GLOBAL UNIQUENESS FOR THE FRACTIONAL SEMILINEAR SCHRÖDINGER EQUATION

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Abstract. We study global uniqueness in an inverse problem for the fractional semilinear Schrödinger equation $(-\Delta)^s u + q(x,u) = 0$ with $s \in (0,1)$. We show that an unknown function $q(x,u)$ can be uniquely determined by the Cauchy data set. In particular, this result holds for any space dimension greater than or equal to 2. Moreover, we demonstrate the comparison principle and provide a L^∞ estimate for this nonlocal equation under appropriate regularity assumptions.

1. Introduction

Let Ω be a bounded domain in $\mathbb{R}^n$, $n \geq 2$ with Lipschitz boundary $\partial\Omega$. We study the nonlocal type inverse problem for the fractional semilinear Schrödinger equation with the exterior Dirichlet data

$$\begin{cases} (-\Delta)^s u + q(x,u) = 0 & \text{in } \Omega, \\ u = g & \text{in } \Omega_e, \end{cases} \tag{1.1}$$

where $s \in (0,1)$, $g \in C_0^3(\Omega_e)$, and

$$\Omega_e := \mathbb{R}^n \setminus \overline{\Omega}$$

is the exterior domain of Ω. Here the fractional Laplacian $(-\Delta)^s$ is defined by

$$(-\Delta)^s u = c_{n,s}\text{P.V.} \int_{\mathbb{R}^n} \frac{u(x)-u(y)}{|x-y|^{n+2s}} dy,$$

for $u \in H^s(\mathbb{R}^n)$, where P.V. is the principal value and

$$c_{n,s} = \frac{\Gamma(\frac{n}{2}+s)}{|\Gamma(-s)|} \frac{4^s}{\pi^{n/2}} \tag{1.2}$$

is a constant that was explicitly calculated in [2]. For the other equivalent definitions of the fractional Laplacian $(-\Delta)^s$, we refer readers to [11].

The study of fractional nonlinear Schrödinger (FNS) equations arises in the investigation of the quantum effects in Bose-Einstein Condensation [17]. In ideal boson systems, the classical Gross-Pitaevskii (GP) equations can describe condensation of weakly interacting boson atoms at a low temperature where the probability density of quantum particles is conserved. However in the inhomogeneous media with long-range (nonlocal) interactions between particles, this yields the density profile no longer retains its shape as in the classical GP equations. This dynamics is described by the fractional GP equations, known as the FNS equation, in which the turbulence and decoherence emerge. It was observed in [10]

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that the turbulence appears from the nonlocal property of the fractional Laplacian; while the local nonlinearity helps maintain coherence of the density profile.

To study the equation (1.1), we assume that the function $q(x,t) : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ fulfilling the following conditions:

$$q(x,t) \text{ and } \partial_t q(x,t) \text{ are continuous for } (x,t) \in \overline{\Omega} \times \mathbb{R}. \tag{1.3}$$

Moreover, suppose that there exist constants $\mu > 0$ and $\delta \in (2, (2n-2s)/(n-2s))$ such that

$$\begin{cases} |q(x,t)| \leq \mu(1+|t|^{\delta-1}) \text{ for all } (x,t) \in \Omega \times \mathbb{R}, \\ \lim_{t\to 0} \dfrac{q(x,t)}{t} = 0 \text{ uniformly in } x \in \Omega, \end{cases} \tag{1.4}$$

and there exist constants $b_0 \in (0,1)$ and $r > 0$ such that

$$0 < \frac{q(x,t)}{t} \leq b_0 \partial_t q(x,t), \text{ for any } x \in \overline{\Omega}, \ |t| \geq r. \tag{1.5}$$

Meanwhile, we further assume that there is a constant $0 < M_0 < \infty$ such that

$$0 \leq \partial_t q(x,t) \leq M_0, \text{ for any } (x,t) \in \overline{\Omega} \times \mathbb{R}. \tag{1.6}$$

The condition (1.6) will be utilized to characterize the well-posedness for the linearized equation of $(-\Delta)^s u + q(x,u) = 0$. For the nonlinear equation (1.1), the weak solution $u \in H^s(\mathbb{R}^n)$ exists provided that the coefficient $q(x,t)$ satisfies (1.3)-(1.5) is discussed in Section 2. However, very little result is known in general about uniqueness of the weak solution u of (1.1). We would like to point out that the uniqueness up to translations of the nontrivial solution of the fractional nonlinear equation holds for certain nonlinearity $q(x,u)$, we refer to [4] and references therein.

