

# FINITE ELEMENT ERROR ANALYSIS FOR ELLIPTIC PARAMETER IDENTIFICATION WITH POWER-TYPE NONLINEARITY

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ABSTRACT. This paper studies the numerical analysis of a parameter identification problem governed by elliptic equations with power-type nonlinearity. We propose a numerical reconstruction via a suitable least-squares minimization problem based on piecewise linear finite elements. As one of our main novelties, we establish conditional stability estimates at the continuous level, which form the theoretical foundation of the present finite element analysis. Our stability analysis relies on tailored analytical tools, including Hardy-type inequalities, fractional Gagliardo–Nirenberg inequalities, and weighted spaces with singular distance weights. By invoking the achieved conditional stability together with the Carstensen quasi-interpolation operator and associated estimates in negative Sobolev spaces, we derive a priori error estimates for the proposed finite element approximation in terms of the mesh size, the regularization parameter, the noise level, and the nonlinearity exponent. Our results extend the recent stability and error estimates for the linear case by Jin et al. [23] and sharpen their error estimates and convergence order under weaker regularity assumptions.

## 1. INTRODUCTION

This paper examines the error analysis of a finite element method (FEM) for a parameter identification problem governed by an elliptic partial differential equation (PDE) involving power-type nonlinearity as follows:

$$(1.1) \quad \begin{cases} -\nabla \cdot (\sigma \nabla u) + qu^m = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

for a bounded Lipschitz domain  $\Omega \subset \mathbb{R}^n$  with  $n \geq 2$  and an odd natural number  $m \geq 1$ . Under this nonlinear PDE-model, we focus on the ill-posed inverse problem for the reconstruction of the zero-order coefficient  $q$  in (1.1). To be more precise, given  $\sigma$  and  $f$  satisfying certain regularity conditions (see Assumption 2.1) and noisy measurement data  $y^\delta$  for the unknown true weak solution  $u^\dagger$  of (1.1), we numerically reconstruct the unknown true coefficient  $q^\dagger$  based on piecewise linear FEM, by solving the following least squares minimization problem:

$$(1.2) \quad \begin{aligned} & \text{Minimize } \frac{1}{2} \|u_h(q_h) - y^\delta\|_{L^2(\Omega)}^2 + \frac{\alpha}{2} (\|q_h\|_{L^2(\Omega)}^2 + \|\nabla q_h\|_{L^2(\Omega)}^2) \\ & \text{subject to } q_h \in S_h \text{ and } \underline{q} \leq q_h \leq \bar{q}. \end{aligned}$$

In the setting of (1.2),  $S_h \subset H^1(\Omega)$  stands for the standard piecewise linear finite element space (see (4.1)),  $u_h(q_h)$  the associated finite element approximation of the forward problem (see (4.3) for its precise definition),  $\alpha > 0$  the Tikhonov regularization parameter, and  $0 \leq \underline{q} \leq \bar{q}$  some prescribed lower and upper bounds for the feasible zero-order coefficient. Our goal is to establish rigorous error estimates for the reconstruction error both in the coefficient and in the corresponding weak PDE-solution with respect to the discretization parameter  $h$ , the regularization parameter  $\alpha$ , the noisy level  $\delta$ , as well as the nonlinearity exponent  $m$ .

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Existing contributions to such a parameter identification problem are predominantly devoted to the linear forward equations (see [4, 9, 10, 18, 19, 21]), focusing primarily on the convergence rate of the Tikhonov regularization at the continuous level. To obtain a computationally tractable formulation, as we propose in (1.2), the regularized output least-squares minimization problem is discretized using FEM due to its adaptability to complex geometries and low regularity. The finite element discretization, however, leads to additional numerical errors in the reconstruction, making a rigorous a priori error analysis essential for the accuracy of the computed approximation. The plain finite element convergence, without error estimates, has been widely studied, see, e.g., [5, 25] for elliptic inverse coefficient problems, [22, 26, 39] for parabolic inverse coefficient problems, and [12] for Maxwell inverse coefficient problems. However, results regarding a priori error estimates for the reconstruction error remain limited.

Wang and Zou [38] were the first to initiate the finite element error analysis and derive weighted-norm estimates for elliptic and parabolic inverse conductivity problems. Their analysis, however, requires full-time measurements and offers no standard estimates in  $L^2$ -norm. Jin and Zhou [24] extended the developed results [38] to a more general setting, allowing measurements on a subinterval  $[T-\sigma, T]$  and reducing the underlying regularity assumptions. More recently, Jin et al. [23] rigorously analyzed the finite element reconstruction of  $q$  in (1.1) for the linear case  $m = 1$  and its linear parabolic counterpart. Their major contributions include conditional stability estimates at the continuous level and finite element error estimates with respect to the noisy level  $\delta$ , the regularization parameter  $\alpha$ , and the mesh size  $h$ . In particular, they introduced a novel tailored test function, different from [38] and [24], serving as the theoretical basis for deriving error estimates in the standard  $L^2$ -norm and the interior  $L^2$ -norm for the finite element solutions.

To the best of our knowledge, this paper is the first to address and explore finite element estimates in the nonlinear case (1.1). Although the linear case  $m = 1$  has readily been investigated in [23], their results, however, cannot be transferred to the nonlinear case  $m > 1$  and require substantial extension. Our main contributions are threefold: First, we establish novel conditional stability estimates for (1.1) (see Theorem 3.9) using a functional-analytic framework based on weighted Sobolev spaces with singular weights, Hardy-type inequalities, and fractional Gagliardo-Nirenberg inequalities. In particular, our analysis avoids the explicit use of a nonstandard test-function required in the previous contribution [23] for the linear case  $m = 1$ . Second, within this functional framework, we not only obtain stronger stability estimates than those available so far only for the linear case [23], but also manage to verify the required positivity condition (see Assumption 3.6) for the nonlinear case  $m > 1$  (see Proposition 3.12). Third, by combining the improved conditional stability with the Carstensen quasi-interpolation operator [2] and suitable estimates in negative Sobolev spaces, we derive in Theorem 4.6 a priori error estimates in terms of the discretization parameter  $h$ , the regularization parameter  $\alpha$ , the noise level  $\delta$ , and the nonlinearity exponent  $m$ . In particular, for the linear case  $m = 1$ , our framework achieves sharper error estimates than [4] under the weaker regularity condition  $q^\dagger \in H^1(\Omega)$  in contrast to the  $H^2$ -regularity assumption used previously in [23]. This relaxation significantly broadens the applicability of the proposed approach (see Remark 4.8).

**Organization of this article.** In Section 2, we review relevant function spaces as well as the well-posedness and regularity results for the semilinear elliptic equation (1.1). Section 3 is devoted to the establishment of the conditional stability results under appropriate assumptions. In Section 4, we provide the rigorous error estimates for the finite element approximation. The final section is devoted to numerical tests that illustrate the theoretical findings achieved.

## 2. PRELIMINARIES

Let us begin by introducing our notation. If  $V$  is a normed vector space, then  $\|\cdot\|_V : V \rightarrow \mathbb{R}_+ \cup \{0\}$  stands for the norm used in the space  $V$ , and  $\langle \cdot, \cdot \rangle_{V^*, V}$  denotes the duality pairing with the corresponding dual space  $V^*$ . If  $V$  is a Hilbert space, then  $(\cdot, \cdot)_V$  and  $\|\cdot\|_V$  denote, respectively, its scalar product and the induced norm. For the special case  $V = \mathbb{R}^n$ , we use the Euclidean scalar product and the Euclidean norm that are denoted by a dot and by  $|\cdot| := \|\cdot\|_{\mathbb{R}^n}$ .

Now, for given  $s \in [0, \infty)$  and  $p \in [1, \infty]$ , we denote by  $W^{s,p}(\Omega)$  the conventional Sobolev space (see, e.g., [1, Definition 8.10.10]). The conjugate exponent of  $p$  is denoted by  $p'$ , i.e.,  $\frac{1}{p} + \frac{1}{p'} = 1$ . We recall that, in the case of  $1 \leq p < \infty$ , the Sobolev norm  $\|\cdot\|_{W^{s,p}(\Omega)}$  is given by

$$(2.1) \quad \|u\|_{W^{s,p}(\Omega)} := \begin{cases} \left( \sum_{|\beta| \leq s} \|D^\beta u\|_{L^p(\Omega)}^p \right)^{\frac{1}{p}} & \text{if } s \in \mathbb{N} \\ \left( \sum_{|\beta| \leq m} \|D^\beta u\|_{L^p(\Omega)}^p + \sum_{|\beta|=m} \int_\Omega \int_\Omega \frac{|D^\beta u(x) - D^\beta u(y)|^p}{|x-y|^{n+\sigma p}} dx dy \right)^{\frac{1}{p}} & \text{if } s = m + \sigma \text{ for } m \in \mathbb{N} \text{ and } \sigma \in (0, 1) \end{cases}$$

where  $D^\beta u$  denoting the weak partial derivative of  $u$  with multi-index  $\beta \in (\mathbb{N} \cup \{0\})^n$ . Similarly,  $W^{s,p}(\Omega)^{n \times n}$  denotes the Sobolev space of  $\mathbb{R}^{n \times n}$ -valued functions. Furthermore,  $C_0^\infty(\Omega)$  stands for the set of all smooth functions  $\varphi$  with a compact support  $\text{supp } \varphi$  contained in  $\Omega$ . By  $W_0^{s,p}(\Omega)$ , we denote the closure of  $C_0^\infty(\Omega)$  with respect to the norm  $\|\cdot\|_{W^{s,p}(\Omega)}$ . Finally,  $W^{-s,p}(\Omega)$  stands for the dual space of  $W_0^{s,p'}(\Omega)$ .

**2.1. Well-posedness.** Let us discuss the well-posedness for the forward problem (1.1) as well as the associated maximum and comparison principles. We begin by formulating the standing assumption for (1.1):

**Assumption 2.1.** *Let  $\Omega \subset \mathbb{R}^n$  with  $n \geq 2$  be a bounded Lipschitz domain,  $m = 2l+1$  with  $l \in \mathbb{N} \cup \{0\}$ ,  $r > n/2$ ,  $f \in L^r(\Omega)$ , and  $\sigma = (\sigma_{ij}(x))_{1 \leq i, j \leq n} \in L^\infty(\Omega)^{n \times n}$  satisfying the following symmetric and elliptic conditions:*

$$(2.2) \quad \begin{cases} \underline{\lambda} |\xi|^2 \leq \sigma(x) \xi \cdot \xi \leq \bar{\lambda} |\xi|^2 & \text{for all } \xi \in \mathbb{R}^n \text{ and a.e. } x \in \Omega \\ \sigma_{ij}(x) = \sigma_{ji}(x) & \text{for all } i, j = 1, \dots, n \end{cases}$$

for some fixed constants  $0 < \underline{\lambda} \leq \bar{\lambda} < \infty$ , independent of  $\xi$  and  $x$ .

**Lemma 2.2** ([31, Theorem 1] and [20, Theorem 0.5]). *Let Assumption 2.1 be satisfied. Then, there exists a real number  $\hat{p} > 2$  such that for every  $p \in (\hat{p}', \hat{p})$  the linear operator*

$$-\nabla \cdot \sigma \nabla : W_0^{1,p}(\Omega) \rightarrow W^{-1,p}(\Omega)$$

is a topological isomorphism.

**Lemma 2.3** (Well-posedness). *Let Assumption 2.1 hold and  $q \in L^r(\Omega)$  be nonnegative. Then, the forward problem (1.1) admits a unique weak solution  $u \in H_0^1(\Omega) \cap C(\bar{\Omega})$  satisfying*

$$(2.3) \quad \|u\|_{H^1(\Omega)} + \|u\|_{C(\bar{\Omega})} \leq C \|f\|_{L^r(\Omega)}$$

with a constant  $C > 0$  independent of  $u$ ,  $f$ , and  $q$ . Furthermore, there exists an  $\bar{p} \in (2, \infty]$  such that the weak solution satisfies the regularity  $u \in W^{1,\bar{p}}(\Omega)$  and

$$(2.4) \quad \|u\|_{W^{1,\bar{p}}(\Omega)} \leq C (\|f\|_{L^r(\Omega)} + \|q\|_{L^r(\Omega)} \|f\|_{L^r(\Omega)}^m)$$

with a constant  $C > 0$  independent of  $u$ ,  $f$ , and  $q$ . Moreover, if  $\Omega$  is convex,  $r > n$ , and  $\sigma \in W^{1,\omega}(\Omega)^{n \times n}$  for some  $\omega > n$ , then the proceeding regularity result and (2.4) hold with  $\bar{p} = \infty$ . If

additionally  $\Omega$  is convex or of class  $C^{1,1}$  and  $\sigma \in W^{1,\infty}(\Omega)^{n \times n}$ , then the weak solution satisfies the additional regularity  $u \in H^2(\Omega)$  and

$$\|u(q)\|_{H^2(\Omega)} \leq C(\|f\|_{L^2(\Omega)} + \|q\|_{L^2(\Omega)}\|f\|_{L^r(\Omega)}^m)$$

with a constant  $C > 0$  independent of  $u$ ,  $f$ , and  $q$ .

