will show that

$$\int_{\partial B_r^{n+1}} |y|^{1-2s}\mu(z)\widetilde{U}(z)\frac{\partial \widetilde{U}}{\partial \nu}(z)dS = D(r) + O(1)H(r). \tag{7.13}$$

By using the equation $\nabla\cdot(|y|^{1-2s}\widetilde{A}\nabla\widetilde{U}) = 0$, we can rewrite $D(r)$ in terms of

$$D(r) = \int_{B_r^{n+1}} \nabla\cdot(|y|^{1-2s}\widetilde{U}\widetilde{A}\nabla\widetilde{U})dz = \int_{\partial B_r^{n+1}} |y|^{1-2s}\widetilde{U}(\widetilde{A}\nu)\cdot\nabla\widetilde{U}dS.$$

We define $\mathcal{T}(z) := \widetilde{A}\nu - \mu(z)\nu \in \mathbb{R}^{n+1}$ and note that

$$\mathcal{T} \cdot \nu = (\widetilde{A}\nu - \mu(z)\nu) \cdot \nu = 0 \text{ on } \partial B_r^{n+1},$$

which means $\mathcal{T}(z)$ is a tangential vector of ∂B_r^{n+1}. From the divergence theorem on ∂B_r^{n+1}, we can derive that

$$\begin{aligned}
&D(r) - \int_{\partial B_r^{n+1}} |y|^{1-2s}\mu(z)\widetilde{U}\frac{\partial \widetilde{U}}{\partial \nu} dS \\
=& \int_{\partial B_r^{n+1}} |y|^{1-2s}\widetilde{U}\nabla\widetilde{U} \cdot (\widetilde{A}\nu - \mu(z)\nu) dS \\
=& -\frac{1}{2}\int_{\partial B_r^{n+1}} |y|^{1-2s}|\widetilde{U}|^2 \nabla \cdot \mathcal{T} dS - \frac{1}{2}\int_{\partial B_r^{n+1}} |\widetilde{U}|^2 \left(\nabla |y|^{1-2s}\right) \cdot \mathcal{T} dS.
\end{aligned}$$

From direct computation, we have $|\nabla_x \cdot \mathcal{T}| \le C_{n,A}$ for some constant $C_{n,A} > 0$ depending on n and $A(x)$ and then

$$\int_{\partial B_r^{n+1}} |y|^{1-2s}|\widetilde{U}|^2 \nabla_x \cdot \mathcal{T} dS = O(1)H(r). \tag{7.14}$$

On the other hand, it is not hard to see that

$$\left|\int_{\partial B_r^{n+1}} |\widetilde{U}|^2 \left(\nabla |y|^{1-2s}\right) \cdot \mathcal{T} dS\right| \le \frac{1}{r}\int_{\partial B_r^{n+1}} |\widetilde{U}|^2 \left|(1-2s)y|y|^{-2s}\left(1-\mu(z)\right)\right| dS$$

and by using $|1-\mu(z)| \le C_A|z|$, for some constant $C_A > 0$, then we can derive

$$\left|\int_{\partial B_r^{n+1}} |\widetilde{U}|^2 \left(\nabla |y|^{1-2s}\right) \cdot \mathcal{T} dS\right| \le C\int_{\partial B_r^{n+1}} |y|^{1-2s}|\widetilde{U}|^2 dS = O(1)H(r).$$

This proves the lemma. $\square$

Lemma 7.5. *The function $D(r)$ is differentiable with*

$$D'(r) = \left(\frac{n-2s}{r} + O(1)\right) D(r) + 2\int_{\partial B_r^{n+1}} |y|^{1-2s}\frac{1}{\mu}\left|(\widetilde{A}\nu) \cdot \nabla\widetilde{U}\right|^2 dS. \tag{7.15}$$

Proof. It is easy to see that

$$D'(r) = \int_{\partial B_r^{n+1}} |y|^{1-2s}\widetilde{A}(z)\nabla\widetilde{U} \cdot \nabla\widetilde{U} dS.$$

