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Leading and second order homogenization of an elastic scattering problem for highly oscillating anisotropic medium

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Abstract We consider the scattering of elastic waves by highly oscillating anisotropic periodic media with bounded support. Applying the two-scale homogenization, we first obtain a constant coefficient second-order partial differential elliptic equation that describes the wave propagation of the effective or overall wave field. We further pursue a higher-order homogenization with the help of complimentary boundary correctors and provide a detailed analysis on the rate of higher-order convergence. Finally we provide preliminary numerical examples to demonstrate the higher-order homogenization.

Keywords Second-order homogenization · Elastic scattering · Periodic media · Wave dispersion · Two-scale homogenization

1 Introduction and summary of results

1.1 Motivation and background

The wave propagation in periodic media is of great interest in cloaking, subwavelength imaging, and noise control, thanks to the underpinning phenomena of frequency-dependent anisotropy and band gaps [19,22,28]. Away from averaging techniques, the effective wave motion can also be obtained using the two-scale method [5] with a perturbation parameter that signifies the ratio between the unit cell of periodicity and wavelength. In the regime of long-wavelength and low-frequency, the leading-order homogenization in particular gives the quasi-static model by a second-order partial differential equation, where the elastic tensor and density are replaced by constant effective elastic tensor and density respectively

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[5,26]. To gain further understanding of the wave dispersion, a higher-order homogenization has been taken into account. A higher-order homogenization was derived for the scalar wave equation in [11,12,9,21,27,15], where a fourth-order partial differential equation was formally derived and the dispersive effect was hence demonstrated. [8] considered a higher-order homogenization of the elastic wave for non-periodic layered media that transcends the usual quasi-staic regime. Alternatively a dispersive model for scalar wave equation was derived using Floquet-Bloch theory and higher-order asymptotic of the Bloch variety [24]. The higher-order homogenization in particular sheds light on sensing the microstructure through dispersion [18]. In the case that the periodic structure was only supported in a bounded domain, contrary to the case that the periodic structure occupies the whole space, the boundary correctors played a role in the homogenization [7]. Our contribution is to investigate the scattering of elastic waves by highly oscillating anisotropic periodic media with bounded support.

The homogenization problem for the second-order elliptic equations or systems with periodic coefficients has been well studied in the literature, for example, see [2,3,5,10,16,17,23,26]. Concerning about the regularity estimates, the authors in [2,3] introduced a famous three-steps compactness method to prove the Hölder estimates for solutions of divergence and non-divergence elliptic systems. Furthermore, in [2], the authors studied the Green function and the Poisson kernel to establish L^p theory of the elliptic homogenization problem. The authors in [17] investigated the asymptotic behaviour of the Green and Neumann functions and derived optimal convergence rates in L^p and $W^{1,p}$ for solutions with Dirichlet or Neumann boundary conditions. They further studied the convergence rates in L^2 of solutions of the elliptic systems in Lipschitz domains in [16]. We refer to [26] for an excellent lecture note for the survey on this research area.

1.2 The elastic scattering problem

Let $\Omega \subset \mathbb{R}^d$ be a bounded simply connected domain with a C^∞-smooth boundary $\partial\Omega$ for $d = 2, 3$. We remark that our study still holds in one dimension where all the tensors become scalar. Let $\epsilon > 0$ be a small parameter and $Y := [0, 1]^d$ be the unit cell. Let $\mathbf{C} = \mathbf{C}(y)$ be an anisotropic elastic fourth-order tensor with $\mathbf{C} = (C_{ijk\ell})_{1\le i,j,k,\ell\le d}$. In this paper, we assume that the elastic tensor $\mathbf{C} = \mathbf{C}(y)$ satisfies the following conditions.

- Periodicity: The elastic tensor $\mathbf{C} = \mathbf{C}(y)$ is Y-periodic,

$$\mathbf{C}(y + z) = \mathbf{C}(y), \text{ for any } y \in \mathbb{R}^d \text{ and } z \in \mathbb{Z}^d.$$

- Strong convexity:

$$\sum_{i,j,k,\ell=1}^{d} C_{ijk\ell}(y)a_{ij}a_{k\ell} \ge c_0 \sum_{i,j=1}^{d} a_{ij}^2, \text{ for any } y \in \mathbb{R}^d, \tag{1}$$

  with some constant $c_0 > 0$ and for any constant symmetric matrix $(a_{ij})_{1\le i,j\le d}$.
- Smoothness: $\mathbf{C}_{ijk\ell}(y) \in C^\infty(\mathbb{R}^d)$, for all $1 \le i, j, k, \ell \le d$.

– Symmetry: The elastic tensor $\mathbf{C} = (C_{ijk\ell})_{1\le i,j,k,\ell\le d}$ satisfies major and minor symmetric condition, that is,

$$C_{ijk\ell} = C_{k\ell ij} \text{ and } C_{ijk\ell} = C_{ij\ell k}, \text{ for all } 1 \le i,j,k,\ell \le d.$$

The symmetric property of the elastic tensor $\mathbf{C} = \mathbf{C}(y)$ plays an important role in the study of the asymptotic analysis of the scattering homogenization problem (see Section 2 and Section 3). Next, let $\rho = \rho(y) \in C^\infty(\mathbb{R}^d)$ be the density of the medium and $\omega \in \mathbb{R}$ be the interrogating frequency. We also assume $\rho(y)$ is Y-periodic, i.e., $\rho(y+z) = \rho(y)$ for any $y \in \mathbb{R}^d$ and $z \in \mathbb{Z}^d$. In the exterior domain $\mathbb{R}^d\backslash\overline{\Omega}$, the medium is homogeneous, isotropic where $\rho = 1$ and the elastic tensor is a constant fourth-order tensor $\mathbf{C}^{(0)} = (C^{(0)}_{ijk\ell})_{1\le i,j,k,\ell\le d}$ given by

$$C^{(0)}_{ijk\ell} = \lambda\delta_{ij}\delta_{k\ell} + \mu(\delta_{ik}\delta_{j\ell} + \delta_{i\ell}\delta_{jk}), \tag{2}$$

where λ and μ are Lamé constants satisfying the strong convexity condition (1), and it is equivalent to

$$\mu > 0 \quad \text{and} \quad d\lambda + 2\mu > 0 \quad \text{where} \quad d = 2, 3.$$

Now let us consider the elastic scattering by highly oscillating periodic media with bounded support. Let $\mathbf{u}(x) = (u_\ell(x))_{\ell=1}^d$ be the displacement vector field, the time-harmonic elastic scattering is modeled by

$$\begin{cases} \nabla\cdot\Big(\mathbf{C}(\frac{x}{\epsilon})\nabla\mathbf{u}\Big) + \omega^2\rho(\frac{x}{\epsilon})\mathbf{u} = 0 & \text{in } \Omega, \\ \Delta^*\mathbf{u}^s + \omega^2\mathbf{u}^s = 0 & \text{in } \mathbb{R}^d\backslash\overline{\Omega}, \\ \mathbf{u}^s + \mathbf{u}^i = \mathbf{u} & \text{on } \partial\Omega, \\ T_{\boldsymbol{\nu}}(\mathbf{u}^s + \mathbf{u}^i) = (\mathbf{C}(\frac{x}{\epsilon})\nabla\mathbf{u})\cdot\boldsymbol{\nu} & \text{on } \partial\Omega, \end{cases}$$

where

$$\begin{cases} \Big(\nabla\cdot(\mathbf{C}(\frac{x}{\epsilon})\nabla\mathbf{u})\Big)_j = \sum_{i,k,\ell=1}^d \frac{\partial}{\partial x_i}\Big(C_{ijk\ell}(\frac{x}{\epsilon})\frac{\partial u_\ell}{\partial x_k}\Big) \text{ for } 1 \le j \le d, \\ \Delta^* = \mu\Delta + (\lambda+\mu)\nabla(\nabla\cdot) = \nabla\cdot(\mathbf{C}^{(0)}\nabla\cdot), \end{cases}$$

and $\boldsymbol{\nu}$ is the unit outward normal on $\partial\Omega$, $\mathbf{u}^s$ is the scattered field, and $\mathbf{u}^{in}$ is an incident field; the operator $T_{\boldsymbol{\nu}}$ stands for the *boundary traction operator* of the isotropic elasticity system (from the exterior domain), which is

$$T_{\boldsymbol{\nu}} u = \begin{cases} 2\mu\frac{\partial\mathbf{u}}{\partial\boldsymbol{\nu}} + \lambda\boldsymbol{\nu}\nabla\cdot\mathbf{u} + \mu\boldsymbol{\nu}^T(\partial_2 u_1 - \partial_1 u_2), & \text{when } d = 2, \\ 2\mu\frac{\partial\mathbf{u}}{\partial\boldsymbol{\nu}} + \lambda\boldsymbol{\nu}\nabla\cdot\mathbf{u} + \mu\boldsymbol{\nu}\times(\nabla\times\mathbf{u}), & \text{when } d = 3. \end{cases} \tag{3}$$

The equation $\Delta^*\mathbf{u}^s + \omega^2\mathbf{u}^s = 0$ with constant coefficients is called the Navier's equation. In particular we consider an incident field that is either a plane shear wave

$$\mathbf{u}^{in}(x) = \mathbf{u}^{in}_s(x) := \mathbf{d}^\perp \exp(i\omega_s x\cdot\mathbf{d}),$$

where $\mathbf{d}, \mathbf{d}^\perp$ are orthonormal vectors in $\mathbb{R}^d$, or a plane pressure wave

$$\mathbf{u}^{in}(x) = \mathbf{u}^{in}_p(x) := \mathbf{d}\exp(i\omega_p x\cdot\mathbf{d}),$$

where ω_s and ω_p are constants denoted by $\omega_s = \dfrac{\omega}{\sqrt{\mu}}$ and $\omega_p = \dfrac{\omega}{\sqrt{\lambda + 2\mu}}$.

Via the well-known Helmoholtz decomposition in $\mathbb{R}^d \backslash \overline{\Omega}$, one can see that the scattered field can be decomposed as

$$\mathbf{u}^s = \mathbf{u}_p^{sc} + \mathbf{u}_s^{sc},$$

with

$$\mathbf{u}_p^{sc} = -\frac{1}{\omega_p^2} \nabla (\nabla \cdot \mathbf{u}^{sc}) \text{ and } \mathbf{u}_s^{sc} = \frac{1}{\omega_s^2} \text{rot}(\text{rot} \mathbf{u}^{sc}),$$

where $\text{rot} = \nabla^T$ represents $\frac{\pi}{2}$ clockwise rotation of the gradient if $d = 2$ and $\text{rot} = \nabla \times$ stands for the curl operator if $d = 3$. The vector functions $\mathbf{u}_p^{sc}$ and $\mathbf{u}_s^{sc}$ are called the pressure (longitudinal) and shear (transversal) parts of the scattered vector field $\mathbf{u}^s$, respectively and they satisfy the Helmholtz equation

$$(\Delta + \omega_p^2) \mathbf{u}_p^{sc} = 0 \text{ and } \text{rot} \mathbf{u}_p^{sc} = 0 \text{ in } \mathbb{R}^d \backslash \overline{\Omega},$$

$$(\Delta + \omega_s^2) \mathbf{u}_s^{sc} = 0 \text{ and } \nabla \cdot \mathbf{u}_s^{sc} = 0 \text{ in } \mathbb{R}^d \backslash \overline{\Omega}.$$

Furthermore, for the elastic scattering problem, the scattered field $\mathbf{u}^s$ satisfies the *Kupradze radiation condition*

$$\lim_{r \to \infty} \left(\frac{\partial \mathbf{u}_p^{sc}}{\partial r} - i \omega_p \mathbf{u}_p^{sc} \right) = 0 \text{ and } \lim_{r \to \infty} \left(\frac{\partial \mathbf{u}_s^{sc}}{\partial r} - i \omega_s \mathbf{u}_s^{sc} \right) = 0, \quad r = |x|, \tag{4}$$

uniformly in all directions $\widehat{x} = \dfrac{x}{|x|}$.

In summary the elastic scattering by highly oscillating periodic media can be formulated as: find the solution $\mathbf{u}^\epsilon \in \left(H^1_{loc}(\mathbb{R}^d) \right)^d$ to

$$\begin{cases} \nabla \cdot \left(\mathbf{C}(\frac{x}{\epsilon}) \nabla \mathbf{u}^\epsilon \right) + \omega^2 \rho(\frac{x}{\epsilon}) \mathbf{u}^\epsilon = 0 & \text{in } \Omega, \\ \Delta^* \mathbf{u}^\epsilon + \omega^2 \mathbf{u}^\epsilon = 0 & \text{in } \mathbb{R}^d \backslash \overline{\Omega}, \\ (\mathbf{u}^\epsilon)^+ - (\mathbf{u}^\epsilon)^- = \mathbf{f} & \text{on } \partial\Omega, \\ (T_{\boldsymbol{\nu}} \mathbf{u}^\epsilon)^+ - \left(\mathbf{C}(\frac{x}{\epsilon}) \nabla \mathbf{u}^\epsilon \right)^- \cdot \boldsymbol{\nu} = \mathbf{g} & \text{on } \partial\Omega, \end{cases} \tag{5}$$

where $\mathbf{u}^\epsilon$ satisfies the Kupradze radiation condition (4) at infinity. Here $\mathbf{u}^\epsilon$ stands for the solution parametrized by ϵ and

$$\mathbf{f} := -\mathbf{u}^{in} \quad \text{and} \quad \mathbf{g} := -T_{\boldsymbol{\nu}} \mathbf{u}^{in} \quad \text{on} \quad \partial\Omega,$$

where $\mathbf{u}^{in}$ is either a shear wave or a pressure wave given as before. The superscripts "+" or "−" stand for the limit from exterior or interior on $\partial\Omega$, respectively. We remark that the highly oscillating periodic media is only supported in Ω.

1.3 Main results and outline

We are interested in the limit behavior or the overall behavior of the solution $\mathbf{u}^\epsilon$ as $\epsilon \to 0$, known as *homogenization.* As $\epsilon \to 0$, we are expecting that $\mathbf{u}^\epsilon \to \mathbf{u}^{(0)}$, where $\mathbf{u}^{(0)}$ is the solution of the homogenized equation

$$\begin{cases} \nabla \cdot (\overline{\mathbf{C}}\nabla \mathbf{u}^{(0)}) + \omega^2 \overline{\rho}\mathbf{u}^{(0)} = 0 & \text{in } \Omega, \\ \Delta^* \mathbf{u}^{(0)} + \omega^2 \mathbf{u}^{(0)} = 0 & \text{in } \mathbb{R}^d \backslash \overline{\Omega}, \\ (\mathbf{u}^{(0)})^+ - (\mathbf{u}^{(0)})^- = \mathbf{f} & \text{on } \partial\Omega, \\ (T_{\boldsymbol{\nu}} \mathbf{u}^{(0)})^+ - (\overline{\mathbf{C}}\nabla \mathbf{u}^{(0)})^- \cdot \boldsymbol{\nu} = \mathbf{g} & \text{on } \partial\Omega, \end{cases} \tag{6}$$

where $\overline{\mathbf{C}} = (\overline{C}_{ijk\ell})$ is the constant four tensor and $\overline{\rho}$ is the constant density given by

$$\begin{cases} \overline{C}_{ijk\ell} = \int_Y \left(C_{ijk\ell} - C_{ijmn} \dfrac{\partial}{\partial y_m} \chi_{nk\ell} \right) dy, \\ \overline{\rho} = \int_Y \rho \, dy, \end{cases}$$

and here we have utilized the Einstein summation convention for the repeated indices; the constant fourth-order tensor $\overline{\mathbf{C}}$ is called the effective fourth-order tensor; the third-order tensor $\boldsymbol{\chi} = (\chi_{nk\ell})_{1 \le n,k,\ell \le d}$ with $\chi_{nk\ell} \in H^1_{per}(Y)$ is uniquely determined by the cell problem

$$\begin{cases} \dfrac{\partial}{\partial y_i} \left(C_{ijmn} - C_{ijk\ell} \dfrac{\partial}{\partial y_k} \chi_{\ell mn} \right) = 0 \text{ in } Y, \\ \int_Y \chi_{\ell mn}(y) \, dy = 0, \end{cases} \tag{7}$$

where we refer to Appendix Section 5 for detailed analysis. $H^1_{per}(Y)$ denotes the periodic Sobolev space consists of H^1 functions defined on the d-dimensional torus $\mathbb{R}^d/\mathbb{Z}^d$ where $Y = [0,1]^d$, we refer readers to [10, Chapter 3] for detailed characterizations.

The constant tensor $\overline{\mathbf{C}}$ also satisfies the strong convexity condition (1) (see Appendix Section 5), which implies that (6) is a well-posed transmission problem. Note that from the standard elliptic regularity theory (see [20] for instance), the corrector $\boldsymbol{\chi}$ is C^∞-smooth due to the smoothness of $\mathbf{C}$.

Now let us look for ansatz $\mathbf{u}^\epsilon$

$$\mathbf{u}^\epsilon = \mathbf{u}^{(0)} + \epsilon \mathbf{u}^{(1)} + \epsilon^2 \mathbf{u}^{(2)} + \cdots$$

as an asymptotic expansion in terms of ϵ, where the functions $\mathbf{u}^{(j)}$ will be characterized in the following sections for $j = 0, 1, 2, \cdots$. Now, we can state our main results in this paper.

Theorem 1 (Convergence in L^2 and H^1) *Let $\mathbf{u}^\epsilon$ and $\mathbf{u}^{(0)}$ be the solutions of* (5) *and* (6)*, respectively. Let $\mathbf{u}^{(1)}$ be the bulk corrector given by* (21) *in Ω with $\mathbf{u}^{(1)} = 0$ in $\mathbb{R}^d \setminus \overline{\Omega}$. Then for any ball B_R with $\Omega \subset B_R$, we have*

$$\|\mathbf{u}^\epsilon - \mathbf{u}^{(0)} - \epsilon \mathbf{u}^{(1)}\|_{H^1(\Omega)} + \|\mathbf{u}^\epsilon - \mathbf{u}^{(0)}\|_{H^1(B_R \setminus \overline{\Omega})} \le C_R \epsilon^{1/2} \|\mathbf{u}^{(0)}\|_{H^2(\Omega)}, \tag{8}$$

and

$$\|\mathbf{u}^\epsilon - \mathbf{u}^{(0)}\|_{L^2(B_R)} \le C_R \epsilon \|\mathbf{u}^{(0)}\|_{H^2(\Omega)}, \tag{9}$$

for some constant $C_R > 0$ independent of ϵ and $\mathbf{u}^{(0)}$.

Furthermore we have the following higher-order convergent rates between solutions $\mathbf{u}^\epsilon$ and $\mathbf{u}^{(0)}$.

Theorem 2 (Higher-order convergence in L^2 and H^1) *Let $\mathbf{u}^\epsilon$ and $\mathbf{u}^{(0)}$ be the solutions of (5) and (6) respectively. Let $\mathbf{u}^{(1)}$ and $\mathbf{u}^{(2)}$ be defined by equations (49) and (55) in Ω, respectively, with $\mathbf{u}^{(1)} = 0$ and $\mathbf{u}^{(2)} = 0$ in $\mathbb{R}^d \setminus \overline{\Omega}$. Let $\boldsymbol{\varphi}^\epsilon$ and $\boldsymbol{\theta}^\epsilon$ be the boundary correctors given by (41) and (64). Then for any ball B_R with $\Omega \subset B_R$, we have*

$$\|\mathbf{u}^\epsilon - (\mathbf{u}^{(0)} + \epsilon \mathbf{u}^{(1)} + \epsilon^2 \mathbf{u}^{(2)} + \epsilon \boldsymbol{\varphi}^\epsilon + \epsilon^2 \boldsymbol{\theta}^\epsilon)\|_{H^1(B_R)} \leq C_R \epsilon^2 \|\mathbf{u}^{(0)}\|_{H^4(\Omega)}, \tag{10}$$

and

$$\|\mathbf{u}^\epsilon - (\mathbf{u}^{(0)} + \epsilon \mathbf{u}^{(1)} + \epsilon \boldsymbol{\varphi}^\epsilon)\|_{L^2(B_R)} \leq C_R \epsilon^2 \|\mathbf{u}^{(0)}\|_{H^4(\Omega)}, \tag{11}$$

where $C_R > 0$ is a constant independent of ϵ and $\mathbf{u}^{(0)}$.

Remark that the above convergence estimates include boundary correctors $\boldsymbol{\varphi}^\epsilon$ and $\boldsymbol{\theta}^\epsilon$ since the periodic media has bounded support, as contrary to the case that the periodic media occupies $\mathbb{R}^d$.

We also remark that as a by-product we obtained a higher-order homogenization that enables to study the anisotropic dispersion of wave propagation in periodic media (in the whole space). Formally the averaged wave field $\mathbf{U}$ of $\mathbf{u}_\epsilon$ up to order ϵ^2 is governed by the fourth-order equation in Ω

$$\begin{aligned}\nabla \cdot \big(\overline{\mathbf{C}} \nabla \mathbf{U}\big) + \omega^2 \overline{\rho} \mathbf{U} = &-\epsilon \big(\mathbf{F} : \nabla^3 \mathbf{U} + \omega^2 \mathbf{G} : \nabla \mathbf{U}\big) \\ &- \epsilon^2 \big(\mathbf{D} : \nabla^4 \mathbf{U} + \omega^2 \mathbf{E} : \nabla^2 \mathbf{U}\big) + O(\epsilon^3),\end{aligned} \tag{12}$$

where $\mathbf{D}$ is a sixth-order tensor, $\mathbf{E}$ is a fourth-order tensor, $\mathbf{F}$ is a fifth-order tensor and $\mathbf{G}$ is a third-order tensor respectively (see Section 3.3 for the definitions and details). The fourth-order partial differential equation (12) in Ω formally introduces the dispersion as is seen from the right hand side of (12). In the low-frequency long-wavelength regime for wave propagation in periodic media, the wave dispersion has been demonstrated by a fourth-order partial differential equation for the acoustic case [7]. In the particular case that the periodic media occupies $\mathbb{R}^d$, (12) models the wave propagation and transcends the quasi-static regime. In the case that the periodic media has bounded support, the boundary correctors play a role. This is a key difference between homogenization in the whole space and in a bounded domain.

This article is further structured as follows. In Section 2, we study the asymptotic analysis for the elastic homogenization problem. From the analysis, we can prove the L^2 convergent rates for the elastic homogenization problem, which shows Theorem 1. In Section 3, we develop the higher-order asymptotic analysis. This gives us important information about the higher-order convergent rates between solutions u^ϵ and its approximations, and we can use it to prove Theorem 2. As a by-product, we formally provide in Section 3.3 a second-order homogenized model for the elastic scattering in periodic media (that occupies in $\mathbb{R}^d$), where the anisotropic dispersion was demonstrated. In Section 4, we provide preliminary numerical examples to illustrate our higher-order homogenization. In Appendix 5, for self-contained proofs, we offer fundamental materials which is used to demonstrate our homogenization theory for the elastic scattering problem.

