Boundary determination of the Lamé moduli for the isotropic elasticity system

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Abstract

We consider the inverse boundary value problem of determining the Lamé moduli of an isotropic, static elasticity equations of system at the boundary from the localized Dirichlet-to-Neumann map. Assuming appropriate local regularity assumptions as weak as possible on the Lamé moduli and on the boundary, we give explicit pointwise reconstruction formulae of the Lamé moduli and their higher order derivatives at the boundary from the localized Dirichlet-to-Neumann map.

Key words: Inverse boundary value problem, Dirichlet-to-Neumann map, isotropic elasticity system, boundary determination, Stroh formalism
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1 Introduction and main result

Let us briefly give our main result before giving its detailed mathematical description. That is we give explicit pointwise reconstruction formulae of Lamé moduli and their derivatives at a given point on the boundary from the measured data called the localized Dirichlet-to-Neumann map for the inverse boundary value problem associated to an isotropic elastic equation in a bounded domain. We will refer this kind of inverse problem by *boundary determination.*

Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with boundary $\partial\Omega$ and $\lambda = \lambda(x), \mu = \mu(x)$ be the Lamé moduli which satisfy

$$\mu > 0,\ 3\lambda + 2\mu > 0 \text{ on } \overline{\Omega}. \tag{1.1}$$

The regularity of $\partial\Omega$ and Lamé moduli will be specified later. Consider the boundary value problem

$$\begin{cases} (\mathcal{L}u)_i := \sum_{j,k,l=1}^3 \dfrac{\partial}{\partial x_j}(\dot{C}_{ijkl}\dfrac{\partial}{\partial x_l}u_k) = 0 \ (i = 1,2,3) & \text{in } \Omega, \\ u = f \in H^{1/2}(\partial\Omega; \mathbb{C}^3) & \text{on } \partial\Omega \end{cases} \tag{1.2}$$

for the displacement vector $u = (u_1, u_2, u_3)$, where

$$\dot{C}_{ijkl} = \dot{C}_{ijkl}(x) = \lambda\delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) \ (1 \le i,j,k,\ell \le 3) \tag{1.3}$$

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are isotropic elastic tensors in terms of the Cartesian coordinates $x = (x_1, x_2, x_3)$ with Kronecker delta δ_{ij}. It is easy to see that $\dot{C}_{ijkl}$ defined by (1.3) satisfies the symmetry given as

$$\dot{C}_{ijkl}(x) = \dot{C}_{klij}(x) = \dot{C}_{jikl}(x)$$

and the strong convexity condition given as

$$\sum_{i,j,k,l=1}^{3} \dot{C}_{ijkl}(x)\varepsilon_{ij}\varepsilon_{kl} \geq c_0 \sum_{i,j=1}^{3} \varepsilon_{ij}^2$$

with some constant $c_0 > 0$ for any $x \in \overline{\Omega}$ and symmetric matrix (ε_{ij}).

Define the Dirichlet-to-Neumann (DN) map $\Lambda_{\mathcal{C}} : H^{1/2}(\partial\Omega) \to H^{-1/2}(\partial\Omega)$ by

$$(\Lambda_{\mathcal{C}} f)_i := \sum_{j,k,l=1}^{3} \nu_j \dot{C}_{ijkl} \frac{\partial u_k}{\partial x_l}|_{\partial\Omega} \quad \text{for} \quad i = 1, 2, 3,$$

where u is the solution of (1.2) and $\nu = (\nu_1, \nu_2, \nu_3)$ is the unit normal of $\partial\Omega$ directed into the exterior of Ω. Let $\epsilon(u) := (\epsilon_{ij}(u))$ be the strain tensor associated to the solution u of (1.2), where

$$\epsilon_{ij}(u) = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \text{ for } i, j = 1, 2, 3.$$

It is well-known that for $f, g \in H^{1/2}(\partial\Omega)$,

$$\langle \Lambda_{\mathcal{C}} f, g \rangle_{H^{-1/2}(\partial\Omega)\times H^{1/2}(\partial\Omega)} = \int_{\Omega} \left(\lambda \operatorname{div} u\, (\overline{\operatorname{div} v}) + 2\mu\, \epsilon(u) : \overline{\epsilon(v)}\right) dy,$$

where $\langle \cdot, \cdot \rangle_{H^{-1/2}(\partial\Omega)\times H^{1/2}(\partial\Omega)}$ is the pairing in $H^{-1/2}(\partial\Omega) \times H^{1/2}(\partial\Omega)$, $v \in H^1(\Omega)$ can be taken whichever satisfies $v = g$ on $\partial\Omega$ and the notation ":" denotes the Frobenius inner product. Let $x_0 \in \partial\Omega$ be an arbitrary point, then the DN map $\Lambda_{\mathcal{C}}$ can be localized near x_0 by restricting the support of f, g in an open neighborhood of x_0 in $\partial\Omega$.

The precise description of the aim of this paper is to recover λ, μ and their higher-order normal derivatives near a given point $x_0 \in \partial\Omega$ by knowing the localized $\Lambda_{\mathcal{C}}$ around x_0, under the regularity assumptions on λ, μ and $\partial\Omega$ near x_0 as weak as possible. More specifically, we will show that for any $m \in \mathbb{N}$, the Lamé moduli and their normal derivatives at $x_0 \in \partial\Omega$ up to order m can be given explicitly from the localized DN map $\Lambda_{\mathcal{C}}$ around x_0. There are some related results on this boundary determination for the elasticity system. For the two dimensional isotropic elastic system, the boundary determination was given by [1] if the Lamé moduli and $\partial\Omega$ are smooth. In [7] and [8] the authors developed layer stripping algorithm in which they solved the boundary determination for the three dimensional isotropic and transversally isotropic elastic systems also for the case the elasticity tensor and $\partial\Omega$ are smooth. But it should be remarked here that a result of boundary determination under regularity assumptions as weak as possible was missing for the elasticity systems and we aimed to provide such a result for the isotropic elasticity system in this paper. For practical application, it is needless to say the importance of such a result. We note that there are related results for both the isotropic and anisotropic conductivity equations using arguments similar as in this paper. For that see [2, 3, 5, 6] and the references there in.

For any $m \in \mathbb{N}$, $p \in (0,1)$, $C^{m,p}(\Omega)$ denotes the standard Hölder space. Then the regularity assumptions on the Lamé moduli λ, μ and $\partial\Omega$ are locally C^{m+2} and $C^{m,p}$ near $x = 0$, respectively.

By introducing the boundary normal coordinates which was used in [3, 4, 6] for the conductivity equation and [12] for the elasticity equation, we can flatten $\partial\Omega$. Also, without loss of generality, we may assume that x_0 can be the origin. In terms of the boundary normal coordinates the displacement vector $u = (u_1, u_2, u_3)$ and isotropic elastic tensor $(\dot{C}_{ijkl})$ will undergo tensorial change which complicates the notations and description of arguments. Hence, in order not to distract reader's attention, we first focus on the reconstruction formulae for the Lamé parameters for the flat boundary case at $0 \in \partial\Omega$. We will illustrate the non flat boundary case in the last section of this paper. It will be shown there that the difference we will have for the non flat boundary case is just coming from the change of coordinates and normal vector.

To begin with assume that $\partial\Omega$ is flat near $0 \in \partial\Omega$ and Ω is locally given as $\{y_3 > 0\}$ in terms of the Cartesian coordinates (y_1, y_2, y_3). For the local determination of the Lamé parameters at $0 \in \partial\Omega$, we only need to assume each $\dot{C}_{ijkl}$ is of $C^{m,p}$ class around the origin. Fix $x = (x_1, x_2, 0) \in \partial\Omega$ and define $\mathcal{C}^{m,x} := (\dot{C}^{m,x}_{ijkl})$ by

$$\dot{C}^{m,x}_{ijkl}(y) := \sum_{b<m} \frac{\partial^b_{y_3}\dot{C}_{ijkl}(y',0)}{b!} y_3^b \text{ for } y \text{ near } x. \tag{1.4}$$

Then extending this $\mathcal{C}^{m,x}$ to $\overline{\Omega}$ without destroying the regularity and strong convexity, we denote the corresponding localized DN map by $\Lambda_{\mathcal{C}^{m,x}}$. Similarly, we define $\lambda^{m,x}$ and $\mu^{m,x}$ by

$$\lambda^{m,x}(y) := \sum_{b<m} \frac{\partial^b_{y_3}\lambda(y',0)}{b!} y_3^b,\ \ \mu^{m,x}(y) := \sum_{b<m} \frac{\partial^b_{y_3}\mu(y',0)}{b!} y_3^b. \tag{1.5}$$

Let $\omega' = (\omega_1, \omega_2, 0)$ be a unit tangent vector of $\partial\Omega$ at 0 and $\eta(y') \in C_0^\infty(\mathbb{R}^2)$ satisfy

$$0 \le \eta \le 1,\ \int_{\mathbb{R}^2} \eta^2 dy' = 1 \text{ and } \operatorname{supp}(\eta) \subset \{|y'| < 1\}.$$

First, by choosing suitably large $\ell \in \mathbb{N}$, we may assume that $\dfrac{m}{\ell} = \dfrac{1}{\widetilde{\rho}}$ for some large $\widetilde{\rho} \in \mathbb{N}$ with $\dfrac{1}{\widetilde{\rho}} < p$ and we also assume that

$$(1 - \frac{1}{\widetilde{\rho}})(m+p) \ge m + \frac{1}{\widetilde{\rho}}. \tag{1.6}$$

For convenience, we denote $\rho = \dfrac{1}{\widetilde{\rho}}$ and for large $N \in \mathbb{N}$, we put $\eta^N(y') := \eta(N^{1-\rho}y')$. For any column vector $\mathbf{a} = (a_1, a_2, a_3) \in \mathbb{C}^3$, let

$$\phi^N(y) := \eta^N(y') \exp(\sqrt{-1}Ny' \cdot \omega')\mathbf{a} \tag{1.7}$$

be the localized Dirichlet data around $0 \in \partial\Omega$, then we have the following theorem.

Theorem 1.1. *(1) Let Ω be of C^1 class near $0 \in \partial\Omega$ and let $\dot{C}_{ijkl}$ be continuous near $y = 0$. Then*

$$\lim_{N\to\infty} \left\langle \Lambda_{\mathcal{C}} \phi^N, \overline{\phi^N} \right\rangle = \sum_{i,j=1}^{3} Z_{ij}(0) a_i \overline{a_j} \tag{1.8}$$

and $Z_{ij} = \overline{Z_{ji}}$ for $1 \le i, j \le 3$, where

$$\begin{aligned} Z_{ii} &= \frac{\mu}{\lambda + 3\mu}\big(2(\lambda + 2\mu) - (\lambda + \mu)\iota_i^2\big), \\ Z_{ij} &= \frac{\mu}{\lambda + 3\mu}\big(- (\lambda + \mu)\iota_i\iota_j + \sqrt{-1}(-1)^k 2\mu\, \iota_k\big),\ 1 \le i < j \le 3 \end{aligned} \tag{1.9}$$

with $(\iota_1, \iota_2, \iota_3) = (\omega_2, -\omega_1, 0)$ and the index $k \in \mathbb{N}$ has to satisfy $1 \le k \le 3$, $k \ne i, j$.