*Proof.* Thanks to Assumption 2.1, since  $q \in L^r(\Omega)$  is a nonnegative function, the existence of a unique weak solution  $u \in H^1(\Omega) \cap C(\bar{\Omega})$  of (1.1) satisfying (2.3) is a well-known result (see, e.g., [37, Theorem 4.10]). By definition, the unique weak solution  $u \in H_0^1(\Omega) \cap C(\bar{\Omega})$  of (1.1) satisfies

$$(2.5) \quad -\nabla \cdot (\sigma \nabla u) = f - qu^m \quad \text{in } \Omega \quad \text{and} \quad u = 0 \quad \text{on } \partial\Omega,$$

in the weak sense. Since  $u \in C(\bar{\Omega})$  and  $f, q \in L^r(\Omega)$ , the Sobolev embedding theorem implies that the right-hand side in (2.5) enjoys the regularity property

$$f - qu^m \in L^r(\Omega) \hookrightarrow W_0^{1,\bar{p}'}(\Omega)^* \quad \text{with} \quad \bar{p} = \begin{cases} \frac{rn}{n-r} & \text{if } r < n \\ \infty & \text{if } r \geq n. \end{cases}$$

Direct computation shows that  $\bar{p} > 2$  holds since  $r > \frac{n}{2}$ . If necessary we may reduce  $\bar{p} > 2$  such that  $\bar{p} \in (2, \hat{p})$  with  $\hat{p} \in (2, \infty)$  as in Lemma 2.2. Thus, applying Lemma 2.2 to (2.5) implies  $u \in W^{1,\bar{p}}(\Omega)$  and

$$\begin{aligned} \|u\|_{W^{1,\bar{p}}(\Omega)} &\leq C\|f - qu^m\|_{W_0^{1,\bar{p}'}(\Omega)^*} \leq C\|f - qu^m\|_{L^r(\Omega)} \\ &\leq C(\|f\|_{L^r(\Omega)} + \|q\|_{L^r(\Omega)}\|u\|_{C(\bar{\Omega})}^m) \stackrel{(2.3)}{\leq} C(\|f\|_{L^r(\Omega)} + \|q\|_{L^r(\Omega)}\|f\|_{L^r(\Omega)}^m) \end{aligned}$$

with a constant  $C > 0$  independent of  $u$ ,  $f$ , and  $q$ .

Now, suppose additionally that  $\Omega$  is convex,  $r > n$ , and  $\sigma \in W^{1,\omega}(\Omega)^{n \times n}$  for some  $\omega > n$ . Then, thanks to [29, Lemma 2.1], the unique weak solution of (1.1) fulfills

$$(2.6) \quad \|\nabla u\|_{L^\infty(\Omega)} \leq C\|\nabla \cdot (\sigma \nabla u)\|_{L^r(\Omega)} \stackrel{(2.5)}{\leq} C\|f - qu^m\|_{L^r(\Omega)} \stackrel{(2.3)}{\leq} C(\|f\|_{L^r(\Omega)} + \|q\|_{L^r(\Omega)}\|f\|_{L^r(\Omega)}^m)$$

with a constant  $C > 0$  independent of  $u$ ,  $f$ , and  $q$ . Combining (2.6) and (2.3), the desired estimate (2.4) is obtained for  $\bar{p} = \infty$ . Analogously, if  $\Omega$  is additionally convex or of class  $C^{1,1}$  and  $\sigma \in W^{1,\infty}(\Omega)^{n \times n}$ , the classical  $H^2$ -regularity results [16, Theorem 3.2.1.2] and [16, Corollary 2.2.2.4] imply  $u \in H^2(\Omega)$  and the corresponding a priori estimate.  $\square$

For the upcoming results, we shall make use of the algebraic identity

$$(2.7) \quad a^m - b^m = \left( \sum_{k=0}^{m-1} a^k b^{m-1-k} \right) (a - b) \quad \forall a, b \in \mathbb{R}.$$

By the mean value theorem, we have  $a^m - b^m = m(a + \theta_{ab}(b - a))^{m-1}(a - b)$  for some  $\theta_{ab} \in (0, 1)$ , and applying this to (2.7), it follows that

$$(2.8) \quad \sum_{k=0}^{m-1} a^k b^{m-1-k} \geq 0 \quad \forall a, b \in \mathbb{R},$$

since  $m$  is an odd natural number.

**Lemma 2.4** (Maximum principle and comparison principle). *Let Assumption 2.1 hold,  $q \in L^r(\Omega)$  be nonnegative, and  $u \in H_0^1(\Omega) \cap C(\bar{\Omega})$  denote the unique weak solution to (1.1). Then, the following claims hold:*

(i) *If  $f(x) \geq 0$  holds for a.e.  $x \in \Omega$ , then  $u(x) \geq 0$  holds for all  $x \in \bar{\Omega}$ .*

- (ii) If  $f(x) > 0$  holds for a.e.  $x \in \Omega$ , then  $u(x) > 0$  holds for all  $x \in \Omega$ .
- (iii) Let  $f_1, f_2 \in L^r(\Omega)$  and  $u_1, u_2 \in H_0^1(\Omega) \cap C(\bar{\Omega})$  denote the unique weak solution to (1.1) with  $f = f_1$  and  $f = f_2$ , respectively. If  $f_1(x) \geq f_2(x)$  holds for a.e.  $x \in \Omega$ , then  $u_1(x) \geq u_2(x)$  holds for all  $x \in \bar{\Omega}$ .

*Proof.* (i) Let us write  $u = u^+ - u^-$  where  $u^+ := \max\{u, 0\} \geq 0$  and  $u^- := \max\{-u, 0\} \geq 0$ . As  $u \in H_0^1(\Omega) \cap C(\bar{\Omega})$ , well-known arguments (for example, see [27, Chapter II, Theorem A.1]) yield that  $u^+, u^- \in H_0^1(\Omega) \cap C(\bar{\Omega})$ . Now, suppose that  $f(x) \geq 0$  holds for a.e.  $x \in \Omega$ . To show that  $u(x) \geq 0$  for all  $x \in \bar{\Omega}$ , we consider  $u^- \in H_0^1(\Omega) \cap C(\bar{\Omega})$  as the test function in the weak formulation of (1.1) to deduce that

$$\begin{aligned} 0 &\stackrel{\substack{\leq \\ f \geq 0 \text{ a.e.}}}{\leq} \int_{\Omega} f u^- dx = \int_{\Omega} \sigma \nabla u \cdot \nabla u^- dx + q u^m u^- dx \\ &= \int_{\Omega} \sigma \nabla (u^+ - u^-) \cdot \nabla u^- dx + \int_{\Omega} q (u^+ - u^-)^m u^- dx \\ &\stackrel{\substack{= \\ u^+ \cdot u^- = 0}}{=} - \int_{\Omega} \sigma \nabla u^- \cdot \nabla u^- dx - \int_{\Omega} q (u^-)^{m+1} dx \stackrel{\substack{\leq \\ u^-, q \geq 0}}{\leq} 0, \end{aligned}$$

from which it follows that  $\|\nabla u^-\|_{L^2(\Omega)}^2 \leq 0$  implies  $u^- \equiv 0$ . Thus, the assertion (i) follows.

(ii) Suppose that  $f(x) > 0$  holds for a.e.  $x \in \Omega$ . We proceed by contradiction and assume that there exists a nonempty open set  $O \subset \Omega$  such that  $u(x) = 0$  for all  $x \in O$ . Then, the weak formulation gives

$$\int_O f \varphi dx = \int_O \sigma \nabla u \cdot \nabla \varphi dx + \int_O q u^m \varphi dx = 0 \quad \forall \varphi \in C_0^\infty(O).$$

Thus, it follows that  $f = 0$  a.e. in  $O$ , which is a contradiction. In conclusion, along with (i), it follows that  $u$  is positive.

(iii) In view of (2.7), we see that  $v := u_1 - u_2 \in H_0^1(\Omega) \cap C(\bar{\Omega})$  is the unique weak solution to

$$(2.9) \quad -\nabla \cdot (\sigma \nabla v) + q \left( \sum_{k=0}^{m-1} u_1^k u_2^{m-1-k} \right) v = f_1 - f_2 \geq 0 \text{ in } \Omega, \quad v = 0 \text{ on } \partial\Omega.$$

Note that (2.9) is a special case of (1.1) with  $m = 1$  and the coefficient  $q \left( \sum_{k=0}^{m-1} u_1^k u_2^{m-1-k} \right) \in L^r(\Omega)$ , which is nonnegative due to (2.8) and  $q \geq 0$ . Therefore, applying (i) to (2.9) implies that  $v(x) \geq 0$  holds for all  $x \in \bar{\Omega}$ , which concludes the proof.  $\square$

### 3. STABILITY ANALYSIS

The goal of this section is to establish conditional stability estimates for (1.1). Our analysis is based on the use of Hardy-type inequalities, Gagliardo-Nirenberg inequalities with fractional negative norms, and specific weighted Sobolev spaces with singular weights related to the distance function (3.1). All these analytical tools are collected in the following subsection:

**3.1. Analytical tools.** In all what follows, let Assumption 2.1 hold. Let us first discuss Hardy-type inequalities relevant for our stability analysis. To simplify the notation, we introduce the following distance function:

$$(3.1) \quad \rho(x) := \text{dist}(x, \partial\Omega) \quad \forall x \in \bar{\Omega},$$

that satisfies the regularity property  $\rho \in W^{1,\infty}(\Omega)$  (see, e.g., [7, Theorem 2.1]). For every  $p \in (1, \infty)$  and  $s \in (-\infty, p-1)$ , it is well-known that the so-called weighted Hardy inequality

$$(3.2) \quad \int_{\Omega} |\varphi(x)|^p \rho(x)^{s-p} dx \leq C \sum_{|\beta|=1} \int_{\Omega} |D^{\beta} \varphi(x)|^p \rho(x)^s dx \quad \forall \varphi \in C_0^{\infty}(\Omega)$$

holds with a constant  $C > 0$  depending only on  $\Omega$ ,  $s$ , and  $p$  (see, e.g., [28, Theorem 1.2] and [32, Theorem 1.6]). Moreover, for  $\sigma \in (0, 1)$  and  $p \in (1, \infty)$ , it was shown in [8, Theorem 1.1] that

$$(3.3) \quad \int_{\Omega} |\varphi(x)|^p \rho(x)^{-\sigma p} dx \leq C \int_{\Omega} \int_{\Omega} \frac{|\varphi(x) - \varphi(y)|^p}{|x - y|^{n+\sigma p}} dx dy \quad \forall \varphi \in C_0^{\infty}(\Omega).$$

A useful consequence of (3.2) and (3.3) is as follows:

**Lemma 3.1.** *Let  $p \in (1, \infty)$  and  $s \in (0, \infty)$ . Then, there exists a constant  $C > 0$  such that*

$$\|\rho^{-s} \varphi\|_{L^p(\Omega)} \leq C \|\varphi\|_{W^{s,p}(\Omega)} \quad \forall \varphi \in C_0^{\infty}(\Omega).$$

*Proof.* Suppose that  $s$  is not an integer. Thus, we have  $s = m + \sigma$  with  $m \in \mathbb{N}$  and  $\sigma \in (0, 1)$ . Let  $\varphi \in C_0^{\infty}(\Omega)$  be arbitrarily fixed. Applying the weighted Hardy inequality (3.2)  $m$ -times, we infer that

$$\begin{aligned} \int_{\Omega} |\varphi|^p \rho^{-sp} dx &= \int_{\Omega} |\varphi|^p \rho^{-mp-\sigma p} dx \leq C \sum_{|\beta|=1} \int_{\Omega} |D^{\beta} \varphi|^p \rho^{(1-m)p-\sigma p} dx \\ &\leq nC^2 \sum_{|\beta|=2} \int_{\Omega} |D^{\beta} \varphi|^p \rho^{(2-m)p-\sigma p} dx \leq \dots \leq n^{m-1} C^m \sum_{|\beta|=m} \int_{\Omega} |D^{\beta} \varphi|^p \rho^{-\sigma p} dx. \end{aligned}$$

Then, applying (3.3) yields the existence of a constant  $C > 0$  such that

$$\int_{\Omega} |\varphi|^p \rho^{-sp} dx \leq C \sum_{|\beta|=m} \int_{\Omega} \int_{\Omega} \frac{|D^{\beta} \varphi(x) - D^{\beta} \varphi(y)|^p}{|x - y|^{n+\sigma p}} dx dy \stackrel{(2.1)}{\leq} C \|\varphi\|_{W^{s,p}(\Omega)}^p.$$

The case of  $s \in \mathbb{N}$  is proved in the same way without the above additional estimate. This completes the proof.  $\square$

