

GAUGE RIGIDITY IN AN INVERSE PROBLEM FOR THE PRESCRIBED GAUSSIAN CURVATURE EQUATION

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ABSTRACT. We prove a uniqueness result for an inverse boundary problem associated with the prescribed Gaussian curvature equation

$$\det D^2 u = K(x)(1 + |\nabla u|^2)^2$$

for graphs over a planar domain. The comparison is made on a common open class of smooth boundary values for which both admissible Dirichlet problems are well posed. We show that, if the corresponding nonlinear Dirichlet-to-Neumann maps agree on this class and the two prescribed curvatures have the same first boundary jet, then the curvatures agree in the whole domain.

The main difficulty comes from a gauge obstruction already present at the first linearization. In logarithmic form, the linearized equation is a two-dimensional non-divergence form elliptic equation with drift. Its boundary data determine the coefficients only up to a boundary-fixing change of variables and a scalar gauge factor. For the prescribed Gaussian curvature equation this gauge is not an artifact of the method: the gradient dependence leaves a residual gauge which cannot be removed by the first variation.

The proof uses the nonlinear structure to remove this remaining gauge. We derive a second-linearized interaction identity in the first-linearized gauge. In this identity, the covariant Hessian of the gauge map appears in the leading part. Testing the identity with two complementary CGO families gives a pair of residual equations for the gauge variables. Combined with the drift and conductivity identities from the first linearization, these equations form a closed system for the residual gauge. A boundary unique continuation argument for this system gives the trivial gauge and hence determines the prescribed curvature.

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1. INTRODUCTION

The prescribed Gaussian curvature equation for graphs is a natural fully nonlinear equation in geometric analysis. If u is locally strictly convex on a planar domain $\Omega \subset \mathbb{R}^2$, the graph $\{(x, u(x)) : x \in \Omega\} \subset \mathbb{R}^3$ has Gaussian curvature $\frac{\det D^2 u}{(1+|\nabla u|^2)^2}$. Prescribing this curvature gives

$$(1.1) \quad \begin{cases} \det D^2 u = K(x)(1 + |\nabla u|^2)^2 & \text{in } \Omega, \\ u = \varphi & \text{on } \partial\Omega. \end{cases}$$

The inverse problem asks whether the boundary response of this nonlinear Dirichlet problem determines the interior function K .

We first specify the forward class. The elliptic branch of (1.1) is tied to convexity, and one should not expect a solution map for arbitrary boundary values without an admissibility condition. We therefore compare the two curvatures only on boundary data for which both Dirichlet problems are well posed. We assume that there is a nonempty open set $\mathcal{U} \subset C^\infty(\partial\Omega)$ such that, for $j = 1, 2$ and $\varphi \in \mathcal{U}$,

$$\begin{cases} \det D^2 u^{(j)} = K_j(x)(1 + |\nabla u^{(j)}|^2)^2 & \text{in } \Omega, \\ u^{(j)} = \varphi & \text{on } \partial\Omega \end{cases}$$

has a unique smooth admissible solution $u_j(\varphi)$ with $D^2 u_j(\varphi) > 0$ on $\overline{\Omega}$. On this common class the nonlinear DN map is $\Lambda_{K_j}(\varphi) = \partial_\nu u_j(\varphi)|_{\partial\Omega}$.

The first linearization does not by itself fix the coordinates. After logarithmic normalization one obtains a planar non-divergence form elliptic equation with drift. The inverse boundary problem for this linearized equation leaves a boundary-fixing diffeomorphism J and a positive density ρ ; equivalently, $L_1(f \circ J) = \rho(L_2 f) \circ J$ for all smooth f . Thus the first variation puts the two linearized problems in the same gauge class, but not in the same gauge.

The divergence form of the same first linearized equation supplies the missing compatibility relation. In the original affine coordinates it gives $\rho = (\det DJ)(K_2 \circ J)/K_1$. After passing to a global isothermal coordinate for the first principal tensor, the conductivity weight is $\widehat{K}_j = (K_j \circ \chi^{-1})/m_\chi$. We write $q = \log(\widehat{K}_1/(\widehat{K}_2 \circ J))$ and $\theta = \log \det DJ - q$, so that $\rho = e^\theta$.

The second variation is then taken with this residual gauge still present. Pulling the second equation back by J gives an interaction identity in which the covariant Hessian of J appears explicitly.

A CGO testing argument gives a residual equation for $\mathbf{r} = 1 - \rho^{-1}$ and $M^\alpha = \rho^{-1}\mathcal{H}^\alpha$, where $\mathcal{H}^\alpha$ is the covariant Hessian of J . The mirror CGO family gives the paired residual equation. The Cauchy-transform terms coming from the critical CGO remainders are kept in the equations and are localized only in the unique continuation argument.

The last step combines the two residual equations with the drift, conductivity, and Jacobian identities from the first linearization. With $W = J - \text{Id}$, $s = q - 2\theta$, and $Z = (\theta, s, W)$, the principal part has the following form: θ satisfies a scalar Laplace equation, s is controlled by the two scalar blocks $\bar{\partial}^2 s$ and $\partial^2 s$, and W satisfies a componentwise drift equation. Boundary Carleman propagation gives $\theta = s = 0$ and $W = 0$. Hence $J = \text{Id}$, $q = 0$, and the two curvatures agree.

We now state the main theorem. Throughout the paper, $\Omega \subset \mathbb{R}^2$ is a bounded simply connected domain with C^∞ -smooth boundary.

Assumption 1.1. *Let $K_j \in C^\infty(\bar{\Omega})$, $j = 1, 2$, satisfy $K_j \geq c_0 > 0$ on $\bar{\Omega}$ for some constant $c_0 > 0$. We assume that there exists a nonempty open set $\mathcal{U} \subset C^\infty(\partial\Omega)$ such that, for each $j = 1, 2$ and each $\varphi \in \mathcal{U}$, the Dirichlet problem*

$$\begin{cases} \det D^2 u^{(j)} = K_j(x)(1 + |\nabla u^{(j)}|^2)^2 & \text{in } \Omega, \\ u^{(j)} = \varphi & \text{on } \partial\Omega \end{cases}$$

has a unique solution $u_j(\varphi) \in C^\infty(\bar{\Omega})$ with $D^2 u_j(\varphi) > 0$ in $\bar{\Omega}$.

Assumption 1.1 fixes the common domain on which the two nonlinear DN maps are defined. It is not used as an additional interior rigidity input. Under this assumption, for each $\varphi \in \mathcal{U}$ we define

$$\Lambda_{K_j}(\varphi) = \partial_\nu u_j(\varphi)|_{\partial\Omega}, \quad j = 1, 2.$$

Theorem 1.1. *Let $\Omega \subset \mathbb{R}^2$ be a bounded simply connected domain with C^∞ -smooth boundary. Let $K_j \in C^\infty(\bar{\Omega})$, $j = 1, 2$, satisfy Assumption 1.1, and let Λ_{K_j} be the nonlinear DN maps defined on the common open set $\mathcal{U}$. Suppose that the boundary derivatives of K_1 and K_2 agree up to order 1, in the sense that, in every boundary normal coordinate system,*

$$(1.2) \quad \partial_\tau^a \partial_\nu^b K_1 = \partial_\tau^a \partial_\nu^b K_2 \quad \text{on } \partial\Omega \quad \text{for all } a, b \geq 0, \quad a + b \leq 1.$$

Suppose also that

$$(1.3) \quad \Lambda_{K_1}(\varphi) = \Lambda_{K_2}(\varphi) \quad \text{for all } \varphi \in \mathcal{U}.$$

Then $K_1 = K_2$ in Ω .

The boundary condition (1.2) is used to fix the boundary representative of the gauge. After that normalization is fixed, the interior argument shows that the residual pair (J, ρ) left by the first linearization is forced to be $(\text{Id}, 1)$.

Earlier literature. Linearization is a standard way to bring nonlinear inverse problems back to linear boundary value problems; an early example is [Isa93]. In many equations the first variation is not enough, and one has to use higher-order terms in the boundary response. Second-order arguments occur, for instance, in [AZ21, CNV19, KN02, Sun96, Sun10, SU97]. Nonlinear interaction methods were developed in a systematic form for wave equations in [KLU18], and they now appear in a wide range of nonlinear inverse problems.