We consider the nonlocal inverse problem with related nonlocal Cauchy data set, instead of the Dirichlet to Neumann (DN) map, $\Lambda_q : u|_{\Omega_e} \to (-\Delta)^s u|_{\Omega_e}$ defined in [6], due to the lack of uniqueness of solutions for (1.1). The Cauchy data set is defined by

$$\mathcal{C}_q^{\Omega_e} = \left\{ (u|_{\Omega_e}, \mathcal{N}_q^s u|_{\Omega_e}) : \ u \in H^s(\mathbb{R}^n) \text{ is a solution of } (1.1) \right\},$$

where

$$\mathcal{N}_q^s u(x) := c_{n,s} \int_\Omega \frac{u(x)-u(y)}{|x-y|^{n+2s}} dy$$

stands for the nonlocal Neumann derivative and the constant $c_{n,s}$ is the same as (1.2). Note that when the equation (1.1) has a unique solution, the inverse problem is to recover $q(x,u)$ from the DN map. For more details about the nonlocal Neumann derivative $\mathcal{N}_q^s$, DN map Λ_q, and their connection, we refer to (2.2) and [3, 6].

In this paper, we focus on the Calderón problem for the fractional semilinear Schrödinger equation, that is, to recover the coefficient $q(x,u)$ from the collected external data set $\mathcal{C}_q^{\Omega_e}$. It is the nonlinear and nonlocal analogue of well-known Calderón problem, the mathematical model of electrical impedance tomography. As a noninvasive type of medical imaging, the electrical conductivity of the object is inferred from voltage and current measurements collected only on the surface of the object. There are several aspects in the Calderón problem, including uniqueness, stability estimates, reconstruction, and numerical algorithms for the known conductivity. For the classical semilinear Schrödinger equation $-\Delta u + q(x,u) = 0$, global uniqueness of an inverse problem with the related DN map on full boundary $\partial\Omega$ is due to [9, 15] when $n \geq 3$ and to [8, 15] when $n = 2$. We refer to the survey paper [16] for recent developments in inverse boundary value problems for linear and nonlinear elliptic equations. The uniqueness result has been studied for fractional Schrödinger equation in [6] and for variable coefficients nonlocal elliptic operators in [5]. The stability estimate for the fractional Schrödinger equation was shown in [13].

For each coefficient $q(x, u)$, we define a set $\mathcal{A}_q \subset \mathbb{R}^n \times \mathbb{R}$ by

$$\mathcal{A}_q := \{(x, u) \in \Omega \times \mathbb{R} : \text{ there exists a solution } u = u(x) \text{ of } (1.1)\}.$$

The main result in this paper is stated in Theorem 1.1, which solves the uniqueness for the fractional semilinear inverse problem with partial data for arbitrary dimension $n \geq 2$.

Theorem 1.1. *Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain and $0 < s < 1$. Suppose that $q_1(x, t)$ and $q_2(x, t)$ satisfy the conditions* (1.3)-(1.6). *Let $W \subset \Omega_e$ be an arbitrary open set. Suppose that the partial Cauchy data sets $\mathcal{C}^W_{q_1} = \mathcal{C}^W_{q_2}$, that is,*

$$\left\{(u_1|_W, \mathcal{N}^s_{q_1} u_1|_W)\right\} = \left\{(u_2|_W, \mathcal{N}^s_{q_2} u_2|_W)\right\}, \tag{1.7}$$

where u_j are solutions to $(-\Delta)^s u_j + q_j(x, u_j) = 0$ in Ω with $u_j = g$ in Ω_e for $j = 1, 2$, for any $g \in C^3_0(W)$. Then $\mathcal{A}_{q_1} = \mathcal{A}_{q_2}$ and

$$q_1(x, u(x)) = q_2(x, u(x)) \text{ in } \mathcal{A}_{q_1}.$$

The proof of Theorem 1.1 starts by showing the comparison principle for the linear fractional Schrödinger equation and the L^∞ estimate for the related solutions. The estimate plays a crucial role in the linearization argument that reduces the inverse problem for the fractional nonlinear equation to the inverse problem for the fractional linear equation. Notice that there exists a strong uniqueness property for the fractional Laplacian, that is, for any u in $\mathbb{R}^n$ satisfies $u|_W = (-\Delta)^s u|_W = 0$ in some open set W, then u is identically zero in $\mathbb{R}^n$ (see [6, Theorem 1.2]).

The paper is organized as follows. In section 2, we introduce fundamental tools of the fractional semilinear equation. The comparison principle and the estimate for solutions are discussed in section 3. In section 4, we prove Theorem 1.1 by utilizing the linearization scheme.

