SIMULTANEOUS RECOVERIES FOR SEMILINEAR PARABOLIC SYSTEMS

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Abstract. In this paper, we study inverse boundary problems associated with semilinear parabolic systems in several scenarios where both the nonlinearities and the initial data can be unknown. We establish several simultaneous recovery results showing that the passive or active boundary Dirichlet-to-Neumann operators can uniquely recover both of the unknowns, even stably in a certain case. It turns out that the nonlinearities play a critical role in deriving these recovery results. If the nonlinear term belongs to a general C^1 class but fulfilling a certain growth condition, the recovery results are established by the control approach via Carleman estimates. If the nonlinear term belongs to an analytic class, the recovery results are established through successive linearization in combination with special CGO (Complex Geometrical Optics) solutions for the parabolic system.

Keywords: Inverse boundary problem, semilinear parabolic equation, passive measurement, active measurement, Carleman estimate, simultaneous recovery, uniqueness, stability

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Contents

1. Introduction

1.1. Mathematical setup and statement of main results. In this paper, we are concerned with inverse problems for semilinear parabolic equations. Depending on the form of the nonlinear term, there are two setups for our study, which shall be discussed separately in what follows.

First, we consider the case that the nonlinear term belongs to a C^1 class fulfilling a certain growth condition. We begin by introducing the forward model. Let $\Omega \subseteq \mathbb{R}^n$ be a bounded domain with a C^∞-smooth boundary Γ for $n \in \mathbb{N}$ and Γ_0 be a nonempty relatively open subset of Γ. For any $T > 0$, we set $Q = \Omega \times (0,T)$ and $\Sigma = \Gamma \times (0,T)$. Assume that $\gamma = (\gamma_{ij}(x,t))_{i,j=1}^n \in C^{2,1}(\overline{Q};\mathbb{R}^{n\times n})$ is a symmetric matrix-valued function in $\overline{Q}$, such that

$$\rho_0|\xi|^2 \le \sum_{i,j=1}^n \gamma_{ij}(x,t)\xi_i\xi_j \le \rho_0^{-1}|\xi|^2, \quad \forall\, (x,t) \in \overline{Q} \quad \text{and} \quad \xi = (\xi_1,\dots,\xi_n) \in \mathbb{R}^n,$$

for some positive constant $\rho_0 \in (0,1)$. Moreover, we denote by $H^{s,r}(Q)$, $H^{s,r}(\Gamma)$, $C^{k+\alpha}(\overline{\Omega})$ and $C^{k+\alpha,\frac{k}{2}+\frac{\alpha}{2}}(\overline{Q})$, respectively, the standard Sobolev spaces and Hölder spaces for $s, r \in \mathbb{R}$, $k \in \mathbb{N}$ and $\alpha \in (0,1)$. We refer to [AF03] and [Eva10] for details of these Banach spaces.

Consider the following semilinear parabolic equation:

$$\begin{cases} u_t - \nabla\cdot(\gamma\nabla u) + a(x,t,u) = 0 & \text{in } \ Q, \\ u = f & \text{on } \ \Sigma, \\ u(x,0) = g(x) & \text{in } \ \Omega, \end{cases} \tag{1.1}$$

where $u_t = \partial_t u = \frac{\partial u}{\partial t}$, ∇ and $\nabla\cdot\zeta$ denote the gradient operator with respect to the spacial variable and the divergence of a vector $\zeta \in \mathbb{R}^n$, $g \in H_0^1(\Omega)$, $f \in L^2(\Sigma)$ and $a = a(x,t,u) : Q\times\mathbb{R} \to \mathbb{R}$ is a given function, satisfying suitable conditions that will be specified later.

For any $g \in H_0^1(\Omega)$ and a suitable function $a : Q \times \mathbb{R} \to \mathbb{R}$, which guarantees the global well-posedness of (1.1) (see Section 2), we introduce the following Dirichlet-to-Neumann (DN for short) operator:

$$\begin{aligned} \Lambda_{a,g} &: \mathcal{E} \to L^2(\Gamma_0 \times (0,T)), \\ & f \mapsto \partial_\nu u\Big|_{\Gamma_0\times(0,T)}. \end{aligned} \tag{1.2}$$

In (1.2), $\partial_\nu u = \frac{\partial u}{\partial \nu}$ denotes the outer normal derivative of u, ν is the unit outer normal vector on Γ, and u is the solution to (1.1) associated to the initial data $g \in H_0^1(\Omega)$ and the boundary data $f \in \mathcal{E}$ with

$$\mathcal{E} = \Big\{ f \in L^2(\Sigma) \ \Big| \ (1.1) \text{ is well-posed associated to } g \text{ and } a, \text{ such that} \\ u \in C([0,T];L^2(\Omega)) \text{ and } \ \partial_\nu u|_{\Gamma_0\times(0,T)} \in L^2(\Gamma_0\times(0,T)) \Big\}.$$

It is known that when $a \in L^\infty(Q;W^{1,\infty}(\mathbb{R}))$ and $g \in H_0^1(\Omega)$,

$$\Big\{ f \in H^{\frac{3}{2},\frac{3}{4}}(\Sigma) \ \Big| \ f(x,0) = 0 \text{ on } \Gamma \Big\} \subseteq \mathcal{E}.$$

When $f \equiv 0$ on Σ, we denote

$$\Lambda_{a,g}^{(0)} := \Lambda_{a,g}(0).$$

In such a case and in the physical situation, the field u is generated by the initial data g, acting as a source, which is assumed to be unknown in our inverse problem study. Hence, the boundary measurement encoded in $\Lambda_{a,g}^{(0)}$ is passively taken by the observer, and in the literature, $\Lambda_{a,g}^{(0)}$ is usually referred to as the *passive measurement*. In contrast, $\Lambda_{a,g}(f)$ associated with the boundary input f is called the *active measurement*, since the field u is actively induced by the observer by imposing boundary inputs in $\mathcal{E}$.

Associated to the forward model (1.1)–(1.2), first, we are interested in the following two inverse problems:

- **Inverse Problem 1.** Can we identify the unknown functions (a,g) by using the passive measurement $\Lambda_{a,g}^0$?

- **Inverse Problem 2.** Can we identify the unknown functions (a, g) by using the active measurement $\Lambda_{a,g}$?

It is emphasized that the principal coefficient $\gamma = \gamma(x,t)$ of our **Inverse Problem 1** and **Inverse Problem 2** can be space-time dependent. In order to study the above problems, we introduce certain a priori conditions on the nonlinear term a to guarantee the well-posedness of the forward problem as well as the feasibility of the inverse problems. Assume that $a : Q \times \mathbb{R} \to \mathbb{R}$ satisfies $a(x,t,\cdot) \in C^1(\mathbb{R})$ in Q and the following growth condition:

$$\limsup_{y\to\infty} \frac{\partial_y a(x,t,y)}{\ln^{\frac{1}{2}}|y|} = 0, \qquad \text{uniformly for } (x,t) \in Q. \tag{1.3}$$

It is clear that any function in $L^\infty(Q; W^{1,\infty}(\mathbb{R}))$ satisfies the condition (1.3). For notational clarity, we set

$$\begin{aligned} \mathcal{A}_T = \Big\{ a : Q \times \mathbb{R} \to \mathbb{R} \ \Big| \ & a(x,t,\cdot) \in C^1(\mathbb{R}) \text{ in } Q,\ a(\cdot,\cdot,0) \in L^2(Q), \\ & \text{and the condition (1.3) is fulfilled} \Big\}. \end{aligned} \tag{1.4}$$

In Section 2, we shall show that for any $g \in H_0^1(\Omega)$, $a \in \mathcal{A}_T$ and $f = 0$, (1.1) has a unique solution $u \in H^{2,1}(Q)$ and therefore, $\partial_\nu u \in L^2(\Sigma)$.

We are in a position to state the first recovery result for the inverse problems introduced above.

Theorem 1.1 (Conditional stability of determining initial data by the passive measurement)**.** *Assume that $a \in \mathcal{A}_T$ and for any $M > 0$, set*

$$\mathcal{G}_M = \Big\{ g \in H_0^1(\Omega) \ \Big| \ \|g\|_{H_0^1(\Omega)} \le M \Big\}.$$

For any $g_j \in \mathcal{G}_M$ $(j = 1,2)$, let Λ^0_{a,g_j} be the passive measurement associated to the following semilinear parabolic equation:

$$\begin{cases} \partial_t u_j - \nabla\cdot(\gamma\nabla u_j) + a(x,t,u_j) = 0 & \text{in } Q, \\ u_j = 0 & \text{on } \Sigma, \\ u_j(x,0) = g_j(x), & \text{in } \Omega. \end{cases} \tag{1.5}$$

Then there exist positive constants C and $\delta_0 \in (0,1)$, depending only on n, T and Ω, such that the following quantitative stability estimate holds:

$$\begin{aligned} \|g_1 - g_2\|^2_{L^2(\Omega)} \le & \frac{C(1+M)}{\delta_0} \|\Lambda^0_{a,g_1} - \Lambda^0_{a,g_2}\|_{L^2(\Gamma_0\times(0,T))} \\ & - \frac{CM^2}{\ln\left(\delta_0 \|\Lambda^0_{a,g_1} - \Lambda^0_{a,g_2}\|_{L^2(\Gamma_0\times(0,T))}\right)}. \end{aligned} \tag{1.6}$$

By Theorem 1.1, it is directly verified that if $\Lambda^0_{a,g_1} = \Lambda^0_{a,g_2}$ on $\Gamma_0 \times (0,T)$, *then* $g_1 = g_2$ in Ω. Theorem 1.1 partially answers **Inverse Problem 1** that if the nonlinear term a belongs to the general class (1.4) and is a priori known, then the initial data g can be uniquely recovered (in a stable manner) by the passive measurement for g in a bounded set $\mathcal{G}_M$.

We proceed to consider **Inverse Problem 2** and introduce another admissible set on a:

$$\begin{aligned} \mathcal{B}_T = \Big\{ a : Q \times \mathbb{R} \to \mathbb{R} \ \Big| \ & a(x,t,u) = a_0(x,t,u)\chi_{[0,T-\epsilon]}(t) + c(x,t,u)\chi_{[T-\epsilon,T]}(t) \\ & \text{for some } \epsilon > 0 \text{ and any given } a_0 \in \mathcal{A}_T, \\ & \text{where } c \in \mathcal{A}_T \text{ and } c(x,t,0) = 0 \text{ in } Q \Big\}, \end{aligned} \tag{1.7}$$

where χ_E is the characteristic function of a set $E \subseteq [0,T]$.

As a corollary of Theorem 1.1, our main unique recovery result for **Inverse Problem 2** is stated as follows.

Theorem 1.2 (Uniqueness of determining initial data by the active measurements)**.** *Assume that $a_j \in \mathcal{B}_T$ and $g_j \in H_0^1(\Omega)$ $(j = 1, 2)$. Let Λ_{a_j,g_j} be the DN map of the semilinear parabolic equation:*

$$\begin{cases} \partial_t u_j - \nabla \cdot (\gamma \nabla u_j) + a_j(x, t, u_j) = 0 & \text{in } Q, \\ u_j = f & \text{on } \Sigma, \\ u_j(x, 0) = g_j(x), & \text{in } \Omega. \end{cases} \tag{1.8}$$

If for any $f \in \mathcal{E}$ with $\mathrm{supp} f \subseteq \Gamma_0 \times [0, T]$*,*

$$\Lambda_{a_1,g_1}(f) = \Lambda_{a_2,g_2}(f) \quad \text{on} \quad \Gamma_0 \times (0, T), \tag{1.9}$$

then one has that

$$g_1 = g_2 \quad \text{in} \ \ \Omega. \tag{1.10}$$

Theorem 1.2 means that the map $\Lambda_{a,g}$ uniquely determines the initial data g, independent of functions $a \in \mathcal{B}_T$.

In the second setup of our study, we consider the case that the nonlinear term a belongs to an analytic class. In such a case, assume that both the initial data and nonlinear term are unknown. Then we can simultaneously recover both of them. To this end, introduce the following class for the nonlinear term.

Definition 1.1 (Admissible class)**.** *Assume that $b = b(x, t, u) : \overline{Q} \times \mathbb{R} \to \mathbb{R}$ satisfies the following conditions:*

$$\begin{cases} \text{the map } u \mapsto b(\cdot, \cdot, u) \text{ is analytic on } \mathbb{R} \text{ with values in } C^{2+\alpha,1+\alpha/2}(\overline{Q}), \\ b(x, t, 0) = 0 \text{ in } Q, \end{cases} \tag{1.11}$$

for some $\alpha \in (0, 1)$. It means that b can be written as the Taylor expansion at any $u_0 \in \mathbb{R}$:

$$b(x, t, u) = \sum_{k=0}^{\infty} \frac{b^{(k)}(x, t, u_0)}{k!}(u - u_0)^k, \tag{1.12}$$

where $\dfrac{b^{(k)}(x, t, u_0)}{k!} = \dfrac{\partial_u^k b(x, t, u_0)}{k!}$ are the Taylor's coefficients at $u_0 \in \mathbb{R}$ for any $k \in \mathbb{N}$.

Next, let $\Omega \subseteq \mathbb{R}^n$ be a bounded domain with a C^∞-smooth boundary Γ, for $n \geq 2$. We introduce the forward model of the following semilinear parabolic equation:

$$\begin{cases} \partial_t u - \Delta u + b(x, t, u) = 0 & \text{in } \ Q, \\ u = f & \text{on } \ \Sigma, \\ u(x, 0) = g(x), & \text{in } \ \Omega, \end{cases} \tag{1.13}$$

where b is the function given in Definition 1.1. It is easily seen that the second condition (1.11) of b implies that $u = 0$ is a trivial solution when the initial and boundary data are both zero. In Section 2, we shall prove the (local) well-posedness of the forward problem (1.13) under the assumption that the coefficient b, initial data g and the boundary data f fulfill the following compatibility condition:

$$g(\cdot) = g_{x_i}(\cdot) = g_{x_i x_j}(\cdot) = f(\cdot, 0) = f_t(\cdot, 0) = 0 \quad \text{on} \ \ \Gamma, \ \ \text{for } i, j = 1, \cdots, n. \tag{1.14}$$

Furthermore, we introduce the boundary measurement associated with (1.13) for our inverse problem study. Let $\mathbb{S}^{n-1}$ be the unit sphere of $\mathbb{R}^n$ and fix $\omega_0 \in \mathbb{S}^{n-1}$. Define

$$\Gamma_{\pm,\omega_0} = \Big\{ x \in \Gamma \ \Big| \ \pm \nu(x) \cdot \omega_0 \geq 0 \Big\} \quad \text{and} \quad \Sigma_{\pm,\omega_0} = \Gamma_{\pm,\omega_0} \times (0, T). \tag{1.15}$$

Let $\mathcal{U}_\pm$ be a neighborhood of $\Gamma_{\pm,\omega_0}$ in Γ and set

$$\mathcal{V}_+ = \mathcal{U}_+ \times (0,T) \quad \text{and} \quad \mathcal{V}_- = \mathcal{U}_- \times (0,T).$$

With these notations and the local well-posedness at hand, the partial DN map $\Lambda^{\mathrm{P}}_{b,g}$ is defined as:

$$\begin{aligned} \Lambda^{\mathrm{P}}_{b,g} : \mathcal{E}_1 &\to C^{1+\alpha,1+\alpha/2}(\mathcal{V}_-), \\ f &\mapsto \partial_\nu u\big|_{\mathcal{V}_-}, \end{aligned} \tag{1.16}$$

where $\mathcal{E}_1 = \left\{ f \in C_0^{2+\alpha,1+\alpha/2}(\mathcal{V}_+) \;\middle|\; \|f\|_{C^{2+\alpha,1+\alpha/2}(\mathcal{V}_+)} < \delta_1 \right\}$ for sufficiently small δ_1 and $g \in C_0^{2+\alpha}(\Omega)$, which satisfy the compatibility conditions (1.14) and guarantee the well-posedness of (1.13), and u is the associated solution to (1.13). Meanwhile, the (full) DN map of the initial-boundary value problem (1.13) is given via

$$\begin{aligned} \Lambda_{b,g} : \mathcal{E}_2 &\to C^{1+\alpha,1+\frac{\alpha}{2}}(\Sigma), \\ f &\mapsto \partial_\nu u\big|_{\Sigma}, \end{aligned} \tag{1.17}$$

where $\mathcal{E}_2 = \left\{ f \in C_0^{2+\alpha,1+\frac{\alpha}{2}}(\Sigma) \;\middle|\; \|f\|_{C^{2+\alpha,1+\frac{\alpha}{2}}(\Sigma)} < \delta_2 \right\}$ for sufficiently small δ_2 and $g \in C_0^{2+\alpha}(\Omega)$, which satisfy the compatibility conditions (1.14) and guarantee the well-posedness of (1.13), and u is the associated solution to (1.13).