(2) For $m \in \mathbb{N}$, let $\partial\Omega$ be of C^{m+2} class near $0 \in \partial\Omega$. Let $\mathcal{C} = (\dot{C}_{ijkl})$ be of $C^{m,p}$ near 0. Then

$$\begin{aligned} &\lim_{N\to\infty} N^m \left\langle (\Lambda_{\mathcal{C}} - \Lambda_{\mathcal{C}^{m,0}})\phi^N, \overline{\phi^N} \right\rangle \qquad (1.10) \\ =&\frac{1}{2^{m+1}} \frac{\partial^m \lambda}{\partial y_3^m}(0) \left(\sqrt{-1} \sum_{i=1}^{2} \omega_i a_i - a_3 \right)^2 \\ &+ \frac{1}{2^m} \frac{\partial^m \mu}{\partial y_3^m}(0) \left[\sum_{i,j=1}^{2} \left(\frac{a_i\omega_j + a_j\omega_i}{2} \right)^2 + 2 \sum_{i=1}^{2} \left(\frac{\sqrt{-1} a_3 \omega_i - a_i}{2} \right)^2 + a_3^2 \right]. \end{aligned}$$

Hence from these formulae, we can recover Lamé moduli and their derivatives up to order m.

Remark 1.2. We remark here that the above boundary determination formulae (1.8) and (1.10) are given in terms of the leading part of the equations of system. Further (1.8) was proved in the Section 2 of [11] and (1.9) was shown in Theorem 1.24 of [11], so we omit their proofs. We will only prove (1.10).

The rest of this paper is organized as follows. In Section 2, we construct an approximate solution of $\mathcal{L}u = 0$ with $u = \phi^N$ on $\partial\Omega$. By using this special solution, we will prove Theorem 1.1 for the flat boundary case in Section 3. Finally, in Section 4, we will demonstrate deriving the reconstruction formulae in terms of the boundary normal coordinates for the non-flat boundary case in the lasts section.

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2 Construction of approximate solutions

In order to prove Theorem 1.1, we need to construct an approximate solution depending on a large parameter N. Let

$$\Omega_N := \left\{ y : |y_1|, |y_2| \leq N^{\rho-1},\ 0 \leq y_3 \leq \frac{1}{\sqrt{N}} \right\}. \tag{2.1}$$

and α be a multi-index such that

$$\alpha = (1-\rho, 1-\rho, 1) \text{ and } N^\alpha y = (N^{1-\rho} y_1, N^{1-\rho} y_2, N y_3) = (N^{1-\rho} y', N y_3).$$

Inspired by [3, 6], we can prove the following lemma.

Lemma 2.1. *For each $N \in \mathbb{N}$, there exists for any column vector $\mathbf{a} = (a_1, a_2, a_3) \in \mathbb{C}^3$ an approximate solution Φ^N of (1.2) near 0 of the form*

$$\Phi^N(y) = e^{\sqrt{-1} N y' \cdot \omega'} e^{-N y_3} \left\{ \eta^N(y')\, \mathbf{a} + \sum_{n=1}^{\frac{m}{\rho}} N^{-n\rho} v_n(N^\alpha y) \right\} \tag{2.2}$$

with $\Phi^N|_{\partial\Omega} = \phi^N = e^{\sqrt{-1} N y' \cdot \omega'} \eta^N(y')\, \mathbf{a}$, where each vector $v_n(N^\alpha y)$ is polynomial in $N y_3$ with coefficients which are C^∞-smooth functions of $N^{1-\rho} y'$ supported in $\{|y'| < N^{\rho-1}\}$ for $n = 1, 2, \cdots, \dfrac{m}{\rho}$ and

$$\left|(\mathcal{L}\Phi^N)(y)\right| \leq C N^{2-m-\rho} \mathcal{P}(N y_3) e^{-N y_3},\ y \in \Omega_N \tag{2.3}$$

for some constant $C = C(m) > 0$. Here $\mathcal{P}(N y_3)$ is a polynomial with non-negative coefficients.

Proof. We look for $\Phi^N = (\Phi_1^N, \Phi_2^N, \Phi_3^N) \in \mathbb{C}^3$ of the form

$$\Phi_k^N(y) = e^{\sqrt{-1} N y' \cdot \omega'} \mathbb{V}_k(N^{1-\rho} y', N y_3). \tag{2.4}$$

Then

$$\begin{aligned}
&(\mathcal{L}\Phi^N)_i \\
&= \sum_{j,k,l=1}^{3} \frac{\partial}{\partial y_j} (\dot{C}_{ijkl} \frac{\partial}{\partial y_l} \Phi_k^N) \\
&= \sum_{k=1}^{3} \Big\{ \sum_{j,l=1}^{2} \dot{C}_{ijkl} (\frac{\partial^2}{\partial y_j \partial y_l} \Phi_k^N) + \sum_{j=1}^{2} \dot{C}_{ijk3} (\frac{\partial^2}{\partial y_j \partial y_3} \Phi_k^N) + \sum_{l=1}^{2} \dot{C}_{i3kl} (\frac{\partial^2}{\partial y_3 \partial y_l} \Phi_k^N) \\
&\quad + \dot{C}_{i3k3} \frac{\partial^2}{\partial y_3^2} \Phi_k^N + \sum_{j=1}^{3} \sum_{l=1}^{2} \big(\frac{\partial}{\partial y_j} \dot{C}_{ijkl} \big) \frac{\partial}{\partial y_l} \Phi_k^N + \sum_{j=1}^{3} \big(\frac{\partial}{\partial y_j} \dot{C}_{ijk3} \big) \frac{\partial}{\partial y_3} \Phi_k^N \Big\}.
\end{aligned} \tag{2.5}$$

Substituting (2.4) into (2.5), we have

$$
\begin{aligned}
&(\mathcal{L}\Phi^N)_i\\
=&e^{\sqrt{-1}Ny'\cdot\omega'}\sum_{k=1}^{3}\Big[-N^2\sum_{j,l=1}^{2}\dot{C}_{ijkl}\omega_j\omega_l+\sqrt{-1}N\Big\{\sum_{j,l=1}^{2}\dot{C}_{ijkl}(\omega_j\frac{\partial}{\partial y_l}+\omega_l\frac{\partial}{\partial y_j})\\
&+\sum_{j=1}^{2}\dot{C}_{ijk3}\omega_j\frac{\partial}{\partial y_3}+\sum_{l=1}^{2}\dot{C}_{i3kl}\omega_l\frac{\partial}{\partial y_3}+\sum_{j,l=1}^{2}(\frac{\partial}{\partial y_j}\dot{C}_{ijkl})\omega_l\Big\}\\
&+\sum_{j=1}^{2}\dot{C}_{ijk3}\frac{\partial^2}{\partial y_j\partial y_3}+\sum_{l=1}^{2}\dot{C}_{i3kl}\frac{\partial^2}{\partial y_l\partial y_3}+\sum_{j,l=1}^{2}\dot{C}_{ijkl}\frac{\partial^2}{\partial y_j\partial y_l}+\dot{C}_{i3k3}\frac{\partial^2}{\partial y_3^2}\\
&+\sum_{j=1}^{3}\sum_{l=1}^{2}(\frac{\partial}{\partial y_j}\dot{C}_{ijkl})\frac{\partial}{\partial y_l}+\sum_{j=1}^{3}(\frac{\partial}{\partial y_j}\dot{C}_{ijk3})\frac{\partial}{\partial y_3}\Big]\mathbb{V}_k(N^{1-\rho}y',Ny_3). \qquad (2.6)
\end{aligned}
$$

Now, we introduce the scaled variables

$$z_i=N^{1-\rho}y_i \text{ for } i=1,2 \text{ and } z_3=Ny_3, \qquad (2.7)$$

which implies

$$\frac{\partial}{\partial y_i}=N^{1-\rho}\frac{\partial}{\partial z_i} \text{ for } i=1,2 \text{ and } \frac{\partial}{\partial y_3}=N\frac{\partial}{\partial z_3}. \qquad (2.8)$$

We refer (2.7) by *scaling*. Then (2.6) becomes

$$
\begin{aligned}
&(\mathcal{L}\Phi^N)_i\\
=&e^{\sqrt{-1}Ny'\cdot\omega'}\sum_{k=1}^{3}\Bigg\{\Bigg[N^2(\dot{C}_{i3k3}\frac{\partial^2}{\partial z_3^2}+\sqrt{-1}(\sum_{j=1}^{2}\dot{C}_{ijk3}\omega_j\frac{\partial}{\partial z_3}+\sum_{l=1}^{2}\dot{C}_{i3kl}\omega_l\frac{\partial}{\partial z_3}\\
&-\sum_{j,l=1}^{2}\dot{C}_{ijkl}\omega_j\omega_l)+N^{2-\rho}(\sqrt{-1}\sum_{j,l=1}^{2}\dot{C}_{ijkl}(\omega_j\frac{\partial}{\partial z_l}+\omega_l\frac{\partial}{\partial z_j})\\
&+\sum_{j=1}^{2}\dot{C}_{ijk3}\frac{\partial^2}{\partial z_j\partial z_3}+\sum_{l=1}^{2}\dot{C}_{i3kl}\frac{\partial^2}{\partial z_l\partial z_3})+N^{2-2\rho}\sum_{j,l=1}^{2}\dot{C}_{ijkl}\frac{\partial^2}{\partial z_j\partial z_l}\\
&+N[\sqrt{-1}\sum_{j,l=1}^{2}\left(\frac{\partial}{\partial y_j}\dot{C}_{ijkl}\right)\omega_l+\sum_{j=1}^{3}\left(\frac{\partial}{\partial y_j}\dot{C}_{ijk3}\right)\frac{\partial}{\partial z_3}]\\
&+N^{1-\rho}\sum_{j=1}^{3}\sum_{l=1}^{2}\left(\frac{\partial}{\partial y_j}\dot{C}_{ijkl}\right)\frac{\partial}{\partial z_l}\Bigg]\mathbb{V}_k(z',z_3)\Bigg\}. \qquad (2.9)
\end{aligned}
$$