For elliptic equations, semilinear problems have been studied in [FO20, LLLS21, LLLS20, LLST22, KU20b, KU20a, FLL23], including partial data results. Quasilinear elliptic problems are treated in [KKU23, CFK⁺21, LW23, LJ26]. The minimal surface equation has been considered in [ABN20, CLLO24, CLLT23, CLT24, Nur24]. Further related results for nonlinear elliptic and fractional elliptic equations can be found in [LL19, LL22, LSX22, HL23, ST23, LL25b]; see also the survey [Las25].

The fully nonlinear case has some different features. The first variation is often a non-divergence form equation, and higher variations naturally involve Hessians of linearized solutions. In [LL25a] an inverse source problem for the Monge–Ampère equation was studied in the plane, and another fully nonlinear inverse source problem was considered in [LLW26]. A related Monge–Ampère inverse source problem is treated in [CG26]. The point of the present paper is different: the gradient dependence in the prescribed Gaussian curvature equation leaves a genuine residual gauge after the first linearization. The second variation is used to eliminate precisely this remaining gauge.

Organization of the article. Sections 2 and 3 contain the forward setup, boundary determination, and the first-linearized gauge. Section 4 derives the fixed-gauge second identity. Sections 5 and 6 extract the plus and mirror residual equations. Section 7 rewrites these equations as an augmented system for (θ, s, W) . Section 8 proves the boundary unique continuation argument and eliminates the gauge.

2. PRELIMINARIES

2.1. The solvability framework. We begin with the forward class used for the linearization argument. Throughout the paper, $\Omega \subset \mathbb{R}^2$ is bounded and simply connected with C^∞ boundary, and the pair K_1, K_2 satisfies Assumption 1.1.

For a fixed curvature K , the equation can be written as $\det D^2 u = \psi(x, u, \nabla u)$ with $\psi(x, z, p) = K(x)(1 + |p|^2)^2$. The function ψ is smooth, positive, and independent of z . On the common solvability class the solutions are smooth and admissible, and the Hessian remains positive up to the boundary. All linearizations below are taken along such nondegenerate branches.

Proposition 2.1 (Local smooth dependence). *Assume that the pair K_1, K_2 satisfies Assumption 1.1. Fix $j \in \{1, 2\}$, $\phi \in \mathcal{U}$, $m \geq 2$, and $\alpha \in (0, 1)$. Then the following assertions hold.*

- (i) *There exists a unique smooth admissible solution $u_j(\phi)$ of*

$$\begin{cases} \det D^2 u_j(\phi) = K_j(x)(1 + |\nabla u_j(\phi)|^2)^2 & \text{in } \Omega, \\ u_j(\phi) = \phi & \text{on } \partial\Omega. \end{cases}$$

Moreover, $u_j(\phi) \in C^\infty(\overline{\Omega})$ and $D^2 u_j(\phi) > 0$ on $\overline{\Omega}$.

- (ii) *There exists a neighborhood $\mathcal{U}_{m,\alpha}$ of ϕ in $C^{m,\alpha}(\partial\Omega)$ such that, for every $\varphi \in \mathcal{U} \cap \mathcal{U}_{m,\alpha}$, there is a unique admissible solution $u_j(\varphi) \in C^{m,\alpha}(\overline{\Omega})$ close to $u_j(\phi)$ satisfying*

$$\begin{cases} \det D^2 u_j(\varphi) = K_j(x)(1 + |\nabla u_j(\varphi)|^2)^2 & \text{in } \Omega, \\ u_j(\varphi) = \varphi & \text{on } \partial\Omega. \end{cases}$$

The map $\varphi \mapsto u_j(\varphi)$ is smooth in each $C^{m,\alpha}$ scale, and therefore smooth in the C^∞ topology.

Proof. The first assertion is the forward solvability assumed in Assumption 1.1. For the local dependence, use the implicit function theorem. Let

$$X := \{w \in C^{m,\alpha}(\overline{\Omega}) : w|_{\partial\Omega} = 0\},$$

and choose a bounded linear right inverse $E : C^{m,\alpha}(\partial\Omega) \rightarrow C^{m,\alpha}(\overline{\Omega})$ of the trace map. For φ close to ϕ , write

$$u = u_j(\phi) + E(\varphi - \phi) + w, \quad w \in X.$$

The equation is equivalent to $\mathcal{F}(w, \varphi) = 0$, where

$$\mathcal{F}(w, \varphi) = \det D^2 u - K_j(x)(1 + |\nabla u|^2)^2.$$

This defines a smooth map

$$\mathcal{F} : X \times C^{m,\alpha}(\partial\Omega) \rightarrow C^{m-2,\alpha}(\overline{\Omega})$$

near $(0, \phi)$.

The derivative in the w variable at $(0, \phi)$ is

$$D_w \mathcal{F}(0, \phi)v = \text{Cof}(D^2 u_j(\phi))^{ij} \partial_{ij} v - 4K_j(x)(1 + |\nabla u_j(\phi)|^2) \nabla u_j(\phi) \cdot \nabla v.$$

Since $D^2 u_j(\phi) > 0$ on $\overline{\Omega}$, the cofactor matrix is uniformly positive definite. Thus, this is a uniformly elliptic non-divergence form operator with no zeroth-order term. The maximum principle gives uniqueness for the homogeneous Dirichlet problem, and the Fredholm alternative, together with Schauder theory, gives an isomorphism

$$D_w \mathcal{F}(0, \phi) : X \rightarrow C^{m-2, \alpha}(\overline{\Omega}).$$

The implicit function theorem gives a unique local admissible branch $\varphi \mapsto u_j(\varphi)$ in $C^{m, \alpha}$. After shrinking the neighborhood, the Hessian remains positive definite. Since Assumption 1.1 gives uniqueness on the common class, this local branch agrees with the solution $u_j(\varphi)$ for $\varphi \in \mathcal{U} \cap \mathcal{U}_{m, \alpha}$. Bootstrapping in the Hölder scale gives smooth dependence in C^∞ . $\square$

2.2. Boundary determination. The next elementary boundary reconstruction fixes the boundary normalization used later in the first linearized inverse problem. The point is that, once $K|_{\partial\Omega}$ is known, the nonlinear DN map gives the boundary values of the logarithmically normalized first linearized coefficients.

Lemma 2.2 (Boundary determination). *Assume the hypotheses of Theorem 1.1. Then, for each $\varphi \in \mathcal{U}$, the nonlinear DN map together with $K_j|_{\partial\Omega}$ determines the boundary values of*

$$A_\varphi^{(j)} = (D^2 u_\varphi^{(j)})^{-1}, \quad B_\varphi^{(j)i} = -\frac{4\partial_i u_\varphi^{(j)}}{1 + |\nabla u_\varphi^{(j)}|^2}, \quad j = 1, 2.$$

In particular, if (1.2) and (1.3) hold, then $A_\varphi^{(1)} = A_\varphi^{(2)}$ on $\partial\Omega$.

Proof. Fix $j \in \{1, 2\}$, $\varphi \in \mathcal{U}$, and $p \in \partial\Omega$. Choose boundary normal coordinates (s, t) near p , where s is arclength on $\partial\Omega$ and t is the inward normal coordinate. We write τ for the unit tangent vector and ν_{in} for the inward unit normal.

The Dirichlet datum gives $u_\varphi^{(j)} = \varphi$ on the boundary, hence all tangential derivatives of $u_\varphi^{(j)}$ there are known. The nonlinear DN map gives the outer normal derivative, and therefore also $\partial_t u_\varphi^{(j)}|_{\partial\Omega}$ after fixing the sign convention. Taking tangential derivatives, we know $u_\varphi^{(j)}$, $\partial_s u_\varphi^{(j)}$, $\partial_t u_\varphi^{(j)}$, $\partial_{ss} u_\varphi^{(j)}$, and $\partial_{st} u_\varphi^{(j)}$ on $\partial\Omega$.

The only second-order boundary component still unknown is the normal-normal derivative. At p , write the Euclidean Hessian in the orthonormal frame (τ, ν_{in}) as

$$H_{\tau\tau} = D^2 u_\varphi^{(j)}(\tau, \tau), \quad H_{\tau\nu} = D^2 u_\varphi^{(j)}(\tau, \nu_{\text{in}}), \quad H_{\nu\nu} = D^2 u_\varphi^{(j)}(\nu_{\text{in}}, \nu_{\text{in}}).$$

The first two quantities are determined by the known first and tangential second boundary jets of $u_\varphi^{(j)}$, together with the known geometry of $\partial\Omega$. The only remaining unknown second-order component is $H_{\nu\nu} = \partial_{tt} u_\varphi^{(j)}$.