By straightforward calculation, we have the following Rellich type identity

$$\begin{aligned}
&\int_{B_r^{n+1}} \left[\nabla \cdot \left(|y|^{1-2s}\overrightarrow{\beta}(\widetilde{A}\nabla\widetilde{U} \cdot \nabla\widetilde{U})\right) - 2\nabla \cdot \left(|y|^{1-2s}(\overrightarrow{\beta} \cdot \nabla\widetilde{U})\widetilde{A}\nabla\widetilde{U}\right)\right] dz \\
=& \int_{B_r^{n+1}} \left[\nabla \cdot (|y|^{1-2s}\overrightarrow{\beta})(\widetilde{A}\nabla\widetilde{U} \cdot \nabla\widetilde{U}) + \sum_{j,k,l=1}^{n+1} y^{1-2s}\beta_l \frac{\partial \widetilde{a}_{jk}}{\partial z_l}\frac{\partial \widetilde{U}}{\partial z_j}\frac{\partial \widetilde{U}}{\partial z_k}\right] dz \\
&- 2\int_{B_r^{n+1}} \sum_{j,k,l=1}^{n+1} |y|^{1-2s}\widetilde{a}_{jk}\frac{\partial \beta_l}{\partial z_k}\frac{\partial \widetilde{U}}{\partial z_j}\frac{\partial \widetilde{U}}{\partial z_k} dz.
\end{aligned} \tag{7.16}$$

Note that $\beta_{n+1} = \dfrac{y}{\mu(z)}$, so we have

$$\begin{aligned}&\int_{B_r^{n+1}} \nabla \cdot (|y|^{1-2s}\overrightarrow{\beta})(\widetilde{A}\nabla\widetilde{U} \cdot \nabla\widetilde{U})dz\\ =&\int_{B_r^{n+1}} (\nabla \cdot \overrightarrow{\beta})(\widetilde{A}\nabla\widetilde{U} \cdot \nabla\widetilde{U})dz + \int_{B_r^{n+1}} (1-2s)\frac{|y|^{1-2s}}{\mu(z)}\widetilde{A}\nabla\widetilde{U} \cdot \widetilde{U}dz. \end{aligned}\tag{7.17}$$

First, for the left hand sides in (7.16), we use the relations $\overrightarrow{\beta} \cdot \nu = r$, $\overrightarrow{\beta} \cdot \nabla\widetilde{U} = \dfrac{r(\widetilde{A}\nu) \cdot \nabla\widetilde{U}}{\mu(z)}$ on ∂B_r^{n+1} and integrate them over B_r^{n+1}, so we get

$$\begin{aligned}&\int_{\partial B_r^{n+1}} |y|^{1-2s}(\widetilde{A}(x)\nabla\widetilde{U} \cdot \widetilde{U})(\overrightarrow{\beta} \cdot \nu)dS - 2\int_{\partial B_r^{n+1}} \left(|y|^{1-2s}(\widetilde{A}\nu \cdot \nabla\widetilde{U}\right)(\beta \cdot \nabla\widetilde{U})dS\\ =&r\int_{\partial B_r^{n+1}} |y|^{1-2s}(\widetilde{A}(x)\nabla\widetilde{U} \cdot \widetilde{U})dS - 2r\int_{\partial B_r^{n+1}} |y|^{1-2s}\frac{\left|\widetilde{A}\nu \cdot \nabla\widetilde{U}\right|^2}{\mu(z)}dS\\ =&rD'(r) - 2r\int_{\partial B_r^{n+1}} |y|^{1-2s}\frac{\left|\widetilde{A}\nu \cdot \nabla\widetilde{U}\right|^2}{\mu(z)}dS. \end{aligned}\tag{7.18}$$

Second, we evaluate the right hand side of (7.16) as follows. For the first term in the right hand side (RHS) of (7.16) can be rewritten as (7.17) and we estimate them separately. By using (7.9), we have $\nabla \cdot \beta = n + 1 + O(r)$ for $z \in B_1^{n+1}$, which implies

$$\int_{B_r^{n+1}} (\nabla \cdot \beta)|y|^{1-2s}(\widetilde{A}(x)\nabla\widetilde{U} \cdot \nabla\widetilde{U}) = (n + 1 + O(r))D(r), \tag{7.19}$$

and we know that $\beta_{n+1} = \dfrac{y}{\mu(z)} = y + \left(1 - \dfrac{1}{\mu(z)}\right)y = y + O(|z|)y$, with $|z| \le r$, hence

$$\begin{aligned}&\int_{B_r^{n+1}} \beta_{n+1}(1-2s)|y|^{-2s}(\widetilde{A}(x)\nabla\widetilde{U} \cdot \nabla\widetilde{U})dz\\ =&\int_{B_r^{n+1}} (y + O(r)y)(1-2s)|y|^{-2s}(\widetilde{A}(x)\nabla\widetilde{U} \cdot \nabla\widetilde{U})dz\\ =&(1 - 2s + O(r))D(r). \end{aligned}\tag{7.20}$$

