GLOBAL RECOVERY OF A SCALAR DIFFUSIVITY FROM A SINGLE FINAL STATE

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Abstract. We prove global uniqueness and Lipschitz stability for a scalar diffusivity in the heat equation from a single final state at any prescribed positive time. On each prescribed uniformly elliptic $C^{1,\alpha}$-bounded class with known coefficient boundary values, we choose a constant initial datum and a spatially constant exponential boundary input that work for the entire class. The result holds on smooth bounded connected domains in every dimension $n \geq 2$ and gives an affirmative answer to [Ale20, Problem 8.2] within these classes. The stability estimate controls the L^2 diffusivity difference by the H^2 final-state difference, and this data norm is optimal within the Sobolev scale for the chosen experiment.

1. Introduction

Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$, be a bounded connected domain with smooth boundary, and fix $T > 0$. We consider the recovery of a positive real scalar diffusivity $a = a(x)$ in

$$\begin{cases} \partial_t u - \operatorname{div}(a\nabla u) = 0 & \text{in } \Omega \times (0,\infty), \\ u = g & \text{on } \partial\Omega \times (0,\infty), \\ u(\cdot,0) = h & \text{in } \Omega. \end{cases}$$

The initial datum h and boundary input g are prescribed, the boundary values of a are known, and the observation is the single final state $u(\cdot,T)$. We write $u(t) = u(\cdot,t)$, and use the same convention for the functions below. The question is whether one can choose h and g independently of the unknown coefficient so that this observation determines a. This is the final-data problem in [Ale20, Section 8, Problem 8.2].

The main difficulty is that the equation at the observation time contains the unmeasured derivative $\partial_t u(T)$. Thus, the final state alone does not provide the right-hand side of an elliptic coefficient equation. A coefficient difference also appears inside a divergence, while the desired estimate controls only its L^2 norm. Critical points of the final state can obstruct a direct transport argument. Moreover, comparing two unknown coefficients changes the evolution operator itself, so the inverse problem remains nonlinear despite the linearity of the forward equation in u.

1.1. Related results.

General accounts of inverse parabolic problems, including coefficient and source recovery, are given in [Cho09, Chapter 3] and [Isa17, Chapter 9]. Final overdetermination for parabolic equations has been studied in [AV85, Isa91]. The formulation in [Ale20, Section 8] distinguishes two questions. Problem 8.1 concerns recovery of a scalar diffusivity in two dimensions from a known source, zero initial and boundary data, and observations at one or more positive times. Problem 8.2 instead asks whether an appropriate

2020 *Mathematics Subject Classification.* Primary 35R30; Secondary 35K05, 35K20, 35B35.

Key words and phrases. Inverse parabolic problem, scalar diffusivity, final state, global uniqueness, Lipschitz stability, experimental design.

choice of the initial datum and boundary input permits recovery from one final state when the coefficient boundary values are known. The latter formulation chooses the experiment part of the uniqueness question.

The elliptic counterpart is the recovery of a diffusion coefficient from a known solution inside the domain. A first-order equation for the coefficient is used in [Ric81] to obtain continuous dependence under suitable assumptions. In two dimensions, [Ale86] studies this identification problem with particular attention to critical points of the observed solution. For elliptic Dirichlet equations with a strictly positive source, [BCD+17] obtains weighted energy estimates and conditional Hölder stability for the L^2 coefficient difference from H^1 interior data under a priori coefficient bounds. In the parabolic problem, applying an elliptic coefficient estimate at time T also requires control of the unobserved term $\partial_t u(T)$ and its dependence on the diffusivity.

With the principal operator prescribed, [Isa91, Theorem 4.1(i)–(ii), equation (4.4)] establishes global uniqueness and Lipschitz stability in Hölder norms for a nonnegative zeroth-order coefficient under positivity and temporal monotonicity assumptions on the boundary input. The stability estimate also requires the final states to be bounded below by a positive constant. For the diffusivity, [Isa91, Corollary 4.2, equation (4.11)] proves uniqueness on an interval, with zero initial data, a prescribed coefficient value at one endpoint, homogeneous Neumann data at that endpoint, and a nonzero Dirichlet input at the other endpoint that is nondecreasing and convex in time and satisfies the stated compatibility conditions.

An early study of zeroth-order coefficient recovery from initial and final states is [Run87]. For a semilinear equation with a prescribed nonlinearity, [Cho94, Theorem 1] proves existence for the recovery of a spatial zeroth-order coefficient on convex domains under small-time and data assumptions. Its continuous dependence estimate in [Cho94, Theorem 2] does not require convexity. With nonzero initial data, [CY97] proves local well-posedness near the zero coefficient under an a priori restriction on its support and additional assumptions on the data. In these problems, the principal diffusion operator is prescribed.

For homogeneous Dirichlet data and zero source, [Tri21, Definition 1.1 and Theorem 1.1] proves global L^2/H^2 Lipschitz stability in dimensions two and three for sufficiently large observation times. The coefficients have prescribed boundary values and a uniform C^1 bound, together with a common quantitative gap condition between distinct Dirichlet eigenvalues. The initial datum satisfies a quantitative dominance condition on its weighted positive and negative parts; in particular, every nonzero nonnegative L^2 initial datum is admissible; see [Tri21, equation (3)]. The proof separates the first spectral mode from a faster decaying remainder and estimates the change in the spectral projections as the diffusivity varies. The decay of the remainder permits absorption for large time.

For equations with a known time-independent source, [JLQZ25, Assumption 1 and Theorem 1] gives a weighted estimate for the diffusivity difference on smooth convex domains in dimensions at most three. The estimate contains a term controlled by the final-state difference and a remainder that decays with the observation time. Under the positivity condition in [JLQZ25, equation (2.7)], sufficiently large time gives conditional Hölder stability, with the L^2 coefficient difference bounded by the square root of the H^1 seminorm of the final-state difference. The argument applies to both standard and time-fractional diffusion and is also used to analyze a numerical reconstruction.

Final and integral observations are also used in [PK93] to study inverse source problems with a prescribed principal operator. For sources of a specified form, that work establishes

the Fredholm property and gives existence, uniqueness, and stability results. A semigroup approach to source recovery from final data is developed in [HS13]. Its representation formula describes the possible nonuniqueness and provides sufficient conditions for uniqueness.

With the diffusion parameter and the remaining forward coefficients prescribed, [CY99, main theorem, p. 243] determines a spatial source factor from one final state. The known factor multiplying the source is positive at the final time, and the stability estimate is in Hölder norms. For every fixed $s_0 > 0$, the result holds for all diffusion parameters $s > s_0$ outside a finite exceptional set. The proof writes the source equation as a compact perturbation of an invertible multiplication operator. The semigroup estimate in [CY99, Lemma 5] gives a contraction for large s, and analytic Fredholm theory extends invertibility to the stated parameter set.

A single final state is used in [CY08] to determine a zeroth-order coefficient, an initial state, and a Robin boundary coefficient. The zeroth-order coefficient is assumed known in an interior subdomain. The initial state is nonnegative, nonzero, and compactly supported, and the known source satisfies sign and time analyticity conditions. The principal diffusion operator is fixed.

Existence for a one-dimensional terminal diffusivity identification problem is studied in [Koz05]. Simultaneous determination of a time-independent source and a diffusivity is considered in [DCJ12]. Its multidimensional uniqueness result assumes that the diffusivity is known in a neighborhood of the boundary, together with regularity bounds on the solution and differential conditions on the final profile; see [DCJ12, Assumption 1 and Theorem 5]. For a prescribed principal part, [IY26, Theorem 1.4] proves global L^2 Lipschitz stability for a zeroth-order coefficient. The estimate uses H^4 norms of the initial and final state differences and an L^2 norm of the time derivative of the conormal flux difference on a relatively open boundary portion near both time endpoints. The coefficient boundary values are prescribed, the compared solutions have the same Dirichlet boundary values, and one of them is assumed to stay uniformly away from zero near both time endpoints.

Carleman estimates provide another classical approach to inverse parabolic problems; see [Yam09, Sections 6 and 8.2]. For the recovery of an anisotropic principal coefficient, [YY09, Theorems 1.1–1.2] chooses finitely many interior sources using a prescribed reference coefficient. Approximate controllability produces intermediate states satisfying a nondegeneracy condition, and a Carleman estimate gives Lipschitz stability from these spatial states together with time-dependent normal derivatives on a boundary subset. The coefficients satisfy a common regularity bound and are prescribed in a neighborhood of the boundary.

1.2. **Main result.** We prove global uniqueness and Lipschitz stability from one final state in every dimension $n \geq 2$, giving an affirmative answer to [Ale20, Problem 8.2] on each prescribed uniformly elliptic $C^{1,\alpha}$-bounded class with known coefficient boundary values. We take $h = 1$ and $g = \mathrm{e}^{\lambda t}$, with λ chosen from the domain, the coefficient bounds, and the prescribed positive observation time.

We write $\|\cdot\|_p$ for the $L^p(\Omega)$ norm and take all spatial norms without an indicated domain on Ω. Fix $0 < \alpha < 1$, $0 < \kappa < 1$, and $0 < M < \infty$, and set

$$\mathcal{A}_{\kappa,M} = \{a \in C^{1,\alpha}(\overline{\Omega};\mathbb{R}) : \kappa \leq a \leq \kappa^{-1},\ \|a\|_{C^{1,\alpha}(\overline{\Omega})} \leq M\}.$$

We use a norm that dominates $\|a\|_\infty$ and $\|\nabla a\|_\infty$. The bounds κ and M are known before the experiment. For $\lambda > 0$, consider

$$\begin{cases} \partial_t u - \operatorname{div}(a\nabla u) = 0 & \text{in } \Omega \times (0,\infty), \\ u = \mathrm{e}^{\lambda t} & \text{on } \partial\Omega \times (0,\infty), \\ u(\cdot,0) = 1 & \text{in } \Omega. \end{cases} \tag{1.1}$$

The initial and boundary values agree at $t = 0$. The solution is constructed in Section 2.2, where its spatial regularity at positive times is also established.

Theorem 1.1. *For every $T_* > 0$, there are $\lambda, C_* > 0$, depending only on $\Omega, n, \alpha, \kappa, M, T_*$, with the following property. If $a_1, a_2 \in \mathcal{A}_{\kappa,M}$ have the same boundary trace and u_j solve (1.1) with this λ, then*

$$\|a_1 - a_2\|_{L^2(\Omega)} \le C_* \|u_1(T) - u_2(T)\|_{H^2(\Omega)}, \quad T \ge T_*. \tag{1.2}$$

In particular, equality of the two final states at any one time $T \ge T_*$ implies $a_1 = a_2$. The same exponential rate and stability constant work for every admissible coefficient pair and every $T \ge T_*$.

For any prescribed observation time $T > 0$, take $T_* = T$. The input is independent of the unknown coefficient within each prescribed class, but may depend on its common regularity bound M.

The choice of a constant initial datum is not essential. Corollary 3.1 allows any prescribed positive spatial profile, with a common exponential rate for profiles having fixed lower and regularity bounds. We also obtain a different type of experiment near a known smooth positive reference coefficient: Theorem 4.1 gives pairwise Lipschitz stability with a time-independent boundary input. Both unknown coefficients may vary near the reference, and the same input is used for the pair.

Theorems 5.1 and 5.2 show that, for the fixed exponential and affine experiments, no uniform L^2 Lipschitz estimate with H^s final data, $0 \le s < 2$, holds even in an arbitrarily small $C^{1,\alpha}$ neighborhood of the constant coefficient 1.

1.3. **Ideas of the proof.** The exponential input introduces a parameter that we can choose before the experiment. Multiplying the solution by $\mathrm{e}^{-\lambda t}$ makes its boundary value constant and shifts the Dirichlet diffusion operator by λ. Subtracting the normalized solution from 1 then gives a profile with zero initial and boundary values and a known constant forcing term. Its evolution formula expresses the unobserved time derivative at T as a function of the shifted operator applied to the final profile. This yields an exact equation for the final state, as shown in Section 2.2.

An elliptic barrier gives a positive lower bound for the Dirichlet operator applied to this profile. When two coefficients have the same boundary values, testing their difference equation with the coefficient difference divided by one of the coefficients removes the boundary terms. Integration by parts then turns the positive lower bound into an L^2 estimate for the coefficient difference. This argument allows the difference to change sign and the final profile to have critical points.