2. Preliminaries

First, let us begin with the fractional Sobolev spaces. Let $0 < s < 1$, $H^s(\mathbb{R}^n) = W^{s,2}(\mathbb{R}^n)$ is the L^2-based fractional Sobolev space with norm

$$\|u\|_{H^s(\mathbb{R}^n)} = \|u\|_{L^2(\mathbb{R}^n)} + \|(-\Delta)^{s/2} u\|_{L^2(\mathbb{R}^n)}.$$

Let $O \subset \mathbb{R}^n$ be an open set (not necessarily bounded), then we define

$$H^s(O) = \{u|_O : u \in H^s(\mathbb{R}^n)\}$$

and

$$H^s_0(O) = \text{closure of } C^\infty_c(O) \text{ in } H^s(O).$$

Next, we characterize the existence of solutions to our main nonlocal problem (1.1).

Lemma 2.1. *(Existence of weak solutions) Let $s \in (0, 1)$ and $q(x, t)$ be a scalar-valued function satisfying* (1.3)-(1.5). *For any $g \in C^3_0(\Omega_e)$, there exists at least one solution $u \in H^s(\mathbb{R}^n)$ of the nonlocal Dirichlet problem* (1.1).

Proof. For any $g \in C_0^3(\Omega_e)$, define the function $\widetilde{g}$ to be an extension of g by

$$\widetilde{g} := \begin{cases} 0 & \text{in } \Omega, \\ g & \text{in } \Omega_e. \end{cases}$$

It is easy to see that $\widetilde{g} \in C_0^2(\mathbb{R}^n) \subset H^2(\mathbb{R}^n)$. Now, consider a function $w := u - \widetilde{g}$ and we have the fact $q(x, u(x)) = q(x, w(x))$ for $x \in \Omega$ since $\widetilde{g} = 0$ in Ω. Then we can rewrite the equation (1.1) as

$$\begin{cases} (-\Delta)^s w + q(x, w) + h(x) = 0 & \text{in } \Omega, \\ w = 0 & \text{in } \Omega_e, \end{cases} \tag{2.1}$$

where $h(x) = (-\Delta)^s \widetilde{g}(x) \in L^2(\mathbb{R}^n)$ (see [6, Remark 2.2]). Therefore, by using [1, Theorem 11.2], one can see that there exists a weak solution $w \in H_0^s(\Omega)$ to the equation (2.1). This implies that there exists a solution $u = w + \widetilde{g} \in H^s(\mathbb{R}^n)$ of (1.1) such that $u = g$ in Ω_e. ☐

Remark 2.1. *In fact, from* [1, Theorem 11.2]*, there exist infinitely many weak solutions* $\{w_k\}_{k\in\mathbb{N}} \subset H_0^s(\Omega)$ *of* (2.1) *such that*

$$\iint_{\mathbb{R}^n\times\mathbb{R}^n} \frac{|w_k(x) - w_k(z)|^2}{|x - z|^{n+2s}} dxdz \to \infty \text{ as } k \to \infty.$$

We only choose $w_\ell \in H_0^s(\Omega)$ *for some* $\ell \in \mathbb{N}$ *to be a weak solution of* (2.1)*. Hence, for this semilinear nonlocal problem, it is more natural to formulate the Calderón problem by using the characterization of the Cauchy data set.*

In this paper, our exterior Dirichlet data g are given in $C_0^3(\Omega_e) \subset H^{2s}(\mathbb{R}^n)$, so that $(-\Delta)^s u \in H^{-s}(\mathbb{R}^n)$, where $u \in H^s(\mathbb{R}^n)$ is a solution of (1.1). By using the relation

$$(-\Delta)^s u|_{\Omega_e} = \mathcal{N}_q^s u|_{\Omega_e} - mu|_{\Omega_e} + (-\Delta)^s(E_0 g)|_{\Omega_e} \tag{2.2}$$

(see Lemma 3.2 in [6]), where $m \in C^\infty(\Omega_e)$ is defined by $m(x) = c_{n,s}\int_\Omega \frac{1}{|x-y|^{n+2s}} dy$ and E_0 is a zero extension in Ω such that $E_0 g(x) = g(x)$ for $x \in \Omega_e$, $E_0 g(x) = 0$ for $x \in \Omega$. Therefore, $\mathcal{N}_q^s u \in H^{-s}(\mathbb{R}^n)$ and the Cauchy data $(u|_{\Omega_e}, \mathcal{N}_q^s u|_{\Omega_e})$ can be regarded as in the function space $H^s(\Omega_e) \times H^{-s}(\Omega_e)$ (indeed, $u|_{\Omega_e} \in H^{2s}(\Omega_e) \subset H^s(\Omega_e)$).