In the frame (τ, ν_{in}) , $\det D^2 u_\varphi^{(j)} = H_{\tau\tau} H_{\nu\nu} - H_{\tau\nu}^2$. Evaluating the equation on $\partial\Omega$ gives

$$(2.1) \quad H_{\tau\tau} \partial_{tt} u_\varphi^{(j)} - H_{\tau\nu}^2 = K_j(1 + |\nabla u_\varphi^{(j)}|^2)^2 \quad \text{on } \partial\Omega.$$

The right-hand side is known from $K_j|_{\partial\Omega}$ and the known first boundary jet of $u_\varphi^{(j)}$. Since the solution is admissible, $D^2 u_\varphi^{(j)} > 0$ on $\overline{\Omega}$, $H_{\tau\tau} > 0$ on $\partial\Omega$. Thus, (2.1) uniquely determines $\partial_{tt} u_\varphi^{(j)}$ on the boundary.

We have recovered the full second-order boundary jet of $u_\varphi^{(j)}$. Hence $A_\varphi^{(j)} = (D^2 u_\varphi^{(j)})^{-1}$ is determined on $\partial\Omega$. The coefficient $B_\varphi^{(j)}$ depends only on the first boundary jet, so it is also determined.

If (1.2) and (1.3) hold, then $K_1 = K_2$ on $\partial\Omega$ and the two nonlinear DN maps give the same first boundary jet for the two solutions with the same boundary value φ . The reconstruction above, therefore, gives the same second-order boundary Hessian for $u_\varphi^{(1)}$ and $u_\varphi^{(2)}$. Hence, $A_\varphi^{(1)} = A_\varphi^{(2)}$ on $\partial\Omega$. $\square$

Remark 2.3. *The full first-order matching in (1.2) is not used in Lemma 2.2. The lemma only uses $K_1 = K_2$ on $\partial\Omega$ to identify the boundary values of the principal coefficients. The first-order boundary matching is used later to obtain the boundary Cauchy data for the residual variables in Section 8.*

3. THE FIRST LINEARIZATION

We first linearize the equation in logarithmic form. This normalization removes a harmless positive factor from the principal coefficient and leaves the gradient dependence as a drift term.

To compare this with Proposition 2.1, differentiate the determinant form. The principal coefficient is $\text{Cof}(D^2 u_\varphi)$. Since $\text{Cof}(D^2 u_\varphi) = (\det D^2 u_\varphi)(D^2 u_\varphi)^{-1}$ and $\det D^2 u_\varphi > 0$ on the admissible branch, multiplication by $(\det D^2 u_\varphi)^{-1}$ gives the operator used below. This multiplication does not change the Dirichlet problem, the first variations, or the linearized boundary data.

For the moment let the curvature be a single function K . If u_φ is the admissible solution with boundary value φ , then the equation is equivalently

$$\log \det D^2 u_\varphi - 2 \log(1 + |\nabla u_\varphi|^2) - \log K = 0.$$

For matrices of the same size we write $A : B = \text{tr}(A^T B)$; for symmetric matrices this is $A : B = A^{ij} B_{ij}$. Repeated indices are summed over 1, 2.

Lemma 3.1 (Formula for the first linearized operator). *Let $\varphi \in \mathcal{U}$ and $h \in C^\infty(\partial\Omega)$. For $|t|$ sufficiently small, $\varphi + th \in \mathcal{U}$. Define $v_h = \frac{d}{dt}|_{t=0} u_{\varphi+th}$, then v_h satisfies*

$$\begin{cases} L_\varphi v_h = 0 & \text{in } \Omega, \\ v_h = h & \text{on } \partial\Omega, \end{cases}$$

where

$$(3.1) \quad L_\varphi = A_\varphi^{ij} \partial_{ij} + B_\varphi^i \partial_i,$$

with

$$(3.2) \quad A_\varphi = (D^2 u_\varphi)^{-1}, \quad B_\varphi^i = -\frac{4\partial_i u_\varphi}{1 + |\nabla u_\varphi|^2}.$$

Proof. Let $u_t := u_{\varphi+th}$. By Proposition 2.1, $v_h \in C^\infty(\overline{\Omega})$. Differentiating $u_t|_{\partial\Omega} = \varphi + th$ at $t = 0$ gives $v_h|_{\partial\Omega} = h$.

Differentiating the logarithmic equation at $t = 0$ gives

$$(D^2 u_\varphi)^{-1} : D^2 v_h - \frac{4\nabla u_\varphi}{1 + |\nabla u_\varphi|^2} \cdot \nabla v_h = 0 \quad \text{in } \Omega.$$

This is exactly $L_\varphi v_h = 0$ with the coefficients in (3.2). $\square$

We shall also use the following geometric form of the first linearized operator. We use the convention

$$\Delta_g v = g^{ij} \partial_{ij} v - g^{ij} \Gamma_{ij}^k(g) \partial_k v.$$

Lemma 3.2. *Let g_φ be the Riemannian metric whose inverse coefficients are $(g_\varphi)^{ij} = A_\varphi^{ij}$, then*

$$L_\varphi = \Delta_{g_\varphi} + Y_\varphi \cdot \nabla,$$

where

$$Y_\varphi = B_\varphi + X_{g_\varphi}, \quad X_{g_\varphi}^k = (g_\varphi)^{ij} \Gamma_{ij}^k(g_\varphi).$$

Proof. By the convention above,

$$\Delta_{g_\varphi} v = A_\varphi^{ij} \partial_{ij} v - A_\varphi^{ij} \Gamma_{ij}^k(g_\varphi) \partial_k v = A_\varphi^{ij} \partial_{ij} v - X_{g_\varphi} \cdot \nabla v.$$

Substituting this into (3.1) gives $L_\varphi v = \Delta_{g_\varphi} v + (B_\varphi + X_{g_\varphi}) \cdot \nabla v$. $\square$

We compare the first linearized operators corresponding to K_1 and K_2 . For $j = 1, 2$, let $u_\varphi^{(j)}$ be the solution with boundary value φ , and let $A_\varphi^{(j)}$, $B_\varphi^{(j)}$, $g_\varphi^{(j)}$, and $Y_\varphi^{(j)}$ denote the corresponding first linearized coefficients and geometric data.

For the linearized inverse problem, the boundary operator is the conormal trace associated with the principal tensor. Thus, if $v_h^{(j)}$ solves

$$\begin{cases} L_\varphi^{(j)} v_h^{(j)} = 0 & \text{in } \Omega, \\ v_h^{(j)} = h & \text{on } \partial\Omega, \end{cases}$$

we define

$$(3.3) \quad \Lambda'_j : C^\infty(\partial\Omega) \rightarrow C^\infty(\partial\Omega), \quad h \mapsto A_\varphi^{(j)} \nabla v_h^{(j)} \cdot \nu|_{\partial\Omega},$$

for $j = 1, 2$. The next lemma identifies this conormal trace with the linearized boundary data coming from the nonlinear Euclidean DN map.

Lemma 3.3 (Identification of the first linearized DN map). *Adopt the assumptions of Theorem 1.1. Fix $\varphi \in \mathcal{U}$. Then (1.2) and (1.3) imply*

$$\Lambda'_1 h = \Lambda'_2 h \quad \text{on } \partial\Omega$$

for all $h \in C^\infty(\partial\Omega)$.

Proof. By Lemma 2.2, the nonlinear DN map together with $K_1 = K_2$ on $\partial\Omega$ determines the boundary values of the logarithmically normalized principal coefficients. Hence,

$$(3.4) \quad A_\varphi^{(1)} = A_\varphi^{(2)} \quad \text{on } \partial\Omega.$$

The first-order boundary matching of the curvature is not needed for this conormal identification; it will be used later to fix the first boundary derivative of the residual curvature ratio.