The remaining issue is to compare the operator functions associated with the two unknown diffusivities. Section 2.3 uses the resolvent identity on a contour common to the coefficient class. The coefficient difference is kept inside a divergence, so the estimate involves its L^2 norm without differentiating it. The resulting bound decays exponentially with the input

rate at every fixed positive time. Since the argument estimates the operator function directly, it does not require a separation of individual eigenvalues.

The decay in this operator estimate dominates the loss in the elliptic lower bound as λ increases. A rate chosen uniformly over the coefficient class therefore allows the term involving the unknown coefficient difference to be absorbed. The remaining term is controlled by the H^2 difference of the measured final states. Section 2.4 carries out this comparison and selects the input rate.

Organization of the paper. In Section 2, we prove Theorem 1.1. Section 3 extends the experiment to positive spatial profiles. Section 4 proves local pairwise stability with time-independent inputs near a known reference. Finally, we prove optimality of the H^2 final-data norm for the exponential and affine experiments in Section 5.

2. Global recovery

2.1. **Uniform elliptic and positive-time estimates.** All constants in this subsection are uniform for $a \in \mathcal{A}_{\kappa,M}$. The inner product on complex $L^2(\Omega)$ is linear in its first argument.

The closed form $\mathfrak{q}_a(v,w) = \int_\Omega a\nabla v \cdot \nabla \overline{w}\, dx$, with domain $H_0^1(\Omega)$, defines the positive self-adjoint Dirichlet operator $L_a = -\operatorname{div}(a\nabla)$. Since $H_0^1(\Omega)$ embeds compactly into $L^2(\Omega)$, this operator has compact resolvent. Uniform ellipticity and Poincaré's inequality give a constant $\rho_* > 0$, depending only on Ω and κ, such that

$$\inf \sigma(L_a) \geq \rho_* \quad \text{for every } a \in \mathcal{A}_{\kappa,M}. \tag{2.1}$$

For a function with nonzero boundary values, we write $-\operatorname{div}(a\nabla v)$ directly.

We use the Dirichlet regularity results in [GT01, Theorems 6.6, 8.12, 9.13 and 9.15]. The weak H^2 regularity theorem applies to $-\operatorname{div}(a\nabla v)$, while the $W^{2,p}$ and Schauder estimates apply to $-a\Delta v - \nabla a \cdot \nabla v$. Their constants are controlled by the prescribed coefficient bounds and the fixed domain. We record the consequences needed below.

Lemma 2.1. *For every $a \in \mathcal{A}_{\kappa,M}$,*

$$\operatorname{Dom}(L_a) = H^2(\Omega) \cap H_0^1(\Omega). \tag{2.2}$$

On this domain,

$$C^{-1}\|v\|_{H^2} \leq \|v\|_2 + \|L_a v\|_2 \leq C\|v\|_{H^2}.$$

For functions with zero boundary trace and the regularity indicated on the left,

$$\|v\|_{W^{2,p}} \leq C_p(\|L_a v\|_p + \|v\|_p), \quad 2 \leq p < \infty, \tag{2.3}$$

and

$$\|v\|_{C^{2,\alpha}} \leq C_\alpha(\|L_a v\|_{C^\alpha} + \|v\|_\infty). \tag{2.4}$$

If $v \in H_0^1(\Omega)$ solves $-\operatorname{div}(a\nabla v) = f$ weakly, then $v \in W^{2,p}$ for $f \in L^p$, $2 \leq p < \infty$, and $v \in C^{2,\alpha}$ for $f \in C^\alpha$, with the corresponding estimates above. The constants depend only on Ω, n, κ, M and, where applicable, p, α.

The semigroup preserves positivity and satisfies

$$\|\mathrm{e}^{-tL_a} f\|_\infty \leq \|f\|_\infty, \quad t \geq 0;$$

see [Ouh05, Corollary 4.3 and Theorem 4.6]. We use the same notation for its consistent extension to $L^p(\Omega)$. By [Ouh05, Theorem 3.13], this extension is holomorphic and contractive

on a sector whose angle depends only on p, for $1 < p < \infty$. Cauchy's formula, as in [Ouh05, equation (1.19)], gives

$$\|\mathrm{e}^{-tL_a} f\|_p + t\|L_a \mathrm{e}^{-tL_a} f\|_p \le C_p \|f\|_p, \quad t > 0.$$

Here L_a on L^p denotes the negative of the generator of the consistent extension. Since the sector angle and the contraction bound are independent of a, so is C_p. Taking $p = 2n$ and applying (2.3) and Sobolev embedding gives the uniform L^∞ to C^α estimate at times $t \ge T_* > 0$.

For bounded f, elliptic regularity gives $v = L_a^{-1} f \in W^{2,2n}(\Omega) \cap H_0^1(\Omega)$, satisfying $a\Delta v + \nabla a \cdot \nabla v = -f$. The weak maximum principle of [GT01, Theorem 9.1, equation (9.4)] gives $\|v\|_\infty \le E\|f\|_\infty$, with E depending only on Ω, n, κ, M. For complex-valued f, apply it to the real and imaginary parts and increase E by a fixed factor if necessary.

Lemma 2.2. *For every $T_* > 0$, there are constants $E, H_*, B_0 < \infty$, depending only on $\Omega, n, \alpha, \kappa, M, T_*$, such that*

$$\|L_a^{-1} f\|_\infty \le E\|f\|_\infty,$$

$$\|\mathrm{e}^{-TL_a} f\|_{C^\alpha} \le H_* \|f\|_\infty, \quad T \ge T_*, \tag{2.5}$$

and

$$\|L_a^{-1}(I - \mathrm{e}^{-TL_a}) f\|_{C^{2,\alpha}} \le B_0 \|f\|_{C^\alpha}, \quad T \ge T_*. \tag{2.6}$$

The first two estimates hold for $f \in L^\infty(\Omega)$, and the third for $f \in C^\alpha(\overline{\Omega})$. The constant E is independent of T_*. No boundary condition on f is required.

To obtain (2.6), set $z = L_a^{-1}(I - \mathrm{e}^{-TL_a})f$, then $L_a z = f - \mathrm{e}^{-TL_a} f$ and $\|z\|_\infty \le 2E\|f\|_\infty$. Applying (2.4) and (2.5) gives

$$\|z\|_{C^{2,\alpha}} \le C_\alpha (1 + H_* + 2E)\|f\|_{C^\alpha}.$$

Thus, one may take $B_0 = C_\alpha(1 + H_* + 2E)$.

2.2. **The final-state equation and positivity.** Fix $\lambda > 0$ and $a \in \mathcal{A}_{\kappa,M}$, and set $A_a = L_a + \lambda I$. For the experiment (1.1), write

$$W_a(t) = \mathrm{e}^{-\lambda t} u_a(t), \quad P_a(t) = 1 - W_a(t). \tag{2.7}$$

Substitution of $u_a = \mathrm{e}^{\lambda t}(1 - P_a)$ gives

$$\begin{cases} \partial_t P_a + A_a P_a = \lambda & \text{in } \Omega \times (0,\infty), \\ P_a = 0 & \text{on } \partial\Omega \times (0,\infty), \\ P_a(\cdot, 0) = 0 & \text{in } \Omega. \end{cases} \tag{2.8}$$

For $r > 0$ and $f \in L^2(\Omega)$, the function $\mathrm{e}^{-rA_a} f$ is the solution at time r of $\partial_t v + A_a v = 0$ with homogeneous Dirichlet boundary conditions and initial datum f. The boundary condition holds at positive times, while the initial datum is attained in $L^2(\Omega)$; thus, f need not have zero boundary trace. Duhamel's formula gives

$$P_a(t) = \lambda \int_0^t \mathrm{e}^{-rA_a} 1 \, dr = \lambda A_a^{-1}(I - \mathrm{e}^{-tA_a})1. \tag{2.9}$$

Formula (2.9) defines $P_a(t) \in \mathrm{Dom}(L_a)$ for $t > 0$, with $P_a(t) \to 0$ in $L^2(\Omega)$ as $t \to 0$. Differentiation gives $\partial_t P_a(t) = \lambda \mathrm{e}^{-tA_a} 1$, for $t > 0$. Since $A_a P_a(t) = \lambda(I - \mathrm{e}^{-tA_a})1$, these

identities show that P_a solves (2.8), and $u_a(t) = \mathrm{e}^{\lambda t}(1 - P_a(t))$ solves (1.1). Positivity and L^∞ contraction give $0 \le \mathrm{e}^{-rL_a}1 \le 1$. Using $\mathrm{e}^{-rA_a} = \mathrm{e}^{-\lambda r}\mathrm{e}^{-rL_a}$ in (2.9), we obtain $0 \le P_a(t) \le 1 - \mathrm{e}^{-\lambda t} \le 1$.

Applying $L_a = A_a - \lambda I$ to (2.9) yields

$$L_a P_a(T) = \lambda(1 - P_a(T) - \mathrm{e}^{-TA_a}1). \tag{2.10}$$

The right-hand side of (2.10) is bounded. Hence, Lemma 2.1 gives $P_a(T) \in W^{2,p}(\Omega)$ for every finite $p \ge 2$. Taking $p = 2n$ and using Sobolev embedding gives $P_a(T) \in C^\alpha(\overline{\Omega})$. Also, $\mathrm{e}^{-TA_a}1 = \mathrm{e}^{-\lambda T}\mathrm{e}^{-TL_a}1 \in C^\alpha(\overline{\Omega})$ by Lemma 2.2. Applying (2.4) to (2.10) now gives $P_a(T) \in C^{2,\alpha}(\overline{\Omega})$. For a fixed λ and $T_* > 0$, the resulting bound for $P_a(T)$ is uniform in $a \in \mathcal{A}_{\kappa,M}$ and $T \ge T_*$. The final state $u_a(T) = \mathrm{e}^{\lambda T}(1 - P_a(T))$ also belongs to $C^{2,\alpha}(\overline{\Omega})$.

The function $v_a(t) = \mathrm{e}^{\lambda t} - u_a(t)$ has zero initial and boundary values and satisfies $\partial_t v_a - \operatorname{div}(a\nabla v_a) = \lambda \mathrm{e}^{\lambda t}$. Thus, the prescribed experiment can also be written as a problem with a known source independent of a. The final-state differences satisfy $v_{a_1}(T) - v_{a_2}(T) = -(u_{a_1}(T) - u_{a_2}(T))$, and $P_a(t) = \mathrm{e}^{-\lambda t}v_a(t)$.

Lemma 2.3. *For $T > 0$, define $F_T(s) = s/(\mathrm{e}^{Ts} - 1)$ for $s > 0$ and set $S_a^\lambda(T) = F_T(A_a)$. This operator is defined by expansion in an orthonormal eigenbasis of A_a: if $A_a\phi = s\phi$, then $S_a^\lambda(T)\phi = F_T(s)\phi$. One has*

$$(A_a + S_a^\lambda(T))P_a(T) = \lambda, \tag{2.11}$$

$$S_a^\lambda(T)P_a(T) = \lambda \mathrm{e}^{-TA_a}1 = \partial_t P_a(T), \tag{2.12}$$

and

$$\|S_a^\lambda(T)\|_{2\to 2} \le T^{-1}. \tag{2.13}$$

Proof. For every $s > 0$, $F_T(s)\frac{1-\mathrm{e}^{-Ts}}{s} = \mathrm{e}^{-Ts}$. Applying this scalar identity to (2.9) by spectral calculus gives

$$S_a^\lambda(T)P_a(T) = \lambda F_T(A_a)A_a^{-1}(I - \mathrm{e}^{-TA_a})1 = \lambda \mathrm{e}^{-TA_a}1 = \partial_t P_a(T).$$

This proves (2.12). Substituting it into (2.8) at $t = T$ proves (2.11). Finally, $\mathrm{e}^{Ts} - 1 \ge Ts$ gives $0 \le F_T(s) \le T^{-1}$, so the spectral theorem yields (2.13). □

Lemma 2.4. *Let $T_* > 0$ and put*

$$d_0 = \sqrt{2/\kappa}, \quad D_0 = d_0 \operatorname{osc}_{\overline{\Omega}} x_1. \tag{2.14}$$

If $\sqrt{\lambda} \ge d_0 M$ and $\lambda T_* - D_0\sqrt{\lambda} \ge \log 2$, then for every $a \in \mathcal{A}_{\kappa,M}$ and $T \ge T_*$,

$$L_a P_a(T) \ge \nu_\lambda := \frac{\lambda}{2}\mathrm{e}^{-D_0\sqrt{\lambda}} \quad \text{on } \overline{\Omega}. \tag{2.15}$$

Proof. Let $V_a \in C^{2,\alpha}(\overline{\Omega})$ solve

$$\begin{cases} -\operatorname{div}(a\nabla V_a) + \lambda V_a = 0 & \text{in } \Omega, \\ V_a = 1 & \text{on } \partial\Omega. \end{cases}$$

Existence follows from [GT01, Theorem 6.14], applied to $a\Delta + \nabla a \cdot \nabla - \lambda$. Comparison with the constant functions 0 and 1 gives $0 \le V_a \le 1$; see [GT01, Theorem 3.3]. Since $1 - V_a$

has zero boundary values and satisfies $A_a(1-V_a)=\lambda$, one has $V_a=1-\lambda A_a^{-1}1$. Therefore, formulas (2.7) and (2.9) give

$$W_a(T)=V_a+\mathrm{e}^{-TA_a}(1-V_a)\ge V_a, \tag{2.16}$$

where the inequality follows from the positivity of the semigroup.