3. L^∞-estimate of weak solutions

In this section, we offer a L^∞-estimate for the solution of the fractional Schrödinger equation under suitable regularity assumptions. This estimate will be used in the linearization scheme of the inverse problem for the fractional semilinear equation. The result of this section is motivated by [12] in which the author considers elliptic integro-differential operators.

3.1. **Comparison principle.** We begin by proving the maximum principle for the fractional Schrödinger equation. The definition of weak solutions is stated as follows.

Definition 3.1. *The function* $u \in H^s(\mathbb{R}^n)$ *is called a weak solution of the fractional Schrödinger equation* $(-\Delta)^s u + au = f$ *in* Ω *with* $u = g$ *in* Ω_e *if*

$$\int_{\mathbb{R}^n} (-\Delta)^{s/2} u \cdot (-\Delta)^{s/2}\phi dx + \int_\Omega au\phi dx = \int_\Omega f\phi dx$$

with $u - g \in \widetilde{H}^s(\Omega)$ *for any* $\phi \in C_c^\infty(\Omega)$*. Here* $\widetilde{H}^s(\Omega)$ *is the closure of* $C_c^\infty(\Omega)$ *in* $H^s(\mathbb{R}^n)$*.*

The comparison principle can be derived directly from the following maximum principle.

Proposition 3.1 (Maximum principle)**.** *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain and $a(x) \in L^\infty(\Omega)$ be a nonnegative potential. Let $u \in H^s(\mathbb{R}^n)$ be a weak solution of*

$$\begin{cases} (-\Delta)^s u + a(x)u = f & \text{in } \Omega, \\ u = g & \text{in } \Omega_e. \end{cases} \tag{3.1}$$

Suppose $0 \leq f \in L^\infty(\Omega)$ in Ω and $0 \leq g \in L^\infty(\Omega_e)$ in Ω_e. Then $u \geq 0$ in Ω.

Proof. If $u \in H^s(\mathbb{R}^n)$ is a weak solution of (3.1), by the weak formulation, we have

$$\int_{\mathbb{R}^n} (-\Delta)^{s/2} u \cdot (-\Delta)^{s/2} \phi dx + \int_\Omega a u \phi dx = \int_\Omega f \phi dx, \tag{3.2}$$

for any $\phi \in \widetilde{H}^s(\Omega)$. Note that

$$\begin{aligned} \int_{\mathbb{R}^n} (-\Delta)^{s/2} u \cdot (-\Delta)^{s/2} \phi dx &= \iint_{\mathbb{R}^{2n}} \frac{(u(x) - u(z))(\phi(x) - \phi(z))}{|x - z|^{n+2s}} dx dz \\ &= \iint_{\mathbb{R}^{2n} \setminus (\Omega_e \times \Omega_e)} \frac{(u(x) - u(z))(\phi(x) - \phi(z))}{|x - z|^{n+2s}} dx dz, \end{aligned}$$

where we have used $\phi \equiv 0$ in Ω_e.

Next, we write $u = u^+ - u^-$ in $\mathbb{R}^n$, where $u^+ = \max\{u, 0\}$ and $u^- = \max\{-u, 0\}$, Notice that $u^+, u^- \in H^s(\mathbb{R}^n)$ due to $u \in H^s(\mathbb{R}^n)$. Recall $u = g \geq 0$ in Ω_e, then we can take $\phi := u^- \in \widetilde{H}^s(\Omega)$ as a test function. We assume that u^- is not identically zero, and we want to prove that it will lead to a contradiction.

Since $f \geq 0$ and $\phi = u^- \geq 0$, the right hand side of (3.2)

$$\int_\Omega f \phi dx \geq 0. \tag{3.3}$$

On the other hand, one has

$$\begin{aligned} &\iint_{\mathbb{R}^{2n} \setminus (\Omega_e \times \Omega_e)} \frac{(u(x) - u(z))(\phi(x) - \phi(z))}{|x - z|^{n+2s}} dx dz \\ =&\iint_{\Omega \times \Omega} \frac{(u(x) - u(z))(u^-(x) - u^-(z))}{|x - z|^{n+2s}} dx dz \\ &+ 2 \int_\Omega \int_{\Omega_e} \frac{(u(x) - g(z)) u^-(x)}{|x - z|^{n+2s}} dz dx \\ =& I + II, \end{aligned}$$

where

$$\begin{aligned} I &:= \iint_{\Omega \times \Omega} \frac{(u(x) - u(z))(u^-(x) - u^-(z))}{|x - z|^{n+2s}} dx dz, \\ II &:= 2 \int_\Omega \int_{\Omega_e} \frac{(u(x) - g(z)) u^-(x)}{|x - z|^{n+2s}} dz dx. \end{aligned}$$