Serving as our second analytical tool, we provide Gagliardo–Nirenberg inequalities for functions on the Lipschitz domain  $\Omega$  with fractional negative norms. To this end, for  $s \in \mathbb{R}$  and  $p, p_1 \in [1, \infty]$ , we denote by  $B_{p,p_1}^s(\Omega)$  the conventional Besov on  $\Omega$  (see [35, Definition 1.95]). It is well-known (see [36, (5) on page 139]) that

$$W^{s,p}(\Omega) = B_{p,p}^s(\Omega) \quad \forall s \in (0, \infty) \setminus \mathbb{N} \quad \forall p \in (1, \infty),$$

which, in particular, implies

$$W_0^{s,p}(\Omega) = \dot{B}_{p,p}^s(\Omega) \quad \forall s \in (0, \infty) \setminus \mathbb{N} \quad \forall p \in (1, \infty)$$

with  $\dot{B}_{p,p}^s(\Omega)$  being the closure of  $C_0^{\infty}(\Omega)$  with respect to the norm of  $B_{p,p}^s(\Omega)$ . Combined with [34, (43)], this ensures that

$$W^{-s,p}(\Omega) = (W_0^{s,p'}(\Omega))^* = \dot{B}_{p',p'}^s(\Omega)^* = B_{p,p}^{-s}(\Omega) \quad \forall s \in (0, \infty) \setminus \mathbb{N} \quad \forall p \in (1, \infty).$$

Recalling the definition  $W^{-k,p}(\Omega) = (W_0^{k,p'}(\Omega))^*$ , we conclude that

$$(3.4) \quad W^{s,p}(\Omega) = B_{p,p}^s(\Omega) \quad \forall s \in \mathbb{R} \setminus \mathbb{Z} \quad \forall p \in (1, \infty).$$

**Lemma 3.2.** *Let  $p \in (1, \infty)$  and  $s_0 > 1$  be real numbers. Then, there exists a constant  $C > 0$ , independent of  $u$ , such that*

$$(3.5) \quad \|u\|_{W^{-1,p}(\Omega)} \leq C \|u\|_{W^{-s_0,p}(\Omega)}^{\frac{1}{s_0}} \|u\|_{L^p(\Omega)}^{\frac{s_0-1}{s_0}} \quad \forall \varphi \in L^p(\Omega).$$

Furthermore, for every  $s_1 > 0$ , there exists a constant  $C > 0$ , independent of  $u$ , such that

$$(3.6) \quad \|u\|_{L^p(\Omega)} \leq C \|u\|_{W^{-s_1,p}(\Omega)}^{\frac{1}{s_1+1}} \|u\|_{W^{1,p}(\Omega)}^{\frac{s_1}{s_1+1}} \quad \forall u \in W^{1,p}(\Omega).$$

*Proof.* By invoking [35, (1.368)], we obtain the following interpolation inequalities

$$(3.7) \quad \|u\|_{B_{p,1}^{-1}(\Omega)} \leq C \|u\|_{B_{p,\infty}^{-s_0}(\Omega)}^{\frac{1}{s_0}} \|u\|_{B_{p,\infty}^0(\Omega)}^{\frac{s_0-1}{s_0}} \quad \forall u \in B_{p,\infty}^0(\Omega)$$

$$(3.8) \quad \|u\|_{B_{p,1}^0(\Omega)} \leq C \|u\|_{B_{p,\infty}^{-s_1}(\Omega)}^{\frac{1}{s_1+1}} \|u\|_{B_{p,\infty}^1(\Omega)}^{\frac{s_1}{s_1+1}} \quad \forall u \in B_{p,\infty}^1(\Omega),$$

where  $C > 0$  is a constant independent of  $u$ . Using (3.4), we deduce from the embedding theorem in [35, (1.299)] that

$$(3.9) \quad B_{p,1}^{-1}(\Omega) \hookrightarrow W^{-1,p}(\Omega), \quad W^{-s_0,p}(\Omega) \hookrightarrow B_{p,\infty}^{-s_0}(\Omega), \quad L^p(\Omega) \hookrightarrow B_{p,\infty}^0(\Omega).$$

Combining (3.9) with (3.7) yields (3.5). Similarly, (3.6) follows from (3.8) together with

$$B_{p,1}^0(\Omega) \hookrightarrow L^p(\Omega), \quad W^{-s_1,p}(\Omega) \hookrightarrow B_{p,\infty}^{-s_1}(\Omega), \quad W^{1,p}(\Omega) \hookrightarrow B_{p,\infty}^1(\Omega).$$

This concludes the proof.  $\square$

Serving as our third analytical tool, we consider the following weighted Sobolev space with a singular weight related to the distance function (3.1).

**Definition 3.3.** *Suppose that  $w \in C(\bar{\Omega}) \cap H_0^1(\Omega) \cap W^{1,\bar{p}}(\Omega)$  for some  $\bar{p} \in (2, \infty]$  satisfies*

$$(3.10) \quad \exists C_\gamma, \gamma > 0 \quad \forall x \in \bar{\Omega}: \quad w(x) \geq C_\gamma \rho(x)^\gamma$$

with the distance function  $\rho(x)$  as defined in (3.1). Then, for every  $\theta \in (1, 2]$ , we denote by  $\mathcal{X}_w^\theta(\Omega)$  the closure of  $C_0^\infty(\Omega)$  under the norm

$$(3.11) \quad \|\varphi\|_{\mathcal{X}_w^\theta(\Omega)} := \|w^{-m}\varphi\|_{W^{1,\theta}(\Omega)} \quad \forall \varphi \in C_0^\infty(\Omega).$$

**Remark 3.4.** *According to Definition 3.3,  $w$  has a vanishing trace so that the weight function  $w^{-m}$  appearing in  $\|\cdot\|_{\mathcal{X}_w^\theta(\Omega)}$  features singularity on the boundary. However, since every  $\varphi \in C_0^\infty(\Omega)$  admits a compact support  $\text{supp}(\varphi) \subset \Omega$ , the condition (3.10) ensures that*

$$0 < w^{-m}(x) \leq C_\gamma^{-m} \rho(x)^{-\gamma m} \leq C_\gamma^{-m} \text{dist}(\text{supp}(\varphi), \partial\Omega)^{-\gamma m} \quad \forall x \in \text{supp}(\varphi) \quad \forall \varphi \in C_0^\infty(\Omega).$$

Therefore,  $\|\cdot\|_{\mathcal{X}_w^\theta(\Omega)}$  defines indeed a norm in  $C_0^\infty(\Omega)$ . In conclusion, the closure of  $C_0^\infty(\Omega)$  with respect to  $\|\cdot\|_{\mathcal{X}_w^\theta(\Omega)}$  is well-defined.

In view of  $w \in C(\bar{\Omega})$  given by Definition 3.3, we see that

$$\|\varphi\|_{L^\theta(\Omega)} \leq \|w\|_{C(\bar{\Omega})}^m \|w^{-m}\varphi\|_{L^\theta(\Omega)} \leq \|w\|_{C(\bar{\Omega})}^m \|w^{-m}\varphi\|_{W^{1,\theta}(\Omega)} = \|w\|_{C(\bar{\Omega})}^m \|\varphi\|_{\mathcal{X}_w^\theta(\Omega)} \quad \forall \varphi \in C_0^\infty(\Omega),$$

implying by Definition 3.3 that

$$(3.12) \quad \mathcal{X}_w^\theta(\Omega) \hookrightarrow L^\theta(\Omega) \quad \implies \quad L^{\theta'}(\Omega) \hookrightarrow \mathcal{X}_w^\theta(\Omega)^*,$$

where  $\theta' = \frac{\theta}{\theta-1}$  denotes the conjugate exponent. To investigate further properties of  $\mathcal{X}_w^\theta(\Omega)$ , we introduce  $\mathcal{Z}_w^\theta(\Omega)$  to be the closure of  $C_0^\infty(\Omega)$  under the norm

$$(3.13) \quad \|\varphi\|_{\mathcal{Z}_w^\theta(\Omega)} := \|\rho^{-(m+1)\gamma}\varphi\|_{L^\theta(\Omega)} + \|\rho^{-m\gamma}\nabla\varphi\|_{L^\theta(\Omega)^n} \quad \forall \varphi \in C_0^\infty(\Omega)$$

with

$$(3.14) \quad \vartheta := (1/\theta - 1/\bar{p})^{-1} \in (1, \infty),$$

since  $\bar{p} \in (2, \infty]$  and  $\theta \in (1, 2]$ .

**Lemma 3.5.** *Let  $w \in C(\bar{\Omega}) \cap H_0^1(\Omega) \cap W^{1,\bar{p}}(\Omega)$  for some  $\bar{p} \in (2, \infty]$  satisfy (3.10) and  $\theta \in (1, 2]$ . Then, it holds that*

$$(3.15) \quad \mathcal{Z}_w^\theta(\Omega) \hookrightarrow \mathcal{X}_w^\theta(\Omega).$$

In other words, there exists a constant  $C > 0$  such that

$$\|\varphi\|_{\mathcal{X}_w^\theta(\Omega)} \leq C \|\varphi\|_{\mathcal{Z}_w^\theta(\Omega)} \quad \forall \varphi \in \mathcal{X}_w^\theta(\Omega).$$

*Proof.* In view of the regularity  $w \in C(\bar{\Omega}) \cap W^{1,\bar{p}}(\Omega)$  and (3.10), we may apply the chain rule to obtain

$$\nabla(w^{-m}\varphi) = -m[\rho^{-(m+1)\gamma}\varphi] \cdot [\rho^{(m+1)\gamma}w^{-m-1}\nabla w] + [\rho^{m\gamma}w^{-m}] \cdot [\rho^{-m\gamma}\nabla\varphi] \quad \forall \varphi \in C_0^\infty(\Omega).$$

Using the above identity as well as

$$\frac{1}{\theta} \underbrace{=}_{(3.14)} \frac{1}{\vartheta} + \frac{1}{\infty} + \frac{1}{\bar{p}} \quad \text{and} \quad \frac{1}{\theta} = \frac{1}{\theta} + \frac{1}{\infty},$$

we deduce from the generalized Hölder's inequality that

$$\begin{aligned} \|\nabla(w^{-m}\varphi)\|_{L^\theta(\Omega)} &\leq m\|\rho^{-(m+1)\gamma}\varphi\|_{L^\vartheta(\Omega)} \|\rho^{(m+1)\gamma}w^{-m-1}\|_{L^\infty(\text{supp}(\varphi))} \|\nabla w\|_{L^{\bar{p}}(\Omega)} \\ &\quad + \|\rho^{-m\gamma}\nabla\varphi\|_{L^\theta(\Omega)} \|\rho^{m\gamma}w^{-m}\|_{L^\infty(\text{supp}(\varphi))} \\ &\stackrel{(3.10)}{\leq} m\|\rho^{-(m+1)\gamma}\varphi\|_{L^\vartheta(\Omega)} C_\gamma^{-(m+1)} \|\nabla w\|_{L^{\bar{p}}(\Omega)} + C_\gamma^{-m} \|\rho^{-m\gamma}\nabla\varphi\|_{L^\theta(\Omega)} \\ &\stackrel{(3.13)}{\leq} \max\{mC_\gamma^{-(m+1)} \|\nabla w\|_{L^{\bar{p}}(\Omega)}, C_\gamma^{-m}\} \|\varphi\|_{\mathcal{Z}_w^\theta(\Omega)} \quad \forall \varphi \in C_0^\infty(\Omega) \end{aligned}$$

and

$$\begin{aligned} \|w^{-m}\varphi\|_{L^\theta(\Omega)} &\leq \|\rho^{-(m+1)\gamma}\varphi\|_{L^\vartheta(\Omega)} \|\rho^{(m+1)\gamma}w^{-(m+1)}\|_{L^\infty(\text{supp}(\varphi))} \|w\|_{C(\bar{\Omega})} \\ &\leq C_\gamma^{-(m+1)} \|w\|_{C(\bar{\Omega})} \|\varphi\|_{\mathcal{Z}_w^\theta(\Omega)} \quad \forall \varphi \in C_0^\infty(\Omega). \end{aligned}$$

In conclusion, the continuous embedding (3.15) holds.  $\square$

**3.2. Conditional stability estimates.** For the rest of this article, let Assumption 2.1 hold and  $\bar{p} \in (2, \infty]$  be as in Lemma 2.3. For every given nonnegative coefficient  $q \in L^r(\Omega)$  of the nonlinearity, we make use of the standing notation  $u = u(q) \in H_0^1(\Omega) \cap C(\bar{\Omega}) \cap W^{1,\bar{p}}(\Omega)$  to denote the unique weak solution of (1.1). Furthermore, in the sequel, we consider the following two admissible sets for the coefficient  $q$ :

$$(3.16) \quad \mathcal{A} := \{q \in L^\infty(\Omega) : \underline{q} \leq q \leq \bar{q} \text{ a.e. in } \Omega\} \quad \text{and} \quad \tilde{\mathcal{A}}_p := \{q \in \mathcal{A} \cap W^{1,p}(\Omega) : \|q\|_{W^{1,p}(\Omega)} \leq \tilde{M}\}$$

for some prescribed constants  $0 \leq \underline{q} \leq \bar{q} < \infty$ ,  $p \in [2, \infty)$  and  $\tilde{M} > 0$ .

**Assumption 3.6.** *Let Assumption 2.1 be satisfied. Suppose that the weak solution  $u(q^\dagger)$  of (1.1) associated with the true coefficient  $q^\dagger \in \mathcal{A}$  satisfies*

$$\exists C_\gamma, \gamma > 0 \quad \forall x \in \bar{\Omega} : \quad u(q^\dagger)(x) \geq C_\gamma \rho(x)^\gamma \quad \forall x \in \bar{\Omega}$$

with the distance function  $\rho(x)$  as defined in (3.1).