On the other hand, expand $\dot{C}_{ijkl}(y)$ of and $\frac{\partial}{\partial y_n}\dot{C}_{ijkl}(y)$ for $n=1,2,3$ into Taylor's series around $y=0$. Let $\beta\in(\mathbb{N}\cup\{0\})^3$ be a multi-index, then we have

$$\dot{C}_{ijkl}(y)=\sum_{|\beta|\le m}\frac{1}{\beta!}\frac{\partial^\beta}{\partial y^\beta}C_{ijkl}(0)y^\beta+O(|y|^{m+p}),$$

and for $n = 1, 2, 3$,

$$\frac{\partial}{\partial y_n}\dot{C}_{ijkl}(y) = \sum_{|\beta|\le m-1} \frac{1}{\beta!}\frac{\partial^\beta}{\partial y^\beta}\left(\frac{\partial}{\partial y_n}\dot{C}_{ijkl}\right)(0)y^\beta + O(|y|^{m-1+p}).$$

Recall that we have posed the condition (1.6), which is $m+\rho \le (1-\rho)(m+p)$. Thus, via (2.7), we obtain

$$\dot{C}_{ijkl}(y) = \sum_{|\beta|\le m} N^{-\beta\cdot\alpha}\frac{1}{\beta!}\frac{\partial^\beta}{\partial z^\beta}\dot{C}_{ijkl}(0)z^\beta + R_1(z), \tag{2.10}$$

where

$$|R_1(z)| = O(N^{-m-\rho}).$$

Similarly, for $n = 1, 2, 3$, we have

$$\frac{\partial}{\partial y_n}\dot{C}_{ijkl}(y) = \sum_{|\beta|\le m} N^{-\beta\cdot\alpha}\frac{1}{\beta!}\frac{\partial^\beta}{\partial z^\beta}\left(\frac{\partial}{\partial y_n}\dot{C}_{ijkl}\right)(0)z^\beta + R_2(z), \tag{2.11}$$

where

$$|R_2(z)| = O(N^{-m-\rho+1}).$$

Note that for the power of N in the expansion (2.10) and (2.11) are of the form $-M\rho$ for some $M \in \mathbb{N}\cup\{0\}$. Thus, we combine (2.9), (2.10), (2.11) and $\mathbb{V} := (\mathbb{V}_1, \mathbb{V}_2, \mathbb{V}_3)^t$, then

$$\mathcal{L}\Phi^N = e^{\sqrt{-1}Ny'\cdot\omega'}\left[\sum_{s=0}^{\frac{m}{\rho}} N^{2-s\rho}L_s + L_R\right]\mathbb{V}, \tag{2.12}$$

where $\mathcal{L}\Phi^N = ((\mathcal{L}\Phi^N)_1, (\mathcal{L}\Phi^N)_2, (\mathcal{L}\Phi^N)_3)^t$ and L_s ($s = 0, 1, 2, \cdots, \frac{m}{\rho}$) are at most second order matrix differential operators in z' and z_3 with coefficients depending on y' and y_3. In particular L_0, L_1, L_2 are given by

$$\begin{aligned}
L_0 &= -\left(\dot{C}_{i3k3}(0)D_3^2 + \left[\sum_{j=1}^{2}\dot{C}_{ijk3}(0)\omega_j \right.\right.\\
&\quad \left.\left. + \sum_{l=1}^{2}\dot{C}_{i3kl}(0)\omega_l\right]D_3 + \sum_{j,l=1}^{2}\dot{C}_{ijkl}(0)\omega_j\omega_l\right)_{1\le i,k\le 3},\\
L_1 &= \left(\sqrt{-1}\sum_{j,l=1}^{2}\dot{C}_{ijkl}(0)\left(\omega_j\frac{\partial}{\partial z_l} + \omega_l\frac{\partial}{\partial z_j}\right)\right.\\
&\quad \left. + \sum_{j=1}^{2}\dot{C}_{ijk3}(0)\frac{\partial^2}{\partial z_j\partial z_3} + \sum_{l=1}^{2}\dot{C}_{i3kl}(0)\frac{\partial^2}{\partial z_l\partial z_3}\right)_{1\le i,k\le 3},\\
L_2 &= \left(\sum_{j,l=1}^{2}\dot{C}_{ijkl}(0)\frac{\partial^2}{\partial z_j\partial z_l}\right)_{1\le i,k\le 3},
\end{aligned}$$

where $D_3 = -\sqrt{-1}\dfrac{\partial}{\partial z_3}$. Moreover, L_R is a second order differential operator in z and its coefficients are of order $O(N^{2-m-\rho})$.

Now, we look for $\mathbb{V}(z', z_3)$ of the form

$$\mathbb{V}(z) = \sum_{n=0}^{\frac{m}{\rho}} N^{-\rho n} V^n(z), \tag{2.13}$$

where $V^n(z) = (V_1^n(z), V_2^n(z), V_3^n(z))^t \in \mathbb{C}^3$. By (2.12) and equating the coefficients in each term of order N^m, we have

$$\begin{aligned}\mathcal{L}\Phi^N &= e^{\sqrt{-1}Ny'\cdot\omega'}\left[\left(\sum_{s=0}^{\frac{m}{\rho}} N^{2-s\rho} L_s\right)\left(\sum_{n=0}^{\frac{m}{\rho}} N^{-\rho n} V^n\right) + L_R\mathbb{V}\right] \qquad (2.14)\\ &= e^{\sqrt{-1}Ny'\cdot\omega'}\left[\sum_{r=0}^{\frac{m}{\rho}} N^{2-r\rho} \sum_{n+s=r} L_s V^n + \mathcal{R}\right],\end{aligned}$$

where

$$\mathcal{R} := \sum_{r=\frac{m}{\rho}+1}^{\frac{2m}{\rho}} N^{2-r\rho} \sum_{n+s=r} L_s V^n + L_R\mathbb{V}.$$

Therefore, we have obtained the following ordinary differential equations of systems (ODE systems) of second order with respect to z_3

$$\begin{aligned}&L_0V^0 = 0,\\ &L_0V^1 + L_1V^0 = 0,\\ &L_0V^2 + L_1V^1 + L_2V^0 = 0,\\ &\cdots\\ &L_0V^{\frac{m}{\rho}} + \cdots + L_{\frac{m}{\rho}}V^0 = 0,\end{aligned} \tag{2.15}$$

with boundary conditions

$$\begin{aligned}&V^0|_{z_3=0} = \eta^N(y')\,\mathbf{a} = \eta(z')\,\mathbf{a},\\ &V^n|_{z_3=0} = 0 \text{ for } n = 1, 2, \cdots, \frac{m}{\rho}.\end{aligned}$$

Note that this undetermined boundary value problem (2.15) can be made determined if we look for solutions which are bounded in $z_3 \in [0, \infty)$.

First, by using the Stroh formalism which is for instance given in [11], we can solve $L_0V^0 = 0$ with $V^0(z', 0) = \eta(z')$ in the following way. The method in solving system of differential equations was consider in [9, 13]. Let $\xi = (\xi_1, \xi_2, \xi_3)$, $\zeta = (\zeta_1, \zeta_2, \zeta_3) \in \mathbb{R}^3$ and we define the 3×3 matrix $\langle \xi, \zeta\rangle$ by

$$\langle \xi, \zeta\rangle = (\langle \xi, \zeta\rangle_{ik}) \text{ with } \langle \xi, \zeta\rangle_{ik} = \sum_{1\le j,\,l\le 3} \dot{C}_{ijkl}(y', y_3)\xi_j\,\zeta_l.$$

Also, if we set $\langle \xi, \zeta\rangle_0 := \langle \xi, \zeta\rangle|_{y=0}$, $e_3 := (0, 0, 1)$ and $\omega := (\omega_1, \omega_2, 0)$, we can rewrite

$$L_0V^0 = -\left[\langle e_3, e_3\rangle_0 D_3^2 + (\langle e_3, \omega\rangle_0 + \langle \omega, e_3\rangle_0) D_3 + \langle \omega, \omega\rangle_0\right] V^0 = 0.$$

Let $W_1^0 = V^0$, $W_2^0 = -\{\langle e_3, e_3\rangle_0 D_3V^0 + \langle e_3, \omega\rangle_0 V^0\}$, then by direct calculation, then we have

$$D_3W_1^0 = -\langle e_3, e_3\rangle_0^{-1} \left[\langle e_3, \omega\rangle_0 W_1^0 - W_0^2\right] \tag{2.16}$$

and

$$D_3W_2^0 = \left[\langle \omega, \omega\rangle_0 - \langle \omega, e_3\rangle_0 \langle e_3, e_3\rangle_0^{-1} \langle e_3, \omega\rangle_0\right] W_1^0 - \langle \omega, e_3\rangle_0 \langle e_3, e_3\rangle_0^{-1} W_2^0. \tag{2.17}$$

Combine (2.16), (2.17) and define the column vector $W^0 := [W_1^0, W_2^0]$, then we obtain

$$D_3W^0 = K^0W^0, \tag{2.18}$$

where

$$K^0 = \begin{bmatrix} -\langle e_3, e_3\rangle_0^{-1} \langle e_3, \omega\rangle_0 & -\langle e_3, e_3\rangle_0^{-1} \\ -\langle \omega, \omega\rangle_0 + \langle \omega, e_3\rangle_0 \langle e_3, e_3\rangle_0^{-1} \langle e_3, \omega\rangle_0 & -\langle \omega, e_3\rangle_0 \langle e_3, e_3\rangle_0^{-1} \end{bmatrix}.$$

Note that K^0 is a 6×6 matrix-valued function independent of z_3 variable and its eigenvalues are determined by

$$\det(\Sigma I_6 - K^0) = 0, \tag{2.19}$$

where I_6 is a 6×6 identity matrix and (2.19) is equivalent to

$$\det\left[\langle e_3, e_3\rangle_0 \Sigma^2 + (\langle e_3, \omega\rangle_0 + \langle \omega, e_3\rangle_0)\Sigma + \langle \omega, \omega\rangle_0\right] = 0. \tag{2.20}$$

By using the results of [11, Chapter 1.8] , we have

$$\begin{aligned} &\det\left[\langle e_3, e_3\rangle_0 \Sigma^2 + (\langle e_3, \omega\rangle_0 + \langle \omega, e_3\rangle_0)\Sigma + \langle \omega, \omega\rangle_0\right] \\ =&\mu^2(0)(\lambda + 2\mu)(0)(1 + \Sigma^2)^3, \end{aligned}$$

which means solving (2.19) is equivalent to solve

$$(1 + \Sigma^2)^3 = 0$$

and use the strong convexity condition (1.1), then it gives that the roots are $\Sigma = \pm\sqrt{-1}$. Moreover, we can find eigenvectors $\{\widetilde{q_1^+}, \widetilde{q_2^+}, \widetilde{q_3^+}, \widetilde{q_1^-}, \widetilde{q_2^-}, \widetilde{q_3^-}\}(0)$ of K^0, i.e.,

$$K^0\widetilde{q_\gamma^\pm} = \pm\sqrt{-1}\widetilde{q_\gamma^\pm} \text{ for } \gamma = 1, 2, 3, \tag{2.21}$$

with $\widetilde{q_\gamma^+}$ being the complex conjugate of $\widetilde{q_\gamma^-}$, or $\widetilde{q_\gamma^+} = \overline{\widetilde{q_\gamma^-}}$ at $y = 0$.