To estimate I, since $(u^+(x) - u^+(z))(u^-(x) - u^-(z)) \leq 0$, we obtain

$$I \leq - \iint_{\Omega \times \Omega} \frac{(u^-(x) - u^-(z))^2}{|x - z|^{n+2s}} dx dz < 0. \tag{3.4}$$

The last strict inequality holds because u^- can not be a constant in Ω. If u^- is a constant, which means $u \equiv -c_0$ is a negative constant in Ω (for some constant $c_0 > 0$). By the definition of the fractional Laplacian, one can see that for $x \in \Omega$,

$$\begin{aligned}(-\Delta)^s u(x) &= c_{n,s}\text{P.V.}\int_{\mathbb{R}^n} \frac{-c_0 - u(z)}{|x-z|^{n+2s}}dz \\ &= c_{n,s}\int_{\Omega_e} \frac{-c_0 - g(z)}{|x-z|^{n+2s}}dz < 0,\end{aligned}$$

since $g(z) \geq 0$ for $z \in \Omega_e$. Therefore, by using (3.1) and $a \geq 0$ in Ω, we know that

$$0 \leq f = (-\Delta)^s u + au < 0 \text{ in } \Omega,$$

which leads to a contradiction. Hence, u^- can not be a constant.

For II, since $g(z) \geq 0$ in Ω_e and $u(x)u^-(x) \leq 0$ in Ω, we deduce that $II \leq 0$. Therefore,

$$\iint_{\mathbb{R}^{2n}\setminus(\Omega_e\times\Omega_e)} \frac{(u(x)-u(z))(\phi(x)-\phi(z))}{|x-z|^{n+2s}}dxdz < 0.$$

which contradicts to (3.2) (because $f \geq 0$ in Ω and $g \geq 0$ in Ω_e). □

With the maximum principle, the comparison principle for the fractional Schrödinger equation follows immediately.

Corollary 3.2 (Comparison principle)**.** *Let u_1 and u_2 be weak solutions of*

$$\begin{cases} (-\Delta)^s u_1 + a(x)u_1 = f_1 & \text{in } \Omega, \\ u_1 = g_1 & \text{in } \Omega_e, \end{cases} \quad \text{and} \quad \begin{cases} (-\Delta)^s u_2 + a(x)u_2 = f_2 & \text{in } \Omega, \\ u_2 = g_2 & \text{in } \Omega_e, \end{cases}$$

respectively. Suppose that $f_1 \geq f_2$ in Ω and $g_1 \geq g_2$ in Ω_e. Then $u_1 \geq u_2$ in Ω.

Proof. Let $u := u_1 - u_2$ and apply proposition 3.1, then we complete the proof. Furthermore, one can conclude that $u_1 \geq u_2$ in $\mathbb{R}^n$. □

Remark 3.1. *From the above comparison principle, once the solution exists, the uniqueness will automatically hold for the fractional linear Schrödinger equation* (3.1)*.*

3.2. L^∞ **bounds for solutions.** The main goal of this section is stated as follows.

Proposition 3.3. *Suppose $f \in L^\infty(\Omega)$ and $g \in L^\infty(\Omega_e)$. Let u be a solution to* (3.1)*, then the following L^∞ estimate*

$$\|u\|_{L^\infty(\Omega)} \leq \|g\|_{L^\infty(\Omega_e)} + C\|f\|_{L^\infty(\Omega)}, \tag{3.5}$$

holds for some constant $C > 0$ independent of u, f, and g.

In order to derive (3.5), we need to construct a barrier function for the fractional Schrödinger equation.

Lemma 3.4 (Barrier)**.** *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$ and $a(x) \in L^\infty(\Omega)$ is a nonnegative potential. Then there exists a function $\varphi \in C_c^\infty(\mathbb{R}^n)$ such that*

$$\begin{cases} (-\Delta)^s \varphi + a(x)\varphi \geq 1 & \text{in } \Omega, \\ \varphi \geq 0 & \text{in } \mathbb{R}^n, \\ \varphi \leq C & \text{in } \Omega, \end{cases} \tag{3.6}$$

where $C > 0$ is a constant depending on n, s, and Ω.