For the second term in the RHS of (7.16), we have $\left|\beta_l\dfrac{\partial\widetilde{a}_{jk}}{\partial z_l}\right| \le C|z| \le Cr$ so that

$$\sum_{j,k,l=1}^{n+1}\int_{B_r^{n+1}} |y|^{1-2s}\beta_l\frac{\partial\widetilde{a}_{jk}}{\partial z_l}\frac{\partial\widetilde{U}}{\partial z_j}\frac{\partial\widetilde{U}}{\partial z_k} = O(r)D(r), \tag{7.21}$$

For the last term in the RHS of (7.16), Now, for the last term in the RHS of (7.16), from $\dfrac{\partial\beta_l}{\partial z_k} = \delta_{lk} + O(r)$ in a bounded region, it is easy to see that

$$\int_{B_r^{n+1}}\sum_{j,k,l=1}^{n+1} |y|^{1-2s}\widetilde{a}_{jk}\frac{\partial\beta_l}{\partial z_k}\frac{\partial\widetilde{U}}{\partial z_j}\frac{\partial\widetilde{U}}{\partial z_k}dz = (1 + O(r))D(r). \tag{7.22}$$

Finally, by plugging (7.18), (7.19), (7.20), (7.21) and (7.22) into (7.16), we finish the proof of this lemma. □

Now, it is ready to prove the doubling inequality.

Lemma 7.6. *(Doubling inequality) Let $\widetilde{U} \in H^1(\mathbb{R}^{n+1}, |y|^{1-2s})$ be a weak solution of $\nabla_{x,y} \cdot (|y|^{1-2s}\widetilde{A}(x)\nabla_{x,y}\widetilde{U}) = 0$ in B_1^{n+1}, then there exists a constant $C > 0$ such that*

$$\int_{B_{2R}^{n+1}} |y|^{1-2s} \left|\widetilde{U}\right|^2 dxdy \leq C \int_{B_R^{n+1}} |y|^{1-2s} \left|\widetilde{U}\right|^2 dxdy, \tag{7.23}$$

whenever $B_{2R}^{n+1} \subset B_1^{n+1}$.

Proof. Since $H(r)$ and $D(r)$ are differentiable, so we can differentiate $N(r)$ with respect to r, then we get

$$N'(r) = N(r) \left\{ \frac{1}{r} + \frac{D'(r)}{D(r)} - \frac{H'(r)}{H(r)} \right\}. \tag{7.24}$$

If we plug (7.10), (7.11), (7.12) and (7.15) into (7.24) and use the Cauchy-Schwartz inequality, then we can deduce that

$$\begin{aligned}
&\frac{1}{r} + \frac{D'(r)}{D(r)} - \frac{H'(r)}{H(r)} \\
\geq& 2\left(\frac{\int_{\partial B_r^{n+1}} |y|^{1-2s} \frac{1}{\mu} \left|\widetilde{A}\nu \cdot \nabla\widetilde{U}\right|^2 dS}{\int_{\partial B_r^{n+1}} |y|^{1-2s} \widetilde{U} \left(\widetilde{A}\nu \cdot \nabla\widetilde{U}\right) dS} - \frac{\int_{\partial B_r^{n+1}} |y|^{1-2s} \widetilde{U} \left(\widetilde{A}\nu \cdot \nabla\widetilde{U}\right) dS}{\int_{\partial B_r^{n+1}} |y|^{1-2s} \mu \left|\widetilde{U}\right|^2 dS} \right) + O(1) \\
\geq& O(1)
\end{aligned}$$

which implies

$$N'(r) \geq -CN(r)$$

for some constant $C > 0$. Moreover, for $R < 1$, we integrate the above inequality over R to 1, then we have

$$\int_R^1 \frac{d}{dr} \log N(r) dr \geq -C(1-R) \geq -C$$

or

$$N(R) \leq e^{-C} N(1). \tag{7.25}$$

Note that (7.12) is equivalent to

$$\frac{d}{dr} \log \frac{H(r)}{r^{n+1-2s}} = 2\frac{N(r)}{r} + O(1),$$

where $O(1)$ is independent of r. After integrating over $(r, 2r)$ and use (7.25), it is easy to see $H(2r) \leq CH(r)$ and integrate this quantity over $(0, R)$, which proves the doubling inequality (7.23).

References

[1] Fuensanta Andreu-Vaillo, José M Mazón, Julio D Rossi, and J Julián Toledo-Melero. *Nonlocal diffusion problems*, volume 165. American Mathematical Society Providence, 2010.