Our third inverse problem is as follows:

- **Inverse Problem 3.** Can we determine the unknown functions (b, g) by using active measurements, either $\Lambda_{b,g}$ or $\Lambda^{\mathrm{P}}_{b,g}$?

The main result established for **Inverse Problem 3** is stated as follows.

Theorem 1.3 (Simultaneous recovery for the semilinear parabolic equation)**.** *Let b_1 and b_2 be in the admissible class. There exists a $\delta > 0$, such that for any $g_j \in C_0^{2+\alpha}(\Omega)$ $(j = 1, 2)$ with $\|g_j\|_{C^{2+\alpha}(\Omega)} < \delta/2$, we denote by Λ_{b_j,g_j} and $\Lambda^{\mathrm{P}}_{b_j,g_j}$ the full and partial DN maps of the semilinear parabolic equation:*

$$\begin{cases} u_t - \Delta u + b_j(x,t,u) = 0 & \text{in } Q, \\ u = f & \text{on } \Sigma, \\ u(x,0) = g_j(x), & \text{in } \Omega, \end{cases} \tag{1.18}$$

for $j = 1, 2$, respectively. Then we have the following results:

(a) *(Full data) If*

$$\Lambda_{b_1,g_1}(f) = \Lambda_{b_2,g_2}(f),$$

for any $f \in \mathcal{E}_2$ with a sufficiently small δ_2, then

$$g_1 = g_2 \text{ in } \Omega \quad \text{and} \quad b_1 = b_2 \text{ in } Q \times \mathbb{R}.$$

(b) *(Partial data) For a domain $\Omega' \subseteq \Omega$ satisfying $\Gamma \subseteq \partial\Omega'$, assume that $b_1 = b_2$ in $\Omega' \times (0,T) \times \mathbb{R}$. If*

$$\Lambda^{\mathrm{P}}_{b_1,g_1}(f) = \Lambda^{\mathrm{P}}_{b_2,g_2}(f),$$

for any $f \in \mathcal{E}_1$ with a sufficiently small δ_1, then

$$g_1 = g_2 \text{ in } \Omega \quad \text{and} \quad b_1 = b_2 \text{ in } Q \times \mathbb{R}.$$

Theorem 1.3 states that **Inverse Problem 3** can be solved under suitable situations. In fact, we can determine $b(\cdot,\cdot,\cdot)$ and $g(\cdot)$ simultaneously by using active measurements with full data. Meanwhile, if we assume $b(\cdot,\cdot,\cdot)$ is known a-priori in a small neighborhood of Σ, then we can also determine $b(\cdot,\cdot,\cdot)$ and $g(\cdot)$ simultaneously with partial measurements.

Remark 1.2. *We would like to point out that*

(1) *The proof of Theorem 1.3 relies on the successive linearization method combining with suitable* complex geometrical optics (*CGO*) *solutions* (*see* [CK18b] *or Appendix A*) *and approximation properties* (*see Section 4*). *We can utilize either full or partial DN maps for the semilinear equation* (1.18) *to determine both coefficients and initial data uniquely. Moreover, the smallness assumptions for both initial and boundary data are needed due to the local well-posedness of the forward problem* (1.18) (*see Section 2*), *but not used to solve the inverse problem.*

(2) *In the statement (b) of Theorem 1.3, the domain Ω' can be chosen as $\Omega' = \Omega \setminus \overline{D}$ with $\overline{D} \subseteq \Omega$ such that $\Omega \setminus \overline{D}$ is connected. Moreover, such a set $D \subseteq \Omega$ can be as large as possible so that the domain Ω' is very "thin". This means, for our partial data result, it is sufficient for us to know the coefficient near the boundary $\Gamma \times (0,T)$ a priori.*

Finally, it would be interesting to consider the linear counterparts of the inverse problems studied in Theorem 1.3. To our best knowledge, the simultaneous recovery results are untouched in the literature even in the linear case, namely

$$b(x,t,u) = q(x,t)u$$

as a linear function with respect to $u \in \mathbb{R}$. For this linear model, the smallness conditions for initial and boundary data are not required, since the well-posedness for general linear parabolic equations have been well understood (for example, see [Eva10, Chapter 7] or [LSU88]). To proceed, let us consider the linear parabolic equation:

$$\begin{cases} \partial_t u - \Delta u + qu = 0 & \text{in } Q, \\ u = f & \text{on } \Sigma, \\ u(x,0) = g(x) & \text{in } \Omega. \end{cases} \tag{1.19}$$

In order to derive the well-posedness of classical solutions to (1.19), we need to impose the following compatibility condition:

$$g(\cdot) = f(\cdot, 0) \quad \text{on } \Gamma. \tag{1.20}$$

Then one has the well-posedness of (1.19) (see [Eva10, Chapter 7]) and therefore, we may define the corresponding partial DN map

$$\begin{aligned} \Lambda^{\mathrm{P}}_{q,g} : C_0^{2+\alpha,1+\alpha/2}(\mathcal{V}_+) &\to C^{1+\alpha,1+\alpha/2}(\mathcal{V}_-), \\ f &\mapsto \partial_\nu u\big|_{\mathcal{V}_-}, \end{aligned} \tag{1.21}$$

and the (full) DN map

$$\begin{aligned} \Lambda_{q,g} : C_0^{2+\alpha,1+\alpha/2}(\Sigma) &\to C^{1+\alpha,1+\alpha/2}(\Sigma), \\ f &\mapsto \partial_\nu u\big|_{\Sigma}. \end{aligned} \tag{1.22}$$

Now, the inverse problem is to determine q and g by using the measurements either $\Lambda^{\mathrm{P}}_{q,g}$ or $\Lambda_{q,g}$. The last main unique recovery result is stated as follows.

Theorem 1.4 (Simultaneous recovery for linear parabolic equations)**.** *Assume that for $j = 1, 2$, $q_j \in C^{2+\alpha,1+\alpha/2}(\overline{Q})$ and $g_j \in C_0^{2+\alpha}(\Omega)$. Denote by Λ_{q_j,g_j} and $\Lambda^{\mathrm{P}}_{q_j,g_j}$ are the full and partial DN maps of the linear parabolic equation:*

$$\begin{cases} \partial_t u - \Delta u + q_j u = 0 & \text{in } Q, \\ u = f & \text{on } \Sigma, \\ u(x,0) = g_j(x), & \text{in } \Omega, \end{cases} \tag{1.23}$$

for $j = 1, 2$, respectively. Then we have the following results:

(a) (*Full data*) *If*
$$\Lambda_{q_1,g_1}(f) = \Lambda_{q_2,g_2}(f),$$
for any $f \in C_0^{2+\alpha,1+\alpha/2}(\Sigma)$, *then*
$$g_1 = g_2 \text{ in } \Omega \quad \text{and} \quad q_1 = q_2 \text{ in } Q.$$

(b) (*Partial data*) *For a domain* $\Omega' \subseteq \Omega$ *satisfying* $\Gamma \subseteq \partial\Omega'$, *assume that* $q_1 = q_2$ *in* $\Omega' \times (0, T)$. *If*
$$\Lambda^{\mathrm{P}}_{q_1,g_1}(f) = \Lambda^{\mathrm{P}}_{q_2,g_2}(f),$$
for any $f \in C_0^{2+\alpha,1+\alpha/2}(\mathcal{V}_+)$, *then*
$$g_1 = g_2 \text{ in } \Omega \quad \text{and} \quad q_1 = q_2 \text{ in } Q.$$

It is noted that when the initial data $g_1 = g_2 = 0$ in Ω, the logarithmic stability result for two potentials of the inverse problem associated with the linear parabolic equation with partial data has been investigated in [CK18b].

1.2. **Background and discussion.** In this paper, we are interested in the study of inverse problems for semilinear parabolic equations. A classical result of inverse boundary value problems for semilinear parabolic equations was proposed by Isakov [Isa93], where a first-order linearization technique was exploited to reduce the inverse problem associated with the nonlinear equation into its counterpart associated with a linear equation. Then one can apply some existing results for the linear equations to investigate related inverse problems for the nonlinear equations. In addition, one can also consider the second-order linearization method, which has been successfully adapted in solving some related inverse problems; see [AZ21, CNV19, KN02, Sun96, SU97] and the references cited therein.

In recent years, various inverse problems for nonlinear hyperbolic equations have been proposed and studied. Some works mentioned above are based on solution properties to inverse problems associated with the linearized equations. It turns out that in the inverse problem study associated with nonlinear hyperbolic equations, one finds that the nonlinear interactions bring more information which enables to solve some inverse problems that are still unsolved in the setting associated with linear equations. In [KLU18], the authors investigated inverse problems for hyperbolic equations with a quadratic nonlinearity on a globally hyperbolic 4-dimensional Lorentzian manifold. For more related works of inverse problems for nonlinear hyperbolic equations, we refer readers to [LUW17, LUW18, CLOP21, dHUW18, KLOU14, WZ19, LLPMT20, LLPMT21, LLL21] and references cited therein. In addition, inverse problems for semilinear elliptic equations have been attracted a lot of attentions in recent years. By utilizing high order linearization approach, it is possible to solve several inverse problems for local and nonlocal nonlinear elliptic equations, and we refer readers to [LLLS21, FO20, LLLS20, LLST22, LL22, LL19, Lin22, LZ20, KU20a, KU20b, KKU22, CK20, CF21] for more detailed discussions.

The study of inverse problems on simultaneously recovering an unknown source and its surrounding inhomogeneous medium has also received considerable attentions recently in the literature due to its connection to many cutting-edge applications, including the photo- and thermo-acoustic tomography [LU15], magnetic anomaly detection via the geomagnetic monitoring [DLL19, DLL20] and quantum mechanics [LLM19, LLM21]. Here, in the setup described in the previous section, say e.g. in (1.13), the initial data g and $b^{(0)}$ for b in (1.12) represent the source terms, whereas the other terms in (1.12) of b represent the medium effects. In [LLL21], the simultaneous recovery for inverse problems associated with semilinear hyperbolic systems with unknown sources and nonlinearities was studied. In this paper, we consider the simultaneous recovery for inverse problems associated with semilinear parabolic systems. It is remarked that we develop new strategies which enable us to deal with more general source and medium configurations in the semilinear parabolic setup than the semilinear hyperbolic case. Finally, we would like to mention in passing some related physical applications that can be described by the semilinear parabolic systems in our study, including the heat diffusion [GSS18], mean-field game theory [Car13, GLL11] and phase field theory [BWBK02, KKL01]. The inverse problems proposed and studied in this paper can be connected to those practical applications.

The rest of the paper is organized as follows. In Section 2, we study the well-posedness of the initial boundary value problems for the semilinear parabolic equations under suitable assumptions. In Section 3, we establish the conditional stability estimates, and show the unique determination by utilizing either passive or active measurements. We prove Theorems 1.3 and 1.4 in Section 4. Finally, for the sake of completeness, we review some basic properties on CGO solutions and weak maximum principle for linear parabolic equations.

2. Well-posedness of the forward problems

This section is devoted to studying the local and global well-posedness for initial-boundary value problems of semilinear parabolic equations, respectively. Let us consider the following semilinear parabolic equation:

$$\begin{cases} u_t - \nabla\cdot(\tilde\gamma\nabla u) + b(x,t,u) = 0 & \text{in } Q,\\ u = \tilde f & \text{on } \Sigma,\\ u(x,0) = \tilde g(x) & \text{in } \Omega, \end{cases} \tag{2.1}$$

where $\tilde\gamma$ is symmetric and uniformly positive definite on $\overline{Q}$ with $\tilde\gamma \in C^{1+\alpha,\alpha/2}(\overline{Q};\mathbb{R}^{n\times n})$ for $\alpha\in(0,1)$, and b satisfies the following conditions:

$$b\in C^2(\overline{Q}\times\mathbb{R}) \quad \text{and} \quad b(\cdot,\cdot,0)=0 \text{ in } Q. \tag{2.2}$$

As a preliminary, we recall the well-posedness result for linear parabolic equations, which can be found in [LSU88].

Lemma 2.1. *Assume that $\tilde\gamma$ is symmetric and uniformly positive with $\tilde\gamma\in C^{1+\alpha,\alpha/2}(\overline{Q};\mathbb{R}^{n\times n})$, and $q\in C^{\alpha,\alpha/2}(\overline{Q})$. For any $\tilde g\in C^{2+\alpha}(\overline{\Omega})$, $\tilde f\in C^{2+\alpha,1+\alpha/2}(\overline{\Sigma})$ and $h\in C^{\alpha,\alpha/2}(\overline{Q})$ with the compatibility conditions:*

$$\tilde g(x) = \tilde f(x,0) \text{ and } \tilde f_t(x,0) = \nabla\cdot(\tilde\gamma(x,0)\nabla\tilde g(x)) - q(x,0)\tilde g(x) + h(x,0) \quad \text{on } \Gamma, \tag{2.3}$$

the following linear parabolic equation:

$$\begin{cases} u_t - \nabla\cdot(\tilde\gamma\nabla u) + qu = h & \text{in } Q,\\ u = \tilde f & \text{on } \Sigma,\\ u(x,0) = \tilde g(x) & \text{in } \Omega, \end{cases} \tag{2.4}$$

admits a unique solution $u\in C^{2+\alpha,1+\alpha/2}(\overline{Q})$. Moreover,

$$\|u\|_{C^{2+\alpha,1+\alpha/2}(\overline{Q})} \le C\Big(\|\tilde f\|_{C^{2+\alpha,1+\alpha/2}(\overline{\Sigma})} + \|\tilde g\|_{C^{2+\alpha}(\overline{\Omega})} + \|h\|_{C^{\alpha,\alpha/2}(\overline{Q})}\Big).$$

Note that, if $h=0$ in Q, $\tilde g\in C^{2+\alpha}(\overline{\Omega})$ with $\tilde g = \tilde g_{x_i} = \tilde g_{x_ix_j} = 0$ $(i,j=1,\cdots,n)$ on Γ and $\tilde f\in C^{2+\alpha,1+\alpha/2}(\overline{\Sigma})$ with $\tilde f(x,0)=\tilde f_t(x,0)=0$ on Γ, then the compatibility condition (2.3) holds.