To obtain a lower bound for V_a, define the barrier function $\phi_\lambda(x)=\exp(d_0\sqrt{\lambda}(x_1-\max_{\overline{\Omega}}x_1))$, then $\phi_\lambda\le 1$ on $\partial\Omega$ and

$$-\operatorname{div}(a\nabla\phi_\lambda)+\lambda\phi_\lambda=(\lambda-ad_0^2\lambda-d_0\sqrt{\lambda}\,\partial_1 a)\phi_\lambda\le(-\lambda+d_0M\sqrt{\lambda})\phi_\lambda\le 0.$$

The comparison principle gives

$$V_a\ge\phi_\lambda\ge\mathrm{e}^{-D_0\sqrt{\lambda}}\quad\text{on }\overline{\Omega}. \tag{2.17}$$

This construction uses only the range of x_1 on $\overline{\Omega}$.

By (2.10), (2.16), (2.17), and $\mathrm{e}^{-TL_a}1\le 1$, we obtain

$$L_aP_a(T)=\lambda W_a(T)-\lambda\mathrm{e}^{-TA_a}1\ge\lambda(\mathrm{e}^{-D_0\sqrt{\lambda}}-\mathrm{e}^{-\lambda T}).$$

The condition $\lambda T_*-D_0\sqrt{\lambda}\ge\log 2$ implies $\mathrm{e}^{-\lambda T}\le\frac{1}{2}\mathrm{e}^{-D_0\sqrt{\lambda}}$ for $T\ge T_*$, proving (2.15). □

2.3. **Perturbation of the smoothing operator.** The next lemma estimates the difference of $S_a^\lambda(T)$ in terms of the L^2 difference of the coefficients. We include $\lambda=0$, which will be used for time-independent inputs in Section 4.

Lemma 2.5. *For every $T_*>0$, there is a constant $B_*>0$, depending only on $\Omega,n,\alpha,\kappa,M,T_*$, such that*

$$\|(S_{a_1}^\lambda(T)-S_{a_2}^\lambda(T))z\|_2\le B_*(1+\lambda)\mathrm{e}^{-\lambda T}\|a_1-a_2\|_2\|z\|_\infty \tag{2.18}$$

for all $\lambda\ge 0$, $T\ge T_*$, $a_1,a_2\in\mathcal{A}_{\kappa,M}$, and $z\in L^\infty(\Omega;\mathbb{C})$, where $S_a^\lambda(T)=F_T(L_a+\lambda I)$. The coefficients may have different boundary values.

Proof. Fix $\theta=\pi/4$ and $r_0=\rho_*/2$, where ρ_* is the lower bound in (2.1), and write $R_a(\zeta)=(\zeta-L_a)^{-1}$. Since $\sigma(L_a)\subset[\rho_*,\infty)$, we estimate the distance $|\zeta-s|$ for real $s\ge\rho_*$. On the rays $\zeta=r\mathrm{e}^{\pm\mathrm{i}\theta}$, $r\ge r_0$, one has

$$|\zeta-s|\ge\sin(\theta/2)(r+s),\quad s\ge\rho_*. \tag{2.19}$$

Indeed,

$$|r\mathrm{e}^{\pm\mathrm{i}\theta}-s|^2=(r-s)^2+4rs\sin^2(\theta/2)\ge\sin^2(\theta/2)(r+s)^2.$$

Thus, the spectral theorem gives

$$\|R_a(\zeta)\|_{2\to2}=\sup_{s\in\sigma(L_a)}\frac{1}{|\zeta-s|}\le\frac{1}{\sin(\theta/2)(r+\rho_*)}\le\frac{C_\theta}{r}, \tag{2.20}$$

where $C_\theta=1/\sin(\theta/2)$.

Fix $a\in\mathcal{A}_{\kappa,M}$ and $z\in L^\infty(\Omega)$, and let $\zeta=r\mathrm{e}^{\pm\mathrm{i}\theta}$ with $r\ge r_0$. Set $v=R_a(\zeta)z$, then $v\in\operatorname{Dom}(L_a)$ and $L_av=\zeta v-z$. The bound (2.20) gives $\|v\|_2\le(C_\theta/r)\|z\|_2$. Applying (2.3) with $p=2$, we obtain

$$\begin{aligned}\|v\|_{W^{2,2}}&\le C(\|L_av\|_2+\|v\|_2)\le C((1+r)\|v\|_2+\|z\|_2)\\&\le C\Big(1+C_\theta\frac{1+r}{r}\Big)\|z\|_2\le C(1+C_\theta(1+r_0^{-1}))|\Omega|^{1/2}\|z\|_\infty,\end{aligned}$$

where we used $(1+r)/r \le 1+r_0^{-1}$ and $\|z\|_2 \le |\Omega|^{1/2}\|z\|_\infty$. Since θ and r_0 are fixed, absorbing these factors into C gives $\|v\|_{W^{2,2}} \le C\|z\|_\infty$, with C independent of a, r, and z.

Set $m = n-1$ and $p_\ell = 2n/(n-\ell)$ for $0 \le \ell \le m$. Thus, $p_0 = 2$ and $p_m = 2n$. For $0 \le \ell < m$, the identity $1/p_\ell - 1/p_{\ell+1} = 1/(2n)$ gives the Sobolev embedding $W^{2,p_\ell}(\Omega) \hookrightarrow W^{1,p_{\ell+1}}(\Omega)$. Suppose that $\|v\|_{W^{2,p_\ell}} \le C(1+r)^\ell \|z\|_\infty$. This embedding implies $v \in L^{p_{\ell+1}}(\Omega)$, and hence $\zeta v - z \in L^{p_{\ell+1}}(\Omega)$. The weak regularity assertion in Lemma 2.1, applied to $L_a v = \zeta v - z$, gives $v \in W^{2,p_{\ell+1}}(\Omega)$. Estimate (2.3) then yields

$$\|v\|_{W^{2,p_{\ell+1}}} \le C((1+r)\|v\|_{p_{\ell+1}} + \|z\|_{p_{\ell+1}}) \le C(1+r)^{\ell+1}\|z\|_\infty.$$

Starting from the $W^{2,2}$ estimate above, induction gives $\|v\|_{W^{2,2n}} \le C(1+r)^m\|z\|_\infty$, and the Sobolev embedding $W^{2,2n}(\Omega) \hookrightarrow W^{1,\infty}(\Omega)$ gives

$$\|\nabla R_a(r\mathrm{e}^{\pm \mathrm{i}\theta})z\|_\infty \le C(1+r)^m\|z\|_\infty. \tag{2.21}$$

On the inner arc $\zeta = r_0\mathrm{e}^{\mathrm{i}\varphi}$, $|\varphi| \le \theta$, the distance from ζ to $\sigma(L_a)$ is at least $\rho_* - r_0 > 0$. The spectral theorem gives $\|R_a(\zeta)\|_{2\to 2} \le (\rho_* - r_0)^{-1}$. Repeating the preceding estimates with $|\zeta| = r_0$ gives

$$\|\nabla R_a(r_0\mathrm{e}^{\mathrm{i}\varphi})z\|_\infty \le C\|z\|_\infty, \quad |\varphi| \le \theta.$$

The constants in these estimates depend only on $\Omega, n, \alpha, \kappa, M$, since θ is fixed and $r_0 = \rho_*/2$ depends only on Ω and κ. They are independent of $a \in \mathcal{A}_{\kappa,M}$, $z \in L^\infty(\Omega)$, and the contour parameters $r \ge r_0$ and $|\varphi| \le \theta$.

For $G \in L^2(\Omega;\mathbb{C}^n)$, define $R_a(\zeta)(-\operatorname{div} G) \in L^2(\Omega)$ by

$$\int_\Omega R_a(\zeta)(-\operatorname{div} G)\,\overline{h}\,dx = \int_\Omega G \cdot \nabla\overline{R_a(\overline{\zeta})h}\,dx, \quad h \in L^2(\Omega).$$

For $w \in \operatorname{Dom}(L_a)$, ellipticity and integration by parts give

$$\kappa\|\nabla w\|_2^2 \le \int_\Omega a|\nabla w|^2\,dx = \langle L_a w, w\rangle = \|L_a^{1/2}w\|_2^2.$$

To apply this estimate with $w = R_a(\overline{\zeta})h$, we estimate $L_a^{1/2}R_a(\overline{\zeta})$ by the functional calculus for the positive self-adjoint operator L_a. This operator corresponds to the function $s \mapsto \sqrt{s}/(\overline{\zeta}-s)$. The same spectral theorem used in (2.20) yields

$$\|L_a^{1/2}R_a(\overline{\zeta})\|_{2\to 2} = \sup_{s\in\sigma(L_a)} \frac{\sqrt{s}}{|\overline{\zeta}-s|} \le \sup_{s\ge\rho_*} \frac{\sqrt{s}}{|\zeta - s|},$$

where we used $\sigma(L_a) \subset [\rho_*, \infty)$ and $|\overline{\zeta} - s| = |\zeta - s|$ for real s. Consequently, for every $h \in L^2(\Omega)$,

$$\|\nabla R_a(\overline{\zeta})h\|_2 \le \kappa^{-1/2}\|L_a^{1/2}R_a(\overline{\zeta})h\|_2 \le \kappa^{-1/2}\sup_{s\ge\rho_*}\frac{\sqrt{s}}{|\zeta-s|}\|h\|_2.$$

This estimate and Cauchy–Schwarz show that the right-hand side of the defining duality identity is a bounded conjugate-linear functional of h. Thus, the Riesz representation theorem justifies the definition of $R_a(\zeta)(-\operatorname{div} G)$ as an L^2 function. Taking the supremum over $\|h\|_2 = 1$ gives

$$\|R_a(\zeta)(-\operatorname{div} G)\|_2 \le \kappa^{-1/2}\sup_{s\ge\rho_*}\frac{\sqrt{s}}{|\zeta-s|}\|G\|_2. \tag{2.22}$$

On the rays, (2.19) gives

$$\sup_{s\ge\rho_*}\frac{\sqrt{s}}{|\zeta-s|}\le C_\theta\sup_{s>0}\frac{\sqrt{s}}{r+s}=\frac{C_\theta}{2\sqrt{r}}.$$

Thus, (2.22) yields $\|R_a(\zeta)(-\operatorname{div}G)\|_2\le C_\theta(2\sqrt{\kappa})^{-1}r^{-1/2}\|G\|_2$ on the rays. On the inner arc, $|\zeta-s|\ge s-r_0\ge s/2$ for $s\ge\rho_*=2r_0$, so the same estimate gives $\|R_a(\zeta)(-\operatorname{div}G)\|_2\le 2(\kappa\rho_*)^{-1/2}\|G\|_2$. If $-\operatorname{div}G\in L^2(\Omega)$, the identity $R_a(\zeta)^*=R_a(\bar\zeta)$ and integration by parts show that the duality definition agrees with the usual resolvent.