**Remark 3.7.** Note that Assumption 3.6 is inspired by [23], who focused on the linear case

$$-\Delta u + qu = f \quad \text{in } \Omega \quad \text{and} \quad u = 0 \quad \text{on } \partial\Omega,$$

and proved in [23, Theorem 2.1] the following conditional stability estimate:

$$\|q - q^\dagger\|_{L^2(\Omega)} \leq C \|u(q) - u(q^\dagger)\|_{H_0^1(\Omega)}^{\frac{1}{2+4\gamma}} \quad \forall q \in \tilde{\mathcal{A}}_2.$$

In the upcoming theorem, we prove our main result on the conditional stability estimate for the nonlinear case (1.1). In particular, for  $m = 1$ , our result improves the convergence order [23, Theorem 2.1] to the order  $\frac{1}{1+2\gamma}$ .

Note that Assumption 3.6 allows us to set  $w = u^\dagger := u(q^\dagger)$  in Definition 3.3. Furthermore, for any given  $p \in [2, \bar{p}] \cap [2, \infty)$ , its conjugate exponent satisfies  $p' = \frac{p}{p-1} \in (1, 2]$  such that the choice  $\theta = p'$  is also allowed for Definition 3.3. Altogether, employing the weighted space  $\mathcal{X}_{u^\dagger}^{p'}(\Omega)$  and all analytical tools from the previous section, we are in a position to prove the following key lemma:

**Lemma 3.8.** Let Assumptions 2.1 and 3.6 be satisfied and  $\kappa := m\gamma + \gamma + n/\bar{p}$  with  $\bar{p}$  from Lemma 2.3. Then, for every  $p \in [2, \bar{p}] \cap [2, \infty)$ , there exists a constant  $C > 0$  such that

$$(3.17) \quad \|\varphi\|_{W^{-1,p}(\Omega)} \leq \begin{cases} C \|\varphi\|_{\mathcal{X}_{u^\dagger}^{p'}(\Omega)^*} & \forall \varphi \in L^p(\Omega) \quad \text{if } \kappa \in (0, 1], \\ C \|\varphi\|_{\mathcal{X}_{u^\dagger}^{p'}(\Omega)^*}^{\frac{1}{\kappa}} \|\varphi\|_{L^p(\Omega)}^{\frac{\kappa-1}{\kappa}} & \forall \varphi \in L^p(\Omega) \quad \text{if } \kappa > 1, \end{cases}$$

and

$$(3.18) \quad \|\varphi\|_{L^{p'}(\Omega)} \leq C \|\varphi\|_{\mathcal{X}_{u^\dagger}^{p'}(\Omega)^*}^{\frac{1}{1+\kappa}} \|\varphi\|_{W^{1,p}(\Omega)}^{\frac{\kappa}{1+\kappa}} \quad \forall \varphi \in W^{1,p}(\Omega).$$

*Proof.* Let  $p \in [2, \bar{p}] \cap [2, \infty)$  and  $\vartheta := (1/p' - 1/\bar{p})^{-1} \in (1, \infty)$ . Then, Lemma 3.1 yields

$$\begin{aligned} \|\rho^{-(m+1)\gamma} \varphi\|_{L^\vartheta(\Omega)} &\leq C \|\varphi\|_{W^{(m+1)\gamma, \vartheta}(\Omega)} \quad \forall \varphi \in C_0^\infty(\Omega) \\ \|\rho^{-m\gamma} \nabla \varphi\|_{L^{p'}(\Omega)^n} &\leq C \|\varphi\|_{W^{m\gamma+1, p'}(\Omega)} \quad \forall \varphi \in C_0^\infty(\Omega) \end{aligned}$$

with a constant  $C > 0$  independent of  $\varphi$ . Thus, along with the definition (3.13) with  $\theta = p'$ , it follows that

$$\|\varphi\|_{\mathcal{X}_{u^\dagger}^{p'}(\Omega)} \leq C (\|\varphi\|_{W^{(m+1)\gamma, \vartheta}(\Omega)} + \|\varphi\|_{W^{m\gamma+1, p'}(\Omega)}) \quad \forall \varphi \in C_0^\infty(\Omega),$$

implying

$$(3.19) \quad W_0^{(m+1)\gamma, \vartheta}(\Omega) \cap W_0^{m\gamma+1, p'}(\Omega) \hookrightarrow \mathcal{Z}_{u^\dagger}^{p'}(\Omega) \hookrightarrow \mathcal{X}_{u^\dagger}^{p'}(\Omega).$$

By definition, we have  $\kappa = (m+1)\gamma + n/\bar{p}$  and  $1/\vartheta = 1/p' - 1/\bar{p}$ , and so  $\kappa - n/p' = (m+1)\gamma - n/\vartheta$ . Thus, the Sobolev embedding theorem (see [35, (1.301)]) implies  $W_0^{\kappa, p'}(\Omega) \hookrightarrow W_0^{(m+1)\gamma, \vartheta}(\Omega)$ . In combination with (3.19), we obtain that

$$W_0^{\kappa, p'}(\Omega) \hookrightarrow W_0^{(m+1)\gamma, \vartheta}(\Omega) \cap W_0^{m\gamma+1, p'}(\Omega) \hookrightarrow \mathcal{X}_{u^\dagger}^{p'}(\Omega),$$

and so

$$(3.20) \quad \mathcal{X}_{u^\dagger}^{p'}(\Omega)^* \hookrightarrow W^{-\kappa, p}(\Omega).$$

In the case of  $\kappa \in (0, 1]$ , the estimate (3.17) follows directly from the embeddings (3.20) and (3.12) (with  $\theta = p'$  and  $w = u^\dagger$ ) as well as  $W^{-\kappa, p}(\Omega) \hookrightarrow W^{-1, p}(\Omega)$ . If  $\kappa > 1$ , it holds that

$$\|\varphi\|_{W^{-1,p}(\Omega)} \stackrel{\text{Lem. 3.2}}{\leq} C \|\varphi\|_{W^{-\kappa, p}(\Omega)}^{\frac{1}{\kappa}} \|\varphi\|_{L^p(\Omega)}^{\frac{\kappa-1}{\kappa}} \stackrel{(3.20)}{\leq} C \|\varphi\|_{\mathcal{X}_{u^\dagger}^{p'}(\Omega)^*}^{\frac{1}{\kappa}} \|\varphi\|_{L^p(\Omega)}^{\frac{\kappa-1}{\kappa}} \quad \forall \varphi \in L^p(\Omega),$$

which is exactly (3.17) for the case of  $\kappa > 1$ . By (3.20) and inequality (3.6) (with  $s_1 = \kappa$ ), the estimate (3.18) follows.  $\square$

**Theorem 3.9.** *Let Assumptions 2.1 and 3.6 be satisfied and  $\kappa := m\gamma + \gamma + n/\bar{p}$  with  $\bar{p}$  as in Lemma 2.3. Then, for every  $p \in [2, \bar{p}] \cap [2, \infty)$ , there exists a constant  $C > 0$  such that*

$$(3.21) \quad \|q - q^\dagger\|_{W^{-1,p}(\Omega)} \leq \begin{cases} C \|u(q) - u(q^\dagger)\|_{W^{1,p}(\Omega)} & \forall q \in \mathcal{A} \text{ if } \kappa \in (0, 1], \\ C \|u(q) - u(q^\dagger)\|_{W^{1,p}(\Omega)}^{\frac{1}{\kappa}} & \forall q \in \mathcal{A} \text{ if } \kappa \in (1, \infty), \end{cases}$$

$$(3.22) \quad \|q - q^\dagger\|_{L^p(\Omega)} \leq C \|u(q) - u(q^\dagger)\|_{W^{1,p}(\Omega)}^{\frac{1}{1+\kappa}} \quad \forall q \in \tilde{\mathcal{A}}_p.$$

*Proof.* Let  $p \in [2, \bar{p}] \cap [2, \infty)$  and  $q \in \mathcal{A}$ . From the weak formulations of  $u(q)$  and  $u(q^\dagger)$ , it follows for all  $v \in W_0^{1,p'}(\Omega)$  that

$$(3.23) \quad \begin{aligned} \int_{\Omega} (q^\dagger - q) u(q^\dagger)^m v \, dx &= \int_{\Omega} [\sigma \nabla(u(q) - u(q^\dagger)) \cdot \nabla v + q(u(q)^m - u(q^\dagger)^m) v] \, dx \\ &\stackrel{(2.7)}{=} \int_{\Omega} \sigma \nabla(u(q) - u(q^\dagger)) \cdot \nabla v + q \left( \sum_{k=0}^{m-1} u(q)^k u(q^\dagger)^{m-1-k} \right) (u(q) - u(q^\dagger)) v \, dx. \end{aligned}$$

Furthermore, by Lemma 2.3, (3.16), we have  $C_{\mathcal{A}} := \sup_{q \in \mathcal{A}} \|u(q)\|_{C(\bar{\Omega})} < \infty$ , which together with (3.23) and (2.2) implies that

$$(3.24) \quad \int_{\Omega} (q^\dagger - q) u(q^\dagger)^m v \, dx \leq \underbrace{\left( n \max_{1 \leq i,j \leq n} \|\sigma_{i,j}\|_{L^\infty(\Omega)} + m\bar{q} C_{\mathcal{A}}^{m-1} \right)}_{=:\hat{C}} \|u(q) - u(q^\dagger)\|_{W^{1,p}(\Omega)} \|v\|_{W^{1,p'}(\Omega)}$$

holds for all  $v \in W_0^{1,p'}(\Omega)$ . In view of Assumption 3.6 and since  $p' = \frac{p}{p-1} \in (1, 2]$ , the weighted space  $\mathcal{X}_{u^\dagger}^{p'}(\Omega)$  as in Definition 3.3 is well-defined for  $w = u(q^\dagger)$  and  $\theta = p'$ . For every arbitrarily fixed  $\varphi \in \mathcal{X}_{u^\dagger}^{p'}(\Omega)$ , the construction of  $\mathcal{X}_{u^\dagger}^{p'}(\Omega)$  (see (3.11) with  $w = u(q^\dagger)$  and  $\theta = p'$ ) implies the existence of a sequence of  $\{\varphi_k\}_{k \in \mathbb{N}} \subset C_0^\infty(\Omega)$  such that  $u(q^\dagger)^{-m} \varphi_k \in W_0^{1,p'}(\Omega)$  and  $\varphi_k \rightarrow \varphi$  in  $\mathcal{X}_{u^\dagger}^{p'}(\Omega)$  as  $k \rightarrow \infty$ . Thus, we may set  $v = u(q^\dagger)^{-m} \varphi_k$  in (3.24) to obtain that

$$(3.25) \quad \left| \int_{\Omega} (q - q^\dagger) \varphi_k \, dx \right| \leq \hat{C} \|u(q) - u(q^\dagger)\|_{W^{1,p}(\Omega)} \|\varphi_k\|_{\mathcal{X}_{u^\dagger}^{p'}(\Omega)} \quad \forall k \in \mathbb{N} \quad \forall q \in \mathcal{A}.$$

Thus, in view of the continuous embedding (3.12), we may pass to the limit  $k \rightarrow \infty$  in (3.25) to obtain

$$\left| \int_{\Omega} (q - q^\dagger) \varphi \, dx \right| \leq \hat{C} \|u(q) - u(q^\dagger)\|_{W^{1,p}(\Omega)} \|\varphi\|_{\mathcal{X}_{u^\dagger}^{p'}(\Omega)} \quad \forall q \in \mathcal{A}.$$

Since  $\varphi \in \mathcal{X}_{u^\dagger}^{p'}(\Omega)$  was chosen arbitrarily, it follows that

$$(3.26) \quad \|q - q^\dagger\|_{\mathcal{X}_{u^\dagger}^{p'}(\Omega)^*} = \sup_{\varphi \neq 0} \frac{\left| \int_{\Omega} (q - q^\dagger) \varphi \, dx \right|}{\|\varphi\|_{\mathcal{X}_{u^\dagger}^{p'}(\Omega)}} \leq \hat{C} \|u(q) - u(q^\dagger)\|_{W^{1,p}(\Omega)} \quad \forall q \in \mathcal{A}.$$

On the one hand, for the case  $\kappa \in (0, 1)$ , the estimate (3.21) follows immediately from (3.26) and (3.17). On the other hand, for the case  $\kappa > 1$ , we have that

$$\begin{aligned} \|q - q^\dagger\|_{W^{-1,p}(\Omega)} &\stackrel{(3.17)}{\leq} C \|q - q^\dagger\|_{\mathcal{X}_{u^\dagger}^{p'}(\Omega)^*}^{\frac{1}{\kappa}} \|q - q^\dagger\|_{L^p(\Omega)}^{\frac{\kappa-1}{\kappa}} \stackrel{(3.26)}{\leq} C \hat{C}^{\frac{1}{\kappa}} \|u(q) - u(q^\dagger)\|_{W^{1,p}(\Omega)}^{\frac{1}{\kappa}} \|q - q^\dagger\|_{L^p(\Omega)}^{\frac{\kappa-1}{\kappa}} \\ &\leq C \hat{C}^{\frac{1}{\kappa}} 2\bar{q} |\Omega|^{\frac{\kappa-1}{\kappa}} \|u(q) - u(q^\dagger)\|_{W^{1,p}(\Omega)}^{\frac{1}{\kappa}} \quad \forall q \in \mathcal{A}, \end{aligned}$$

leading to (3.21). Similarly, the combination of (3.26) and (3.18) shows for all  $q \in \tilde{\mathcal{A}}$  that

$$\|q - q^\dagger\|_{L^p(\Omega)} \stackrel{(3.18)}{\leq} C \|q - q^\dagger\|_{\mathcal{X}_{u^\dagger}^{p'}(\Omega)^*}^{\frac{1}{1+\kappa}} \|q - q^\dagger\|_{W^{1,p}(\Omega)}^{\frac{\kappa}{1+\kappa}} \stackrel{(3.26)}{\leq} C \hat{C}^{\frac{1}{1+\kappa}} (2\tilde{M})^{\frac{\kappa}{1+\kappa}} \|u(q) - u(q^\dagger)\|_{W^{1,p}(\Omega)}^{\frac{1}{1+\kappa}},$$

where we have used the uniform boundedness of  $q \in \tilde{\mathcal{A}}_p$  in  $W^{1,p}(\Omega)$ . This gives (3.22) and hence completes the proof.  $\square$

**3.3. Verification of Assumption 3.6.** This subsection is devoted to the verification of Assumption 3.6 on a convex domain. The proof relies on the following pointwise lower bound estimate.