By Lemma 2.1 and the fixed-point method, we have the following local well-posedness for (2.1).

Theorem 2.1 (Local well-posedness)**.** *Assume that $\tilde\gamma$ is symmetric and uniformly positive with $\tilde\gamma\in C^{1+\alpha,\alpha/2}(\overline{Q};\mathbb{R}^{n\times n})$, and b satisfies the condition (2.2). Then there exists a positive* *constant δ, such that for any $(\tilde{f},\tilde{g}) \in V_\delta$, the equation (2.1) has a unique solution $u \in C^{2+\alpha,1+\alpha/2}(\overline{Q})$, where*

$$V_\delta = \Big\{(\tilde{f},\tilde{g}) \in C^{2+\alpha,1+\alpha/2}(\overline{\Sigma}) \times C^{2+\alpha}(\overline{\Omega}) \ \Big|\ \tilde{f}(x,0) = \tilde{f}_t(x,0) = 0 \text{ on } \Gamma, \\ \tilde{g} = \tilde{g}_{x_i} = \tilde{g}_{x_ix_j} = 0,\ i,j = 1,\cdots,n \text{ on } \Gamma, \\ \text{and } \|\tilde{f}\|_{C^{2+\alpha,1+\alpha/2}(\overline{\Sigma})} + \|\tilde{g}\|_{C^{2+\alpha}(\overline{\Omega})} \le \delta\Big\}.$$

Proof. The proof can be accomplished by the fixed-point technique. First, we set

$$K = \Big\{z \in C^{\alpha,\alpha/2}(\overline{Q}) \ \Big|\ \|z\|_{C^{\alpha,\alpha/2}(\overline{Q})} \le 1, z(\cdot,0) = \tilde{g} \text{ in } \Omega \text{ and } z = \tilde{f} \text{ on } \Sigma\Big\},$$

where $(\tilde{f},\tilde{g}) \in V_\delta$ for a sufficiently small $\delta > 0$. It is straightfoward to show that K is a nonempty convex and compact subset in $L^2(Q)$. Also, we define

$$q(x,t,s) := \begin{cases} \dfrac{b(x,t,s)}{s} & \text{for } s \neq 0, \\ b_s(x,t,0) & \text{for } s = 0. \end{cases}$$

For any $z \in K$, let us consider the following linear parabolic equation:

$$\begin{cases} u_t - \nabla\cdot(\tilde{\gamma}\nabla u) + q_z(x,t)u = 0 & \text{in } Q, \\ u = \tilde{f} & \text{on } \Sigma, \\ u(x,0) = \tilde{g}(x) & \text{in } \Omega, \end{cases} \tag{2.5}$$

where $q_z(x,t) = q(x,t,z(x,t))$, and define the following map:

$$\Psi(z) = u, \quad \forall\, z \in K,$$

where u is the solution to (2.5) associated to q_z. By Lemma 2.1, it follows that $u \in C^{2+\alpha,1+\frac{\alpha}{2}}(\overline{Q})$. Moreover,

$$\|u\|_{C^{2+\alpha,1+\alpha/2}(\overline{Q})} \le C(\tilde{\gamma},b,n,\Omega,T)\Big(\|\tilde{f}\|_{C^{2+\alpha,1+\alpha/2}(\overline{\Sigma})} + \|\tilde{g}\|_{C^{2+\alpha}(\overline{\Omega})}\Big) \le C(\tilde{\gamma},b,n,\Omega,T)\delta,$$

where $C(\tilde{\gamma},b,n,\Omega,T)$ denotes a positive constant, depending only on $\tilde{\gamma}$, b, n, Ω and T. Hence, when δ is sufficiently small, $\|u\|_{C^{2+\alpha,1+\alpha/2}(\overline{Q})} \le 1$, and therefore, $\Psi(K) \subseteq K$. By the Schauder fixed-point theorem, it is ready to show that Ψ has a fixed point in K, which is the solution to (2.1). The proof is complete.

□

Remark 2.2. *Regarding the local well-posedness, we give several remarks.*

(a) *The condition (2.2) on $b = b(x,t,u)$ is not essential and it is for convenience to express compatibility conditions. Also, the admissible condition on $b(x,t,u)$ is not used in the proof of the local well-posedness, but it will be utilized in the proof of our simultaneously recovering inverse problem.*

(b) *In [Isa93], it was assumed that the coefficient $b = b(x,u)$ is independent of t and $\partial_u b(x,u) \ge 0$ for any $u \in \mathbb{R}$. In contrast, we provide different time-dependent nonlinearities and utilize different techniques to study related inverse problems for semilinear parabolic equations.*

(c) *In order to apply the higher order linearization method, we need the infinite differentiability of the equation with respect to the given lateral boundary data f, which can be shown by applying the implicit function theorem in Banach spaces. To see* *this, let us define the following spaces. Set*

$$X_1 = \Big\{(f,g) \in C^{2+\alpha,1+\alpha/2}(\overline{\Sigma})\times C^{2+\alpha}(\overline{\Omega}) \;\Big|\; f(x,0) = f_t(x,0) = 0 \quad \text{on } \Gamma, \\ g = g_{x_i} = g_{x_ix_j} = 0 \quad \text{on } \Gamma \text{ for } i,j = 1,\cdots,n\Big\},$$

$$X_2 = \Big\{u \in C^{2+\alpha,1+\alpha/2}(\overline{Q}) \;\Big|\; u(x,0) = u_t(x,0) = 0 \quad \text{on } \Gamma, \\ u(x,0) = u_{x_i}(x,0) = u_{x_ix_j}(x,0) = 0 \quad \text{on } \Gamma \text{ for } i,j = 1,\cdots,n, \\ u_t(x,0) - \nabla\cdot(\gamma(x,0)\nabla u(x,0)) = 0 \quad \text{on } \Gamma\Big\},$$

$$\text{and } X_3 = \Big\{h \in C^{\alpha,\alpha/2}(\overline{Q}) \;\Big|\; h(x,0) = 0 \quad \text{on } \Gamma\Big\} \times X_1.$$

We consider the map $\mathcal{G} : X_1 \times X_2 \to X_3$ by

$$\mathcal{G}(f,g,u) = \Big(u_t - \nabla\cdot(\gamma\nabla u) + b(x,t,u), u\Big|_\Sigma - f, u(x,0) - g\Big).$$

Then $\mathcal{G}(0,0,0) = 0$ and $\mathcal{G}_u(0,0,0) : X_2 \to X_3$ is given by

$$\mathcal{G}_u(0,0,0)v = \Big(v_t - \nabla\cdot(\gamma\nabla v) + b_u(\cdot,\cdot,0)v, v\Big|_\Sigma, v(x,0)\Big).$$

It is straightforward to show that $\mathcal{G}_u(0,0,0)$ is a linear isomorphism from X_2 to X_3 by Lemma 2.1. By the implicit function theorem in Banach spaces, there exists a positive constant δ, and a holomorphic map $S : V_\delta \to C^{2+\alpha,1+\alpha/2}(\overline{Q})$, such that for any $(f,g) \in V_\delta$, we have $\mathcal{G}(f,g,S(f,g)) = 0$. Set $u = S(f,g)$ and this implies the local well-posedness of (2.1). Notice that in the above proof, we use the condition that $b = b(x,t,u)$ is in the admissible class in Definition 1.1. Also, the map of boundary data to the solution is C^∞-Fréchet differentiable. Hence, we can also derive the corresponding DN map is also C^∞-Fréchet differentiable.

Next, for a different nonlinearity, let us consider the global well-posedness of the semilinear parabolic equation:

$$\begin{cases} u_t - \nabla\cdot(\gamma\nabla u) + a(x,t,u) = 0 & \text{in } Q, \\ u = f & \text{on } \Sigma, \\ u(x,0) = g(x) & \text{in } \Omega, \end{cases} \tag{2.6}$$

where γ is symmetric and uniformly positive definite with $\gamma \in C^{1,0}(\overline{Q};\mathbb{R}^{n\times n})$, and $a : Q\times\mathbb{R} \to \mathbb{R}$ satisfies

$$a(\cdot,\cdot,0) \in L^2(Q), \quad a(x,t,\cdot) \in C^1(\mathbb{R}) \tag{2.7}$$

and the increasing condition (1.3).

The global well-posedness result of (2.6) is stated as follows.

Theorem 2.2 (Global well-posedness)**.** *Assume that a satisfies (2.7) and (1.3). Then for any $g \in H_0^1(\Omega)$ and $f \in H^{\frac{3}{2},\frac{3}{4}}(\Sigma)$ with $f(\cdot,0) = 0$ on Γ, the semilinear parabolic equation (2.6) admits a unique strong solution $u \in H^{2,1}(Q)$.*

Proof. First, let us set

$$q(x,t,s) := \begin{cases} \dfrac{a(x,t,s) - a(x,t,0)}{s} & \text{for } s \neq 0, \\ a_s(x,t,0) & \text{for } s = 0. \end{cases}$$

For any $z \in L^2(Q)$, consider the following linear parabolic equation:

$$\begin{cases} u_t - \nabla \cdot (\gamma \nabla u) + a_z(x,t)u + a(x,t,0) = 0 & \text{in } \ Q, \\ u = f & \text{on } \ \Sigma, \\ u(x,0) = g(x) & \text{in } \ \Omega, \end{cases} \tag{2.8}$$

where $a_z(x,t) = q(x,t,z(x,t))$. By the condition (1.3), we have that $a_z(\cdot,\cdot) \in L^{n+2}(Q)$. Indeed, there exist positive constants, which are denoted by C and may be different in one place or another, such that

$$\begin{aligned} &\int_Q |a_z(x,t)|^{n+2}\, dxdt = \int_0^T \|a_z(\cdot,t)\|^{n+2}_{L^{n+2}(\Omega)}\, dt \\ \le& C + C\int_0^T e^{\|a_z(\cdot,t)\|^2_{L^{n+2}(\Omega)}}\, dt = C + C\int_0^T \sum_{j=0}^{\infty} \frac{1}{j!}\|a_z(\cdot,t)\|^{2j}_{L^{n+2}(\Omega)}\, dt \\ =& C + C\int_0^T \sum_{j=0}^{n+2} \frac{1}{j!}\|a_z(\cdot,t)\|^{2j}_{L^{n+2}(\Omega)}\, dt + C\int_0^T \sum_{j=n+3}^{\infty} \frac{1}{j!}\|a_z(\cdot,t)\|^{2j}_{L^{n+2}(\Omega)}\, dt \\ \le& C + C\int_0^T \sum_{j=n+3}^{\infty} \frac{1}{j!}\|a_z(\cdot,t)\|^{2j}_{L^{n+2}(\Omega)}\, dt \\ =& C + C\int_0^T \sum_{j=n+3}^{\infty} \frac{1}{j!}\left(\int_\Omega |a_z(x,t)|^{n+2} dx\right)^{\frac{2j}{n+2}}\, dt \\ \le& C + C\int_0^T \sum_{j=n+3}^{\infty} \frac{C^j}{j!}\int_\Omega |a_z(x,t)|^{2j}\, dxdt \le C + C\int_Q e^{C|a_z(x,t)|^2} dxdt. \end{aligned} \tag{2.9}$$

By the condition (1.3), for any $\epsilon > 0$, there always is a positive constant C_ϵ, such that for any $z \in L^2(Q)$, it holds that

$$|a_s(x,t,z(x,t))|^2 \le \epsilon \ln|z(x,t)| + C_\epsilon.$$

Hence, for a sufficient small ϵ,

$$\begin{aligned} &\int_Q e^{C|a_z(x,t)|^2}\, dxdt \le \int_Q e^{C[\epsilon \ln(1+|z(x,t)|)+C_\epsilon]}\, dxdt \\ \le& C\int_Q (1+|z(x,t)|)^{C\epsilon}\, dxdt \le C\Big(1 + \|z\|^2_{L^2(Q)}\Big). \end{aligned} \tag{2.10}$$

(2.9) and (2.10) imply that $a_z \in L^{n+2}(Q)$.

By [LSU88], the linear parabolic equation (2.8) admits a unique strong solution $u \in H^{2,1}(Q)$. Moreover, by the energy estimate, it holds that

$$\begin{aligned} &\|u\|^2_{L^2(0,T;H^1(\Omega))\cap C([0,T];L^2(\Omega))} + \|u_t\|^2_{L^2(0,T;H^{-1}(\Omega))} \\ \le& Ce^{T\|a_z\|^2_{L^{n+2}(Q)}} \left(\|a(\cdot,\cdot,0)\|^2_{L^2(Q)} + \|g\|^2_{L^2(\Omega)} + \|f\|^2_{H^{1,0}(\Sigma)}\right). \end{aligned} \tag{2.11}$$

Define the following map:

$$\mathcal{G} : L^2(Q) \to L^2(Q)$$

by

$$\mathcal{G}(z) = u,$$

where u is the solution to the equation (2.8) associated to a_z. Obviously, $\mathcal{G}$ is well-posed and compact. Define

$$V = \Big\{ z \in L^2(Q) \ \Big| \ \|z\|_{L^2(Q)} \le C^* \Big\},$$

where C^* will be specified later. By (2.9)-(2.11),

$$\|u\|_{L^2(Q)}^2 \leq C\left(\|a(\cdot,\cdot,0)\|_{L^2(Q)}^2 + \|g\|_{L^2(\Omega)}^2 + \|f\|_{H^{1,0}(\Sigma)}^2\right)\left(1 + \|z\|_{L^2(Q)}\right).$$

Indeed, we may choose $\epsilon = 1/C$ in (2.10). It follows that there exists a $C^* > 0$, such that $\mathcal{G}(V) \subseteq V$. By the Schauder fixed point theorem, it is easy to check that $\mathcal{G}$ has a fixed point in V, which is the solution to (2.6) in $H^{2,1}(Q)$. □

3. Unique determination of initial data

In this section, we present proofs of Theorems 1.1 and 1.2 concerning the first two inverse problems of this paper.

3.1. Carleman estimates.

3.1. Carleman estimates. In order to prove Theorem 1.1, we first present two Carleman estimates for the following linear parabolic equation:

$$\begin{cases} u_t - \nabla\cdot(\gamma\nabla u) + A(x,t)u = F(x,t) & \text{in } Q, \\ u = 0 & \text{on } \Sigma, \\ u(x,0) = g(x) & \text{in } \Omega, \end{cases} \tag{3.1}$$

where γ is the same as the one in (1.1), $A \in L^\infty(0,T;L^{2n}(\Omega))$, $F \in L^2(Q)$ and $g \in H_0^1(\Omega)$.