Let $u_\varepsilon^{(j)}$ be the solution corresponding to K_j with boundary value $\varphi + \varepsilon h$. Since $\mathcal{U}$ is open, this path lies in $\mathcal{U}$ for all sufficiently small $|\varepsilon|$. The equality of the nonlinear DN maps gives $\partial_\nu u_\varepsilon^{(1)} = \partial_\nu u_\varepsilon^{(2)}$ on $\partial\Omega$. Differentiating at $\varepsilon = 0$ yields

$$(3.5) \quad \partial_\nu v_h^{(1)} = \partial_\nu v_h^{(2)} \quad \text{on } \partial\Omega.$$

Since both first variations have the same Dirichlet value h , their tangential derivatives agree on the boundary. Together with (3.5), this gives $\nabla v_h^{(1)} = \nabla v_h^{(2)}$ on $\partial\Omega$. Combining this gradient equality with (3.4) gives $A_\varphi^{(1)} \nabla v_h^{(1)} \cdot \nu = A_\varphi^{(2)} \nabla v_h^{(2)} \cdot \nu$ on $\partial\Omega$. This is exactly $\Lambda'_1 h = \Lambda'_2 h$. $\square$

We use the following planar result for non-divergence form equations. This is the only input from the first-linearized inverse problem; the later sections use the second variation to remove the gauge that appears here.

In Lemma 3.4, the DN map is the conormal map in (3.3). For $L_j = L_\varphi^{(j)}$, Lemma 3.3 gives equality of these conormal maps. Lemma 2.2 fixes the boundary values of the logarithmically normalized

principal tensors. With this boundary representative fixed, the tensor conormal trace $A_j \nabla v \cdot \nu$ and the metric normal trace used in [LL25a, Lemma 4.2] differ by the same known nonzero boundary factor for $j = 1, 2$. Indeed, if $g_j^{-1} = A_j$, then $A_j \nabla v \cdot \nu = (A_j \nu \cdot \nu)^{1/2} \partial_{\nu_{g_j}} v$ on $\partial\Omega$. The factor $(A_j \nu \cdot \nu)^{1/2}$ is known from the fixed boundary principal tensor and is the same for $j = 1, 2$. Thus the two DN conventions give the same gauge information.

Lemma 3.4 (Planar non-divergence gauge theorem). *Let $\Omega \subset \mathbb{R}^2$ be bounded and simply connected with C^∞ boundary. For $j = 1, 2$, let $L_j = A_j^{ab} \partial_{ab} + B_j^a \partial_a$ be a smooth, uniformly elliptic operator on $\bar{\Omega}$ with no zeroth-order term. Equivalently, write $L_j = \Delta_{g_j} + Y_j \cdot \nabla$, where $g_j^{ab} = A_j^{ab}$. Assume that the conormal DN maps of L_1 and L_2 agree, and that the boundary representative of the gauge class is fixed as above. Then there exists a smooth diffeomorphism $J : \bar{\Omega} \rightarrow \bar{\Omega}$ and a smooth positive function ρ on $\bar{\Omega}$, with $J|_{\partial\Omega} = \text{Id}$ and $\rho|_{\partial\Omega} = 1$, such that*

$$(3.6) \quad L_1(f \circ J) = \rho(L_2 f) \circ J \text{ for all } f \in C^\infty(\bar{\Omega}).$$

In particular,

$$(3.7) \quad DJ A_1 DJ^T = \rho(A_2 \circ J),$$

and, for $\alpha = 1, 2$,

$$(3.8) \quad A_1^{ij} \partial_{ij} J^\alpha + B_1^i \partial_i J^\alpha = \rho(B_2^\alpha \circ J).$$

Proof. Set

$$\mathcal{L}_j := -L_j = -\Delta_{g_j} - Y_j \cdot \nabla = -\Delta_{g_j} + X_j \cdot \nabla, \quad X_j := -Y_j.$$

Multiplying the equation by -1 does not change the Dirichlet solutions or the equality of the boundary measurements. Thus, with the DN convention fixed before the lemma, the equality of the conormal DN maps for L_1 and L_2 gives the corresponding equality of boundary data for $\mathcal{L}_1$ and $\mathcal{L}_2$ in the convention of [LL25a, Lemma 4.2].

Applying [LL25a, Lemma 4.2] to $\mathcal{L}_j = -\Delta_{g_j} + X_j \cdot \nabla$ gives a boundary-fixing diffeomorphism J and a positive function c , with $c|_{\partial\Omega} = 1$, such that

$$g_1 = cJ^*g_2, \quad X_1 = c^{-1}J^*X_2.$$

Since $X_j = -Y_j$, this is equivalent to

$$g_1 = cJ^*g_2, \quad Y_1 = c^{-1}J^*Y_2.$$

Define $\rho := c^{-1}$, then $\rho|_{\partial\Omega} = 1$. We first derive the operator identity. In dimension two, multiplying a metric by c multiplies the scalar Laplacian by c^{-1} . Hence, for any $f \in C^\infty(\bar{\Omega})$,

$$\Delta_{g_1}(f \circ J) = c^{-1} \Delta_{J^*g_2}(f \circ J) = c^{-1}(\Delta_{g_2} f) \circ J = \rho(\Delta_{g_2} f) \circ J.$$

Also, by the definition of the pullback of a vector field,

$$Y_1 \cdot \nabla(f \circ J) = c^{-1}(J^*Y_2) \cdot \nabla(f \circ J) = c^{-1}(Y_2 \cdot \nabla f) \circ J = \rho(Y_2 \cdot \nabla f) \circ J.$$

Adding the two identities gives (3.6).

The coefficient identities follow by comparing terms. The relation $g_1 = cJ^*g_2$ means, in coordinates,

$$(g_1)_{ij} = c \partial_i J^\alpha \partial_j J^\beta (g_2)_{\alpha\beta} \circ J.$$

Taking inverses gives $A_1 = c^{-1}(DJ)^{-1}(A_2 \circ J)(DJ)^{-T}$. Since $\rho = c^{-1}$, multiplying this identity by DJ on the left and by DJ^T on the right gives (3.7).

Finally, the chain rule gives

$$\partial_i(f \circ J) = (\partial_\alpha f) \circ J \partial_i J^\alpha, \quad \partial_{ij}(f \circ J) = ((\partial_{\alpha\beta} f) \circ J) \partial_i J^\alpha \partial_j J^\beta + ((\partial_\alpha f) \circ J) \partial_{ij} J^\alpha.$$

Substituting these into (3.6) and comparing the coefficients of $\partial_\alpha f$ gives (3.8). $\square$

A *boundary-fixing gauge pair* is a pair $(J_\varphi, \rho_\varphi)$ where $J_\varphi : \overline{\Omega} \rightarrow \overline{\Omega}$ is a diffeomorphism with $J_\varphi|_{\partial\Omega} = \text{Id}$, and $\rho_\varphi \in C^\infty(\overline{\Omega})$ is positive with $\rho_\varphi|_{\partial\Omega} = 1$. We write Id for the identity map and I for the 2×2 identity matrix.

The logarithmically normalized first linearized operator is in the class of Lemma 3.4. The drift is part of the operator. Lemma 3.3 gives the required equality of conormal data, while Lemma 2.2 fixes the boundary representative of the principal tensor. Hence Lemma 3.4 applies for each fixed $\varphi \in \mathcal{U}$.

Proposition 3.5 (Unique determination up to gauge). *Adopt the assumptions of Theorem 1.1. Then, for any $\varphi \in \mathcal{U}$, there exists a boundary-fixing gauge pair $(J_\varphi, \rho_\varphi)$ such that*

$$(3.9) \quad DJ_\varphi A_\varphi^{(1)} DJ_\varphi^T = \rho_\varphi (A_\varphi^{(2)} \circ J_\varphi).$$

Moreover, for $\alpha = 1, 2$,

$$(3.10) \quad A_\varphi^{(1)ij} \partial_{ij} J_\varphi^\alpha + B_\varphi^{(1)i} \partial_i J_\varphi^\alpha = \rho_\varphi (B_\varphi^{(2)\alpha} \circ J_\varphi).$$

Equivalently,

$$(3.11) \quad L_\varphi^{(1)}(f \circ J_\varphi) = \rho_\varphi (L_\varphi^{(2)} f) \circ J_\varphi \quad \text{for all } f \in C^\infty(\overline{\Omega}).$$

In addition,

$$(3.12) \quad J_\varphi|_{\partial\Omega} = \text{Id}, \quad \rho_\varphi|_{\partial\Omega} = 1, \quad A_\varphi^{(1)} = A_\varphi^{(2)} \quad \text{on } \partial\Omega.$$

Proof. By Lemma 3.3, the DN maps of the first linearized equations agree. By Lemma 3.2, each first linearized operator has the form $L_\varphi^{(j)} = \Delta_{g_\varphi^{(j)}} + Y_\varphi^{(j)} \cdot \nabla$. Applying Lemma 3.4 gives a boundary-fixing gauge pair $(J_\varphi, \rho_\varphi)$ satisfying (3.11). The coefficient identities (3.9) and (3.10) are exactly (3.7) and (3.8) applied to $L_\varphi^{(1)}$ and $L_\varphi^{(2)}$. Finally, $A_\varphi^{(1)} = A_\varphi^{(2)}$ on $\partial\Omega$ follows from Lemma 2.2. $\square$

Corollary 3.6 (Boundary first jet of the gauge). *Under the assumptions of Proposition 3.5, one has $DJ_\varphi = I$ on $\partial\Omega$.*

Proof. By (3.9) and (3.12),

$$DJ_\varphi A_\varphi^{(1)} DJ_\varphi^T = A_\varphi^{(1)} \quad \text{on } \partial\Omega.$$

Fix $p \in \partial\Omega$. Let τ be the unit tangent vector and ν the outward unit normal at p . Since $J_\varphi = \text{Id}$ on $\partial\Omega$, we have $DJ_\varphi(p)\tau = \tau$. Hence, in the ordered orthonormal basis (τ, ν) ,

$$DJ_\varphi(p) = \begin{pmatrix} 1 & \alpha \\ 0 & \beta \end{pmatrix}$$

for some $\alpha, \beta \in \mathbb{R}$. Since J_φ maps the interior side to the interior side and fixes the boundary pointwise, $\beta > 0$.