[2] Eemeli Blåsten, O Yu Imanuvilov, and Masahiro Yamamoto. Stability and uniqueness for a two-dimensional inverse boundary value problem for less regular potentials. *arXiv preprint arXiv:1504.02207*, 2015.

[3] Krzysztof Bogdan. The boundary Harnack principle for the fractional Laplacian. *Studia Mathematica*, 123(1):43–80, 1997.

[4] Claudia Bucur and Enrico Valdinoci. *Nonlocal diffusion and applications*, volume 20. Springer, 2016.

[5] Alexander Bukhgeim. Recovering a potential from Cauchy data in the two-dimensional case. *Journal of Inverse and Ill-posed Problems, V*, 16(1):19–33, 2008.

[6] Xavier Cabré and Yannick Sire. Nonlinear equations for fractional Laplacians, I: Regularity, maximum principles, and Hamiltonian estimates. *Annales de l'Institut Henri Poincare (C) Non Linear Analysis*, 31(1):23–53, 2014.

[7] Luis Caffarelli and Luis Silvestre. An extension problem related to the fractional Laplacian. *Communications in partial differential equations*, 32(8):1245–1260, 2007.

[8] Luis A Caffarelli and Pablo Raúl Stinga. Fractional elliptic equations, Caccioppoli estimates and regularity. *Annales de l'Institut Henri Poincare (C) Non Linear Analysis*, 33(3):767–807, 2016.

[9] Alberto P Calderón. On an inverse boundary value problem. *Computational & Applied Mathematics*, 25(2-3):133–138, 2006.

[10] Sagun Chanillo. A problem in electrical prospection and an n-dimensional borg-levinson theorem. *Proceedings of the American Mathematical Society*, 108(3):761–767, 1990.

[11] Zhen-Qing Chen, Masatoshi Fukushima, and Jiangang Ying. Traces of symmetric Markov processes and their characterizations. *The Annals of Probability*, 34(3):1052–1102, 2006.

[12] Edward Brian Davies. *Heat kernels and spectral theory*, volume 92. Cambridge university press, 1990.

[13] Eleonora Di Nezza, Giampiero Palatucci, and Enrico Valdinoci. Hitchhiker's guide to the fractional Sobolev spaces. *Bulletin des Sciences Mathématiques*, 136(5):521–573, 2012.

[14] Serena Dipierro, Xavier Ros-Oton, and Enrico Valdinoci. Nonlocal problems with Neumann boundary conditions. *arXiv preprint arXiv:1407.3313*, 2014.

[15] Serena Dipierro, Ovidiu Savin, and Enrico Valdinoci. All functions are locally s-harmonic up to a small error. *Journal of the European Mathematical Society*, 19(4):957–966, 2017.

[16] Eugene B Fabes, Carlos Kenig, and Raul P Serapioni. The local regularity of solutions of degenerate elliptic equations. *Communications in Statistics-Theory and Methods*, 7(1):77–116, 1982.

[17] David Dos Santos Ferreira, Carlos Kenig, Mikko Salo, and Gunther Uhlmann. Limiting Carleman weights and anisotropic inverse problems. *Inventiones mathematicae*, 178(1):119–171, 2009.

[18] Tuhin Ghosh, Mikko Salo, and Gunther Uhlmann. The Calderón problem for the fractional Schrödinger equation. *arXiv preprint arXiv:1609.09248*, 2016.

[19] David Gilbarg and Neil S Trudinger. *Elliptic partial differential equations of second order*. springer, 2015.

[20] Alexander Grigor'yan. *Heat kernel and analysis on manifolds*, volume 47. American Mathematical Society Providence, 2009.

[21] Gerd Grubb. Fractional Laplacians on domains, a development of Hörmander's theory of μ-transmission pseudodifferential operators. *Advances in Mathematics*, 268:478–528, 2015.

[22] Colin Guillarmou and Leo Tzou. Calderón inverse problem with partial data on Riemann surfaces. *Duke Mathematical Journal*, 158(1):83–120, 2011.

[23] Oleg Imanuvilov, Gunther Uhlmann, and Masahiro Yamamoto. The Calderón problem with partial data in two dimensions. *Journal of the American Mathematical Society*, 23(3):655–691, 2010.

[24] Victor Isakov. *Inverse source problems*. Number 34. American Mathematical Soc., 1990.

[25] Carlos Kenig and Mikko Salo. The Calderón problem with partial data on manifolds and applications. *Analysis & PDE*, 6(8):2003–2048, 2014.