As preliminaries, for two parameters $\lambda, \mu \geq 1$, we introduce the following functions:

$$\eta(x,t) = \frac{e^{\mu\psi(x)} - e^{2\mu\|\psi\|_{C(\overline{\Omega})}}}{t^2(T-t)^2},\ \varphi(x,t) = \frac{e^{\mu\psi(x)}}{t^2(T-t)^2} \ \text{ and } \ \theta_1(x,t) = e^{\lambda\eta(x,t)},$$

where $\psi(\cdot) \in C^4(\overline{\Omega})$ satisfies that $\psi(x) > 0$ in Ω, $|\nabla\psi(x)| > 0$ in $\overline{\Omega}$ and

$$\sum_{i,j=1}^n \gamma_{ij}\psi_{x_i}\nu_j \leq 0 \quad \text{on } \ (\Gamma\setminus\Gamma_0)\times(0,T).$$

Also, for any $L > 0$, there exist $t_0 \in (0,T)$ and $K > 0$, such that

$$K + t_0 < \min\left\{1, \frac{1}{2L}\right\}.$$

Set $\theta_2(t) = \dfrac{1}{K + t_0 - t}$ for $t \in [0, t_0]$.

The first Carleman estimate is stated as follows.

Lemma 3.1. *There exist positive constants λ_0, μ_0 and C, such that for any $\lambda \geq \lambda_0$ and $\mu \geq \mu_0$, the following estimate holds for any solution to (3.1):*

$$\begin{aligned} &\int_Q \theta_1^2\left(\lambda\mu^2\varphi|\nabla u|^2 + \lambda^3\mu^4\varphi^3u^2\right)dxdt \\ \leq &C\int_Q \theta_1^2F^2\,dxdt + C\int_0^T\int_{\Gamma_0}\theta_1^2\lambda\mu\varphi\,|\partial_\nu u|^2\,dSdt. \end{aligned} \tag{3.2}$$

Proof. The proof is inspired by [Yua17, Theorem 2.2]. In fact, when $A \equiv 0$, the estimate (3.2) holds true for any solution to (3.1). If $A \in L^\infty(0,T;L^{2n}(\Omega))$, we have that

$$\begin{aligned} &\int_Q \theta_1^2\left(\lambda\mu^2\varphi|\nabla u|^2 + \lambda^3\mu^4\varphi^3u^2\right)dxdt \\ \leq &C\int_Q \theta_1^2(F - Au)^2\,dxdt + C\int_0^T\int_{\Gamma_0}\theta_1^2\lambda\mu\varphi\,|\partial_\nu u|^2\,dSdt. \end{aligned}$$

Notice that when $n \geq 3$,

$$\int_Q \theta_1^2 A^2 u^2 \, dxdt \leq \int_0^T \|A\|^2_{L^{2n}(\Omega)} \|\theta_1 u\|_{L^2(\Omega)} \|\theta_1 u\|_{L^{\frac{2n}{n-2}}(\Omega)} \, dt$$
$$\leq C \int_0^T \left(\|\theta_1 u\|^2_{L^2(\Omega)} + \|\nabla(\theta_1 u)\|^2_{L^2(\Omega)} \right) dt \leq C \int_Q \theta_1^2 \left(|\nabla u|^2 + \lambda^2 \mu^2 \varphi^2 u^2 \right) dxdt.$$

When $n = 1$ and $n = 2$, the term $\|\theta_1 u\|_{L^{\frac{2n}{n-2}}(\Omega)}$ can be replaced by $\|\theta_1 u\|_{L^\infty(\Omega)}$ and $\|\theta_1 u\|_{L^4(\Omega)}$, respectively. Hence, when μ_0 is sufficiently large, (3.2) holds for any solution to (3.1). □

The second Carleman estimate is given as follows.

Lemma 3.2. *Assume that $T \in (0,1)$. Then there exists a positive constant L_0, such that for any $L \geq L_0$, $t_0 \in (0,T)$ and $K > 0$ with*

$$K + t_0 < \min\left\{1, \frac{1}{2L}\right\},$$

one can always find positive constants λ_0 and C, so that for any $\lambda \geq \lambda_0$, the following estimate holds for any solution to (3.1):

$$\begin{aligned}
&\int_0^{t_0} \int_\Omega \theta_2^{2\lambda} \Big(\lambda \theta_2^2 u^2 + L \sum_{i,j=1}^n \gamma_{ij} u_{x_i} u_{x_j} \Big) \, dxdt + \int_\Omega \frac{\lambda}{(K+t_0)^{2\lambda+1}} u^2(x,0) \, dx \\
&\leq \int_\Omega \frac{\lambda}{K^{2\lambda+1}} u^2(x,t_0) dx + \int_\Omega \frac{1}{(K+t_0)^{2\lambda}} \sum_{i,j=1}^n \gamma_{ij}(x,0) u_{x_i}(x,0) u_{x_j}(x,0) \, dx \\
&\quad + C \int_0^{t_0} \int_\Omega \theta_2^{2\lambda} F^2 \, dxdt.
\end{aligned} \tag{3.3}$$

Proof. The proof can be adapted from that of [Yu21, Theorem 2.4.1] for the Carleman estimate of stochastic degenerate parabolic equations. We sketch the necessary modifications in what follows. First, for any $\lambda \geq 1$, we set $z = \theta_2^\lambda(t) u$. Then it is straightforward to show that

$$\begin{aligned}
&2\theta_2^\lambda \left[-\lambda \theta_2 z - \sum_{i,j=1}^n (\gamma_{ij} z_{x_i})_{x_j} \right] \left[u_t - \sum_{i,j=1}^n (\gamma_{ij} u_{x_i})_{x_j} \right] \\
&= -(\lambda \theta_2 z^2)_t + \lambda \theta_2^2 z^2 - 2 \sum_{i,j=1}^n (\gamma_{ij} z_{x_i} z_t)_{x_j} + \sum_{i,j=1}^n (\gamma_{ij} z_{x_i} z_{x_j})_t - \sum_{i,j=1}^n \gamma_{ij,t} z_{x_i} z_{x_j} \\
&\quad + 2 \left[\lambda \theta_2 z + \sum_{i,j=1}^n (\gamma_{ij} z_{x_i})_{x_j} \right]^2.
\end{aligned}$$

Integrating the above equality on $\Omega\times(0,t_0)$, we obtain that

$$\begin{aligned}&\int_0^{t_0}\int_\Omega \lambda\theta_2^2 z^2\,dxdt+\int_\Omega\sum_{i,j=1}^n\gamma_{ij}(x,t_0)z_{x_i}(x,t_0)z_{x_j}(x,t_0)\,dx+\int_\Omega\lambda\theta_2(0)z^2(x,0)\,dx\\ \le&\int_\Omega\sum_{i,j=1}^n\gamma_{ij}(x,0)z_{x_i}(x,0)z_{x_j}(x,0)\,dx+\int_\Omega\lambda\theta_2(t_0)z^2(x,t_0)\,dx\\ &+\int_0^{t_0}\int_\Omega\Big|\sum_{i,j=1}^n\gamma_{ij,t}z_{x_i}z_{x_j}\Big|\,dxdt+\int_0^{t_0}\int_\Omega\theta_2^{2\lambda}(F-Au)^2\,dxdt.\end{aligned}\tag{3.4}$$

On the other hand, notice that

$$2\theta_2^{2\lambda}u\Big[u_t-\sum_{i,j=1}^n(\gamma_{ij}u_{x_i})_{x_j}\Big]=(\theta_2^{2\lambda}u^2)_t-2\sum_{i,j=1}^n(\gamma_{ij}z_{x_i}z)_{x_j}-2\lambda\theta_2^{2\lambda+1}u^2+2\sum_{i,j=1}^n\gamma_{ij}u_{x_i}u_{x_j}.$$

This implies that for any $L>0$,

$$\begin{aligned}&2L\int_0^{t_0}\int_\Omega\sum_{i,j=1}^n\gamma_{ij}z_{x_i}z_{x_j}\,dxdt+L\int_\Omega\theta_2^{2\lambda}(t_0)u^2(x,t_0)\,dx\\ \le&2L\lambda\int_0^{t_0}\int_\Omega\theta_2^{2\lambda+1}u^2\,dxdt+L\int_\Omega\theta_2^{2\lambda}(0)u^2(x,0)\,dx+2L\int_0^{t_0}\int_\Omega\theta_2^{2\lambda}u(F-Au)\,dxdt.\end{aligned}\tag{3.5}$$

By (3.4), (3.5) and the definition of θ_2, it follows that

$$\begin{aligned}&\int_0^{t_0}\int_\Omega\Big(\lambda\theta_2^{2\lambda+2}u^2+2L\theta_2^{2\lambda}\sum_{i,j=1}^n\gamma_{ij}u_{x_i}u_{x_j}\Big)\,dxdt\\ &+\int_\Omega\frac{\lambda}{(K+t_0)^{2\lambda+1}}u^2(x,0)\,dx+\int_\Omega\frac{1}{K^{2\lambda}}\sum_{i,j=1}^n\gamma_{ij}(x,t_0)u_{x_i}(x,t_0)u_{x_j}(x,t_0)\,dx\\ \le&L\int_\Omega\frac{1}{(K+t_0)^{2\lambda}}u^2(x,0)\,dx+\int_\Omega\frac{1}{(K+t_0)^{2\lambda}}\sum_{i,j=1}^n\gamma_{ij}(x,0)u_{x_i}(x,0)u_{x_j}(x,0)\,dx\\ &+\int_\Omega\frac{\lambda}{K^{2\lambda+1}}u^2(x,t_0)\,dx+\int_0^{t_0}\int_\Omega\Big(2L\lambda\theta_2^{2\lambda+1}u^2+\Big|\sum_{i,j=1}^n\gamma_{ij,t}z_{x_i}z_{x_j}\Big|\Big)\,dxdt\\ &+\int_0^{t_0}\int_\Omega\theta_2^{2\lambda}\Big[Lu^2+(L+1)(F-Au)^2\Big]\,dxdt.\end{aligned}$$

Furthermore, we notice that $\theta_2(t)\ge\frac{1}{K+t_0}>2L$. Also, for any $\epsilon>0$,

$$\begin{aligned}&\int_0^{t_0}\int_\Omega\theta_2^{2\lambda}A^2u^2\,dxdt\\ \le&\int_0^{t_0}\theta_2^{2\lambda}\|A\|^2_{L^{2n}(\Omega)}\|u\|_{L^2(\Omega)}\|u\|_{L^{\frac{2n}{n-2}}(\Omega)}\,dt\\ \le&\epsilon\int_0^{t_0}\int_\Omega\theta_2^{2\lambda}|\nabla u|^2\,dxdt+C\int_0^{t_0}\int_\Omega\theta_2^{2\lambda}u^2\,dxdt.\end{aligned}$$

Hence, for sufficiently large L and λ, it follows that

$$\begin{aligned}&\int_0^{t_0}\int_\Omega\Big(\lambda\theta_2^{2\lambda+2}u^2+2L\theta_2^{2\lambda}\sum_{i,j=1}^n\gamma_{ij}u_{x_i}u_{x_j}\Big)\,dxdt+\int_\Omega\frac{\lambda}{(K+t_0)^{2\lambda+1}}u^2(x,0)\,dx\\ \le&\int_\Omega\frac{1}{(K+t_0)^{2\lambda}}\sum_{i,j=1}^n\gamma_{ij}(x,0)u_{x_i}(x,0)u_{x_j}(x,0)\,dx\\ &+\int_\Omega\frac{\lambda}{K^{2\lambda+1}}u^2(x,t_0)\,dx+C\int_0^{t_0}\int_\Omega\theta_2^{2\lambda}F^2\,dxdt.\end{aligned}$$

This implies the desired estimate (3.3). The proof is complete. □

3.2. **Determination of initial data.** Based on Lemmas 3.1 and 3.2, one has the following conditional stability result for the inverse source problem of (3.1).

Lemma 3.3. *For any $M>0$, if*

$$\|g\|_{H_0^1(\Omega)}+\|F\|_{L^2(Q)}\le M, \tag{3.6}$$

there exist positive constants C and $\delta_0\in(0,1)$, depending only on n,T and Ω, such that the following estimate holds for any solution to (3.1):

$$\|u(\cdot,0)\|_{L^2(\Omega)}^2\le\frac{C(M+1)}{\delta_0}\|(F,\partial_\nu u)\|-\frac{CM^2}{\ln[\delta_0\|(F,\partial_\nu u)\|]}, \tag{3.7}$$

where $\|(F,\partial_\nu u)\|=\big(\|F\|_{L^2(Q)}^2+\|\partial_\nu u\|_{L^2(\Gamma_0\times(0,T))}^2\big)^{1/2}$.

Proof. Without loss of generality, we assume that $T<1$. For any $t_1\in(0,T)\cap(0,\frac{2}{3})\cap(0,\frac{1}{3L})$ with L being the constant in Lemma 3.2, choose $K=\frac{t_1}{2}$ and $t_0\in[\frac{t_1}{2},t_1]$. Then,

$$K+t_0\le\frac{3}{2}t_1<\min\Big\{1,\frac{1}{2L}\Big\}$$

and

$$\Big(\frac{t_1+2t_0}{2}\Big)^{-2\lambda}=(K+t_0)^{-2\lambda}\le\theta_2^{2\lambda}(t)\le\Big(\frac{2}{t_1}\Big)^{2\lambda},\quad\text{for any }\lambda\ge\lambda_0\text{ and }t\in[0,t_0].$$

By Lemma 3.2,

$$\begin{aligned}&\lambda\int_\Omega\Big(\frac{t_1+2t_0}{2}\Big)^{-2\lambda-1}u^2(x,0)\,dx\\ \le&C\int_\Omega\Big(\frac{t_1+2t_0}{2}\Big)^{-2\lambda}|\nabla u(x,0)|^2\,dx+C\lambda\Big(\frac{2}{t_1}\Big)^{2\lambda+1}\Big[\int_\Omega u^2(x,t_0)dx+\int_Q F^2(x,t)\,dxdt\Big].\end{aligned}$$

This implies that

$$\begin{aligned}&\int_\Omega u^2(x,0)\,dx\\ \le&\frac{C}{\lambda}\int_\Omega|\nabla u(x,0)|^2\,dx\\ &+C\Big(\frac{t_1+2t_0}{2}\Big)^{2\lambda}\Big(\frac{2}{t_1}\Big)^{2\lambda+1}\Big[\int_\Omega u^2(x,t_0)\,dx+\int_Q F^2(x,t)\,dxdt\Big]\\ \le&\frac{C}{\lambda}\int_\Omega|\nabla u(x,0)|^2\,dx+C9^\lambda\|(F,t_0)\|^2,\end{aligned} \tag{3.8}$$

where $\|(F,t_0)\|^2:=\int_\Omega u^2(x,t_0)\,dx+\int_Q F^2(x,t)\,dxdt.$

On the other hand, by Lemma 3.1, for fixed parameters λ and μ, it holds that

$$\int_{\frac{t_0}{2}}^{t_0}\int_\Omega \left(u^2+|\nabla u|^2\right)\,dxdt \le C\int_Q F^2dxdt + C\int_0^T\int_{\Gamma_0}|\partial_\nu u|^2\,dSdt.$$