Set $b=a_1-a_2$. Since the operators have the common domain (2.2), the resolvent identity gives

$$R_{a_1}(\zeta)-R_{a_2}(\zeta)=R_{a_1}(\zeta)[-\operatorname{div}(b\nabla)]R_{a_2}(\zeta).$$

For the estimate, apply (2.22) with $G=b\nabla R_{a_2}(\zeta)z$. Its L^2 norm is at most $\|b\|_2\|\nabla R_{a_2}(\zeta)z\|_\infty$. Together with (2.21), this proves

$$\|(R_{a_1}(\zeta)-R_{a_2}(\zeta))z\|_2\le Cr^{-1/2}(1+r)^m\|b\|_2\|z\|_\infty \tag{2.23}$$

on the rays. On the inner arc, the bound is $C\|b\|_2\|z\|_\infty$. In particular, these estimates involve the L^2 norm of b without its derivatives.

Let Γ consist of the lower ray directed outwards, the upper ray directed inwards, and the inner arc directed clockwise from angle θ to $-\theta$. Adding an outer arc from $-\theta$ to θ gives the positively oriented boundary of the truncated sector outside the disk of radius r_0. Set $c_0=r_0\cos\theta>0$. The function F_T extends meromorphically to $\mathbb{C}$, with poles at $2\pi\mathrm{i}j/T$, $j\in\mathbb{Z}\setminus\{0\}$, and a removable singularity at zero. Hence, $F_T(\lambda+\zeta)$ is holomorphic in the sector enclosed by Γ for every $\lambda\ge0$. For $\zeta\in\Gamma$ and $T\ge T_*$,

$$|F_T(\lambda+\zeta)|\le\frac{(\lambda+|\zeta|)\mathrm{e}^{-\lambda T}\mathrm{e}^{-T_*\operatorname{Re}\zeta}}{1-\mathrm{e}^{-T_*c_0}}. \tag{2.24}$$

Indeed, $|\mathrm{e}^{T(\lambda+\zeta)}-1|\ge\mathrm{e}^{T(\lambda+\operatorname{Re}\zeta)}-1$, and $\operatorname{Re}\zeta\ge c_0$.

We use the representation

$$S_a^\lambda(T)=\frac{1}{2\pi\mathrm{i}}\int_\Gamma F_T(\lambda+\zeta)R_a(\zeta)\,d\zeta. \tag{2.25}$$

The resolvent bounds and (2.24) make the integral converge in operator norm. For each fixed $s\ge\rho_*$, close the contour by an outer arc of radius $R>2s$. On this arc, $\operatorname{Re}\zeta\ge R\cos\theta$ and $|\zeta-s|\ge R/2$, so the integral of $F_T(\lambda+\zeta)/(\zeta-s)$ over the arc tends to zero as $R\to\infty$. Cauchy's formula gives $F_T(\lambda+s)$ for the scalar integral over Γ. The integrable operator-norm bound permits interchange with the spectral integral and proves (2.25). The same contour is used for every coefficient in the class.

Subtracting the representations (2.25) and applying (2.23) and (2.24), we bound the contribution of the two rays by a constant multiple of

$$\mathrm{e}^{-\lambda T}\|b\|_2\|z\|_\infty\int_{r_0}^\infty(\lambda+r)r^{-1/2}(1+r)^m\mathrm{e}^{-T_*r\cos\theta}\,dr.$$

The integral is bounded by $C_*(1+\lambda)$, with C_* independent of λ and T. The inner arc has fixed length, and its contribution is also bounded by $C_*(1+\lambda)\mathrm{e}^{-\lambda T}\|b\|_2\|z\|_\infty$. This proves (2.18). □

2.4. **Pairwise comparison and choice of the input.** The following coefficient estimate follows from integration by parts.

Lemma 2.6. *Let $a \in \mathcal{A}_{\kappa,M}$ and $q \in C^{2,\alpha}(\overline{\Omega})$ be real-valued, with $-\operatorname{div}(a\nabla q) \geq \nu > 0$ in Ω. For every real-valued $b \in C^1(\overline{\Omega})$ with zero boundary trace,*

$$\|b\|_2 \leq \frac{2}{\kappa\nu}\|\operatorname{div}(b\nabla q)\|_2. \tag{2.26}$$

The function q may have nonzero boundary values and critical points.

Proof. Put $r = b/a$ and $D = \operatorname{div}(b\nabla q) = \operatorname{div}(ar\nabla q)$. Since $a \geq \kappa$ and b has zero boundary trace, one has $r \in C^1(\overline{\Omega}) \cap H_0^1(\Omega)$. Integration by parts gives

$$-\int_\Omega rD\,dx = \int_\Omega ar\nabla q\cdot\nabla r\,dx = \frac{1}{2}\int_\Omega a\nabla q\cdot\nabla(r^2)\,dx = \frac{1}{2}\int_\Omega[-\operatorname{div}(a\nabla q)]r^2\,dx \geq \frac{\nu}{2}\|r\|_2^2.$$

Both boundary terms vanish because r has zero boundary trace. The Cauchy–Schwarz inequality implies

$$\frac{\nu}{2}\|r\|_2^2 \leq -\int_\Omega rD\,dx \leq \|r\|_2\|D\|_2.$$

If $r = 0$, then $b = 0$ and the conclusion is immediate. Otherwise, division by $\|r\|_2$ and the bound $a \leq \kappa^{-1}$ yield $\|b\|_2 \leq \kappa^{-1}\|r\|_2 \leq \frac{2}{\kappa\nu}\|D\|_2$, which proves (2.26). □

Proof of Theorem 1.1. Fix $T_* > 0$ and increase B_* in Lemma 2.5 to at least 1. Choose λ so that

$$\lambda \geq \max\{1, d_0^2M^2\}, \quad \lambda T_* - D_0\sqrt{\lambda} \geq H_0, \quad H_0 = \max\{\log 2, \log(16B_*/\kappa)\}, \tag{2.27}$$

where d_0, D_0 are defined in (2.14). For example, take $\lambda = x^2$ with

$$x \geq \max\left\{1, d_0M, \frac{D_0 + \sqrt{D_0^2 + 4T_*H_0}}{2T_*}\right\}. \tag{2.28}$$

This choice depends only on the prescribed parameters.

Take $a_1, a_2 \in \mathcal{A}_{\kappa,M}$ with the same boundary values and fix $T \geq T_*$. For simplicity, let us write $u_j = u_{a_j}$, $P_j = P_{a_j}(T)$, $S_j = S_{a_j}^\lambda(T)$, $b = a_1 - a_2$, and $d = P_1 - P_2$. Lemma 2.3 gives $(A_{a_j} + S_j)P_j = \lambda$ for $j = 1, 2$. Since $(A_{a_1} - A_{a_2})P_2 = -\operatorname{div}(b\nabla P_2)$, subtraction yields

$$0 = (A_{a_1} + S_1)P_1 - (A_{a_2} + S_2)P_2 = (A_{a_1} + S_1)d - \operatorname{div}(b\nabla P_2) + (S_1 - S_2)P_2.$$

Rearranging yields

$$\operatorname{div}(b\nabla P_2) = (S_1 - S_2)P_2 + (A_{a_1} + S_1)d. \tag{2.29}$$

The functions P_1, P_2, d belong to the common domain $H^2(\Omega)\cap H_0^1(\Omega)$ of the Dirichlet operators, and all terms in (2.29) belong to $L^2(\Omega)$.

The assumption that a_1 and a_2 have the same boundary values gives $b|_{\partial\Omega} = 0$. The choice (2.27) satisfies the hypotheses of Lemma 2.4, so (2.15) gives $L_{a_2}P_2 \geq \nu_\lambda = \frac{\lambda}{2}e^{-D_0\sqrt{\lambda}}$. Applying Lemma 2.6 with $a = a_2$, $q = P_2$, and $\nu = \nu_\lambda$, we obtain

$$\|b\|_2 \leq \frac{2}{\kappa\nu_\lambda}\|\operatorname{div}(b\nabla P_2)\|_2 = \frac{4}{\kappa\lambda}e^{D_0\sqrt{\lambda}}\|\operatorname{div}(b\nabla P_2)\|_2. \tag{2.30}$$

We estimate the two terms on the right-hand side of (2.29) separately. For the first term, (2.18) with $z = P_2$ and the bound $\|P_2\|_\infty \leq 1$ give

$$\|(S_1 - S_2)P_2\|_2 \leq B_*(1+\lambda)e^{-\lambda T}\|b\|_2\|P_2\|_\infty \leq B_*(1+\lambda)e^{-\lambda T}\|b\|_2.$$

Substituting (2.29) into (2.30) and using this bound, we obtain

$$\begin{aligned}\|b\|_2 &\le \frac{4}{\kappa\lambda}\mathrm{e}^{D_0\sqrt{\lambda}}\Big(\|(S_1-S_2)P_2\|_2+\|(A_{a_1}+S_1)d\|_2\Big)\\ &\le \frac{4B_*}{\kappa}\frac{1+\lambda}{\lambda}\mathrm{e}^{D_0\sqrt{\lambda}-\lambda T}\|b\|_2+\frac{4}{\kappa\lambda}\mathrm{e}^{D_0\sqrt{\lambda}}\|(A_{a_1}+S_1)d\|_2.\end{aligned}$$

By (2.27), we have $\lambda\ge 1$ and $\lambda T_*-D_0\sqrt{\lambda}\ge H_0$. Since $T\ge T_*$, it follows that

$$\frac{4B_*}{\kappa}\frac{1+\lambda}{\lambda}\mathrm{e}^{D_0\sqrt{\lambda}-\lambda T}\le\frac{8B_*}{\kappa}\mathrm{e}^{D_0\sqrt{\lambda}-\lambda T_*}\le\frac{8B_*}{\kappa}\mathrm{e}^{-H_0}\le\frac{1}{2},$$

where the last inequality uses $H_0\ge\log(16B_*/\kappa)$. Therefore,

$$\|b\|_2\le\frac{1}{2}\|b\|_2+\frac{4}{\kappa\lambda}\mathrm{e}^{D_0\sqrt{\lambda}}\|(A_{a_1}+S_1)d\|_2.$$

Subtracting $\frac{1}{2}\|b\|_2$ from both sides and multiplying by 2 gives

$$\|b\|_2\le\frac{8}{\kappa\lambda}\mathrm{e}^{D_0\sqrt{\lambda}}\|(A_{a_1}+S_1)d\|_2. \tag{2.31}$$

It remains to estimate $(A_{a_1}+S_1)d$ in terms of the measured final states. For every $a\in\mathcal{A}_{\kappa,M}$ and $v\in\mathrm{Dom}(L_a)$, the identity $L_av=-a\Delta v-\nabla a\cdot\nabla v$ gives

$$\|L_av\|_2\le\kappa^{-1}\|\Delta v\|_2+M\|\nabla v\|_2\le C_L\|v\|_{H^2},$$

where C_L depends only on n,κ,M. Also, (2.13) gives $\|S_1\|_{2\to2}\le T^{-1}\le T_*^{-1}$. Since $A_{a_1}=L_{a_1}+\lambda I$ and $d\in\mathrm{Dom}(L_{a_1})$, these estimates imply

$$\|(A_{a_1}+S_1)d\|_2\le\|L_{a_1}d\|_2+\lambda\|d\|_2+\|S_1d\|_2\le(C_L+\lambda+T_*^{-1})\|d\|_{H^2}.$$

The normalization $P_j=1-\mathrm{e}^{-\lambda T}u_j(T)$ gives $d=-\mathrm{e}^{-\lambda T}(u_1(T)-u_2(T))$ and $\|d\|_{H^2}=\mathrm{e}^{-\lambda T}\|u_1(T)-u_2(T)\|_{H^2}$. Substituting these two estimates into (2.31), we obtain

$$\|a_1-a_2\|_2\le\frac{8}{\kappa\lambda}\mathrm{e}^{D_0\sqrt{\lambda}-\lambda T}(C_L+\lambda+T_*^{-1})\|u_1(T)-u_2(T)\|_{H^2}. \tag{2.32}$$

Since $T\ge T_*$, estimate (1.2) follows with $C_*=\frac{8}{\kappa\lambda}\mathrm{e}^{D_0\sqrt{\lambda}-\lambda T_*}(C_L+\lambda+T_*^{-1})$. The selected λ and this constant depend only on $\Omega,n,\alpha,\kappa,M,T_*$. In particular, they are independent of the coefficient pair and of $T\ge T_*$. If $u_1(T)=u_2(T)$, estimate (1.2) gives $\|a_1-a_2\|_2=0$, and hence $a_1=a_2$ as desired. □

Remark 2.7. The factor $\mathrm{e}^{-\lambda T}$ in (2.32) comes from the relation $u_a(T)=\mathrm{e}^{\lambda T}W_a(T)$. For differences of the normalized profiles $W_a(T)$, this factor is absent from the stability estimate. Multiplying the initial datum and boundary input by the same constant $c>0$ multiplies the final state by c and divides the corresponding stability constant by c.