Write $A_\varphi^{(1)}(p) = \begin{pmatrix} a & b \\ b & c \end{pmatrix}$. Since $A_\varphi^{(1)}(p)$ is positive definite, $c > 0$. A direct computation gives

$$DJ_\varphi(p) A_\varphi^{(1)}(p) DJ_\varphi(p)^T = \begin{pmatrix} a + 2\alpha b + \alpha^2 c & \beta(b + \alpha c) \\ \beta(b + \alpha c) & \beta^2 c \end{pmatrix}.$$

Comparing with $A_\varphi^{(1)}(p)$, the $(2, 2)$ entry gives $\beta^2 c = c$. Since $c > 0$ and $\beta > 0$, we get $\beta = 1$. The $(1, 2)$ entry then gives $b + \alpha c = b$, hence $\alpha = 0$. Therefore $DJ_\varphi(p) = I$. Since p was arbitrary, $DJ_\varphi = I$ on $\partial\Omega$. $\square$

Remark 3.7. *The first-order coefficient does not transform by a pure pullback. The term $(A_\varphi^{(1)})^{ij} \partial_{ij} J_\varphi^\alpha$ in (3.10) is the chain-rule contribution from the principal part. In the second linearization, the same mechanism produces the Hessian terms involving $D^2 J_\varphi^\alpha$.*

3.1. The conductivity form and the curvature ratio gauge equations. Fix the background boundary value and suppress the subscript φ . Recall that

$$L_j v = A_j^{ab} \partial_{ab} v + B_j^a \partial_a v, \quad A_j = (D^2 u_j)^{-1}, \quad B_j^a = -\frac{4 \partial_a u_j}{1 + |\nabla u_j|^2}.$$

Let $\sigma_j = 1 + |\nabla u_j|^2$. Since $\text{Cof}(D^2 u_j)^{ab} = (\det D^2 u_j) A_j^{ab}$ and $\det D^2 u_j = K_j \sigma_j^2$, one has $K_j A_j^{ab} = \text{Cof}(D^2 u_j)^{ab} / \sigma_j^2$. The Piola identity gives $\partial_a \text{Cof}(D^2 u_j)^{ab} = 0$, and hence,

$$\partial_a (K_j A_j^{ab}) = -2 \sigma_j^{-3} (\partial_a \sigma_j) \text{Cof}(D^2 u_j)^{ab}.$$

Using $\partial_a \sigma_j = 2 \partial_m u_j \partial_{am} u_j$, we get

$$\partial_a (K_j A_j^{ab}) = -4 \sigma_j^{-3} (\det D^2 u_j) A_j^{ab} \partial_m u_j \partial_{am} u_j = -4 K_j \frac{\partial_b u_j}{\sigma_j} = K_j B_j^b.$$

Thus,

$$L_j v = K_j^{-1} \partial_a (K_j A_j^{ab} \partial_b v).$$

The associated conductivity is $\Sigma_j = K_j A_j$.

This divergence form identifies the scalar density ρ in the fixed-gauge intertwining identity. The computation uses the same boundary-fixing map J as above; no second diffeomorphism is introduced.

Lemma 3.8 (The scalar density from the divergence form). *Let J and ρ be the boundary-fixing gauge obtained from (3.6). Then*

$$(3.13) \quad DJ A_1 DJ^T = \rho A_2 \circ J.$$

Moreover,

$$(3.14) \quad \rho = (\det DJ) \frac{K_2 \circ J}{K_1}.$$

Proof. Write

$$(3.15) \quad N = DJ, \quad d = \det N, \quad \Sigma_j = K_j A_j.$$

Since J is a diffeomorphism and $DJ = I$ on $\partial\Omega$, we have $d > 0$ on $\overline{\Omega}$.

Comparing the second-order coefficients in the intertwining identity gives (3.13), namely $NA_1 N^T = \rho(A_2 \circ J)$. We now write the same identity in divergence form. Let $y = J(x)$. The divergence transformation gives $\text{div}_x (\Sigma_1 \nabla_x (f \circ J)) = d [\text{div}_y (d^{-1} N \Sigma_1 N^T \nabla_y f)] \circ J$. Thus,

$$L_1(f \circ J) = K_1^{-1} d [\text{div}_y (d^{-1} N (K_1 A_1) N^T \nabla_y f)] \circ J.$$

Set $\tilde{K}_1 = K_1 \circ J^{-1}$, $\tilde{\rho} = \rho \circ J^{-1}$ and $\tilde{d} = d \circ J^{-1}$, using (3.13), the above identity becomes

$$L_1(f \circ J) = \left[\frac{\tilde{d}}{\tilde{K}_1} \text{div}_y \left(\frac{\tilde{K}_1 \tilde{\rho}}{\tilde{d}} A_2 \nabla_y f \right) \right] \circ J.$$

On the other hand,

$$\rho(L_2 f) \circ J = \left[\frac{\tilde{\rho}}{\tilde{K}_2} \text{div}_y (K_2 A_2 \nabla_y f) \right] \circ J.$$

Comparing the first-order coefficients gives

$$A_2 \nabla_y \log \left(\frac{\tilde{K}_1 \tilde{\rho}}{\tilde{d} K_2} \right) = 0.$$

Since A_2 is invertible, the function $\frac{\tilde{K}_1 \tilde{\rho}}{dK_2}$ is constant. On $\partial\Omega$, we have $J = \text{Id}$, $DJ = I$, $d = 1$, $\rho = 1$, and $K_1 = K_2$. Hence, the constant is 1. Returning to the x variable gives $\frac{K_1 \rho}{d(K_2 \circ J)} = 1$, which is exactly (3.14). $\square$

Adopting the notations (3.15), define the affine-coordinate curvature ratio by

$$q_{\text{aff}} := \log \frac{K_1}{K_2 \circ J}, \quad \theta := \log d - q_{\text{aff}}.$$

Then (3.14) gives $\rho = e^\theta$, $\rho^{-1} = e^{-\theta}$, and therefore $\mathbf{r} = 1 - \rho^{-1} = 1 - e^{-\theta}$.

3.2. The drift mismatch and the equation for J . We now extract the equation for the gauge map. The affine-coordinate curvature ratio is $q_{\text{aff}} = \log(K_1/(K_2 \circ J))$. The background boundary value is fixed, and the subscript φ is suppressed.

Recall that $B_j^a = -4\partial_a u_j / (1 + |\nabla u_j|^2)$, and set $Z_j = -B_j$. Let $(g_j)_{ab} = \partial_{ab} u_j$ and $(g_j)^{ab} = A_j^{ab}$. Then g_j is the Hessian metric of u_j , and $A_j = g_j^{-1}$. With the convention $\Delta_{g_j} v = A_j^{ab} \partial_{ab} v - A_j^{ab} \Gamma(g_j)_{ab}^k \partial_k v$, we have $L_j = \Delta_{g_j} + Y_j \cdot \nabla$ with $Y_j = X_{g_j} - Z_j$, where $X_{g_j}^k = A_j^{ab} \Gamma(g_j)_{ab}^k$. We first relate Z_j to X_{g_j} and K_j .