1.4 Notation

1. We use sub-index to represent the component of a tensor, in particular the $i_1 i_2 \cdots i_n$ component of a n-th order tensor $\boldsymbol{\chi}$ is represented by $\chi_{i_1 i_2 \cdots i_n}$.
2. We use Einstein summation convention for the repeated indices.
3. $\cdot$ denotes the standard inner product and $:$ denotes the standard contraction between two tensors.
4. $C_c^\infty(\Omega)$ denotes the space consists of C^∞ functions that are compactly supported in Ω.

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2 Asymptotic analysis of the transmission problem

To begin with, we recall the two-scale homogenization method for the elasticity scattering in periodic media.

2.1 Basic asymptotic analysis

Let us consider x and $y = \dfrac{x}{\epsilon}$ that are the slow and fast variables, respectively. Let $\mathbf{u}^\epsilon$ be a solution of (5) and we rewrite (5) to a first-order system

$$\begin{cases} \mathbf{v}^\epsilon - C(\frac{x}{\epsilon})\nabla \mathbf{u}^\epsilon = 0, \\ \nabla \cdot \mathbf{v}^\epsilon + \omega^2 \rho(\frac{x}{\epsilon})\mathbf{u}^\epsilon = 0, \end{cases} \quad \text{for } x \in \Omega. \tag{13}$$

Note that $\mathbf{u}^\epsilon = (u_i^\epsilon)_{1\le i\le d}$ is a vector-valued function and $\mathbf{v}^\epsilon = (v_{ij}^\epsilon)_{1\le i,j\le d}$ is a matrix-valued function. The two-scale homogenization method begins with ansatz $\mathbf{u}^\epsilon = \mathbf{u}^\epsilon(x,y)$ and $\mathbf{v}^\epsilon = \mathbf{v}^\epsilon(x,y)$ such that

$$\begin{aligned} \mathbf{u}^\epsilon(x,y) &= \mathbf{u}^{(0)}(x,y) + \epsilon \mathbf{u}^{(1)}(x,y) + \epsilon^2 \mathbf{u}^{(2)}(x,y) + \cdots = \sum_{k=0}^\infty \epsilon^k \mathbf{u}^{(k)}(x,y), \\ \mathbf{v}^\epsilon(x,y) &= \mathbf{v}^{(0)}(x,y) + \epsilon \mathbf{v}^{(1)}(x,y) + \epsilon^2 \mathbf{v}^{(2)}(x,y) + \cdots = \sum_{k=0}^\infty \epsilon^k \mathbf{v}^{(k)}(x,y). \end{aligned} \tag{14}$$

Furthermore since there are no microstructure in the exterior domain $\mathbb{R}^d \backslash \overline{\Omega}$, the asymptotic expansions of $\mathbf{u}^\epsilon$ and $\mathbf{v}^\epsilon$ are nothing but

$$\mathbf{u}^\epsilon = \mathbf{u}^{(0)}(x) \text{ and } \mathbf{v}^\epsilon = \mathbf{v}^{(0)}(x).$$

Proceeding with the ansatz, we consider x and $y = \dfrac{x}{\epsilon}$ as independent variables and correspondingly

$$\nabla = \nabla_x + \frac{1}{\epsilon}\nabla_y. \tag{15}$$

Use the formal expansion and combine (13), (14) and (15), then we have

$$\begin{cases} \sum_{m=0}^{\infty} \epsilon^m \mathbf{v}^{(m)}(x,y) - C(y)\Big(\nabla_x + \frac{1}{\epsilon}\nabla_y\Big)\Big(\sum_{m=0}^{\infty} \epsilon^m \mathbf{u}^{(m)}(x,y)\Big) = 0, \\ \Big(\nabla_x + \frac{1}{\epsilon}\nabla_y\Big)\cdot\Big(\sum_{m=0}^{\infty} \epsilon^m \mathbf{v}^{(m)}(x,y)\Big) + \omega^2\rho(y)\sum_{m=0}^{\infty} \epsilon^m \mathbf{u}^{(m)}(x,y) = 0. \end{cases} \tag{16}$$

By collecting the ϵ^m terms in equation (16) with $m = -1, 0, 1\cdots$,

$$O(\epsilon^{-1}): \qquad C\nabla_y \mathbf{u}^{(0)} = 0, \tag{17}$$

$$\nabla_y \cdot \mathbf{v}^{(0)} = 0. \tag{18}$$

$$O(1): \qquad \mathbf{v}^{(0)} - \mathbf{C}\left(\nabla_x \mathbf{u}^{(0)} + \nabla_y \mathbf{u}^{(1)}\right) = 0, \tag{19}$$

$$\left(\nabla_x \cdot \mathbf{v}^{(0)} + \nabla_y \cdot \mathbf{v}^{(1)}\right) + \omega^2 \rho \mathbf{u}^{(0)} = 0. \tag{20}$$

Via (17), we get $\mathbf{u}^{(0)} = \mathbf{u}^{(0)}(x)$. From (18), (19) and the cell function $\chi_{\ell mn}$ solving (7), we can find that the bulk corrector $\mathbf{u}^{(1)} = (u_\ell^{(1)})_{1\le\ell\le d}$ is given componentwisely by

$$u_\ell^{(1)}(x,y) = -\chi_{\ell mn}(y)\frac{\partial u_n^{(0)}}{\partial x_m}(x). \tag{21}$$

We call $\mathbf{u}^{(1)}(x,y)$ the first-order corrector for the periodic homogenization problem. Plugging equation (21) directly into (19) yields

$$\begin{aligned} v_{ij}^{(0)}(x,y) &= \left(\mathbf{C}(y)\left(\nabla_x \mathbf{u}^{(0)}(x) + \nabla_y \mathbf{u}^{(1)}(x,y)\right)\right)_{ij} \\ &= C_{ijkl}(y)\frac{\partial u_\ell^{(0)}}{\partial x_k} - C_{ijkl}\frac{\partial \chi_{\ell mn}}{\partial y_k}\frac{\partial u_n^{(0)}}{\partial x_m}, \end{aligned} \tag{22}$$

and consequently the Y-average of $\mathbf{v}_0$ is

$$\langle \mathbf{v}^{(0)}\rangle := \overline{\mathbf{v}}^{(0)} = \int_Y \mathbf{v}^{(0)}(x,y)dy = \overline{\mathbf{C}}\nabla \mathbf{u}^{(0)}.$$

In order to solve $\mathbf{v}^{(1)}$, we introduce the following partial differential equation in the unit cell. Let $\mathbf{q}(x,y)$ be a solution to

$$\mathrm{rot}_y(\mathbf{q}) = \mathbf{v}^{(0)} - \overline{\mathbf{C}}\nabla \mathbf{u}^{(0)}, \tag{23}$$

where $\mathbf{q}(x,y)$ is

$$\begin{cases} \mathbf{q} = (q_1, q_2) \text{ when } d = 2 \text{ and } q_i\text{'s are scalar functions for } i = 1, 2, \\ \mathbf{q} = (q_1, q_2, q_3) \text{ when } d = 3 \text{ and } q_i\text{'s are column vectors for } i = 1, 2, 3, \end{cases}$$

where each component of $\mathbf{q}(x,y)$ belongs to $H^1_{per}(Y)$ as a function of the y variable. Let $\boldsymbol{\gamma}(y) \in \left(H^1_{per}(Y)\right)^{d\times d}$ solve the following equation

$$\begin{cases} \dfrac{\partial}{\partial y_i}\Big(C_{ijk\ell}\dfrac{\partial \gamma_{m\ell}}{\partial y_k}\Big) = (\overline{\rho} - \rho)\delta_{jm}, \\ \int_Y \gamma_{m\ell}(y)dy = 0. \end{cases} \tag{24}$$

We remark that the right hand side of (24) has zero mean (i.e., $\int_Y (\overline{\rho} - \rho)\, dy = 0$), and then compatibility condition is satisfied, which means (24) is solvable.

From (23), (24), (20) and note that $\nabla_y \cdot (\mathrm{rot}_x \mathbf{q}(x, y)) = -\nabla_x \cdot (\mathrm{rot}_y \mathbf{q}(x, y))$, one candidate for $v^{(1)}$ is $\widetilde{\mathbf{v}}^{(1)} = (\widetilde{v}^{(1)}_{ij})_{1\le i,j\le d}$ defined by

$$\widetilde{v}^{(1)}_{ij}(x, y) := (\mathrm{rot}_x(q(x, y)))_{ij} + \omega^2 C_{ijk\ell}(y) \frac{\partial \gamma_{m\ell}}{\partial y_k}(y) u^{(0)}_m(x), \text{ for } 1 \le i, j \le d. \quad (25)$$

We remark that the function $\widetilde{\mathbf{v}}^{(1)}$ in the form (25) is convenient in proving Theorem 1, and we will choose another form of $\widetilde{\mathbf{v}}^{(1)}$ to derive higher-order estimate.

2.2 Rates of convergence in H^1 and L^2

Now let us introduce the boundary corrector $\widetilde{\boldsymbol{\varphi}}^\epsilon$ that solves

$$\begin{cases} \nabla \cdot \Big(\mathbf{C}(\frac{x}{\epsilon}) \nabla \widetilde{\boldsymbol{\varphi}}^\epsilon\Big) + \omega^2 \rho(\frac{x}{\epsilon}) \widetilde{\boldsymbol{\varphi}}^\epsilon = 0 & \text{in } \Omega, \\ \Delta^* \widetilde{\boldsymbol{\varphi}}^\epsilon + \omega^2 \widetilde{\boldsymbol{\varphi}}^\epsilon = 0 & \text{in } \mathbb{R}^d \setminus \overline{\Omega}, \\ (\widetilde{\boldsymbol{\varphi}}^\epsilon)^+ - (\widetilde{\boldsymbol{\varphi}}^\epsilon)^- = \mathbf{u}^{(1)} & \text{on } \partial\Omega, \\ (T_{\boldsymbol{\nu}} \widetilde{\boldsymbol{\varphi}}^\epsilon)^+ - (\mathbf{C}(\frac{x}{\epsilon}) \nabla \widetilde{\boldsymbol{\varphi}}^\epsilon)^- \cdot \boldsymbol{\nu} = \left(\dfrac{\mathbf{v}^{(0)} - \overline{\mathbf{v}}^{(0)}}{\epsilon} + \widetilde{\mathbf{v}}^{(1)}\right) \cdot \boldsymbol{\nu} & \text{on } \partial\Omega, \end{cases} \quad (26)$$

where $\widetilde{\boldsymbol{\varphi}}^\epsilon$ satisfies the Kupradze radiation condition (4). By plugging (21) and (25) into (26), one can see that the transmission conditions on $\partial\Omega$ is

$$\begin{cases} \Big((\widetilde{\boldsymbol{\varphi}}^\epsilon)^+ - (\widetilde{\boldsymbol{\varphi}}^\epsilon)^-\Big)_j = -\chi_{jmn}(y) \dfrac{\partial u^{(0)}_n}{\partial x_m}, \text{ for } 1 \le \alpha \le d & \text{on } \partial\Omega \\ (T_{\boldsymbol{\nu}} \widetilde{\boldsymbol{\varphi}}^\epsilon)^+ - (\mathbf{C}(\frac{x}{\epsilon})) \nabla \widetilde{\boldsymbol{\varphi}}^\epsilon)^- \cdot \boldsymbol{\nu} = \left[\mathrm{rot}\ \mathbf{q} + \omega^2 (\mathbf{C} : \nabla \boldsymbol{\gamma})(y) \mathbf{u}^{(0)}\right] \cdot \boldsymbol{\nu} & \text{on } \partial\Omega, \end{cases} \quad (27)$$

where $\mathrm{rot} = \mathrm{rot}_x + \frac{1}{\epsilon} \mathrm{rot}_y$.

Before proceeding with the following lemma, we remark that the solution $\mathbf{u}^{(0)}$ to (6) is sufficiently smooth, since (6) is a well-posed transmission problem and moreover the tensor $\overline{\mathbf{C}}$ and density $\overline{\rho}$ are constants and the boundary data is sufficiently smooth.

Lemma 1 *Let $\mathbf{u}^\epsilon$ and $\mathbf{u}^{(0)}$ be the solutions of* (5) *and* (6)*, respectively. Let $\mathbf{u}^{(1)}$ be the bulk corrector given by* (21) *in Ω with $\mathbf{u}^{(1)} = 0$ in $\mathbb{R}^d \setminus \overline{\Omega}$ and $\widetilde{\boldsymbol{\varphi}}^\epsilon$ be the boundary corrector given by* (26)*. Then for any ball B_R with $\Omega \subset B_R$, we have*

$$\|\mathbf{u}^\epsilon - (\mathbf{u}^{(0)} + \epsilon \mathbf{u}^{(1)} + \epsilon \widetilde{\boldsymbol{\varphi}}^\epsilon)\|_{H^1(B_R)} \le C_R \epsilon \|\mathbf{u}^{(0)}\|_{H^2(\Omega)}, \quad (28)$$

where $C_R > 0$ is a constant independent of ϵ and $\mathbf{u}^{(0)}$.

Proof Consider the error functions in D given by

$$\mathbf{w}^\epsilon := \mathbf{u}^\epsilon - \mathbf{u}^{(0)} - \epsilon \mathbf{u}^{(1)}, \quad (29)$$

and

$$\boldsymbol{\zeta}^\epsilon := \mathbf{C}(\frac{x}{\epsilon}) \nabla \mathbf{u}^\epsilon - \mathbf{v}^{(0)} - \epsilon \widetilde{\mathbf{v}}^{(1)}, \quad (30)$$

where $\mathbf{w}^\epsilon$ is a vector-valued function and $\boldsymbol{\zeta}^\epsilon$ is a matrix-valued function. From straightforward calculations, we can get

$$\begin{cases} \mathbf{C}(\frac{x}{\epsilon})\nabla \mathbf{w}^\epsilon - \boldsymbol{\zeta}^\epsilon = \epsilon\big(\widetilde{\mathbf{v}}^{(1)} - \mathbf{C}(y)\nabla_x \mathbf{u}^{(1)}\big), \\ (\nabla\cdot\boldsymbol{\zeta}^\epsilon)_j + \omega^2\rho(y)w_j^\epsilon = -\epsilon\omega^2\Big(\rho(y)u_j^{(1)} + C_{ijk\ell}(y)\frac{\partial\gamma_{m\ell}}{\partial y_k}\frac{\partial u_m^{(0)}}{\partial x_i}\Big), \text{ for } 1\le j\le d, \end{cases} \tag{31}$$

which is a first order differential system of $(\mathbf{w}^\epsilon, \boldsymbol{\zeta}^\epsilon)$ such that their right hand sides are of order $O(\epsilon)$.

While outside Ω we simply consider the error functions $\mathbf{w}^\epsilon := \mathbf{u}^\epsilon - \mathbf{u}^{(0)}$ and $\boldsymbol{\zeta}^\epsilon := \mathbf{C}^{(0)}\nabla\mathbf{w}^\epsilon$, and then they satisfy

$$-\nabla\cdot\boldsymbol{\zeta}^\epsilon = \omega^2\mathbf{w}^\epsilon.$$

Let B_R be a sufficiently large ball that contains $\overline{\Omega}$ and $\boldsymbol{\phi} \in \big(C_c^\infty(B_R)\big)^d$ be a vector-valued test function, we shall derive an estimate for

$$\int_{B_R}(\mathbf{w}^\epsilon - \epsilon\widetilde{\boldsymbol{\varphi}}^\epsilon)\cdot\boldsymbol{\phi}\,dx. \tag{32}$$

To estimate the above quantity we consider another auxiliary function $\boldsymbol{\Phi}^\epsilon \in \big(H^1_{loc}(\mathbb{R}^d)\big)^d$ by

$$\begin{cases} \nabla\cdot(\mathbf{C}(\frac{x}{\epsilon})\nabla\boldsymbol{\Phi}^\epsilon) + \omega^2\rho(\frac{x}{\epsilon})\boldsymbol{\Phi}^\epsilon = \phi & \text{in } \Omega, \\ \Delta^*\boldsymbol{\Phi}^\epsilon + \omega^2\boldsymbol{\Phi}^\epsilon = \boldsymbol{\phi} & \text{in } \mathbb{R}^d\setminus\overline{\Omega}, \\ (\boldsymbol{\Phi}^\epsilon)^+ - (\boldsymbol{\Phi}^\epsilon)^- = 0 & \text{on } \partial\Omega, \\ (T_{\boldsymbol{\nu}}\boldsymbol{\Phi}^\epsilon)^+ - (\mathbf{C}(\frac{x}{\epsilon}))\nabla\boldsymbol{\Phi}^\epsilon)^-\cdot\boldsymbol{\nu} = 0 & \text{on } \partial\Omega, \end{cases} \tag{33}$$

where $\boldsymbol{\phi}$ is the test function as we mentioned before and $\boldsymbol{\Phi}^\epsilon$ further satisfies the Kupradze radiation condition (4) at infinity. Now we replace $\boldsymbol{\phi}$ in (32) by (33)

$$\begin{aligned} \int_{B_R}(\mathbf{w}^\epsilon - \epsilon\widetilde{\boldsymbol{\varphi}}^\epsilon)\cdot\boldsymbol{\phi}\,dx &= \int_\Omega(\mathbf{w}^\epsilon - \epsilon\widetilde{\boldsymbol{\varphi}}^\epsilon)\cdot\Big(\nabla\cdot(\mathbf{C}(\frac{x}{\epsilon})\nabla\boldsymbol{\Phi}^\epsilon) + \omega^2\rho(\frac{x}{\epsilon})\boldsymbol{\Phi}^\epsilon\Big)\,dx \\ &\quad + \int_{B_R\setminus\Omega}(\mathbf{w}^\epsilon - \epsilon\widetilde{\boldsymbol{\varphi}}^\epsilon)\cdot(\Delta^*\boldsymbol{\Phi}^\epsilon + \omega^2\boldsymbol{\Phi}^\epsilon)dx \\ &= -\int_\Omega\big(\mathbf{C}(\frac{x}{\epsilon})\nabla\mathbf{w}^\epsilon\big):\nabla\boldsymbol{\Phi}^\epsilon dx + \epsilon\int_\Omega\big(\mathbf{C}(\frac{x}{\epsilon})\nabla\widetilde{\boldsymbol{\varphi}}^\epsilon\big):\nabla\boldsymbol{\Phi}^\epsilon dx \\ &\quad + \int_\Omega\omega^2\rho(\frac{x}{\epsilon})(\mathbf{w}^\epsilon - \epsilon\widetilde{\boldsymbol{\varphi}}^\epsilon)\cdot\boldsymbol{\Phi}^\epsilon dx + \int_{\partial\Omega}T_{\boldsymbol{\nu}}(\mathbf{w}^\epsilon - \epsilon\widetilde{\boldsymbol{\varphi}}^\epsilon)^+\cdot\boldsymbol{\Phi}^\epsilon dS \\ &\quad + \int_{\partial B_R}(\mathbf{w}^\epsilon - \epsilon\widetilde{\boldsymbol{\varphi}}^\epsilon)\cdot T_{\boldsymbol{\nu}}\boldsymbol{\Phi}^\epsilon dS - \int_{\partial B_R}T_{\boldsymbol{\nu}}(\mathbf{w}^\epsilon - \epsilon\widetilde{\boldsymbol{\varphi}}^\epsilon)\cdot\boldsymbol{\Phi}^\epsilon dS, \end{aligned} \tag{34}$$

where we used the integration by parts formula once for the interior Ω and twice for the exterior $B_R\setminus\overline{\Omega}$, and the function $\mathbf{w}^\epsilon - \epsilon\widetilde{\boldsymbol{\varphi}}^\epsilon$ has no jumps across $\partial\Omega$. Furthermore the last two terms of (34) are zero, due to the Kupradze radiation condition; indeed from equations (91), (92) in Appendix 5.1 on the discussion of Kupradze radiation condition, a direct calculation yields

$$\int_{\partial B_R}(\mathbf{w}^\epsilon-\epsilon\widetilde{\boldsymbol{\varphi}}^\epsilon)\cdot T_{\boldsymbol{\nu}}\boldsymbol{\Phi}^\epsilon dS-\int_{\partial B_R}T_{\boldsymbol{\nu}}(\mathbf{w}^\epsilon-\epsilon\widetilde{\boldsymbol{\varphi}}^\epsilon)\cdot\boldsymbol{\Phi}^\epsilon dS=0.$$

Now we have

$$\begin{aligned}\int_{B_R}(\mathbf{w}^\epsilon-\epsilon\widetilde{\boldsymbol{\varphi}}^\epsilon)\cdot\phi dx=&-\int_\Omega\big(\mathbf{C}(\frac{x}{\epsilon})\nabla\mathbf{w}^\epsilon\big):\nabla\boldsymbol{\Phi}^\epsilon dx+\epsilon\int_\Omega\big(\mathbf{C}(\frac{x}{\epsilon})\nabla\widetilde{\boldsymbol{\varphi}}^\epsilon\big):\nabla\boldsymbol{\Phi}^\epsilon dx\\&+\int_\Omega\omega^2\rho(\frac{x}{\epsilon})(\mathbf{w}^\epsilon-\epsilon\widetilde{\boldsymbol{\varphi}}^\epsilon)\cdot\boldsymbol{\Phi}^\epsilon dx+\int_{\partial\Omega}T_{\boldsymbol{\nu}}(\mathbf{w}^\epsilon-\epsilon\widetilde{\boldsymbol{\varphi}}^\epsilon)^+\cdot\boldsymbol{\Phi}^\epsilon dS.\end{aligned}$$

Since $\widetilde{\boldsymbol{\varphi}}^\epsilon$ satisfies equation (26), then from integration by parts

$$\begin{aligned}\int_{B_R}(\mathbf{w}^\epsilon-\epsilon\widetilde{\boldsymbol{\varphi}}^\epsilon)\cdot\phi dx=&-\int_\Omega\big(\mathbf{C}(\frac{x}{\epsilon})\nabla\mathbf{w}^\epsilon\big):\nabla\boldsymbol{\Phi}^\epsilon dx+\epsilon\int_{\partial\Omega}\big(\mathbf{C}(\frac{x}{\epsilon})\nabla\widetilde{\boldsymbol{\varphi}}^\epsilon\big)^-\boldsymbol{\nu}\cdot\boldsymbol{\Phi}^\epsilon dx\\&+\omega^2\int_\Omega\rho(\frac{x}{\epsilon})\mathbf{w}^\epsilon\cdot\boldsymbol{\Phi}^\epsilon dx+\int_{\partial\Omega}\big(T_{\boldsymbol{\nu}}(\mathbf{w}^\epsilon-\epsilon\widetilde{\boldsymbol{\varphi}}^\epsilon)\big)^+\cdot\boldsymbol{\Phi}^\epsilon dS.\end{aligned}$$