According to the result in [11], the eigenvalue problem for K^0 is *degenerate* and there are generalized eigenvectors. More precisely, let

$$\widetilde{q_1^+} = \begin{pmatrix} \omega_2 \\ -\omega_1 \\ 0 \\ \sqrt{-1}\mu\omega_2 \\ -\sqrt{-1}\mu\omega_1 \\ 0 \end{pmatrix}(0), \ \widetilde{q_2^+} = \begin{pmatrix} \omega_1 \\ \omega_2 \\ \sqrt{-1} \\ -2\mu\omega_1 \\ -2\mu\omega_2 \\ 2\sqrt{-1} \end{pmatrix}(0)$$

and

$$\widetilde{q_3^+} = \begin{pmatrix} 0 \\ 0 \\ -\frac{\lambda+3\mu}{\lambda+\mu} \\ -\frac{2\mu^2}{\lambda+\mu}\omega_1 \\ -\frac{2\mu^2}{\lambda+\mu}\omega_2 \\ -\sqrt{-1}\frac{2\mu(\lambda+2\mu)}{\lambda+\mu} \end{pmatrix}(0)$$

such that

$$K^0\widetilde{q_3^+} - \sqrt{-1}\widetilde{q_3^+} = \widetilde{q_2^+}.$$

and define

$$\widetilde{Q} := \left(\widetilde{q_1^+}, \widetilde{q_2^+}, \widetilde{q_3^+}, \widetilde{q_1^-}, \widetilde{q_2^-}, \widetilde{q_3^-}\right),$$

which is a non-singular matrix giving the Jordan canonical form

$$\widetilde{Q}^{-1}K^0\widetilde{Q} = \begin{pmatrix} \sqrt{-1} & & & & & \\ & \sqrt{-1} & 1 & & & \\ & & \sqrt{-1} & & & \\ & & & -\sqrt{-1} & & \\ & & & & -\sqrt{-1} & 1 \\ & & & & & -\sqrt{-1} \end{pmatrix}.$$

Since we want to have a general form of solution of (2.18) which is bounded for $z_3 \in [0, \infty)$, we take $\zeta = \sqrt{-1}$. Further we take linearly independent vectors $\sigma_1 = \begin{pmatrix} \omega_2 \\ -\omega_1 \\ 0 \end{pmatrix}$, $\sigma_2 = \begin{pmatrix} \omega_1 \\ \omega_2 \\ \sqrt{-1} \end{pmatrix}$ and $\sigma_3 = \begin{pmatrix} 0 \\ 0 \\ -\frac{\lambda+3\mu}{\lambda+\mu}(0) \end{pmatrix}$. Then, for any given $\mathbf{a} = (a_1, a_2, a_3)$, there exists constants $c_\beta \in \mathbb{C}$ $(\beta = 1, 2, 3)$ such that $\mathbf{a} = \sum_{s=1}^{3} c_s\sigma_s$. Therefore, as in [11, Lemma 1.6 and (2.66)], $V^0(z', z_3)$ is given as

$$\begin{aligned} V^0(z', z_3) &= e^{-z_3}\eta(z')\big(\sum_{s=1}^{3} c_s\sigma_s - \sqrt{-1}c_3\,\sigma_2\,z_3\big) \\ &= e^{-z_3}\eta(z')\left(\mathbf{a} - \sqrt{-1}c_3\,z_3\,\sigma_2\right), \end{aligned} \tag{2.22}$$

which is a C^∞-smooth solution of $L_0V^0 = 0$ with $V^0(z', 0) = \eta(z')\,\mathbf{a}$.

Next, we solve

$$L_0V^1 + L_1V^0 = 0 \text{ with } V^1(z', 0) = 0. \tag{2.23}$$

Since

$$\begin{aligned} L_1 &= \Big(\sqrt{-1}\sum_{j,l=1}^{2} \dot{C}_{ijkl}(0)(\omega_j\frac{\partial}{\partial z_l} + \omega_l\frac{\partial}{\partial z_j}) \\ &\quad + \sum_{j=1}^{2} \dot{C}_{ijk3}(0)\frac{\partial^2}{\partial z_j\partial z_3} + \sum_{l=1}^{2} \dot{C}_{i3kl}(0)\frac{\partial^2}{\partial z_l\partial z_3}\Big)_{1\le i,k\le 3}, \end{aligned}$$

and

$$L_1V^0(z', z_3) = e^{-z_3}\sum_{d=0}^{1} P_0^d(z')z_3^d,$$

where $P_0^d(z')$ are C^∞-smooth vector-valued function depending on $\partial_{z'}^\beta \eta(z')$ for multi-indices $|\beta| \le 1$ for $d = 0, 1$. It is worth mentioning the following observation. That is for any $d \in \mathbb{N}$, we have by direct computation

$$L_0 \left(z_3^d e^{-z_3} \right) = e^{-z_3} \left(z_3^{d-1} R_1^d + z_3^{d-2} R_2^d \right)$$

with invertible matrices

$$\begin{aligned} R_1^d &= d \left[2 \langle e_3, e_3 \rangle_0 - \sqrt{-1} \left(\langle \omega, e_3 \rangle_0 + \langle e_3, \omega \rangle_0 \right) \right], \\ R_2^d &= -d(d-1) \langle e_3, e_3 \rangle_0 . \end{aligned}$$

Based on this we look for $V^1(z', z_3)$ in the following form

$$V^1(z', z_3) = e^{-z_3} \sum_{d=1}^{2} z_3^d P_1^d(z'), \tag{2.24}$$

where $P_1^d(z') \in \mathbb{C}^3$ are vector-valued functions which will be determined later. By straightforward computation, we can have

$$L_0 V^1 = z_3 \, e^{-z_3} \, R_1^2 + e^{-z_3} \left\{ R_2^2 P_1^2(z') + R_1^1 P_1^1(z') \right\}. \tag{2.25}$$

Then by equating the equation $L_0 V^1 = -L_1 V^0$, we have

$$P_0^1(z') = -R_1^2 P_1^2(z'), \tag{2.26}$$

$$P_0^0(z') = -R_2^2 \, P_1^2(z') - R_1^1 \, P_1^1(z'). \tag{2.27}$$

In order to solve $P_1^d(z')$ explicitly for $d = 1, 2$, first, we can invert the right hand side of (2.26) to find $P_1^2(z')$ and plug it into (2.27) to know $P_1^1(z')$. Thus we have (2.24).

Further for each $n \ge 2$, we can express the solution $V^n(z', z_3)$ of (2.15) with $V^n(z', 0) = 0$ inductively as

$$V^n(z) = \sum_{d=1}^{n+1} z_3^d P_n^d(z') e^{-z_3},$$

where $P_n^d(z')$ are smooth vector-valued functions depending on $\mu(0), \lambda(0), \eta(z')$ and supported in $\{|z'| < 1\}$ for $n = 1, 2, \cdots, \dfrac{m}{\rho}$. Finally, from (2.14) and (2.15), we have

$$\mathcal{L}\Phi^N = L_R \mathbb{V}.$$

Here note that the coefficients of L_R are of the form $O(N^{2-m-\rho})$ multiplied with polynomials in z_3. Therefore, there exists $C = C(m)$ such that

$$|L_R \mathbb{V}| \le C N^{2-m-\rho} \mathcal{P}(z_3) e^{-z_3},$$

where $\mathcal{P}(z_3)$ is a polynomial in z_3 with non-negative coefficients, which completes the proof. $\square$

3 Proof of Theorem 1.1, (2)

In this section, we prove item (2) of Theorem 1.1. Our ideas are initiated from [3, 6]. Let $\zeta(y_3) \in C^\infty([0,\infty))$ satisfy $0 \le \zeta \le 1$, $\zeta(y_3) = 1$ for $0 \le y_3 \le \dfrac{1}{2}$, $\zeta(y_3) = 0$ for $y_3 \ge 1$, and put

$$\zeta_N(y_3) = \zeta(\sqrt{N} y_3).$$

Given $\epsilon > 0$, choose large $N \in \mathbb{N}$,

$$\mathrm{supp}(\eta^N \zeta_N) \subset \Omega_\epsilon := \{|x| \le \epsilon\}.$$

For $m \in \mathbb{N}$, recall that the regularity of $\partial\Omega$ is of C^{m+2} class. For convenience, denote $\mathcal{C}^m = \mathcal{C}^{m,0}$, $\lambda^m = \lambda^{m,0}$ and $\mu^m = \mu^{m,0}$, where $\mathcal{C}^{m,x}$, $\lambda^{m,x}$ and $\mu^{m,x}$ were introduced in (1.4) and (1.5). Let $u^N = (u_1^N, u_2^N, u_3^N) \in H^1(\Omega; \mathbb{C}^3)$ be the solution to

$$\begin{cases} \sum_{j,k,l=1}^3 \dfrac{\partial}{\partial x_j}(\dot{C}_{ijkl} \dfrac{\partial}{\partial x_l} u_k^N) = 0 \ \ (1 \le i \le 3) & \text{in } \Omega, \\ u^N = \phi^N & \text{on } \partial\Omega, \end{cases} \tag{3.1}$$

and let $v^N = (v_1^N, v_2^N, v_3^N) \in H^1(\Omega; \mathbb{C}^3)$ be the solution to

$$\begin{cases} \sum_{j,k,l=1}^3 \dfrac{\partial}{\partial x_j}(\dot{C}^m_{ijkl} \dfrac{\partial}{\partial x_l} v_k^N) = 0 \ \ (1 \le i \le 3) & \text{in } \Omega, \\ v^N = \phi^N & \text{on } \partial\Omega. \end{cases} \tag{3.2}$$

Let $\zeta_N \, \Phi^N$ and $\zeta_N \, \Psi^N$ be approximate solutions of u^N and v^N with $\zeta_N \, \Phi^N|_{\partial\Omega} = \zeta_N \, \Psi^N|_{\partial\Omega} = \phi^N$, respectively. Likewise the construction in Section 2, we can express Ψ^N as

$$\Psi^N(y) = e^{\sqrt{-1}Ny'\cdot\omega'} e^{-Ny_3} \left\{ \eta^N(y')\,\mathbf{a} + \sum_{n=1}^{\frac{m}{\rho}} N^{-n\rho} v_n^m(N^\alpha y) \right\}, \tag{3.3}$$

where $v_n^m(N^\alpha y)$ are polynomials in Ny_3 depending on $\dot{C}^m_{ijkl}(0)$ and their coefficients are C^∞-smooth functions of $N^\alpha y$ supported in $\{|y'| < N^{\rho-1}\}$ for $n = 1, 2, \cdots, \dfrac{m}{\rho}$.