Since $(g_j)_{ab} = \partial_{ab} u_j$, the Christoffel symbols of g_j are

$$\Gamma(g_j)_{ab}^k = \frac{1}{2} A_j^{k\ell} (\partial_a (g_j)_{b\ell} + \partial_b (g_j)_{a\ell} - \partial_\ell (g_j)_{ab}).$$

Substituting $(g_j)_{ab} = \partial_{ab} u_j$ gives $\Gamma(g_j)_{ab}^k = \frac{1}{2} A_j^{k\ell} (\partial_{ab\ell} u_j + \partial_{ba\ell} u_j - \partial_{\ell ab} u_j)$. By the symmetry of third derivatives, the first two terms are equal, and the last term is the same third derivative; then we obtain $\Gamma(g_j)_{ab}^k = \frac{1}{2} A_j^{k\ell} \partial_{ab\ell} u_j$. This implies

$$X_{g_j}^k = A_j^{ab} \Gamma(g_j)_{ab}^k = \frac{1}{2} A_j^{k\ell} A_j^{ab} \partial_{ab\ell} u_j.$$

Since mixed derivatives commute, $\partial_{ab\ell} u_j = \partial_{\ell ab} u_j$, and we also have $\partial_\ell \log \det D^2 u_j = A_j^{ab} \partial_{\ell ab} u_j$. Thus,

$$X_{g_j}^k = \frac{1}{2} A_j^{k\ell} \partial_\ell \log \det D^2 u_j.$$

Using $\det D^2 u_j = K_j (1 + |\nabla u_j|^2)^2$, we get $\partial_\ell \log \det D^2 u_j = \partial_\ell \log K_j + 2\partial_\ell \log(1 + |\nabla u_j|^2)$. Moreover, since $\partial_\ell (1 + |\nabla u_j|^2) = 2\partial_m u_j \partial_{\ell m} u_j$, this becomes $\partial_\ell \log \det D^2 u_j = \partial_\ell \log K_j + \frac{4\partial_m u_j \partial_{\ell m} u_j}{1 + |\nabla u_j|^2}$. Substituting this into the formula for $X_{g_j}^k$ gives $X_{g_j}^k = \frac{1}{2} A_j^{k\ell} \partial_\ell \log K_j + \frac{2A_j^{k\ell} \partial_{\ell m} u_j \partial_m u_j}{1 + |\nabla u_j|^2}$. Since $A_j = (D^2 u_j)^{-1}$, we have $A_j^{k\ell} \partial_{\ell m} u_j = \delta_m^k$. Hence,

$$X_{g_j}^k = \frac{1}{2} A_j^{k\ell} \partial_\ell \log K_j + \frac{2\partial_k u_j}{1 + |\nabla u_j|^2}.$$

Since $Z_j^k = \frac{4\partial_k u_j}{1 + |\nabla u_j|^2}$, we obtain

$$(3.16) \quad Z_j^k = 2X_{g_j}^k - A_j^{k\ell} \partial_\ell \log K_j.$$

Expanding the intertwining identity (3.6) and comparing the coefficients of $\partial_\alpha f$ gives

$$(3.17) \quad A_1^{ij} \partial_{ij} J^\alpha - Z_1^i \partial_i J^\alpha = -\rho Z_2^\alpha(J).$$

We also use the corresponding identity for the Laplace–Beltrami parts. The principal gauge relation means that the metric g_1 is the pullback of the conformal metric $\rho^{-1}g_2$ by J . Since the scalar Laplacian is conformally covariant in dimension two,

$$\Delta_{g_1}(f \circ J) = \rho(\Delta_{g_2} f) \circ J.$$

Comparing the coefficients of $\partial_\alpha f$ gives

$$(3.18) \quad A_1^{ij} \partial_{ij} J^\alpha - X_{g_1}^i \partial_i J^\alpha = -\rho X_{g_2}^\alpha(J).$$

Subtracting (3.18) from (3.17) yields

$$Z_1^i \partial_i J^\alpha - \rho Z_2^\alpha(J) = X_{g_1}^i \partial_i J^\alpha - \rho X_{g_2}^\alpha(J).$$

Insert (3.16) into both sides. The terms involving $2X_{g_j}$ cancel. Using the principal gauge relation, we obtain

$$(3.19) \quad Z_1^i \partial_i J^\alpha - \rho Z_2^\alpha(J) = \partial_i J^\alpha A_1^{ij} \partial_j q_{\text{aff}}.$$

Combining (3.17) and (3.19) gives, in the original Euclidean affine coordinate,

$$A_1^{ij} \partial_{ij} J^\alpha = \partial_i J^\alpha A_1^{ij} \partial_j q_{\text{aff}}.$$

Equivalently, this is the tensorial identity

$$(3.20) \quad A_1^{ij} (\nabla dJ)_{ij}^\alpha = \partial_i J^\alpha A_1^{ij} \partial_j q_{\text{aff}},$$

where the covariant Hessian of the map J is taken with respect to the flat Euclidean connection in the domain and target.

We now choose the isothermal coordinate used in the rest of the proof. Let $A_{1,X}$ be the affine-coordinate first principal tensor and set $g_{1,X} = A_{1,X}^{-1}$. Since $g_{1,X}$ is smooth and uniformly positive definite on $\bar{\Omega}$, and since Ω is simply connected, the global isothermal coordinate theorem gives a smooth diffeomorphism $\chi : \bar{\Omega} \rightarrow \tilde{\Omega}$, smooth up to the boundary, such that the transformed first principal tensor is conformal to the identity; see [PSU23, Theorem 3.4.16] and the classical references [Ahl06, AIM09]. We write $x = \chi(X)$ and $\tilde{\Omega} = \chi(\Omega)$. After composing χ with a positive dilation, we also assume $\text{diam}(\tilde{\Omega}) < 2$. This normalization is used only to keep the translated noncritical CGO critical points outside the translated domain. In this coordinate,

$$A_1(x) = \gamma(x)I, \quad \gamma(x) > 0.$$

To keep the affine and transformed coefficients separate, write $K_j^x = K_j \circ \chi^{-1}$ only in this paragraph, and let $A_{j,X}$ denote the affine-coordinate principal tensor. If $V(X) = v(\chi(X))$, then

$$A_j(x) = D\chi(X) A_{j,X}(X) D\chi(X)^T, \quad X = \chi^{-1}(x).$$

The divergence transformation gives

$$K_j(X)^{-1} \text{div}_X(K_j(X) A_{j,X} \nabla_X V) = \hat{K}_j(x)^{-1} \text{div}_x(\hat{K}_j(x) A_j(x) \nabla_x v),$$

where $\hat{K}_j = K_j^x / m_\chi$ and $m_\chi(x) = |\det D\chi(\chi^{-1}(x))|$. Thus the conductivity weight in the isothermal coordinate is $\hat{K}_j$, not K_j^x . After this paragraph we again suppress the superscript x .

The density identity becomes

$$\rho = (\det DJ) \frac{\hat{K}_2 \circ J}{\hat{K}_1},$$

where DJ and $\det DJ$ are computed in the x -coordinate. From this point on,

$$q = \log \frac{\hat{K}_1}{\hat{K}_2 \circ J}, \quad \theta = \log \det DJ - q,$$

and $\rho = e^\theta$. The curvature ratio in the tensorial drift identity is still the unweighted ratio. In the present coordinate,

$$q_{\text{aff}} = q + \log m_\chi - \log(m_\chi \circ J).$$

The connection in (3.20), however, is still the pulled-back flat Euclidean connection; hence, its Christoffel symbols need not vanish in the x -coordinate. Dividing by the positive factor γ , the transformed drift identity becomes

$$(3.21) \quad \delta^{ij} \mathcal{H}_{ij}^\alpha = \partial_i J^\alpha \partial_i \left(q + \log m_\chi - \log(m_\chi \circ J) \right),$$

where

$$(3.22) \quad \mathcal{H}_{ij}^\alpha = \partial_{ij} J^\alpha - \Gamma_{ij}^k(x) \partial_k J^\alpha + \Gamma_{\beta\gamma}^\alpha(J(x)) \partial_i J^\beta \partial_j J^\gamma.$$

Here, Γ denotes the Christoffel symbols of the pulled-back flat Euclidean connection, evaluated at the domain point x in the second term and at the target point $J(x)$ in the third term.