3. Positive spatial profiles

We replace the initial datum 1 and boundary input $\mathrm{e}^{\lambda t}$ in Theorem 1.1 by k and $\mathrm{e}^{\lambda t}k$, respectively, where k is a known positive spatial profile. The exponential rate can be chosen uniformly for all such profiles with a prescribed positive lower bound and a prescribed $C^{2,\alpha}$ norm bound. For $0<m\le K<\infty$, set

$$\mathcal{K}_{m,K}=\{k\in C^{2,\alpha}(\overline{\Omega}):\ k\ge m,\ \|k\|_{C^{2,\alpha}}\le K\}.$$

We take the $C^{2,\alpha}$ norm to dominate the sup norms of the function and its derivatives through order two.

Corollary 3.1. *Fix $T_* > 0$, $0 < m \le K < \infty$, and the coefficient class $\mathcal{A}_{\kappa,M}$. There are $\lambda > 0$ and $C_* < \infty$, depending only on $\Omega, n, \alpha, \kappa, M, T_*, m, K$, such that the following holds. For any $k \in \mathcal{K}_{m,K}$ and any $a_1, a_2 \in \mathcal{A}_{\kappa,M}$ with the same boundary values, let u_j solve*

$$\begin{cases} \partial_t u_j - \operatorname{div}(a_j \nabla u_j) = 0 & \text{in } \Omega \times (0,\infty), \\ u_j = \mathrm{e}^{\lambda t} k & \text{on } \partial\Omega \times (0,\infty), \\ u_j(\cdot,0) = k & \text{in } \Omega. \end{cases}$$

Then, for every $T \ge T_*$,

$$\|a_1 - a_2\|_{L^2(\Omega)} \le C_* \|u_1(\cdot,T) - u_2(\cdot,T)\|_{H^2(\Omega)}.$$

The same exponential rate and stability constant work for every $k \in \mathcal{K}_{m,K}$ and every admissible coefficient pair.

Proof. Take $\lambda \ge \max\{1, d_0^2 M^2\}$, where d_0 is defined in (2.14); a further lower bound will be chosen below. Fix $k \in \mathcal{K}_{m,K}$. For $a \in \mathcal{A}_{\kappa,M}$, let u_a denote the solution with initial datum k and boundary input $\mathrm{e}^{\lambda t}k$, and set $W_a(t) = \mathrm{e}^{-\lambda t}u_a(t)$, $P_a(t) = k - W_a(t)$, and $f_a = -\operatorname{div}(a\nabla k) + \lambda k$. The function P_a satisfies $\partial_t P_a + A_a P_a = f_a$ with initial and boundary values 0, where $A_a = L_a + \lambda I$. The solution formula and the scalar identities used in Lemma 2.3 give

$$P_a(T) = A_a^{-1}(I - \mathrm{e}^{-TA_a})f_a, \quad (A_a + S_a^\lambda(T))P_a(T) = f_a. \tag{3.1}$$

The normalized function satisfies $\partial_t W_a - \operatorname{div}(a\nabla W_a) + \lambda W_a = 0$, with initial and boundary values k. Comparison with the constant functions 0 and K gives $0 \le W_a(t) \le K$. Since $0 \le k \le K$, it follows that $\|P_a(t)\|_\infty \le K$. Formula (3.1) places $P_a(T)$ in $\operatorname{Dom}(L_a)$ for $T > 0$. Its equation $L_a P_a(T) = f_a - \mathrm{e}^{-TA_a}f_a - \lambda P_a(T)$, together with $f_a \in C^\alpha(\overline{\Omega})$ and Lemmas 2.1 and 2.2, gives $P_a(T) \in C^{2,\alpha}(\overline{\Omega})$ by the same argument as for (2.10). Thus, $u_a(T)$ also belongs to $C^{2,\alpha}(\overline{\Omega})$.

Since $\operatorname{div}(a\nabla k) = a\Delta k + \nabla a \cdot \nabla k$, the coefficient bounds give $\|\operatorname{div}(a\nabla k)\|_\infty \le C_0 K$, where we may take $C_0 = n(\kappa^{-1} + M)$. Since $\lambda \ge 1$, $\|f_a\|_\infty \le (1 + C_0)K\lambda$. Let V_a be the solution of $-\operatorname{div}(a\nabla V_a) + \lambda V_a = 0$ in Ω, with $V_a = 1$ on $\partial\Omega$, used in the proof of Lemma 2.4. It satisfies $0 \le V_a \le 1$ and, by (2.17), $V_a \ge \mathrm{e}^{-D_0\sqrt{\lambda}}$. The stationary function mV_a solves the equation for W_a. On the boundary it is at most k, and initially $mV_a \le m \le k$. The parabolic comparison principle gives $W_a(t) \ge mV_a$.

Set $q_a(T) = \mathrm{e}^{-TA_a}f_a$. Lemma 2.2 gives $\|q_a(T)\|_\infty \le \mathrm{e}^{-\lambda T}\|f_a\|_\infty$. Applying A_a to the first identity in (3.1) and substituting the definition of f_a, we obtain

$$-\operatorname{div}(a\nabla(P_a(T) - k)) = \lambda W_a(T) - q_a(T) \ge \lambda(m\mathrm{e}^{-D_0\sqrt{\lambda}} - (1 + C_0)K\mathrm{e}^{-\lambda T}). \tag{3.2}$$

The estimate uses only the absolute bound for $q_a(T)$, and it holds when f_a changes sign.

Increase B_* in Lemma 2.5 to at least 1. Define $H_{m,K}$ and choose λ by

$$H_{m,K} = \max\left\{\log\frac{2(1+C_0)K}{m},\ \log\frac{16B_*K}{\kappa m}\right\},\ \lambda \ge \max\{1, d_0^2M^2\},\ \lambda T_* - D_0\sqrt{\lambda} \ge H_{m,K}.$$

Such a rate is obtained from (2.28) with H_0 replaced by $H_{m,K}$, and depends only on the stated parameters. The first term in the maximum defining $H_{m,K}$ ensures that (3.2) gives

$$-\operatorname{div}(a\nabla(P_a(T) - k)) \ge \frac{m\lambda}{2}\mathrm{e}^{-D_0\sqrt{\lambda}}, \quad T \ge T_*.$$

Fix $T \geq T_*$ and write $P_j = P_{a_j}(T)$, $S_j = S^{\lambda}_{a_j}(T)$, $b = a_1 - a_2$, and $d = P_1 - P_2$. Since $f_{a_1} - f_{a_2} = -\operatorname{div}(b\nabla k)$, subtraction of the second identity in (3.1) gives

$$D := \operatorname{div}\Big(b\nabla(P_2 - k)\Big) = (S_1 - S_2)P_2 + (A_{a_1} + S_1)d.$$

The function $P_2 - k$ has boundary values $-k$ and belongs to $C^{2,\alpha}(\overline{\Omega})$. Since b has zero boundary trace, Lemma 2.6 applies to this profile and yields $\|b\|_2 \leq \frac{4}{\kappa m\lambda}\mathrm{e}^{D_0\sqrt{\lambda}}\|D\|_2$. Using Lemma 2.5 and $\|P_2\|_\infty \leq K$, the coefficient of $\|b\|_2$ arising from $(S_1 - S_2)P_2$ is bounded by

$$\frac{4B_*K}{\kappa m}\frac{1+\lambda}{\lambda}\mathrm{e}^{D_0\sqrt{\lambda}-\lambda T} \leq \frac{8B_*K}{\kappa m}\mathrm{e}^{-H_{m,K}} \leq \frac{1}{2}.$$

Moving this term to the left and using the bound for $A_{a_1} + S_1$ from the proof of Theorem 1.1, we obtain $\|b\|_2 \leq (8/(\kappa m\lambda))\mathrm{e}^{D_0\sqrt{\lambda}}(C_L + \lambda + T_*^{-1})\|d\|_{H^2}$. Since the two experiments use the same k, one has $d = -\mathrm{e}^{-\lambda T}(u_1(T) - u_2(T))$, and hence

$$\|a_1 - a_2\|_2 \leq \frac{8}{\kappa m\lambda}\mathrm{e}^{D_0\sqrt{\lambda}-\lambda T}(C_L + \lambda + T_*^{-1})\|u_1(T) - u_2(T)\|_{H^2}.$$

Replacing $\mathrm{e}^{-\lambda T}$ by $\mathrm{e}^{-\lambda T_*}$ proves the assertion, with $C_* = (8/(\kappa m\lambda))\mathrm{e}^{D_0\sqrt{\lambda}-\lambda T_*}(C_L + \lambda + T_*^{-1})$. □

4. Time-independent inputs near a known reference

4.1. **The local pairwise estimate.** Near a known reference coefficient, one can use the same spatial profile for the initial datum and the time-independent boundary input. Consider

$$\begin{cases} \partial_t u - \operatorname{div}(a\nabla u) = 0 & \text{in } \Omega \times (0,\infty), \\ u = k & \text{on } \partial\Omega \times (0,\infty), \\ u(\cdot,0) = k & \text{in } \Omega. \end{cases}$$

We denote its solution by $u_{a,k}$ and write $u_{a,k}(t) = u_{a,k}(\cdot,t)$.

Theorem 4.1. *Let $a_0 \in C^\infty(\overline{\Omega})$ be real-valued and positive. There is a real-valued $h_0 \in C^\infty(\overline{\Omega})$ satisfying $\operatorname{div}(a_0\nabla h_0) = 0$ and $\inf_{\overline{\Omega}}|\nabla h_0| > 0$ with the following property. For every $T_* > 0$, there are $\delta_* > 0$, $\varepsilon_* > 0$, and $C_* > 0$ such that, whenever real positive $a_j \in C^{1,\alpha}(\overline{\Omega})$ satisfy*

$$\|a_j - a_0\|_{C^{1,\alpha}} \leq \delta_*, \quad a_j|_{\partial\Omega} = a_0|_{\partial\Omega}, \quad j = 1,2,$$

and real $k \in C^{2,\alpha}(\overline{\Omega})$ satisfies $\|k - h_0\|_{C^{2,\alpha}} \leq \varepsilon_*$, one has

$$\|a_1 - a_2\|_{L^2(\Omega)} \leq C_*\|u_{a_1,k}(T) - u_{a_2,k}(T)\|_{H^2(\Omega)}, \quad T \geq T_*. \tag{4.1}$$

The constants depend only on $\Omega, n, \alpha, a_0, h_0, T_*$ and the norm convention. The same constants work for every such coefficient pair, input, and observation time. Equality of the two final states at any one time $T \geq T_*$ implies $a_1 = a_2$.

The harmonicity condition is imposed only on h_0; the common input k may vary throughout the prescribed $C^{2,\alpha}$ neighborhood. For a constant reference, an affine profile suffices.

Corollary 4.2. *Let $a_0 = 1$ and $h_0(x) = x_1 + c$, where $c \in \mathbb{R}$ is fixed. For every $T_* > 0$, the conclusion of Theorem 4.1 holds with this h_0.*

4.2. **The final-state comparison and transport estimate.** For $a \in \mathcal{A}_{\kappa,M}$ and real $k \in C^{2,\alpha}(\overline{\Omega})$, set $f_{a,k} = \operatorname{div}(a\nabla k)$ and $z_{a,k}(t) = u_{a,k}(t) - k$, then $z_{a,k}$ has zero initial and boundary values and satisfies $\partial_t z_{a,k} + L_a z_{a,k} = f_{a,k}$. Consequently,

$$z_{a,k}(T) = L_a^{-1}(I - \mathrm{e}^{-TL_a})f_{a,k}. \tag{4.2}$$

Since $f_{a,k} \in C^{\alpha}(\overline{\Omega})$, Lemma 2.2 gives $z_{a,k}(T) \in C^{2,\alpha}(\overline{\Omega})$ and the uniform estimate (2.6) for $T \geq T_*$. The formula applies without a boundary condition on $f_{a,k}$.