Note that $\left\langle \Lambda_{\mathcal{C}^m} \phi^N, \overline{\phi^N} \right\rangle$ is real and hence we have

$$\left\langle \Lambda_{\mathcal{C}^m} \phi^N, \overline{\phi^N} \right\rangle = \left\langle \overline{\Lambda_{\mathcal{C}^m} \phi^N}, \phi^N \right\rangle.$$

By direct calculation, we have

$$\begin{aligned} &\left\langle (\Lambda_{\mathcal{C}} - \Lambda_{\mathcal{C}^m})\phi^N, \overline{\phi^N} \right\rangle \\ =& \int_\Omega \left[\lambda \mathrm{div}\, u^N (\overline{\mathrm{div}\,(\zeta^N \Psi^N)}) + 2\mu\epsilon(u^N) : \overline{\epsilon(\zeta^N\Psi^N)} \right] dy \\ &- \int_\Omega \left[\lambda^m \mathrm{div}\, v^N (\overline{\mathrm{div}\,(\zeta^N \Phi^N)}) + 2\mu^m\epsilon(v^N) : \overline{\epsilon(\zeta^N\Phi^N)} \right] dy. \end{aligned}$$

Let

$$u^N = \Phi^N + f^N \text{ and } v^N = \Psi^N + g^N$$

with

$$f^N|_{\partial\Omega} = g^N|_{\partial\Omega} = 0.$$

Then we have

$$\begin{aligned}
&\left\langle (\Lambda_{\mathcal{C}} - \Lambda_{\mathcal{C}^m})\phi^N, \overline{\phi^N} \right\rangle \\
=&\int_\Omega \left[\lambda \mathrm{div}\, \Phi^N (\overline{\mathrm{div}\, (\zeta^N \Psi^N)}) + 2\mu\epsilon(\Phi^N) : \overline{\epsilon(\zeta^N \Psi^N)}\right] dy \\
&- \int_\Omega \left[\lambda^m \mathrm{div}\, \Psi^N (\overline{\mathrm{div}\, (\zeta^N \Phi^N)}) + 2\mu^m\epsilon(\Psi^N) : \overline{\epsilon(\zeta^N \Phi^N)}\right] dy \\
&+ \int_\Omega \left[\lambda \mathrm{div}\, f^N (\overline{\mathrm{div}\, (\zeta^N \Psi^N)}) + 2\mu\epsilon(f^N) : \overline{\epsilon(\zeta^N \Psi^N)}\right] dy \\
&- \int_\Omega \left[\lambda^m \mathrm{div}\, g^N (\overline{\mathrm{div}\, (\zeta^N \Phi^N)}) + 2\mu^m\epsilon(g^N) : \overline{\epsilon(\zeta^N \Phi^N)}\right] dy \\
:=& I + II + III,
\end{aligned}$$

where

$$\begin{aligned}
I &= \int_\Omega \left[\lambda \mathrm{div}\, \Phi^N (\overline{\mathrm{div}\, (\zeta^N \Psi^N)}) + 2\mu\epsilon(\Phi^N) : \overline{\epsilon(\zeta^N \Psi^N)}\right] dy \\
&\quad - \int_\Omega \left[\lambda^m \mathrm{div}\, \Psi^N (\overline{\mathrm{div}\, (\zeta^N \Phi^N)}) + 2\mu^m\epsilon(\Psi^N) : \overline{\epsilon(\zeta^N \Phi^N)}\right] dy, \\
II &= \int_\Omega \left[\lambda \mathrm{div}\, f^N (\overline{\mathrm{div}\, (\zeta^N \Psi^N)}) + 2\mu\epsilon(f^N) : \overline{\epsilon(\zeta^N \Psi^N)}\right] dy, \\
III &= - \int_\Omega \left[\lambda^m \mathrm{div}\, g^N (\overline{\mathrm{div}\, (\zeta^N \Phi^N)}) + 2\mu^m\epsilon(g^N) : \overline{\epsilon(\zeta^N \Phi^N)}\right] dy.
\end{aligned}$$

We will estimate I, II and III separately in the next subsections.

3.1 Estimate of I

Let

$$\Omega'_N := \left\{ y : |y_1|, |y_2| \le N^{\rho-1},\ \frac{1}{2\sqrt{N}} \le y_3 \le \frac{1}{\sqrt{N}} \right\} \text{ and } D_N := \Omega_N \backslash \Omega'_N.$$

Then we can rewrite I as

$$\begin{aligned}
I &= \int_{D_N} \left[(\lambda - \lambda^m) \mathrm{div}\, \Phi^N (\overline{\mathrm{div}\, \Psi^N}) + 2(\mu - \mu^m)\epsilon(\Phi^N) : \overline{\epsilon(\Psi^N)}\right] dy \\
&\quad + \int_{\Omega'_N} \left[\lambda \mathrm{div}\, \Phi^N (\overline{\mathrm{div}\, (\zeta^N \Psi^N)}) + 2\mu\epsilon(\Phi^N) : \overline{\epsilon(\zeta^N \Psi^N)}\right] dy \\
&\quad - \int_{\Omega'_N} \left[\lambda^m \mathrm{div}\, \Psi^N (\overline{\mathrm{div}\, (\zeta^N \Phi^N)}) + 2\mu^m\epsilon(\Psi^N) : \overline{\epsilon(\zeta^N \Phi^N)}\right] dy \\
&:= I_1 + I_2,
\end{aligned}$$

where

$$
\begin{aligned}
I_1 &= \int_{D_N} (\lambda-\lambda^m)\mathrm{div}\,\Phi^N(\overline{\mathrm{div}\,\Psi^N}) + 2(\mu-\mu^m)\epsilon(\Phi^N):\overline{\epsilon(\Psi^N)}dy,\\
I_2 &= \int_{\Omega'_N} \lambda \mathrm{div}\,\Phi^N(\overline{\mathrm{div}\,(\zeta^N\Psi^N)}) + 2\mu\epsilon(\Phi^N):\overline{\epsilon(\zeta^N\Psi^N)}dy,\\
&\quad -\int_{\Omega'_N} \lambda^m \mathrm{div}\,\Psi^N(\overline{\mathrm{div}\,(\zeta^N\Phi^N)}) - 2\mu^m\epsilon(\Psi^N):\overline{\epsilon(\zeta^N\Phi^N)}dy.
\end{aligned}
$$

Recall that $\mathbf{a}=(a_1,a_2,a_3)\in\mathbb{C}^3$, then for I_1, by (2.2), (3.3), and direct calculation, we have

$$
\mathrm{div}\,\Phi^N = Ne^{\sqrt{-1}Ny'\cdot\omega'}e^{-Ny_3}\left(\sqrt{-1}\sum_{i=1}^{2}\omega_i a_i - a_3\right)\eta^N(y') + O(N^{1-\rho})e^{-c_0Ny_3},
$$

with some constant $c_0>0$. Hereafter c_0 denotes a general constant which may differ time to time. Also for each $\epsilon_{ij}(\Phi^N)$ of $\epsilon(\Phi^N)=(\epsilon_{ij}(\Phi^N))$ we have

$$
\begin{aligned}
\epsilon_{ij}(\Phi^N) &= \sqrt{-1}Ne^{\sqrt{-1}Ny'\cdot\omega'}e^{-Ny_3}\frac{a_i\omega_j+a_j\omega_i}{2}\eta^N(y')\\
&\quad + O(N^{1-\rho})e^{-c_0Ny_3},\\
\epsilon_{i3}(\Phi^N) &= Ne^{\sqrt{-1}Ny'\cdot\omega'}e^{-Ny_3}\frac{\sqrt{-1}a_3\omega_i - a_i}{2}\eta^N(y') + O(N^{1-\rho})e^{-c_0Ny_3},\\
\epsilon_{33}(\Phi^N) &= -Ne^{\sqrt{-1}Ny'\cdot\omega'}e^{-Ny_3}\eta^N(y')a_3 + O(N^{1-\rho})e^{-c_0Ny_3}.
\end{aligned}
$$

Similarly, we have

$$
\mathrm{div}\,\Psi^N = Ne^{\sqrt{-1}Ny'\cdot\omega'}e^{-Ny_3}\left(\sqrt{-1}\sum_{i=1}^{2}\omega_i\, a_i - a_3\right)\eta^N(y') + O(N^{1-\rho})e^{-c_0Ny_3}
$$

and for $i,j=1,2$, we have

$$
\begin{aligned}
\epsilon_{ij}(\Psi^N) &= \sqrt{-1}Ne^{\sqrt{-1}Ny'\cdot\omega'}e^{-Ny_3}\frac{a_i\omega_j+a_j\omega_i}{2}\eta^N(y')\\
&\quad + O(N^{1-\rho})e^{-c_0Ny_3},\\
\epsilon_{i3}(\Psi^N) &= Ne^{\sqrt{-1}Ny'\cdot\omega'}e^{-Ny_3}\frac{\sqrt{-1}a_3\omega_i - a_i}{2}\eta^N(y') + O(N^{1-\rho})e^{-c_0Ny_3},\\
\epsilon_{33}(\Psi^N) &= Ne^{\sqrt{-1}Ny'\cdot\omega'}e^{-Ny_3}\eta^N(y')a_3 + O(N^{1-\rho})e^{-c_0Ny_3}.
\end{aligned}
$$

Recall that

$$
\epsilon(\Phi^N):\epsilon(\Psi^N) = \sum_{1\le i,j\le 3}\epsilon_{ij}(\Phi^N)\epsilon_{ij}(\Psi^N),
$$

by straightforward calculation, then we have

$$
\begin{aligned}
I_1 &= N^2\int_0^{\frac{1}{2\sqrt{N}}}\int_{|y'|\le N^{\rho-1}} e^{-2Ny_3}\Big\{\eta^N(y')^2(\lambda-\lambda^m)\\
&\quad \times\Big(\sqrt{-1}\sum_{i=1}^{2}\omega_i a_i - a_3\Big)^2 + 2\Big(\mu-\mu^m\Big)\eta^N(y')^2\\
&\quad \times\left[\sum_{i,j=1}^{2}\Big(\frac{a_i\omega_j+a_j\omega_i}{2}\Big)^2 + 2\sum_{i=1}^{2}\Big(\frac{\sqrt{-1}a_3\omega_i-a_i}{2}\Big)^2 + a_3^2\right]\Big\}dy'\,dy_3\\
&\quad + O(N^{2-\rho})\int_0^{\frac{1}{2\sqrt{N}}}\int_{|y'|\le N^{\rho-1}} e^{-2c_0Ny_3}\big(|\lambda-\lambda^m|+|\mu-\mu^m|\big)\,dy'\,dy_3. \qquad (3.4)
\end{aligned}
$$

For further argument we need the following lemma which was proved in [6].