Hence, there exists a $\hat t \in (\frac{t_0}{2}, t_0)$, such that

$$\int_\Omega \left(u^2(x,\hat t)+|\nabla u(x,\hat t)|^2\right)\,dx \le C\int_Q F^2dxdt + C\int_0^T\int_{\Gamma_0}|\partial_\nu u|^2\,dSdt.$$

By the standard energy estimate,

$$\begin{aligned}&\int_\Omega u^2(x,t_0)\,dx\\ \le& C\int_\Omega u^2(x,\hat t)\,dx + C\int_{\hat t}^{t_0}\int_\Omega (u^2+F^2)\,dxdt \\ \le& C\int_Q F^2\,dxdt + C\int_0^T\int_{\Gamma_0}|\partial_\nu u|^2\,dSdt.\end{aligned} \tag{3.9}$$

By (3.8) and (3.9), it holds that

$$\int_\Omega u^2(x,0)\,dx \le \frac{C}{\lambda}\int_\Omega |\nabla u(x,0)|^2\,dx + C9^\lambda\|(F,u_\nu)\|^2. \tag{3.10}$$

Take

$$\delta_0\in(0,e^{-\lambda_0\ln 9}) \quad \text{and} \quad \lambda=\frac{1}{\ln 9}\ln\left(\frac{\|(F,u_\nu)\|+1}{\delta_0\|(F,u_\nu)\|}\right),$$

where λ_0 is the constant in Lemma 3.2. Then, $\lambda\ge\lambda_0$. Set

$$\hat u = \frac{\delta_0}{\|(F,u_\nu)\|+1}u \quad \text{and} \quad \hat F=\frac{\delta_0}{\|(F,u_\nu)\|+1}F.$$

By (3.10), it follows that

$$\begin{aligned}&\int_\Omega \hat u^2(x,0)\,dx\\ \le&\frac{C}{\lambda}\int_\Omega|\nabla\hat u(x,0)|^2\,dx + C9^\lambda\frac{\delta_0^2}{(\|(F,u_\nu)\|+1)^2}\|(F,u_\nu)\|^2\\ \le&\frac{C}{\ln\left(\frac{\|(F,u_\nu)\|+1}{\delta_0\|(F,u_\nu)\|}\right)}\int_\Omega|\nabla\hat u(x,0)|^2\,dx + C\frac{\delta_0\|(F,u_\nu)\|}{\|(F,u_\nu)\|+1}.\end{aligned}$$

This implies that

$$\int_\Omega u^2(x,0)\,dx \le \frac{C}{\ln\left(\frac{\|(F,u_\nu)\|+1}{\delta_0\|(F,u_\nu)\|}\right)}\int_\Omega|\nabla u(x,0)|^2\,dx + C\frac{[\|(F,u_\nu)\|+1]\|(F,u_\nu)\|}{\delta_0}.$$

For any $M>0$ given in (3.6), by the well-posedness of linear parabolic equations, we have that

$$\|(F,u_\nu)\|\le CM.$$

Hence,

$$\int_\Omega u^2(x,0)\,dx \le \frac{CM^2}{\ln\left(\frac{\|(F,u_\nu)\|+1}{\delta_0\|(F,u_\nu)\|}\right)} + C\frac{(M+1)\|(F,u_\nu)\|}{\delta_0}.$$

This implies the desired estimate (3.7). □

Now, we give a proof of Theorem 1.1.

Proof of Theorem 1.1. For any $a \in \mathcal{A}_T$ and two initial values $g_1, g_2 \in H_0^1(\Omega)$, let $\widetilde{u} = u_1 - u_2$, where u_j $(j = 1,2)$ are the solutions to (1.5) associated to g_j. Then $\widetilde{u} \in H^{2,1}(Q)$ is the solution to the following parabolic equation:

$$\begin{cases} \widetilde{u}_t - \nabla \cdot (\gamma \nabla \widetilde{u}) + A(x,t)\widetilde{u} = 0 & \text{in } Q, \\ \widetilde{u} = 0 & \text{on } \Sigma, \\ \widetilde{u}(x,0) = g_1 - g_2, & \text{in } \Omega, \end{cases} \tag{3.11}$$

with

$$A(x,t)\widetilde{u} = a(x,t,u_1) - a(x,t,u_2) = \Big(\int_0^1 a_u(x,t,su_1 + (1-s)u_2)\, ds \Big) \cdot \widetilde{u}.$$

with $A(x,t) = \displaystyle\int_0^1 a_u(x,t,su_1 + (1-s)u_2)\, ds$. Similar to [LLL21, Theorem 3.2], we can prove that $A \in L^\infty(0,T;L^{2n}(\Omega))$. By Lemma 3.3, for any $M > 0$, if $\|g_1 - g_2\|_{H_0^1(\Omega)} \le M$, there exist positive constants C and $\delta_0 \in (0,1)$, depending only on n, T and Ω, such that

$$\|\widetilde{u}(\cdot,0)\|^2_{L^2(\Omega)} \le \frac{C(M+1)}{\delta_0} \|\partial_\nu \widetilde{u}\|_{L^2(\Gamma_0 \times (0,T))} - \frac{CM^2}{\ln\big(\delta_0 \|\partial_\nu \widetilde{u}\|_{L^2(\Gamma_0\times(0,T))}\big)}.$$

This proves the desired estimate (1.6). □

Furthermore, there is a counterexample showing that if a is unknown, the passive measurement cannot uniquely determine all unknowns.

Theorem 3.1 (Non-uniqueness)**.** *Suppose that $\gamma \in C^{2,1}(\overline{Q}; \mathbb{R}^{n\times n})$ is symmetric and uniformly positive definite, $a_j \in \mathcal{A}_T$ and $g_j \in H_0^1(\Omega)$ for $j = 1,2$. Denote by $\Lambda^0_{a_j,g_j}$ the passive measurement of the following semilinear parabolic equation:*

$$\begin{cases} \partial_t u_j - \nabla \cdot (\gamma \nabla u_j) + a_j(x,t,u_j) = 0 & \text{in } Q, \\ u_j = 0 & \text{on } \Sigma, \\ u_j(x,0) = g_j(x), & \text{in } \Omega. \end{cases} \tag{3.12}$$

Then there exist two groups of unknown sources $(g_1,a_1), (g_2,a_2) \in H_0^1(\Omega) \times \mathcal{A}_T$, such that

$$(g_1,a_1) \ne (g_2,a_2),$$

but

$$\Lambda^0_{g_1,a_1} = \Lambda^0_{g_2,a_2}.$$

Proof. Assume that two functions $u_1, u_2 \in C^\infty(\overline{Q})$ satisfy that

$$u_1(\cdot,0) \ne u_2(\cdot,0) \text{ in a measurable set of } \Omega \text{ with positive measure,}$$

$$\text{and } u_1(x,t) = u_2(x,t) = 0 \text{ in } \Omega_\epsilon \times [0,T],$$

where $\Omega_\epsilon = \Big\{ x \in \Omega \;\Big|\; \mathrm{dist}(x,\Gamma) < \epsilon \Big\}$. Set

$$A_j(x,t) = -\partial_t u_j(x,t) + \nabla \cdot (\gamma \nabla u_j(x,t)), \qquad \text{for } j = 1,2 \text{ and } (x,t) \in Q.$$

It is easy to show that u_j $(j = 1,2)$ are solutions to (3.12) associated to

$$g_j(x) = u_j(x,0) \quad \text{ and } \quad a_j(x,t,u_j) = A_j(x,t).$$

Then,

$$(g_1,a_1) \ne (g_2,a_2),$$

but

$$\partial_\nu u_1\Big|_{\Gamma_0\times(0,T)} = \Lambda^0_{g_1,a_1} = \Lambda^0_{g_2,a_2} = \partial_\nu u_2\Big|_{\Gamma_0\times(0,T)} = 0.$$

□

Finally, as a corollary of Theorem 1.1, we prove Theorem 1.2 under the condition that $a \in \mathcal{B}_T$ (see (1.7)).

Proof of Theorem 1.2. For any $a_j \in \mathcal{B}_T$ $(j = 1, 2)$,

$$a_j(x,t,y) = a_0(x,t,u)\chi_{[0,T-\epsilon]}(t) + c_j(x,t,u)\chi_{[T-\epsilon,T]}(t),$$

where $\epsilon > 0$, $a_0 \in \mathcal{A}_T$ and $c_1, c_2 \in \mathcal{A}_T$ with $c_1(x,t,0) = c_2(x,t,0) = 0$ in Q. By the condition (1.9),

$$\Lambda_{a_1,g_1}(0) = \Lambda_{a_2,g_2}(0) \quad \text{on} \quad \Gamma_0 \times (0,T).$$

Hence,

$$\Lambda^0_{a_1,g_1} = \Lambda^0_{a_2,g_2} \quad \text{on} \quad \Gamma_0 \times (0,T-\epsilon).$$

By the results in Theorem 1.1 for $a = a_0$ in the time period $[0, T-\epsilon]$, we get the assertion in Theorem 1.2. □

4. Simultaneous recovery results for inverse problems

In this section, we present the proofs of Theorems 1.3 and 1.4 on the simultaneous recovery results for the inverse problems. We first derive the unique determination of the coefficient for the linear parabolic equation. To that end, let us prove some useful properties, which will be needed in the proofs of Theorems 1.3 and 1.4.

4.1. **Approximation and denseness properties.** Let us begin with the Runge approximation properties for linear parabolic equations. The following approximation property will be used in the proof of Theorems 1.3 and 1.4 with full data.

Lemma 4.1 (Runge approximation with full data)**.** *Let $q \in C^{2+\alpha,1+\alpha/2}(\overline{Q})$. Then for any solutions $v_\pm \in L^2(0,T;H^1(\Omega)) \cap H^1(0,T;H^{-1}(\Omega))$ to*

$$\begin{cases} \partial_t v_+ - \Delta v_+ + q v_+ = 0 & \text{in } Q, \\ v_+(x,0) = 0 & \text{in } \Omega, \end{cases} \tag{4.1}$$

and

$$\begin{cases} -\partial_t v_- - \Delta v_- + q v_- = 0 & \text{in } Q, \\ v_-(x,T) = 0 & \text{in } \Omega, \end{cases} \tag{4.2}$$

and any $\eta > 0$, there exist solutions $V_\pm \in C^{2+\alpha,1+\alpha/2}(\overline{Q})$ to

$$\begin{cases} \partial_t V_+ - \Delta V_+ + q V_+ = 0 & \text{in } Q, \\ V_+(x,0) = 0 & \text{in } \Omega, \end{cases} \tag{4.3}$$

and

$$\begin{cases} -\partial_t V_- - \Delta V_- + q V_- = 0 & \text{in } Q, \\ V_-(x,T) = 0 & \text{in } \Omega, \end{cases} \tag{4.4}$$

such that

$$\|V_\pm - v_\pm\|_{L^2(Q)} < \eta.$$

Proof. We only prove the case for the forward parabolic equation, and the backward one can be proved similarly. Define

$$X = \left\{ V \in C^{2+\alpha,1+\alpha/2}(\overline{Q}) \,\middle|\, V \text{ is a solution to (4.3)} \right\}$$

and

$$Y = \left\{ v \in L^2(0,T;H^1(\Omega)) \cap H^1(0,T;H^{-1}(\Omega)) \,\middle|\, v \text{ is a solution to (4.1)} \right\}.$$

We aim to show that X is dense in Y. By the Hahn-Banach theorem, it suffices to prove the following statement: If $f \in L^2(Q)$ satisfies

$$\int_Q fV\, dxdt = 0, \qquad \text{for any } V \in X,$$

then

$$\int_Q fv\, dxdt = 0, \qquad \text{for any } v \in Y.$$

To this end, let $f \in L^2(Q)$ and suppose $\int_Q fV\, dxdt = 0$, for any $V \in X$. Consider

$$\begin{cases} -\partial_t \overline{V} - \Delta \overline{V} + q\overline{V} = f & \text{in } \ Q, \\ \overline{V} = 0 & \text{on } \ \Sigma, \\ \overline{V}(x,T) = 0 & \text{in } \ \Omega \end{cases} \tag{4.5}$$

and its solution is in $H^{2,1}(Q)$. For any $V \in X$, one has

$$0 = \int_Q fV\, dxdt = \int_Q (-\partial_t \overline{V} - \Delta \overline{V} + q\overline{V})V\, dxdt = \int_\Sigma \partial_\nu \overline{V} V\, dSdt.$$

Since $V|_\Sigma$ can be arbitrary function, which is compactly supported on Σ, we must have $\partial_\nu \overline{V} = 0$ on Σ. Thus, for any $v \in Y$,

$$\int_Q fv\, dxdt = \int_Q (-\partial_t \overline{V} - \Delta \overline{V} + q\overline{V})v\, dxdt = \int_\Sigma \partial_\nu \overline{V} v\, dSdt = 0,$$

which verifies the assertion. □

Let $\Omega \subset \mathbb{R}^n$ be a connected domain, and Ω' be a connected open subset of Ω such that $\partial\Omega \subset \partial\Omega'$. Define $Q' = (\Omega \backslash \Omega') \times (0,T)$. Meanwhile, for given $\varepsilon > 0$ and $\omega \in \mathbb{S}^{n-1}$, we set

$$\begin{aligned} &\Gamma_{+,\omega,\varepsilon} := \left\{ x \in \Gamma \ \middle|\ \nu(x)\cdot\omega > \varepsilon \right\}, \\ &\Gamma_{-,\omega,\varepsilon} := \left\{ x \in \Gamma \ \middle|\ -\nu(x)\cdot\omega > \varepsilon \right\}, \\ &\text{and } \Sigma_{\pm,\omega,\varepsilon} := \Gamma_{\pm,\omega,\varepsilon} \times (0,T). \end{aligned}$$

The following approximation property will be used to prove Theorems 1.3 and 1.4 with partial data.