From (26), (31) and integration by parts we can further obtain

$$\begin{aligned}&\int_{B_R}(\mathbf{w}^\epsilon-\epsilon\widetilde{\boldsymbol{\varphi}}^\epsilon)\cdot\phi dx\\=&-\int_\Omega\boldsymbol{\zeta}^\epsilon\cdot\nabla\boldsymbol{\Phi}^\epsilon dx+\omega^2\int_\Omega\rho(\frac{x}{\epsilon})\mathbf{w}_\epsilon\cdot\boldsymbol{\Phi}^\epsilon dx+\int_{\partial\Omega}(T_{\boldsymbol{\nu}}\mathbf{w}^\epsilon)^+\cdot\boldsymbol{\Phi}^\epsilon dS\\&+\epsilon\int_\Omega\big(-\widetilde{\mathbf{v}}^{(1)}+\mathbf{C}(\frac{x}{\epsilon})\nabla_x\mathbf{u}^{(1)}\big):\nabla\boldsymbol{\Phi}^\epsilon dx+\int_{\partial\Omega}(\overline{\mathbf{v}}^{(0)}-\mathbf{v}^{(0)}-\epsilon\widetilde{\mathbf{v}}^{(1)})\boldsymbol{\nu}\cdot\boldsymbol{\Phi}^\epsilon dS\\=&-\epsilon\omega^2\int_\Omega\Big(\rho(\frac{x}{\epsilon})u_j^{(1)}+C_{ijk\ell}(\frac{x}{\epsilon})\frac{\partial\gamma_{m\ell}}{\partial y_k}\frac{\partial u_m^{(0)}}{\partial x_i}\Big)\Phi_j^\epsilon dx\\&+\epsilon\int_\Omega\Big(-\widetilde{\mathbf{v}}^{(1)}+\mathbf{C}(\frac{x}{\epsilon})\nabla_x\mathbf{u}^{(1)}\Big):\nabla\boldsymbol{\Phi}^\epsilon dx\\&+\int_{\partial\Omega}\Big((\overline{\mathbf{v}}^{(0)}-\mathbf{v}^{(0)}-\epsilon\widetilde{\mathbf{v}}^{(1)})^-\cdot\boldsymbol{\nu})-(\boldsymbol{\zeta}^\epsilon)^-\cdot\boldsymbol{\nu}+(T_{\boldsymbol{\nu}}\mathbf{w}^\epsilon)^+\Big)\cdot\boldsymbol{\Phi}^\epsilon dS.\end{aligned}$$

From equations (5), (6), (26), (29) and (30) we can obtain that the the last term in the above equality is zero. Thus,

$$\begin{aligned}&\int_{B_R}(\mathbf{w}^\epsilon-\epsilon\widetilde{\boldsymbol{\varphi}}^\epsilon)\cdot\phi dx\\=&-\epsilon\omega^2\int_\Omega\Big(\rho(\frac{x}{\epsilon})u_j^{(1)}+C_{ijk\ell}(\frac{x}{\epsilon})\frac{\partial\gamma_{m\ell}}{\partial y_k}\frac{\partial u_m^{(0)}}{\partial x_i}\Big)\Phi_j^\epsilon dx\\&+\epsilon\int_\Omega\Big(-\widetilde{\mathbf{v}}^{(1)}+\mathbf{C}(\frac{x}{\epsilon})\nabla_x\mathbf{u}^{(1)}\Big):\nabla\boldsymbol{\Phi}^\epsilon dx,\end{aligned}\tag{35}$$

Note that the function $\mathbf{q}$ solves (23), then one can choose $\mathbf{q}$ such that

$$\sup_{y\in Y}|\widetilde{\mathbf{v}}^{(1)}|\le C\left(2\sum_{i,j}\left|\frac{\partial\mathbf{u}^{(0)}}{\partial x_i\partial x_j}\right|+|\mathbf{u}^{(0)}|\right),$$

where $C > 0$ is a constant independent of ϵ. Recall that $\mathbf{u}^{(1)}$ is represented by (21), then we obtain

$$\left\|\rho u_j^{(1)} + C_{ijk\ell}\frac{\gamma_{m\ell}}{\partial y_k}\frac{\partial u_m^{(0)}}{\partial x_i}\right\|_{L^2(\Omega)} \le C\|\mathbf{u}^{(0)}\|_{H^2(\Omega)}$$
$$\left\|C_{ijk\ell}\frac{\partial u_\ell^{(1)}}{\partial x_k}\right\|_{L^2(\Omega)} \le C\|\mathbf{u}^{(0)}\|_{H^2(\Omega)},$$

where the first inequality holds for $1 \le j \le d$ and the second inequality holds for $1 \le i, j \le d$. Appling the Cauchy-Schwartz inequality to (35) we can obtain

$$\left|\int_{B_R}(\mathbf{w}^\epsilon - \epsilon\widetilde{\boldsymbol{\varphi}}^\epsilon)\cdot\boldsymbol{\phi}\,dx\right| \le C\epsilon\|\mathbf{u}^{(0)}\|_{H^2(\Omega)}\|\boldsymbol{\Phi}^\epsilon\|_{H^1(\Omega)},$$

for some constant $C > 0$ independent of ϵ. Finally from the standard estimate for the elliptic system (see [20] for instance),

$$\|\boldsymbol{\Phi}^\epsilon\|_{H^1(\Omega)} \le C\|\boldsymbol{\phi}\|_{H^{-1}(B_R)},$$

where $C > 0$ is a constant depends on the coefficients and R. Finally the proof follows from the duality argument. □

Now we can have the following theorem.

Theorem 3 *Let $\mathbf{u}^\epsilon$ and $\mathbf{u}^{(0)}$ be the solutions of* (5) *and* (6)*, respectively. Let $\mathbf{u}^{(1)}$ be the bulk correction given by* (21) *in Ω with $\mathbf{u}^{(1)} = 0$ in $\mathbb{R}^d \setminus \overline{\Omega}$, then we have*

$$\|\mathbf{u}^\epsilon - \mathbf{u}^{(0)} - \epsilon\mathbf{u}^{(1)}\|_{H^1(\Omega)} + \|\mathbf{u}^\epsilon - \mathbf{u}^{(0)}\|_{H^1(B_R\setminus\overline{\Omega})} \le C_R\epsilon^{1/2}\|\mathbf{u}^{(0)}\|_{H^2(\Omega)},$$

for some constant $C_R > 0$ independent of ϵ and $\mathbf{u}^{(0)}$.

Proof From the elastic transmission problem (26), the function $\widetilde{\boldsymbol{\varphi}}^\epsilon$ satisfies the following H^1 estimate (see Theorem 7 in Appendix Section 5),

$$\begin{aligned}&\|\widetilde{\boldsymbol{\varphi}}^\epsilon\|_{H^1(\Omega)} + \|\widetilde{\boldsymbol{\varphi}}^\epsilon\|_{H^1(B_R\setminus\overline{\Omega})}\\ \le& C_R\left(\|\mathbf{u}^{(1)}\|_{H^{1/2}(\partial\Omega)} + \left\|\left(\frac{\overline{\mathbf{v}}^{(0)} - \mathbf{v}^{(0)}}{\epsilon} - \widetilde{\mathbf{v}}^{(1)}\right)\cdot\boldsymbol{\nu}\right\|_{H^{-1/2}(\partial\Omega)}\right),\end{aligned} \tag{36}$$

for some constant $C_R > 0$ independent of ϵ. Recall that $u_\ell^{(1)}(x, \frac{x}{\epsilon}) = \chi_{\ell mn}(\frac{x}{\epsilon})\frac{\partial u_n^{(0)}}{\partial x_m}(x)$, by a standard argument of the trace theorem and cutoff techniques in the homogenization theorem (see [10, Chapter 7] for instance), we have

$$\|\mathbf{u}^{(1)}\|_{H^{1/2}(\partial\Omega)} \le C\epsilon^{-1/2}\|\mathbf{u}^{(0)}\|_{H^2(\Omega)}, \tag{37}$$

where $C > 0$ is a constant independent of ϵ and $\mathbf{u}^{(0)}$. From equations (26) and (27),

$$\left(\frac{\overline{\mathbf{v}}^{(0)} - \mathbf{v}^{(0)}}{\epsilon} - \widetilde{\mathbf{v}}^{(1)}\right)\cdot\boldsymbol{\nu} = \left(\operatorname{rot}\,\mathbf{q} + \omega^2\mathbf{C}(y)\nabla\boldsymbol{\gamma}(y)\mathbf{u}^{(0)}\right)\cdot\boldsymbol{\nu},$$

where $O(\frac{1}{\epsilon})$ term is absorbed.

Now let $\boldsymbol{\phi}$ be any arbitrary smooth vector-valued test function, then when $d=2$, we have

$$\int_{\partial\Omega} \mathrm{rot}(q_j)\cdot\boldsymbol{\nu}\boldsymbol{\phi}\,dS = -\int_{\partial\Omega} q_j\ \mathrm{rot}(\boldsymbol{\phi})\cdot\boldsymbol{\nu}\,dS,\ \text{for } j=1,2,$$

and when $d=3$,

$$\int_{\partial D} \mathrm{rot}(q_j)\cdot\boldsymbol{\nu}\boldsymbol{\phi}\,dS = -\int_{\partial D} (q_j\times\nabla\boldsymbol{\phi})\cdot\boldsymbol{\nu}\,dS,\ \text{for } j=1,2,3.$$

From the governing equation (23) of $q_j(y)$ for $1\le j\le d$, we can see that the $H^1_{per}(Y)$-norm of $\mathbf{q}=(q_j)_{1\le j\le d}$ is bounded by $\|\mathbf{u}^{(0)}\|_{H^2(\Omega)}$. From the trace theorem we can obtain

$$\|q_j\|_{L^2(\partial\Omega)}\le C\|\mathbf{u}^{(0)}\|_{H^2(\Omega)}\quad\text{and}\quad \|q_j\|_{H^1(\partial\Omega)}\le C\epsilon^{-1}\|\mathbf{u}^{(0)}\|_{H^2(\Omega)},\ \text{for } 1\le j\le d,$$

where $C>0$ is a constant independent of ϵ. Then from the above inequalities and the duality argument,

$$\Big\|\Big(\frac{\mathbf{v}^{(0)}-\overline{\mathbf{v}}^{(0)}}{\epsilon}+\widetilde{\mathbf{v}}^{(1)}\Big)\cdot\boldsymbol{\nu}\Big\|_{H^{-1}(\partial\Omega)}\le C\|\mathbf{u}^{(0)}\|_{H^2(\Omega)}, \tag{38}$$

and

$$\Big\|\Big(\frac{\mathbf{v}^{(0)}-\overline{\mathbf{v}}^{(0)}}{\epsilon}+\widetilde{\mathbf{v}}^{(1)}\Big)\cdot\boldsymbol{\nu}\Big\|_{L^2(\partial\Omega)}\le C\epsilon^{-1}\|\mathbf{u}^{(0)}\|_{H^2(\Omega)}, \tag{39}$$

where $C>0$ is a constant independent of ϵ. Therefore by interpolating between (38) and (39), and combining with (36), (37), we obtain

$$\|\widetilde{\boldsymbol{\varphi}}^\epsilon\|_{H^1(\Omega)}+\|\widetilde{\boldsymbol{\varphi}}^\epsilon\|_{H^1(B_R\setminus\overline{\Omega})}\le C_R\epsilon^{-1/2}\|\mathbf{u}^{(0)}\|_{H^2(\Omega)}, \tag{40}$$

for some constant $C_R>0$ independent of ϵ. Finally from (28) and (40) we obtain

$$\|\mathbf{u}^\epsilon-\mathbf{u}^{(0)}-\epsilon\mathbf{u}^{(1)}\|_{H^1(\Omega)}+\|\mathbf{u}^\epsilon-\mathbf{u}^{(0)}\|_{H^1(B_R\setminus\overline{\Omega})}\le C_R\epsilon^{1/2}\|\mathbf{u}^{(0)}\|_{H^2(\Omega)},$$

where $C_R>0$ is some constant independent of ϵ and this completes the proof. This also proves (8) in Theorem 1. □

Consider another boundary corrector function as follows. Let $\boldsymbol{\varphi}^\epsilon\in\big(H^1_{loc}(\mathbb{R}^d)\big)^d$ be a boundary corrector that solves the following equation

$$\begin{cases}\nabla\cdot\big(\mathbf{C}(\frac{x}{\epsilon})\nabla\boldsymbol{\varphi}^\epsilon\big)+\omega^2\rho(\frac{x}{\epsilon})\boldsymbol{\varphi}^\epsilon=0 & \text{in } \Omega,\\ \Delta^*\boldsymbol{\varphi}^\epsilon+\omega^2\boldsymbol{\varphi}^\epsilon=0 & \text{in } \mathbb{R}^d\setminus\overline{\Omega},\\ (\boldsymbol{\varphi}^\epsilon)^+-(\boldsymbol{\varphi}^\epsilon)^-=\mathbf{u}^{(1)} & \text{on } \partial\Omega,\\ (T_{\boldsymbol{\nu}}\boldsymbol{\varphi}^\epsilon)^+-\big(\mathbf{C}(\frac{x}{\epsilon})\nabla\boldsymbol{\varphi}^\epsilon\cdot\boldsymbol{\nu}\big)^-=\big(\frac{\mathbf{v}^{(0)}-\overline{\mathbf{v}}^{(0)}}{\epsilon}+\mathbf{v}^{(1)}\big)\cdot\boldsymbol{\nu} & \text{on } \partial\Omega,\end{cases} \tag{41}$$

where $\boldsymbol{\varphi}^\epsilon$ further satisfies the Kupradze radiation condition (4) and $\mathbf{v}^{(1)}$ is the function given in the asymptotic expansion (14). Here we remark that $\mathbf{v}^{(1)}$ might not be the same function as $\widetilde{\mathbf{v}}^{(1)}$.

Lemma 2 *Let B_R be an arbitrary ball in $\mathbb{R}^d$ such that $\Omega \subset B_R$. Let $\mathbf{u}^{(0)} \in H^2(B_R)$ be the solution of (6) and $\boldsymbol{\varphi}^\epsilon$ be the solution of (41), then we have*

$$\|\boldsymbol{\varphi}^\epsilon\|_{L^2(B_R)} \leq C_R \|\mathbf{u}^{(0)}\|_{H^2(\Omega)},$$

for some constant $C_R > 0$ independent of ϵ.

Proof Let us consider a test function $\boldsymbol{\phi} \in \left(L^2(B_R)\right)^d$ such that $\boldsymbol{\phi} \equiv 0$ outside B_R and let $\boldsymbol{\Phi}^\epsilon$ be the solution to the transmission problem (33). We begin with the estimate of $\widetilde{\boldsymbol{\varphi}}^\epsilon$. It is similar to the proof of Lemma 1 and indeed we can obtain

$$\begin{aligned}\int_{B_R} \widetilde{\boldsymbol{\varphi}}^\epsilon \cdot \boldsymbol{\phi} dx = & -\int_\Omega \left(\mathbf{C}(\frac{x}{\epsilon}) \nabla \widetilde{\boldsymbol{\varphi}}^\epsilon\right) : \nabla \boldsymbol{\Phi}^\epsilon dx + \omega^2 \int_\Omega \rho(\frac{x}{\epsilon}) \widetilde{\boldsymbol{\varphi}}^\epsilon \cdot \boldsymbol{\Phi}^\epsilon dx \\ & + \int_{\partial\Omega} \Big(\left(\mathbf{C}(\frac{x}{\epsilon}) \nabla \boldsymbol{\Phi}^\epsilon\right)^- \cdot \boldsymbol{\nu} \Big) \cdot (\widetilde{\boldsymbol{\varphi}}^\epsilon)^- dS \\ & + \int_{\partial\Omega} \Big(\left(\nabla \widetilde{\boldsymbol{\varphi}}^\epsilon\right)^+ \cdot \boldsymbol{\nu} \Big) \cdot (\boldsymbol{\Phi}^\epsilon)^+ dS - \int_{\partial\Omega} \Big(\left(\nabla \boldsymbol{\Phi}^\epsilon\right)^+ \cdot \boldsymbol{\nu} \Big) \cdot (\widetilde{\boldsymbol{\varphi}}^\epsilon)^+ dS.\end{aligned}$$

By using the integration by parts in Ω, the equation for $\widetilde{\boldsymbol{\varphi}}^\epsilon$, Kupradze radiation condition (4) for $\widetilde{\boldsymbol{\varphi}}^\epsilon$ and continuous transmission boundary conditions for $\boldsymbol{\Phi}^\epsilon$ (see (33) again), we can derive

$$\begin{aligned} & \int_{B_R} \widetilde{\boldsymbol{\varphi}}^\epsilon \cdot \boldsymbol{\phi} dx \\ = & \int_{\partial\Omega} \Big(\big(\frac{\mathbf{v}^{(0)} - \overline{\mathbf{v}}^{(0)}}{\epsilon} + \widetilde{\mathbf{v}}^{(1)} \big) \cdot \boldsymbol{\nu} \Big) \cdot (\boldsymbol{\Phi}^\epsilon)^+ dS - \int_{\partial\Omega} \mathbf{u}^{(1)} \cdot \Big(\left(\nabla \boldsymbol{\Phi}^\epsilon\right)^+ \cdot \boldsymbol{\nu} \Big) dS \\ = & \int_{\partial\Omega} \bigg(\Big(\left(\mathrm{rot}\mathbf{q}\right)_{ij} + \omega^2 C_{ijk\ell}(y) \frac{\partial \gamma_{m\ell}}{\partial y_k}(y) u_m^{(0)} \Big) \nu_i \bigg) \cdot (\Phi_j^\epsilon)^+ dS \\ & + \int_{\partial\Omega} \chi_{\ell mn}(\frac{x}{\epsilon}) \frac{\partial u_n^{(0)}}{\partial x_m} \Big(\left(\nabla \Phi^\epsilon\right)^+ \cdot \nu \Big)_\ell dS.\end{aligned}$$

Let $\boldsymbol{\Phi}^{(0)}$ be the solution of the leading-order homogenized transmission problem with respect to $\boldsymbol{\Phi}^\epsilon$, and $\Phi_\ell^{(1)}(x,y) = \chi_{\ell mn}(y) \dfrac{\partial \Phi_n^{(0)}}{\partial x_m}(x)$ be the first order corrector term corresponding to $\boldsymbol{\Phi}^\epsilon$. Furthermore let $\boldsymbol{\Psi}^\epsilon$ be the bulk corrector of $\boldsymbol{\Phi}^\epsilon$ as the role of $\widetilde{\boldsymbol{\varphi}}^\epsilon$ playing for $\mathbf{u}^\epsilon$. Following the same proof of Lemma 1, we can derive that

$$\|\boldsymbol{\Phi}^\epsilon - (\boldsymbol{\Phi}^{(0)} + \epsilon \boldsymbol{\Phi}^{(1)} + \epsilon \boldsymbol{\Psi}^\epsilon)\|_{H^1(B_R)} \leq C_R \epsilon \|\boldsymbol{\Phi}^{(0)}\|_{H^2(\Omega)},$$

where $C_R > 0$ is a constant independent of ϵ and $\boldsymbol{\Phi}^{(0)}$. Since the bulk correction of $\boldsymbol{\Phi}^\epsilon$ is zero outside $\overline{\Omega}$, then in particular we have

$$\|\boldsymbol{\Phi}^\epsilon - (\boldsymbol{\Phi}^{(0)} + \epsilon \boldsymbol{\Psi}^\epsilon)\|_{H^1(B_R \setminus \overline{\Omega})} \leq C_R \epsilon \|\boldsymbol{\Phi}^{(0)}\|_{H^2(\Omega)}.$$

From the definitions of $\boldsymbol{\Phi}^\epsilon$, $\boldsymbol{\Phi}^{(0)}$ and $\boldsymbol{\Psi}^\epsilon$ in $B_R \setminus \overline{\Omega}$, we know that $\nabla \cdot (\mathbf{C}^{(0)} \nabla \boldsymbol{\Phi}^\epsilon)$, $\nabla \cdot (\mathbf{C}^{(0)} \nabla \boldsymbol{\Phi}^{(0)})$ and $\nabla \cdot (\mathbf{C}^{(0)} \nabla \boldsymbol{\Psi}^\epsilon)$ belong to $L^2(B_R \setminus \overline{\Omega})$, then from Appendix Section 5

$$\left\| \nabla \Big((\boldsymbol{\Phi}^\epsilon)^+ - (\boldsymbol{\Phi}^{(0)})^+ - \epsilon (\boldsymbol{\Psi}^\epsilon)^+ \Big) \cdot \boldsymbol{\nu} \right\|_{H^{-1/2}(\partial\Omega)} \leq C_R \epsilon \|\boldsymbol{\Phi}^{(0)}\|_{H^2(\Omega)}. \tag{42}$$

Therefore we can now obtain

$$\begin{aligned}
&\left|\int_{\partial\Omega}\left(\left(\left(\mathrm{rot}\mathbf{q}\right)_{ij}+\omega^2 C_{ijk\ell}(y)\frac{\partial\gamma_{m\ell}}{\partial y_k}(y)u_m^{(0)}\right)\nu_i\right)\cdot(\Phi_j^{(0)})^+dS\right|\\
&\leq C\|\mathbf{q}\|_{H^{-1}(\partial\Omega)}\|\nabla\boldsymbol{\Phi}^{(0)}\|_{H^1(\partial\Omega)}+\omega^2\left|\int_{\partial\Omega}C_{ijk\ell}(y)\frac{\partial\gamma_{m\ell}}{\partial y_k}(y)u_m^{(0)}\nu_i\ (\Phi_j^{(0)})^+dS\right|\\
&\leq C\|\mathbf{u}^{(0)}\|_{H^2(\Omega)}\|\boldsymbol{\Phi}^{(0)}\|_{H^2(\Omega)}, \qquad (43)
\end{aligned}$$

where we have used $\|q_j\|_{H^{-1}(\partial\Omega)}\leq C\|\mathbf{u}^{(0)}\|_{H^2(\Omega)}$ with the constant $C>0$ independent of ϵ (by (23)) and $C_{ijk\ell}\dfrac{\partial\gamma_{m\ell}}{\partial y_k}$ is bounded in $H^{1/2}(\partial\Omega)$ independent of ϵ for all $1\leq i,j,m\leq d$ (see (24)). We also know that

$$\|\mathbf{u}^{(1)}\|_{L^2(\partial\Omega)}\leq C\|\mathbf{u}^{(0)}\|_{H^2(\Omega)},$$

for some constant $C>0$ independent of ϵ and hence,

$$\begin{aligned}
\left|\int_{\partial\Omega}\left((\nabla\boldsymbol{\Phi}^{(0)})^+\cdot\boldsymbol{\nu}\right)\cdot\mathbf{u}^{(1)}dS\right| &\leq C\|\mathbf{u}^{(1)}\|_{L^2(\partial\Omega)}\|\nabla\boldsymbol{\Phi}^{(0)}\|_{L^2(\partial\Omega)}\\
&\leq C\|\mathbf{u}^{(0)}\|_{H^2(\Omega)}\|\boldsymbol{\Phi}^{(0)}\|_{H^2(\Omega)}. \qquad (44)
\end{aligned}$$

