THE CALDERÓN PROBLEM FOR THE LOGARITHMIC SCHRÖDINGER EQUATION

BASTIAN HARRACH, YI-HSUAN LIN, AND TOBIAS WETH

Abstract. We study the Calderón problem for a logarithmic Schrödinger type operator of the form $L_\Delta + q$, where L_Δ denotes the logarithmic Laplacian, which arises as formal derivative $\frac{d}{ds}\big|_{s=0}(-\Delta)^s$ of the family of fractional Laplacian operators. This operator enjoys remarkable nonlocal properties, such as the unique continuation and Runge approximation. Based on these tools, we can uniquely determine bounded potentials using the Dirichlet-to-Neumann map. Additionally, we can build a constructive uniqueness result by utilizing the monotonicity method. Our results hold for any space dimension.

Contents

1. Introduction 1
2. Preliminaries 6
2.1. The logarithmic Laplacian and associated function spaces 6
2.2. The energy form of the logarithmic Schrödinger type operator 10
3. The forward problem 11
3.1. The Dirichlet problem 12
3.2. The DN map 13
4. Proof of Theorem 1.1 14
5. Constructive uniqueness 15
5.1. Localized potentials 15
5.2. Monotonicity relations 16
References 18

1. Introduction

Calderón's pioneer work [Cal80] investigated the inverse conductivity problem, which asks whether the conductivity can be determined from its boundary electrical voltage and current measurements. Using a suitable reduction scheme shows that a closely related problem is the Calderón problem for the classical Schrödinger equation, which can be stated as follows. Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with Lipschitz boundary $\partial\Omega$, and $q \in L^\infty(\Omega)$. Consider the Dirichlet boundary value problem

$$\begin{cases} (-\Delta + q)\, u = 0 & \text{in } \Omega, \\ u = f & \text{on } \partial\Omega. \end{cases} \tag{1.1}$$

Assume that 0 is not a Dirichlet eigenvalue of the Schrödinger operator $-\Delta + q$ in Ω, which implies that there exists a unique solution $u \in H^1(\Omega)$ to (1.1), for any

2020 *Mathematics Subject Classification.* 35R30; 26A33; 35J70.

Key words and phrases. Logarithmic Laplacian, Calderón's problem, unique continuation, Runge approximation, localized potentials, monotonicity method.

given Dirichlet boundary data $f \in H^{1/2}(\partial\Omega)$. Then the boundary measurement is called the (full) *Dirichlet-to-Neumann* (DN) map, which is well-defined and can be encoded by

$$\Lambda_q^{(1)} : H^{1/2}(\partial\Omega) \to H^{-1/2}(\partial\Omega), \quad f \mapsto \partial_\nu u_f|_{\partial\Omega}\,, \tag{1.2}$$

where $u_f \in H^1(\Omega)$ is the solution to (1.1). The Calderón problem for the Schrödinger equation (1.1) is to ask whether the DN map Λ_q determines the potential $q \in L^\infty(\Omega)$ or not. The result is positive for a more regular potential q with the full boundary DN map, and we refer readers to the articles [SU87] for $n \geq 3$ and [Buk08] for $n = 2$. The article [Uhl09] provides a comprehensive introduction and review in this direction.

After several decades of developments on Calderón's problems, this type of problem has been generalized to the nonlocal scenario. The Calderón problem for the *fractional Schrödinger equation* has first been studied in [GSU20]. The underlying problem was proposed as an exterior value problem due to its natural nonlocality: Given $s \in (0,1)$, let $\Omega \subset \mathbb{R}^n$ be a bounded open set with Lipschitz boundary for $n \in \mathbb{N}$, and let $q \in L^\infty(\Omega)$. Consider the Dirichlet exterior value problem

$$\begin{cases} ((-\Delta)^s + q)\, u = 0 & \text{in } \Omega, \\ u = f & \text{in } \Omega_e, \end{cases} \tag{1.3}$$

where

$$\Omega_e := \mathbb{R}^n \setminus \overline{\Omega}.$$

Similarly as in the case $s = 1$, let us assume that 0 is not a Dirichlet eigenvalue of $(-\Delta)^s + q$ in Ω. Then there exists a unique solution $u \in H^s(\mathbb{R}^n)$ to (1.3), for any given Dirichlet exterior data f in a suitable function space, such as $C_c^\infty(\Omega_e)$. The exterior (partial) DN map can be formally defined by

$$\Lambda_q^{(s)} : C_c^\infty(W_1) \ni f \mapsto (-\Delta)^s u_f|_{W_2}\,, \tag{1.4}$$

where $W_1, W_2 \Subset \Omega_e$ can be arbitrary nonempty open subsets.

In the foundational work [GSU20], the authors showed that the DN map (1.4) can determine the bounded potential q in Ω uniquely, which holds for any spatial dimension $n \in \mathbb{N}$. Moreover, inverse problems for fractional equations have more profound properties than their local counterparts, such as the *unique continuation property* (UCP) and *Runge approximation property*. Roughly speaking, the UCP for $(-\Delta)^s$ $(0 < s < 1)$ states that given a nonempty open subset $\mathcal{O} \subset \mathbb{R}^n$,

$$u = (-\Delta)^s u = 0 \text{ in } \mathcal{O} \quad \text{implies that} \quad u \equiv 0 \text{ in } \mathbb{R}^n,$$

where u belongs to appropriate function spaces (e.g., u belongs to some fractional Sobolev space $H^r(\mathbb{R}^n)$ for some $r \in \mathbb{R}$). Moreover, the Runge approximation property states that any L^2 function can be approximated by solutions of the fractional Schrödinger equation.

Based on the properties as mentioned above, fractional-type inverse problems have received rapidly growing attention in recent years. In the work [CLR20], the authors demonstrated that both drift and potential can be determined uniquely, which cannot be true for the local case in general. Moreover, simultaneous recovering for both the obstacle and surrounded medium has been studied in [CLL19], and the determination of bounded potentials for anisotropic nonlocal Schrödinger equation was investigated in [GLX17]. These two problems remain open for the case $s = 1$ and $n \geq 3$. Therefore, one could expect that the nonlocality is beneficial in the study of related inverse problems.

For more related work on both linear and nonlinear nonlocal inverse problems, we refer readers to [HL19, HL20, LL23, LL22, GRSU20, CMRU22, RS20, RS18,

GRSU20, LLR20, Lin22, LZ23, GU21, LLU22] and references therein. In particular, one can also determine the interior coefficients, by using either a reduction of the Caffarelli-Silvestre extension (see e.g. [CGRU23, Rül23, LLU23, LZ24]) or the heat semigroup on closed Riemannian manifolds (see e.g. [FGKU21, Fei24, FKU24, Lin24]), where the UCP may not be a necessary tool for some cases. Very recently, an entanglement principle for nonlocal operators has also been investigated in [FKU24, FL24], and it may significantly influence new nonlocal inverse problems.

Loosely speaking, most of the existing studies of inverse problems are related to the classical Laplacian $-\Delta$ or the fractional Laplacian $(-\Delta)^s$, $s \in (0,1)$. We want to point out that the key tools in solving classical and fractional inverse problems:

- **Classical case.** Suitable integral identities and complex geometrical optics (CGO) solutions ($(-\Delta)^s$ for $s = 1$).
- **Fractional case.** Suitable integral identities and Runge approximation property ($(-\Delta)^s$ for $0 < s < 1$).

The above-mentioned tools are usually used to recover lower-order coefficients in related mathematical problems. Furthermore, a recent work [CdHS24] investigates geometric optics solutions for the fractional Helmholtz equation, which combines both local and nonlocal features.

While the nonlocality distinguishes the fractional case from the classical one, it is important to note that the fractional Laplacian $(-\Delta)^s$ is still an elliptic operator of positive order $2s$ and therefore admits a highly useful elliptic regularity theory. In the present paper, we wish to study an inverse problem beyond the framework of operators of positive order. To motivate our problem, recall that

$$\lim_{s\to 1^-}(-\Delta)^s u = -\Delta u \quad \text{and} \quad \lim_{s\to 0^+}(-\Delta)^s u = u \qquad \text{pointwisely in } \mathbb{R}^n \tag{1.5}$$

for any $u \in C_c^2(\mathbb{R}^n)$. For the second limit, one may identify a well-defined first-order correction term given as the formal derivative $\log(-\Delta)u := \frac{d}{ds}\big|_{s=0}(-\Delta)^s u$. The associated *logarithmic Laplacian operator*

$$L_\Delta := \log(-\Delta)$$

has been introduced in [CW19] in the context of Dirichlet problems, and it has attracted growing interest in recent years due to its usefulness in the analysis of order-dependent families of linear and nonlinear fractional Dirichlet problems and their asymptotic limits as $s \to 0^+$, see e.g. [CW19, JSnW20, FJW22, HSSn22, LW21]. It also appears in the context of the 0-fractional perimeter, see [DLNP21].

We note that the operator L_Δ has the Fourier symbol $2\log|\xi|$, which can be seen by differentiating the symbol $|\xi|^{2s}$ with respect to s and evaluate at $s = 0$. More precisely, by [CW19, Theorem 1.1], there holds

$$\log(-\Delta)u := \mathcal{F}^{-1}\left(2\log|\xi|\,\widehat{u}(\xi)\right), \quad \text{for} \quad u \in C_c^\alpha(\mathbb{R}^n), \tag{1.6}$$

for some $\alpha > 0$, where $\widehat{u}$ is the Fourier transform of u, and $\mathcal{F}^{-1}$ is the inverse Fourier transform. Due to the weak growth of the logarithmic symbol, L_Δ is sometimes called a (near) zero-order operator.

Very recently, the problem of finding an extension problem for the logarithmic Laplacian has been solved in [CHW23], and the authors used it to prove the UCP for L_Δ. The UCP will play a major role in the proof of our main results.

Mathematical formulations and main results. Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with Lipschitz boundary $\partial\Omega$, and $q \in L^\infty(\Omega)$. Consider

$$\begin{cases} (L_\Delta + q)\, u = 0 & \text{in } \Omega, \\ u = f & \text{in } \Omega_e. \end{cases} \tag{1.7}$$

To obtain the well-posedness of (1.7), let us recall the eigenvalue problem of the logarithmic Laplacian.