Set $S_a(T) = F_T(L_a)$. The scalar identity $(s + F_T(s))(1 - \mathrm{e}^{-Ts})/s = 1$, applied to (4.2), gives $(L_a + S_a(T))z_{a,k}(T) = f_{a,k}$. For two coefficients and the same k, write $z_j = z_{a_j,k}(T)$, $b = a_1 - a_2$, and $d = z_1 - z_2$. Since $f_{a_1,k} - f_{a_2,k} = \operatorname{div}(b\nabla k)$, subtraction yields

$$\operatorname{div}(b(\nabla k + \nabla z_2)) = (S_{a_1}(T) - S_{a_2}(T))z_2 + (L_{a_1} + S_{a_1}(T))d. \tag{4.3}$$

The case $\lambda = 0$ of Lemma 2.5 gives

$$\|(S_{a_1}(T) - S_{a_2}(T))z\|_2 \leq B_*\|b\|_2\|z\|_\infty, \quad T \geq T_*.$$

Lemma 4.3. *Let real $h_0 \in C^{2,\alpha}(\overline{\Omega})$ satisfy $|\nabla h_0| \geq \mu > 0$, and set*

$$\gamma = \frac{2 + \|\Delta h_0\|_\infty}{\mu^2}, \quad C_0 = 2\mathrm{e}^{\gamma \operatorname{osc}_{\overline{\Omega}} h_0}.$$

Choose $G_0 > 0$ such that $\|\gamma\nabla h_0 \cdot \nabla r + \Delta r\|_\infty \leq G_0\|r\|_{C^{2,\alpha}}$ for every $r \in C^{2,\alpha}(\overline{\Omega})$. If real $r \in C^{2,\alpha}(\overline{\Omega})$ satisfies $G_0\|r\|_{C^{2,\alpha}} \leq 1$, then

$$\|b\|_2 \leq C_0\|\operatorname{div}(b(\nabla h_0 + \nabla r))\|_2, \quad b \in H_0^1(\Omega;\mathbb{R}). \tag{4.4}$$

Proof. Put $X = \nabla h_0 + \nabla r$ and $w = \mathrm{e}^{-\gamma h_0}$. Integration by parts, using the zero trace of b, gives

$$\int_\Omega wb[\operatorname{div}(bX)]\,dx = \frac{1}{2}\int_\Omega w(\gamma X \cdot \nabla h_0 + \operatorname{div} X)b^2\,dx.$$

For $b \in H_0^1$, the identity follows by approximation with smooth compactly supported functions, since $X \in W^{1,\infty}$. The choice of γ gives $\gamma|\nabla h_0|^2 + \Delta h_0 \geq 2$, while the assumption on r gives $|\gamma\nabla r \cdot \nabla h_0 + \Delta r| \leq 1$. Hence $\gamma X \cdot \nabla h_0 + \operatorname{div} X \geq 1$. Applying the Cauchy–Schwarz inequality yields

$$\frac{1}{2}(\min_{\overline{\Omega}} w)\|b\|_2^2 \leq (\max_{\overline{\Omega}} w)\|b\|_2\|\operatorname{div}(bX)\|_2.$$

The ratio $2\max w/\min w$ equals C_0, which proves (4.4). □

Proposition 4.4. *Let $a_0 \in C^{1,\alpha}(\overline{\Omega})$ be a positive function and $h_0 \in C^{2,\alpha}(\overline{\Omega})$ be a real function satisfying*

$$\operatorname{div}(a_0\nabla h_0) = 0, \quad \inf_{\overline{\Omega}} |\nabla h_0| = \mu > 0. \tag{4.5}$$

For every $T_* > 0$, there are $\delta_* > 0$, $\varepsilon_* > 0$, and $C_* > 0$ such that (4.1) holds for real $a_j \in C^{1,\alpha}(\overline{\Omega})$ and $k \in C^{2,\alpha}(\overline{\Omega})$ satisfying $\|a_j - a_0\|_{C^{1,\alpha}} \leq \delta_*$, $a_1|_{\partial\Omega} = a_2|_{\partial\Omega}$, and $\|k - h_0\|_{C^{2,\alpha}} \leq \varepsilon_*$. The constants depend only on $\Omega, n, \alpha, a_0, h_0, T_*$ and the norm convention, and work for every $T \geq T_*$.

Proof. Choose $r_* > 0$ with $r_* \le \min\{1, \frac{1}{2}\min_{\overline{\Omega}} a_0\}$. The ball $\|a - a_0\|_{C^{1,\alpha}} \le r_*$ then lies in $\mathcal{A}_{\kappa,M}$, by taking κ sufficiently small and $M \ge \|a_0\|_{C^{1,\alpha}} + r_*$. We fix these values of κ and M throughout the proof. The elliptic and semigroup estimates hold with the same constants for every coefficient in this ball.

Write $\eta = k - h_0$. The stationary equation in (4.5) gives

$$f_{a,k} = \operatorname{div}((a - a_0)\nabla h_0) + \operatorname{div}(a\nabla\eta).$$

The product rule and the C^α product estimate give constants $A_h, A_a > 0$, depending only on the fixed coefficient class and h_0, such that

$$\|f_{a,k}\|_{C^\alpha} \le A_h\|a - a_0\|_{C^{1,\alpha}} + A_a\|k - h_0\|_{C^{2,\alpha}}.$$

Only the first derivatives of a enter this estimate. By (2.6), $\|z_{a,k}(T)\|_{C^{2,\alpha}} \le B_0\|f_{a,k}\|_{C^\alpha}$ for $T \ge T_*$. Thus, for coefficients and inputs in balls of radii δ and ε, respectively, one has $\|z_{a,k}(T)\|_{C^{2,\alpha}} \le q$, where $q = B_0(A_h\delta + A_a\varepsilon)$.

Let γ, C_0, G_0 be as in Lemma 4.3. The vector field in (4.3) equals $\nabla h_0 + \nabla(\eta + z_2)$, and $b = a_1 - a_2 \in H_0^1(\Omega)$ because the coefficients have the same boundary values. Suppose that

$$\delta \le r_*, \quad G_0(\varepsilon + q) \le 1, \quad C_0B_*q \le \tfrac{1}{2}. \tag{4.6}$$

The transport estimate, (4.3), and Lemma 2.5 then give

$$\|b\|_2 \le C_0B_*\|b\|_2\|z_2\|_\infty + C_0\|(L_{a_1} + S_{a_1}(T))d\|_2 \le \tfrac{1}{2}\|b\|_2 + C_0\|(L_{a_1} + S_{a_1}(T))d\|_2.$$

The uniform bound $\|L_a d\|_2 \le C_L\|d\|_{H^2}$ and the scalar inequality $F_T(s) \le T^{-1}$ give $\|(L_{a_1} + S_{a_1}(T))d\|_2 \le (C_L + T_*^{-1})\|d\|_{H^2}$. Since $d = u_{a_1,k}(T) - u_{a_2,k}(T)$, absorption proves (4.1) with $C_* = 2C_0(C_L + T_*^{-1})$.

To satisfy (4.6), take all constants positive and set

$$\begin{aligned} Q &= \min\{(2G_0)^{-1}, (2C_0B_*)^{-1}\}, \\ \delta_* &= \min\{r_*, Q/(2B_0A_h)\}, \\ \varepsilon_* &= \min\{(2G_0)^{-1}, Q/(2B_0A_a)\}. \end{aligned}$$

These choices give $q \le Q$, $G_0(\varepsilon_* + q) \le 1$, and $C_0B_*q \le \frac{1}{2}$. They are independent of the coefficient pair, the chosen input, and $T \ge T_*$. □

The proposition assumes only equality of the two coefficient boundary values. It therefore applies when both equal the reference boundary values, as in Theorem 4.1.

4.3. **Construction of the reference profile.**

Lemma 4.5. *Every real positive $a_0 \in C^\infty(\overline{\Omega})$ admits a real $h_0 \in C^\infty(\overline{\Omega})$ satisfying (4.5), in every dimension $n \ge 2$.*

Proof. Extend $\log a_0$ smoothly to a neighborhood of $\overline{\Omega}$, multiply the extension by a smooth cutoff equal to 1 near $\overline{\Omega}$, and exponentiate. This gives a smooth positive extension of a_0 to $\mathbb{R}^n$, equal to 1 outside a ball. We keep the notation a_0 for this extension and choose a fixed ball B containing both $\overline{\Omega}$ and the support of $q_0 = -\Delta\sqrt{a_0}/\sqrt{a_0}$.

The complex geometrical optics (CGO) construction in [BBMT13, Section 5.1, proof of Lemma 3.3], valid for $n \ge 2$, gives solutions $v_\rho = \mathrm{e}^{\rho\cdot x}(1 + \psi_\rho)$ of $\Delta v_\rho + q_0 v_\rho = 0$ for sufficiently large $|\rho|$, with $\rho \cdot \rho = 0$. Since $q_0 \in C_c^\infty(\mathbb{R}^n)$, [BU10, Propositions 3.1 and 3.3, equation (25)] gives $\|\psi_\rho\|_{C^1(\overline{B})} \le C/|\rho|$, where B contains the entire support of the extended potential, and

the constants depend only on the fixed extension and B. The function $U_\rho = a_0^{-1/2} v_\rho$ satisfies $\operatorname{div}(a_0 \nabla U_\rho) = 0$.

Differentiation gives

$$\sqrt{a_0}\nabla U_\rho = \mathrm{e}^{\rho\cdot x}(\rho + \phi_\rho), \quad \phi_\rho = \rho\psi_\rho + \nabla\psi_\rho - (1+\psi_\rho)\nabla \log\sqrt{a_0}.$$

Hence, $\|\phi_\rho\|_{L^\infty(\Omega)} \le C_1$ independently of sufficiently large $|\rho|$. Choose $\rho = \tau(e_1 + \mathrm{i}e_2)$, where e_1, e_2 are the first two coordinate vectors, and put $h_0 = \operatorname{Re} U_\rho$. Since a_0 is real, h_0 is a_0-harmonic. Moreover,

$$\sqrt{a_0}\nabla h_0 = \mathrm{e}^{\tau x_1}[\tau(e_1\cos(\tau x_2) - e_2\sin(\tau x_2)) + \operatorname{Re}(\mathrm{e}^{\mathrm{i}\tau x_2}\phi_\rho)].$$

The leading real vector has length τ. Taking τ above the CGO threshold and with $\tau \ge 2C_1$ gives

$$|\nabla h_0(x)| \ge \frac{\tau \mathrm{e}^{\tau x_1}}{2\sqrt{a_0(x)}} \ge \frac{\tau \mathrm{e}^{\tau \min_{\overline{\Omega}} x_1}}{2\sqrt{\max_{\overline{\Omega}} a_0}} > 0.$$

The solution is smooth on a neighborhood of $\overline{\Omega}$ by interior elliptic regularity; see [GT01, Theorem 9.19]. This proves the lemma. □

Proof of Theorem 4.1 and Corollary 4.2. Choose h_0 by Lemma 4.5 and apply Proposition 4.4 for each $T_* > 0$. The profile is fixed using only the known reference. When $a_0 = 1$, take $h_0 = x_1 + c$. It satisfies (4.5) since $\Delta h_0 = 0$ and $|\nabla h_0| = 1$, so the proposition also proves the corollary. □

Adding a constant to h_0 preserves (4.5). After adding a sufficiently large constant and, if necessary, reducing the input radius, the center profile and every admissible input are positive.

5. Optimality of the final-data norm

Throughout this section, let $L = -\Delta$ with $\operatorname{Dom}(L) = H^2(\Omega) \cap H_0^1(\Omega)$, and write $\langle v, w\rangle = \int_\Omega v\overline{w}\,dx$ for the L^2 inner product.