To proceed, we can use similar arguments as before to get

$$\|\boldsymbol{\Psi}^\epsilon\|_{H^{1/2}(\partial\Omega)}\leq C\epsilon^{-1/2}\|\boldsymbol{\Phi}^{(0)}\|_{H^2(\Omega)},$$

and

$$\left\|\frac{\mathbf{v}^{(0)}-\overline{\mathbf{v}}^{(0)}}{\epsilon}+\widetilde{\mathbf{v}}^{(1)}\right\|_{H^{-1/2}(\partial\Omega)}\leq C\epsilon^{-1/2}\|\mathbf{u}^{(0)}\|_{H^2(\Omega)},$$

which implies that

$$\left|\int_{\partial\Omega}\left(\frac{\mathbf{v}^{(0)}-\overline{\mathbf{v}}^{(0)}}{\epsilon}+\widetilde{\mathbf{v}}^{(1)}\right)\cdot\epsilon\boldsymbol{\Psi}_\epsilon dS\right|\leq C\|\mathbf{u}^{(0)}\|_{H^2(\Omega)}\|\boldsymbol{\Phi}^{(0)}\|_{H^2(\Omega)}, \qquad (45)$$

where $C>0$ is a constant independent of ϵ. Similar arguments give

$$\epsilon\left|\int_{\partial\Omega}\mathbf{u}^{(1)}\cdot((\nabla\boldsymbol{\Psi}^\epsilon)^+\cdot\boldsymbol{\nu})dS\right|\leq C\|\mathbf{u}^{(0)}\|_{H^2(\Omega)}\|\boldsymbol{\Phi}^{(0)}\|_{H^2(\Omega)}, \qquad (46)$$

for some constant $C>0$ independent of ϵ, where we have utilized the fact that

$$\|\mathbf{u}^{(1)}\|_{H^{1/2}(\partial\Omega)}=\left\|\boldsymbol{\chi}(\frac{x}{\epsilon})\nabla\mathbf{u}^{(0)}\right\|_{H^{1/2}(\partial\Omega)}\leq C\epsilon^{-1/2}\|\mathbf{u}^{(0)}\|_{H^2(\Omega)}.$$

Finally from (40) we can obtain that

$$\begin{aligned}
&\epsilon\Big|-\int_\Omega\left(\mathbf{C}(\frac{x}{\epsilon})\nabla\widetilde{\boldsymbol{\varphi}}^\epsilon\right):\nabla\boldsymbol{\Psi}^\epsilon dx+\omega^2\int_\Omega\rho(\frac{x}{\epsilon})\widetilde{\boldsymbol{\varphi}}^\epsilon\cdot\boldsymbol{\Psi}^\epsilon dx\\
&\quad+\int_{\partial\Omega}\left((\mathbf{C}(\frac{x}{\epsilon})(\nabla\boldsymbol{\Psi}^\epsilon)^-)\cdot\boldsymbol{\nu}\right)\cdot\widetilde{\boldsymbol{\varphi}}^\epsilon dS\\
&\quad+\int_{\partial\Omega}((\nabla\widetilde{\boldsymbol{\varphi}}^\epsilon)^+\cdot\boldsymbol{\nu})\cdot(\boldsymbol{\Psi}^\epsilon)^+dS-\int_{\partial\Omega}((\nabla\boldsymbol{\Psi}^\epsilon)^+\cdot\boldsymbol{\nu})\cdot(\widetilde{\boldsymbol{\varphi}}^\epsilon)^+dS\Big|\\
&\leq C\epsilon\|\boldsymbol{\varphi}^\epsilon\|_{H^1(B_R)}\|\boldsymbol{\Psi}^\epsilon\|_{H^1(B_R)}\ \leq\ C\|\mathbf{u}^{(0)}\|_{H^2(\Omega)}\|\boldsymbol{\Phi}^{(0)}\|_{H^2(\Omega)}, \qquad (47)
\end{aligned}$$

for some constants $C > 0$ independent of ϵ, $\mathbf{u}^{(0)}$ and $\boldsymbol{\Phi}^{(0)}$. Then by combining (43)-(47) and the remainder term of (42) if of order ϵ

$$\|\widetilde{\boldsymbol{\varphi}}^{\epsilon}\|_{L^2(B_R)}\|\boldsymbol{\phi}\|_{L^2(B_R)} \leq C\|\mathbf{u}^{(0)}\|_{H^2(\Omega)}\|\boldsymbol{\Phi}^{(0)}\|_{H^2(\Omega)}.$$

Furthermore since $\|\boldsymbol{\Phi}^{(0)}\|_{H^2(\Omega)} \leq C\|\boldsymbol{\phi}\|_{L^2(B_R)}$, then there exists a constant $C > 0$ independent of ϵ such that

$$\|\widetilde{\boldsymbol{\varphi}}^{\epsilon}\|_{L^2(B_R)} \leq C\|\mathbf{u}^{(0)}\|_{H^2(\Omega)}. \tag{48}$$

Finally since the difference between $\widetilde{\boldsymbol{\varphi}}^{\epsilon}$ and $\boldsymbol{\varphi}^{\epsilon}$ only appears in the jump conormal derivative across the boundary ∂D (see equations (26) and (41)), this proves the theorem. □

Now, we are ready to prove the rates of convergence of $\|\mathbf{u}^{\epsilon} - \mathbf{u}^{(0)}\|_{L^2(B_R)}$.

Proof (Proof of Theorem 1) It is easy to see that

$$\|\mathbf{u}^{(1)}\|_{L^2(B_R)} = \|\mathbf{u}^{(1)}\|_{L^2(\Omega)} \leq C\|\mathbf{u}^{(0)}\|_{H^2(\Omega)},$$

by using the definition of $\mathbf{u}^{(1)}$ and the smoothness of $\boldsymbol{\chi}(y)$. From (28) and (48), one can see that

$$\begin{aligned}\|\mathbf{u}^{\epsilon} - \mathbf{u}^{(0)}\|_{L^2(B_R)} &\leq C_R\epsilon\|\mathbf{u}^{(0)}\|_{H^2(\Omega)} + \epsilon\|\mathbf{u}^{(1)}\|_{L^2(B_R)} + \epsilon\|\widetilde{\boldsymbol{\varphi}}^{\epsilon}\|_{L^2(B_R)}\\ &\leq C_R\epsilon\|\mathbf{u}^{(0)}\|_{H^2(\Omega)},\end{aligned}$$

for some constant $C_R > 0$ independent of ϵ and $\mathbf{u}^{(0)}$. This completes the proof. □

3 Higher-order asymptotic analysis of the transmission problem

There are recent interests on higher-order two-scale homogenization of wave propagation in periodic meida [1,7,9,21,27]. In the case that the periodic structure was only supported in a bounded domain, contrary to the case that the periodic structure occupies $\mathbb{R}^d$, the boundary correctors played a role both in the leading-order and second-order homogenization as demonstrated in [7] for scalar wave equation. In this section we study the higher-order homogenization of the elastic scattering problem where the periodic media has bounded support.

3.1 Higher-order asymptotic expansion

Recall in asymptotic expansion (16) the first order term $\mathbf{u}^{(1)}$ was given by (21), in this section we consider a more general form of $\mathbf{u}^{(1)} = \mathbf{u}^{(1)}(x, y)$ given by

$$u_{\ell}^{(1)}(x, y) = -\chi_{\ell mn}(y)\frac{\partial u_n^{(0)}}{\partial x_m}(x) + \widetilde{u}_{\ell}^{(1)}(x). \tag{49}$$

From the ansatz we further obtain

$$O(\epsilon):\ \mathbf{v}^{(1)} - \mathbf{C}(y)\big(\nabla_x\mathbf{u}^{(1)} + \nabla_y\mathbf{u}^{(2)}\big) = 0, \tag{50}$$

$$\big(\nabla_x \cdot \mathbf{v}^{(1)} + \nabla_y \cdot \mathbf{v}^{(2)}\big) + \omega^2\rho(y)\mathbf{u}^{(1)} = 0. \tag{51}$$

Now we first derive a representation for u_2. Applying divergence $\nabla_y\cdot$ to equation (50) and using (20) yield

$$\nabla_y\cdot\big(\mathbf{C}(y)\nabla_y\mathbf{u}^{(2)}\big)+\nabla_y\cdot\big(\mathbf{C}(y)\nabla_x\mathbf{u}^{(1)}\big)=\nabla_y\cdot\mathbf{v}^{(1)}=-\nabla_x\cdot\mathbf{v}^{(0)}-\omega^2\rho(y)\mathbf{u}^{(0)}. \tag{52}$$

From equations (49), (52), (22) and direct computations, we obtain the governing equation for $u^{(2)}$

$$\begin{aligned}\frac{\partial}{\partial y_i}\Big(C_{ijk\ell}(y)\frac{\partial u_\ell^{(2)}}{\partial y_k}\Big)=&\Big(-C_{ijk\ell}+C_{ijmn}\frac{\partial\chi_{nk\ell}}{\partial y_m}+\frac{\partial}{\partial y_m}\big(\chi_{ni\ell}C_{mjkn}\big)+\overline{C}_{ijk\ell}\Big)\frac{\partial^2u_\ell^{(0)}}{\partial x_k\partial x_i}\\&-\frac{C_{ijk\ell}}{\partial y_i}\frac{\partial\widetilde{u}_\ell^{(1)}}{\partial x_k}+\omega^2(\overline{\rho}-\rho)u_j^{(0)}.\end{aligned} \tag{53}$$

Let us set

$$b_{ijk\ell}=-C_{ijk\ell}+C_{ijmn}\frac{\partial\chi_{nk\ell}}{\partial y_m}+\frac{\partial}{\partial y_m}\big(\chi_{ni\ell}C_{mjkn}\big),$$

and note that $\int_Y b_{ijk\ell}\,dy=-\overline{C}_{ijk\ell}$. Besides, due to the symmetric properties of $C_{ijk\ell}$, we know that $\overline{b}_{ijk\ell}$ also has the major and minor symmetry.

Now we introduce higher-order cell functions $\chi_{ik\ell q}\in H^1_{per}(Y)$ that is Y-periodic and solves

$$\frac{\partial}{\partial y_\alpha}\left(C_{\alpha j\beta q}(y)\frac{\partial\chi_{ik\ell q}}{\partial y_\beta}\right)=b_{ijk\ell}-\int_Y b_{ijk\ell}\,dy. \tag{54}$$

In addition with the help of the cell functions $\chi_{\ell mn}$ defined by (7) and $\gamma_{m\ell}$ defined by (24), one can directly obtain from equation (53) that

$$u_p^{(2)}=\chi_{mnqp}\frac{\partial^2u_q^{(0)}}{\partial x_n\partial x_m}-\chi_{pmn}\frac{\partial\widetilde{u}_n^{(1)}}{\partial x_m}+\omega^2\gamma_{mp}(y)u_m^{(0)}+\widetilde{u}_p^{(2)}(x),\ \text{for } 1\le p\le d, \tag{55}$$

where the function $\widetilde{u}_p^{(2)}$ will be determined later. It is not hard to see that $\mathbf{u}^{(2)}$ is a solution of (50) (due to $\nabla_y\widetilde{\mathbf{u}}^{(2)}(x)=0$). From (50)

$$\mathbf{v}^{(1)}=\mathbf{C}(y)\big(\nabla_x\mathbf{u}^{(1)}+\nabla_y\mathbf{u}^{(2)}\big), \tag{56}$$

then applying the divergence $\nabla_x\cdot$ to (56) and note that $\mathbf{u}^{(1)}$ and $\mathbf{u}^{(2)}$ are given by (49) and (55) respectively,

$$\begin{aligned}(\nabla_x\cdot\mathbf{v}^{(1)})_j=&\left(-C_{ijn\ell}\chi_{\ell mq}+C_{ijk\ell}\frac{\partial\chi_{mnq\ell}}{\partial y_k}\right)\frac{\partial^3u_q^{(0)}}{\partial x_i\partial x_m\partial x_n}\\&+\omega^2C_{mjk\ell}\frac{\partial\gamma_{n\ell}}{\partial y_k}\frac{\partial u_n^{(0)}}{\partial x_m}+\left(C_{ijk\ell}-C_{ijmn}\frac{\partial\chi_{nk\ell}}{\partial y_m}\right)\frac{\partial^2\widetilde{u}_\ell^{(1)}}{\partial x_i\partial x_k}.\end{aligned} \tag{57}$$

Applying the divergence $\nabla_x\cdot$ to (56) and then averaging that over Y yield

$$\int_Y\nabla_x\cdot\mathbf{v}^{(1)}dy-\int_Y\nabla_x\cdot\Big(\mathbf{C}(y)\big(\nabla_x\mathbf{u}^{(1)}+\nabla_y\mathbf{u}^{(2)}\big)\Big)dy=0,$$

note that $\mathbf{u}^{(1)}$, $\mathbf{u}^{(2)}$ and $\nabla_x \cdot \mathbf{v}^{(1)}$ are given by (49), (55) and (57) respectively, then a direct calculation yields

$$\begin{aligned}\overline{C}_{ijk\ell}\frac{\partial^2 \widetilde{u}_\ell^{(1)}}{\partial x_i \partial x_k} + \omega^2 \overline{\rho}\widetilde{u}_j^{(1)} &= -\left(\frac{\partial^3 u_q^{(0)}}{\partial x_i \partial x_m \partial x_n}\right)\int_Y \left(-C_{ijn\ell}\chi_{\ell mq} + C_{ijk\ell}\frac{\partial \chi_{mnq\ell}}{\partial y_k}\right) dy \\ &\quad -\omega^2 \frac{\partial u_n^{(0)}}{\partial x_m}\int_Y \left(-\rho\chi_{jmn} + C_{mjk\ell}\frac{\partial \gamma_{n\ell}}{\partial y_k}\right) dy. \end{aligned} \tag{58}$$

We will show that the function $\widetilde{\mathbf{u}}^{(1)}(x)$ cannot be chosen as zero in the elastic homogenization case, which is different from the scalar case [7] (the function $\widetilde{\mathbf{u}}^{(1)}$ can be taken by zero in the scalar case). Via integration by parts and periodic conditions of $C_{ijk\ell}$, $\chi_{mnq\ell}$ and the cell problem (7), one can see that

$$\begin{aligned}&\int_Y C_{ijk\ell}\frac{\partial}{\partial y_k}\chi_{mnq\ell}dy = -\int_Y \chi_{mnq\ell}\frac{\partial}{\partial y_k}C_{k\ell ij}dy \\ &= -\int_Y \chi_{mnq\ell}\frac{\partial}{\partial y_k}\left(C_{k\ell\alpha\beta}\frac{\partial}{\partial y_\alpha}\chi_{\beta ij}\right)dy = -\int_Y \chi_{\beta ij}\frac{\partial}{\partial y_\alpha}\left(C_{k\ell\alpha\beta}\frac{\partial}{\partial y_k}\chi_{mnq\ell}\right)dy,\end{aligned}$$

where we have used integration by parts twice in the last equality. From the symmetric condition of the fourth-order tensor $C_{k\ell\alpha\beta} = C_{\alpha\beta k\ell}$ and equation (54), we can get

$$\begin{aligned}&\int_Y C_{ijk\ell}\frac{\partial}{\partial y_k}\chi_{mnq\ell}dy = -\int_Y \chi_{\beta ij}\frac{\partial}{\partial y_\alpha}\left(C_{\alpha\beta k\ell}\frac{\partial}{\partial y_k}\chi_{mnq\ell}\right)dy \\ &= -\int_Y \chi_{\beta ij}(b_{m\beta nq} - \int_Y b_{m\beta nq}\, dy)dy \\ &= -\int_Y \chi_{\beta ij}\left(-C_{m\beta nq} + C_{m\beta\alpha\gamma}\frac{\partial \chi_{\gamma nq}}{\partial y_\alpha} + \frac{\partial}{\partial y_\alpha}(\chi_{\gamma mq}C_{\alpha\beta n\gamma}) + \overline{C}_{m\beta nq}\right)dy \\ &= \int_Y \chi_{\beta ij}C_{m\beta nq}dy - \int_Y \chi_{\beta ij}C_{m\beta\alpha\gamma}\frac{\partial \chi_{\gamma nq}}{\partial y_\alpha}dy + \int_Y \chi_{\gamma mq}C_{\alpha\beta n\gamma}\frac{\partial \chi_{\beta ij}}{\partial y_\alpha}dy,\end{aligned} \tag{59}$$

where we used the integration by parts and $\int_Y \chi_{\beta ij}dy = 0$ in the last equality. Therefore from (59) we can obtain

$$\begin{aligned}&\left(\frac{\partial^3 u_q^{(0)}}{\partial x_i \partial x_m \partial x_n}\right)\left(\int_Y C_{ijk\ell}\frac{\partial}{\partial y_k}\chi_{mnq\ell}dy - \int_Y C_{ijn\ell}\chi_{\ell mq}dy\right) \\ &= \left(\frac{\partial^3 u_q^{(0)}}{\partial x_i \partial x_m \partial x_n}\right)\int_Y \Big(-\chi_{\ell mq}C_{ijn\ell} + \chi_{\beta ij}C_{m\beta nq} - \chi_{\beta ij}C_{m\beta\alpha\gamma}\frac{\partial \chi_{\gamma nq}}{\partial y_\alpha} \\ &\qquad + \chi_{\gamma mq}C_{\alpha\beta n\gamma}\frac{\partial \chi_{\beta ij}}{\partial y_\alpha}\Big)dy.\end{aligned}$$

From the above representation, it is easy to see that the above quantity may not be zero, since the index q induces non-symmetry among the indices q, i, m, n even though the indices i, m, n can be interchanged freely.

For the second term in the right hand side of (58), from equation (24) governing γ and integration by parts, we have

$$\begin{aligned}
&\int_Y C_{mjk\ell}\frac{\partial}{\partial y_k}\gamma_{n\ell}dy = -\int_Y \gamma_{n\ell}\frac{\partial}{\partial y_k}C_{mjk\ell}dy = -\int_Y \gamma_{n\ell}\frac{\partial}{\partial y_k}C_{k\ell mj}dy\\
&= -\int_Y \frac{\partial}{\partial y_k}(C_{k\ell pq}\frac{\partial}{\partial y_p}\chi_{qmj})\gamma_{n\ell}dy = -\int_Y \frac{\partial}{\partial y_p}(C_{k\ell pq}\frac{\partial}{\partial y_k}\gamma_{n\ell})\chi_{qmj}dy\\
&= -\int_Y \frac{\partial}{\partial y_p}(C_{pqk\ell}\frac{\partial}{\partial y_k}\gamma_{n\ell})\chi_{qmj}dy = -\int_Y \chi_{qmj}(\overline{\rho}-\rho)\delta_{qn}dy = \int_Y \rho\chi_{nmj}dy,
\end{aligned}$$

whereby

$$\int_Y \left(-\rho\chi_{jmn} + C_{mjk\ell}\frac{\partial}{\partial y_k}\gamma_{n\ell}\right)dy = \int_Y \left(-\rho\chi_{jmn} + \rho\chi_{nmj}\right)dy.$$

The fact that χ_{jmn} has symmetries with respect to m and n may not yield the above quantity to be zero. Note that for the scalar case (see [7]), the right hand side of (58) is zero, thus one can choose $\widetilde{\mathbf{u}}^{(1)} = 0$ without loss of generality in the scalar case, but for the elastic case, we simply keep $\widetilde{\mathbf{u}}^{(1)}(x)$ in the following analysis.

Now let us seek for the a formula for $\mathbf{v}^{(2)}$ and we denote such a function by $\widehat{\mathbf{v}}^{(2)}$ in this section. In particular from equation (51)

$$\nabla_y \cdot \mathbf{v}^{(2)} = -\nabla_x \cdot \mathbf{v}^{(1)} - \omega^2\rho(y)\mathbf{u}^{(1)}.$$

From equations (49), (55) and (56), one can derive the equation for $\mathbf{v}^{(2)}$

$$\begin{aligned}
(\nabla_y \cdot \mathbf{v}^{(2)})_j =& \Big(C_{ijn\ell}\chi_{\ell mq} - C_{ijk\ell}\frac{\partial\chi_{mnq\ell}}{\partial y_k}\Big)\frac{\partial^3 u_q^{(0)}}{\partial x_i\partial x_m\partial x_n}\\
&+\omega^2\Big(-C_{mjk\ell}\frac{\partial\gamma_{n\ell}}{\partial y_k} + \rho\chi_{jmn}\Big)\frac{\partial u_n^{(0)}}{\partial x_m}\\
&+\Big(-C_{ijkq} + C_{ijmn}\frac{\partial\chi_{nkq}}{\partial y_m}\Big)\frac{\partial^2\widetilde{u}_q^{(1)}}{\partial x_k\partial x_i} - \omega^2\rho\widetilde{u}_j^{(1)}.
\end{aligned}$$

From the governing equation (58) for $\tilde{u}_1$, one can further simplify the above equation to

$$\begin{aligned}
(\nabla_y \cdot \mathbf{v}^{(2)})_j =& \Big(C_{ijn\ell}\chi_{\ell mq} - C_{ijk\ell}\frac{\partial\chi_{mnq\ell}}{\partial y_k}\\
&\qquad -\int_Y \big(C_{ijn\ell}\chi_{\ell mq} - C_{ijk\ell}\frac{\partial\chi_{mnq\ell}}{\partial y_k}\big)dy\Big)\frac{\partial^3 u_q^{(0)}}{\partial x_i\partial x_m\partial x_n}\\
&+\omega^2\Big(-C_{mjk\ell}\frac{\partial\gamma_{n\ell}}{\partial y_k} + \rho\chi_{jmn} - \int_Y\big(-C_{mjk\ell}\frac{\partial\gamma_{n\ell}}{\partial y_k} + \rho\chi_{jmn}\big)dy\Big)\frac{\partial u_n^{(0)}}{\partial x_m}\\
&+\Big(-C_{ijkq} + C_{ijmn}\frac{\partial\chi_{nkq}}{\partial y_m} - \int_Y\big(-C_{ijkq} + C_{ijmn}\frac{\partial\chi_{nkq}}{\partial y_m}\big)dy\Big)\frac{\partial^2\widetilde{u}_q^{(1)}}{\partial x_k\partial x_i}\\
&-(\rho-\overline{\rho})\omega^2\widetilde{u}_j^{(1)}.
\end{aligned}\tag{60}$$

Now we introduce the following higher-order cell function to construct $\mathbf{v}^{(2)}$ such that equation (60) holds. In particular we construct $\widehat{\chi}_{inmq\ell}$, $\widehat{\gamma}_{ikq\ell}$ and $\widehat{\gamma}_{\ell mn}$ that belong to $H^1_{per}(Y)$ and solve

$$\begin{cases} \dfrac{\partial}{\partial y_\alpha}\left(C_{\alpha j\beta\ell}(y)\dfrac{\partial\widehat{\chi}_{inmq\ell}}{\partial y_\beta}\right) & = d_{ijnmq} - \fint_Y d_{ijnmq}\,dy, \\ \dfrac{\partial}{\partial y_\alpha}\left(C_{\alpha j\beta\ell}(y)\dfrac{\partial\widehat{\gamma}_{ikq\ell}}{\partial y_\beta}\right) & = -C_{ijkq} + C_{ijmn}\dfrac{\partial\chi_{nkq}}{\partial y_m} \\ & \quad - \fint_Y\left(-C_{ijk\ell} + C_{ijmn}\dfrac{\partial\chi_{nkq}}{\partial y_m}\right)dy, \\ \dfrac{\partial}{\partial y_\alpha}\left(C_{\alpha j\beta\ell}(y)\dfrac{\partial\widehat{\gamma}_{\ell mn}}{\partial y_\beta}\right) & = -C_{mjk\ell}\dfrac{\partial\gamma_{n\ell}}{\partial y_k} + \rho\chi_{jmn} \\ & \quad - \fint_Y\left(-C_{mjk\ell}\dfrac{\partial\gamma_{n\ell}}{\partial y_k} + \rho\chi_{jmn}\right)dy, \end{cases} \tag{61}$$

where d_{ijnmq} is defined by

$$d_{ijnmq} = C_{ijn\ell}\chi_{\ell mq} - C_{ijk\ell}\frac{\partial\chi_{mnq\ell}}{\partial y_k}. \tag{62}$$

Now let us construct a solution $\widehat{\mathbf{v}}^{(2)}$ to (60) whose $\alpha\beta$-th component is given by

$$\begin{aligned} \widehat{v}^{(2)}_{\alpha\beta} = & C_{\alpha\beta k\ell}\Big(\frac{\partial\widehat{\chi}_{inmq\ell}}{\partial y_k}\frac{\partial^3 u^{(0)}_q}{\partial x_i\partial x_m\partial x_n} + \frac{\partial\widehat{\gamma}_{ipq\ell}}{\partial y_k}\frac{\partial^2\widetilde{u}^{(1)}_q}{\partial x_i\partial x_p} \\ & + \omega^2\frac{\partial\widehat{\gamma}_{\ell mn}}{\partial y_k}\frac{\partial u^{(0)}_n}{\partial x_m} + \omega^2\frac{\partial\gamma_{m\ell}}{\partial y_k}\widetilde{u}^{(1)}_m\Big). \end{aligned} \tag{63}$$

Then from equation (58), (61) and (62), one can directly verify that $\widehat{\mathbf{v}}^{(2)}$ satisfies equation (51).