By [CW19, Theorem 1.2], it is known that L_Δ in Ω has eigenvalues

$$\lambda_1(\Omega) < \lambda_2(\Omega) \leq \ldots \leq \lambda_k(\Omega) \leq \ldots \nearrow \infty \tag{1.8}$$

in a bounded domain Ω. Any of these eigenvalues $\lambda_k(\Omega)$ may be positive, zero, or negative, depending on the size and shape of Ω. In order to have a well-posed forward problem (1.7), let us impose the condition

$$\lambda_1(\Omega) + q(x) \geq \lambda_0 > 0, \text{ for all } x \in \Omega, \tag{1.9}$$

for some positive constant λ_0, where $\lambda_1(\Omega)$ is the first Dirichlet eigenvalue of L_Δ in Ω. Lower bounds for $\lambda_1(\Omega)$ in terms of $|\Omega|$, the measure of Ω, are given in [CW19, LW21], see in particular [LW21, Corollary 4.3 and Theorem 4.4][1].

With the eigenvalue condition (1.9) at hand, we can prove (1.7) is well-posed for any f contained in the associated *trace space* $\mathbb{H}_T(\Omega_e)$ which is defined in Section 3 below. Hence, the DN map of (1.7) can be defined as a map

$$\Lambda_q : \mathbb{H}_T(\Omega_e) \mapsto \mathbb{H}_T(\Omega_e)^*.$$

In particular, if $W_1, W_2 \Subset \Omega_e$ are open bounded subsets, then for every $f \in C_c^\infty(W_1)$ and $g \in C_c^\infty(W_1)$ we have

$$\langle \Lambda_q f, g\rangle = \int_{W_2} (L_\Delta u_f) g\, dx = 2\int_{\mathbb{R}^n} (\log|\xi|)\widehat{u_f}(\xi)\overline{\widehat{g}(\xi)}\, d\xi,$$

where u_f is the unique (weak) solution to (1.7), and $\langle\cdot,\cdot\rangle$ denotes the natural duality pairing between $\mathbb{H}_T(\Omega_e)$ and $\mathbb{H}_T(\Omega_e)^*$, see Section 3 below. The key question studied in this paper is the following.

(IP) Inverse Problem. Can one determine q via the DN map Λ_q?

The following main result gives an affirmative answer to **(IP)**.

Theorem 1.1 (Global uniqueness)**.** *Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain for $n \in \mathbb{N}$, and $W_1, W_2 \Subset \Omega_e$ be nonempty bounded open sets. Assume that $q_j \in L^\infty(\Omega)$ satisfies (1.9) for $j = 1, 2$. Let Λ_{q_j} be the DN map of*

$$\begin{cases} (\log(-\Delta) + q_j)\, u_j = 0 & \text{in } \Omega, \\ u_j = f & \text{in } \Omega_e, \end{cases} \tag{1.10}$$

for $j = 1, 2$. Suppose that

$$\langle \Lambda_{q_1} f, g\rangle = \langle \Lambda_{q_2} f, g\rangle \quad \text{for any } f \in C_c^\infty(W_1),\ g \in C_c^\infty(W_2). \tag{1.11}$$

Then there holds $q_1 = q_2$ in Ω.

The proof of the preceding theorem is based on the UCP and Runge approximation for the logarithmic Schrödinger equation (1.7).

On top of that, we also have a constructive uniqueness for q using a *monotonicity test*. Indeed, similarly as in the case of the fractional problem considered in [HL19], one can derive an if-and-only-if monotonicity relation between bounded potentials and associated DN maps. More precisely, we have the following.

[1] Note that $\frac{1}{2}L_\Delta$ is considered in place of L_Δ in [LW21], so the bounds there apply to $\frac{\lambda_1(\Omega)}{2}$ in place of $\lambda_1(\Omega)$.

Theorem 1.2 (If-and-only-if monotonicity)**.** *Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain, and assume that $q_j \in L^\infty(\Omega)$ satisfies (1.9) for $j = 1, 2$. Then the following are equivalent:*

(i) $q_2 \geq q_1$ *a.e. in* Ω*;*
(ii) $\langle (\Lambda_{q_2} - \Lambda_{q_1}) f, f \rangle \geq 0$ *for all* $f \in \mathbb{H}_T(\Omega_e)$*;*
(iii) There exists a nonempty bounded open set $W \Subset \Omega_e$ with the property that

$$\langle (\Lambda_{q_1} - \Lambda_{q_2}) f, f \rangle \geq 0, \qquad \text{for all } f \in C_c^\infty(W). \tag{1.12}$$

In the following, if $W \Subset \Omega_e$ is a nonempty bounded open set, we say that

$$\Lambda_{q_1} \leq \Lambda_{q_2} \ \text{ in } \ W \tag{1.13}$$

if (1.12) holds.

Remark 1.3. *In the special case $W_1 = W_2 = W$, Theorem 1.1 can be viewed as a corollary of Theorem 1.2 since in this case condition (1.11) implies that both $\Lambda_{q_1} \leq \Lambda_{q_2}$ and $\Lambda_{q_2} \leq \Lambda_{q_1}$ hold in Ω and therefore $q_1 = q_2$ in Ω by Theorem 1.2.*

As indicated above, the if-and-only-if monotonicity relation given by Theorem 1.2 yields a constructive global uniqueness result in the case where $\lambda_1(\Omega) > 0$ and $q \in L^\infty(\Omega)$ is nonnegative. For this purpose, let us recall that a point $x \in \mathbb{R}^n$ is called (Lebesgue) *density one* for a measurable set E if

$$\lim_{r \to 0} \frac{|B_r(x) \cap E|}{|B_r(x)|} = 1, \tag{1.14}$$

where $B_r(x)$ denotes the ball of radius r and centered at x. The space of *density one simple functions* can be defined by

$$\Sigma := \left\{ \varphi = \sum_{j=1}^m a_j \chi_{E_j} \,:\, a_j \in \mathbb{R},\ E_j \subseteq \Omega \text{ is a density one set} \right\},$$

where we call a subset $E \subseteq \Omega$ a *density one set* provided that E is nonempty, measurable and has density one for all $x \in E$. It is not hard to find that density one sets have a positive measure, and finite intersections of density one sets are density one. Let, moreover, $\Sigma_{+,0} \subseteq \Sigma$ denote the subset of *nonnegative* density one simple functions.

Theorem 1.4 (Constructive uniqueness)**.** *Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain with $\lambda_1(\Omega) > 0$, and let $q \in L^\infty(\Omega)$ be nonnegative. Moreover, let $W \Subset \Omega_e$ be an arbitrary nonempty bounded open set. Then the potential q can be determined uniquely via the formula*

$$q(x) = \sup \left\{ \varphi(x) \,:\, \varphi \in \Sigma_{+,0} \text{ and } \Lambda_\varphi \leq \Lambda_q \text{ in } W \right\}, \text{ for a.e. } x \in \Omega, \tag{1.15}$$

where the relation $\Lambda_\varphi \leq \Lambda_q$ in W is defined as above.

Note that the proofs of both Theorems 1.1 and 1.4 rely on the Runge approximation and its applications for the logarithmic Schrödinger equation.

Organization of this article. In Section 2, we introduce several basic function spaces and a rigorous definition of the logarithmic Laplacian. In Section 3, we demonstrate the well-posedness of the exterior value problem (1.7) for suitable function spaces. The existence of the DN map can be guaranteed by the well-posedness of (1.7). We prove Theorem 1.1 in Section 4, by using suitable integral identity and the Runge approximation. Finally, in Section 5, we derive the monotonicity relations between the DN maps and nonnegative potentials as given in Theorem 1.2, which will be applied to show the constructive uniqueness as stated in Theorem 1.4.

2. Preliminaries

In this section, let us introduce and recall several useful properties for our study. Our notation for the Fourier transform is

$$\widehat{f}(\xi) = \mathcal{F}f(\xi) = \int_{\mathbb{R}^n} e^{-\mathrm{i}x\cdot\xi} f(x)\,dx, \tag{2.1}$$

where $\mathrm{i} = \sqrt{-1}$ denotes the imaginary unit. In what follows, let us always consider $u : \mathbb{R}^n \to \mathbb{R}$ as a (Lebesgue) measurable function, and $\Omega \subset \mathbb{R}^n$ be a bounded domain with Lipschitz boundary. Given $s \in (0,1)$, the fractional Sobolev space $H^s(\mathbb{R}^n)$ is the standard L^2-based Sobolev space with the norm $\|u\|_{H^s(\mathbb{R}^n)} = \left\| (1+|\xi|)^{s/2}\,\widehat{u}\right\|_{L^2(\mathbb{R}^n)}$.

2.1. The logarithmic Laplacian and associated function spaces.

Given $s \in (0,1)$, recall that the fractional Laplacian $(-\Delta)^s$ can be defined via the Fourier transform

$$(-\Delta)^s u := \mathcal{F}^{-1}\big(|\xi|^{2s}\,\widehat{u}(\xi)\big), \quad \text{for} \quad u \in \mathcal{S}(\mathbb{R}^n), \tag{2.2}$$

where $\mathcal{S}(\mathbb{R}^n)$ stands for the Schwartz space. Alternatively, the fractional Laplacian can be written as a hypersingular integral operator of the form

$$(-\Delta)^s u(x) = \text{P.V.} \int_{\mathbb{R}^n} (u(x) - u(z))\,\mathcal{K}_s(x-z)\,dz \tag{2.3}$$

with the symmetric kernel

$$\mathcal{K}_s(z) := \frac{C_{n,s}}{|z|^{n+2s}}, \tag{2.4}$$

and

$$C_{n,s} := \frac{4^s\Gamma\left(\frac{n}{2}+s\right)}{\pi^{n/2}\,|\Gamma(-s)|}. \tag{2.5}$$

The logarithmic Laplacian L_Δ appears in the study of $(-\Delta)^s$ in the limit $s \to 0$. Given $u \in C_c^\alpha(\mathbb{R}^n)$ for some $\alpha > 0$, $L_\Delta u(x)$ can be uniquely defined by the asymptotic expansion[2]

$$(-\Delta)^s u(x) = u(x) + sL_\Delta u(x) + o(s), \quad \text{as} \quad s \to 0^+, \tag{2.6}$$

so $L_\Delta u$ appears as a linear correction term in the second limit of (1.5). Formally, we thus have

$$\frac{d}{ds}\Big|_{s=0} (-\Delta)^s u(x) = L_\Delta u(x), \tag{2.7}$$

and $L_\Delta u$ has the symbol $2\log|\xi|$ given by (1.6).