Lemma 4.2 (Runge approximation with partial data)**.** *Let $q \in C^{2+\alpha,1+\alpha/2}(\overline{Q})$. Then for any solutions $W_\pm \in L^2(0,T;H^1(\Omega)) \cap H^1(0,T;H^{-1}(\Omega))$ to*

$$\begin{cases} \partial_t W_+ - \Delta W_+ + qW_+ = 0 & \text{in } \ Q, \\ W_+(x,0) = 0 & \text{in } \ \Omega \end{cases} \tag{4.6}$$

and

$$\begin{cases} -\partial_t W_- - \Delta W_- + qW_- = 0 & \text{in } \ Q, \\ W_-(x,T) = 0 & \text{in } \ \Omega, \end{cases} \tag{4.7}$$

and any $\eta > 0$, there exist solutions $v_\pm \in C^{2+\alpha,1+\alpha/2}(\overline{Q})$ to

$$\begin{cases} \partial_t v_+ - \Delta v_+ + qv_+ = 0 & \text{in } \ Q, \\ v_+ = 0 & \text{on } \ \Gamma_{-,\omega,\varepsilon} \times (0,T), \\ v_+(x,0) = 0 & \text{in } \ \Omega, \end{cases} \tag{4.8}$$

and

$$\begin{cases} -\partial_t v_- - \Delta v_- + q v_- = 0 & \text{in } Q, \\ v_- = 0 & \text{on } \Gamma_{+,\omega,\varepsilon} \times (0,T), \\ v_-(x,T) = 0 & \text{in } \Omega, \end{cases} \tag{4.9}$$

such that

$$\|W_\pm - v_\pm\|_{L^2(Q')} < \eta.$$

Proof. We may only prove the case for forward parabolic equations. Define

$$X' = \left\{ v \in C^{2+\alpha,1+\alpha/2}(\overline{Q}) \,\middle|\, v \text{ is a solution to } (4.8) \right\}$$

and

$$Y' = \left\{ W \in L^2(0,T;H^1(\Omega)) \cap H^1(0,T;H^{-1}(\Omega)) \,\middle|\, V \text{ is a solution to } (4.6) \right\}.$$

We aim to show that X' is dense in Z. By the Hahn-Banach theorem again, it suffices to claim that if $f \in L^2(Q')$ satisfies

$$\int_{Q'} f v \, dxdt = 0, \text{ for any } v \in X',$$

then

$$\int_{Q'} f W \, dxdt = 0, \text{ for any } W \in Y'.$$

Let $f \in L^2(Q')$ satisfy that $\int_{Q'} f v \, dxdt = 0, \ \forall v \in X'$. We extend f to Q by letting $f = 0$ outside Q'.

Consider

$$\begin{cases} -\partial_t \overline{v} - \Delta \overline{v} + q \overline{v} = f & \text{in } Q, \\ \overline{v} = 0 & \text{on } \Sigma, \\ \overline{v}(x,T) = 0 & \text{in } \Omega, \end{cases} \tag{4.10}$$

and its solution is in $H^{2,1}(Q)$. Then for any $v \in X'$,

$$0 = \int_Q f v \, dxdt = \int_Q (-\partial_t \overline{v} - \Delta \overline{v} + q \overline{v}) v \, dxdt = \int_\Sigma \partial_\nu \overline{v} v \, dSdt.$$

Since $v|_\Sigma$ can be arbitrary function, which is compactly supported on $\Sigma \backslash (\Gamma_{-,\omega,\varepsilon} \times (0,T))$ and $v = 0$ on $\Gamma_{-,\omega,\varepsilon} \times (0,T)$, we have that $\partial_\nu \overline{v} = 0$ on $\Sigma \backslash (\Gamma_{-,\omega,\varepsilon} \times (0,T))$.

Next, let Ω_1 be a nonempty open set such that $(\Omega_1 \cap \partial\Omega) \subset (\Gamma \backslash \Gamma_{-,\omega,\varepsilon})$. Then $\overline{v} = 0$ on $\Omega_1 \times (0,T)$. Notice that

$$-\partial_t \overline{v} - \Delta \overline{v} + q \overline{v} = 0 \text{ in } (\Omega' \cup \Omega_1) \times (0,T).$$

Since $\Omega' \cup \Omega_1$ is open and connected, by the unique continuation principle for linear parabolic equations (for instance, see [SS87]), we have $\overline{v} = 0$ on $\Omega' \times (0,T)$. Hence, $\overline{v}\big|_{\partial\Omega' \times (0,T)} = \partial_\nu \overline{v}\big|_{\partial\Omega' \times (0,T)} = 0$, and it follows that

$$\overline{v}\big|_{\partial(\Omega \backslash \Omega') \times (0,T)} = \partial_\nu \overline{v}\big|_{\partial(\Omega \backslash \Omega') \times (0,T)} = 0.$$

Hence, for any $W \in Y'$,

$$\int_{Q'} f W \, dxdt = \int_{Q'} (-\partial_t \overline{v} - \Delta \overline{v} + q \overline{v}) W \, dxdt = \int_{\partial(\Omega \backslash \Omega') \times (0,T)} \partial_\nu \overline{v} W \, dSdt = 0.$$

This completes the proof. □

Remark 4.3. *Let us refer readers to some related approximation property for some different diffusion equations, such as* [CK18a, Lemma 5.3]*. Since the proofs of the global uniqueness results with either full data or partial data are similar, we focus on presenting the arguments for the full data case and remark the necessary modifications for the partial data case, and vice versa.*

Lemma 4.4 (Denseness property)**.** *Let $q_1, q_2 \in L^\infty(Q)$. Assume that $f \in L^\infty(Q)$, such that*

$$\int_Q f v_1 v_2 \, dx dt = 0,$$

for any v_1 and v_2, which satisfy $v_1 v_2 \in L^1(Q)$, and are, respectively, solutions to

$$\begin{cases} \partial_t v_1 - \Delta v_1 + q_1 v_1 = 0 & \text{in } Q, \\ v_1(x,0) = 0 & \text{in } \Omega, \end{cases} \tag{4.11}$$

and

$$\begin{cases} \partial_t v_2 + \Delta v_2 - q_2 v_2 = 0 & \text{in } Q, \\ v_2(x,T) = 0 & \text{in } \Omega. \end{cases} \tag{4.12}$$

Then $f = 0$. In other words, the linear span of products of solutions to forward and backward parabolic equations are dense in $L^1(Q)$.

Proof. Since $q_j \in L^\infty(Q)$ for $j = 1, 2$, without loss of generality, we may assume that there exists a positive number m, such that $q_1, q_2 \in \left\{ q \in L^\infty(Q) \;\middle|\; \|q\|_{L^\infty(Q)} < m \right\}$. First, let us fix $\omega \in \mathbb{S}^{n-1}$. Consider $\rho > 0$ to be sufficiently large, and $(\xi, \tau) \in M := \left\{ (\xi, \tau) \in \mathbb{R}^{n+1} \middle| \, \xi \cdot \omega = 0 \right\}$ with $|(\xi, \tau)|^2 < \rho - 1$. Then by Proposition A.1, there is a solution $v_{1,\rho}(\cdot, \cdot; \xi, \tau)$ to (4.11) such that

$$v_{1,\rho} = \psi_{-,\rho}(\theta_{+,\rho} + z_{+,\rho,q_1})$$

with $\|z_{+,\rho,q_1}\|_{L^2(Q)} \to 0$ as $\rho \to \infty$. Similarly, there is a solution $v_{2,\rho}(\cdot, \cdot)$ to the backward parabolic equation (4.12) such that

$$v_{2,\rho} = \psi_{+,\rho}(\theta_{-,\rho} + z_{-,\rho,q_2})$$

with $\|z_{-,\rho,q_2}\|_{L^2(Q)}$ tending to 0, as $\rho \to \infty$. Then

$$\begin{aligned} v_{1,\rho} v_{2,\rho} &= \theta_{+,\rho}\theta_{-,\rho} + \theta_{+,\rho} z_{-,\rho,q_2} + z_{+,\rho,q_1}\theta_{-,\rho} + z_{+,\rho,q_1} z_{-,\rho,q_2} \\ &= \varphi_\rho(t) e^{-\mathrm{i}(x,t)\cdot(\xi,\tau)} + \theta_{+,\rho} z_{-,\rho,q_2} + z_{+,\rho,q_1}\theta_{-,\rho} + z_{+,\rho,q_1} z_{-,\rho,q_2}, \end{aligned}$$

where $\varphi_\rho(t) = 1 - \exp(-\rho^{3/4} t) - \exp(-\rho^{3/4}(T - t)) + \exp(-\rho^{3/4} T)$. Note that $\theta_{+,\rho}$ and $\theta_{-,\rho}$ are bounded with respect to $\rho > 0$. Hence, letting $\rho \to +\infty$ in $\int_Q f v_{1,\rho} v_{2,\rho} \, dx dt = 0$, we have that

$$\int_Q f e^{-\mathrm{i}(x,t)\cdot(\xi,\tau)} \, dx dt = 0. \tag{4.13}$$

Therefore, for a fixed $\omega \in \mathbb{S}^{n-1}$, (4.13) holds in any compact subset of M. Clearly, M is an n-dimensional subspace of $\mathbb{R}^{n+1}$. Notice that f has compact support as a distribution and its Fourier transform is analytic. The Fourier transform of f is zero in any compact subset of M as shown, and therefore by changing $\omega \in \mathbb{S}^{n-1}$ in a small conic neighborhood, we can conclude it is zero in $\mathbb{R}^{n+1}$. This implies $f = 0$ in Q as desired. □

In the application of the preceding denseness result with full data, we are able to derive the following global uniqueness result as follows.

Corollary 4.5 (Global uniqueness with full data). *Let $q_1, q_2 \in L^\infty(Q)$. Let Λ_{q_j} be the full DN map of the linear heat equation:*

$$\begin{cases} \partial_t v_j - \Delta v_j + q_j v_j = 0 & \text{in } Q, \\ v_j(x,0) = 0 & \text{in } \Omega, \end{cases} \tag{4.14}$$

for $j = 1, 2$, respectively. Assume that

$$\Lambda_{q_1}(f) = \Lambda_{q_2}(f) \text{ on } \Sigma, \tag{4.15}$$

for any $f \in L^2(0,T; H^{1/2}(\Gamma))$, then $q_1 = q_2$ in Q.

Proof. This result can be regarded as an application of [CK18b], and we offer the proof for the sake of completeness. Let $\hat{v}$ be a solution to the backward heat equation:

$$\begin{cases} \partial_t \hat{v} + \Delta \hat{v} - q_2 \hat{v} = 0 & \text{in } Q, \\ \hat{v}(x,T) = 0 & \text{in } \Omega. \end{cases} \tag{4.16}$$

Subtracting (4.14) with $j = 1, 2$, then we have

$$\begin{cases} \partial_t \widetilde{v} - \Delta \widetilde{v} + q_2 \widetilde{v} = (q_2 - q_1) v_1 & \text{in } Q, \\ \widetilde{v}(x,0) = 0 & \text{in } \Omega, \end{cases} \tag{4.17}$$

where $\widetilde{v} = v_1 - v_2$ in Q. Multiplying (4.17) by the solution $\hat{v}$ of (4.16), with the condition (4.15) at hand, it is easy to derive that

$$\int_Q (q_2 - q_1) v_1 \hat{v} \, dxdt = 0. \tag{4.18}$$

Therefore, by applying (4.4), one can conclude that $q_1 = q_2$ in Q as desired.

∎

Lemma 4.6 (Global uniqueness with partial data). *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with C^∞-smooth boundary Γ. For any $q_j \in C^{2+\alpha, 1+\frac{\alpha}{2}}(\overline{Q})$ $(j = 1, 2)$, assume that $\Lambda^{\mathrm{P}}_{q_j}$ are the partial DN maps of the linear parabolic equation:*

$$\begin{cases} (\partial_t - \Delta + q_j) u = 0 & \text{in } Q, \\ u = f & \text{on } \Sigma, \\ u(x,0) = 0, & \text{in } \Omega, \end{cases} \tag{4.19}$$

and

$$\Lambda^{\mathrm{P}}_{q_1}(f) = \Lambda^{\mathrm{P}}_{q_2}(f) \text{ in } \mathcal{V}_-,$$

for any $f \in C_0^{2+\alpha, 1+\alpha/2}(\mathcal{V}_+)$. If $q_1 = q_2$ in $\Omega' \times (0,T)$, where Ω' is an arbitrarily given connected open subset of Ω with $\Gamma \subset \partial\Omega'$, then

$$q_1 = q_2 \text{ in } Q.$$

Proof. By Proposition A.1, there is a solution

$$v_1(\cdot, \cdot; \rho, \xi, \tau, \omega) = \psi_{-,\rho}(\theta_{+,\rho} + z_{+,\rho,q_1}) \in L^2(0,T; H^1(\Omega)) \cap H^1(0,T; H^{-1}(\Omega))$$

to the forward parabolic equation (4.19) with respect to q_1 such that

$$\lim_{\rho\to\infty} \|z_{+,\rho,q_1}\|_{L^2(Q)} = 0.$$

For $j \in \{1, 2\}$, let us define

$$S_j = \left\{ v \in L^2(0,T; H^1(\Omega)) \cap H^1(0,T; H^{-1}(\Omega)) \,\middle|\, (\partial_t - \Delta + q_j) v = 0 \text{ in } Q, v(x,0) = 0 \text{ in } \Omega \right\},$$

and the map $\mathcal{M} : S_1 \to S_2$ is defined by

$$\mathcal{M}(v_1) = v_2,$$

where v_2 is the solution to

$$\begin{cases} (\partial_t - \Delta + q_2)v_2 = 0 & \text{in } \ Q, \\ v_2 = v_1 & \text{on } \ \Sigma, \\ v_2(x,0) = 0, & \text{in } \ \Omega. \end{cases} \tag{4.20}$$

By using the trace theorem, $v_1\big|_\Sigma \in L^2(0,T;H^{1/2}(\Gamma))$ and the map $\mathcal{M}$ is well-defined. Now we have

$$\begin{cases} (\partial_t - \Delta + q_2)(v_1 - v_2) = (q_2 - q_1)v_1 & \text{in } \ Q, \\ v_1 - v_2 = 0 & \text{on } \ \Sigma, \\ (v_1 - v_2)(x,0) = 0 & \text{in } \ \Omega. \end{cases} \tag{4.21}$$

Consider a solution $\hat{v}$ to the backward parabolic equation (A.2) of the form that we have constructed in Proposition A.1 with $q = q_2$. Then by Lemma 4.2, there are two sequences of functions $\{v_1^k\}_{k=1}^\infty$, $\{\hat{v}^k\}_{k=1}^\infty \in C^{2+\alpha,1+\frac{\alpha}{2}}(\overline{Q})$, such that v_1^k are solutions to (4.8), $\hat{v}^k$ are solutions to (4.9), and $v_1^k \to v_1$, $\hat{v}^k \to \hat{v}$ in $L^2(Q')$ as $k \to \infty$. Hence, we have

$$\begin{cases} (\partial_t - \Delta + q_2)(v_1^k - \mathcal{M}(v_1^k)) = (q_2 - q_1)v_1^k & \text{in } \ Q, \\ v_1^k - \mathcal{M}(v_1^k) = 0 & \text{on } \ \Sigma, \\ (v_1^k - \mathcal{M}(v_1^k))(x,0) = 0 & \text{in } \ \Omega. \end{cases} \tag{4.22}$$

Let $v_2^k = \mathcal{M}(v_1^k)$. Multiplying by the functions $\hat{v}^k$ on the both sides of the above equation and integration by parts implies

$$\int_Q (q_2 - q_1)\, v_1^k \hat{v}^k \; dxdt = \int_\Sigma \hat{v}^k \partial_\nu (v_1^k - v_2^k)\, dSdt.$$

Since $\mathcal{U}_\pm$ is a neighborhood of $\Gamma_{\pm,\omega_0}$ (recalling $\mathcal{V}_\pm = \mathcal{U}_\pm \times (0,T)$), there is an $\varepsilon > 0$, such that[1]

$$\Big\{x \in \Gamma \,\Big|\, 0 < \omega_0 \cdot \nu(x) < 2\varepsilon\Big\} \times (0,T) \subset \mathcal{V}_-,$$
$$\Big\{x \in \Gamma \,\Big|\, \omega_0 \cdot \nu(x) > -2\varepsilon\Big\} \times (0,T) \subset \mathcal{V}_+.$$

Therefore, by choosing

$$\omega \in \Big\{\omega \in \mathbb{S}^{n-1} \,\Big|\, |\omega - \omega_0| < \varepsilon\Big\},$$

we get that

$$\begin{aligned} \operatorname{supp} v_1^k\big|_\Sigma &\subset \Big\{x \in \Gamma \,\Big|\, \omega \cdot \nu(x) \geq -\varepsilon\Big\} \times (0,T) \\ &\subset \Big\{x \in \Gamma \,\Big|\, \omega_0 \cdot \nu(x) > -2\varepsilon\Big\} \times (0,T) \subset \mathcal{V}_+, \end{aligned}$$

and

$$\Big\{x \in \Gamma \,\Big|\, \omega_0 \cdot \nu(x) \geq 2\varepsilon\Big\} \subset \Gamma_{+,\omega,\varepsilon}.$$

[1]We also utilize the same parameter ε to construct the solution v_1.