Poincaré's inequality and the elliptic estimate in Lemma 2.1, applied to $L = -\Delta$, give

$$\|L^{-1}f\|_{H^2} \le C\|f\|_2, \quad f \in L^2(\Omega),$$

where C depends only on Ω.

5.1. **The exponential experiment.** Fix $\lambda, T > 0$ and set $\mathcal{F}_{\lambda,T}(a) = u_a(T)$, where u_a solves the exponential experiment with initial datum 1 and boundary input $\mathrm{e}^{\lambda t}$.

Theorem 5.1. *Let $n \ge 2$ and fix $\lambda, T > 0$. For every $0 \le s < 2$, there are no $\delta > 0$ and $C < \infty$ such that*

$$\|a_1 - a_2\|_2 \le C\|\mathcal{F}_{\lambda,T}(a_1) - \mathcal{F}_{\lambda,T}(a_2)\|_{H^s}$$

for all real positive $a_j \in C^{1,\alpha}(\overline{\Omega})$ satisfying $a_j|_{\partial\Omega} = 1$ and $\|a_j - 1\|_{C^{1,\alpha}} \le \delta$.

Proof. Let $U = u_1$ be the solution for the constant coefficient 1. Extend U to negative times by setting $U(x,t) = 1$ for $t < 0$. Since $U(t) \to 1$ in $L^2(\Omega)$ as $t \downarrow 0$ and $\Delta 1 = 0$, the extension satisfies $\partial_t U - \Delta U = 0$ in distributions across $t = 0$ in the interior of Ω. Interior regularity for the heat equation therefore gives bounded spatial derivatives of every fixed order of U and $\partial_t U$ on compact subsets of Ω for $0 \le t \le T$. This also gives $\nabla U(0) = 0$.

We first prove convergence of the directional difference quotient in H^2. Fix real $b \in C_c^\infty(\Omega)$, set $a_\varepsilon = 1 + \varepsilon b$, and define $Q_\varepsilon(t) = (u_{a_\varepsilon}(t) - U(t))/\varepsilon$ for small nonzero ε. Then Q_ε has zero

initial and boundary values and satisfies $\partial_t Q_\varepsilon + L_{a_\varepsilon} Q_\varepsilon = f_b(t)$, where $f_b(t) = \operatorname{div}(b\nabla U(t))$. Let Q_0 solve this problem with L in place of L_{a_ε}. The source is smooth, compactly supported in a fixed subset of Ω, and uniformly bounded for $0 \le t \le T$ in $\|f_b(t)\|_2 + \|L^j f_b(t)\|_2$ for every fixed $j \ge 1$. The mild formula gives

$$L^j Q_0(t) = \int_0^t \mathrm{e}^{-(t-r)L} L^j f_b(r)\, dr.$$

Semigroup contraction and successive interior elliptic estimates give uniform bounds on every fixed spatial derivative of Q_0 on compact subsets; see [GT01, Theorem 9.19].

Put $R_\varepsilon = Q_\varepsilon - Q_0$ and $g_b(t) = \operatorname{div}(b\nabla Q_0(t))$. Subtraction gives $\partial_t R_\varepsilon + L_{a_\varepsilon} R_\varepsilon = \varepsilon g_b(t)$, with zero initial and boundary values. The source $g_b(t)$ is supported in $\operatorname{supp} b$ and is uniformly bounded in $H^2(\Omega)$ for $0 \le t \le T$. In particular, its boundary trace is zero, so $g_b(t) \in H^2(\Omega) \cap H_0^1(\Omega) = \operatorname{Dom}(L_{a_\varepsilon})$. For fixed b and small ε, Lemma 2.1 uniformly compares the H^2 norm with $\|v\|_2 + \|L_{a_\varepsilon} v\|_2$. The semigroup contracts both terms of the latter norm, so

$$R_\varepsilon(T) = \varepsilon \int_0^T \mathrm{e}^{-(T-t)L_{a_\varepsilon}} g_b(t)\, dt, \quad \|R_\varepsilon(T)\|_{H^2} \le C_{b,T}|\varepsilon|.$$

We write $D\mathcal{F}_{\lambda,T}(1)b = Q_0(T)$ for this directional derivative. The convergence is in H^2 for each fixed b, and $C_{b,T}$ may depend on b.

By (2.9), the function $P(T) = 1 - \mathrm{e}^{-\lambda T} U(T)$ is nonzero and has zero boundary trace: the scalar multiplier $(1 - \mathrm{e}^{-Ts})/s$ is strictly positive on the spectrum of $L + \lambda I$. Hence, $U(T)$ is not constant. Connectedness of Ω implies that $\nabla U(T)$ is nonzero at some interior point. The flow-box theorem for the smooth vector field $\nabla U(T)$ gives local coordinates $x = \Phi(r, y)$ in a neighborhood $V \Subset \Omega$, with $\partial_r \Phi(r, y) = \nabla U(T)(\Phi(r, y))$ and $y \in \mathbb{R}^{n-1}$. Since $n \ge 2$, define $\phi(\Phi(r, y)) = y_1$. Differentiation in r and y_1 gives $\nabla\phi \cdot \nabla U(T) = 0$ and $\nabla\phi(\Phi(r, y)) \cdot \partial_{y_1} \Phi(r, y) = 1$, so $|\nabla\phi| > 0$ on V. Put $Y = \nabla\phi/|\nabla\phi|^2$, so $Y \cdot \nabla\phi = 1$, and choose nonzero real $\beta \in C_c^\infty(V)$. Define $b_N = \beta \cos(N\phi)$ on V and extend it by zero to Ω. Integration by parts along Y gives $\|b_N\|_2^2 \to \frac{1}{2}\|\beta\|_2^2 > 0$ as $N \to \infty$.

Set $q_N = D\mathcal{F}_{\lambda,T}(1)b_N$. Its source is $f_N(t) = \operatorname{div}(b_N \nabla U(t))$. At $t = T$, the orthogonality condition on ϕ gives $f_N(T) = g \cos(N\phi)$, where $g = \operatorname{div}(\beta\nabla U(T)) \in C_c^\infty(V)$ is independent of N. Also $f_N(0) = 0$. Integrating the Duhamel formula by parts in time yields

$$q_N = L^{-1} f_N(T) - \int_0^T L^{-1} \mathrm{e}^{-(T-t)L} \operatorname{div}(b_N \nabla \partial_t U(t))\, dt. \tag{5.1}$$

For each fixed N, the source is continuously differentiable in L^2 and compactly supported, which justifies this identity in L^2.

Let $A \in H^2(V)$ be supported in $\operatorname{supp}\beta$. Since $Y \cdot \nabla\phi = 1$, we have $Y \cdot \nabla \sin(N\phi) = N\cos(N\phi)$ and $Y \cdot \nabla \cos(N\phi) = -N \sin(N\phi)$. Two integrations by parts give

$$\int_V A \cos(N\phi)\, dx = -\frac{1}{N} \int_V \operatorname{div}(AY) \sin(N\phi)\, dx = -\frac{1}{N^2} \int_V \operatorname{div}\Big(Y \operatorname{div}(AY)\Big) \cos(N\phi)\, dx.$$

There are no boundary terms because A is compactly supported in V. The expression $\operatorname{div}(Y \operatorname{div}(AY))$ contains derivatives of A of order at most two, with coefficients bounded on $\operatorname{supp}\beta$. The Cauchy–Schwarz inequality gives

$$\left| \int_V A \cos(N\phi)\, dx \right| \le C N^{-2} \|A\|_{H^2(V)},$$

where C depends only on the fixed support and ϕ. Smooth approximation justifies the identities for $A \in H^2(V)$. For $\psi \in L^2(\Omega)$, set $v = L^{-1}\psi$. Applying this observation to $g\overline{v}$ and

using the Dirichlet elliptic estimate gives

$$|\langle L^{-1} f_N(T), \psi\rangle| = |\langle g\cos(N\phi), v\rangle| \le CN^{-2}\|v\|_{H^2} \le CN^{-2}\|\psi\|_2.$$

The amplitude is supported in V, so there are no boundary terms; smooth approximation justifies the integrations for $v \in H^2$. Also, $\|L^{-1} f_N(T)\|_{H^2} \le C\|g\|_2$ independently of N.

Denote the integral in (5.1) by r_N. For $\tau > 0$, let $v_\tau = L^{-1}\mathrm{e}^{-\tau L}\psi$. Since $V \Subset \Omega$ and $Lv_\tau = \mathrm{e}^{-\tau L}\psi$, interior elliptic regularity gives

$$\|v_\tau\|_{H^3(V)} \le C(\|\mathrm{e}^{-\tau L}\psi\|_{H^1(\Omega)} + \|v_\tau\|_2) \le C(1+\tau^{-1/2})\|\psi\|_2;$$

see [GT01, Theorem 9.19]. The second inequality follows from the Dirichlet inverse estimate, L^2 contraction, and

$$\|\nabla \mathrm{e}^{-\tau L}\psi\|_2 = \|L^{1/2}\mathrm{e}^{-\tau L}\psi\|_2 \le \Big(\sup_{\xi>0}\sqrt{\xi}\mathrm{e}^{-\tau\xi}\Big)\|\psi\|_2 \le C\tau^{-1/2}\|\psi\|_2.$$

Pair r_N with ψ and integrate the divergence by parts. Applying the preceding oscillatory estimate with $A = \beta\nabla\partial_t U(t)\cdot\nabla\overline{v_{T-t}}$ gives

$$|\langle r_N, \psi\rangle| \le CN^{-2}\int_0^T \|v_{T-t}\|_{H^3(V)}\,dt \le C_T N^{-2}\|\psi\|_2.$$

The spatial derivatives of $\partial_t U$ are uniformly bounded on $\operatorname{supp}\beta \Subset V$, and $(T-t)^{-1/2}$ is integrable.

For the H^2 bound, let $G \in L^2(\Omega;\mathbb{C}^n)$ and $\tau > 0$. Define $\mathrm{e}^{-\tau L}\operatorname{div} G$ by duality:

$$\langle \mathrm{e}^{-\tau L}\operatorname{div} G, \psi\rangle = -\int_\Omega G\cdot\nabla\overline{\mathrm{e}^{-\tau L}\psi}\,dx, \quad \psi \in L^2(\Omega).$$

The preceding gradient estimate gives

$$|\langle \mathrm{e}^{-\tau L}\operatorname{div} G, \psi\rangle| \le C\tau^{-1/2}\|G\|_2\|\psi\|_2.$$

Thus the pairing defines an L^2 function and agrees with the usual semigroup action when $\operatorname{div} G \in L^2(\Omega)$. The Dirichlet inverse estimate gives

$$\|L^{-1}\mathrm{e}^{-\tau L}\operatorname{div} G\|_{H^2} \le C\|\mathrm{e}^{-\tau L}\operatorname{div} G\|_2 \le C\tau^{-1/2}\|G\|_2.$$

Apply this estimate with $G = b_N\nabla\partial_t U(t)$ and $\tau = T-t$. The L^2 norm of G is bounded independently of N and t, and $(T-t)^{-1/2}$ is integrable. Hence the integral defining r_N converges in $H^2(\Omega)\cap H_0^1(\Omega)$ and satisfies $\|r_N\|_{H^2} \le C_T$. Together with the preceding L^2 bounds, this gives

$$\|q_N\|_2 \le C_T N^{-2}, \quad \|q_N\|_{H^2} \le C_T.$$

Sobolev interpolation yields $\|q_N\|_{H^s} \le C_{T,s}N^{s-2}$ for $0 \le s < 2$.

Suppose the asserted nonlinear Lipschitz estimate held. For each fixed N, take $a_1 = 1+\varepsilon b_N$ and $a_2 = 1$. They satisfy the coefficient assumptions for sufficiently small $|\varepsilon|$, with the allowable size depending on N. Divide by $|\varepsilon|$ and let $\varepsilon \to 0$. The H^2 directional convergence gives $\|b_N\|_2 \le C\|q_N\|_{H^s}$. Letting $N \to \infty$ contradicts $\|b_N\|_2^2 \to \frac{1}{2}\|\beta\|_2^2 > 0$. The directional limit is taken with N fixed throughout. □

5.2. **The affine experiment.**

Theorem 5.2. *Let $n \geq 2$, fix $T > 0$, and set $h_0(x) = x_1 + c$ with fixed $c \in \mathbb{R}$. For every $0 \leq s < 2$, there are no $\delta > 0$ and $C < \infty$ such that*

$$\|a_1 - a_2\|_2 \leq C\|u_{a_1,h_0}(T) - u_{a_2,h_0}(T)\|_{H^s} \tag{5.2}$$

for all real positive $a_j \in C^{1,\alpha}(\overline{\Omega})$ satisfying $a_j|_{\partial\Omega} = 1$ and $\|a_j - 1\|_{C^{1,\alpha}} \leq \delta$.