Moreover, from [CW19, Theorem 1.1] again, it is known that the logarithmic Laplacian admits an integral representation

$$L_\Delta u(x) = c_n \text{P.V.} \int_{B_1(x)} \frac{u(x)-u(z)}{|x-z|^n}\,dz - c_n \int_{\mathbb{R}^n\setminus B_1(x)} \frac{u(z)}{|x-z|^n}\,dz + \rho_n u(x), \tag{2.8}$$

for $x \in \mathbb{R}^n$ and $u \in C_c^\alpha(\mathbb{R}^n)$, for some $\alpha > 0$, where $B_r(x)$ is the ball center at x with radius $r > 0$, and

$$c_n := \pi^{-n/2}\Gamma\left(\frac{n}{2}\right) = \frac{2}{|\mathbb{S}^{n-1}|}, \quad \rho_n := 2\log 2 + \psi\left(\frac{n}{2}\right) - \gamma. \tag{2.9}$$

Here $|\mathbb{S}^{n-1}|$, $\gamma := -\Gamma'(1)$ and $\psi = \frac{\Gamma'}{\Gamma}$ are the $(n-1)$-dimensional volume of the unit sphere in $\mathbb{R}^n$, the Euler Mascheroni constant, and the Digamma function, respectively.

[2] Here $o(s)$ denotes the small o, which satisfies $\frac{o(s)}{s} \to 0$ as $s \to 0^+$.

In the distributional sense, the logarithmic Laplacian is defined on the function space

$$L^1_0(\mathbb{R}^n) := \left\{ u \in L^1_{\mathrm{loc}}(\mathbb{R}^n) : \int_{\mathbb{R}^n} \frac{|u(x)|}{(1+|x|)^n}\, dx < \infty \right\}. \tag{2.10}$$

More precisely, for a function $u \in L^1_0(\mathbb{R}^n)$, the (distributional) logarithmic Laplacian is well-defined by setting

$$(L_\Delta u)(\phi) = \int_{\mathbb{R}^n} u\,(L_\Delta \phi)\, dx, \quad \text{for } \phi \in C^\infty_c(\mathbb{R}^n).$$

On $\mathbb{R}^n$, the natural energy space associated with the logarithmic Laplacian is defined by

$$\mathbb{H}(\mathbb{R}^n) = \left\{ u \in L^2(\mathbb{R}^n) \ : \ \int_{\mathbb{R}^n} \big|\log|\xi|\big| |\widehat{u}(\xi)|^2\, d\xi < \infty \right\}.$$

It is a Hilbert space with a scalar product

$$(v,w) \mapsto \langle v,w\rangle_{\mathbb{H}(\mathbb{R}^n)} := \langle v,w\rangle_{L^2(\mathbb{R}^n)} + \int_{\mathbb{R}^n} \big|\log|\xi|\big| \widehat{v}(\xi)\overline{\widehat{w}(\xi)}\, d\xi \tag{2.11}$$

and induced norm $\|v\|_{\mathbb{H}(\mathbb{R}^n)} = \sqrt{\langle v,v\rangle_{\mathbb{H}(\mathbb{R}^n)}}$.

Next, the bilinear form associated with the logarithmic Laplacian is given by

$$\begin{aligned} &B_0 : \mathbb{H}(\mathbb{R}^n) \times \mathbb{H}(\mathbb{R}^n) \to \mathbb{R}, \\ &B_0(v,w) := 2\int_{\mathbb{R}^n} \log|\xi| \widehat{v}(\xi)\overline{\widehat{w}(\xi)} d\xi. \end{aligned} \tag{2.12}$$

Indeed this bilinear form is well-defined and continuous on $\mathbb{H}(\mathbb{R}^n)$ since

$$\begin{aligned} \int_{\mathbb{R}^n} \big|\log|\xi|\big| |\widehat{v}(\xi)||\overline{\widehat{w}(\xi)}| d\xi &\le \Big(\int_{\mathbb{R}^n} \big|\log|\xi|\big| |\widehat{v}(\xi)|^2\, \xi\Big)^{1/2} \Big(\int_{\mathbb{R}^n} \big|\log|\xi|\big| |\widehat{v}(\xi)|^2\, \xi\Big)^{1/2} \\ &\le \|u\|_{\mathbb{H}(\mathbb{R}^n)} \|v\|_{\mathbb{H}(\mathbb{R}^n)} \quad \text{for } u,v \in \mathbb{H}(\mathbb{R}^n). \end{aligned} \tag{2.13}$$

The bilinear form B_0 allows to define L_Δ in the weak sense as an operator

$$L_\Delta : \mathbb{H}(\mathbb{R}^n) \to \mathbb{H}(\mathbb{R}^n)^*$$

by

$$\langle L_\Delta v, w\rangle = B_0(v,w)$$

for $v,w \in \mathbb{H}(\mathbb{R}^n)$, where $\langle \cdot,\cdot\rangle$ denotes the duality pairing between $\mathbb{H}(\mathbb{R}^n)^*$ and $\mathbb{H}(\mathbb{R}^n)$. By the definition of (2.12), we have the symmetry property

$$B_0(v,w) = B_0(w,v), \quad \text{for } v,w \in \mathbb{H}(\mathbb{R}^n). \tag{2.14}$$

Finally, for an open subset $\Omega \subset \mathbb{R}^n$, we consider the closed subspace

$$\mathbb{H}_0(\Omega) := \{u \in \mathbb{H}(\mathbb{R}^n) : \ u = 0 \text{ in } \Omega_e\} \tag{2.15}$$

of $\mathbb{H}(\mathbb{R}^n)$, which is again a Hilbert space. We note the following observation.

Lemma 2.1. *Let $\Omega \subset \mathbb{R}^n$ be an open set of finite measure. Then the inner product*

$$(v,w) \mapsto \langle v,w\rangle_{\mathbb{H}_0(\Omega)} := \int_{|x-z|\le 1} \frac{(v(x)-v(z))\,(w(x)-w(z))}{|x-z|^n}\, dxdz \tag{2.16}$$

induces an equivalent norm

$$v \mapsto \|v\|_{\mathbb{H}_0(\Omega)} = \sqrt{\langle v,v\rangle_{\mathbb{H}_0(\Omega)}} \text{ on } \mathbb{H}_0(\Omega).$$

Moreover, $\mathbb{H}_0(\Omega)$ is compactly embedded into $L^2(\Omega)$.

Proof. In the following, the letter $C > 0$ stands for different positive constants. We shall use the symbol $\xi \mapsto \log(1+|\xi|^2)$, associated with the logarithmic Schrödinger operator $\log(-\Delta+1)$, as a comparison function. The operator $\log(-\Delta+1)$ is a singular integral operator with kernel function

$$z \mapsto \ell(z) = C_n K_{n/2}(|z|)\,|z|^{-n/2}\,,$$

for some constant C_n depending only on $n \in \mathbb{N}$, where K_ν denotes the modified Bessel function of second kind with index ν (see e.g. [Feu23, Section 1]). As a consequence, for every $\phi \in L^2(\mathbb{R}^n)$ we have

$$\int_{\mathbb{R}^n} \log(1+|\xi|^2)|\widehat{\phi}(\xi)|^2\,d\xi = \int_{\mathbb{R}^n\times\mathbb{R}^n} \ell(|x-z|)\,(\phi(x)-\phi(z))^2\,dxdz, \tag{2.17}$$

where the finiteness of one side of this equality implies the finiteness of the other. We also note that, by the positivity of K_ν and its asymptotics at $z=0$ (see e.g. [Feu23, Section 1] again), we have

$$\frac{\ell(z)}{C_0} \le |z|^{-n} \le C_0\ell(z), \quad \text{for } z \in B_1(0)\setminus\{0\} \text{ with a constant } C_0 > 0.$$

Therefore, for every $\phi \in \mathbb{H}_0(\Omega) \subset \mathbb{H}(\mathbb{R}^n)$, we now have the estimate

$$\begin{aligned}
\int_{|x-z|\le 1} \frac{(\phi(x)-\phi(z))^2}{|x-z|^n}\,dxdz &\le C\int_{\mathbb{R}^n\times\mathbb{R}^n} \ell(|x-z|)\,(\phi(x)-\phi(z))^2\,dxdz \\
&= C\int_{\mathbb{R}^n} \log(1+|\xi|^2)|\widehat{\phi}(\xi)|^2\,d\xi \\
&\le C\int_{\mathbb{R}^n} (1+\big|\log|\xi|\big|)|\widehat{\phi}(\xi)|^2\,d\xi \\
&= C\|\phi\|^2_{\mathbb{H}(\mathbb{R}^n)}.
\end{aligned} \tag{2.18}$$