Now let us introduce the boundary corrector function $\boldsymbol{\theta}^\epsilon \in \left(H^1_{loc}(\mathbb{R}^d)\right)^d$ that solves

$$\begin{cases} \nabla\cdot\left(\mathbf{C}(\frac{x}{\epsilon})\nabla\boldsymbol{\theta}^\epsilon\right) + \omega^2\rho(\frac{x}{\epsilon})\boldsymbol{\theta}^\epsilon = 0 & \text{in } \Omega, \\ \Delta^*\boldsymbol{\theta}^\epsilon + \omega^2\boldsymbol{\theta}^\epsilon = 0 & \text{in } \mathbb{R}^d\setminus\overline{\Omega}, \\ (\boldsymbol{\theta}^\epsilon)^+ - (\boldsymbol{\theta}^\epsilon)^- = \mathbf{u}^{(2)} & \text{on } \partial\Omega, \\ (T_{\boldsymbol{\nu}}\boldsymbol{\theta}^\epsilon)^+ - (\mathbf{C}(\frac{x}{\epsilon}))\nabla\boldsymbol{\theta}^\epsilon)^-\cdot\boldsymbol{\nu} = \widehat{\mathbf{v}}^{(2)}\cdot\boldsymbol{\nu} & \text{on } \partial\Omega, \end{cases} \tag{64}$$

where $\boldsymbol{\theta}^\epsilon$ satisfies the Kupradze radiation condition (4).

3.2 Rates of convergence in L^2 and H^1: The higher-order case

Via previous discussions on higher-order asymptotic analysis, we have

Theorem 4 *Let $\mathbf{u}^\epsilon$ and $\mathbf{u}^{(0)}$ be the solutions of (5) and (6) respectively. Let $\mathbf{u}^{(1)}$ and $\mathbf{u}^{(2)}$ be defined by equations (49) and (55), respectively, with $\mathbf{u}^{(1)} = 0$ and $\mathbf{u}^{(2)} = 0$ in $\mathbb{R}^d\setminus\overline{\Omega}$. Let $\boldsymbol{\varphi}^\epsilon$ and $\boldsymbol{\theta}^\epsilon$ be the boundary correctors given by (41) and (64). Then for any ball B_R with $\Omega\subset B_R$, we have*

$$\|\mathbf{u}^\epsilon - (\mathbf{u}^{(0)} + \epsilon\mathbf{u}^{(1)} + \epsilon^2\mathbf{u}^{(2)} + \epsilon\boldsymbol{\varphi}^\epsilon + \epsilon^2\boldsymbol{\theta}^\epsilon)\|_{H^1(B_R)} \le C_R\epsilon^2\|\mathbf{u}^{(0)}\|_{H^4(\Omega)},$$

where $C_R > 0$ is a constant independent of ϵ.

Proof The proof is similar to the proof of Theorem 1. Again consider error functions in D defined by

$$\mathbf{w}^\epsilon := \mathbf{u}^\epsilon - \mathbf{u}^{(0)} - \epsilon\mathbf{u}^{(1)} - \epsilon^2\mathbf{u}^{(2)},$$

and

$$\boldsymbol{\zeta}^\epsilon := \mathbf{C}(\frac{x}{\epsilon})\nabla\mathbf{u}^\epsilon - \mathbf{v}^{(0)} - \epsilon\mathbf{v}^{(1)} - \epsilon^2\widehat{\mathbf{v}}^{(2)},$$

where $\mathbf{v}^{(1)}$, $\widehat{\mathbf{v}}^{(2)}$ are defined by (56), (63) with $\mathbf{v}^{(1)} = 0$ and $\widehat{\mathbf{v}}^{(2)} = 0$ in $\mathbb{R}^d \setminus \overline{\Omega}$. Here $\mathbf{w}^\epsilon$ is a vector-valued function and $\boldsymbol{\zeta}^\epsilon$ is a matrix-valued function. In this proof we conveniently use the same notations as in the proof of Theorem 1, since it is clear from the context. From straightforward computations, we can get

$$\begin{cases} \mathbf{C}(\frac{x}{\epsilon})\nabla\mathbf{w}^\epsilon - \boldsymbol{\zeta}^\epsilon = \epsilon^2(\mathbf{v}^{(2)} - \mathbf{C}(y)\nabla_x\mathbf{u}^{(2)}), \\ \nabla\cdot\boldsymbol{\zeta}^\epsilon + \omega^2\rho(y)\mathbf{w}^\epsilon = -\epsilon^2\big[\omega^2\rho\mathbf{u}^{(2)} + \nabla_x\cdot\widehat{\mathbf{v}}^{(2)}\big], \end{cases} \tag{65}$$

and moreover

$$\nabla\cdot\boldsymbol{\zeta}^\epsilon + \omega^2\rho\mathbf{w}^\epsilon = -\epsilon^2\big(\nabla_x\cdot\widehat{\mathbf{v}}^{(2)} + \omega^2\rho\mathbf{u}^{(2)}\big). \tag{66}$$

Outside D we simply define the error functions by $\mathbf{w}^\epsilon := \mathbf{u}^\epsilon - \mathbf{u}^{(0)}$ and $\boldsymbol{\zeta}^\epsilon := \nabla\mathbf{w}^\epsilon$, this directly gives

$$-\nabla\cdot\boldsymbol{\zeta}^{(\epsilon)} = \omega^2\mathbf{w}^\epsilon.$$

Let $\boldsymbol{\phi} \in \big(C_c^\infty(B_R)\big)^d$ be a vector-valued test function and consider an auxiliary function $\boldsymbol{\Phi}^\epsilon \in \big(H^1_{loc}(\mathbb{R}^d)\big)^d$ that solves

$$\begin{cases} \nabla\cdot\Big(\mathbf{C}(\frac{x}{\epsilon})\nabla\boldsymbol{\Phi}^\epsilon\Big) + \omega^2\rho(\frac{x}{\epsilon})\boldsymbol{\Phi}^\epsilon = \boldsymbol{\phi} & \text{in } \Omega, \\ \Delta^*\boldsymbol{\Phi}^\epsilon + \omega^2\boldsymbol{\Phi}^\epsilon = \boldsymbol{\phi} & \text{in } \mathbb{R}^d\setminus\overline{\Omega}, \\ (\boldsymbol{\Phi}^\epsilon)^+ - (\boldsymbol{\Phi}^\epsilon)^- = 0 & \text{on } \partial\Omega, \\ (T_{\boldsymbol{\nu}}\boldsymbol{\Phi}^\epsilon)^+ - (\mathbf{C}(\frac{x}{\epsilon}))\nabla\boldsymbol{\Phi}^\epsilon)^-\cdot\boldsymbol{\nu} = 0 & \text{on } \partial\Omega, \end{cases} \tag{67}$$

where $\boldsymbol{\Phi}^\epsilon$ satisfies the Kupradze radiation condition (4). Thus from the same argument as in the proof of Theorem 1, one can get

$$\begin{aligned} &\int_{B_R}(\mathbf{w}^\epsilon - \epsilon\boldsymbol{\varphi}^\epsilon - \epsilon^2\boldsymbol{\theta}^\epsilon)\cdot\boldsymbol{\phi}dx \\ =&\int_\Omega(\mathbf{w}^\epsilon - \epsilon\boldsymbol{\varphi}^\epsilon - \epsilon^2\boldsymbol{\theta}^\epsilon)\cdot\Big(\nabla\cdot(\mathbf{C}(\frac{x}{\epsilon})\nabla\boldsymbol{\Phi}^\epsilon) + \omega^2\rho(\frac{x}{\epsilon})\boldsymbol{\Phi}^\epsilon\Big)dx \\ &+\int_{B_R\setminus\Omega}(\mathbf{w}^\epsilon - \epsilon\boldsymbol{\varphi}^\epsilon - \epsilon^2\boldsymbol{\theta}^\epsilon)\cdot(\Delta^*\boldsymbol{\Phi}^\epsilon + \omega^2\boldsymbol{\Phi}^\epsilon)dx \\ =&-\int_\Omega\big(\mathbf{C}(\frac{x}{\epsilon})\nabla\mathbf{w}^\epsilon\big):\nabla\boldsymbol{\Phi}^\epsilon dx + \epsilon\int_\Omega\big(\mathbf{C}(\frac{x}{\epsilon})\nabla\boldsymbol{\varphi}^\epsilon\big):\nabla\boldsymbol{\Phi}^\epsilon dx \\ &+\epsilon^2\int_\Omega\big(\mathbf{C}(\frac{x}{\epsilon})\nabla\boldsymbol{\theta}^\epsilon\big):\nabla\boldsymbol{\Phi}^\epsilon dx + \int_\Omega\omega^2\rho(\frac{x}{\epsilon})(\mathbf{w}^\epsilon - \epsilon\boldsymbol{\varphi}^\epsilon - \epsilon^2\boldsymbol{\theta}^\epsilon)\cdot\boldsymbol{\Phi}^\epsilon dx \\ &+\int_{\partial\Omega}T_{\boldsymbol{\nu}}(\mathbf{w}^\epsilon - \epsilon\boldsymbol{\varphi}^\epsilon - \epsilon^2\boldsymbol{\theta}^\epsilon)^+\cdot\boldsymbol{\Phi}^\epsilon dS. \end{aligned}$$

From integration by parts, one can further obtain

$$\begin{aligned}
&\int_{B_R} (\mathbf{w}^\epsilon - \epsilon\boldsymbol{\varphi}^\epsilon - \epsilon^2\boldsymbol{\theta}^\epsilon)\cdot\phi dx\\
=&-\int_\Omega \big(\mathbf{C}(\frac{x}{\epsilon})\nabla\mathbf{w}^\epsilon\big):\nabla\boldsymbol{\Phi}^\epsilon dx + \epsilon\int_{\partial\Omega}\big(\mathbf{C}(\frac{x}{\epsilon})(\nabla\boldsymbol{\varphi}^\epsilon)^-\cdot\boldsymbol{\nu}\big)\cdot\boldsymbol{\Phi}^\epsilon dS + \int_\Omega \omega^2\rho(\frac{x}{\epsilon})\mathbf{w}^\epsilon\cdot\boldsymbol{\Phi}^\epsilon dx\\
&+\epsilon^2\int_{\partial\Omega}\big(\mathbf{C}(\frac{x}{\epsilon})(\nabla\boldsymbol{\theta}^\epsilon)^-\cdot\boldsymbol{\nu}\big)\cdot\boldsymbol{\Phi}^\epsilon dS + \int_{\partial\Omega} T_{\boldsymbol{\nu}}(\mathbf{w}^\epsilon - \epsilon\boldsymbol{\varphi}^\epsilon - \epsilon^2\boldsymbol{\theta}^\epsilon)^+\cdot\boldsymbol{\Phi}^\epsilon dS.
\end{aligned}$$

Then from equations (41), (64), (65) and (66), and integration by parts one can obtain

$$\begin{aligned}
&\int_{B_R} (\mathbf{w}^\epsilon - \epsilon\boldsymbol{\varphi}^\epsilon - \epsilon^2\boldsymbol{\theta}^\epsilon)\cdot\phi dx\\
=&-\int_\Omega \boldsymbol{\zeta}^\epsilon:\nabla\boldsymbol{\Phi}^\epsilon dx + \omega^2\int_D \rho(\frac{x}{\epsilon})\mathbf{w}^\epsilon\cdot\boldsymbol{\Phi}^\epsilon dx - \epsilon\int_{\partial\Omega}\Big((T_{\boldsymbol{\nu}}\boldsymbol{\varphi}^\epsilon)^+ - \mathbf{C}(\frac{x}{\epsilon})(\nabla\boldsymbol{\varphi}^\epsilon)^-\cdot\boldsymbol{\nu}\Big)\cdot\boldsymbol{\Phi}^\epsilon dS\\
&+\int_{\partial\Omega} T_{\boldsymbol{\nu}}\mathbf{w}^{\epsilon+}\cdot\boldsymbol{\Phi}^\epsilon dS - \epsilon^2\int_{\partial\Omega}\Big((T_{\boldsymbol{\nu}}\boldsymbol{\theta}^\epsilon)^+ - \mathbf{C}(\frac{x}{\epsilon})(\nabla\boldsymbol{\theta}^\epsilon)^-\cdot\boldsymbol{\nu}\Big)\cdot\boldsymbol{\Phi}^\epsilon dS\\
&-\epsilon^2\int_\Omega\big(\widehat{\mathbf{v}}^{(2)} - \mathbf{C}(\frac{x}{\epsilon})\nabla_x\mathbf{u}^{(2)}\big):\nabla\boldsymbol{\Phi}^\epsilon dx\\
=&-\epsilon^2\int_\Omega\big(\widehat{\mathbf{v}}^{(2)} - \mathbf{C}(\frac{x}{\epsilon})\nabla_x\mathbf{u}^{(2)}\big):\nabla\boldsymbol{\Phi}^\epsilon dx - \epsilon^2\int_\Omega\big(\nabla_x\cdot\widehat{\mathbf{v}}^{(2)} + \omega^2\rho\mathbf{u}^{(2)}\big)dx. \qquad (68)
\end{aligned}$$

From the equations of $\mathbf{u}^{(1)}$, $\mathbf{u}^{(2)}$, $\mathbf{v}^{(1)}$ and $\widehat{\mathbf{v}}^{(2)}$ defined by (49), (55), (56) and (63), one can obtain

$$\begin{aligned}
\Big\|\widehat{\mathbf{v}}^{(2)} - \mathbf{C}(\frac{x}{\epsilon})\nabla_x\mathbf{u}^{(2)}\Big\|_{L^2(\Omega)} &\le C\|\mathbf{u}^{(0)}\|_{H^4(\Omega)},\\
\Big\|\nabla_x\cdot\widehat{\mathbf{v}}^{(2)} + \omega^2\rho\mathbf{u}^{(2)}\Big\|_{L^2(\Omega)} &\le C\|\mathbf{u}^{(0)}\|_{H^4(\Omega)},
\end{aligned}$$

where C is a constant. Furthermore apply the Cauchy-Schwartz inequality on (68), then we obtain

$$\left|\int_{B_R} (\mathbf{w}^\epsilon - \epsilon\boldsymbol{\varphi}^\epsilon - \epsilon^2\boldsymbol{\theta}^\epsilon)\cdot\phi dx\right| \le C\epsilon^2\|\mathbf{u}^{(0)}\|_{H^4(\Omega)}\|\boldsymbol{\Phi}^\epsilon\|_{H^1(\Omega)},$$

for some constant $C > 0$ independent of ϵ. Again we utilize the standard estimate for the elliptic system (67) (see [20] for instance), then we can obtain

$$\|\boldsymbol{\Phi}^\epsilon\|_{H^1(\Omega)} \le C\|\phi\|_{H^{-1}(B_R)},$$

where $C > 0$ is a constant depends on the coefficients and R, but independent of ϵ. By the duality arguments in the Sobolev space, then we complete the proof. This proves (10) in Theorem 2. □

Recall that $\mathbf{u}^{(1)} = 0$, $\mathbf{u}^{(2)} = 0$, $\mathbf{v}^{(1)} = 0$ and $\widehat{\mathbf{v}}^{(2)} = 0$ in $\mathbb{R}^d\setminus\overline{\Omega}$, without loss of generality, we can choose $\widetilde{\mathbf{u}}^{(1)}$ solves the constant coefficient elliptic system (58) in Ω with $\widetilde{\mathbf{u}}^{(1)} = 0$ on $\partial\Omega$. Therefore, $\widetilde{\mathbf{u}}^{(1)}$ is a smooth solution in Ω, by using the elliptic estimate of (58) again, then we obtain $\|\widetilde{\mathbf{u}}^{(1)}\|_{L^2(B_R)} = \|\widetilde{\mathbf{u}}^{(1)}\|_{L^2(\Omega)} \le C\|\mathbf{u}^{(0)}\|_{H^4(\Omega)}$ for some constant $C > 0$.

Now, we are ready to prove Theorem 2.

Proof (Proof of Theorem 2) By using the same reason and arguments as before, one can easily see that

$$\|\mathbf{u}^{(2)}\|_{L^2(B_R)} \le C_R \|\mathbf{u}^{(0)}\|_{H^4(\Omega)} \quad \text{and} \quad \|\boldsymbol{\theta}^{\epsilon}\|_{L^2(B_R)} \le C_R \|\mathbf{u}^{(0)}\|_{H^4(\Omega)},$$

for some constants $C_R > 0$ independent of ϵ. Therefore, the proof is nothing but a straightforward corollary by combining previous lemmas. Therefore, we prove (11) and complete our proof of Theorem 2. □

3.3 A second-order homogenization and wave dispersion

As a by-product of our higher-order homogenization in highly oscillating anisotropic media, in this section we illustrate formally the higher-order effective wave equation that can demonstrate dispersion of wave propagation in periodic media that occupies the whole space. To begin with let us recall from asymptotic expansion (16), we get

$$O(\epsilon^2): \ \mathbf{v}^{(2)} - \mathbf{C}(y)\big(\nabla_x \mathbf{u}^{(2)} + \nabla_y \mathbf{u}^{(3)}\big) = 0, \tag{69}$$

$$\big(\nabla_x \cdot \mathbf{v}^{(2)} + \nabla_y \cdot \mathbf{v}^{(3)}\big) + \omega^2 \rho(y) \mathbf{u}^{(2)} = 0. \tag{70}$$

Here we seek for a $\mathbf{v}^{(2)}$ different from equation (63). This is to be realized by finding $\mathbf{u}^{(3)}$ first. Indeed taking the divergence of equation (69) respect to the y variable and noting (51),

$$\nabla_y \cdot \big(\mathbf{C}(y)\nabla_y \mathbf{u}^{(3)}\big) + \nabla_y \cdot \big(\mathbf{C}(y)\nabla_x \mathbf{u}^{(2)}\big) = \nabla_y \cdot \mathbf{v}^{(2)} = -\nabla_x \cdot \mathbf{v}^{(1)} - \omega^2 \rho(y) \mathbf{u}^{(1)}.$$

With the help of (49), (55) and (56), A direct calculation yields

$$\begin{aligned}
\frac{\partial}{\partial y_\alpha}\left(C_{\alpha j \beta \ell}(y)\frac{\partial u_\ell^{(3)}}{\partial y_\beta}\right) =& \Big(-\frac{\partial}{\partial y_p}\big(C_{pji\ell}\chi_{mnq\ell}\big) + \big(C_{ijn\ell}\chi_{\ell mq} - C_{ijk\ell}\frac{\partial \chi_{mnq\ell}}{\partial y_k}\big) \\
& - \int_Y \big(C_{ijn\ell}\chi_{\ell mq} - C_{ijk\ell}\frac{\partial \chi_{mnq\ell}}{\partial y_k}\big) dy\Big) \frac{\partial^3 u_q^{(0)}}{\partial x_i \partial x_m \partial x_n} \\
& + \Big(\frac{\partial}{\partial y_m}\big(C_{mjkn}\chi_{niq}\big) + \big(-C_{ijkq} + C_{ijmn}\frac{\partial \chi_{nkq}}{\partial y_m}\big) \\
& - \int_Y \big(-C_{ijkq} + C_{ijmn}\frac{\partial \chi_{nkq}}{\partial y_m}\big)\, dy\Big) \frac{\partial^2 \widetilde{u}_q^{(1)}}{\partial x_k \partial x_i} \\
& + \omega^2 \Big(-\frac{\partial}{\partial y_p}\big(C_{pjm\ell}\gamma_{n\ell}\big) + \big(-C_{mjk\ell}\frac{\partial \gamma_{n\ell}}{\partial y_k} + \rho \chi_{jmn}\big) \\
& - \int_Y \big(-C_{mjk\ell}\frac{\partial \gamma_{n\ell}}{\partial y_k} + \rho\chi_{jmn}\big)\, dy\Big) \frac{\partial u_n^{(0)}}{\partial x_m} \\
& - \frac{\partial C_{ijkq}}{\partial y_i}\frac{\partial \widetilde{u}_q^{(2)}}{\partial x_k} - (\rho - \overline{\rho})\omega^2 \widetilde{u}_j^{(1)}
\end{aligned} \tag{71}$$

Let us introduce the higher-order cell functions $\chi_{inmq\ell}$ and $\gamma_{\ell mn}$ that belong to $H^1_{per}(Y)$ and solve

$$\begin{aligned}\frac{\partial}{\partial y_\alpha}\left(C_{\alpha j\beta\ell}(y)\frac{\partial \chi_{inmq\ell}}{\partial y_\beta}\right) &= -\frac{\partial}{\partial y_p}\big(C_{pji\ell}\chi_{mnq\ell}\big) + \big(C_{ijn\ell}\chi_{\ell mq} - C_{ijk\ell}\frac{\partial \chi_{mnq\ell}}{\partial y_k}\big)\\ &\quad - \int_Y \big(C_{ijn\ell}\chi_{\ell mq} - C_{ijk\ell}\frac{\partial \chi_{mnq\ell}}{\partial y_k}\big)dy \qquad (72)\\ \frac{\partial}{\partial y_\alpha}\left(C_{\alpha j\beta\ell}(y)\frac{\partial \gamma_{\ell mn}}{\partial y_\beta}\right) &= -\frac{\partial}{\partial y_p}\big(C_{pjm\ell}\gamma_{n\ell}\big) + \big(-C_{mjk\ell}\frac{\partial \gamma_{n\ell}}{\partial y_k} + \rho\chi_{jmn}\big)\\ &\quad - \int_Y \big(-C_{mjk\ell}\frac{\partial \gamma_{n\ell}}{\partial y_k} + \rho\chi_{jmn}\big)\,dy. \qquad (73)\end{aligned}$$

By changing the index one can see that the governing equation (54) for $\chi_{ikq\ell}$ can be written as

$$\begin{aligned}\frac{\partial}{\partial y_\alpha}\left(C_{\alpha j\beta\ell}(y)\frac{\partial \chi_{ikq\ell}}{\partial y_\beta}\right) &= \frac{\partial}{\partial y_m}\big(C_{mjkn}\chi_{niq}\big) + \big(-C_{ijkq} + C_{ijmn}\frac{\partial \chi_{nkq}}{\partial y_m}\big)\\ &\quad - \int_Y \big(-C_{ijkq} + C_{ijmn}\frac{\partial \chi_{nkq}}{\partial y_m}\big)\,dy. \qquad (74)\end{aligned}$$