Lemma 3.1. *[6] For any $k \in \mathbb{N}$, $f(y) = f(y', y_3) \in C^k$ around $y = 0$, we have*

$$\lim_{N\to\infty} N^{2+k} \int_0^{\frac{1}{2\sqrt{N}}} \int_{|y'|\le N^{\rho-1}} \eta^N(y')^2 e^{-2Ny_3} \left(f(y) - f^k(y)\right) dy' dy_3$$
$$= \frac{1}{2^{k+1}} \frac{\partial^k f}{\partial y_3^k}(0), \tag{3.5}$$

where $f^k(y) = \sum_{n=0}^{k-1} \frac{1}{n!} \frac{\partial^n}{\partial y_3^n} f(y', 0) y_3^n$. Also for each $d \in \mathbb{N}$, we have

$$\lim_{N\to\infty} N^{2-\rho+k} \int_0^{\frac{1}{2\sqrt{N}}} \int_{|y'|\le N^{\rho-1}} \psi(\sqrt{N}y')(Ny_3)^d e^{-2Ny_3} \left| f(y) - f^k(y) \right| dy' dy_3$$
$$= 0, \tag{3.6}$$

where $\psi = \psi(\sqrt{N}y')$ is a C^∞ function supported in $\{|y'| < \frac{1}{\sqrt{N}}\}$.

Take the limit $N \to \infty$ on (3.4) and use (3.5) and (3.6), then we can see that

$$\lim_{N\to\infty} N^m I_1 \tag{3.7}$$
$$= \frac{1}{2^{m+1}} \frac{\partial^m \lambda}{\partial y_3^m}(0) \left(\sqrt{-1}\sum_{i=1}^{2} \omega_i a_i - a_3\right)^2$$
$$+ \frac{1}{2^m} \frac{\partial^m \mu}{\partial y_3^m}(0) \left[\sum_{i,j=1}^{2} \left(\frac{a_i\omega_j + a_j\omega_i}{2}\right)^2 + 2\left(\frac{\sqrt{-1}a_3\omega_i - a_i}{2}\right)^2 + a_3^2 \right].$$

For I_2, via (2.2) and (3.3), we have $|\Phi^N| + |\Psi^N| \le \exp(-c_1 N y_3)$ for some constant $c_1 > 0$ and

$$|I_2| \le c_0 \exp\left(-c_1 \frac{N^{\frac{1}{2}}}{2}\right). \tag{3.8}$$

Combining (3.7) and (3.8), we have

$$\lim_{N\to\infty} N^m I$$
$$= \frac{1}{2^{m+1}} \frac{\partial^m \lambda}{\partial y_3^m}(0) \left(\sqrt{-1}\sum_{i=1}^{2} \omega_i a_i - a_3\right)^2$$
$$+ \frac{1}{2^m} \frac{\partial^m \mu}{\partial y_3^m}(0) \left[\sum_{i,j=1}^{2} \left(\frac{a_i\omega_j + a_j\omega_i}{2}\right)^2 + 2\left(\frac{\sqrt{-1}a_3\omega_i - a_i}{2}\right)^2 + a_3^2 \right].$$

3.2 Estimates of II and III

In this section, we will prove

$$\lim_{N\to\infty} N^m II = \lim_{N\to\infty} N^m III = 0.$$

Since we can use the same method to prove II and III, we only prove the case

$$\lim_{N\to\infty} N^m II = 0. \tag{3.9}$$

By direct calculation,

$$\begin{aligned} II &= \int_\Omega \Big[\lambda \mathrm{div}\, f^N(\overline{\mathrm{div}\,(\zeta^N\Psi^N)}) + 2\mu\epsilon(f^N) : \overline{\epsilon(\zeta^N\Psi^N)}\Big]\, dy \\ &= \int_{D_N} \Big[\lambda \mathrm{div}\, f^N(\overline{\mathrm{div}\,(\Psi^N)}) + 2\mu\epsilon(f^N) : \overline{\epsilon(\Psi^N)}\Big]\, dy \\ &\quad + \int_{\Omega'_N} \Big[\lambda \mathrm{div}\, f^N(\overline{\mathrm{div}\,(\zeta^N\Psi^N)}) + 2\mu\epsilon(f^N) : \overline{\epsilon(\zeta^N\Psi^N)}\Big]\, dy \\ &:= II_1 + II_2, \end{aligned}$$

where

$$\begin{aligned} II_1 &= \int_{D_N} \Big[\lambda \mathrm{div}\, f^N(\overline{\mathrm{div}\,(\Psi^N)}) + 2\mu\epsilon(f^N) : \overline{\epsilon(\Psi^N)}\Big]\, dy, \\ II_2 &= \int_{\Omega'_N} \Big[\lambda \mathrm{div}\, f^N(\overline{\mathrm{div}\,(\zeta^N\Psi^N)}) + 2\mu\epsilon(f^N) : \overline{\epsilon(\zeta^N\Psi^N)}\Big]\, dy. \end{aligned}$$

From (2.2) (2.3) and (3.3), it is not hard to see

$$|II_2| = O(e^{-\frac{\sqrt{N}}{2}}) \text{ as } N \to \infty.$$

Hence it remains to show that

$$\lim_{N\to\infty} N^m II_1 = 0$$

with

$$II_1 = II_3 + II_4,$$

where

$$\begin{aligned} II_3 &= \int_{D_N} \Big[\lambda \mathrm{div}\, f^N(\overline{\mathrm{div}\,(\Phi^N)}) + 2\mu\epsilon(f^N) : \overline{\epsilon(\Phi^N)}\Big]\, dy, \\ II_4 &= \int_{D_N} \Big[\lambda \mathrm{div}\, f^N(\overline{\mathrm{div}\,(\Psi^N - \Phi^N)}) + 2\mu\epsilon(f^N) : \overline{\epsilon(\Psi^N - \Phi^N)}\Big]. \end{aligned}$$

Note that $f^N \in H_0^1(\Omega; \mathbb{R}^3)$ satisfies $f^N = u^N - \Phi^N$ and

$$\mathcal{L} f^N = -\mathcal{L}\Phi^N \text{ in } \Omega. \tag{3.10}$$

By using the standard elliptic regularity theory, we have

$$\|f^N\|_{H_0^1(\Omega)} \le C\|\mathcal{L}\Phi^N\|_{H^{-1}(\Omega)} \le C \left\| \Big(\sum_{j,k,l=1}^{3} \dot{C}_{ijkl} \frac{\partial}{\partial x_l} \Phi_k^N \Big)_{i=1}^3 \right\|_{L^2(\Omega)}$$

for some constant $C > 0$. By straightforward computation and (2.2), we have the following lemma.

Lemma 3.2. *Let $k, l = 1, 2, 3$. For each $b \in \mathbb{N} \cup \{0\}$, there exists a constant $C_b > 0$ such that*

$$\left\| y_3^b \frac{\partial}{\partial y_l} \Phi_k^N \right\|_{L^2(\Omega_N)} \leq C_b N^{-\frac{1}{2}+\rho-b}. \tag{3.11}$$

By taking $b = 0$, (3.11) will imply that

$$\|f^N\|_{H_0^1(\Omega)} \leq CN^{-\frac{1}{2}+\rho}.$$

Now, $\partial D_N = \Gamma_1 \cup \Gamma_2$, where

$$\begin{aligned} \Gamma_1 &:= \left\{ y : |y_1| = N^{\rho-1} \text{ or } |y_2| = N^{\rho-1},\ 0 \leq y_3 \leq \frac{1}{2\sqrt{N}} \right\}, \\ \Gamma_2 &:= \left\{ y : |y_1|, |y_2| \leq N^{\rho-1},\ y_3 = \frac{1}{2\sqrt{N}} \right\}. \end{aligned}$$

For $k, l = 1, 2, 3$, it is easy to see that

$$\frac{\partial}{\partial y_l} \Phi_k^N(y) = 0 \text{ on } \Gamma_1 \text{ and } \frac{\partial}{\partial y_l} \Phi_k^N(y) = O(e^{-\frac{1}{2}N^{\frac{1}{2}}}) \text{ on } \Gamma_2 \text{ as } N \to \infty.$$

Integration by parts yields that

$$II_3 = -\sum_{i=1}^{3} \int_{D_N} f_i^N (\mathcal{L}\Phi^N)_i dy + O(e^{-\frac{1}{2}N^{\frac{1}{2}}}) \text{ as } N \to \infty.$$

Thus, for $i = 1, 2, 3$, by using the Hardy's inequality for $f_i^N \in H_0^1(\Omega)$ which was also used in [2, 3, 6, 10], we obtain

$$\begin{aligned} \int_{D_N} f_i^N (\mathcal{L}\Phi^N)_i dy &\leq \|y_3(\mathcal{L}\Phi^N)_i\|_{L^2(D_N)} \|y_3^{-1} f_i^N\|_{L^2(D_N)} \\ &\leq C\|y_3(\mathcal{L}\Phi^N)_i\|_{L^2(D_N)} \|f_i^N\|_{H^1(D_N)} \\ &\leq C\|y_3(\mathcal{L}\Phi^N)_i\|_{L^2(D_N)} N^{-\frac{1}{2}+\rho}. \end{aligned} \tag{3.12}$$

By (2.3), we can see that

$$\begin{aligned} \|y_3(\mathcal{L}\Phi^N)_i\|_{L^2(D_N)} &\leq CN^{2-m-\rho} \|y_3 \mathcal{P}(Ny_3) e^{-Ny_3}\|_{L^2(D_N)} \\ &\leq CN^{-m-1}. \end{aligned} \tag{3.13}$$

By (3.12) and (3.13), we get

$$II_3 = O(N^{-m+\rho-3/2}) \text{ as } N \to \infty,$$

which implies

$$\lim_{N\to\infty} N^m II_3 = 0.$$

Finally, we need to show that

$$\lim_{N\to\infty} N^m II_4 = 0. \tag{3.14}$$

Notice that for $i = 1, 2, 3$,

$$\Phi_i^N - \Psi_i^N = 0 \text{ on } \Gamma_1 \text{ and } \Phi_i^N - \Psi_i^N = O(e^{-\frac{1}{2}N^{\frac{1}{2}}}) \text{ as } N \to \infty.$$

By using the integration by parts and (3.10), we have

$$
\begin{aligned}
II_4 &= -\sum_{i=1}^{3}\int_{D_N}(\mathcal{L}f^N)_i(\Phi_i^N-\Psi_i^N)dy+O(e^{-\frac{1}{2}N^{\frac{1}{2}}}) \\
&= \sum_{i=1}^{3}\int_{D_N}(\mathcal{L}\Phi^N)_i(\Phi_i^N-\Psi_i^N)dy+O(e^{-\frac{1}{2}N^{\frac{1}{2}}}).
\end{aligned}
$$

We can use the same arguments for II_3 to show (3.14), which finishes the proof of Theorem 1.1, (2).