Moreover, by (2.17) we have

$$\begin{aligned}
\|\phi\|^2_{\mathbb{H}(\mathbb{R}^n)} &= \int_{\mathbb{R}^n} (1+|\log|\xi||)|\widehat{\phi}(\xi)|^2\,d\xi \\
&\le \int_{\mathbb{R}^n} \big(1-1_{B_1(0)}(\xi)\log|\xi| + \log(1+|\xi|^2)\big)|\widehat{\phi}(\xi)|^2\,d\xi \\
&\le \|\widehat{\phi}\|^2_{L^2(\mathbb{R}^n)} + \|\widehat{\phi}\|^2_{L^\infty(B_1(0)}\int_{B_1(0)}(-\log|\xi|)d\xi \\
&\quad + C\int_{\mathbb{R}^n\times\mathbb{R}^n} \ell(|x-z|)\,(\phi(x)-\phi(z))^2\,dxdz \\
&\le \|\phi\|^2_{L^2(\Omega)} + C\|\phi\|^2_{L^1(\Omega)} + C\int_{|x-z|\le 1} \frac{(\phi(x)-\phi(z))^2}{|x-y|^n}\,dxdz \\
&\quad + C\int_{|x-z|>1} \ell(x-z)\,\big(\phi^2(x)+\phi^2(z)\big)\,dxdz \\
&\le C\Big(\|\phi\|^2_{L^2(\Omega)} + \|\phi\|^2_{\mathbb{H}_0(\Omega)} + \|\phi\|^2_{L^2(\mathbb{R}^n)}\int_{\mathbb{R}^n\setminus B_1(0)} \ell(y)\,dy\Big) \\
&\le C\big(\|\phi\|^2_{L^2(\Omega)} + \|\phi\|^2_{\mathbb{H}_0(\Omega)}\big),
\end{aligned}$$

where $1_{B_1(0)}(\xi) = \begin{cases} 1 & \text{for } \xi \in B_1(0) \\ 0 & \text{otherwise} \end{cases}$ is the characteristic function. Here we used the assumption that Ω has finite measure $|\Omega| \in (0,\infty)$, such that the Hölder inequality

$$\|\phi\|_{L^1(\Omega)} \le |\Omega|^{1/2}\|\phi\|_{L^2(\Omega)}$$

holds. Combining the above estimates with the obvious bound

$$\|\phi\|_{L^2(\Omega)} = \|\phi\|_{L^2(\mathbb{R}^n)} = \|\widehat{\phi}\|_{L^2(\mathbb{R}^n)} \le \|\phi\|_{\mathbb{H}(\mathbb{R}^n)} \quad \text{for } \phi \in \mathbb{H}_0(\Omega),$$

we deduce that

$$\phi \mapsto \|\phi\|_* := \big(\|\phi\|^2_{L^2(\Omega)} + \|\phi\|^2_{\mathbb{H}_0(\Omega)}\big)^{\frac{1}{2}}$$

is an equivalent norm to $\|\cdot\|_{\mathbb{H}(\mathbb{R}^n)}$ on $\mathbb{H}_0(\Omega)$. Moreover, by [JW20, Theorem 1.2], the embedding $(\mathbb{H}_0(\Omega), \|\cdot\|_*) \hookrightarrow L^2(\Omega)$ is compact. From this compactness and the fact that $\|\phi\|_{\mathbb{H}_0(\Omega)} > 0$ for all $\phi \in \mathbb{H}_0(\Omega) \setminus \{0\}$, it follows by a standard argument that $\|\phi\|_{L^2(\Omega)} \le C\|\phi\|_{\mathbb{H}_0(\Omega)}$ for all $\phi \in \mathbb{H}_0(\Omega)$, so the norms $\|\cdot\|_*$ and $\|\cdot\|_{\mathbb{H}_0(\Omega)}$ are equivalent on $H^1_0(\Omega)$. This proves the result. □

We also have the following useful estimate.

Lemma 2.2. *For $\alpha > 0$, we have $C^\alpha_c(\mathbb{R}^n) \subset \mathbb{H}(\mathbb{R}^n)$. Moreover, for every nonempty bounded open set $W \subset \mathbb{R}^n$, there exists a constant $C = C(W, \alpha) > 0$ with*

$$\|f\|_{\mathbb{H}(\mathbb{R}^n)} \le C\|f\|_{C^\alpha(W)} \qquad \text{for all } f \in C^\alpha_c(W). \tag{2.19}$$

Proof. It clearly suffices to prove (2.19). Since $W \subset B_R(0)$ for R chosen sufficiently large, we may assume without loss of generality that $W = B_R(0)$ for some $R > 1$. Since $\|\cdot\|_{\mathbb{H}(\mathbb{R}^n)}$ is equivalent to $\|\cdot\|_{\mathbb{H}_0(W)}$ on $\mathbb{H}_0(W)$, it suffices, by approximation, to prove the estimate with $\|\cdot\|_{\mathbb{H}_0(W)}$ in place of $\|\cdot\|_{\mathbb{H}(\mathbb{R}^n)}$.

Let $f \in C^\alpha_c(W)$, then we can write, similar to [CW19, Proposition 3.2],

$$\|f\|^2_{\mathbb{H}_0(W)} = \int_{\substack{x,z\in W\\ |x-z|<1}} \frac{(f(x)-f(z))^2}{|x-z|^n}\,dxdz + 2\int_W |f(x)|^2\kappa_W(x)\,dx \tag{2.20}$$

with

$$\kappa_W(x) = \int_{(\mathbb{R}^n\setminus\overline{W})\cap B_1(x)} |y-x|^{-n}\,dx. \tag{2.21}$$

It is easy to compute that

$$\kappa_W(x) \le |\mathbb{S}^{n-1}|\left[\log\frac{1}{\operatorname{dist}(x,\partial\Omega)}\right]_+ \le |\mathbb{S}^{n-1}|\left[\log\frac{1}{R-|x|}\right]_+,$$

for $x \in W = B_R(0)$, which implies that

$$\begin{aligned}\int_W |f(x)|^2\kappa_W(x)\,dx &\le |\mathbb{S}^{n-1}|\|f\|^2_{L^\infty(W)}\int_{B_R(0)}\left[\log\frac{1}{R-|x|}\right]_+ dx\\ &\le |\mathbb{S}^{n-1}|^2\|f\|^2_{C^\alpha(W)}\int_{R-1}^R r^{n-1}\log\frac{1}{R-r}\,dr\\ &\le C\|f\|^2_{C^\alpha(W)},\end{aligned} \tag{2.22}$$

with a constant $C = C(W) > 0$. Moreover, we have

$$\begin{aligned}\int_{\substack{x,z\in W\\ |x-z|<1}} \frac{(f(x)-f(z))^2}{|x-z|^n}\,dxdz &\le \|f\|^2_{C^\alpha(W)}\int_{\substack{x,z\in W\\ |x-z|<1}} |x-z|^{\alpha-n}\,dxdz\\ &\le |W|\|f\|^2_{C^\alpha(W)}\int_{B_1(0)}|y|^{\alpha-n}\,dy\\ &\le \frac{|W||\mathbb{S}^{n-1}|}{\alpha}\|f\|^2_{C^\alpha(W)}.\end{aligned} \tag{2.23}$$

Combining (2.20), (2.22) and (2.23), we obtain (2.19) with a constant $C = C(W, \alpha) > 0$. This proves the assertion. □

Throughout the remainder of this paper, let us denote the L^2-inner product by

$$(\phi,\psi)_{L^2(A)} := \int_A \phi\psi\,dx, \tag{2.24}$$

for any $\phi,\psi \in L^2(A)$ and for any $A \subset \mathbb{R}^n$.

2.2. **The energy form of the logarithmic Schrödinger type operator.** Let $\Omega \subset \mathbb{R}^n$ be a bounded open set with Lipschitz boundary and $q \in L^\infty(\Omega)$. We consider the bilinear form associated with the operator $L_\Delta + q$ given by

$$\begin{aligned} B_q &: \mathbb{H}(\mathbb{R}^n)\times\mathbb{H}(\mathbb{R}^n) \to \mathbb{R}, \\ B_q(v,w) &:= B_0(v,w) + (qv,w)_{L^2(\Omega)}\,, \end{aligned} \tag{2.25}$$

where B_0 is defined in (2.12). We recall that the variational characterization of $\lambda_1(\Omega)$, the first Dirichlet-eigenvalue of L_Δ on Ω, is then given by

$$\lambda_1(\Omega) = \inf_{\substack{u\in\mathbb{H}_0(\Omega)\\ u\neq 0}} \frac{B_0(u,u)}{\|u\|^2_{L^2(\Omega)}}. \tag{2.26}$$

Hence the eigenvalue condition, if satisfied, implies that

$$B_q(u,u) \geq \lambda_0\|u\|^2_{L^2(\Omega)} \qquad \text{for all } u \in \mathbb{H}_0(\Omega). \tag{2.27}$$

We also note the following useful estimates.

Lemma 2.3. *Let $B_q(\cdot,\cdot)$ be the bilinear form given by (2.25), then we have*

$$|B_q(u,v)| \leq (2+\|q\|_{L^\infty(\Omega)})\|u\|_{\mathbb{H}(\mathbb{R}^n)}\|v\|_{\mathbb{H}(\mathbb{R}^n)} \quad \textit{for } u,v \in \mathbb{H}(\mathbb{R}^n). \tag{2.28}$$

and

$$B_q(u,u) \geq 2\|u\|^2_{\mathbb{H}(\mathbb{R}^n)} - C\|u\|^2_{L^2(\Omega)} \qquad \textit{for } u \in \mathbb{H}_0(\Omega) \tag{2.29}$$

with a constant $C>0$. If moreover (1.9) holds, then

$$B_q(u,u) \geq C\|u\|^2_{\mathbb{H}(\mathbb{R}^n)} \qquad \textit{for } u \in \mathbb{H}_0(\Omega) \tag{2.30}$$

with a constant $C>0$.

Proof. For $u,v \in \mathbb{H}(\mathbb{R}^n)$ we have, by (2.13),

$$\begin{aligned} |B_q(u,v)| &\leq |B_0(u,v)| + |\,(qv,w)_{L^2(\Omega)}\,| \\ &\leq 2\int_{\mathbb{R}^n} |\log|\xi||\widehat{u}(\xi)||\widehat{v}(\xi)|d\xi + \|q\|_{L^\infty(\Omega)}\|u\|_{L^2(\Omega)}\|v\|_{L^2(\Omega)} \\ &\leq 2\|u\|_{\mathbb{H}(\mathbb{R}^n)}\|v\|_{\mathbb{H}(\mathbb{R}^n)} + \|q\|_{L^\infty(\Omega)}\|u\|_{L^2(\mathbb{R}^n)}\|v\|_{L^2(\mathbb{R}^n)} \\ &\leq (2+\|q\|_{L^\infty(\Omega)})\|u\|_{\mathbb{H}(\mathbb{R}^n)}\|v\|_{\mathbb{H}(\mathbb{R}^n)}, \end{aligned}$$

as claimed in (2.28). Moreover, we have

$$\begin{aligned} B_0(u,u) &= 2\int_{\mathbb{R}^n} \log|\xi||\widehat{u}(\xi)|^2\,d\xi \\ &= 2\int_{\mathbb{R}^n} (1+\big|\log|\xi|\big|)|\widehat{u}(\xi)|^2\,d\xi - 2\int_{\mathbb{R}^n} |\widehat{u}(\xi)|^2\,d\xi \\ &\quad + 4\int_{B_1(0)} (\log|\xi|)|\widehat{u}(\xi)|^2\,d\xi \\ &\geq 2\|u\|^2_{\mathbb{H}(\mathbb{R}^n)} - 2\|u\|^2_{L^2(\Omega)} + 4\|\widehat{u}\|^2_{L^\infty(\mathbb{R}^n)}\int_{B_1(0)} \log|\xi|\,d\xi \\ &\geq 2\|u\|^2_{\mathbb{H}(\mathbb{R}^n)} - 2\|u\|^2_{L^2(\Omega)} - C\|u\|^2_{L^1(\Omega)} \\ &\geq 2\|u\|^2_{\mathbb{H}(\mathbb{R}^n)} - C\|u\|^2_{L^2(\Omega)}, \quad \text{for } u \in \mathbb{H}_0(\Omega). \end{aligned}$$

Here we used again the fact that $\|u\|_{L^1(\Omega)} \leq C\|u\|_{L^2(\Omega)}$ thanks to Ω has finite measure. We conclude that

$$B_q(u,u) \geq B_0(u,u) - \|q\|_{L^\infty(\Omega)}\|u\|^2_{L^2(\Omega)} \geq 2\|u\|^2_{\mathbb{H}(\mathbb{R}^n)} - C\|u\|^2_{L^2(\Omega)},$$

for $u \in \mathbb{H}_0(\Omega)$, as claimed in (2.29).