Note that $v_1^k\Big|_\Sigma = v_2^k\Big|_\Sigma \in C_0^{2+\alpha,1+\alpha/2}(\mathcal{V}_+)$ and recall $\hat{v}^k = 0$ on $\Gamma_{+,\omega,\varepsilon}$. Then we have

$$\begin{aligned}
&\left|\int_\Sigma \hat{v}^k \partial_\nu (v_1^k - v_2^k)\, dSdt\right| \\
=&\left|\int_{\{\omega_0\cdot\nu\geq 2\varepsilon\}} \hat{v}^k \partial_\nu (v_1^k - v_2^k)\, dSdt\right| + \left|\int_{\{0<\omega_0\cdot\nu<2\varepsilon\}} \hat{v}^k \partial_\nu (v_1^k - v_2^k)\, dSdt\right| \\
&+\left|\int_{\{\omega_0\cdot\nu\leq 0\}} \hat{v}^k \partial_\nu (v_1^k - v_2^k)\, dSdt\right| \\
=&0
\end{aligned}$$

Then

$$\int_{Q'} (q_2 - q_1)\, v_1^k \hat{v}^k\ dxdt + \int_{Q\setminus Q'} (q_2 - q_1)\, v_1^k \hat{v}^k\ dxdt = 0.$$

Since we assume $q_1 = q_2$ in $Q \setminus Q'$, it follows that

$$\int_{Q'} (q_2 - q_1)\, v_1^k \hat{v}^k\, dxdt = 0.$$

Therefore, by similar arguments as in Corollary 4.5, letting $\rho \to \infty$, one has that

$$\int_{Q'} (q_2 - q_1) e^{-\mathrm{i}(x,t)\cdot(\xi,\tau)}\, dxdt = 0,$$

where $\mathrm{i} = \sqrt{-1}$. Since $\omega \in \left\{\omega \in \mathbb{S}^{n-1} \,\Big|\, |\omega - \omega_0| < \varepsilon\right\}$, it can be changed in a small conic neighborhood. By by using similar arguments as in Corollary 4.5, we have

$$q_1 = q_2 \text{ in } Q$$

as desired. □

Remark 4.7. *For the full data case of Theorem 1.3, we can use Lemma 4.1 to get an approximation of CGO solutions instead of Lemma 4.2, since the boundary inputs f needs to belong to the Hölder space. However, our CGO solutions only in the space $L^2(0,T;H^1(\Omega))\cap H^1(0,T;H^{-1}(\Omega))$, so we need to utilize the approximation property to connect these two different solution spaces. In other words, Lemma 4.1 is necessary even for the full data case in this paper. We do not need to assume $q_1 = q_2$ in $\Omega' \times (0,T)$, and we also point out that we cannot apply Corollary 4.5 to get the result for full data because we need to control the trace of solution on Σ.*

4.2. **Proof of Theorem 1.3.** With Lemma 4.6 at hand, combining with the higher order linearization method, we are able to prove Theorem 1.3.

Proof of Theorem 1.3. Let us first remark that the proofs of (a) and (b) in Theorem 1.3 are similar, so it suffices to show the global uniqueness result with partial data. The whole proof is divided into five parts.

Step 1. Initiation

Let us introduce the following boundary value

$$f(x,t;\epsilon) = \sum_{\ell=1}^{M} \epsilon_\ell f_\ell \quad \text{on } \Sigma, \tag{4.23}$$

where $M \in \mathbb{N}$, $f_1, \cdots, f_M \in C_0^{2+\alpha,1+\alpha/2}(\mathcal{V}_+)$ and $\epsilon = (\epsilon_1, \ldots, \epsilon_M)$ is a parameter vector in $\mathbb{R}^M$ with $|\epsilon| = \sum_{\ell=1}^{M} |\epsilon_\ell|$ small enough, such that $\left\| \sum_{\ell=1}^{M} \epsilon_\ell f_\ell \right\|_{C_0^{2+\alpha,1+\alpha/2}(\overline{\Sigma})}$ is sufficiently small. For $j = 1, 2$, by the local well-posedenss property in Section 2, there exist unique solutions $u_j = u_j(x, t; \epsilon) \in C^{2+\alpha,1+\alpha/2}(\overline{Q})$ to

$$\begin{cases} u_{j,t} - \Delta u_j + b_j(x, t, u_j) = 0 & \text{in } Q, \\ u_j = \sum_{\ell=1}^{M} \epsilon_\ell f_\ell & \text{on } \Sigma, \\ u_j(x, 0) = g_j(x) & \text{in } \Omega, \end{cases} \tag{4.24}$$

where $g_j \in C_0^{2+\alpha}(\Omega)$ with $\|g_j\|_{C^{2+\alpha}(\Omega)} < \frac{\delta}{2}$ being sufficiently small, and $b_j(x, t, z)$ are admissible coefficients defined in Section 1. For the sake of convenience, when $\epsilon = 0$, let $\widetilde{u}_j = u_j(\cdot, \cdot; 0)$ be the solutions to

$$\begin{cases} \widetilde{u}_{j,t} - \Delta \widetilde{u}_j + b_j(x, t, \widetilde{u}_j) = 0 & \text{in } Q, \\ \widetilde{u}_j = 0 & \text{on } \Sigma, \\ \widetilde{u}_j(x, 0) = g_j, & \text{in } \Omega. \end{cases} \tag{4.25}$$

By utilizing the higher order linearization to (4.24) around the solution $\widetilde{u}_j$ to (4.25), we will determine information on b_j for $j = 1, 2$.

Step 2. The first order linearization $(M = 1)$

One can linearize the equation (4.24) around $\widetilde{u}_j$, where $\widetilde{u}_j$ is the solution to (4.25), for $j = 1, 2$. Due to Remark 2.2, direct computations demonstrate that for $j = 1, 2$ and $\ell = M = 1$[2],

$$v_j^{(\ell)}(x, t) = \lim_{\epsilon \to 0} \frac{u_j(x, t) - \widetilde{u}_j(x, t)}{\epsilon_\ell}$$

satisfies the following parabolic equation:

$$\begin{cases} v_{j,t}^{(\ell)} - \Delta v_j^{(\ell)} + q_j v_j^{(\ell)} = 0 & \text{in } Q, \\ v_j^{(\ell)} = f_\ell & \text{on } \Sigma, \\ v_j^{(\ell)}(x, 0) = 0 & \text{in } \Omega, \end{cases} \tag{4.26}$$

where

$$q_j(x, t) := b_{j,u}(x, t, \widetilde{u}_j(x, t)) \text{ in } Q \quad \text{and} \quad q_j \in C^{2+\alpha,1+\frac{\alpha}{2}}(\overline{Q}).$$

We need to point out that both $\widetilde{u}_j$ and $v_j^{(\ell)}$ in (4.25) and (4.26) are still unknown, respectively, since they solve parabolic equations with unknown coefficients and initial data. In this step, we will show that

$$q_1(x, t) = q_2(x, t) \text{ in } Q. \tag{4.27}$$

With the same partial DN maps at hand

$$\Lambda^{\mathrm{P}}_{b_1,g_1}(f) = \Lambda^{\mathrm{P}}_{b_2,g_2}(f), \quad \text{for any sufficiently small } f \in C_0^{2+\alpha,1+\frac{\alpha}{2}}(\mathcal{V}_+),$$

such that we have

$$v_1^{(\ell)}(x, 0) = v_2^{(\ell)}(x, 0), \quad v_1^{(\ell)}\Big|_\Sigma = v_2^{(\ell)}\Big|_\Sigma, \quad \partial_\nu v_1^{(\ell)}\Big|_{\mathcal{V}_-} = \partial_\nu v_2^{(\ell)}\Big|_{\mathcal{V}_-}, \tag{4.28}$$

for $\ell = M = 1$.

[2]In fact, the arguments hold for all $\ell = 1, \ldots, M$, and we will use in steps 2-5.

Now, subtracting (4.26) with $j=1,2$, we have

$$\begin{cases} v_t^{(\ell)} - \Delta v^{(\ell)} + q_2 v^{(\ell)} = (q_2 - q_1) v_1^{(\ell)} & \text{in } Q, \\ v^{(\ell)} = 0 & \text{on } \Sigma, \\ v^{(\ell)}(x,0) = 0 & \text{in } \Omega, \end{cases} \tag{4.29}$$

where $v^{(\ell)} := v_1^{(\ell)} - v_2^{(\ell)}$. Let $\tilde{v}_2^{(\ell)}$ be a solution to the following backward parabolic equation:

$$\begin{cases} \tilde{v}_{2,t}^{(\ell)} + \Delta \tilde{v}_2^{(\ell)} - q_2 \tilde{v}_2^{(\ell)} = 0 & \text{in } Q, \\ \tilde{v}_2(x,T) = 0 & \text{in } \Omega. \end{cases} \tag{4.30}$$

Multiplying both sides of the first equation in (4.29) by $\tilde{v}_2^{(\ell)}$, by (4.28), an integration by parts yields that

$$\int_Q (q_2 - q_1)\, v_1^{(\ell)} \tilde{v}_2^{(\ell)} \, dxdt = \int_\Sigma \tilde{v}_2^{(\ell)} \partial_\nu v_1^{(\ell)} \, dSdt. \tag{4.31}$$

Moreover, with the condition $b_1 = b_2$ in $\Omega' \times (0,T) \times \mathbb{R}$ at hand, by applying Lemma 4.6, one can easily see that the claim (4.27) holds. Furthermore, as $q_1 = q_2$ in Q, $v_1^{(\ell)}$ and $v_2^{(\ell)}$ satisfy the same parabolic equation (4.26), by the uniqueness of solutions, we obtain that

$$v^{(\ell)} := v_1^{(\ell)} = v_2^{(\ell)} \text{ in } Q. \tag{4.32}$$

Step 3. The second order linearization ($M=2$)

For the second linearization ($m=2$), one can differentiate (4.24) with respect to different parameters ϵ_1 and ϵ_2. A direct computation shows that $w_j^{(2)}$ $(j=1,2)$ satisfy

$$\begin{cases} w_{j,t}^{(2)} - \Delta w_j^{(2)} + q w_j^{(2)} + b_{j,uu}(x,t,\widetilde{u}_j) v^{(1)} v^{(2)} = 0 & \text{in } Q, \\ w_j^{(2)} = 0 & \text{on } \Sigma, \\ w_j^{(2)}(x,0) = 0 & \text{in } \Omega, \end{cases} \tag{4.33}$$

where $q = q_1 = q_2$, $b_{j,uu}(\cdot,\cdot,\tilde{u}_j) \in C^{2+\alpha,1+\frac{\alpha}{2}}(\overline{Q})$ and $v^{(1)}, v^{(2)} \in C^{2+\alpha,1+\frac{\alpha}{2}}(\overline{Q})$ satisfy

$$\begin{cases} v_t^{(\ell)} - \Delta v^{(\ell)} + q(x,t) v^{(\ell)} = 0 & \text{in } Q, \\ v^{(\ell)} = f_\ell & \text{on } \Sigma, \\ v^{(\ell)}(x,0) = 0 & \text{in } \Omega, \end{cases}$$

here f_1 and f_2 can be arbitrarily chosen.

Next, we will prove that

$$b_{1,uu}(x,t,\widetilde{u}_1(x,t)) = b_{2,uu}(x,t,\widetilde{u}_2(x,t)) \quad \textit{in} \quad Q. \tag{4.34}$$

With the same DN map at hand, by differentiating ϵ_1 and ϵ_2, we have

$$w_1^{(2)}(x,0) = w_2^{(2)}(x,0), \quad w_1^{(2)}\Big|_\Sigma = w_2^{(2)}\Big|_\Sigma, \quad \partial_\nu w_1^{(2)}\Big|_{\mathcal{V}_-} = \partial_\nu w_2^{(2)}\Big|_{\mathcal{V}_-}. \tag{4.35}$$

Let $v^{(0)}$ be any solution to the backward parabolic equation:

$$\begin{cases} v_t^{(0)} + \Delta v^{(0)} - q v^{(0)} = 0 & \text{in } Q, \\ v^{(0)}(x,T) = 0 & \text{in } \Omega. \end{cases} \tag{4.36}$$

By subtracting the equations (4.33) associated to $j=1,2$, an integration by parts yields

$$\int_Q \Big[b_{1,uu}(x,t,\widetilde{u}_1(x,t)) - b_{2,uu}(x,t,\widetilde{u}_2(x,t)) \Big] v^{(0)} v^{(1)} v^{(2)} \, dxdt = 0 \tag{4.37}$$

We next choose a nonzero boundary data f_2 such that $f_2 \geq 0$ on Σ and $f_2 > 0$ on $D_t \times (0,T)$, where $D_t \subset \Gamma$ is a relative open subset for any $t \in (0,T)$. Via the condition $f_2 = v^{(2)}|_\Sigma \in L^\infty(\Sigma)$ at any time $t \in (0,T)$, by applying the maximum principle for parabolic equation (for example, see [Eva10, Chapter 7] or Appendix A), we have a bounded positive solution $v^{(2)}$ in Q. Now, by selecting $v^{(1)}$ and $v^{(0)}$ as the CGO solutions of forward and backward parabolic equations, via Corollary 4.5, we get

$$\big[b_{1,uu}(x,t,\widetilde{u}_1(x,t)) - b_{2,uu}(x,t,\widetilde{u}_2(x,t))\big]v^{(2)} = 0 \text{ in } Q.$$

With the positivity of $v^{(2)}$ in Q at hand, we have (4.34) as desired. Furthermore, by the uniqueness of solutions to (4.33), one can immediately obtain

$$w_1^{(2)} = w_2^{(2)} \quad \text{ in } Q.$$

Step 4. The higher order linearization $(M > 2)$

By utilizing the higher order linearization with the induction hypothesis, we are able to find M-th order derivative of (4.24) and prove that

$$\partial_u^M b_1(x,t,\widetilde{u}_1(x,t)) = \partial_u^M b_2(x,t,\widetilde{u}_2(x,t)) \quad \text{ in } Q, \tag{4.38}$$

for any $M = 3, 4, \cdots$. Let us first assume that

$$\partial_u^k b_1(x,t,\widetilde{u}_1(x,t)) = \partial_u^k b_2(x,t,\widetilde{u}_2(x,t)) \text{ in } Q, \quad \text{for any } k = 1, \ldots, M-1.$$