**Lemma 3.10.** *Let Assumption 2.1 hold and  $q \in C(\bar{\Omega}) \cap C^\vartheta(\Omega)$  for some  $\vartheta > 0$  be nonnegative. Suppose additionally that  $\Omega$  is convex,  $f \in C(\bar{\Omega}) \cap C^\vartheta(\Omega)$  is non-negative,  $m \geq 2$ , and  $\sigma \in C^\infty(\Omega)^{n \times n}$ . Then, the unique weak solution  $u(q)$  to (1.1) satisfies*

$$u(q)(x) \geq H(x) \left( 1 + (m-1) \int_{\Omega} G(x,y) H(y)^m q(y) dy \right)^{-\frac{1}{m-1}} \quad \forall x \in \Omega,$$

where  $H(x) := \int_{\Omega} G(x,y) f(y) dy$ .

*Proof.* By Lemma 2.3, it follows that  $u(q) \in H_0^1(\Omega) \cap W^{1,\infty}(\Omega)$ . Thanks to the regularity properties  $u(q) \in W^{1,\infty}(\Omega)$  and  $q, f \in C(\bar{\Omega}) \cap C^\vartheta(\Omega)$ , we see that the weak solution  $u(q)$  satisfies

$$-\nabla \cdot (\sigma \nabla u(q)) = f - q u(q)^m \in C(\bar{\Omega}) \cap C^\vartheta(\Omega)$$

Consequently, Schauder's estimate [14, Thm 6.13] implies  $u(q) \in C^{2,\theta}(\Omega)$ . Altogether, we may apply [15, Theorem 3.1] to the solution  $u(q) \in C^{2,\theta}(\Omega)$  and complete the proof.  $\square$

**Remark 3.11.** *The smooth requirement  $\sigma \in C^\infty(\bar{\Omega})^{n \times n}$  in the above lemma can be weakened to  $\sigma \in W^{1,\beta}(\Omega)^{n \times n}$  for some  $\beta > n$ . This is possible by a perturbation argument and the pointwise estimates of Green functions on convex domains by [17].*

**Proposition 3.12.** *Let Assumption 2.1 hold. Assume additionally that  $\Omega$  is convex with dimension  $n = 2, 3$ ,  $\sigma \in C^\infty(\bar{\Omega})^{n \times n}$ , and there exists a constant  $c_f > 0$  such that  $f(x) \geq c_f$  holds for a.e.  $x \in \Omega$ . Then, there exists  $c^\dagger > 0$  such that*

$$(3.27) \quad u(q)(x) \geq \begin{cases} c^\dagger \rho(x)^{2+\frac{1}{m-1}} & \text{for all } x \in \bar{\Omega}, q \in \mathcal{A}, \text{ as } m > 1 \\ c^\dagger \rho(x)^2 & \text{for all } x \in \bar{\Omega}, q \in \mathcal{A}, \text{ as } m = 1. \end{cases}$$

*In particular, Assumption 3.6 is satisfied.*

*Proof.* Since the case  $m = 1$  has been treated in [23] (i.e., the linear case), we will prove the rest. Let us divide the proof into three steps:

(Step 1.) Let  $q \in \mathcal{A}$  be arbitrarily fixed. By  $\bar{u}, u_c \in H_0^1(\Omega) \cap C(\bar{\Omega})$ , we denote, respectively, the unique weak solutions to

$$\begin{cases} -\nabla \cdot (\sigma \nabla \bar{u}) + \bar{q} \bar{u}^m = c_f & \text{in } \Omega \\ \bar{u} = 0 & \text{on } \partial\Omega \end{cases}, \quad \begin{cases} -\nabla \cdot (\sigma \nabla u_c) + q u_c^m = c_f & \text{in } \Omega \\ u_c = 0 & \text{on } \partial\Omega \end{cases}.$$

Since  $c_f > 0$ , the maximum principle (Lemma 2.4) applied to above systems yield

$$(3.28) \quad \bar{u}(x) > 0 \quad \text{and} \quad u_c(x) > 0 \quad \forall x \in \Omega.$$

Moreover, since  $f(x) \geq c_f$  holds for a.e.  $x \in \Omega$ , the comparison principle (Lemma 2.4 (iii)) implies that

$$(3.29) \quad u(q)(x) \geq u_c(x) \quad \forall x \in \bar{\Omega}.$$

Now, we see that  $e := u_c - \bar{u} \in H_0^1(\Omega) \cap C(\bar{\Omega})$  is the unique weak solution to

$$(3.30) \quad -\nabla \cdot (\sigma \nabla e) + q \left( \sum_{k=0}^{m-1} u_c^k \bar{u}^{m-1-k} \right) e = (\bar{q} - q) \bar{u}^m \quad \text{in } \Omega \quad \text{and} \quad e = 0 \quad \text{on } \partial\Omega.$$

In view of (3.28) and since  $q(x) \leq \bar{q}$  holds for a.e.  $x \in \Omega$ , we have

$$(\bar{q}(x) - q(x)) \bar{u}^m(x) \geq 0 \quad \text{for a.e. } x \in \Omega.$$

Thus, applying the maximum principle (Lemma 2.4) to (3.30) gives

$$e_u(x) \geq 0 \quad \forall x \in \bar{\Omega} \implies u_c(x) \geq \bar{u}(x), \quad \forall x \in \bar{\Omega}.$$

Combining the above inequality with (3.29) and since  $q \in \mathcal{A}$  was chosen arbitrarily, we infer that

$$u(q)(x) \geq \bar{u}(x), \quad \forall x \in \bar{\Omega} \quad \forall q \in \mathcal{A}.$$

In conclusion, the claim (3.27) is a consequence of

$$(3.31) \quad \bar{u}(x) \geq c^\dagger \rho(x)^{2+\frac{1}{m-1}} \quad \forall x \in \bar{\Omega},$$

for some constant  $c^\dagger > 0$ . We will then prove (3.31) by the following steps.

(Step 2.) Let us denote by  $G$  the Green function (see [13]) for the elliptic operator  $-\nabla \cdot \sigma \nabla : W_0^{1,p}(\Omega) \rightarrow W_0^{1,p'}(\Omega)^*$  with  $p \in (2, \bar{p})$  as in Lemma 2.2. Furthermore, we set

$$H \in C(\bar{\Omega}) \cap W_0^{1,p}(\Omega), \quad H(x) := \int_{\Omega} G(x, y) c_f dy \quad \forall x \in \Omega,$$

where the regularity  $H \in C(\bar{\Omega})$  is obtained due to the construction that it is the unique weak solution of

$$-\nabla \cdot (\sigma \nabla H) = c_f \quad \text{in } \Omega \quad \text{and} \quad H = 0 \quad \text{on } \partial\Omega.$$

Moreover, one can ensure  $H \in C^2(\Omega)$  using the Schauder estimate introduced in [14, Chapter 6].

Let us show that

$$(3.32) \quad \exists C_f \in (0, \infty) \quad \forall x \in \Omega : \quad C_f^{-1} \rho(x)^2 \leq H(x) \leq C_f.$$

To this end, on the one hand, the right-hand side estimate in (3.32) obviously holds for any constant being larger than or equal to  $\|H\|_{C(\bar{\Omega})}$ . On the other hand, since  $n \geq 2$ , [23, Theorem 2.2] implies the existence of a constant  $C_\Omega > 0$  such that

$$G(x, y) \geq \frac{C_\Omega^{-1}}{|x - y|^{n-2}} \quad \forall x, y \in \Omega \quad \text{with} \quad |x - y| \leq \frac{1}{2} \rho(x).$$

The above estimate, along with the nonnegativity of the Green function  $G$ , implies finally

$$H(x) \geq \frac{c_f}{C_\Omega} \int_{B(x, \rho_x)} |x - y|^{2-n} dy = \frac{c_f}{8C_\Omega} \rho(x)^2 |\mathbb{S}_{n-1}| \quad \forall x \in \Omega,$$

where  $B(x, \frac{1}{2}\rho_x)$  denotes the open ball with radius  $\frac{1}{2}\rho_x$  centered at  $x$ , and  $|\mathbb{S}_{n-1}|$  is the Lebesgue measure of  $(n-1)$ -dimensional unit sphere. This proves the claim (3.32) with  $C_f := \max\left(\|H\|_{C(\bar{\Omega})}, \frac{c_f}{8C_\Omega} |\mathbb{S}_{n-1}|\right)$ .

(Step 3) The application of Lemma 3.10 shows that

$$(3.33) \quad \bar{u}(x) \geq H(x) \left( 1 + (m-1) \frac{1}{H(x)} \int_{\Omega} G(x, y) H^m(y) \bar{q} dy \right)^{-\frac{1}{m-1}}, \quad \forall x \in \Omega.$$

By the definition of the Green function, the function  $w := \int_{\Omega} G(\cdot, y) H^m(y) \bar{q} dy$  solves the linear elliptic equation

$$-\nabla \cdot (\sigma \nabla w) = H^m \bar{q} \quad \text{in } \Omega \quad \text{and} \quad w = 0 \quad \text{on } \partial\Omega.$$

Thus, since  $H^m \bar{q} \in C(\bar{\Omega})$ ,  $\Omega$  is convex, and  $\sigma \in C^\infty(\Omega)^{n \times n}$ , it follows that  $w \in W^{1, \infty}(\Omega)$ , and then the fact that  $w$  vanishes on  $\partial\Omega$  further implies

$$w(x) \leq \|\nabla w\|_{L^\infty(\Omega)} \rho(x) \leq C_{p,r} \|H^m \bar{q}\|_{L^r(\Omega)} \rho(x), \quad \text{for all } x \in \Omega,$$

where  $C_{p,r} > 0$  is the constant given in the estimate (2.4). Combing this with (3.32), we have that

$$\frac{1}{H(x)} \int_{\Omega} G(x, y) H^m(y) \bar{q} dy \leq \frac{C_f C_{p,r} \|H^m \bar{q}\|_{L^r(\Omega)}}{\rho(x)} \quad \forall x \in \Omega,$$

which, together with (3.33), shows that

$$\bar{u}(x) \geq c_0 \rho(x)^{2+\frac{1}{m-1}} (c_0 \rho(x) + (m-1)c_1)^{-\frac{1}{m-1}} \geq c^\dagger \rho(x)^{2+\frac{1}{m-1}},$$

where  $c_1 = C_{p,r} \|H^m \bar{q}\|_{L^r(\Omega)}$ . Here, the constant  $c^\dagger > 0$  depends on  $c_0$ , but is independent of the value  $m$ . This completes the proof, since  $\rho(x) \leq \text{diam}(\Omega) < \infty$  is uniformly bounded.  $\square$

#### 4. FINITE ELEMENT APPROXIMATIONS

This section analyzes the numerical reconstruction of  $q$  based on the piecewise linear FEM as readily proposed in the introduction. Let us begin by introducing the standing assumption for our numerical analysis:

**Assumption 4.1.** *Let Assumption 2.1 be satisfied with  $\Omega$  being either a bounded convex polygonal domain in  $\mathbb{R}^2$ , or a bounded convex polyhedral domain in  $\mathbb{R}^3$ . let additionally  $\sigma \in W^{1, \infty}(\Omega)^{n \times n}$  and  $r > n$ . Furthermore, the true coefficient  $q^\dagger$  is assumed to satisfy*

$$q^\dagger \in H^1(\Omega) \quad \text{and} \quad \underline{q} \leq q^\dagger(x) \leq \bar{q} \quad \text{for a.e. } x \in \Omega.$$

**Remark 4.2.** *The proposed regularity condition for  $q^\dagger$  for our error analysis is significantly weaker than the  $H^2(\Omega)$ -regularity assumption considered in [23] for the linear case  $m = 1$ . This relaxation is achieved by employing the Carstensen quasi-interpolation operator [2] in combination with the  $L^2$  projection operator and corresponding error estimates in  $H^{-1}(\Omega)$ .*

From Lemma 2.3, it follows that Assumption 4.1 implies that  $u(q) \in H_0^1(\Omega) \cap H^2(\Omega) \cap W^{1, \infty}(\Omega)$  holds for all  $q \in \mathcal{A}$ . Supposing that Assumption 4.1 holds, in all what follows, let  $\{\mathcal{T}_h\}_{h>0}$  be a quasi-uniform family of simplicial triangulations of  $\Omega$  and

$$(4.1) \quad S_h := \{u_h \in C(\bar{\Omega}) : u_h|_T \text{ is a polynomial of degree 1 on } T \text{ for all } T \in \mathcal{T}_h\}, \quad S_h^0 := S_h \cap H_0^1(\Omega).$$

In view of the quasi uniformity of  $\{\mathcal{T}_h\}_{h>0}$ , the following inequalities (see [11, Corollary 1.141]) are obtained:

$$(4.2) \quad \|v_h\|_{H^l(\Omega)} \leq Ch^{-(l-k)} \|v_h\|_{H^k(\Omega)} \quad \forall v_h \in S_h \quad \forall 0 \leq k \leq l \leq 1.$$

We consider the finite element approximation of (1.1) as follows: Given  $q \in \mathcal{A}$ , let  $u_h(q) \in S_h^0$  denote the unique solution to

$$(4.3) \quad a(u_h(q), v) + (qu_h(q)^m, v)_{L^2(\Omega)} = (f, v)_{L^2(\Omega)} \quad \forall v \in S_h^0$$

with  $a : H_0^1(\Omega) \times H_0^1(\Omega) \rightarrow \mathbb{R}$ ,  $a(u, v) := \int_{\Omega} \sigma \nabla u \cdot \nabla v \, dx$ . The well-posedness of the discrete variational problem (4.3) follows standard arguments (cf. [3, p. 3]).