Proof. Let B_R be an arbitrarily large ball such that $\Omega \Subset B_R$ and $\eta \in C_c^\infty(B_R)$ be a smooth cutoff function such that

$$0 \leq \eta \leq 1 \text{ in } \mathbb{R}^n, \ \eta \equiv 1 \text{ in } \Omega.$$

For any $x \in \Omega$, it is clear that $\eta(x) = 1$ that is the maximum value of η. Thus, one has

$$2\eta(x) - \eta(x+y) - \eta(x-y) \geq \eta(x) - \eta(x+y) \geq 0. \tag{3.7}$$

Recall that for any function u in the Schwartz space, we can also represent the fractional Laplacian as (see [2] for instance)

$$(-\Delta)^s u(x) = \frac{c_{n,s}}{2} \int_{\mathbb{R}^n} \frac{2u(x) - u(x+y) - u(x-y)}{|y|^{n+2s}} dy \ \text{ for all } x \in \mathbb{R}^n,$$

where $c_{n,s}$ is the constant in (1.2). For any $x \in \Omega$, from (3.7) and by using the change of variables $z = x + y$, one has

$$\begin{aligned}
(-\Delta)^s \eta + a(x)\eta &\geq \frac{1}{2} c_{n,s} \int_{\mathbb{R}^n} \frac{\eta(x) - \eta(z)}{|x-z|^{n+2s}} dz + a(x)\eta \\
&\geq \frac{1}{2} c_{n,s} \int_{\mathbb{R}^n \setminus B_R} \frac{1}{|x-z|^{n+2s}} dz \\
&\geq \lambda
\end{aligned}$$

for some constant $\lambda > 0$. Here we have utilized the fact that $\eta(x) - \eta(z) = 1$ for $z \in \mathbb{R}^n \setminus B_R$, then we have

$$\int_{\mathbb{R}^n \setminus B_R} \frac{1}{|x-z|^{n+2s}} dz = \int_{\mathbb{R}^n \setminus (x + B_R)} \frac{1}{|y|^{n+2s}} dy \geq \int_{\mathbb{R}^n \setminus B_{2R}} \frac{1}{|y|^{n+2s}} dy = c > 0,$$

where $c > 0$ is a positive constant. Now, we let $\varphi(x) := \eta(x)/\lambda$, then we complete the proof. □

It remains to prove the L^∞ bound for the solution u.

Proof of Proposition 3.3. Let

$$v(x) := \|g\|_{L^\infty(\Omega_e)} + \|f\|_{L^\infty(\Omega)} \varphi(x),$$

where $\varphi(x)$ is the barrier given by Lemma 3.4. From $a(x) \geq 0$ and (3.6), we deduce that

$$(-\Delta)^s u + a(x)u = f \leq (-\Delta)^s v + a(x)v \text{ in } \Omega$$

and $g \leq v$ in Ω_e. Applying the comparison principle in Corollary 3.2, we obtain

$$u \leq \|g\|_{L^\infty(\Omega_e)} + C\|f\|_{L^\infty(\Omega)} \text{ in } \Omega,$$

where we use $\varphi \leq C$ in Ω. Similarly, the same argument will hold for $-u$, which finishes the proof.

□

4. Proof of Theorem 1.1

In this section, we apply the linearization scheme to transfer the inverse problem for the nonlocal semilinear Schrödinger equation to the inverse problem for the nonlocal linear equation.

4.1. **Linearization.** This linearization method was used in the local type inverse problem, see [7, 9, 14, 15].

Theorem 4.1. *Let $n \geq 2$ and $0 < s < 1$. Let g and h be in $C_0^3(\Omega_e)$ and η be in $\mathbb{R}$. Suppose that $u_{g+\eta h}$ is the solution of (1.1) with $u_{g+\eta h} = g + \eta h$ in Ω_e. Suppose that u^* is the unique solution of the linearized equation*

$$\begin{cases} (-\Delta)^s u^* + \partial_t q(x, u_g) u^* = 0 & \text{in } \Omega, \\ u^* = h & \text{in } \Omega_e, \end{cases} \tag{4.1}$$

then we have

$$\lim_{\eta \to 0} \left\| \frac{u_{g+\eta h} - u_g}{\eta} - u^* \right\|_{H^s(\mathbb{R}^n)} = 0.$$

Proof. First, by using (1.6), i.e., $0 \leq \partial_t q(x,t) \leq M_0$ for $(x,t) \in \overline{\Omega} \times \mathbb{R}$ gives the well-posedness of the fractional linear Schrödinger equation, i.e., for a given function $h \in C_0^3(\Omega_e) \subset H^s(\Omega_e)$, there exists a unique weak solution $u^* \in H^s(\mathbb{R}^n)$ such that u^* solves (4.1).