For this fixed profile, define $\mathcal{F}_T(a) = u_{a,h_0}(T)$. The reference solution at $a = 1$ is stationary: $u_{1,h_0}(t) = h_0$. Hence the source of the linearized equation in a direction $b \in C_c^\infty(\Omega;\mathbb{R})$ is $\operatorname{div}(b\nabla h_0) = \partial_1 b$, and

$$D\mathcal{F}_T(1)b = L^{-1}(I - \mathrm{e}^{-TL})\partial_1 b. \tag{5.3}$$

The H^2 directional convergence follows from the argument in the proof of Theorem 5.1, with $U = h_0$. In this case, the source $\partial_1 b$ is time-independent, smooth, and compactly supported, so it satisfies all the regularity conditions used in that argument.

Proof of Theorem 5.2. Choose a nonzero real function $\beta \in C_c^\infty(\Omega)$ and set $b_N = \beta\sin(Nx_2)$ for positive integers N. Since $\sin^2(Nx_2) = (1 - \cos(2Nx_2))/2$,

$$\|b_N\|_2^2 = \frac{1}{2}\|\beta\|_2^2 - \frac{1}{2}\int_\Omega \beta^2\cos(2Nx_2)\,dx.$$

Integration by parts in x_2, with no boundary term because β is compactly supported, gives

$$\left|\int_\Omega \beta^2\cos(2Nx_2)\,dx\right| = \left|-\frac{1}{2N}\int_\Omega \partial_2(\beta^2)\sin(2Nx_2)\,dx\right| \leq \frac{1}{2N}\|\partial_2(\beta^2)\|_{L^1} \longrightarrow 0.$$

Thus, $\|b_N\|_2^2 \to \frac{1}{2}\|\beta\|_2^2 > 0$. In particular, there is an integer N_0 such that $\|b_N\|_2 \geq c_\beta := \frac{1}{2}\|\beta\|_2 > 0$ for all $N \geq N_0$.

Put $f_N = \partial_1 b_N = (\partial_1\beta)\sin(Nx_2)$ and $w_N = L^{-1}f_N$. For $\varphi \in L^2(\Omega)$, let $v = L^{-1}\varphi$. Self-adjointness of L^{-1} and two integrations by parts in x_2 give

$$\langle w_N, \varphi\rangle = \langle f_N, v\rangle = \int_\Omega (\partial_1\beta)\overline{v}\sin(Nx_2)\,dx = -\frac{1}{N^2}\int_\Omega \partial_2^2((\partial_1\beta)\overline{v})\sin(Nx_2)\,dx.$$

The product $(\partial_1\beta)\overline{v}$ belongs to $H^2(\Omega)$ and is compactly supported in Ω, so these integrations by parts produce no boundary terms. The product rule and the elliptic estimate for L^{-1} imply

$$|\langle w_N, \varphi\rangle| \leq CN^{-2}(\|v\|_2 + \|\partial_2 v\|_2 + \|\partial_2^2 v\|_2) \leq CN^{-2}\|v\|_{H^2} \leq CN^{-2}\|\varphi\|_2.$$

Taking the supremum over $\|\varphi\|_2 = 1$ yields $\|w_N\|_2 \leq CN^{-2}$.

By (5.3), write

$$q_N := D\mathcal{F}_T(1)b_N = (I - \mathrm{e}^{-TL})w_N.$$

The semigroup is an L^2 contraction, preserves $\mathrm{Dom}(L)$, and commutes with L on this domain. Consequently,

$$\|q_N\|_2 \leq 2\|w_N\|_2 \leq CN^{-2}, \quad \|Lq_N\|_2 = \|(I - \mathrm{e}^{-TL})f_N\|_2 \leq 2\|f_N\|_2 \leq 2\|\partial_1\beta\|_2.$$

The elliptic estimate therefore gives $\|q_N\|_{H^2} \leq C(\|Lq_N\|_2 + \|q_N\|_2) \leq C$. Interpolation between L^2 and H^2 now yields

$$\|D\mathcal{F}_T(1)b_N\|_{H^s} = \|q_N\|_{H^s} \leq C_s\|q_N\|_2^{1-s/2}\|q_N\|_{H^2}^{s/2} \leq C_{T,s}N^{s-2}, \quad 0 \leq s < 2, \tag{5.4}$$

where $C_{T,s}$ is independent of N.

Suppose that (5.2) holds with a constant C_{Lip} independent of the coefficient pair. Fix N and take $a_1 = 1 + \varepsilon b_N$ and $a_2 = 1$. Both coefficients equal 1 near $\partial\Omega$, and

$$\|a_1 - 1\|_{C^{1,\alpha}} = |\varepsilon| \|b_N\|_{C^{1,\alpha}} \longrightarrow 0$$

as $\varepsilon \to 0$. Hence both coefficients are admissible for sufficiently small nonzero real ε. Applying the assumed estimate and dividing by $|\varepsilon|$ gives

$$\|b_N\|_2 \le C_{\mathrm{Lip}} \left\| \frac{\mathcal{F}_T(1 + \varepsilon b_N) - \mathcal{F}_T(1)}{\varepsilon} \right\|_{H^s}.$$

Keeping N fixed, the directional differentiability established above gives

$$\frac{\mathcal{F}_T(1 + \varepsilon b_N) - \mathcal{F}_T(1)}{\varepsilon} \longrightarrow D\mathcal{F}_T(1) b_N \quad \text{in } H^2(\Omega)$$

as $\varepsilon \to 0$, and hence also in $H^s(\Omega)$. Passing to this limit, using (5.4) and the lower bound for $\|b_N\|_2$, we obtain

$$0 < c_\beta \le \|b_N\|_2 \le C_{\mathrm{Lip}} \|D\mathcal{F}_T(1) b_N\|_{H^s} \le C_{\mathrm{Lip}} C_{T,s} N^{s-2}, \quad N \ge N_0.$$

We now let $N \to \infty$. Since $s < 2$, the right-hand side tends to zero, contradicting $c_\beta > 0$. □

Statements and declarations

Data availability statement. No datasets were generated or analyzed in this study.

Conflict of interest. The authors declare no conflict of interest.

Acknowledgments. Y.-H. Lin is partially supported by the National Science and Technology Council (NSTC), Taiwan, under the project 113-2115-M-A49-017-MY3. He also acknowledges financial support from the Alexander von Humboldt Foundation through the Henriette Herz Scouting Programme, hosted by Universität Duisburg-Essen, Germany. The main results, arguments, and statements of this paper were developed without the use of AI. AI tools were used for auxiliary mathematical discussions and for assistance with parts of the writing. The authors take full responsibility for the final mathematical statements and proofs.

References

[Ale86] Giovanni Alessandrini. An identification problem for an elliptic equation in two variables. *Annali di Matematica Pura ed Applicata*, 145:265–295, 1986.

[Ale20] Giovanni Alessandrini. A small collection of open problems. *Rend. Istit. Mat. Univ. Trieste*, 52:591–600, 2020.

[AV85] Giovanni Alessandrini and Sergio Vessella. Error estimates in an identification problem for a parabolic equation. *Boll. Un. Mat. Ital. C (6)*, 4(1):183–203, 1985.

[BBMT13] Guillaume Bal, Eric Bonnetier, François Monard, and Faouzi Triki. Inverse diffusion from knowledge of power densities. *Inverse Probl. Imaging*, 7(2):353–375, 2013.

[BCD+17] Andrea Bonito, Albert Cohen, Ronald DeVore, Guergana Petrova, and Gerrit Welper. Diffusion coefficients estimation for elliptic partial differential equations. *SIAM Journal on Mathematical Analysis*, 49(2):1570–1592, 2017.

[BU10] Guillaume Bal and Gunther Uhlmann. Inverse diffusion theory of photoacoustics. *Inverse Problems*, 26(8):085010, 2010.

[Cho94] Mourad Choulli. An inverse problem for a semilinear parabolic equation. *Inverse Problems*, 10(5):1123–1132, 1994.

[Cho09] Mourad Choulli. *Une introduction aux problèmes inverses elliptiques et paraboliques*, volume 65 of *Mathématiques et Applications*. Springer, Berlin, Heidelberg, 2009.

[CY97] Mourad Choulli and Masahiro Yamamoto. An inverse parabolic problem with non-zero initial condition. *Inverse Problems*, 13(1):19–27, 1997.

[CY99] Mourad Choulli and Masahiro Yamamoto. Generic well-posedness of a linear inverse parabolic problem with diffusion parameters. *J. Inverse Ill-Posed Probl.*, 7(3):241–254, 1999.

[CY08] Mourad Choulli and Masahiro Yamamoto. Uniqueness and stability in determining the heat radiative coefficient, the initial temperature and a boundary coefficient in a parabolic equation. *Nonlinear Analysis: Theory, Methods & Applications*, 69(11):3983–3998, 2008.

[DCJ12] A. De Cezaro and B. T. Johansson. A note on uniqueness in the identification of a spacewise dependent source and diffusion coefficient for the heat equation, 2012. arXiv:1210.7346.

[GT01] David Gilbarg and Neil S. Trudinger. *Elliptic Partial Differential Equations of Second Order*. Classics in Mathematics. Springer, Berlin, Heidelberg, 2 edition, 2001.

[HS13] Alemdar Hasanov and Marián Slodička. An analysis of inverse source problems with final time measured output data for the heat conduction equation: A semigroup approach. *Applied Mathematics Letters*, 26(2):207–214, 2013.

[Isa91] Victor Isakov. Inverse parabolic problems with the final overdetermination. *Comm. Pure Appl. Math.*, 44(2):185–209, 1991.

[Isa17] Victor Isakov. *Inverse Problems for Partial Differential Equations*, volume 127 of *Applied Mathematical Sciences*. Springer, Cham, 3 edition, 2017.

[IY26] O. Y. Imanuvilov and Masahiro Yamamoto. Stability and uniqueness for inverse problems for parabolic equations with final overdetermination. *Inverse Problems*, 42(9):095004, 2026.

[JLQZ25] Bangti Jin, Xiliang Lu, Qimeng Quan, and Zhi Zhou. Numerical recovery of the diffusion coefficient in diffusion equations from terminal measurement. *Numer. Math.*, 157:2323–2355, 2025.

[Koz05] A. I. Kozhanov. Solvability of the inverse problem of finding thermal conductivity. *Siberian Math. J.*, 46(5):841–856, 2005.

[Ouh05] El Maati Ouhabaz. *Analysis of Heat Equations on Domains*, volume 31 of *London Mathematical Society Monographs Series*. Princeton University Press, Princeton, NJ, 2005.

[PK93] A. I. Prilepko and A. B. Kostin. On certain inverse problems for parabolic equations with final and integral observation. *Russian Academy of Sciences. Sbornik. Mathematics*, 75(2):473–490, 1993.

[Ric81] Gerard R. Richter. An inverse problem for the steady state diffusion equation. *SIAM Journal on Applied Mathematics*, 41(2):210–221, 1981.

[Run87] William Rundell. The determination of a parabolic equation from initial and final data. *Proceedings of the American Mathematical Society*, 99(4):637–642, 1987.

[Tri21] Faouzi Triki. Coefficient identification in parabolic equations with final data. *J. Math. Pures Appl. (9)*, 148:342–359, 2021.

[Yam09] Masahiro Yamamoto. Carleman estimates for parabolic equations and applications. *Inverse Problems*, 25(12):123013, 2009.

[YY09] Ganghua Yuan and Masahiro Yamamoto. Lipschitz stability in the determination of the principal part of a parabolic equation. *ESAIM: Control, Optimisation and Calculus of Variations*, 15(3):525–554, 2009.

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