Finally, if (1.9) holds, then by (2.27) and (2.29) we have, for $u \in \mathbb{H}_0(\Omega)$ and $\epsilon \in (0,1)$,

$$\begin{aligned} B_q(u,u) &= (1-\epsilon)B_q(u,u) + \epsilon B_q(u,u) \\ &\geq (1-\epsilon)\lambda_0\|u\|^2_{L^2(\Omega)} + \epsilon\big(2\|u\|^2_{\mathbb{H}(\mathbb{R}^n)} - C\|u\|^2_{L^2(\Omega)}\big) \\ &= 2\epsilon\|u\|^2_{\mathbb{H}(\mathbb{R}^n)} + \big((1-\epsilon)\lambda_0 - C\epsilon\big)\|u\|^2_{L^2(\Omega)}. \end{aligned}$$

Since $\lambda_0 > 0$, we can then choose $\epsilon \in (0,1)$ such that $(1-\epsilon)\lambda_0 - C\epsilon \geq 0$. Then (2.30) holds with $C = 2\epsilon$. This concludes the proof. □

3. The forward problem

Let $\Omega \subset \mathbb{R}^n$ be a bounded open set with Lipschitz boundary, and let $q \in L^\infty(\Omega)$ satisfy (1.9). In this section, we study the forward Dirichlet problem for the logarithmic Schrödinger type equation $(L_\Delta + q)\,u = F$ in Ω. For this, we need some preparations. We define the trace space

$$\mathbb{H}_T(\Omega_e) := \big\{ f\big|_{\Omega_e} \,:\, f \in \mathbb{H}(\mathbb{R}^n) \big\}.$$

To define a suitable norm on $\mathbb{H}_T(\Omega_e)$, we note the following result.

Lemma 3.1. *For every function $f \in \mathbb{H}_T(\Omega_e)$, the infimum*

$$c_f := \inf\{\|g\|_{\mathbb{H}(\mathbb{R}^n)} \,:\, g \in \mathbb{H}(\mathbb{R}^n),\ g\big|_{\Omega_e} = f\} \tag{3.1}$$

admits a minimizer $\tilde{f} \in \mathbb{H}(\mathbb{R}^n)$. Moreover, $\tilde{f}$ is uniquely determined by the property that

$$\langle \tilde{f}, h\rangle_{\mathbb{H}(\mathbb{R}^n)} = 0 \qquad \text{for all } h \in \mathbb{H}_0(\Omega), \tag{3.2}$$

and the map $f \mapsto \tilde{f}$ is linear.

Proof. Let $f \in \mathbb{H}_T(\Omega_e)$ By definition of $\mathbb{H}_T(\Omega_e)$, the set

$$M_f := \big\{ g \in \mathbb{H}(\mathbb{R}^n) \,:\, g\big|_{\Omega_e} = f \big\}$$

is nonempty. Let $(f_k)_{k\in\mathbb{N}}$ be a minimizing sequence in M_f for the infimum in (3.1). Since f_k is bounded in $\mathbb{H}(\mathbb{R}^n)$, we may pass to a subsequence such that $f_k \rightharpoonup \tilde{f} \in \mathbb{H}(\mathbb{R}^n)$ and therefore

$$\|\tilde{f}\|_{\mathbb{H}(\mathbb{R}^n)} \leq \lim_{k\to\infty} \|f_k\|_{\mathbb{H}(\mathbb{R}^n)} = c_f.$$

Moreover, we have $h_k := f_k - f_1$ in $\mathbb{H}_0(\Omega)$ for all $k \in \mathbb{N}$, and $h_k \rightharpoonup \tilde{f} - f_1$. Since $\mathbb{H}_0(\Omega) \subset \mathbb{H}(\mathbb{R}^n)$ is a closed subspace, it follows that $\tilde{f} - f_1 \in \mathbb{H}_0(\Omega)$, hence $\tilde{f} \in M_f$ and $\tilde{f}$ is a minimizer for (3.1). It then follows that for any $h \in \mathbb{H}_0(\Omega)$ we have

$$0 \leq \lim_{t\to 0} \frac{1}{t}\big(\|\tilde{f} \pm th\|^2_{\mathbb{H}(\mathbb{R}^n)} - \|\tilde{f}\|^2_{\mathbb{H}(\mathbb{R}^n)}\big) = \pm 2\langle \tilde{f}, h\rangle_{\mathbb{H}(\mathbb{R}^n)}$$

and therefore (3.2) follows. Moreover, if $\tilde{f}_* \in M_f$ is another function satisfying (3.2), then $\tilde{f} - \tilde{f}_* \in \mathbb{H}_0(\Omega) \cap \big(\mathbb{H}_0(\Omega)\big)^\perp = \{0\}$ and therefore $\tilde{f} = \tilde{f}_*$. This shows that $\tilde{f}$ is uniquely determined by (3.2), and from this, the linearity of the map $f \to \tilde{f}$ follows. This proves the assertion □

Lemma 3.1 shows that a norm $\|\cdot\|_{\mathbb{H}_T(\Omega_e)}$ is well-defined by setting

$$\|f\|_{\mathbb{H}_T(\Omega_e)} := \|\tilde{f}\|_{\mathbb{H}(\mathbb{R}^n)} = \inf\{\|g\|_{\mathbb{H}(\mathbb{R}^n)} \,:\, g \in \mathbb{H}(\mathbb{R}^n),\ g\big|_{\Omega_e} = f\}, \tag{3.3}$$

for $f \in \mathbb{H}_T(\Omega_e)$.

3.1. **The Dirichlet problem.** For $f \in \mathbb{H}_T(\Omega_e)$ and $F \in L^2(\Omega)$, we now consider the Dirichlet problem

$$\begin{cases} (L_\Delta + q)\, u = F & \text{in } \Omega, \\ u = f & \text{in } \Omega_e. \end{cases} \tag{3.4}$$

To define the notion of weak solutions, we use the bilinear form B_q defined in (2.25).

Definition 3.2 (Weak solutions)**.** *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set with Lipschitz boundary. Given $f \in \mathbb{H}_T(\Omega_e)$ and $F \in L^2(\Omega)$, a function $u \in \mathbb{H}(\mathbb{R}^n)$ is called a weak solution to (3.4) if $u \equiv f$ in Ω_e and*

$$B_q(u, \varphi) = (F, \varphi)_{L^2(\Omega)}\,, \quad \text{for any } \varphi \in \mathbb{H}_0(\Omega). \tag{3.5}$$

We then have the following well-posedness result.

Lemma 3.3 (Well-posedness)**.** *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set with Lipschitz boundary, and $q \in L^\infty(\Omega)$ satisfy (1.9). Then, for any $f \in \mathbb{H}_T(\Omega_e)$ and $F \in L^2(\Omega)$, there exists a unique weak solution $u \in \mathbb{H}(\mathbb{R}^n)$ to (3.4). In addition, there holds*

$$\|u\|_{\mathbb{H}(\mathbb{R}^n)} \le C\left(\|F\|_{L^2(\Omega)} + \|f\|_{\mathbb{H}_T(\Omega_e)}\right), \tag{3.6}$$

for some constant $C > 0$ independent of u, F, f.

Proof of Lemma 3.3. Let $\tilde{f}$ be the $\|\cdot\|_{\mathbb{H}(\mathbb{R}^n)}$-minimizing extension of f given by Lemma 3.1, which satisfies $\|\tilde{f}\|_{\mathbb{H}(\mathbb{R}^n)} = \|f\|_{\mathbb{H}_T(\Omega_e)}$. Then $u \in \mathbb{H}(\mathbb{R}^n)$ is a weak solution to (3.4) if and only if $v = u - \tilde{f} \in \mathbb{H}_0(\Omega)$, and v satisfies

$$B_q(v, \varphi) = (F, \varphi)_{L^2(\Omega)} - B_q(\tilde{f}, \varphi), \quad \text{for any } \varphi \in \mathbb{H}_0(\Omega). \tag{3.7}$$

The existence of a unique $v \in \mathbb{H}_0(\Omega)$ with this property follows from the Lax-Milgram theorem, since, by Lemma 2.3, B_q is a continuous bilinear form on $\mathbb{H}_0(\Omega)$ satisfying (2.30). This shows the existence of a unique weak solution $u \in \mathbb{H}(\mathbb{R}^n)$ to (3.4), and $u = v + \tilde{f}$ with $v \in \mathbb{H}_0(\Omega)$ as above. Moreover, by (2.30) and (2.28) we have, with a constant $C > 0$,

$$\begin{aligned} \|v\|^2_{\mathbb{H}(\mathbb{R}^n)} &\le C B_q(v, v) \\ &= C\big((F, v)_{L^2(\Omega)} - B_q(\tilde{f}, v)\big) \\ &\le C\big(\|v\|_{L^2(\Omega)}\|F\|_{L^2(\Omega)} + \|\tilde{f}\|_{\mathbb{H}(\mathbb{R}^n)}\|v\|_{\mathbb{H}(\mathbb{R}^n)}\big) \\ &\le C\|v\|_{\mathbb{H}(\mathbb{R}^n)}\big(\|F\|_{L^2(\Omega)} + \|\tilde{f}\|_{\mathbb{H}(\mathbb{R}^n)}\big) \end{aligned}$$

and therefore

$$\|v\|_{\mathbb{H}(\mathbb{R}^n)} \le C\big(\|F\|_{L^2(\Omega)} + \|\tilde{f}\|_{\mathbb{H}(\mathbb{R}^n)}\big). \tag{3.8}$$