Next, consider

$$w := \frac{u_{g+\eta h} - u_g}{\eta} \in H^s(\mathbb{R}^n),$$

for $\eta \in \mathbb{R}$, then w is a solution of

$$\begin{cases} (-\Delta)^s w + Q(x) w = 0 & \text{in } \Omega, \\ w = h & \text{in } \Omega_e, \end{cases}$$

where

$$Q(x) = \int_0^1 \partial_t q(x, \xi u_{g+\eta h}(x) + (1-\xi) u_g(x)) d\xi \geq 0.$$

Let $v := w - u^*$, then v solves

$$\begin{cases} (-\Delta)^s v + Q(x) v = -\left(Q(x) - \partial_t q(x, u_g)\right) u^* & \text{in } \Omega, \\ v = 0 & \text{in } \Omega_e. \end{cases} \tag{4.2}$$

By multiplying v on both sides of (4.2), we can see that

$$\begin{aligned} \|v\|_{H^s(\mathbb{R}^n)} &\leq C \| \left(Q(x) - \partial_t q(x, u_g)\right) u^* \|_{L^2(\mathbb{R}^n)} \\ &\leq C \|Q(x) - \partial_t q(x, u_g)\|_{L^\infty(\Omega)} \|u^*\|_{L^2(\mathbb{R}^n)}, \end{aligned} \tag{4.3}$$

for some constant $C > 0$ independent of v and u^*.

Now, since $u_{g+\eta h} - u_g \in H^s(\mathbb{R}^n)$ solves

$$\begin{cases} (-\Delta)^s (u_{g+\eta h} - u_g) + Q(x)(u_{g+\eta h} - u_g) = 0 & \text{in } \Omega, \\ u_{g+\eta h} - u_g = \eta h & \text{in } \Omega_e, \end{cases}$$

where $h \in C_0^3(\Omega_e) \subset L^\infty(\Omega_e)$. By the L^∞ estimate (3.5), we have

$$\|u_{g+\eta h} - u_g\|_{L^\infty(\Omega)} \leq |\eta| \|h\|_{L^\infty(\Omega_e)} \to 0,$$

as $\eta \to 0$.

By the continuity of $\partial_t q(x,t)$, it implies that

$$\|\partial_t q(x, \xi u_{g+\eta f} + (1-\xi) u_g) - \partial_t q(x, u_g)\|_{L^\infty(\Omega)} \to 0$$

as $\eta \to 0$ that leads to

$$\|Q(x) - \partial_t q(x,u_g)\|_{L^\infty(\Omega)} \leq \int_0^1 \|\partial_t q(x, \xi u_{g+\eta f} + (1-\xi)u_g) - \partial_t q(x,u_g)\|_{L^\infty(\Omega)} d\xi \to 0,$$

whenever $\eta \to 0$. Combining with (4.3), the proof is complete.

□

4.2. **Proof of Theorem 1.1.** Now, we are ready to prove our main theorem.

Proof of Theorem 1.1. For any $g \in C_0^3(W) \subset C_0^3(\Omega_e)$, let $u_g^{(1)} \in H^s(\mathbb{R}^n)$ be a solution of

$$\begin{cases} (-\Delta)^s u_g^{(1)} + q_1(x, u_g^{(1)}) = 0 & \text{in } \Omega, \\ u_g^{(1)} = g & \text{in } \Omega_e. \end{cases}$$

From the Cauchy data assumption (1.7), there exists a solution $u_g^{(2)} \in H^s(\mathbb{R}^n)$ such that $u_g^{(2)} = u_g^{(1)}$, $\mathcal{N}_{q_2}^s u_g^{(2)} = \mathcal{N}_{q_1}^s u_g^{(1)}$ in W and $u_g^{(2)}$ solves

$$\begin{cases} (-\Delta)^s u_g^{(2)} + q_2(x, u_g^{(2)}) = 0 & \text{in } \Omega, \\ u_g^{(2)} = g & \text{in } \Omega_e. \end{cases}$$

For any $h \in C_0^3(W) \subset C_0^3(\Omega_e)$, using the Cauchy data assumption again, there are solutions $u_{g+\eta h}^{(j)} \in H^s(\mathbb{R}^n)$ of

$$\begin{cases} (-\Delta)^s u_{g+\eta h}^{(j)} + q_j(x, u_{g+\eta h}^{(j)}) = 0 & \text{in } \Omega, \\ u_{g+\eta h}^{(j)} = g + \eta h & \text{in } \Omega_e, \end{cases}$$

for $j = 1, 2$, such that

$$\mathcal{N}_{q_1}^s u_{g+\eta h}^{(1)} = \mathcal{N}_{q_2}^s u_{g+\eta h}^{(2)} \text{ in } W \subset \Omega_e$$

for any $\eta \in \mathbb{R}$. We differentiate the above equation with respect to η at $\eta = 0$ and use (2.2) with Theorem 4.1, then we obtain that

$$\mathcal{N}_{q_1}^s \dot{u}_{g,h}^{(1)} = \mathcal{N}_{q_2}^s \dot{u}_{g,h}^{(2)} \text{ in } W, \tag{4.4}$$