4 Non-flat boundary case

In this section we will consider the boundary determination for the non-flat boundary case. By using the boundary normal coordinates to flatten $\partial\Omega$, we will show the necessary change we need for the non-flat boundary case based on the boundary determination argument we gave for the flat boundary case. Similar argument was given in [12, Section 3] for the isotropic elasticity system.

Given any boundary point $x_0 \in \partial\Omega$, for all $x \in \Omega$ near $x_0 \in \partial\Omega$, let $y = F(x) : \mathbb{R}^3 \to \mathbb{R}^3$ be a C^1-diffeomorphism which induces the boundary normal coordinates $y = (y', y_3)$ such that $F(x_0) = 0$ and $\nabla F(x_0) = I_3$ (a 3×3 identity matrix). Let us define the Jacobian matrix $J := \nabla F = \left(\dfrac{\partial y_a}{\partial x_r}\right)_{a,r=1}^{3}$ and denote $G = JJ^T = (g_{ai})$, where J^T is the transpose of J and $G(x_0) = I_3$. In addition, near $x_0 \in \partial\Omega$,

$$
g_{ai}(x) = \sum_{r=1}^{3}\frac{\partial x^a}{\partial x_r}(x)\frac{\partial x^i}{\partial x_r}(x)
$$

satisfying

$$
g_{33} = 1,\ g_{a3} = g_{3a} = 0 \text{ for } a = 1, 2.
$$

Now, we have the following push-forward relations of the elastic tensor $\dot{\mathcal{C}}$ by

$$
\widetilde{\mathcal{C}} := F_*\dot{\mathcal{C}} = J\dot{\mathcal{C}}J^T|_{x=F^{-1}(y)}, \tag{4.1}
$$

or componentwisely, $\widetilde{\mathcal{C}} = (\widetilde{\mathcal{C}}_{iqkp})_{1\le i,q,k,p\le 3}$ with

$$
\widetilde{\mathcal{C}}_{iqkp}(y) = \left.\left\{\sum_{j,l=1}^{3}\dot{C}_{ijkl}(x)\frac{\partial y_p}{\partial x_l}\frac{\partial y_q}{\partial x_j}\right\}\right|_{x=F^{-1}(y)}.
$$

It is easy to check that under such localized boundary normal coordinates, the isotropic elastic equation (1.2) will become

$$
\begin{cases}
(\widetilde{\mathcal{L}}u)_i := \sum_{q,k,p=1}^{3}\dfrac{\partial}{\partial y_q}(\widetilde{\mathcal{C}}_{iqkp}\dfrac{\partial}{\partial y_p}\widetilde{u}_k) = 0 & \text{in } \{y_3 > 0\},\ \text{for } i = 1, 2, 3, \\
\widetilde{u} = \widetilde{f} & \text{on } \{y_3 = 0\},
\end{cases} \tag{4.2}
$$

where $\widetilde{u} = (F^{-1})^*u := u\circ F^{-1}$ and $\widetilde{f} = f\circ F^{-1}$. Similar as in Section 2, we can find an approximate solution $\widetilde{\Phi}^N(y)$ of 4.6 with the localized boundary data

$\widetilde{\Phi}^N(y',0) = \phi^N(y') := \eta^N(y')\mathbf{a}$, where $\eta^N(y')\mathbf{a} \in \mathbb{C}^3$ was given by (1.7) with arbitrary $\mathbf{a} \in \mathbb{C}^3$.

As in [12, Section 3], by denoting $\zeta := (\zeta', \zeta_3)$, we can define

$$\begin{aligned}
\widetilde{T}(y,\zeta') &:= \left(\widetilde{C}_{i3k3}(y)\right)_{1\le i,k\le 3},\\
\widetilde{R}(y,\zeta') &:= \left(\sum_{p=1}^{2} \widetilde{C}_{ipk3}(y)\zeta_j\right)_{1\le i,k\le 3},\\
\widetilde{Q}(y,\zeta') &:= \left(\sum_{p,q=1}^{2} \widetilde{C}_{ipkq}(y)\zeta_j\zeta_l\right)_{1\le i,k\le 3}.
\end{aligned}$$

Likewise Section 2, we need to find a solution of the second order ordinary differential system with constant matrix variables

$$\begin{cases}\widetilde{T}(0)D_3^2U_0 + \left(\widetilde{R}(0) + \widetilde{R}^T(0)\right) D_3U_0 + \widetilde{Q}(0)U_0 = 0,\\ V^0|_{x^3=0} = \phi^N(y').\end{cases} \tag{4.3}$$

For that repeat the argument given in Section 2 and need to consider the following eigenvalue problem

$$\det\left[\widetilde{T}(0)\Sigma^2 + \left(\widetilde{R}(0) + \widetilde{R}^T(0)\right)\Sigma + \widetilde{Q}(0)\right] = 0, \tag{4.4}$$

which is similar to 2.20.

By the transformation rule of tensor, we have

$$\left(\sum_{p,q=1}^{3} \widetilde{C}_{iqkp}\zeta_q\zeta_p\right)_{i,k=1}^3 = J\left(\sum_{j,l=1}^{3} \dot{C}_{ijkl}\xi_j\xi_l\right)_{i,k=1}^3 J^T \tag{4.5}$$

for any x near $x_0 \in \partial\Omega$ (or for any y near $0 \in \partial\widetilde{\Omega}$, where $\widetilde{\Omega} = F(\Omega)$). In addition, for any $x \in \partial\Omega$ near x_0, we can choose a unit vector $\nu(x) = (\nu_1, \nu_2, \nu_3) \in \mathbb{R}^3$ such that for any $\xi = (\xi_1, \xi_2, \xi_3) \in \mathbb{R}^3$ can be represented as $\xi(x) = q\nu(x) + \omega(x,\xi)$ for some $q \in \mathbb{R}$ and $\nu \perp \omega$ and we define

$$\begin{aligned}
\dot{T} &:= \left(\sum_{j,l=1}^{3} \dot{C}_{ijkl}\nu_j\nu_l\right)_{1\le i,k\le 3},\\
\dot{R} &:= \left(\sum_{j,l=1}^{3} \dot{C}_{ijkl}\nu_j\omega_l\right)_{1\le i,k\le 3},\\
\dot{Q} &:= \left(\sum_{j,l=1}^{3} \dot{C}_{ijkl}\omega_j\omega_l\right)_{1\le i,k\le 3}.
\end{aligned}$$

By 4.5, we also have the following relations

$$\widetilde{T} = J\dot{T}J^T,\ \widetilde{R} = J\dot{R}J^T \text{ and } \widetilde{Q} = J\dot{Q}J^T \tag{4.6}$$

in a small neighborhood of $x_0 \in \partial\Omega$. For solving the eigenvalue problem 4.4, use the relation 4.6 and J is an invertible Jacobian matrix, then it is equivalent to solve

$$\det\left[\dot{T}\Sigma^2 + \left(\dot{R} + \dot{R}^T\right)\Sigma + \dot{Q}\right]_{x=x_0} = 0. \tag{4.7}$$

Since $\dot{C}_{ijkl}$ is isotropic, $\dot{T}$, $\dot{R}$ and $\dot{Q}$ will not change the forms if we rotate the Cartesian coordinates associated to this $\xi(x)$, therefore, for any fixed x, we may assume $\xi(x) = (\xi', \xi_3)(x)$, $\nu(x) = (0, 0, 1)$ and $\omega(x, \xi) = (\xi', 0)$.

In addition, we can construct an approximate solution in terms of this Cartesian coordinates as we did in Section 2. Hence, we can give an explicit reconstruction formulae for the Lamé moduli $\lambda(x)$, $\mu(x)$ and their derivatives from the localized DN map at any $x_0 \in \partial\Omega$ with C^{m+2}-smooth boundary. To be more precise, we will give the reconstruction formulae to identify the Lamé moduli and their first order derivatives at the boundary for the non-flat boundary case in which the effect coming from the transformation of coordinates and normal vector can be seen very clearly. The reconstruction formulae are given as follows: For any $\mathbf{a} = (a_1, a_2, a_3) \in \mathbb{C}^3$, let $y = F(x)$ be the map given above, then for any Dirichlet boundary data $\phi^N = \phi^N(F(x)\big|_{\partial\Omega})$, where $\phi^N(y') = \eta^N(y')\exp(\sqrt{-1}Ny'\cdot\omega')\mathbf{a}$, we have the following approximate solution

$$\Phi^N(y) = e^{\sqrt{-1}Ny'\cdot\omega'}e^{-Ny_3}\left\{\eta^N(y')\mathbf{a} + \sum_{n=1}^{\frac{m}{\rho}} N^{-n\rho}v_n(N^\alpha y)\right\},$$

with

$$\Phi^N(F(x))|_{\partial\Omega} = \phi^N$$

1. When $\partial\Omega \in C^1$ and $\widetilde{C}_{ijkl}$ is continuous at $x_0 \in \partial\Omega$, we have

$$\lim_{N\to\infty}\left\langle\Lambda_{\widetilde{C}}\phi^N, \overline{\phi^N}\right\rangle = \sum_{i,j=1}^{3} Z_{ij}(x_0)a_i\overline{a_j}, \tag{4.8}$$

where (Z_{ij}) is the rank 2 tensor appeared in Theorem 1.1.