Equivalently, (3.21) can be written with ordinary partial derivatives as

$$(3.23) \quad \Delta J^\alpha = \partial_i J^\alpha \partial_i q + \mathfrak{R}^\alpha(x, J, DJ),$$

where $\Delta = \delta^{ij} \partial_{ij}$ and

$$(3.24) \quad \mathfrak{R}^\alpha(x, J, DJ) = \delta^{ij} \Gamma_{ij}^k(x) \partial_k J^\alpha - \delta^{ij} \Gamma_{\beta\gamma}^\alpha(J(x)) \partial_i J^\beta \partial_j J^\gamma + \partial_i J^\alpha \partial_i (\log m_\chi - \log(m_\chi \circ J)).$$

The remainder $\mathfrak{R}^\alpha$ is smooth in (x, J, DJ) , contains no second derivatives of J , and satisfies

$$(3.25) \quad \mathfrak{R}^\alpha(x, \text{Id}, I) = 0.$$

Indeed, the last term in (3.24) also vanishes at $(J, DJ) = (\text{Id}, I)$. The term $\mathfrak{R}^\alpha$ is kept in the later partial derivative drift reduction. Since it contains no second derivatives of J , it does not change the second-order Hessian block extracted from the second linearization.

Finally, we record the corresponding pullback formulae for first and second jets. If $\tilde{v} = v \circ J$, then $\nabla_x \tilde{v} = DJ^T (\nabla_y v) \circ J$ and

$$\nabla_x^2 \tilde{v} = DJ^T ((\nabla_y^2 v) \circ J) DJ + ((\partial_\alpha v) \circ J) \mathcal{H}^\alpha.$$

Equivalently,

$$(\nabla_y v) \circ J = DJ^{-T} \nabla_x \tilde{v}, \quad (\nabla_y^2 v) \circ J = DJ^{-T} (\nabla_x^2 \tilde{v} - (DJ^{-T} \nabla_x \tilde{v})_\alpha \mathcal{H}^\alpha) DJ^{-1}.$$

Remark 3.9. *The chain-rule term detected by the second linearization is the covariant Hessian $\mathcal{H}^\alpha = (\nabla dJ)^\alpha$. In the original Euclidean affine coordinate, this is simply $D^2 J^\alpha$, while in the global isothermal coordinate it is given by (3.22).*

4. ANALYSIS OF THE SECOND LINEARIZATION

We next derive the second identity used to detect the remaining gauge. Fix $\varphi \in \mathcal{U}$ and write $u_{j,0} = u_\varphi^{(j)}$ for the corresponding background solutions. All first linearized operators are taken in logarithmic normalization, so in the original Euclidean coordinate

$$L_j v = A_j^{ab} \partial_{ab} v + B_j^a \partial_a v, \quad A_j = (D^2 u_{j,0})^{-1}, \quad B_j^a = -\frac{4\partial_a u_{j,0}}{1 + |\nabla u_{j,0}|^2}.$$

By Proposition 3.5 and Corollary 3.6, there is a boundary-fixing gauge pair $(J_\varphi, \rho_\varphi)$ such that

$$DJ_\varphi A_1 DJ_\varphi^T = \rho_\varphi A_2 \circ J_\varphi, \quad L_1(f \circ J_\varphi) = \rho_\varphi(L_2 f) \circ J_\varphi \quad \text{for all smooth } f,$$

and

$$J_\varphi|_{\partial\Omega} = \text{Id}, \quad DJ_\varphi = I \quad \text{on } \partial\Omega.$$

4.1. The fixed isothermal coordinate. We keep the global isothermal coordinate introduced in Subsection 3.2. Thus $x = \chi(X)$, the first principal tensor satisfies

$$A_1(x) = \gamma(x)I, \quad \gamma(x) > 0,$$

and all objects below are transformed by the same fixed coordinate change χ . We write

$$J = \chi \circ J_\varphi \circ \chi^{-1}, \quad \rho = \rho_\varphi \circ \chi^{-1}.$$

Then the transformed intertwining identity and the transformed principal gauge identity have the same form as before. Moreover,

$$(4.1) \quad J = \text{Id}, \quad DJ = I \quad \text{on } \partial\tilde{\Omega}.$$

The connection ∇ used below is the pullback of the original Euclidean connection in the X -variables. Thus, if $U(X)$ is the original function and $u(x) = U(\chi^{-1}(x))$, then $\nabla^2 u$ is the original Euclidean Hessian written in the x -coordinate, and the transformed principal tensor is $A_j = (\nabla^2 u_{j,0})^{-1}$.

The transformed logarithmic equation is written using this covariant Hessian. Indeed, $D_X^2 U = (D\chi)^T (\nabla^2 u) D\chi$, so $\det D_X^2 U = (\det D\chi)^2 \det(\nabla^2 u)$. The positive factor $(\det D\chi)^2$ is independent of u and is absorbed into the fixed zeroth-order term in the transformed logarithmic equation.

Let ∇ denote the flat Euclidean connection written in the global isothermal coordinate. For a scalar function f , write

$$(\nabla^2 f)_{ij} := \partial_{ij} f - \Gamma_{ij}^k \partial_k f,$$

where Γ_{ij}^k are the Christoffel symbols of this flat connection in the x -coordinate. For the map J , define its covariant Hessian by

$$(4.2) \quad \mathcal{H}_{ij}^\alpha := \partial_{ij} J^\alpha - \Gamma_{ij}^k \partial_k J^\alpha + \Gamma_{\beta\gamma}^\alpha(J) \partial_i J^\beta \partial_j J^\gamma.$$

Equivalently, $\mathcal{H}^\alpha = (\nabla dJ)^\alpha$ is the second covariant derivative of the map J with respect to the same flat connection in the domain and target.

Throughout this section, $\mathcal{J}^2 v$ denotes the full two-jet of v . This notation is used to avoid confusion with the gauge map J .

4.2. The second identity. Let $d\mu$ be the smooth positive density in the global isothermal coordinate. All formal adjoints in this subsection are taken with respect to $d\mu$.

The fixed-gauge Green identity is the same operator-level second-variation identity as in [LL25a, Section 7.1]. The CGO construction used later comes from [LL25a, Section 5]. Once the first-linearized gauge is fixed, the second equation is pulled back by the same boundary-fixing map J and tested against a solution of the adjoint first-linearized equation. The boundary terms cancel because the nonlinear DN maps agree after differentiating twice in the two boundary directions, the mixed second variation has zero Dirichlet trace, and $J = \text{Id}$, $DJ = I$ on the boundary.

Only the quadratic source changes from the Monge–Ampère calculation. In the present logarithmic normalization it is the mixed second variation of the transformed prescribed curvature equation. Thus the Green identity has the same form as in [LL25a, Section 7.1], while the quadratic form $\mathcal{Q}_j$ is computed below.

Let $X = (X^1, X^2)$ be the original Euclidean coordinate and let $x = (x^1, x^2) = \chi(X)$ be the global isothermal coordinate. If $U(X)$ is the original function and $u(x) = U(\chi^{-1}(x))$, then $\partial_{X^i} U = \partial_a u \partial_{X^i} x^a$. We write

$$\mathbf{g}^{ab}(x) = \delta^{ij} \partial_{X^i} x^a \partial_{X^j} x^b,$$

so that $|\nabla_X U|^2 = \mathbf{g}^{ab} \partial_a u \partial_b u$. The connection ∇ is the pullback of the flat Euclidean connection, and $(\nabla^2 u)_{ab} = \partial_{ab} u - \Gamma_{ab}^c \partial_c u$.

Let $\mathcal{Q}_j$ be the quadratic part in the two first variations of the full mixed second variation at $u_{j,0}$. With this convention, if w is the mixed second variation of the solution, then $L_j w + \mathcal{Q}_j = 0$. The next lemma gives the required formula.

Lemma 4.1 (Full transformed second variation). *Set*

$$\sigma_j := 1 + \mathbf{g}^{ab} \partial_a u_{j,0} \partial_b u_{j,0}.$$

Then the transformed logarithmic equation can be written in the form

$$\log \det(\nabla^2 u) - 2 \log(1 + \mathbf{g}^{ab} \partial_a u \partial_b u) + \beta_j(x) = 0,$$

where β_j is a fixed smooth function independent of u . For two variations v and $\tilde{v}$, the full mixed second variation at $u_{j,0}$ has quadratic part

$$(4.3) \quad \mathcal{Q}_j(\mathcal{J}^2 v, \mathcal{J}^2 \tilde{v}) = -\operatorname{tr}(A_j \nabla^2 v A_j \nabla^2 \tilde{v}) - \frac{4}{\sigma_j} \mathbf{g}^{ab} \partial_a v \partial_b \tilde{v} + \frac{8}{\sigma_j^2} (\mathbf{g}^{ab} \partial_a u_{j,0} \partial_b v) (\mathbf{g}^{cd} \partial_c u_{j,0} \partial_d \tilde{v}).$$

In particular, once the transformed equation is written using the covariant Hessian ∇^2 , there are no Hessian-gradient mixed terms in the full mixed second variation.