From (71), (72), (73) and (74), one can obtain that

$$\begin{aligned}u^{(3)}_\ell = {}& \chi_{inmq\ell}\frac{\partial^3 u^{(0)}_q}{\partial x_i\partial x_m\partial x_n} + \chi_{ikq\ell}\frac{\partial^2 \widetilde{u}^{(1)}_q}{\partial x_i\partial x_k} + \omega^2\gamma_{\ell mn}(y)\frac{\partial u^{(0)}_n}{\partial x_m}\\ &- \chi_{\ell kq}\frac{\partial \widetilde{u}^{(2)}_q}{\partial x_k} + \omega^2\gamma_{m\ell}\widetilde{u}^{(1)}_m. \qquad (75)\end{aligned}$$

Now we have from equation (69) that

$$\mathbf{v}^{(2)} = C(y)\big(\nabla_x \mathbf{u}^{(2)} + \nabla_y \mathbf{u}^{(3)}\big),$$

then from the representation of $\mathbf{u}^{(2)}$ and $\mathbf{u}^{(3)}$ in equations (55) and (75) respectively, one can obtain

$$\begin{aligned}v^{(2)}_{\alpha\beta} = {}& C_{\alpha\beta k\ell}\big(\frac{\partial \chi_{inmq\ell}}{\partial y_k} + \chi_{mnq\ell}\delta_{ki}\big)\frac{\partial^3 u^{(0)}_q}{\partial x_i\partial x_m\partial x_n}\\ &+ C_{\alpha\beta k\ell}\big(\frac{\partial \chi_{imq\ell}}{\partial y_k} - \chi_{\ell mq}\delta_{ki}\big)\frac{\partial^2 \widetilde{u}^{(1)}_q}{\partial x_i\partial x_m}\\ &+ \omega^2 C_{\alpha\beta k\ell}\big(\frac{\partial \gamma_{\ell mq}}{\partial y_k} + \gamma_{q\ell}\delta_{mk}\big)\frac{\partial u^{(0)}_q}{\partial x_m} + C_{\alpha\beta k\ell}\big(-\frac{\partial \chi_{\ell mq}}{\partial y_k} + \delta_{q\ell}\delta_{mk}\big)\frac{\partial \widetilde{u}^{(2)}_q}{\partial x_m}\\ &+ \omega^2 C_{\alpha\beta k\ell}\frac{\partial \gamma_{q\ell}}{\partial y_k}\widetilde{u}^{(1)}_q. \qquad (76)\end{aligned}$$

Taking the Y-average of equation (70) yields

$$\int_Y \nabla_x\cdot \mathbf{v}^{(2)}dy + \omega^2\int_Y \rho\mathbf{u}^{(2)}dy = 0,$$

note further that $\mathbf{v}^{(2)}$ and $\mathbf{u}^{(2)}$ are given by (76) and (55) respectively, then from a direct calculation one can obtain the following equation for $\widetilde{\mathbf{u}}^{(2)}$

$$
\begin{aligned}
&\overline{C}_{\alpha\beta mq}\frac{\partial^2 \widetilde{u}_q^{(2)}}{\partial x_\alpha \partial x_m} + \omega^2 \overline{\rho}\widetilde{u}_\beta^{(2)} \\
=& -\frac{\partial^4 u_q^{(0)}}{\partial x_\alpha \partial x_i \partial x_m \partial x_n}\int_Y \big(C_{\alpha\beta k\ell}\frac{\partial \chi_{inmq\ell}}{\partial y_k} + C_{\alpha\beta i\ell}\chi_{mnq\ell}\big)\, dy \\
&- \omega^2 \frac{\partial^2 u_q^{(0)}}{\partial x_\alpha \partial x_m}\Big(\int_Y \big(C_{\alpha\beta k\ell}\frac{\partial \gamma_{\ell mq}}{\partial y_k} + C_{\alpha\beta m\ell}\gamma_{q\ell}\big)\, dy \\
&\qquad + \int_Y \rho\chi_{m\alpha q\beta}\, dy - (\overline{\rho})^{-1}\overline{C}_{\alpha jmq}\int_Y \rho\gamma_{m\beta}\, dy\Big) \\
&- \frac{\partial^3 \widetilde{u}_q^{(1)}}{\partial x_\alpha \partial x_i \partial x_m}\int_Y \big(C_{\alpha\beta k\ell}\frac{\partial \chi_{imq\ell}}{\partial y_k} - C_{\alpha\beta i\ell}\chi_{\ell mq}\big)\, dy. \\
&+ \omega^2 \frac{\partial \widetilde{u}_q^{(1)}}{\partial x_m}\int_Y \big(\rho\chi_{\beta mq} - C_{m\beta k\ell}\frac{\partial \gamma_{q\ell}}{\partial y_k}\big)\, dy
\end{aligned} \tag{77}
$$

Now let us recall that the solution $\mathbf{u}^\epsilon$ to (13) has the following anstaz

$$
\mathbf{u}^\epsilon(x,y) = \mathbf{u}^{(0)}(x,y) + \epsilon\mathbf{u}^{(1)}(x,y) + \epsilon^2\mathbf{u}^{(2)}(x,y) + \cdots = \sum_{k=0}^{\infty}\epsilon^k\mathbf{u}^{(k)}(x,y).
$$

Note that all the cell functions are Y-periodic and their averages over Y are zero, then from equations (6), (58) and change of index, we can summarize the governing equations for $\overline{\mathbf{u}}^{(0)}$ and $\overline{\mathbf{u}}^{(1)}$ in Ω, where $\overline{\mathbf{u}}^{(0)}$, $\overline{\mathbf{u}}^{(1)}$ are the averages of $\mathbf{u}^{(0)}$ and $\mathbf{u}^{(1)}$ respectively in the unit cell Y (recall that $\mathbf{u}^{(0)}(x,y) = \mathbf{u}^{(0)}(x)$ so that $\overline{\mathbf{u}}^{(0)} = \mathbf{u}^{(0)}$),

$$
\nabla\cdot(\overline{\mathbf{C}}\nabla\overline{\mathbf{u}}^{(0)}) + \omega^2\overline{\rho}\overline{\mathbf{u}}^{(0)} = 0, \tag{78}
$$

$$
\begin{aligned}
\overline{C}_{ijk\ell}\frac{\partial^2 \overline{u}_\ell^{(1)}}{\partial x_i \partial x_k} + \omega^2\overline{\rho}\overline{u}_j^{(1)} =& -\left(\frac{\partial^3 \overline{u}_q^{(0)}}{\partial x_i \partial x_m \partial x_n}\right)\int_Y \left(-C_{ijn\ell}\chi_{\ell mq} + C_{ijk\ell}\frac{\partial}{\partial y_k}\chi_{mnq\ell}\right) dy \\
&-\omega^2\frac{\partial \overline{u}_n^{(0)}}{\partial x_m}\int_Y \left(-\rho\chi_{jmn} + C_{mjk\ell}\frac{\partial}{\partial y_k}\gamma_{n\ell}\right) dy.
\end{aligned} \tag{79}
$$

Let $\mathbf{U} := \overline{\mathbf{u}}^{(0)} + \epsilon\overline{\mathbf{u}}^{(1)} + \epsilon^2\overline{\mathbf{u}}^{(2)}$. Now multiply equation (77) and equation (79) by ϵ^2 and ϵ respectively, and sum them with equation (78), then it is seen that $\mathbf{U} = (U_\beta)_{1\le\beta\le d}$ satisfies the following fourth-order partial differential equation

$$
\begin{aligned}
&\overline{C}_{\alpha\beta mq}\frac{\partial^2 U^q}{\partial x_\alpha \partial x_m} + \omega^2 \overline{\rho} U^\beta \\
=& -\epsilon^2 \frac{\partial^4 u_q^{(0)}}{\partial x_\alpha \partial x_i \partial x_m \partial x_n} \int_Y \big(C_{\alpha\beta k\ell}\frac{\partial \chi_{inmq\ell}}{\partial y_k} + C_{\alpha\beta i\ell}\chi_{mnq\ell}\big)\, dy \\
&-\ \epsilon^2\omega^2 \frac{\partial^2 u_q^{(0)}}{\partial x_\alpha \partial x_m}\Big(\int_Y \big(C_{\alpha\beta k\ell}\frac{\partial \gamma_{\ell mq}}{\partial y_k} + C_{\alpha\beta m\ell}\gamma_{q\ell}\big)\, dy \\
&\qquad + \int_Y \rho\chi_{m\alpha q\beta}\, dy - (\overline{\rho})^{-1}\overline{C}_{\alpha jmq}\int_Y \rho\gamma_{m\beta}\, dy\Big) \\
&-\ \epsilon \frac{\partial^3(\epsilon \widetilde{u}_q^{(1)} + u_q^{(0)})}{\partial x_\alpha \partial x_i \partial x_m}\int_Y \big(C_{\alpha\beta k\ell}\frac{\partial \chi_{imq\ell}}{\partial y_k} - C_{\alpha\beta i\ell}\chi_{\ell mq}\big)\, dy. \\
&+\ \epsilon\omega^2 \frac{\partial(\epsilon \widetilde{u}_q^{(1)} + u_q^{(0)})}{\partial x_m}\int_Y \big(\rho\chi_{\beta mq} - C_{m\beta k\ell}\frac{\partial \gamma_{q\ell}}{\partial y_k}\big)\, dy,
\end{aligned}
$$

and therefore the governing equation of $\mathbf{U}$ up to order ϵ^3 reads

$$
\begin{aligned}
&\overline{C}_{\alpha\beta mq}\frac{\partial^2 U^q}{\partial x_\alpha \partial x_m} + \omega^2 \overline{\rho} U^\beta \\
=& -\epsilon^2 \frac{\partial^4 U^q}{\partial x_\alpha \partial x_i \partial x_m \partial x_n} \int_Y \big(C_{\alpha\beta k\ell}\frac{\partial \chi_{inmq\ell}}{\partial y_k} + C_{\alpha\beta i\ell}\chi_{mnq\ell}\big)\, dy \\
&-\ \epsilon^2\omega^2 \frac{\partial^2 U^q}{\partial x_\alpha \partial x_m}\Big(\int_Y \big(C_{\alpha\beta k\ell}\frac{\partial \gamma_{\ell mq}}{\partial y_k} + C_{\alpha\beta m\ell}\gamma_{q\ell}\big)\, dy \\
&\qquad + \int_Y \rho\chi_{m\alpha q\beta}\, dy - (\overline{\rho})^{-1}\overline{C}_{\alpha jmq}\int_Y \rho\gamma_{m\beta}\, dy\Big) \\
&-\ \epsilon \frac{\partial^3 U^q}{\partial x_\alpha \partial x_i \partial x_m}\int_Y \big(C_{\alpha\beta k\ell}\frac{\partial \chi_{imq\ell}}{\partial y_k} - C_{\alpha\beta i\ell}\chi_{\ell mq}\big)\, dy. \\
&-\ \epsilon\omega^2 \frac{\partial U^q}{\partial x_m}\int_Y \big(-\rho\chi_{\beta mq} + C_{m\beta k\ell}\frac{\partial \gamma_{q\ell}}{\partial y_k}\big)\, dy + O(\epsilon^3).
\end{aligned}
$$

Hence the fourth-order equation can be conveniently casted as

$$
\begin{aligned}
&\nabla\cdot\big(\overline{\mathbf{C}}\nabla\mathbf{U}\big) + \omega^2\overline{\rho}\mathbf{U} \\
=& -\epsilon^2\big(\mathbf{D} : \nabla^4\mathbf{U} + \omega^2\mathbf{E} : \nabla^2\mathbf{U}\big) - \epsilon\big(\mathbf{F} : \nabla^3\mathbf{U} + \omega^2\mathbf{G} : \nabla\mathbf{U}\big) + O(\epsilon^3), \qquad (80)
\end{aligned}
$$

where $\mathbf{D} = (D_{\beta\alpha imnq})$ is a sixth-order tensor, $\mathbf{E} = (E_{\beta\alpha mq})$ is a fourth-order tensor, $\mathbf{F} = (F_{\beta\alpha imq})$ is a fifth-order tensor and $\mathbf{G} = (G_{\beta mq})$ is a third-order tensors, respectively defined by

$$
\begin{aligned}
D_{\beta\alpha imnq} &= \int_Y \big(C_{\alpha\beta k\ell}\frac{\partial \chi_{inmq\ell}}{\partial y_k} + C_{\alpha\beta i\ell}\chi_{mnq\ell}\big)\ dy,\\
E_{\beta\alpha mq} &= \int_Y \big(C_{\alpha\beta k\ell}\frac{\partial \gamma_{\ell mq}}{\partial y_k} + C_{\alpha\beta m\ell}\gamma_{q\ell}\big)\ dy + \int_Y \rho\chi_{m\alpha q\beta}\ dy\\
&\quad -(\overline{\rho})^{-1}\overline{C}_{\alpha jmq}\int_Y \rho\gamma_{m\beta}\ dy,\\
F_{\beta\alpha imq} &= \int_Y \big(C_{\alpha\beta k\ell}\frac{\partial \chi_{imq\ell}}{\partial y_k} - C_{\alpha\beta i\ell}\chi_{\ell mq}\big)\ dy,\\
G_{\beta mq} &= \int_Y \big(-\rho\chi_{\beta mq} + C_{m\beta k\ell}\frac{\partial \gamma_{q\ell}}{\partial y_k}\big)\ dy.
\end{aligned}
$$

The fourth-order partial differential equation (80) in Ω formally introduce the dispersion as is seen from the right hand side of (80). We remark that our formal derivation holds for $\mathbf{C}$ and ρ that belong to $L^\infty(\Omega)$. In the low-frequency long-wavelength regime for wave propagation in periodic media, the dispersive wave equation has been demonstrated by a fourth-order partial differential equation for the acoustic case [7].

In the particular case that the periodic media occupies $\mathbb{R}^d$, (80) models the wave propagation that transcends the quasi-static regime. For more details we can recast the higher-order homogenization result in [11,12] when the one-dimensional unit cell $Y=(0,1)$ is composed of two homogeneous phases:

$$
\begin{aligned}
\rho(x) = \widetilde{\rho}_1,\, C(x) = \widetilde{C}_1 \quad &\text{for} \quad 0 < x < \alpha,\\
\rho(x) = \widetilde{\rho}_2,\, C(x) = \widetilde{C}_2 \quad &\text{for} \quad \alpha < x < 1,
\end{aligned}
$$

where $\widetilde{C}_1$, $\widetilde{C}_2$, $\widetilde{\rho}_1$, $\widetilde{\rho}_2$ are all constants, and $0<\alpha<1$. With these notations, the elastic scattering problem in a one-dimensional domain $\Omega=(0,L)$, where the homogenized constants $\overline{\rho}$, $\overline{C}$, are given by [11,12]

$$
\overline{\rho} = \alpha\widetilde{\rho}_1 + (1-\alpha)\widetilde{\rho}_2 \quad \text{and} \quad \overline{C} = \frac{\widetilde{C}_1\widetilde{C}_2}{\alpha\widetilde{C}_2 + (1-\alpha)\widetilde{C}_1}.
$$

Furthermore, constants D, E, F, and G can be explicitly calculated as

$$
\begin{aligned}
D &= \frac{\alpha^2(1-\alpha)^2\overline{C}(\widetilde{C}_1-\widetilde{C}_2)(\widetilde{C}_1\widetilde{\rho}_1-\widetilde{C}_2\widetilde{\rho}_2)}{12\overline{\rho}(\alpha\widetilde{C}_2+(1-\alpha)\widetilde{C}_1)^2},\\
E &= \frac{\alpha^2(1-\alpha)^2\overline{C}(\widetilde{\rho}_1-\widetilde{\rho}_2)(\widetilde{C}_2\widetilde{\rho}_2-\widetilde{C}_1\widetilde{\rho}_1)}{12\overline{\rho}\widetilde{C}_2\widetilde{C}_1},\\
F &= G = 0.
\end{aligned}
$$

Therefore, for the one-dimensional case, (80) can be read as a second order constant coefficients ordinary differential equation:

$$
\overline{C}\frac{\partial^2 U}{\partial^2 x} + \omega^2\overline{\rho}U = -\epsilon^2\left(D\frac{\partial^4 U}{\partial x^4} + \omega^2 E\frac{\partial^2 U}{\partial x^2}\right) + O(\epsilon^3), \tag{81}
$$

where constants $\overline{C}$, $\overline{\rho}$, D, E are given as above. Equation (81) is the time-harmonic analog to the higher-order homogenization in both space and time [11,12].

To further understand the high-order homogenization for periodic media in bounded domains where boundary correctors play an important role, we provide in the next section the numerical study in one dimension.

4 Numerical examples

In this section, we illustrate the higher-order asymptotics by examples in one dimension. We do not use bold symbols throughout this section. We consider the one dimensional periodic structure where ρ and C are Y-periodic with $Y=(0,1)$. In the unit cell Y, ρ and C are given by:

$$\rho(x)=\widetilde{\rho}_1,\, C(x)=\widetilde{C}_1 \quad \text{for} \quad 0<x<\alpha, \tag{82}$$

$$\rho(x)=\widetilde{\rho}_2,\, C(x)=\widetilde{C}_2 \quad \text{for} \quad \alpha<x<1, \tag{83}$$

where $\widetilde{C}_1$, $\widetilde{C}_2$ are constants, while $\widetilde{\rho}_1$, $\widetilde{\rho}_2$ are constants or functions, and $0<\alpha<1$. Such examples has been studies in [11,12] where the periodic media occupies the whole space. The periodic structure only occupies in a bounded domain $\Omega=(0,L)$ in our case, and it will be shown that the boundary correctors play an important role. With these notations, the elastic scattering problem in a one-dimensional domain Ω is

$$\begin{cases} \frac{\partial}{\partial x}\left(C(\frac{x}{\epsilon})\frac{\partial u^\epsilon}{\partial x}\right)+\omega^2\rho(\frac{x}{\epsilon})u^\epsilon=0 & \text{in } \Omega,\\ \frac{\partial^2}{\partial x^2}u^\epsilon+\omega^2u^\epsilon=0 & \text{in } \mathbb{R}\backslash\overline{\Omega},\\ (u^\epsilon+u^i)^+=(u^\epsilon)^- & \text{at } x=0 \text{ and } x=L,\\ \left(\frac{\partial}{\partial x}(u^\epsilon+u^i)\cdot\nu\right)^+=\left(C(\frac{x}{\epsilon})\frac{\partial u^\epsilon}{\partial x}\cdot\nu\right)^- & \text{at } x=0 \text{ and } x=L, \end{cases} \tag{84}$$

where u^ϵ satisfies the radiation condition at ∞. Here u^ϵ restricted in Ω represents the total wave field, while u^ϵ restricted in $\mathbb{R}\backslash\overline{\Omega}$ represents the scattered wave field. Recall that the solution u^ϵ has the following asymptotic expansion

$$u^\epsilon=u^{(0)}+\epsilon u^{(1)}+\epsilon^2u^{(2)}+\cdots.$$

Following from equation (6), we first write down the governing equation for $u^{(0)}$

$$\begin{cases} \frac{\partial}{\partial x}\left(\overline{C}\frac{\partial u^{(0)}}{\partial x}\right)+\omega^2\overline{\rho}u^{(0)}=0 & \text{in } \Omega,\\ \frac{\partial^2}{\partial x^2}u^{(0)}+\omega^2u^{(0)}=0 & \text{in } \mathbb{R}\backslash\overline{\Omega},\\ (u^{(0)}+u^i)^+=(u^{(0)})^- & \text{at } x=0 \text{ and } x=L,\\ \left(\frac{\partial}{\partial x}(u^{(0)}+u^i)\cdot\nu\right)^+=\left(\overline{C}\frac{\partial u^{(0)}}{\partial x}\cdot\nu\right)^- & \text{at } x=0 \text{ and } x=L, \end{cases} \tag{85}$$

where the constants of homogenization $\overline{\rho}$ and $\overline{C}$ can be seen from [11,12] as

$$\overline{\rho}=\int_Y\rho(y)dy,\qquad \overline{C}=\frac{\widetilde{C}_1\widetilde{C}_2}{\alpha\widetilde{C}_2+(1-\alpha)\widetilde{C}_1}.$$

From Theorem 1 we have that

$$\|u^\epsilon-u^{(0)}\|_{L^2(B_R)}\le C_R\epsilon\|u^{(0)}\|_{H^2(\Omega)}, \tag{86}$$

where $\Omega\subset B_R$.

Let us further write down the expressions of $u^{(1)}$ and the boundary corrector φ^ϵ. In the case when $\widetilde{C}_1 = \widetilde{C}_2 = C$, the function χ given by equation (7) is zero, and consequently $u^{(1)}$ defined by (49) is zero since $\tilde{u}^{(1)}$ is zero in one dimension. Now the governing equation (41) for the boundary corrector φ^ϵ can be simplified to

$$\begin{cases} \nabla\cdot\big(C\frac{\partial\varphi^\epsilon}{\partial x}\big) + \omega^2\rho(\frac{x}{\epsilon})\varphi^\epsilon = 0 & \text{in } \Omega, \\ \frac{\partial^2}{\partial x^2}\varphi^\epsilon + \omega^2\varphi^\epsilon = 0 & \text{in } \mathbb{R}\setminus\overline{\Omega}, \\ (\ \varphi^\epsilon)^+ - (\varphi^\epsilon)^- = 0 & \text{at } x=0 \text{ and } x=L, \\ (\frac{\partial}{\partial x}\varphi^\epsilon\cdot\nu)^+ - \big(\overline{C}\frac{\partial}{\partial x}\varphi^\epsilon\cdot\nu\big)^- = \omega^2 C\frac{\partial\gamma(0)}{\partial x}u^{(0)}(0)\cdot\nu & \text{at } x=0, \\ (\frac{\partial}{\partial x}\varphi^\epsilon\cdot\nu)^+ - \big(\overline{C}\frac{\partial}{\partial x}\varphi^\epsilon\cdot\nu\big)^- = \omega^2 C\frac{\partial\gamma(1)}{\partial x}u^{(0)}(1)\cdot\nu & \text{at } x=L, \end{cases} \tag{87}$$

where γ is given by equation (24), i.e.

$$\begin{cases} \dfrac{\partial}{\partial y}\Big(C\dfrac{\partial\gamma}{\partial y}\Big) = \big(\overline{\rho}-\rho\big) \quad \text{in} \quad Y, \\ \int_Y \gamma(y)dy = 0. \end{cases} \tag{88}$$

From Theorem 2 we have that

$$\|u^\epsilon - u^{(0)} - \epsilon u^{(1)} - \epsilon\varphi^\epsilon\|_{L^2(B_R)} \leq C_R\epsilon^2\|u^{(0)}\|_{H^4(\Omega)}, \tag{89}$$

where $\Omega\subset B_R$.