Similar to previous steps, we differentiate (4.24) with respect to $\epsilon_1, \ldots, \epsilon_{M-1}$ and ϵ_M, then we have

$$\int_Q \Big[\partial_u^M b_1(x,t,\widetilde{u}_1(x,t)) - \partial_u^M b_2(x,t,\widetilde{u}_2(x,t))\Big] v^{(0)}v^{(1)}\cdots v^{(M)}\,dxdt = 0,$$

where $v^{(0)}$ is the solution to the backward parabolic equation (4.36), and $v^{(\ell)}$ $(\ell = 1, 2, \cdots, M)$ are solutions to the forward parabolic equation (4.26). Similar to Step 3, let us choose $v^{(0)}$ and $v^{(1)}$ as CGO solutions, and $v^{(2)}, \ldots, v^{(M)}$ are bounded positive solutions in Q

$$\partial_u^M b_1(x,t,\widetilde{u}_1(x,t)) = \partial_u^M b_2(x,t,\widetilde{u}_2(x,t)) \text{ in } Q, \quad \text{for any } M \in \mathbb{N}. \tag{4.39}$$

Step 5. The determination of initial data and coefficients

Recall that $\widetilde{u}_j$ $(j = 1, 2)$ are the solutions to the semilinear parabolic equation:

$$\begin{cases} \widetilde{u}_{j,t} - \Delta \widetilde{u}_j + b_j(x,t,\widetilde{u}_j) = 0 & \text{in } \ Q, \\ \widetilde{u}_j = 0 & \text{on } \ \Sigma, \\ \widetilde{u}_j(x,0) = g_j, & \text{in } \ \Omega. \end{cases}$$

As in the proof of [LLL21, Theorem 1.3], by the admissible property of b_1 and b_2,

$$\begin{aligned} & b_1(x,t,\widetilde{u}_1(x,t)) - b_2(x,t,\widetilde{u}_2(x,t)) \\ = \ & \sum_{k=1}^{\infty} \frac{\partial_u^k b_2(x,t,\widetilde{u}_2(x,t))}{k!}\Big[-\widetilde{u}_2(x,t)\Big]^k - \sum_{k=1}^{\infty} \frac{\partial_u^k b_1(x,t,\widetilde{u}_1(x,t))}{k!}\Big[-\widetilde{u}_1(x,t)\Big]^k \\ = \ & \sum_{k=1}^{\infty} \frac{\partial_u^k b_1(x,t,\widetilde{u}_1(x,t))(-1)^k}{k!}\Big\{\Big[\widetilde{u}_2(x,t)\Big]^k - \Big[\widetilde{u}_1(x,t)\Big]^k\Big\}. \end{aligned} \tag{4.40}$$

Since both $\widetilde{u}_1$ and $\widetilde{u}_2$ are bounded, set $R = \|\widetilde{u}_1\|_{L^\infty(Q)} + \|\widetilde{u}_2\|_{L^\infty(Q)}$. Then, for any $L > 0$ and $(x,t) \in Q$,

$$\begin{aligned}
&\left|\frac{b_1(x,t,\widetilde{u}_1(x,t)) - b_2(x,t,\widetilde{u}_2(x,t))}{\widetilde{u}_1(x,t) - \widetilde{u}_2(x,t)}\right| \\
=&\left|\sum_{k=1}^{\infty} \frac{\partial_u^k b_1(x,t,\widetilde{u}_1(x,t))}{k!}(-1)^{k+1}\left\{\left[\widetilde{u}_1(x,t)\right]^{k-1} + \left[\widetilde{u}_1(x,t)\right]^{k-2}\widetilde{u}_2(x,t) + \cdots\right.\right. \\
&\left.\left. + \widetilde{u}_1(x,t)\left[\widetilde{u}_2(x,t)\right]^{k-2} + \left[\widetilde{u}_2(x,t)\right]^{k-1}\right\}\right| \\
\le& \sum_{k=1}^{\infty}\left|\partial_u^k b_1(x,t,\widetilde{u}_1(x,t))\right| \frac{R^{k-1}}{(k-1)!} \\
\le& \sum_{k=1}^{\infty} \frac{kR^{k-1}}{L^k} \sup_{|z-\widetilde{u}_1(x,t)|=L} |b_1(x,t,z)|.
\end{aligned}$$

Choose $L = 2(R+1)$. By the admissibility of b_1 and b_2,

$$G(\cdot,\cdot) = \frac{b_1(\cdot,\cdot,\widetilde{u}_1(\cdot,\cdot)) - b_2(\cdot,\cdot,\widetilde{u}_2(\cdot,\cdot))}{\widetilde{u}_1(\cdot,\cdot) - \widetilde{u}_2(\cdot,\cdot)} \in L^\infty(Q).$$

Set $w = \widetilde{u}_1 - \widetilde{u}_2$. It is easy to see that

$$\begin{cases} w_t - \Delta w + Gw = 0 & \text{in } Q, \\ w = 0 & \text{on } \Sigma, \\ w(x,0) = g_1 - g_2 & \text{in } \Omega. \end{cases}$$

By $\Lambda_{b_1,g_1}(0) = \Lambda_{b_2,g_2}(0)$ and Lemma 3.3, we have

$$g_1 = g_2 \text{ in } \Omega \quad \text{and} \quad \widetilde{u}_1 = \widetilde{u}_2 \text{ in } Q.$$

By (4.40),

$$b_1(x,t,\widetilde{u}_1(x,t)) = b_2(x,t,\widetilde{u}_2(x,t)) \quad \text{in } Q.$$

In addition, note that for $j = 1,2$ and any $(x,t,z) \in Q \times \mathbb{R}$,

$$b_j(x,t,z) = b_j(x,t,\widetilde{u}_j(x,t)) + \sum_{k=1}^{\infty} \frac{\partial_u^k b_j(x,t,\widetilde{u}_j(x,t))}{k!}\left(z - \widetilde{u}_j(x,t)\right)^k,$$

which implies that $b_1(x,t,z) = b_2(x,t,z)$ in $Q \times \mathbb{R}$. This proves the assertion. □

4.3. **Proof of Theorem 1.4.** Similar to the proof of Theorem 1.3, we are ready to prove Theorem 1.4.

Proof of Theorem 1.4. The argument is similar to the proof of Theorem 1.3, and we prove this result with the full data. Let us divide the proof into two steps.

Step 1. Unique determination of coefficients

Let $u_j = u_j(x,t)$ be the solution to

$$\begin{cases} u_{j,t} - \Delta u_j + q_j u_j = 0 & \text{in } Q, \\ u_j = f & \text{on } \Sigma, \\ u_j(x,0) = g_j(x) & \text{in } \Omega, \end{cases}$$

and let $\widetilde{u}_j = \widetilde{u}_j(x,t)$ be the solution to

$$\begin{cases} \widetilde{u}_{j,t} - \Delta \widetilde{u}_j + q_j \widetilde{u}_j = 0 & \text{in } Q, \\ \widetilde{u}_j = 0 & \text{on } \Sigma, \\ \widetilde{u}_j(x,0) = g_j(x) & \text{in } \Omega, \end{cases} \tag{4.41}$$

for $j = 1, 2$. With the same DN maps on the lateral boundary at hand, we have

$$\partial_\nu u_1 = \partial_\nu u_2 \quad \text{and} \quad \partial_\nu \widetilde{u}_1 = \partial_\nu \widetilde{u}_2 \quad \text{on} \quad \Sigma. \tag{4.42}$$

We next consider $v_j := u_j - \widetilde{u}_j$ for $j = 1, 2$, then v_j is the solution of

$$\begin{cases} v_{j,t} - \Delta v_j + q_j v_j = 0 & \text{in } Q, \\ v_j = f & \text{on } \Sigma, \\ v_j(x,0) = 0 & \text{in } \Omega. \end{cases} \tag{4.43}$$

Subtracting (4.43) with respect to $j = 1, 2$, we get

$$\begin{cases} v_t - \Delta v + q_2 v = (q_2 - q_1) v_1 & \text{in } Q, \\ v = \partial_\nu v = 0 & \text{on } \Sigma, \\ v(x,0) = 0 & \text{in } \Omega, \end{cases} \tag{4.44}$$

where $v = v_1 - v_2$. Moreover, via the condition (4.42), we have $\partial_\nu v = 0$ on Σ. On the other hand, let $\widetilde{v}_2$ be a solution to the backward parabolic equation

$$\begin{cases} \widetilde{v}_{2,t} + \Delta \widetilde{v}_2 + q_2 \widetilde{v}_2 = 0 & \text{in } Q, \\ \widetilde{v}_2(x,T) = 0 & \text{in } \Omega. \end{cases}$$

Multiplying (4.44) by the function $\widetilde{v}_2$, an integration by parts yields that

$$\int_Q (q_2 - q_1) v_1 \widetilde{v}_2 \, dxdt = 0. \tag{4.45}$$

By applying the global uniqueness result with full data (Corollary 4.5), then we have $q_1 = q_2$ as desired.

Step 2. Unique determination of initial data

Recalling that $\widetilde{u}_j$ is the solution of (4.41), by using the uniqueness $q_1 = q_2$, we can subtract (4.41) with respect to $j = 1, 2$, then we obtain

$$\begin{cases} u_t - \Delta u + qu = 0 & \text{in } Q, \\ u = 0 & \text{on } \Sigma, \\ u(x,0) = g_1 - g_2 & \text{in } \Omega, \end{cases} \tag{4.46}$$

where $q = q_1 - q_2$ and $u = \widetilde{u}_1 - \widetilde{u}_2$. Via the condition (4.42) again, we have $\partial_\nu u = 0$ on Σ. Finally, by applying the quantitative stability estimate (1.6), we can obtain the uniqueness of the initial data $g_1 = g_2$ in Ω. This proves the assertion. □

Remark 4.8. *One can find that when the initial and boundary data are small enough, Theorem 1.4 can be regarded as a corollary of Theorem 1.3, where we can simply take $b_j(x,t,u) := q_j(x,t)u$ for $j = 1, 2$. In order to distinguish the statements of Theorems 1.3 and 1.4, we provide two complete proofs of both theorems.*

Appendix A. Auxiliary results

In the end of this paper, for the sake of self-containedness, we review some properties for linear hear equations, which were used in our proofs.

A.1. **Complex geometrical optics solutions.** We first prove a density result for the product of solutions to forward and backward parabolic equations in $L^1(Q)$. It depends on the construction of CGO solutions, which vanish at initial or final time. They were constructed in [CK18a], and we summarize the results as the following propositions. To make the explanation clear, we split the procedure into two parts.

For any $\rho > 0$, we define

$$\begin{cases} \psi_{+,\rho}(x,t) = \exp(-(\rho\omega \cdot x + \rho^2 t)), \\ \psi_{-,\rho}(x,t) = \exp(\rho\omega \cdot x + \rho^2 t), \end{cases}$$

and

$$\begin{cases} \theta_{+,\rho}(x,t;\xi,\tau) = \left(1 - \exp(-\rho^{3/4} t)\right) \exp(-\mathrm{i}(x,t)\cdot(\xi,\tau)), \\ \theta_{-,\rho}(x,t) = 1 - \exp(-\rho^{3/4}(T-t)), \end{cases}$$

where $\xi \in \mathbb{R}^n$ with $\xi \cdot \omega = 0$ and $\tau \in \mathbb{R}$.

The following proposition was demonstrated in [CK18a, Propositions 4.3, 4.4], and we state the result without proofs for the sake of convenience.

Proposition A.1. *Let $m, \varepsilon > 0$ and $\omega \in \mathbb{S}^{n-1}$. There is a positive constant C, depending only on Q, m and ε, such that for any $q \in \left\{ q \in L^\infty(Q) \;\middle|\; \|q\|_{L^\infty(Q)} < m \right\}$. Then we have*

(a) *There exists a CGO solution $u \in L^2(0,T;H^1(\Omega)) \cap H^1(0,T;H^{-1}(\Omega))$ to the forward parabolic equation*

$$\begin{cases} (\partial_t - \Delta + q)u = 0 & \text{in } Q, \\ u(x,0) = 0 & \text{in } \Omega, \end{cases} \tag{A.1}$$

of the form

$$u(\cdot,\cdot;\rho,\xi,\tau) = \psi_{-,\rho}(\theta_{+,\rho} + z_{+,\rho,q}),$$

where $z_{+,\rho,q} \in L^2(0,T;H^1(\Omega)) \cap H^1(0,T;H^{-1}(\Omega))$

$$\lim_{\rho\to\infty} \|z_{+,\rho,q}\|_{L^2(Q)} = 0$$

(b) *There exists a CGO solution $u \in L^2(0,T;H^1(\Omega)) \cap H^1(0,T;H^{-1}(\Omega))$ to the backward parabolic equation*

$$\begin{cases} (-\partial_t - \Delta + q)u = 0 & \text{in } Q, \\ u = 0 & \text{on } \Gamma_{+,\omega,\varepsilon} \times (0,T), \\ u(x,T) = 0 & \text{in } \Omega, \end{cases} \tag{A.2}$$

of the form

$$u(\cdot,\cdot;\rho) = \psi_{+,\rho}(\theta_{-,\rho} + z_{-,\rho,q}),$$

where $z_{-,\rho,q} \in L^2(0,T;H^1(\Omega)) \cap H^1(0,T;H^{-1}(\Omega))$ and it satisfies the decay condition:

$$\lim_{\rho\to\infty} \|z_{-,\rho,q}\|_{L^2(Q)} = 0$$

A.2. **Maximum principle.** Finally, let us show the maximum principle for a linear parabolic equation.

Lemma A.2 (Strong maximum principle)**.** *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with smooth boundary Γ for $n \in \mathbb{N}$. Let $q \in C(\overline{Q})$ and $v \in C^{2,1}(Q) \cap C(\overline{Q})$ be a solution to*

$$\begin{cases} v_t - \Delta v + qv = 0 & \text{in } Q, \\ v = f & \text{on } \Sigma, \\ v(x,0) = 0 & \text{in } \Omega. \end{cases} \tag{A.3}$$

Suppose that $f \geq 0$ *on* Σ *and* $f > 0$ *on* $D_t \times (0, T)$ *with* $D_t \subset \Gamma$ *being a relative open subset for any* $t \in (0, T)$*, then* $v > 0$ *in* Q*.*

Proof. Without loss of generality, we assume that $q \geq 0$ in Q. Otherwise, let $u = e^{-\lambda t} v$, where $\lambda > 0$ is a sufficiently large positive parameter. If there exists a pair $(x_0, t_0) \in Q$, such that $v(x_0, t_0) = 0$. Then by [Eva10, Chapter 7], $v \equiv 0$ in $\Omega \times (0, t_0)$. It contradicts with the fact that $f > 0$ on $D_t \times (0, t_0)$. Hence, $v > 0$ in Q.

□

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