[26] Carlos Kenig, Johannes Sjöstrand, and Gunther Uhlmann. The Calderón problem with partial data. *Annals of Mathematics*, 165(2):567–591, 2007.

[27] Alois Kufner and Anna-Margarete Sändig. *Some applications of weighted Sobolev spaces*, volume 100. BG Teubner Gmbh, 1987.

[28] Peter Lax. A stability theorem for solutions of abstract differential equations, and its application to the study of the local behavior of solutions of elliptic equations. *Communications on Pure and Applied Mathematics*, 9(4):747–766, 1956.

[29] Fang-Hua Lin. Nodal sets of solutions of elliptic and parabolic equations. *Communications on Pure and Applied Mathematics*, 44(3):287–308, 1991.

[30] William Charles Hector McLean. *Strongly elliptic systems and boundary integral equations.* Cambridge University Press, 2000.

[31] Benjamin Muckenhoupt. Weighted norm inequalities for the Hardy maximal function. *Transactions of the American Mathematical Society*, 165:207–226, 1972.

[32] Adrian I Nachman. Inverse scattering at fixed energy. *Mathematical Physics X*, pages 434–441, 1992.

[33] Aleš Nekvinda. Characterization of traces of the weighted Sobolev space $W^{1,p}(\Omega, d_M^\epsilon)$ on M. *Czechoslovak Mathematical Journal*, 43(4):695–711, 1993.

[34] Amnon Pazy. *Semigroups of linear operators and applications to partial differential equations*, volume 44. Springer Science & Business Media, 2012.

[35] Petteri Piiroinen and Martin Simon. Probabilistic interpretation of the Calderón problem. *arXiv preprint arXiv:1503.05949*, 2015.

[36] Frigyes Riesz and Béla Sz Nagy. *Functional Analysis. Translated from the 2nd French edition by Leo F. Boron.* Dover Publications Inc., New York, 1990.

[37] Xavier Ros-Oton. Nonlocal elliptic equations in bounded domains: a survey. *arXiv preprint arXiv:1504.04099*, 2015.

[38] Xavier Ros-Oton and Joaquim Serra. The Pohozaev identity for the fractional Laplacian. *arXiv preprint arXiv:1207.5986*, 2012.

[39] Xavier Ros-Oton and Joaquim Serra. The Dirichlet problem for the fractional Laplacian: regularity up to the boundary. *Journal de Mathématiques Pures et Appliquées*, 101(3):275–302, 2014.

[40] Walter Rudin. *Functional analysis.* McGraw-Hill Series in Higher Mathematics. New York: McGraw-Hill Book Company.

[41] Angkana Rüland. Unique continuation for fractional Schrödinger equations with rough potentials. *Communications in Partial Differential Equations*, 40(1):77–114, 2015.

[42] R. T. Seeley. Complex powers of an elliptic operator. 1967 Singular Integrals (Proc. Sympos. Pure Math., Chicago, Ill., 1966) pp. 288–307 Amer. *Math. Soc., Providence, RI.*

[43] Luis Silvestre. Regularity of the obstacle problem for a fractional power of the Laplace operator. *Communications on Pure and Applied Mathematics*, 60(1):67–112, 2007.

[44] Pablo Raúl Stinga and José Luis Torrea. Extension problem and Harnack's inequality for some fractional operators. *Communications in Partial Differential Equations*, 35(11):2092–2122, 2010.

[45] John Sylvester and Gunther Uhlmann. A global uniqueness theorem for an inverse boundary value problem. *Annals of Mathematics*, 125(1):153–169, 1987.

[46] Xiangxing Tao and Songyan Zhang. Weighted doubling properties and unique continuation theorems for the degenerate Schrödinger equations with singular potentials. *Journal of Mathematical Analysis and Applications*, 339(1):70–84, 2008.

[47] Michael E Taylor. Functions of several self-adjoint operators. *Proceedings of the American Mathematical Society*, 19(1):91–98, 1968.

[48] Alexander Ivanovich Tyulenev. Description of traces of functions in the Sobolev space with a Muckenhoupt weight. *Proceedings of the Steklov Institute of Mathematics*, 1(284):280–295, 2014.

[49] Gunther Uhlmann. Electrical impedance tomography and Calderón's problem. *Inverse problems*, 25(12):123011, 2009.

[50] Gunther Uhlmann. Inverse problems: seeing the unseen. *Bulletin of Mathematical Sciences*, 4(2):209–279, 2014.

[51] Hui Yu. Unique continuation for fractional orders of elliptic equations. *arXiv preprint arXiv:1609.01376*, 2016.

□