In the following we illustrate the performance of our higher-order homogenization. We use NGSolve [25] to compute the exact solution u^ϵ to (84), the leading-order approximation $u^{(0)}$ to (85), and the boundary corrector φ^ϵ to (87). In all the numerical examples we choose $\Omega = (0,1)$, $\epsilon = 0.1$, $\alpha = \frac{1}{2}$, $\omega = 1$, and $\tilde{C}_1 = \tilde{C}_2 = 1$. We choose $B_R = (-1,2)$, and the computational domain is $(-2,3)$ where PML was implemented in $(-2,-1)\cup(2,3)$. For the computation of γ to (88), we impose periodic boundary conditions on Y and compute it using NGSolve.

We plot the exact solution u^ϵ, leading-order approximation $u^{(0)}$, and higher-order approximation $u^{(0)} + \epsilon u^{(1)} + \epsilon\varphi^\epsilon$ in the domain $\Omega = (0,1)$. The first numerical example (Fig. 1) is for $\tilde{\rho}_1 = 2.4$ and $\tilde{\rho}_2 = 0.8$, the computed error for the leading-order approximation is $\|u^\epsilon - u^{(0)}\|_{L^2(B_R)} = 1.78e-2$, and the computed error for the higher-order approximation is $\|u^\epsilon - u^{(0)} - \epsilon u^{(1)} - \epsilon\varphi^\epsilon\|_{L^2(B_R)} = 1.90e-4$. This agrees with the ϵ-convergence in the leading-order homogenization as indicated by (86) and ϵ^2-convergence in the higher-order homogenization as indicated by (89). The second numerical example (Fig. 2) is for $\tilde{\rho}_1 = 2+\sin(2\pi x)/2$ and $\tilde{\rho}_2 = 2+\sin(2\pi x)/2$, the computed error for the leading-order approximation is $\|u^\epsilon - u^{(0)}\|_{L^2(B_R)} = 6.36e{-3}$, and the computed error for the higher-order approximation is $\|u^\epsilon - u^{(0)} - \epsilon u^{(1)} - \epsilon\varphi^\epsilon\|_{L^2(B_R)} = 7.76e{-05}$. The third numerical example (Fig. 3) is for $\tilde{\rho}_1 = 1$ and $\tilde{\rho}_2 = 0.4$, the computed error for the leading-order approximation is $\|u^\epsilon - u^{(0)}\|_{L^2(B_R)} = 7.84e-3$, and the computed error for the higher-order approximation is $\|u^\epsilon - u^{(0)} - \epsilon u^{(1)} - \epsilon\varphi^\epsilon\|_{L^2(B_R)} = 7.37e-05$.

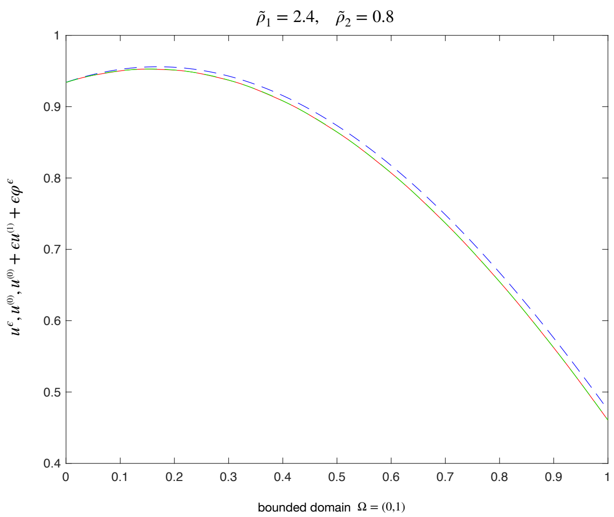

Fig. 1 $\tilde{\rho}_1 = 2.4$ and $\tilde{\rho}_2 = 0.8$. Red: exact solution u^ϵ; Blue: leading-order approximation $u^{(0)}$; Green: higher-order approximation $u^{(0)} + \epsilon u^{(1)} + \epsilon\varphi^\epsilon$.

We also remark that the boundary corrector φ^ϵ can be approximated by its leading-order approximation $\varphi^{(0)}$ satisfying

$$\begin{cases} \nabla\cdot\left(\overline{C}\frac{\partial\varphi^{(0)}}{\partial x}\right) + \omega^2\overline{\rho}\varphi^{(0)} = 0 & \text{in } \Omega, \\ \frac{\partial^2}{\partial x^2}\varphi^{(0)} + \omega^2\varphi^{(0)} = 0 & \text{in } \mathbb{R}\setminus\overline{\Omega}, \\ (\,\varphi^{(0)})^+ - (\varphi^{(0)})^- = 0 & \text{at } x=0 \text{ and } x=L, \\ (\frac{\partial}{\partial x}\varphi^{(0)}\cdot\nu)^+ - \left(\overline{C}\frac{\partial}{\partial x}\varphi^{(0)}\cdot\nu\right)^- = \omega^2 C\frac{\partial\gamma(0)}{\partial x}u^{(0)}(0)\cdot\nu & \text{at } x=0, \\ (\frac{\partial}{\partial x}\varphi^{(0)}\cdot\nu)^+ - \left(\overline{C}\frac{\partial}{\partial x}\varphi^{(0)}\cdot\nu\right)^- = \omega^2 C\frac{\partial\gamma(1)}{\partial x}u^{(0)}(1)\cdot\nu & \text{at } x=L, \end{cases} \tag{90}$$

We illustrate this approximation by the fourth numerical example (Fig. 4), where all the setup is the same as the first example except that we approximate φ^ϵ by $\varphi^{(0)}$, the computed error for the leading-order approximation is $\|u^\epsilon - u^{(0)}\|_{L^2(B_R)} = 1.78e-2$, and the computed error for the higher-order approximation is $\|u^\epsilon - u^{(0)} - \epsilon u^{(1)} - \epsilon\varphi^{(0)}\|_{L^2(B_R)} = 3.41e-4$. These preliminary examples clearly demonstrate the performance of our high-order homogenization.

5 Appendix

In the end of this paper, we offer basic materials in analysing the elastic scattering in periodic media.

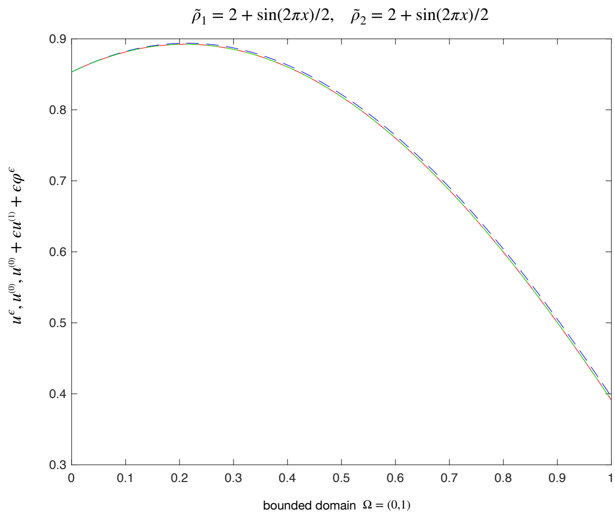

Fig. 2 $\tilde{\rho}_1 = 2+\sin(2\pi x)/2$ and $\tilde{\rho}_2 = 2+\sin(2\pi x)/2$. Red: exact solution u^ϵ; Blue: leading-order approximation $u^{(0)}$; Green: higher-order approximation $u^{(0)} + \epsilon u^{(1)} + \epsilon\varphi^\epsilon$.

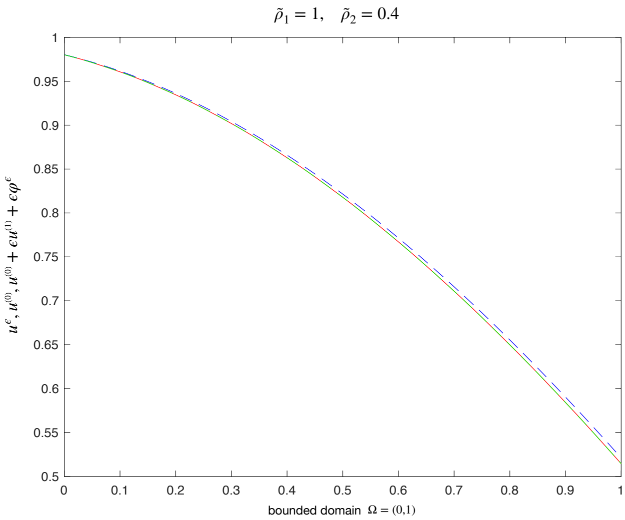

Fig. 3 $\tilde{\rho}_1 = 1$ and $\tilde{\rho}_2 = 0.4$. Red: exact solution u^ϵ; Blue: leading-order approximation $u^{(0)}$; Green: higher-order approximation $u^{(0)} + \epsilon u^{(1)} + \epsilon\varphi^\epsilon$.

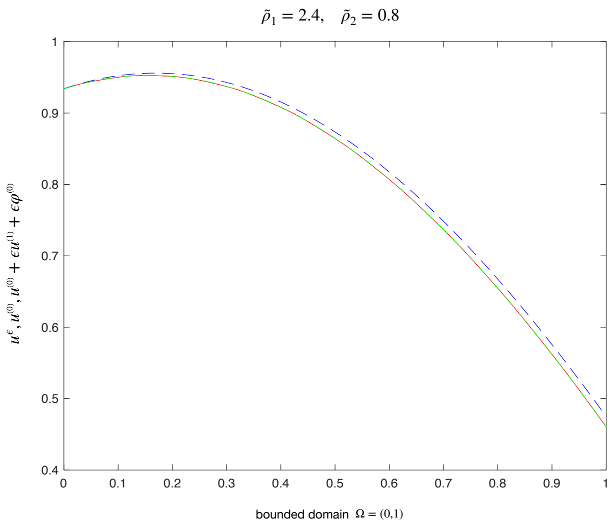

Fig. 4 $\tilde{\rho}_1 = 2.4$ and $\tilde{\rho}_2 = 0.8$. Red: exact solution u^ϵ; Blue: leading-order approximation $u^{(0)}$; Green: higher-order approximation $u^{(0)} + \epsilon u^{(1)} + \epsilon\varphi^{(0)}$.

5.1 The Dirichlet to Neumann map

Let $\mathbf{u}$ satisfy the Navier's equation in the exterior domain

$$\Delta^* \mathbf{u} + \omega^2 \mathbf{u} = 0 \text{ in } \mathbb{R}^d \backslash \overline{\Omega},$$

and $\mathbf{u}$ has a decomposition that satisfies the *Kupradze radiation condition*. Let B_R be a sufficiently large ball such that $\Omega \subset B_R$. In the case that $\Omega \subset \mathbb{R}^3$, we introduce the polar coordinates r, θ, ϕ and the unit vectors $\widehat{r}, \widehat{\theta}, \widehat{\phi}$. The θ coordinate corresponds to the angle from the z-axis, $\theta \in [0, \pi]$, and the ϕ coordinate corresponds to the angle in the (x, y)-plane, $\phi \in [0, 2\pi]$. Let Y_{nm} be the spherical harmonic

$$Y_{nm}(\theta, \phi) = \sqrt{\frac{(2n+1)(n-|m|)!}{4\pi(n+|m|!)}} P_n^{|m|}(\cos\theta) e^{im\phi}, \quad n \geq 0, \quad |m| < n.$$

Now we let U_{nm} and V_{nm} be the vector spherical harmonics defined by

$$U_{nm}(\theta, \phi) = \frac{1}{\sqrt{\lambda_n}} \big(\frac{\partial Y_{nm}}{\partial \theta} \widehat{\theta} + \frac{1}{\sin\theta} \frac{\partial Y_{nm}}{\partial \phi} \widehat{\phi} \big), \quad n \geq 1,$$
$$V_{nm}(\theta, \phi) = \widehat{r} \times U_{nm} = \frac{1}{\sqrt{\lambda_n}} \big(-\frac{1}{\sin\theta} \frac{\partial Y_{nm}}{\partial \phi} \widehat{\theta} + \frac{\partial Y_{nm}}{\partial \theta} \widehat{\phi} \big), \quad n \geq 1,$$

where $\lambda_n = n(n+1)$. The vectors $Y_{nm}\widehat{r}, U_{nm}, V_{nm}$ form an orthonormal basis for $L^2(S)$ where S denotes the unit sphere. Then $\mathbf{u}$ on ∂B_R has the following series expansion

$$\mathbf{u}=\sum_{n=0}^{\infty}\sum_{|m|<n}\Big((u|_{\partial B_R},V_{nm})V_{nm}+(u|_{\partial B_R},U_{nm})U_{nm}\qquad(91)$$
$$+(u|_{\partial B_R},Y_{nm}\hat{r})Y_{nm}\hat{r}\Big),$$

where $(\cdot,\cdot)$ denotes the $L^2(S)$ inner product. One can correspondingly express $T_{\boldsymbol{\nu}}\mathbf{u}$ on ∂B_R as (see [13])

$$T_{\boldsymbol{\nu}}\mathbf{u}=\sum_{n=0}^{\infty}\sum_{|m|<n}\Big(a_n(u|_{\partial B_R},V_{nm})V_{nm}+\big[b_n(u|_{\partial B_R},U_{nm})+c_n(u|_{\partial B_R},Y_{nm}\widehat{r})\big]U_{nm}$$
$$+\big[c_n(u|_{\partial B_R},u_{nm})+d_n(u|_{\partial B_R},Y_{nm}\widehat{r})\big]Y_{nm}\widehat{r}\Big)\quad\text{on}\quad B_R.\qquad(92)$$

The coefficients a_n,b_n,c_n,d_n are given by

$$a_n=\mu_0(\gamma_s-\frac{1}{R}),$$
$$b_n=\Big(2\mu_0\sqrt{\lambda_n}(\gamma_p-\frac{1}{R})\Big)\frac{B_n^{(1,1)}}{R}+\mu_0\Big(2\gamma_s+R\omega_s^2+2(1-\lambda_n)\frac{1}{R}\Big)\frac{B_n^{(2,1)}}{R},$$
$$c_n=\Big(2\mu_0\sqrt{\lambda_n}(-\gamma_s+\frac{1}{R})\Big)\frac{B_n^{(2,1)}}{R}+\Big(2\mu_0(-2\gamma_p+\frac{\lambda_n}{R})\Big)\frac{B_n^{(1,1)}}{R-\mu_0\omega_s^2},$$
$$d_n=-\Big(2\mu_0\sqrt{\lambda_n}(-\gamma_s+\frac{1}{R})\Big)\frac{B_n^{(1,1)}}{R}+\Big(2\mu_0(-2\gamma_p+\frac{\lambda_n}{R})\Big)\frac{B_n^{(1,2)}}{R-\mu_0\omega_s^2},$$

where

$$\gamma_s=\omega_s\frac{h_n^{'}(\omega_sR)}{h_n(\omega_sR)},\quad\gamma_p=\omega_p\frac{h_n^{'}(\omega_pR)}{h_n(\omega_pR)},\quad B_n^{(1,1)}=-\frac{\sqrt{\lambda_n}R}{R\gamma_p(R\gamma_s+1)-\lambda_n},$$
$$B_n^{(1,2)}=\frac{R(1+R\gamma_s)}{R\gamma_p(R\gamma_s+1)-\lambda_n},\quad B_n^{(1,1)}=-\frac{R^2\gamma_p}{R\gamma_p(R\gamma_s+1)-\lambda_n}.$$

Now for any functions $\mathbf{w}$ and $\mathbf{u}$ that satisfy the Kupradze radiation condition (4), one can directly obtain from (91) and (92) that

$$\int_{\partial B_R}T_{\boldsymbol{\nu}}\mathbf{u}\cdot\mathbf{w}dS-\int_{\partial B_R}T_{\boldsymbol{\nu}}\mathbf{w}\cdot\mathbf{u}dS=0.$$

We remark that when $\Omega\subset\mathbb{R}^2$, the above equality can be derived in a similar way [4].

Let $B_R\subset\mathbb{R}^d$ be a ball of radius $R>0$, then the Dirichlet to Neumann (DN) map was given by [4].

Definition 1 For any $\mathbf{g}\in\big(H^{1/2}(\partial B_R)\big)^d$, the DN map

$$\Lambda:\big(H^{1/2}(\partial B_R)\big)^d\to\big(H^{-1/2}(\partial B_R)\big)^d\quad\text{with}\quad\Lambda\mathbf{g}|_{\partial B_R}=T_{\boldsymbol{\nu}}\mathbf{u}|_{\partial B_R},\qquad(93)$$

where $\mathbf{u}\in\big(H^1_{loc}(\mathbb{R}^d\setminus\overline{B}_R)\big)^d$ is a solution of the Navier's equation $\Delta^*\mathbf{u}+\omega^2\mathbf{u}=0$ in $\mathbb{R}^d\setminus\overline{B_R}$ and $\mathbf{u}$ satisfies the Kupradze radiation condition (4) at infinity.

Notice that the DN map Λ is a bounded operator, so that it helps to reduce the scattering problem in unbounded domain to a bounded domain, and we refer readers to [4, Section 2] for detailed discussions.

5.2 Derivation of the homogenized equation

Consider the simplest linear elliptic system of the homogenization theory. The periodic homogenization theory was studied by [10,14] and we refer readers to these references for the comprehensive study. We are concerned with the divergence form second order elliptic operators with rapidly oscillating periodic coefficients,

$$\mathcal{L}_\epsilon := -\nabla\cdot\left(\mathbf{A}(\frac{x}{\epsilon})\nabla\right) = -\frac{\partial}{\partial x_i}\left(a_{ijk\ell}(\frac{x}{\epsilon})\frac{\partial}{\partial x_k}\right), \quad \epsilon>0.$$

We assume the coefficients $\mathbf{A}(y)=(a_{ijk\ell}(y))$ with $1\le i,j,k,\ell\le d$ for the dimension $d\ge 2$ is real, bounded and measurable such that $\mathbf{A}$ satisfies

$$\text{ellipticity: } \mu\sum_{i,j=1}^{d}|\varepsilon_{ij}|^2 \le a_{ijk\ell}(y)\varepsilon_{ij}\varepsilon_{k\ell} \le \frac{1}{\mu}\sum_{i,j=1}^{d}|\varepsilon_{ij}|^2, \tag{94}$$

for all symmetric matrix $(\varepsilon_{ij})_{1\le i,j\le d}$, and

$$\text{Y-periodicity: } \mathbf{A}(y+z)=\mathbf{A}(y) \text{ for all } y\in\mathbb{R}^d,\ z\in Y:=[0,1]^d,$$

for some constant $\mu>0$.

Given $\mathbf{F}\in\left(H^{-1}(\Omega)\right)^d$, let $\mathbf{u}^\epsilon\in\left(H_0^1(\Omega)\right)^d$ be a solution of

$$\mathcal{L}_\epsilon\mathbf{u}^\epsilon=\mathbf{F} \text{ in } \Omega, \tag{95}$$

where Ω is a bounded Lipschitz domain in $\mathbb{R}^d$. By the Lax-Milgram theorem, we have

$$\|\mathbf{u}^\epsilon\|_{H_0^1(\Omega)}\le C\|\mathbf{F}\|_{H^{-1}(\Omega)},$$

where the constant C independent of ϵ. Note that $\mathbf{u}^\epsilon\in\left(H_0^1(\Omega)\right)^d$ is a weak solution of (95) if for all $\boldsymbol{\varphi}\in\left(H_0^1(\Omega)\right)^d$, we have

$$\int_\Omega\left(\mathbf{A}(\frac{x}{\epsilon})\nabla\mathbf{u}^\epsilon\right):\nabla\boldsymbol{\varphi}\,dx=\langle\mathbf{F},\boldsymbol{\varphi}\rangle_{H^{-1}(\Omega)\times H_0^1(\Omega)}.$$

Next, we want to derive the homogenized equation by using the following asymptotic analysis. We consider $\mathbf{u}^\epsilon$ to be the perturbation of $\mathbf{u}^{(0)}$ with respect to ϵ-parameter. Moreover, by observing the elliptic operator $\mathcal{L}_\epsilon$, we introduce the famous *two-scale homogenization method* in the homogenization theory: Let us regard $x=x$, and $y=\frac{x}{\epsilon}$ as two independent parameters. Let

$$\mathbf{u}^\epsilon:=\mathbf{u}^{(0)}+\epsilon\mathbf{u}^{(1)}+\epsilon^2\mathbf{u}^{(2)}+\cdots$$

be the asymptotic expansion of u_ϵ, where

$$\mathbf{u}^{(j)}:=\mathbf{u}^{(j)}(x,y)=\mathbf{u}^{(j)}(x,\frac{x}{\epsilon}).$$

In addition,

$$\nabla\mathbf{u}^{(j)}=\nabla_x\mathbf{u}^{(j)}(x,y)+\frac{1}{\epsilon}\nabla_y\mathbf{u}^{(j)}(x,y), \text{ as } y=\frac{x}{\epsilon},$$

which means under our two-scaled method, the operator $\nabla = \nabla_x + \frac{1}{\epsilon}\nabla_y$. Therefore, (95) will become

$$-\left(\nabla_x + \frac{1}{\epsilon}\nabla_y\right)\cdot\left\{\mathbf{A}(y)\left[\left(\nabla_x + \frac{1}{\epsilon}\nabla_y\right)\left(\mathbf{u}^{(0)} + \epsilon\mathbf{u}^{(1)} + \epsilon^2\mathbf{u}^{(2)} + \cdots\right)\right]\right\} = \mathbf{F}(x) \text{ in } \Omega. \tag{96}$$

We point out that the derivation of the homogenized equation did not need to take care of the boundary condition of certain equations. Expand (96) and compare it with the same ϵ^N-orders (for $N = 0, -1, -2$), so we get

$$\begin{aligned} O(\frac{1}{\epsilon^2}): &\quad -\nabla_y \cdot (\mathbf{A}(y)\nabla_y \mathbf{u}^{(0)}(x,y)) = 0, \\ O(\frac{1}{\epsilon}): &\quad -\nabla_y \cdot (\mathbf{A}(y)\nabla_y \mathbf{u}^{(1)}(x,y)) = \nabla_y \cdot (\mathbf{A}(y)\nabla_x \mathbf{u}^{(1)}) + \nabla_x \cdot (\mathbf{A}(y)\nabla_y \mathbf{u}^{(0)}), \\ O(1): &\quad -\nabla_y \cdot (\mathbf{A}(y)\nabla_y \mathbf{u}^{(2)}(x,y)) = \nabla_y \cdot (\mathbf{A}(y)\nabla_x \mathbf{u}^{(1)}) + \nabla_x \cdot (\mathbf{A}(y)\nabla_y \mathbf{u}^{(1)}) \\ &\qquad\qquad + \nabla_x \cdot (\mathbf{A}(y)\nabla_x \mathbf{u}^{(0)}) + \mathbf{F}(x). \end{aligned} \tag{97}$$

Recall that for the periodic elliptic equation

$$-\nabla \cdot (\mathbf{A}(y)\nabla \mathbf{v}(y)) = \mathbf{h}(y), \text{ whenever } \mathbf{A}(y) \text{ is Y-periodic,}$$

then we have

$$\int_Y \mathbf{h}(y)dy = 0,$$

by using the divergence theorem. For $O(\frac{1}{\epsilon^2})$ term, this equation is solvable because the right hand side is zero. In further, we multiply $\mathbf{u}^{(0)}(x,y)$ on both sides and integrate by parts, which will imply

$$0 = \int_Y \left(\mathbf{A}(y)\nabla_y \mathbf{u}^{(0)}\right) : \nabla_y \mathbf{u}^{(0)} \geq \mu \int_Y |\nabla_y \mathbf{u}^{(0)}(x,y)|^2 dy \geq 0,$$

which gives us the information that

$$\mathbf{u}^{(0)}(x,y) \equiv \mathbf{u}^{(0)}(x)$$

and we know that $\mathbf{u}_0$ is independent of y.