Consequently,

$$\begin{aligned} \|u\|_{\mathbb{H}(\mathbb{R}^n)} &\le \|v\|_{\mathbb{H}(\mathbb{R}^n)} + \|\tilde{f}\|_{\mathbb{H}(\mathbb{R}^n)} \\ &\le C\big(\|F\|_{L^2(\Omega)} + \|\tilde{f}\|_{\mathbb{H}(\mathbb{R}^n)}\big) \\ &= C\big(\|F\|_{L^2(\Omega)} + \|f\|_{\mathbb{H}_T(\Omega_e)}\big), \end{aligned}$$

as claimed. □

Corollary 3.4. *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set with Lipschitz boundary, let $q \in L^\infty(\Omega)$ satisfy (1.9), and let $W \subset \Omega_e$ be a nonempty bounded open set. Then for every $f \in C_c^\alpha(W)$, there exists a unique weak solution $u \in \mathbb{H}(\mathbb{R}^n)$ to (3.4). In addition, there holds*

$$\|u\|_{\mathbb{H}(\mathbb{R}^n)} \leq C\left(\|F\|_{L^2(\Omega)} + \|f\|_{C^\alpha(W)}\right), \tag{3.9}$$

for some constant $C > 0$ independent of u, F, f.

Proof. The result follows by combining Lemma 2.2 and Lemma 3.3, since $f \in C_c^\alpha(W) \subset \mathbb{H}(\mathbb{R}^n)$ and

$$\left\|f|_{\Omega_e}\right\|_{\mathbb{H}_T(\Omega_e)} \leq \|f\|_{\mathbb{H}(\mathbb{R}^n)} \leq C\|f\|_{C^\alpha(W)}$$

by definition of $\|\cdot\|_{\mathbb{H}_T(\Omega_e)}$. □

3.2. **The DN map.** With Theorem 3.3 at hand, the DN map of (1.7) can be defined rigorously via the bilinear form B_q defined in (2.25).

Lemma 3.5 (The DN map)**.** *Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain, and let $q \in L^\infty(\Omega)$ fulfill (1.9). Define*

$$\langle \Lambda_q f, g\rangle := B_q(u_f, v_g), \tag{3.10}$$

for any $f, g \in \mathbb{H}_T(\Omega_e)$, where $u_f \in \mathbb{H}(\mathbb{R}^n)$ is the solution to (1.7), and $v_g \in \mathbb{H}(\mathbb{R}^n)$ can be any representative function with $v_g|_{\Omega_e} = g$. Then

$$\Lambda_q : \mathbb{H}_T(\Omega_e) \to \mathbb{H}_T(\Omega_e)^* \tag{3.11}$$

is a bounded linear operator, which satisfies the symmetry property

$$\langle \Lambda_q f, g\rangle = \langle \Lambda_q g, f\rangle \qquad \text{for any } f, g \in \mathbb{H}_T(\Omega_e). \tag{3.12}$$

Proof. First, if there are two functions $v_g, \widetilde{v}_g \in \mathbb{H}(\mathbb{R}^n)$ such that $v_g|_{\Omega_e} = \widetilde{v}_g|_{\Omega_e}$, then $v_g - \widetilde{v}_g \in \mathbb{H}_0(\Omega)$. Therefore, by the linearity of $B_q(u_f, \cdot)$, one has

$$B_q(u_f, \widetilde{v}_g) = \underbrace{B_q(u_f, v_g) + B_q(u_f, \widetilde{v}_g - v_g) = B_q(u_f, v_g)}_{\text{since } u_f \text{ solves (1.7) and } \widetilde{v}_g - v_g \in \mathbb{H}_0(\Omega)}, \tag{3.13}$$

which shows (3.10) is well-defined. To show the boundedness of Λ_q, we let $\tilde{g} \in \mathbb{H}(\mathbb{R}^n)$ be the $\|\cdot\|_{\mathbb{H}(\mathbb{R}^n)}$-minimizing extension of $g \in \mathbb{H}_T(\Omega_e)$ given by Lemma 3.1, so that $\|\tilde{g}\|_{\mathbb{H}(\mathbb{R}^n)} = \|g\|_{\mathbb{H}_T(\Omega_e)}$. Then, by Lemma 2.3 and (3.6) applied with $F = 0$, we have

$$|\langle \Lambda_q f, g\rangle| = |B_q(u_f, \tilde{g})| \leq C\|u_f\|_{\mathbb{H}(\mathbb{R}^n)}\|\tilde{g}\|_{\mathbb{H}(\mathbb{R}^n)} \leq C\|f\|_{\mathbb{H}_T(\Omega_e)}\|g\|_{\mathbb{H}_T(\Omega_e)}.$$

This shows that $\Lambda_q : \mathbb{H}_T(\Omega_e) \to \mathbb{H}_T(\Omega_e)^*$ is bounded. Now, the symmetry of the DN map can be seen from the symmetry of the bilinear form B_q. This proves the assertion. □

We can also derive the integral identity.

Lemma 3.6 (Integral identity)**.** *Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain, and let $q_j \in L^\infty(\Omega)$ satisfy (1.9) for $j = 1, 2$. For any $f_1, f_2 \in \mathbb{H}_T(\Omega_e)$, we then have*

$$\langle (\Lambda_{q_1} - \Lambda_{q_2})\, f_1, f_2\rangle = \int_\Omega (q_1 - q_2)\, u_{f_1} u_{f_2}\, dx, \tag{3.14}$$

where u_{f_j} is the unique weak solution to

$$\begin{cases} (L_\Delta + q_j)\, u_{f_j} = 0 & \text{in } \Omega, \\ u_{f_j} = f_j & \text{in } \Omega_e, \end{cases} \tag{3.15}$$

for $j = 1, 2$.

Proof. Via (3.12), one can obtain

$$\begin{aligned}\langle(\Lambda_{q_1}-\Lambda_{q_2})\,f_1,f_2\rangle &= \langle\Lambda_{q_1}(f_1),f_2\rangle-\langle f_1,\Lambda_{q_2}(f_2)\rangle\\ &= B_{q_1}(u_{f_1},u_{f_2})-B_{q_2}(u_{f_1},u_{f_2})\\ &= \int_\Omega (q_1-q_2)\,u_{f_1}u_{f_2}\,dx,\end{aligned}\tag{3.16}$$

where we used (2.14) in the last equality. □

4. Proof of Theorem 1.1

As we discussed before, in solving nonlocal inverse problems, one usually needs the Runge approximation for nonlocal operators. The proof of the (qualitative) Runge approximation for L_Δ is based on the following unique continuation property (UCP) for the logarithmic Laplacian. Recall the definition of the space $L^1_0(\mathbb{R}^n)$ given in (2.10).

Proposition 4.1 ([CHW23, Theorem 5.1])**.** *Let $D\subset\mathbb{R}^n$ be a nonempty open set. If $u\in L^1_0(\mathbb{R}^n)$ satisfies*

$$u=L_\Delta u=0 \text{ in } D \text{ in distributional sense,}\tag{4.1}$$

then $u\equiv 0$ in $\mathbb{R}^n$.

Let $\Omega\subset\mathbb{R}^n$ be a bounded nonempty Lipschitz domain, and let $q\in L^\infty(\Omega)$ satisfy (1.9). Then the solution operator associated with (1.7) is defined as

$$P_q:\mathbb{H}_T(\Omega_e)\to\mathbb{H}(\mathbb{R}^n),\quad f\mapsto u_f,\tag{4.2}$$

where $u_f\in\mathbb{H}(\mathbb{R}^n)$ is the solution to (1.7), so the solution of (3.4) with $F=0$. With Proposition 4.1 at hand, we can show the Runge approximation.

Proposition 4.2 (Runge approximation)**.** *Let $\Omega\subset\mathbb{R}^n$ be a bounded Lipschitz domain, let $q\in L^\infty(\Omega)$ satisfy (1.9), and consider the solution operator P_q given by (4.2). Then for every nonempty open set $W\Subset\Omega_e$, the set*

$$\mathcal{R}:=\left\{P_qf\big|_\Omega:\,f\in C_c^\infty(W)\right\},\tag{4.3}$$

is dense in $L^2(\Omega)$.

Proof. By the Hahn-Banach theorem, it suffices to show that for any $w\in L^2(\Omega)$, which satisfies

$$(P_qf,w)_{L^2(\Omega)}=0\quad\text{for any}\quad f\in C_c^\infty(W),\tag{4.4}$$

it results that $w\equiv 0$. Let $\varphi\in\mathbb{H}_0(\Omega)$ be the solution to

$$\begin{cases}(L_\Delta+q)\,\varphi=w & \text{in }\Omega,\\ \varphi=0 & \text{in }\Omega_e.\end{cases}\tag{4.5}$$

The well-posedness of the above equation was guaranteed in Section 3. Hence, we have

$$B_q(\varphi,f)=B_q(\varphi,f-P_qf)=(w,f-P_qf)_{L^2(\Omega)}=-\,(w,P_qf)_{L^2(\Omega)}=0,\tag{4.6}$$

for any $f\in C_c^\infty(W)$, where we used (4.4) in the last identity. Hence for any $f\in C_c^\infty(W)$ we have, since $f\equiv 0$ in Ω,

$$\begin{aligned}\int_{\mathbb{R}^n}\varphi L_\Delta f\,dx &= 2\int_{\mathbb{R}^n}(\log|\xi|)\widehat{\varphi}(\xi)\widehat{f}(\xi)\,d\xi\\ &= B_q(\varphi,f)+\int_\Omega q(x)\varphi(x)f(x)\,dx\\ &= 0.\end{aligned}$$

This yields that $L_\Delta \varphi \equiv 0$ in W in distributional sense, while also $\varphi \equiv 0$ in W. Note that $\varphi \in L^1_0(\mathbb{R}^n)$ since $\mathbb{H}_0(\Omega) \subset L^1_0(\mathbb{R}^n)$, and this allows us to apply the UCP of Proposition 4.1. Therefore, Proposition 4.1 implies that $\varphi \equiv 0$ in $\mathbb{R}^n$ and therefore $w \equiv 0$ as desired. This concludes the proof. □

Remark 4.3. *As we have mentioned earlier in Section 1, the logarithmic Laplacian is a near zero-order operator, so we do not expect that the above-mentioned Runge approximation possesses a higher regularity approximation property.*

We are now ready to prove the global uniqueness result by using the Runge approximation.