On the basis of (4.3), we numerically reconstruct  $q^\dagger$ , defined by a minimizer  $q_{h,\alpha}^\delta \in \mathcal{A}_h := \mathcal{A} \cap S_h$  (see (3.16)) of the following least squares minimization problem:

$$(4.4) \quad \min_{q_h \in \mathcal{A}_h} \mathcal{J}_{\alpha,h}^\delta(q_h) := \frac{1}{2} \|u_h(q_h) - y^\delta\|_{L^2(\Omega)}^2 + \frac{\alpha}{2} (\|q_h\|_{L^2(\Omega)}^2 + \|\nabla q_h\|_{L^2(\Omega)}^2)$$

with noise measurement  $y^\delta \in L^2(\Omega)$  satisfying

$$(4.5) \quad \|u(q^\dagger) - y^\delta\|_{L^2(\Omega)} \leq \delta.$$

Note that the existence of a minimizer  $q_{q,h}^\delta$  for every fixed  $h, \delta, \alpha$  follows from the Weierstrass lemma, as the feasible set  $\mathcal{A}_h$  is compact and  $u_h : \mathcal{A}_h \subset S_h \rightarrow S_h$  is continuous (see Remark 4.5).

**4.1. Error estimates.** Our goal is to establish error estimates between the finite element approximation  $q_{h,\alpha}^\delta$  and the true solution  $q^\dagger$ . Let us start by introducing the operators  $A : H_0^1(\Omega) \rightarrow H^{-1}(\Omega)$  and  $A_h : S_h^0 \rightarrow S_h^0$  associated with the bilinear form  $a$ , respectively, for the continuous and discrete setting by

$$\langle Au, v \rangle_{H^{-1}(\Omega), H_0^1(\Omega)} := a(u, v) \quad \forall u, v \in H_0^1(\Omega), \quad (A_h w_h, v_h)_{L^2(\Omega)} := a(w_h, v_h) \quad \forall w_h, v_h \in S_h^0.$$

By  $P_h : L^2(\Omega) \rightarrow S_h$  and  $R_h : W_0^{1,1}(\Omega) \rightarrow S_h^0$ , we denote, respectively, the standard  $L^2$  projection onto  $S_h$  and the Ritz projection associated with the elliptic operator  $A$ , i.e., they satisfy

$$(P_h u, v_h)_{L^2(\Omega)} = (u, v_h)_{L^2(\Omega)} \quad \forall u \in L^2(\Omega), v_h \in S_h, \quad a(R_h w, v_h) = a(w, v_h) \quad \forall w \in H_0^1(\Omega), v_h \in S_h^0.$$

For every  $k = 0, 1, p \in (1, \infty]$ ,  $m = 1, 2$ , it is well-known that there is a constant  $C > 0$ , depending only on  $\Omega, \sigma$  and shape regularity constants, such that (see[11, Proposition 1.134] and [33, Theorem 2.2]):

$$(4.6) \quad \|u - P_h u\|_{W^{k,p}(\Omega)} \leq Ch^{m-k} \|u\|_{W^{m,p}(\Omega)} \quad \forall u \in W^{m,p}(\Omega) \quad \forall h > 0$$

$$\|u - R_h u\|_{H^k(\Omega)} \leq Ch^{m-k} \|u\|_{H^m(\Omega)} \quad \forall u \in H^m(\Omega) \quad \forall h > 0$$

$$(4.7) \quad \|P_h u\|_{L^p(\Omega)} \leq C \|u\|_{L^p(\Omega)}, \quad \|R_h u\|_{W^{1,p}(\Omega)} \leq C \|u\|_{W^{1,p}(\Omega)} \quad \forall u \in W^{1,p}(\Omega) \quad \forall h > 0.$$

Our error analysis also relies on the use of the Carstensen quasi-interpolation operator [2], defined by

$$\mathcal{I}_h : L^1(\Omega) \rightarrow S_h, \quad \mathcal{I}_h(u) := \sum_{i=1}^{N_{\mathcal{T}}} \pi_i(u) \phi_i \quad \pi_i(u) := \frac{\int_{\omega_i} \phi_i u \, dx}{\int_{\omega_i} \phi_i \, dx} \quad \text{with } \omega_i := \text{supp } \phi_i,$$

where  $\{\phi_i\}_{i=1}^{N_{\mathcal{T}}} \subset S_h$  denotes the canonical nodal basis of  $S_h$ . It is well-known (see [2, Theorem 3.1]) that  $\mathcal{I}_h : L^1(\Omega) \rightarrow S_h$  enjoys the  $H^1$ -stability: There exists a constant  $C > 0$  independent of  $h > 0$  such that

$$(4.8) \quad \|\nabla \mathcal{I}_h u\|_{L^2(\Omega)} \leq C \|\nabla u\|_{L^2(\Omega)} \quad \forall u \in H^1(\Omega).$$

By definition, it also satisfies

$$(4.9) \quad \underline{q} \leq q(x) \leq \bar{q} \text{ for a.e. } x \in \Omega \implies \underline{q} \leq \mathcal{I}_h(q)(x) \leq \bar{q} \text{ for a.e. } x \in \Omega.$$

Furthermore, according to [6, Lemma 4.4] along with

$$\|u\|_{H^{-1}(\Omega)} = \sup_{\varphi \in H_0^1(\Omega) \setminus \{0\}} \frac{(u, \varphi)_{L^2(\Omega)}}{\|\varphi\|_{H^1(\Omega)}} \leq \sup_{\varphi \in H^1(\Omega) \setminus \{0\}} \frac{(u, \varphi)_{L^2(\Omega)}}{\|\varphi\|_{H^1(\Omega)}} = \|u\|_{H^1(\Omega)^*} \quad \forall u \in L^2(\Omega),$$

there exists a constant  $C > 0$ , independent of  $h$ , such that

$$(4.10) \quad \|\mathcal{I}_h(u) - u\|_{H^{-1}(\Omega)} \leq Ch^2 \|u\|_{H^1(\Omega)} \quad \forall u \in H^1(\Omega).$$

Moreover, a priori error estimation for (4.3) is obtained as follows:

**Lemma 4.3** ([3, Theorem 2]). *Let Assumption 4.1 be satisfied. Then, there exist constants  $h_0, C > 0$  such that*

$$(4.11) \quad \|u(q) - u_h(q)\|_{L^2(\Omega)} + h\|u(q) - u_h(q)\|_{H^1(\Omega)} \leq Ch^2 \quad \forall h \in (0, h_0) \quad \forall q \in \mathcal{A}$$

$$(4.12) \quad C_{\mathcal{A}} := \sup_{h \in (0, h_0)} \sup_{q \in \mathcal{A}} (\|u_h(q)\|_{H^1(\Omega)} + \|u_h(q)\|_{C(\bar{\Omega})}) < \infty.$$

**Lemma 4.4.** *Let Assumption 4.1 hold, and  $h_0 > 0$  be as in Lemma 4.3. Then, there is a constant  $C > 0$ , independent of  $h$ , such that*

$$(4.13) \quad \|u(q^\dagger) - u_h(\mathcal{I}_h q^\dagger)\|_{L^2(\Omega)} + h\|\nabla(u(q^\dagger) - u_h(\mathcal{I}_h q^\dagger))\|_{L^2(\Omega)} \leq Ch^2 \quad \forall h \in (0, h_0).$$

*Proof.* By (4.11), it suffices to show that there is a constant  $C > 0$  such that

$$(4.14) \quad \|u_h(q^\dagger) - u_h(\mathcal{I}_h q^\dagger)\|_{L^2(\Omega)} + h\|\nabla(u_h(q^\dagger) - u_h(\mathcal{I}_h q^\dagger))\|_{L^2(\Omega)} \leq Ch^2 \quad \forall h \in (0, h_0).$$

Thanks to Assumption 4.1 and (4.9), it holds that  $\mathcal{I}_h q^\dagger \in \mathcal{A}$  for all  $h > 0$ . Let us next prove that

$$(4.15) \quad C_\infty := \sup_{h \in (0, h_0)} \|u_h(q^\dagger)\|_{W^{1,\infty}(\Omega)} < \infty.$$

Using the  $L^2$  projection operator  $P_h$  and the Ritz projection operator  $R_h$ , it follows from (4.3) that

$$\begin{aligned} A_h u_h(q^\dagger) + P_h q^\dagger u_h(q^\dagger) = P_h f & \implies u_h(q^\dagger) = A_h^{-1} P_h (f - q^\dagger u_h(q^\dagger)^m) \\ & \implies u_h(q^\dagger) = R_h A^{-1} (f - q^\dagger u_h(q^\dagger)^m). \\ & \underbrace{A_h^{-1} P_h = R_h A^{-1}} \end{aligned}$$

On the other hand, we know that  $A^{-1} \in \mathcal{L}(L^r(\Omega), W^{1,\infty}(\Omega))$  due to the convexity assumption on  $\Omega$  and the regularity  $\sigma \in C^{0,1}(\bar{\Omega})^{3 \times 3}$  (see (2.6)). Thus, along with Hölder's inequality and (4.7), it follows that

$$\begin{aligned} \|u_h(q^\dagger)\|_{W^{1,\infty}(\Omega)} &= \|R_h A^{-1} (f - q^\dagger u_h(q^\dagger)^m)\|_{W^{1,\infty}(\Omega)} \\ &\stackrel{(4.7)}{\leq} \|A^{-1} (f - q^\dagger u_h(q^\dagger)^m)\|_{W^{1,\infty}(\Omega)} \leq C \|f - q^\dagger u_h(q^\dagger)^m\|_{L^r(\Omega)} \\ &\leq C (\|f\|_{L^r(\Omega)} + \|q^\dagger\|_{L^\infty(\Omega)} \|u_h(q^\dagger)\|_{C(\bar{\Omega})}^m) \\ &\stackrel{(4.12)}{\leq} C (\|f\|_{L^r(\Omega)} + \bar{q} C_{\mathcal{A}}^m) \quad \forall h \in (0, h_0). \end{aligned}$$

In conclusion, (4.15) is valid.