where $\dot{u}_{g,h}^{(1)}$ and $\dot{u}_{g,h}^{(2)}$ are solutions of

$$(-\Delta)^s \dot{u}_{g,h}^{(1)} + \partial_t q_1(x, u_g^{(1)}) \dot{u}_{g,h}^{(1)} = 0 \text{ in } \Omega \text{ with } \dot{u}_{g,h}^{(1)} = h \text{ in } \Omega_e, \tag{4.5}$$

and

$$(-\Delta)^s \dot{u}_{g,h}^{(2)} + \partial_t q_2(x, u_g^{(2)}) \dot{u}_{g,h}^{(2)} = 0 \text{ in } \Omega \text{ with } \dot{u}_{g,h}^{(2)} = h \text{ in } \Omega_e. \tag{4.6}$$

Recall that $\partial_t q_j \in L^\infty$ and $\partial_t q_j \geq 0$ in Ω. Thus, by using the well-posedness of the linearized fracional Schrödinger equation (see Section 2 in [6]), nonlocal DN maps $\Lambda_{\partial_t q_j}$ exist and are defined by

$$\Lambda_{\partial_t q_j(x, u_g^{(j)})} : \dot{u}_{g,h}^{(j)}|_{\Omega_e} \to (-\Delta)^s \dot{u}_{g,h}^{(j)}|_{\Omega_e}, \text{ for } j = 1, 2.$$

For a fixed $g \in C_0^3(W)$, since (4.4) holds for any $h \in C_0^3(W)$, by using (2.2), we derive

$$\Lambda_{\partial_t q_1(x, u_g^{(1)})} h|_W = \Lambda_{\partial_t q_2(x, u_g^{(2)})} h|_W \text{ for any } h \in C_0^3(W).$$

Thus, we can use global uniqueness of the fractional linear Schrödinger equation (see Theorem 1.1 in [6]), then we can conclude

$$\partial_t q_1(x, u_g^{(1)}) = \partial_t q_2(x, u_g^{(2)}) \text{ in } \Omega. \tag{4.7}$$

Via (4.7), we know that $\dot{u}_{g,h}^{(1)}$ and $\dot{u}_{g,h}^{(2)}$ solve (4.5) and (4.6) (with the same coefficients), respectively. From the well-posedness for the fractional linear Schrödinger equation again, one can see that the weak solution will be unique, that is,

$$\dot{u}_{g,h}^{(1)} = \dot{u}_{g,h}^{(2)} \text{ in } H^s(\mathbb{R}^n).$$

In particular, we take the original g by ηg and h by g, then we have

$$\dot{u}_{\eta g,g}^{(1)} = \dot{u}_{\eta g,g}^{(2)} \text{ in } H^s(\mathbb{R}^n), \text{ for any } \eta \in \mathbb{R}.$$

By using Theorem 4.1, this implies that

$$\frac{d}{d\eta} u_{\eta g}^{(1)} = \frac{d}{d\eta} u_{\eta g}^{(2)} \text{ in } H^s(\mathbb{R}^n) \text{ for any } \eta \in \mathbb{R}.$$

Hence, there exists a function $\psi = \psi(x) \in H^s(\mathbb{R}^n)$ independent of η such that

$$u_g^{(1)} = u_g^{(2)} + \psi \text{ in } H^s(\mathbb{R}^n) \tag{4.8}$$

for all $g \in C_0^3(\Omega_e)$.

Now, by using the assumption on Cauchy data (1.7),

$$\{(u_g^{(1)}|_W, \mathcal{N}_{q_1}^s u_g^{(1)}|_W)\} = \{(u_g^{(2)}|_W, \mathcal{N}_{q_2}^s u_g^{(2)}|_W)\},$$

we can obtain $\psi \in H^s(\mathbb{R}^n)$ such that

$$\psi = (-\Delta)^s \psi = 0 \text{ in } W \subset \Omega_e. \tag{4.9}$$

We apply Theorem 1.2 in [6], then the function $\psi \equiv 0$ in $\mathbb{R}^n$. Finally, we substitute $u_g^{(1)}(x) = u_g^{(2)}(x)$ for $x \in \mathbb{R}^n$ into the fractional semilinear Schrödinger equation

$$(-\Delta)^s u_g^{(1)} + q_1(x, u_g^{(1)}) = (-\Delta)^s u_g^{(2)} + q_2(x, u_g^{(2)}) \text{ in } \Omega,$$

which implies

$$q_1(x, u) = q_2(x, u) \text{ for } x \in \Omega.$$

This completes the proof of our main result. □

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