Proof of Theorem 1.1. With the condition (1.11), the symmetry of the operators Λ_{q_i} and integral identity (3.14) imply

$$\int_\Omega (q_1 - q_2)\, u_f u_g \, dx = 0 \qquad \text{for all } f \in C_c^\infty(W_1),\ g \in C_c^\infty(W_2), \tag{4.7}$$

where $u_f = P_{q_1} f$ and $u_g = P_{q_2} g \in \mathbb{H}(\mathbb{R}^n)$ are the solutions to

$$\begin{cases} (L_\Delta + q_1)\, u_f = 0 & \text{in } \Omega, \\ u_f = f & \text{in } \Omega_e, \end{cases} \tag{4.8}$$

and

$$\begin{cases} (L_\Delta + q_2)\, u_g = 0 & \text{in } \Omega, \\ u_g = g & \text{in } \Omega_e, \end{cases} \tag{4.9}$$

respectively. By the Runge approximation in Proposition 4.2, given any $h \in L^2(\Omega)$, there exist sequences of functions $f_k \in C_c^\infty(W_1)$, $g_k \in C_c^\infty(W_2)$ with $P_{q_1} f_k \to h$ and $P_{q_2} g_k \to 1$ in $L^2(\Omega)$ as $k \to \infty$. By (4.7), we then conclude that

$$\int_\Omega (q_1 - q_2)\, h \, dx = \lim_{k\to\infty} \int_\Omega (q_1 - q_2)\, (P_{q_1} f_k)(P_{q_2} g_k)\, dx = 0. \tag{4.10}$$

Due to the arbitrariness of $h \in L^2(\Omega)$, there must hold $q_1 = q_2$ a.e. in Ω. This proves the assertion. □

Remark 4.4. *It would be interesting to ask if one can prove Theorem 1.1 using a single measurement. It is clear that if the potential q is regular enough (e.g. continuous potentials), one may directly apply the existing UCP of Proposition 4.1 for L_Δ. However, for the rough potential case, one needs to study a measurable UCP result for the logarithmic Laplacian, i.e., the UCP for L_Δ holds on a set of positive measures.*

5. Constructive uniqueness

To prove Theorem 1.4, we will utilize the monotonicity method combined with the localized potentials for (1.7). This shows that increasing the potential q increases the DN map Λ_q in the sense of quadratic forms, and vice versa. To accomplish the complete arguments, we need the following localized potentials.

5.1. Localized potentials.

With the Runge approximation at hand, we can immediately derive the existence of the localized potentials.

Lemma 5.1 (Localized potentials)**.** *Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain $q \in L^\infty(\Omega)$, and $W \Subset \Omega_e$ be a nonempty open set. For every measurable set*

$M \subset \Omega$, there exists a sequence $f^k \in C_c^\infty(W)$, such that the corresponding solutions $u^k \in \mathbb{H}(\mathbb{R}^n)$ of

$$\begin{cases} (L_\Delta + q)\, u^k = 0 & \text{in } \Omega, \\ u^k = f^k & \text{in } \Omega_e, \end{cases} \tag{5.1}$$

for all $k \in \mathbb{N}$, satisfy that

$$\int_M \left|u^k\right|^2 dx \to \infty \quad \text{and} \quad \int_{\Omega\setminus M} \left|u^k\right|^2 dx \to 0 \quad \text{as} \quad k \to \infty.$$

Proof. Applying the Runge approximation of Proposition 4.2, we find a sequence of functions $\widetilde{f}^k \in C_c^\infty(W)$ so that the corresponding solutions $\widetilde{u}^k\big|_\Omega$ converge to $\frac{\chi_M}{\sqrt{\int_M 1\, dx}}$ in $L^2(\Omega)$, and

$$\left\|\widetilde{u}^k\right\|^2_{L^2(M)} = \int_M \left|\widetilde{u}^k\right|^2 dx \to 1, \quad \text{and} \quad \left\|\widetilde{u}^k\right\|^2_{L^2(\Omega\setminus M)} = \int_{\Omega\setminus M} \left|\widetilde{u}^k\right|^2 dx \to 0$$

as $k \to \infty$. We may assume for all $k \in \mathbb{N}$ that $\widetilde{u}^k \not\equiv 0$ without loss of generality, so that $\left\|\widetilde{u}^{(k)}\right\|_{L^2(\Omega\setminus M)} > 0$ follows from the UCP of Proposition 4.1 for L_Δ. Taking normalizing

$$f^k := \frac{\widetilde{f}^k}{\left\|\widetilde{u}^k\right\|^{1/2}_{L^2(\Omega\setminus M)}},$$

the sequence of corresponding solutions $u^k \in \mathbb{H}(\mathbb{R}^n)$ of (5.1) possesses the desired property

$$\left\|u^k\right\|^2_{L^2(M)} = \frac{\left\|\widetilde{u}^k\right\|^2_{L^2(M)}}{\left\|\widetilde{u}^k\right\|_{L^2(\Omega\setminus M)}} \to \infty, \quad \text{and} \quad \left\|u^k\right\|^2_{L^2(\Omega\setminus M)} = \left\|\widetilde{u}^k\right\|_{L^2(\Omega\setminus M)} \to 0,$$

as $k \to \infty$. This completes the proof. □

5.2. **Monotonicity relations.** Here we complete the proof of Theorems 1.2 and 1.4.

Lemma 5.2 (Monotonicity relations)**.** *Let $\Omega \subset \mathbb{R}^n$ be a bounded open Lipschitz domain and $f \in \mathbb{H}_T(\Omega_e)$. Moreover, for $j = 1, 2$, let $q_j \in L^\infty(\Omega)$ satisfy (1.9), and let $u_j \in \mathbb{H}(\mathbb{R}^n)$ be the unique solutions of*

$$\begin{cases} (L_\Delta + q_j)\, u_j = 0 & \text{in } \Omega, \\ u_j = f & \text{in } \Omega_e. \end{cases} \tag{5.2}$$

Then we have the monotonicity relations

$$\langle (\Lambda_{q_2} - \Lambda_{q_1})\, f, f\rangle \le \int_\Omega (q_2 - q_1)\, |u_1|^2\, dx, \tag{5.3}$$

and

$$\langle (\Lambda_{q_2} - \Lambda_{q_1} f, f\rangle \ge \int_\Omega (q_2 - q_1)\, |u_2|^2\, dx. \tag{5.4}$$

Proof. The definition of the DN map (3.10) implies that

$$\langle \Lambda_{q_1} f, f\rangle = B_{q_1}(u_1, u_1) \qquad \text{and} \qquad \langle \Lambda_{q_2} f, f\rangle = B_{q_2}(u_2, u_2) = B_{q_2}(u_2, u_1).$$

By the symmetry property of the bilinear form, we thus have

$$\begin{aligned} 0 &\leq B_{q_2}(u_2-u_1,u_2-u_1) \\ &= B_{q_2}(u_2,u_2)-2B_{q_2}(u_2,u_1)+B_{q_2}(u_1,u_1) \\ &= -\langle \Lambda_{q_2} f, f\rangle + B_{q_2}(u_1,u_1) \\ &= \langle (\Lambda_{q_1}-\Lambda_{q_2}) f, f\rangle + B_{q_2}(u_1,u_1) - B_{q_1}(u_1,u_1) \\ &= \langle (\Lambda_{q_1}-\Lambda_{q_2}) f, f\rangle + \int_\Omega (q_2-q_1)\,|u_1|^2\,dx, \end{aligned} \tag{5.5}$$

which proves (5.3). Interchanging q_1 and q_2 in the above computations yields (5.4). This proves the assertion. □

Proof of Theorem 1.2. We have to show the equivalence of Properties (i)–(iii). Suppose first that (i) holds, i.e., we have $q_1 \leq q_2$ a.e. in Ω. Then (5.4) readily implies that $\langle (\Lambda_{q_2}-\Lambda_{q_1}) f, f\rangle \geq 0$ for every $f \in \mathbb{H}_T(\Omega_e)$, so (ii) holds. From (ii) it directly follows that (1.12) holds for any nonempty bounded open set $W \Subset \Omega_e$, since $C_c^\infty(W) \subset \mathbb{H}_T(\Omega_e)$. Therefore (ii) implies (iii). Finally, let us show the implication (iii) $\Longrightarrow$ (i). So we assume that (1.12) holds for any nonempty bounded open set $W \Subset \Omega_e$ for some nonempty bounded open set $W \Subset \Omega_e$, and we need to show that $q_1 \leq q_2$ a.e. in Ω. Arguing by contradiction, let us assume that $q_1 \leq q_2$ does not hold a.e. in Ω. Then there exists $\delta > 0$ and a positive measurable set $M \subset \Omega$ such that $q_1 - q_2 \geq \delta > 0$ on $M \subseteq \Omega$. Using the sequence of localized potentials from Lemma 5.1 for the coefficient q_1, and the monotonicity inequality from Lemma 5.2, one can get

$$\begin{aligned} \left\langle (\Lambda_{q_2}-\Lambda_{q_1}) f^k, f^k\right\rangle &\leq \int_\Omega (q_2-q_1)\left|u_1^k\right|^2 dx \\ &\leq \|q_2-q_1\|_{L^\infty(\Omega\setminus M)} \left\|u_1^k\right\|^2_{L^2(\Omega\setminus M)} - \delta\left\|u_1^k\right\|^2_{L^2(M)} \\ &\to -\infty, \text{ as } k\to\infty, \end{aligned} \tag{5.6}$$

which contradicts (1.12). Therefore, the assertion is proved. □

Finally, let us prove Theorem 1.4.