2. When $\partial\Omega \in C^3$ and $\dot{C}_{ijkl} \in C^{1,p}$ near $x_0 \in \partial\Omega$, we have

$$\begin{aligned}
&\lim_{N\to\infty} N\left\langle(\Lambda_{\dot{C}} - \Lambda_{\dot{C}^1})\phi^N, \overline{\phi^N}\right\rangle \\
=&\frac{1}{2}\sum_{i,q,k,p=1}^{3}\frac{\partial}{\partial y_3}\left(\sum_{j,l=1}^{3}\dot{C}_{ijkl}(x)\frac{\partial y_p}{\partial x_l}\frac{\partial y_q}{\partial x_j}\right)\Bigg|_{x=F^{-1}(0)} A_{kp}A_{iq} \\
=&\frac{1}{4}\frac{\partial\lambda}{\partial y_3}(x_0)\left(\sqrt{-1}\sum_{i=1}^{2}\omega_i a_i - a_3\right)^2 \\
&+\frac{1}{2}\frac{\partial\mu}{\partial y_3}(x_0)\left[\sum_{i,j=1}^{2}\left(\frac{a_i\omega_j + a_j\omega_i}{2}\right)^2 + 2\sum_{i=1}^{2}\left(\frac{\sqrt{-1}a_3\omega_i - a_i}{2}\right)^2 + a_3^2\right] \\
&+\frac{1}{2}\sum_{\substack{\alpha+\beta+\gamma=1\\0\le\alpha,\beta,\gamma\le1}}\sum_{i,j,k,l,p,q=1}^{3}\left(\left(\frac{\partial^\alpha}{\partial y_3^\alpha}\dot{C}_{ijkl}\right)\frac{\partial^\beta}{\partial y_3^\beta}\left(\frac{\partial y_p}{\partial x_l}\right)\frac{\partial^\gamma}{\partial y_3^\gamma}\left(\frac{\partial y_q}{\partial x_j}\right)\right)\Bigg|_{x=x_0} A_{kp}A_{iq},
\end{aligned} \tag{4.9}$$

where the elastic tensor $\dot{\mathcal{C}}^1$ is given by

$$\dot{\mathcal{C}}^1 = F^*(\widetilde{\mathcal{C}}^{1,0}) \text{ with } \widetilde{\mathcal{C}}^{1,0} = \widetilde{\mathcal{C}}(y',0),$$

for $y = (y', y_3)$ near $0 \in \partial F(\Omega)$ and A_{kp} is a constant rank 2 tensor defined by

$$A_{kp} = \begin{cases} \sqrt{-1}\omega_p a_k, & \text{for } k = 1,2,3 \text{ and } p = 1,2, \\ -\omega_3 a_k, & \text{for } k = 1,2,3 \text{ and } p = 3. \end{cases} \tag{4.10}$$

We remark here that the boundary determination formulae for the Lame moduli and their normal derivatives are given in terms of the leading part of the equations of system. This is really an advantage of scaling (2.7) we introduced before.

Since (4.8) easily follows by taking into account on the arguments given before the previous paragraph of this section and $J(x_0) = I$, we will focus on (4.9). This formula can be derived by using the integration by parts and the representation formula of (4.1). By using the same argument given in Section 3, we know that the limit with respect to N as $N \to \infty$ of the difference of DN maps only depends on the highest order term with respect to N, which means we have the following relation

$$\lim_{N\to\infty} N \left\langle (\Lambda_{\widetilde{\mathcal{C}}} - \Lambda_{\widetilde{\mathcal{C}}^{1,0}})\phi^N, \overline{\phi^N} \right\rangle = \lim_{N\to\infty} N \int_\Omega (\mathcal{C} - \dot{\mathcal{C}}^1)\nabla\Phi^N : \nabla\Psi^N dx, \tag{4.11}$$

where $\Phi^N(F(x))$, $\Psi^N(F(x))$ are approximate solutions of the differential operators $\nabla\cdot(\dot{\mathcal{C}}\nabla)$ and $\nabla\cdot(\dot{\mathcal{C}}^1\nabla)$ with the same boundary data $\Phi^N(F(x))|_{\partial\Omega} = \Psi^N(F(x))|_{\partial\Omega} = \phi^N$, respectively.

We will further compute the right hand side of (4.11) to obtain (4.9). By the change of variable $y = F(x)$ and the chain rule, we have

$$\begin{aligned} &\int_\Omega \sum_{i,j,k,l=1}^3 \dot{C}_{ijkl}(x)\frac{\partial\Phi_k^N}{\partial x_l}\frac{\partial\Psi_i^N}{\partial x_j}dx \\ &= \int_{F(\Omega)} \sum_{i,p,k,p=1}^3 \sum_{j,l=1}^3 \left(\dot{C}_{ijkl}(x)\frac{\partial y_p}{\partial x_l}\frac{\partial y_q}{\partial x_j}\right)\Big|_{x=F^{-1}(y)} \frac{\partial\Phi_k^N}{\partial y_p}\frac{\partial\Psi_i^N}{\partial y_q}dy \\ &= \int_{F(\Omega)} \sum_{i,p,k,p=1}^3 \widetilde{C}_{ipkq}(y)\frac{\partial\Phi_k^N}{\partial y_p}\frac{\partial\Psi_i^N}{\partial y_q}dy, \end{aligned} \tag{4.12}$$

and similarly

$$\int_\Omega \sum_{i,j,k,l=1}^3 \dot{C}^1_{ijkl}(x)\frac{\partial\Phi_k^N}{\partial x_l}\frac{\partial\Psi_i^N}{\partial x_j}dx = \int_{F(\Omega)} \sum_{i,q,k,p=1}^3 \widetilde{C}_{ipkp}(y',0)\frac{\partial\Phi_k^N}{\partial y_p}\frac{\partial\Psi_i^N}{\partial y_q}dy, \tag{4.13}$$

where $\widetilde{x}$ is the point such that $F(\widetilde{x}) = (y', 0)$. Then it is easy to see

$$\begin{aligned}
&\int_{F(\Omega)} (\widetilde{\mathcal{C}}(y) - \widetilde{\mathcal{C}}^{1,0}(y',0))\nabla_y \Phi^N : \nabla_y \Psi^N dy \\
=&\int_{F(\Omega)} \sum_{i,q,k,p=1}^{3} \widetilde{C}_{ipkp}(y) \frac{\partial \Phi_k^N}{\partial y_p} \frac{\partial \Psi_i^N}{\partial y_q} dy \\
&- \int_{F(\Omega)} \sum_{i,q,k,p=1}^{3} \widetilde{C}_{ipkp}(y',0) \frac{\partial \Phi_k^N}{\partial y_p} \frac{\partial \Psi_i^N}{\partial y_q} dy
\end{aligned} \tag{4.14}$$

From direct calculation for the approximate solutions, we have

$$\nabla_y \Phi_j^N = N \begin{pmatrix} \sqrt{-1}\omega' \\ -1 \end{pmatrix} e^{\sqrt{-1}Ny'\cdot\omega'} e^{-Ny_3} \eta^N(y') a_j + O(N^{1-\rho}) \tag{4.15}$$

and

$$\nabla_y \Phi_j^N = N \begin{pmatrix} \sqrt{-1}\omega' \\ -1 \end{pmatrix} e^{\sqrt{-1}Ny'\cdot\omega'} e^{-Ny_3} \eta^N(y') a_j + O(N^{1-\rho}). \tag{4.16}$$

Substitute (4.13) and (4.13) into (4.14), by using similar arguments as in Section 3, then we have

$$\begin{aligned}
&\int_{\Omega} (\dot{\mathcal{C}}(x) - \dot{\mathcal{C}}^1(x))\nabla_y \Phi^N : \nabla_y \Psi^N dx \\
=&\int_{F(\Omega)} (\widetilde{\mathcal{C}}(y) - \widetilde{\mathcal{C}}(y',0))\nabla_y \Phi^N : \nabla_y \Psi^N dy \\
=&\sum_{i,q,k,p=1}^{3} \int_{F(\Omega)} (\widetilde{C}_{iqkp} - \widetilde{C}_{iqkp}(y',0)) \frac{\partial \Phi_k^N}{\partial y_p} \frac{\partial \Psi_i^N}{\partial y_q} dy.
\end{aligned} \tag{4.17}$$

Now, we substitute (4.15) and (4.16) into (4.17), and use (3.5) again, then we have

$$\begin{aligned}
&\lim_{N\to\infty} \int_{\Omega} (\dot{\mathcal{C}}(x) - \dot{\mathcal{C}}^1(x))\nabla_y \Phi^N : \nabla_y \Psi^N dx \\
=&\frac{1}{2} \sum_{i,q,k,p=1}^{3} \left(\frac{\partial}{\partial y_3} \widetilde{C}_{iqkp} \right)(0) A_{kp} A_{iq}.
\end{aligned} \tag{4.18}$$

Note that the quantity A_{kp} is obtained from the representation of the approximate solution and straightforward calculation.

It remains to give the explicit formula for (4.18) in terms of the elastic tensor

$\dot{\mathcal{C}}$. By the straightforward calculation for (4.18), it is not hard to see that

$$
\begin{aligned}
&\frac{1}{2}\sum_{i,q,k,p=1}^{3}\left(\frac{\partial}{\partial y_3}\widetilde{C}_{iqkp}\right)(0)A_{kp}A_{iq}\\
=&\frac{1}{2}\sum_{i,q,k,p=1}^{3}\frac{\partial}{\partial y_3}\left(\sum_{j,l=1}^{3}\dot{C}_{ijkl}(x)\frac{\partial y_p}{\partial x_l}\frac{\partial y_q}{\partial x_j}\right)\Bigg|_{x=F^{-1}(0)}A_{kp}A_{iq}\\
=&\frac{1}{4}\frac{\partial\lambda}{\partial y_3}(x_0)\left(\sqrt{-1}\sum_{i=1}^{2}\omega_i a_i-a_3\right)^2\\
&+\frac{1}{2}\frac{\partial\mu}{\partial y_3}(x_0)\left[\sum_{i,j=1}^{2}\left(\frac{a_i\omega_j+a_j\omega_i}{2}\right)^2+2\sum_{i=1}^{2}\left(\frac{\sqrt{-1}a_3\omega_i-a_i}{2}\right)^2+a_3^2\right]\\
&+\frac{1}{2}\sum_{\substack{\alpha+\beta+\gamma=1\\0\le\alpha,\beta,\gamma\le1}}\sum_{i,j,k,l,p,q=1}^{3}\left(\left(\frac{\partial^\alpha}{\partial y_3^\alpha}\dot{C}_{ijkl}\right)\frac{\partial^\beta}{\partial y_3^\beta}\left(\frac{\partial y_p}{\partial x_l}\right)\frac{\partial^\gamma}{\partial y_3^\gamma}\left(\frac{\partial y_q}{\partial x_j}\right)\right)\Bigg|_{x=x_0}A_{kp}A_{iq}
\end{aligned}
$$

which proves the reconstruction formula to identify the first order derivatives of the Lamé moduli at the boundary for the non-flat boundary case.

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