Now, for the second term $O(\frac{1}{\epsilon})$, the second term on the right hand side should be zero since $\nabla_y \mathbf{u}^{(0)}(x) = 0$. Solve the equation

$$-\nabla_y \cdot (\mathbf{A}(y)\nabla_y \mathbf{u}^{(1)}(x,y)) = \nabla_y \cdot (\mathbf{A}(y)\nabla_x \mathbf{u}^{(0)}) = (\nabla_y \cdot \mathbf{A}(y))(\nabla_x \mathbf{u}^{(0)})$$

formally. Note that since $\mathbf{A}(y)$ is Y-periodic, then the equation is solvable for $\mathbf{u}^{(1)}$ if

$$\int_Y (\nabla_y \cdot \mathbf{A}(y)) \cdot (\nabla_x \mathbf{u}^{(0)})dy = \int_{\partial Y} (\mathbf{A}(y)\nabla_x \mathbf{u}^{(0)}) \cdot \boldsymbol{\nu}(y)dS(y) = 0.$$

By using the separation of variables, we put the *ansatz*

$$\mathbf{u}^{(1)}(x,y) = \boldsymbol{\chi}(y) \cdot (\nabla_x \mathbf{u}^{(0)}(x))$$

with $\mathbf{u}^{(1)} = (u_\alpha^{(1)})_{1\le\alpha\le d}$ such that

$$u_\alpha^{(1)}(x,y) = \chi_{\alpha j\beta}(y)\frac{\partial u_\beta^{(0)}}{\partial x_j}(x).$$

Moreover, the corrector $\chi_{\alpha j\beta}$ is Y-periodic and solves the cell problem

$$\begin{cases} \dfrac{\partial}{\partial y_i}\left(a_{ijmn} - a_{ijk\ell}\dfrac{\partial}{\partial y_k}\chi_{\ell mn}\right) = 0 \text{ in } Y, \\ \int_Y \chi_{\ell mn}(y)dy = 0, \end{cases}$$

and plug $\mathbf{u}^{(1)}$ to the $O(\frac{1}{\epsilon})$ equation (97) to obtain

$$-\nabla\cdot(\mathbf{A}(y)\nabla_y\boldsymbol{\chi}(y))(\nabla_x\mathbf{u}^{(0)}) = (\nabla_y\cdot\mathbf{A}(y))(\nabla_x\mathbf{u}^{(0)}).$$

Finally plug $\mathbf{u}^{(1)}(x,y) = \boldsymbol{\chi}(y)\nabla_x\mathbf{u}^{(0)}$ into the $O(1)$ equation and examine the solvability condition for $\mathbf{u}^{(2)}(x,y)$, we have

$$\begin{aligned} 0 &= \int_Y \left[\nabla_y\cdot(\mathbf{A}(y)\nabla_x\mathbf{u}^{(1)}) + \nabla_x\cdot(\mathbf{A}(y)\nabla_y\mathbf{u}^{(1)}) + \nabla_x\cdot(\mathbf{A}(y)\nabla_x\mathbf{u}^{(0)}) + \mathbf{F}(x)\right]dy \\ &= \nabla_x\cdot\left\{\left[\int_Y \mathbf{A}(y)(\nabla_y\boldsymbol{\chi}(y))dy\right]\nabla_x\mathbf{u}^{(0)}\right\} + \nabla_x\cdot\left\{\left[\int_Y \mathbf{A}(y)dy\right]\nabla_x\mathbf{u}^{(0)}\right\} + \mathbf{F}(x), \end{aligned}$$

where the first term vanishes by the periodicity of $\mathbf{A}$ and $\boldsymbol{\chi}$. Thus, we can obtain that $\mathbf{u}^{(0)} \in \left(H_0^1(\Omega)\right)^d$ is a solution of

$$\overline{\mathcal{L}}\mathbf{u}^{(0)} := -\nabla\cdot(\overline{\mathbf{A}}\nabla\mathbf{u}^{(0)}) = \mathbf{F}(x) \text{ in } \Omega, \tag{98}$$

where

$$\overline{\mathbf{A}} = \int_Y \{\mathbf{A}(y) + \mathbf{A}(y)(\nabla_y\boldsymbol{\chi}(y))\}\,dy,$$

where $\overline{\mathbf{A}}$ is the (constant) homogenized operator and we call (98) to be the *homogenized equation*. In addition, $\overline{\mathbf{A}} = (\overline{a}_{ijk\ell})_{1\le i,j,k,\ell\le d}$ and

$$\overline{a}_{ijk\ell} = \int_Y \left(a_{ijk\ell} - a_{ijmn}\frac{\partial}{\partial y_m}\chi_{nk\ell}\right)dy. \tag{99}$$

For the rigorous derivation of the homogenized equation, we need to use a famous result, which is called the Div-Curl lemma. We skip the rigorous analysis here and refer readers to the lecture note [26] for more details.

Note that $\overline{\mathcal{L}} := -\nabla\cdot(\overline{\mathbf{A}}\nabla)$ is the homogenized second order elliptic operator with respect to $\mathbf{A}$ and we want to prove $\overline{\mathcal{L}}$ is an elliptic operator with constant coefficients.

Theorem 5 *The homogenized operator $\overline{\mathcal{L}}$ satisfies that*

1. $\overline{\mathcal{L}}$ is an elliptic operator, which means

$$\mu_1\sum_{i,j=1}^d |\varepsilon_{ij}|^2 \le \overline{a}_{ijk\ell}(y)\varepsilon_{ij}\varepsilon_{k\ell} \le \frac{1}{\mu_1}\sum_{i,j=1}^d |\varepsilon_{ij}|^2, \tag{100}$$

for some constant $\mu_1 > 0$.

2. The effective coefficient $\overline{a}_{ijk\ell}$ is major and minor symmetric provided $a_{\alpha\beta\gamma\delta}$ is major and minor symmetric.

Proof It is easy to see that $|\overline{a}_{ijk\ell}| \leq C$ by using (99) and the ellipticity of $A(y)$, for some constant $C > 0$. It remains to show $\overline{a}_{ijk\ell}\varepsilon_{ij}\varepsilon_{k\ell} \geq \mu_1 \sum_{i,j=1}^{d} |\varepsilon_{ij}|^2$ for some constant $\mu_1 > 0$. We can rewrite (99) as

$$\overline{a}_{ijk\ell} = \int_Y \frac{\partial}{\partial y_\alpha} \{\delta_{\beta j} y_i + \chi_{\beta ij}\} \cdot a_{\alpha\beta\gamma\delta} \cdot \frac{\partial}{\partial y_\gamma} \{\delta_{\delta\ell} y_k + \chi_{\delta k\ell}\} \, dy,$$

where $\delta_{s\alpha}$ is the standard Kronecker delta (i.e., $\delta_{s\alpha} = 1$ if $s = \alpha$, and $\delta_{s\alpha} = 0$ otherwise). Hence, for $\varepsilon = (\varepsilon_{ij}) \in \mathbb{R}^{d\times d}$, we have

$$\begin{aligned} \overline{a}_{ijk\ell}\varepsilon_{ij}\varepsilon_{k\ell} &= \int_Y \frac{\partial}{\partial y_\alpha} \{\delta_{\beta j} y_i \varepsilon_{ij} + \chi_{\beta ij}\varepsilon_{ij}\} \cdot a_{\alpha\beta\gamma\delta} \cdot \frac{\partial}{\partial y_\gamma} \{\delta_{\delta\ell} y_k \varepsilon_{k\ell} + \chi_{\delta k\ell}\varepsilon_{k\ell}\} \, dy \\ &\geq \mu \sum_{\beta=1}^{d} \int_Y |\nabla(y_i\varepsilon_{i\beta} + \chi_{\beta ij}\varepsilon_{ij})|^2 dy \geq 0. \end{aligned}$$

If $\overline{a}_{ijk\ell}\varepsilon_{ij}\varepsilon_{k\ell} = 0$ for some $\varepsilon = (\varepsilon_{ij}) \in \mathbb{R}^{d\times d}$, then $y_i\varepsilon_{i\beta} + \chi_{\beta ij}$ must be a constant. Recall that $\chi_{\beta ij}(y)$ is Y-periodic, so this implies that $\varepsilon = 0$. This means that there exists $\mu_1 > 0$ such that (100) holds. □

5.3 Tools and estimates

In the last part, for the completeness of this paper, we provide some elliptic estimate where we have utilized in previous sections. The following theorem was proved in [6, Theorem 5.7] for the scalar case. It will hold for the vector case. For completeness, we provide the theorem and its proof as follows.

Theorem 6 *(Trace Theorem) Let* $\mathbf{A} = (a_{ijk\ell})_{1\leq i,j,k,\ell\leq d}$ *be a four tensor satisfying the ellipticity condition* (100) *and* $\Omega \subset \mathbb{R}^d$ *be a bounded domain with a* C^∞*-smooth boundary, for* $d \geq 2$*. The (cornormal) mapping* $Tr : \mathbf{u} \to \frac{\partial \mathbf{u}}{\partial \boldsymbol{\nu}_{\mathbf{A}}} :=$ $(\mathbf{A}\nabla\mathbf{u})\cdot\boldsymbol{\nu}$ *defined in* $C^\infty(\overline{\Omega})$ *can be continuously extended to a linearly continuous mapping (still denote by* Tr*) from* $H^1(\Omega, \mathbf{A})$ *to* $H^{-1/2}(\partial\Omega)$*, where* $H^1(\Omega, \mathbf{A})$ *is the space equipped with the graph norm*

$$\|\mathbf{u}\|^2_{H^1(\Omega,\mathbf{A})} := \|\mathbf{u}\|^2_{H^1(\Omega)} + \|\nabla\cdot(\mathbf{A}\nabla\mathbf{u})\|^2_{L^2(\Omega)}.$$

Proof Let $\boldsymbol{\varphi} \in \left(C^\infty(\overline{\Omega})\right)^d$ be a test function and $\mathbf{u} \in C^\infty(\overline{\Omega}; \mathbb{R}^d)$. The integration by parts formula gives

$$\int_{\partial\Omega} (\mathbf{A}\nabla\mathbf{u}\cdot\boldsymbol{\nu})\cdot\boldsymbol{\varphi}\, dS = \int_\Omega (\mathbf{A}\nabla\mathbf{u}) : \nabla\boldsymbol{\varphi} dx + \int_\Omega \nabla\cdot(\mathbf{A}\nabla\mathbf{u})\cdot\boldsymbol{\varphi}\, dx.$$

By the standard density arguments, the above equation holds for $\boldsymbol{\varphi} \in \left(H^1(\Omega)\right)^d$ so that

$$\left|\int_{\partial\Omega} (\mathbf{A}\nabla\mathbf{u}\cdot\boldsymbol{\nu})\cdot\boldsymbol{\varphi} dS\right| \leq C\|\mathbf{u}\|_{H^1(\Omega,\mathbf{A})}\|\boldsymbol{\varphi}\|_{H^1(\Omega)}, \tag{101}$$

for any $\boldsymbol{\varphi} \in \left(H^1(\Omega)\right)^d$, $\mathbf{u} \in \left(C^\infty(\overline{\Omega})\right)^d$, where constant $C > 0$ is a constant independent of $\boldsymbol{\varphi}$ and $\mathbf{u}$. Let $\mathbf{g} \in \left(H^{1/2}(\partial D)\right)^d$, by using the trace theorem, then there exists a function $\boldsymbol{\varphi} \in \left(H^1(\Omega)\right)^d$ such that $\gamma_{\partial\Omega}\boldsymbol{\varphi} = \mathbf{f}$, where $\gamma_{\partial\Omega}$ stands for the trace operator. Continuing the inequality (101) and the trace theorem,

$$\left| \int_{\partial\Omega} (\mathbf{A}\nabla u \cdot \boldsymbol{\nu}) \cdot \mathbf{f} dS \right| \leq C\|\mathbf{u}\|_{H^1(\Omega,\mathbf{A})}\|\mathbf{f}\|_{H^{1/2}(\partial\Omega)},$$

for any $\mathbf{f} \in \left(H^{1/2}(\partial)\right)^d$, $\mathbf{u} \in \left(C^\infty(\overline{\Omega})\right)^d$.

Hence, the mapping

$$\mathbf{f} \to \int_{\partial\Omega} (\mathbf{A}\nabla \mathbf{u} \cdot \boldsymbol{\nu}) \cdot \mathbf{f} dS, \text{ for any } \mathbf{f} \in \left(H^{1/2}(\partial\Omega)\right)^d$$

defines a continuous linear operator and from the duality argument,

$$\|(\mathbf{A}\nabla) \cdot \boldsymbol{\nu}\|_{H^{-1/2}(\partial\Omega)} \leq C\|\mathbf{u}\|_{H^1(\Omega,\mathbf{A})}.$$

Therefore, the linear mapping $Tr : \mathbf{u} \to (\mathbf{A}\nabla u) \cdot \boldsymbol{\nu}$ defined on $\left(C^\infty(\overline{\Omega})\right)^d$ is continuous under the norm $H^1(\Omega, \mathbf{A})$. Thus, the assertion follows from the density arguments. □

Let $\mathbf{C} = (C_{ijk\ell})$ be an anisotropic elastic four tensor and $\mathbf{C}_0$ be a constant isotropic elastic tensor defined by (2), which satisfy all the conditions given in Section 1. Next, we provide the stability estimate for the following transmission problem. The scalar case was demonstrated in [6, Section 5] and here we generalize the result to a system version.

Theorem 7 *Let $\Omega \subset \mathbb{R}^d$ be a bounded C^∞-smooth domain. Given $\mathbf{f} \in \left(H^{1/2}(\partial\Omega)\right)^d$ and $\mathbf{g} \in \left(H^{-1/2}(\partial\Omega)\right)^d$. Let $\mathbf{u} \in \left(H^1(\Omega)\right)^d$ and $\mathbf{v} \in \left(H^1_{loc}(\mathbb{R}^d \setminus \overline{\Omega})\right)^d$ be the solutions of the following transmission problem*

$$\begin{cases} \nabla \cdot (\mathbf{C}\nabla \mathbf{u}) + \omega^2 \rho \mathbf{u} = 0 & \text{in } \Omega, \\ \Delta^* \mathbf{v} + \omega^2 \mathbf{v} = 0 & \text{in } \mathbb{R}^d \setminus \overline{\Omega}, \\ \mathbf{u} - \mathbf{v} = \mathbf{f} & \text{on } \partial\Omega, \\ T_{\boldsymbol{\nu}}\mathbf{u} - (\mathbf{C}\nabla \mathbf{u}) \cdot \boldsymbol{\nu} = \mathbf{g} & \text{on } \partial\Omega, \end{cases} \tag{102}$$

where $T_{\boldsymbol{\nu}}$ is the boundary traction operator given by (3)*, $\omega \in \mathbb{R}$ is not an eigenvalue of the transmission problem* (102) *and v satisfies the Kupradze radiation condition* (4)*. Then for any ball B_R with $\Omega \subset B_R$, there exists a constant $C_R > 0$ such that*

$$\|\mathbf{u}\|_{H^1(\Omega)} + \|\mathbf{v}\|_{H^1(B_R \setminus \overline{\Omega})} \leq C_R \left\{ \|\mathbf{f}\|_{H^{1/2}(\partial\Omega)} + \|\mathbf{g}\|_{H^{-1/2}(\partial\Omega)} \right\}. \tag{103}$$

Proof Firstly, by using similar arguments in [4, Section 2] and [6, Section 5], the elastic scattering problem (102) is equivalent to the following transmission problem: Let $\mathbf{u} \in \left(H^1(\Omega)\right)^d$ and $\mathbf{v} \in \left(H^1(B_R \setminus \overline{\Omega})\right)^d$ be the solutions of

$$\begin{cases} \nabla \cdot (\mathbf{C}\nabla \mathbf{u}) + \omega^2 \rho \mathbf{u} = 0 & \text{in } \Omega, \\ \Delta^* \mathbf{v} + \omega^2 \mathbf{v} = 0 & \text{in } B_R \setminus \overline{\Omega}, \\ \mathbf{u} - \mathbf{v} = \mathbf{f} & \text{on } \partial\Omega, \\ T_{\boldsymbol{\nu}}\mathbf{u} - (\mathbf{C}\nabla \mathbf{u}) \cdot \boldsymbol{\nu} = \mathbf{g} & \text{on } \partial\Omega, \\ T_{\boldsymbol{\nu}}\mathbf{v} = \Lambda \mathbf{v} & \text{on } \partial B_R, \end{cases} \tag{104}$$

where Λ is the DN map defined by (93) on ∂B_R. Furthermore, by using [4, Lemma 2.8], the DN map Λ is a bounded operator and Λ can decomposed into $\Lambda = \Lambda_1 + \Lambda_2$, where $-\Lambda_1$ is a positive operator and Λ_2 is a compact operator from $\left(H^{1/2}(\partial B_R)\right)^d$ to $\left(H^{-1/2}(\partial B_R)\right)^d$.

Next, let $\mathbf{v_f} \in \left(H^1(B_R \setminus \overline{\Omega})\right)^d$ be the unique solution of the Navier's equation in the exterior domain

$$\begin{cases} \Delta^* \mathbf{v_f} + \omega^2 \mathbf{v_f} = 0 & \text{in } B_R \setminus \overline{\Omega}, \\ \mathbf{v_f} = \mathbf{f} & \text{on } \partial\Omega, \\ \mathbf{v_f} = 0 & \text{on } \partial B_R. \end{cases}$$

By straight forward calculation, it is not hard to see that the variational formula of (104) can be written as follows: Find a function $w \in H^1(B_R)$ such that

$$\begin{aligned} &\int_\Omega \Big((\mathbf{C}\nabla \mathbf{w}) : \nabla \boldsymbol{\phi} - \omega^2 \rho \mathbf{w} \cdot \boldsymbol{\phi} \Big)\, dx + \int_{B_R \setminus \overline{\Omega}} \Big((\mathbf{C}_0 \nabla \mathbf{w}) : \nabla \boldsymbol{\phi} - \omega^2 \mathbf{w} \cdot \boldsymbol{\phi} \Big)\, dx \\ &\quad - \int_{\partial B_R} \boldsymbol{\phi} \cdot T_{\boldsymbol{\nu}} \mathbf{w}\, dS + \int_{\partial B_R} \boldsymbol{\phi} \cdot T_{\boldsymbol{\nu}} \mathbf{v_f}\, dS \\ &= \int_{\partial\Omega} \mathbf{g} \cdot \boldsymbol{\phi}\, dS + \int_{B_R \setminus \overline{\Omega}} \Big((\mathbf{C}_0 \nabla \mathbf{v_f}) : \nabla \boldsymbol{\phi} - \omega^2 \mathbf{v_f} \cdot \boldsymbol{\phi} \Big)\, dx, \end{aligned} \tag{105}$$

for any test function $\boldsymbol{\phi} \in \left(H^1(B_R)\right)^d$, where $\mathbf{C}_0$ is a constant elastic tensor defined by (2). By using the integration by parts, one can easily see that $\mathbf{u} = \mathbf{w}|_\Omega$ and $\mathbf{v} = \mathbf{w}|_{B_R \setminus \overline{\Omega}} - \mathbf{v_f}$ satisfy (104).

Now, let us consider two bilinear forms

$$\begin{aligned} b_1(\boldsymbol{\psi}, \boldsymbol{\phi}) &:= \int_\Omega \left((\mathbf{C}\nabla \boldsymbol{\psi}) : \nabla \boldsymbol{\phi} + \boldsymbol{\psi} \cdot \boldsymbol{\phi} \right) dx + \int_{B_R \setminus \overline{\Omega}} \left((\boldsymbol{C}_0 \nabla \mathbf{w}) : \nabla \boldsymbol{\phi} + \mathbf{w} \cdot \boldsymbol{\phi} \right) dx \\ &\quad - \int_{\partial B_R} \boldsymbol{\phi} \cdot (\Lambda_1 \boldsymbol{\psi})\, dS, \\ b_2(\boldsymbol{\psi}, \boldsymbol{\phi}) &:= - \int_\Omega \Big(\omega^2 \rho + 1 \Big) \boldsymbol{\phi} \cdot \boldsymbol{\psi} dx - \int_{B_R \setminus \overline{\Omega}} (\omega^2 + 1) \boldsymbol{\phi} \cdot \boldsymbol{\psi} dx \\ &\quad - \int_{\partial B_R} \boldsymbol{\phi} \cdot (\Lambda_2 \boldsymbol{\psi})\, dS, \qquad \text{for all } \boldsymbol{\phi}, \boldsymbol{\psi} \in \left(H^1(B_R)\right)^d, \end{aligned}$$

and

$$F(\boldsymbol{\phi}) := \int_{\partial\Omega} \mathbf{g} \cdot \boldsymbol{\phi}\, dS - \int_{\partial B_R} \boldsymbol{\phi} \cdot T_{\boldsymbol{\nu}} \mathbf{v_f}\, dS + \int_{B_R \setminus \overline{\Omega}} \Big((\mathbf{C}_0 \nabla \mathbf{v_f}) : \nabla \boldsymbol{\phi} - \omega^2 \mathbf{v_f} \cdot \boldsymbol{\phi} \Big)\, dx.$$

Then we can rewrite the problem (105) as finding a function $\mathbf{w} \in \left(H^1(B_R)\right)^d$ such that

$$b_1(\mathbf{w}, \boldsymbol{\phi}) + b_2(\mathbf{w}, \boldsymbol{\phi}) = F(\boldsymbol{\phi}), \qquad \text{for any } \boldsymbol{\phi} \in \left(H^1(B_R)\right)^d.$$

Since $-\Lambda_1$ is a positive operator, one can conclude that $b_1(\cdot, \cdot)$ is strictly coercive. Therefore, from the Lax-Milgram theorem, one can see that the operator $A : \left(H^1(B_R)\right)^d \to \left(H^1(B_R)\right)^d$ defined by $b_1(\mathbf{w}, \boldsymbol{\phi}) = (A\mathbf{w}, \boldsymbol{\phi})_{H^1(B_R)}$ is invertible and has a bounded inverse. On the other hand, since Λ_2 is a compact operator from $\left(H^{1/2}(\partial B_R)\right)^d \to \left(H^{-1/2}(\partial B_R)\right)^d$ and $\left(H^1(B_R)\right)^d \to \left(L^2(B_R)\right)^d$ is a compact embedding, then it is not hard to see that the operator $B : \left(H^1(B_R)\right)^d \to \left(H^1(B_R)\right)^d$ defined by $b_2(\mathbf{w}, \boldsymbol{\phi}) = (B\mathbf{w}, \boldsymbol{\phi})_{H^1(B_R)}$ is compact. Hence, by using [6, Theorem 5.16], one can derive that the existence of the transmission problem (104) from the uniqueness of (104) and the stability estimate (103) holds automatically. □

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