With this result at hand, we now derive (4.14). Subtracting the weak formulations for  $u_h(q^\dagger)$  and  $u_h(\mathcal{I}_h q^\dagger)$  (cf. (4.3)) from each other and then using the algebraic identity (2.7), we deduce with  $W_h := \sum_{k=0}^{m-1} u_h(\mathcal{I}_h q^\dagger)^k u_h(q^\dagger)^{m-1-k}$  that

$$(4.16) \quad \begin{aligned} &a(u_h(\mathcal{I}_h q^\dagger) - u_h(q^\dagger), v_h) + (\mathcal{I}_h q^\dagger W_h (u_h(\mathcal{I}_h q^\dagger) - u_h(q^\dagger)), v_h)_{L^2(\Omega)} \\ &= ((q^\dagger - \mathcal{I}_h q^\dagger) u_h(q^\dagger)^m, v_h)_{L^2(\Omega)} \quad \forall v_h \in S_h^0. \end{aligned}$$

On the other hand, thanks to (2.8) and the non-negativity of  $\mathcal{I}_h q^\dagger \in \mathcal{A}$ , we have that  $\mathcal{I}_h q^\dagger W_h \geq 0$ . Thus, it follows from (4.16) and Poincaré inequality that

$$(4.17) \quad \|u_h(\mathcal{I}_h q^\dagger) - u_h(q^\dagger)\|_{H^1(\Omega)} \leq C \|(\mathcal{I}_h q^\dagger - q^\dagger) u_h(q^\dagger)^m\|_{H^{-1}(\Omega)} \quad \forall h \in (0, h_0)$$

with  $C > 0$  depending only on  $\Omega$ . By the definition of  $H^{-1}(\Omega)$ , it follows that

$$\begin{aligned} \|(\mathcal{I}_h q^\dagger - q^\dagger)u_h(q^\dagger)^m\|_{H^{-1}(\Omega)} &:= \sup_{\|\varphi\|_{H_0^1(\Omega)}=1} \{((\mathcal{I}_h q^\dagger - q^\dagger)u_h(q^\dagger)^m, \varphi)_{L^2(\Omega)}\} \\ &\leq \|\mathcal{I}_h q^\dagger - q^\dagger\|_{H^{-1}(\Omega)} \sup_{\|\varphi\|_{H_0^1(\Omega)}=1} \{\|u_h(q^\dagger)^m \varphi\|_{H_0^1(\Omega)}\} \quad \forall h \in (0, h_0). \end{aligned}$$

Combined with (4.10) and the chain rule, this finally leads to

$$(4.18) \quad \|(\mathcal{I}_h q^\dagger - q^\dagger)u_h(q^\dagger)^m\|_{H^{-1}(\Omega)} \leq Ch^2 \|u(q_h)^m\|_{W^{1,\infty}(\Omega)} \leq CC_\infty^m h^2 \quad \forall h \in (0, h_0).$$

Applying (4.17) to (4.18) implies finally (4.14). This completes the proof.  $\square$

**Remark 4.5.** By the same reasoning leading to (4.17), we obtain for every  $h > 0$  and  $q_h, q'_h \in \mathcal{A}_h$

$$\|u_h(q_h) - u_h(q'_h)\|_{H^1(\Omega)} \leq C \|(q_h - q'_h)u_h(q_h)^m\|_{H^{-1}(\Omega)} \leq C \|q_h - q'_h\|_{L^2(\Omega)} \|u_h(q_h)\|_{L^\infty(\Omega)}^m$$

with a constant  $C > 0$  independent of  $h > 0$ . For every fixed  $h > 0$ , this estimate implies in particular the continuity of the mapping  $u_h : \mathcal{A}_h \subset S_h \rightarrow S_h$ .

**Theorem 4.6.** Let Assumptions 3.6 and 4.1 be satisfied, and  $h_0 > 0$  be as in Lemma 4.3. Then, there is a constant  $C > 0$ , independent of  $h, \delta$ , and  $\alpha$ , such that

$$(4.19) \quad \|u^\dagger - u_h(q_{h,\alpha}^\delta)\|_{L^2(\Omega)} \leq C \overbrace{(h^2 + \delta + \sqrt{\alpha})}^{=: \eta}$$

$$(4.20) \quad \|q^\dagger - q_{h,\alpha}^\delta\|_{L^2(\Omega)} \leq C(\eta + h + \min\{h + h^{-1}(\delta + \sqrt{\alpha}), 1\})^{\frac{1}{1+(m+1)\gamma}} (1 + \alpha^{-\frac{1}{2}\eta})^{\frac{(m+1)\gamma}{1+(m+1)\gamma}}$$

hold for every  $h \in (0, h_0)$ ,  $\alpha, \delta > 0$ , and every minimizer  $q_{h,\alpha}^\delta$  of (4.4).

*Proof.* The proof is divided into the following steps:

(Step 1.) Let us prove the existence of a constant  $C > 0$ , independent of  $h, \delta$ , and  $\alpha$ , such that

$$(4.21) \quad \|u_h(q_{h,\alpha}^\delta) - P_h u(q^\dagger)\|_{L^2(\Omega)} \leq C\eta = C(h^2 + \delta + \sqrt{\alpha}) \quad \forall h \in (0, h_0) \quad \forall \alpha, \delta > 0.$$

To this aim, let  $h \in (0, h_0)$  and  $\alpha, \delta > 0$  be arbitrarily fixed. Since  $q_{h,\alpha}^\delta \in \mathcal{A}_h$  is a minimizer of (4.4), we obtain from (4.9) that

$$(4.22) \quad \begin{aligned} &\|u_h(q_{h,\alpha}^\delta) - y^\delta\|_{L^2(\Omega)}^2 + \alpha(\|q_{h,\alpha}^\delta\|_{L^2(\Omega)}^2 + \|\nabla q_{h,\alpha}^\delta\|_{L^2(\Omega)}^2) = \mathcal{J}_{\alpha,h}^\delta(q_{h,\alpha}^\delta) \\ &\leq \mathcal{J}_{\alpha,h}^\delta(\mathcal{I}_h q^\dagger) = \|u_h(\mathcal{I}_h q^\dagger) - y^\delta\|_{L^2(\Omega)}^2 + \alpha(\|\mathcal{I}_h q^\dagger\|_{L^2(\Omega)}^2 + \|\nabla \mathcal{I}_h q^\dagger\|_{L^2(\Omega)}^2). \end{aligned}$$

By the triangle inequality, (4.5), (4.8), (4.9), and (4.13), the above right hand side can be estimated by

$$\begin{aligned} &\|u_h(\mathcal{I}_h q^\dagger) - y^\delta\|_{L^2(\Omega)}^2 + \alpha(\|\mathcal{I}_h q^\dagger\|_{L^2(\Omega)}^2 + \|\nabla \mathcal{I}_h q^\dagger\|_{L^2(\Omega)}^2) \\ &\leq 2\|u_h(\mathcal{I}_h q^\dagger) - u(q^\dagger)\|_{L^2(\Omega)}^2 + 2\|u(q^\dagger) - y^\delta\|_{L^2(\Omega)}^2 + \alpha(\|\mathcal{I}_h q^\dagger\|_{L^2(\Omega)}^2 + \|\nabla \mathcal{I}_h q^\dagger\|_{L^2(\Omega)}^2) \\ &\stackrel{(4.5), (4.13)}{\leq} C(h^4 + \delta^2) + \alpha(\|\mathcal{I}_h q^\dagger\|_{L^2(\Omega)}^2 + \|\nabla \mathcal{I}_h q^\dagger\|_{L^2(\Omega)}^2) \stackrel{(4.8), (4.9)}{\leq} C(h^4 + \delta^2 + \alpha) \leq C\eta^2 \end{aligned}$$

with a constant  $C > 0$ , independent of  $h, \delta$ , and  $\alpha$ . Applying the above estimate to (4.22) implies

$$(4.23) \quad \|u_h(q_{h,\alpha}^\delta) - y^\delta\|_{L^2(\Omega)}^2 + \alpha(\|q_{h,\alpha}^\delta\|_{L^2(\Omega)}^2 + \|\nabla q_{h,\alpha}^\delta\|_{L^2(\Omega)}^2) \leq C\eta^2.$$

Now, using the triangle inequality, the desired estimate (4.21) is obtained as follows:

$$\|u_h(q_{h,\alpha}^\delta) - P_h u(q^\dagger)\|_{L^2(\Omega)} \leq \|u_h(q_{h,\alpha}^\delta) - y^\delta\|_{L^2(\Omega)} + \|y^\delta - u(q^\dagger)\|_{L^2(\Omega)} + \|u(q^\dagger) - P_h u(q^\dagger)\|_{L^2(\Omega)}$$

$$\underbrace{\leq}_{(4.5),(4.6),(4.23)} C\eta + \delta + h^2 \|u(q^\dagger)\|_{H^2(\Omega)} \leq (C + 1 + \|u(q^\dagger)\|_{H^2(\Omega)})\eta.$$

As a consequence of (4.21),  $u^\dagger = u(q^\dagger) \in H^2(\Omega)$ , and (4.6), we conclude that (4.19) is valid.

(Step 2.) Let us prove the existence of a constant  $C > 0$ , independent of  $h, \delta$ , and  $\alpha$ , such that

$$(4.24) \quad \|q^\dagger - q_{h,\alpha}^\delta\|_{\mathcal{X}_{u^\dagger}^2(\Omega)^*} \leq C(\eta + h + \min\{h + h^{-1}(\delta + \sqrt{\alpha}), 1\}).$$

To this end, let  $h \in (0, h_0)$  and  $\alpha, \delta > 0$  be arbitrarily fixed. We notice that the variational formulations of  $u(q^\dagger)$  and  $u_h(q_{h,\alpha}^\delta)$

$$\begin{aligned} a(u(q^\dagger), v) + (q^\dagger u(q^\dagger)^m, v)_{L^2(\Omega)} &= (f, v)_{L^2(\Omega)} \quad \forall v \in H_0^1(\Omega) \\ a(u_h(q_{h,\alpha}^\delta), v_h) + (q_{h,\alpha}^\delta u_h(q_{h,\alpha}^\delta)^m, v_h)_{L^2(\Omega)} &= (f, v_h)_{L^2(\Omega)} \quad \forall v_h \in S_h^0 \end{aligned}$$

imply

$$(4.25) \quad \int_{\Omega} (q^\dagger - q_{h,\alpha}^\delta) u(q^\dagger)^m v_h \, dx = a(u_h(q_{h,\alpha}^\delta) - u(q^\dagger), v_h) + (q_{h,\alpha}^\delta (u_h(q_{h,\alpha}^\delta)^m - u(q^\dagger)^m), v_h)_{L^2(\Omega)} \quad \forall v_h \in S_h^0.$$

Now, let  $v \in H_0^1(\Omega)$  be arbitrarily fixed. Then, applying the identity

$$\int_{\Omega} (q^\dagger - q_{h,\alpha}^\delta) u(q^\dagger)^m v \, dx = \int_{\Omega} (q^\dagger - q_{h,\alpha}^\delta) u(q^\dagger)^m P_h v \, dx + \int_{\Omega} (q^\dagger - q_{h,\alpha}^\delta) u(q^\dagger)^m (I - P_h) v \, dx$$

to (4.25) with  $v_h = P_h v$  yields that

$$(4.26) \quad \begin{aligned} \int_{\Omega} (q^\dagger - q_{h,\alpha}^\delta) u(q^\dagger)^m v \, dx &= a(u_h(q_{h,\alpha}^\delta) - u(q^\dagger), P_h v) + (q_{h,\alpha}^\delta (u_h(q_{h,\alpha}^\delta)^m - u(q^\dagger)^m), P_h v) \\ &\quad + \int_{\Omega} (q^\dagger - q_{h,\alpha}^\delta) u(q^\dagger)^m (I - P_h) v \, dx =: \mathbf{J}_1 + \mathbf{J}_2 + \mathbf{J}_3. \end{aligned}$$

In the sequel, we estimate these three terms separately. To estimate  $\mathbf{J}_1$ , we first deduce from (4.6) and the inverse estimate (4.2) that

$$(4.27) \quad \begin{aligned} \mathbf{J}_1 &\leq \|\nabla(u_h(q_{h,\alpha}^\delta) - u(q^\dagger))\|_{L^2(\Omega)} \|\nabla P_h v\|_{L^2(\Omega)} \\ &\leq C(\|\nabla(u_h(q_{h,\alpha}^\delta) - P_h u(q^\dagger))\|_{L^2(\Omega)} + \|\nabla(P_h u(q^\dagger) - u(q^\dagger))\|_{L^2(\Omega)}) \|v\|_{H^1(\Omega)} \\ &\leq C(h^{-1} \|u_h(q_{h,\alpha}^\delta) - P_h u(q^\dagger)\|_{L^2(\Omega)} + h \|u(q^\dagger)\|_{H^2(\Omega)}) \|v\|_{H^1(\Omega)} \\ &\leq \underbrace{C}_{(4.21)} (h + h^{-1}(\delta + \sqrt{\alpha})) \|v\|_{H^1(\Omega)}. \end{aligned}$$

On the other hand, Lemma 4.3 along with  $q_{h,\alpha}^\delta \in \mathcal{A}$  yields that  $\|u_h(q_{h,\alpha}^\delta)\|_{H^1(\Omega)} \leq C_{\mathcal{A}}$ , and consequently

$$\|\nabla(u_h(q) - u(q^\dagger))\|_{L^2(\Omega)} \leq C_{\mathcal{A}} + \|u(q^\dagger)\|_{H^1(\Omega)}.$$

Combined with (4.27), this estimate shows that

$$(4.28) \quad \mathbf{J}_1 \leq C \min\{h + h^{-1}(\delta + \sqrt{\alpha}), 1\} \|v\|_{H^1(\Omega)}.$$

For the second term  $\mathbf{J}_2$ , we deduce that

$$\begin{aligned}
\mathbf{J}_2 &\leq C \|u_h(q_{h,\alpha}^\delta)^m - u(q^\dagger)^m\|_{L^2(\Omega)} \|P_h v\|_{L^2(\Omega)} \\
(4.29) \quad &\stackrel{(2.7)}{\leq} \underbrace{\left\| \sum_{k=0}^{m-1} u_h(q_{h,\alpha}^\delta)^{m-1-k} u(q^\dagger)^k \right\|_{L^\infty(\Omega)}}_{(4.12),(4.7)} \|u_h(q_{h,\alpha}^\delta) - u(q^\dagger)\|_{L^2(\Omega)} \|P_h v\|_{L^2(\Omega)} \\
&\leq C(m+1) C_{\mathcal{A}}^{m-1} \|u_h(q_{h,\alpha}^\delta) - u(q^\dagger)\|_{L^2(\Omega)} \|v\|_{L^2(\Omega)} \stackrel{(4.21)}{\leq} C\eta \|v\|_{H^1(\Omega)}.
\end{aligned}$$

To estimate the final term  $\mathbf{J}_3$ , we utilize (4.6) to obtain

$$(4.30) \quad \mathbf{J}_3 \leq \|(q^\dagger - q_{h,\alpha}^\delta)u(q^\dagger)^m\|_{L^2(\Omega)} \|(I - P_h)v\|_{L^2(\Omega)} \leq 2\bar{q} \|u(q^\dagger)\|_{C(\bar{\Omega})}^m Ch \|v\|_{H^1(\Omega)}.$$

Since  $v \in H_0^1(\Omega)$  was chosen arbitrarily, it follows from (4.26), (4.28), (4.29) and (4.30) that

$$(4.31) \quad \int_{\Omega} (q^\dagger - q_{h,\alpha}^\delta)u(q^\dagger)^m v \, dx \leq C(\eta + h + \min\{h + h^{-1}(\delta + \sqrt{\alpha}), 1\}) \|v\|_{H^1(\Omega)} \quad \forall v \in H_0^1(\Omega).$$

Applying the same argument as in Theorem 3.9 to (4.31) gives the desired estimate (4.24).