Proof of Theorem 1.4. By [HL19, Lemma 4.4], it is known that given any nonnegative $q \in L^\infty(\Omega)$, then one has

$$q(x) = \sup\{\varphi(x) : \varphi \in \Sigma_{+,0},\ \varphi \leq q\} \text{ for a.e. } x \in \Omega. \tag{5.7}$$

Moreover, since we assume that $\lambda_1(\Omega) > 0$, the relation (1.9) is satisfied, and it is also satisfied for φ in place of q if $\varphi \in \Sigma_{+,0}$. We may therefore combine (5.7) with Theorem 1.2, applied to $q_1 = \varphi \in \Sigma_{+,0}$ and $q_2 = q$, to complete the proof. □

Remark 5.3. *With the if-and-only-if monotonicity relations, inverse problems have more possible applications. For instance, one can also study inverse obstacle problems, which recover the unknown inclusion via the monotonicity test (see e.g.* [HL19, HPS19]*). The stability result could also be interesting (see e.g.* [HL20]*).*

Acknowledgments. Y.-H. Lin is partially supported by the National Science and Technology Council (NSTC) Taiwan, under the projects 113-2628-M-A49-003. Y.-H. Lin is also a Humboldt research fellow.

References

[Buk08] Alexander Bukhgeim. Recovering a potential from Cauchy data in the two-dimensional case. *J. Inverse Ill-Posed Probl.*, 16(1):19–33, 2008.

[Cal80] Alberto-P. Calderón. On an inverse boundary value problem. In *Seminar on Numerical Analysis and its Applications to Continuum Physics (Rio de Janeiro, 1980)*, pages 65–73. Soc. Brasil. Mat., Rio de Janeiro, 1980.

[CdHS24] Giovanni Covi, Maarten V. de Hoop, and Mikko Salo. Geometrical optics for the fractional Helmholtz equation and applications to inverse problems. *arXiv preprint arXiv:2412.14698*, 2024.

[CGRU23] Giovanni Covi, Tuhin Ghosh, Angkana Rüland, and Gunther Uhlmann. A reduction of the fractional Calderón problem to the local Calderón problem by means of the Caffarelli-Silvestre extension. *arXiv preprint arXiv:2305.04227*, 2023.

[CHW23] Huyuan Chen, Daniel Hauer, and Tobias Weth. An extension problem for the logarithmic Laplacian. *arXiv preprint arXiv:2312.15689*, 2023.

[CLL19] Xinlin Cao, Yi-Hsuan Lin, and Hongyu Liu. Simultaneously recovering potentials and embedded obstacles for anisotropic fractional Schrödinger operators. *Inverse Probl. Imaging*, 13(1):197–210, 2019.

[CLR20] Mihajlo Cekić, Yi-Hsuan Lin, and Angkana Rüland. The Calderón problem for the fractional Schrödinger equation with drift. *Calc. Var. Partial Differential Equations*, 59(3):Paper No. 91, 46, 2020.

[CMRU22] Giovanni Covi, Keijo Mönkkönen, Jesse Railo, and Gunther Uhlmann. The higher order fractional Calderón problem for linear local operators: Uniqueness. *Adv. Math.*, 399:Paper No. 108246, 2022.

[CW19] Huyuan Chen and Tobias Weth. The Dirichlet problem for the logarithmic Laplacian. *Comm. Partial Differential Equations*, 44(11):1100–1139, 2019.

[DLNP21] Lucia De Luca, Matteo Novaga, and Marcello Ponsiglione. The 0-fractional perimeter between fractional perimeters and Riesz potentials. *Ann. Sc. Norm. Super. Pisa Cl. Sci. (5)*, 22(4):1559–1596, 2021.

[Fei24] Ali Feizmohammadi. Fractional Calderón problem on a closed Riemannian manifold. *Trans. Amer. Math. Soc.*, 377(4):2991–3013, 2024.

[Feu23] Pierre Aime Feulefack. The logarithmic Schrödinger operator and associated Dirichlet problems. *J. Math. Anal. Appl.*, 517(2):Paper No. 126656, 33, 2023.

[FGKU21] Ali Feizmohammadi, Tuhin Ghosh, Katya Krupchyk, and Gunther Uhlmann. Fractional anisotropic Calderón problem on closed Riemannian manifolds. *arXiv:2112.03480*, 2021.

[FJW22] Pierre Aime Feulefack, Sven Jarohs, and Tobias Weth. Small order asymptotics of the Dirichlet eigenvalue problem for the fractional Laplacian. *J. Fourier Anal. Appl.*, 28(2):Paper No. 18, 44, 2022.

[FKU24] Ali Feizmohammadi, Katya Krupchyk, and Gunther Uhlmann. Calderón problem for fractional Schrödinger operators on closed Riemannian manifolds. *arXiv preprint arXiv:2407.16866*, 2024.

[FL24] Ali Feizmohammadi and Yi-Hsuan Lin. Entanglement principle for the fractional Laplacian with applications to inverse problems. *arXiv preprint arXiv:2412.13118*, 2024.

[GLX17] Tuhin Ghosh, Yi-Hsuan Lin, and Jingni Xiao. The Calderón problem for variable coefficients nonlocal elliptic operators. *Comm. Partial Differential Equations*, 42(12):1923–1961, 2017.

[GRSU20] Tuhin Ghosh, Angkana Rüland, Mikko Salo, and Gunther Uhlmann. Uniqueness and reconstruction for the fractional Calderón problem with a single measurement. *J. Funct. Anal.*, 279(1):108505, 42, 2020.

[GSU20] Tuhin Ghosh, Mikko Salo, and Gunther Uhlmann. The Calderón problem for the fractional Schrödinger equation. *Anal. PDE*, 13(2):455–475, 2020.

[GU21] Tuhin Ghosh and Gunther Uhlmann. The Calderón problem for nonlocal operators. *arXiv:2110.09265*, 2021.

[HL19] Bastian Harrach and Yi-Hsuan Lin. Monotonicity-based inversion of the fractional Schrödinger equation I. Positive potentials. *SIAM J. Math. Anal.*, 51(4):3092–3111, 2019.

[HL20] Bastian Harrach and Yi-Hsuan Lin. Monotonicity-based inversion of the fractional Schödinger equation II. General potentials and stability. *SIAM J. Math. Anal.*, 52(1):402–436, 2020.

[HPS19] Bastian Harrach, Valter Pohjola, and Mikko Salo. Monotonicity and local uniqueness for the Helmholtz equation. *Anal. PDE*, 12(7):1741–1771, 2019.

[HSSn22] Víctor Hernández Santamaría and Alberto Saldaña. Small order asymptotics for nonlinear fractional problems. *Calc. Var. Partial Differential Equations*, 61(3):Paper No. 92, 26, 2022.

[JSnW20] Sven Jarohs, Alberto Saldaña, and Tobias Weth. A new look at the fractional Poisson problem via the logarithmic Laplacian. *J. Funct. Anal.*, 279(11):108732, 50, 2020.

[JW20] Sven Jarohs and Tobias Weth. Local compactness and nonvanishing for weakly singular nonlocal quadratic forms. *Nonlinear Anal.*, 193:111431, 15, 2020.

[Lin22] Yi-Hsuan Lin. Monotonicity-based inversion of fractional semilinear elliptic equations with power type nonlinearities. *Calc. Var. Partial Differential Equations*, 61(5):Paper No. 188, 30, 2022.

[Lin24] Yi-Hsuan Lin. The fractional anisotropic Calderón problem for a nonlocal parabolic equation on closed Riemannian manifolds. *arXiv preprint arXiv:2410.17750*, 2024.

[LL22] Ru-Yu Lai and Yi-Hsuan Lin. Inverse problems for fractional semilinear elliptic equations. *Nonlinear Anal.*, 216:Paper No. 112699, 21, 2022.

[LL23] Yi-Hsuan Lin and Hongyu Liu. Inverse problems for fractional equations with a minimal number of measurements. *Commun. Anal. Comput.*, 1(1):72–93, 2023.

[LLR20] Ru-Yu Lai, Yi-Hsuan Lin, and Angkana Rüland. The Calderón problem for a space-time fractional parabolic equation. *SIAM J. Math. Anal.*, 52(3):2655–2688, 2020.

[LLU22] Ching-Lung Lin, Yi-Hsuan Lin, and Gunther Uhlmann. The Calderón problem for nonlocal parabolic operators. *arXiv preprint arXiv:2209.11157*, 2022.

[LLU23] Ching-Lung Lin, Yi-Hsuan Lin, and Gunther Uhlmann. The Calderón problem for nonlocal parabolic operators: A new reduction from the nonlocal to the local. *arXiv preprint arXiv:2308.09654*, 2023.

[LW21] Ari Laptev and Tobias Weth. Spectral properties of the logarithmic Laplacian. *Anal. Math. Phys.*, 11(3):Paper No. 133, 24, 2021.

[LZ23] Yi-Hsuan Lin and Philipp Zimmermann. Unique determination of coefficients and kernel in nonlocal porous medium equations with absorption term. *arXiv preprint arXiv:2305.16282*, 2023.

[LZ24] Yi-Hsuan Lin and Philipp Zimmermann. Approximation and uniqueness results for the nonlocal diffuse optical tomography problem. *arXiv preprint arXiv:2406.06226*, 2024.

[RS18] Angkana Rüland and Mikko Salo. Exponential instability in the fractional Calderón problem. *Inverse Problems*, 34(4):045003, 21, 2018.

[RS20] Angkana Rüland and Mikko Salo. The fractional Calderón problem: low regularity and stability. *Nonlinear Anal.*, 193:111529, 56, 2020.

[Rül23] Angkana Rüland. Revisiting the anisotropic fractional Calderón problem using the Caffarelli-Silvestre extension. *arXiv preprint arXiv:2309.00858*, 2023.

[SU87] John Sylvester and Gunther Uhlmann. A global uniqueness theorem for an inverse boundary value problem. *Ann. of Math. (2)*, 125(1):153–169, 1987.

[Uhl09] Gunther Uhlmann. Electrical impedance tomography and Calderón's problem. *Inverse Problems*, 25(12):123011, 39, 2009.

Institute for Mathematics, Goethe University Frankfurt, Germany
Email address: harrach@math.uni-frankfurt.de

Department of Applied Mathematics, National Yang Ming Chiao Tung University, Hsinchu, Taiwan & Fakultät für Mathematik, University of Duisburg-Essen, Essen, Germany
Email address: yihsuanlin3@gmail.com

Institute for Mathematics, Goethe University Frankfurt, Germany
Email address: weth@math.uni-frankfurt.de