(Step 3.) From (4.23), it follows that

$$\|q^\dagger - q_{h,\alpha}^\delta\|_{H^1(\Omega)} \leq \|q^\dagger\|_{H^1(\Omega)} + \|q_{h,\alpha}^\delta\|_{H^1(\Omega)} \leq \|q^\dagger\|_{H^1(\Omega)} + C\alpha^{-\frac{1}{2}}\eta.$$

Finally, invoking the interpolation inequality (3.18) of Lemma 3.8 with  $p = 2$ ,  $\bar{p} = \infty$ , and  $\kappa = m\gamma + \gamma$  yields

$$\begin{aligned}
\|q^\dagger - q_{h,\alpha}^\delta\|_{L^2(\Omega)} &\leq C \|q^\dagger - q_{h,\alpha}^\delta\|_{\mathcal{X}_{u^\dagger}^{2(\Omega)^*}}^{\frac{1}{1+\kappa}} \|q^\dagger - q_{h,\alpha}^\delta\|_{H^1(\Omega)}^{\frac{\kappa}{1+\kappa}} \\
&\leq C(\eta + h + \min\{h + h^{-1}(\delta + \sqrt{\alpha}), 1\})^{\frac{1}{1+\kappa}} (1 + \alpha^{-\frac{1}{2}}\eta)^{\frac{\kappa}{1+\kappa}} \\
&= C(\eta + h + \min\{h + h^{-1}(\delta + \sqrt{\alpha}), 1\})^{\frac{1}{1+(m+1)\gamma}} (1 + \alpha^{-\frac{1}{2}}\eta)^{\frac{(m+1)\gamma}{1+(m+1)\gamma}}.
\end{aligned}$$

Thus, the claim is valid.  $\square$

**Remark 4.7.** Under the settings  $h \sim \sqrt{\delta}$  and  $\sqrt{\alpha} \sim \delta$ , Theorem 4.6 implies the following error estimates

$$\|u^\dagger - u(q_{h,\alpha}^\delta)\|_{L^2(\Omega)} \leq C\delta \quad \text{and} \quad \|q^\dagger - q_{h,\alpha}^\delta\|_{L^2(\Omega)} \leq C\delta^{\frac{1}{2(1+(m+1)\gamma)}}.$$

**Remark 4.8.** Theorem 4.6 significantly improves the existing error estimate by Jin et al. [23] for the linear case  $m = 1$  with regards to two aspects. First, as pointed out in Remark 4.2, the error estimation (4.20) is readily obtained under the  $H^1(\Omega)$ -regularity assumption on the true coefficient, omitting the  $H^2(\Omega)$ -regularity assumption from the prior contribution. Secondly, the achieved error estimate (4.20) improves the convergence order [23, Corollary 3.2] to the power of two. This improved convergence rate was readily observed in the numerical tests by [23]. Our numerical tests (see below) confirm our theoretical finding.

## 5. NUMERICAL SIMULATION AND DISCUSSIONS

In this section, we present two numerical tests to validate our theoretical analysis. All implementations were done in Python using the open-source FEniCS [30] on a Mac mini (Apple M4 Pro chip, 24 GB RAM) running macOS. Furthermore, the discrete minimization problem (4.4) is solved by the BFGS-B algorithm available through the `scipy.optimize` package in Python with the initial

guess  $q_0 \equiv 1.0$ . Regarding the data for our tests, we set  $\sigma \equiv 1$ ,  $\underline{q} = 0$ ,  $\bar{q} = 2$ ,  $m = 1$ , and the noisy data  $y^\delta$  is generated by

$$y^\delta = u^\dagger + \varepsilon \|u^\dagger\|_{L^\infty(\Omega)} \xi$$

with  $\xi$  being the standard uniform distribution. Two scenarios for the true coefficient  $q^\dagger$  and the corresponding true solution  $u^\dagger$  are considered as follows:

**Example 5.1.**

(a)  $\Omega := (0, 1)$ ,  $u^\dagger = \sin(\pi x)$ ,  $q^\dagger(x) = 1.5 - |x - 0.5|$ .

(b)  $\Omega := (0, 1)^2$ ,  $u^\dagger = \sin(\pi x_1) \sin(\pi x_2)$ ,  $q^\dagger(x) := 1 + 0.5(1 - 2 \max\{|x_1 - 0.5|, |x_2 - 0.5|\})$ .

Note that  $q^\dagger \in H^1(\Omega) \setminus H^2(\Omega)$  holds in both cases of Example 5.1. The discretization for Example 5.1 uses a uniform mesh of mesh size  $h$ . Moreover, the noisy level and the regularization parameter are set to be

$$\begin{cases} \delta = h^2 & \text{and} & \alpha = 10^{-2} \times \delta^2 & \text{for case (a),} \\ \delta = h^2 & \text{and} & \alpha = 5 \times \delta^2 & \text{for case (b).} \end{cases}$$

This choice corresponds to Remark 4.7.

Tables 1-2 depict the computed numerical error

$$e_q(\delta) := \|u^\dagger - u_h(q_{h,\alpha}^\delta)\|_{L^2(\Omega)} \quad \text{and} \quad e_u(\delta) := \|q^\dagger - q_{h,\alpha}^\delta\|_{L^2(\Omega)}$$

as well as the corresponding experimental order of convergence

$$\text{ECO}_u := \frac{\log(e_u(\delta_1)) - \log(e_u(\delta_2))}{\log \delta_1 - \log \delta_2} \quad \text{and} \quad \text{ECO}_q := \frac{\log(e_q(\delta_1)) - \log(e_q(\delta_2))}{\log \delta_1 - \log \delta_2}$$

for two consecutive noise level  $\delta_1$  and  $\delta_2$ . As pointed out in Remark 4.7, our theoretical finding predicts  $\text{ECO}_u = 1$  at best and  $\text{ECO}_q = 0.5$  at best, respectively. The numerical results for the one-dimensional and two-dimensional cases of Example 5.1 confirm these predictions. A clear convergence of the errors  $e_u(\delta)$  and  $e_q(\delta)$  is observed as  $\delta \rightarrow 0$ , with the experimental order of convergence closely matching the theoretical ones. Finally, the reconstructed coefficient  $q_{h,\alpha}^\delta$  is visually demonstrated in the plots (Figs. 5.1 and 5.3), which illustrate its convergence to  $q^\dagger$  as  $h \rightarrow 0$ .

TABLE 1. Examples 5.1 (a): Convergence behavior of numerical solutions

$h$	$\delta$	$e_u(\delta)$	$\text{EOC}_u$	$e_q(\delta)$	$\text{EOC}_q$
$1.56 \times 10^{-2}$	$2.44 \times 10^{-4}$	$9.34 \times 10^{-5}$	–	$7.12 \times 10^{-2}$	–
$7.81 \times 10^{-3}$	$6.1 \times 10^{-5}$	$2.59 \times 10^{-5}$	0.925	$3.28 \times 10^{-2}$	0.560
$3.91 \times 10^{-3}$	$1.53 \times 10^{-5}$	$6.2 \times 10^{-6}$	1.031	$1.31 \times 10^{-2}$	0.664
$1.95 \times 10^{-3}$	$3.81 \times 10^{-6}$	$1.55 \times 10^{-6}$	1.002	$6.26 \times 10^{-3}$	0.530

TABLE 2. Examples 5.1 (b): Convergence behavior of numerical solutions

$h$	$\delta$	$e_u(\delta)$	$\text{EOC}_u$	$e_q(\delta)$	$\text{EOC}_q$
$1.41 \times 10^{-1}$	$2.00 \times 10^{-2}$	$3.44 \times 10^{-3}$	–	$2.41 \times 10^{-1}$	–
$6.43 \times 10^{-2}$	$4.13 \times 10^{-3}$	$9.42 \times 10^{-4}$	0.822	$1.14 \times 10^{-1}$	0.474
$4.16 \times 10^{-2}$	$1.73 \times 10^{-3}$	$3.75 \times 10^{-4}$	1.059	$7.14 \times 10^{-2}$	0.537
$2.83 \times 10^{-2}$	$8.00 \times 10^{-4}$	$1.77 \times 10^{-4}$	0.972	$5.16 \times 10^{-2}$	0.420

FIGURE 5.1. Exact and recovered coefficients for Example 5.1 (a)

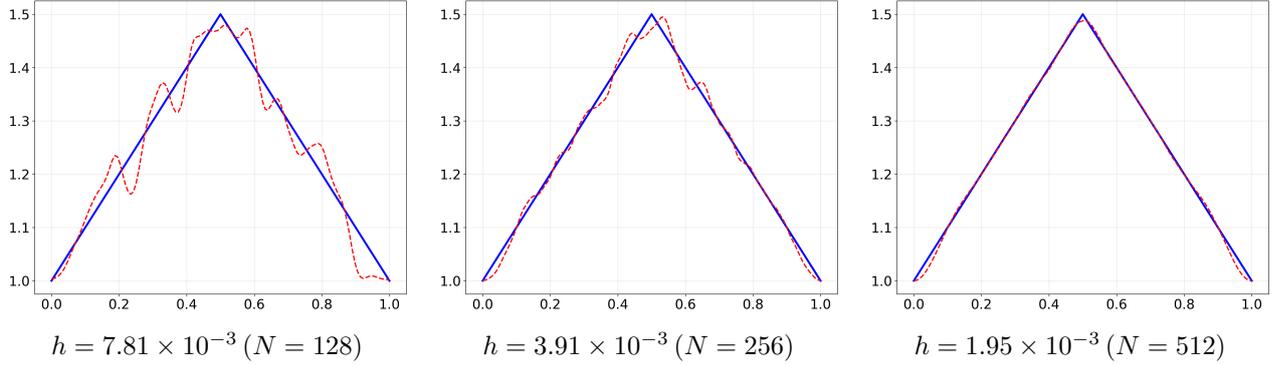
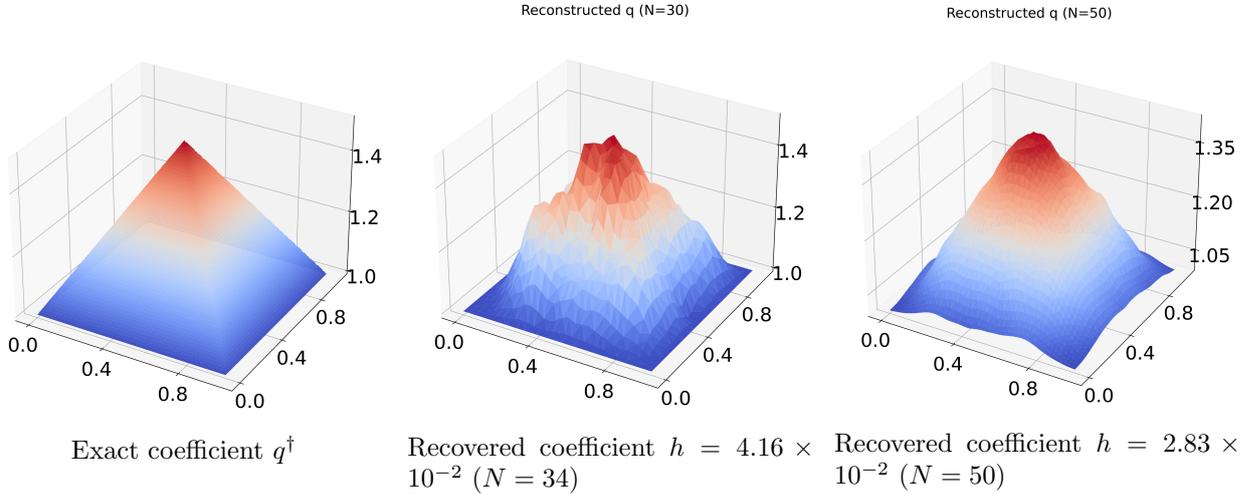


FIGURE 5.3. Exact and recovered coefficients for Example 5.1 (b)



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