Proof. We first compute the Hessian transformation. Since the covariant Hessian is tensorial, applying the chain rule to the second derivatives gives

$$\partial_{X^i X^j} U = (\nabla^2 u)_{ab} \partial_{X^i} x^a \partial_{X^j} x^b.$$

Indeed, the terms containing first derivatives of u are exactly canceled by the Christoffel correction in the pulled-back flat connection.

In matrix form, if $D\chi = (\partial_{X^i} x^a)_{a,i}$, then $D_X^2 U = (D\chi)^T (\nabla^2 u) D\chi$. Therefore, $\det D_X^2 U = (\det D\chi)^2 \det(\nabla^2 u)$. The factor $(\det D\chi)^2$ is smooth, positive, and independent of u . Thus, the original logarithmic equation $\log \det D_X^2 U - 2 \log(1 + |\nabla_X U|^2) - \log K_j(X) = 0$ becomes, in the x -coordinate,

$$\log \det(\nabla^2 u) - 2 \log(1 + \mathbf{g}^{ab} \partial_a u \partial_b u) + \beta_j(x) = 0,$$

where $\beta_j(x) = 2 \log |\det D\chi(\chi^{-1}(x))| - \log K_j(\chi^{-1}(x))$. In particular, β_j is a fixed smooth function independent of u . This also shows that the only u -dependent second derivative term in the transformed equation is $\log \det(\nabla^2 u)$.

We now compute the mixed second variation. Since the connection is fixed, it is independent of the solution. Hence, for $u(t, s) = u_{j,0} + tv + s\tilde{v}$, $\nabla^2 u(t, s) = \nabla^2 u_{j,0} + t\nabla^2 v + s\nabla^2 \tilde{v}$. Let

$$M(t, s) = \nabla^2 u_{j,0} + t\nabla^2 v + s\nabla^2 \tilde{v}.$$

Then $\partial_t \log \det M(t, s) = \operatorname{tr}(M(t, s)^{-1} \nabla^2 v)$. Differentiating in s and evaluating at $t = s = 0$ gives

$$\partial_s \partial_t \log \det M(t, s) \big|_{t=s=0} = -\operatorname{tr}(A_j \nabla^2 v A_j \nabla^2 \tilde{v}),$$

where $A_j = (\nabla^2 u_{j,0})^{-1}$ in the transformed coordinate.

It remains to compute the mixed second variation of the gradient term. Set

$$S(u) = 1 + \mathbf{g}^{ab} \partial_a u \partial_b u.$$

At $u_{j,0}$, we have

$$\partial_t S(u_{j,0} + tv) \big|_{t=0} = 2\mathbf{g}^{ab} \partial_a u_{j,0} \partial_b v \quad \text{and} \quad \partial_s \partial_t S(u_{j,0} + tv + s\tilde{v}) \big|_{t=s=0} = 2\mathbf{g}^{ab} \partial_a v \partial_b \tilde{v}.$$

Since $\sigma_j = S(u_{j,0})$, we obtain

$$\begin{aligned} \partial_s \partial_t [-2 \log S(u_{j,0} + tv + s\tilde{v})] \big|_{t=s=0} &= -2 \left[\frac{2\mathbf{g}^{ab} \partial_a v \partial_b \tilde{v}}{\sigma_j} - \frac{(2\mathbf{g}^{ab} \partial_a u_{j,0} \partial_b v)(2\mathbf{g}^{cd} \partial_c u_{j,0} \partial_d \tilde{v})}{\sigma_j^2} \right] \\ &= -\frac{4}{\sigma_j} \mathbf{g}^{ab} \partial_a v \partial_b \tilde{v} + \frac{8}{\sigma_j^2} (\mathbf{g}^{ab} \partial_a u_{j,0} \partial_b v) (\mathbf{g}^{cd} \partial_c u_{j,0} \partial_d \tilde{v}). \end{aligned}$$

Combining the log-determinant and gradient contributions gives (4.3). $\square$

We next derive the fixed-gauge second identity from the nonlinear boundary measurements. We include the boundary cancellation in the proof: the equality of the nonlinear DN maps is differentiated twice in the two boundary directions, and the relation (4.1) is used to remove the boundary terms after pulling back the second equation.

Lemma 4.2 (The second integral identity). *Let $v^{(1)}$ and $v^{(2)}$ solve $L_1 v^{(1)} = L_1 v^{(2)} = 0$ in $\tilde{\Omega}$, and set $v_2^{(1)} = v^{(1)} \circ J^{-1}$, $v_2^{(2)} = v^{(2)} \circ J^{-1}$. Then $L_2 v_2^{(1)} = L_2 v_2^{(2)} = 0$. If v^* solves the formal adjoint equation $L_{1,d\mu}^* v^* = 0$ in $\tilde{\Omega}$, then*

$$(4.4) \quad \int_{\tilde{\Omega}} v^* [\mathcal{Q}_1(\mathcal{J}^2 v^{(1)}, \mathcal{J}^2 v^{(2)}) - \rho(\mathcal{Q}_2(\mathcal{J}^2 v_2^{(1)}, \mathcal{J}^2 v_2^{(2)}) \circ J)] d\mu = 0.$$

The identity is understood complex bilinearly when the three solutions are complex-valued.

Proof. Since $v^{(k)} = v_2^{(k)} \circ J$ for $k = 1, 2$, the fixed-gauge intertwining identity (3.6) gives

$$0 = L_1 v^{(k)} = L_1(v_2^{(k)} \circ J) = \rho(L_2 v_2^{(k)}) \circ J.$$

Since ρ is positive, $L_2 v_2^{(k)} = 0$ in $\tilde{\Omega}$.

Let $h_k := v^{(k)}|_{\partial\tilde{\Omega}}$. Because $J = \text{Id}$ on $\partial\tilde{\Omega}$, $v_2^{(k)}|_{\partial\tilde{\Omega}} = v^{(k)} \circ J^{-1}|_{\partial\tilde{\Omega}} = h_k$. Hence, the first variations for the two equations correspond to the same boundary perturbations h_1, h_2 . The C^∞ smooth dependence of the solution map from Proposition 2.1 justifies differentiating the nonlinear equation and the nonlinear DN map twice in these directions.

Let $u_{j,s,t}$ be the j -th nonlinear solution with boundary value $\varphi + sh_1 + th_2$, and let $w_j = \partial_s \partial_t u_{j,s,t}|_{s=t=0}$. For $|s| + |t|$ sufficiently small, the boundary path remains in $\mathcal{U}$, since $\mathcal{U}$ is open. Since the boundary path is affine in s and t , we have $w_j = 0$ on $\partial\tilde{\Omega}$. In the transformed logarithmic equation, the mixed second variations satisfy

$$L_j w_j + \mathcal{Q}_j(\mathcal{J}^2 v_j^{(1)}, \mathcal{J}^2 v_j^{(2)}) = 0.$$

Pulling the second equation back by J and using the transformed fixed-gauge intertwining relation gives

$$L_1(w_2 \circ J) = \rho(L_2 w_2) \circ J = -\rho(\mathcal{Q}_2(\mathcal{J}^2 v_2^{(1)}, \mathcal{J}^2 v_2^{(2)}) \circ J).$$

Subtracting this pulled-back equation from the first equation gives

$$L_1(w_1 - w_2 \circ J) + \mathcal{Q}_1(\mathcal{J}^2 v^{(1)}, \mathcal{J}^2 v^{(2)}) - \rho(\mathcal{Q}_2(\mathcal{J}^2 v_2^{(1)}, \mathcal{J}^2 v_2^{(2)}) \circ J) = 0.$$

We check the boundary cancellation. Both w_1 and w_2 have zero Dirichlet data, and since $J = \text{Id}$ on $\partial\tilde{\Omega}$, also $w_2 \circ J = 0$ on $\partial\tilde{\Omega}$. The equality of the nonlinear DN maps gives, for the two-parameter boundary value $\varphi + sh_1 + th_2$, $\partial_\nu u_{1,s,t} = \partial_\nu u_{2,s,t}$ on $\partial\Omega$. Differentiating this identity in s and t at $(s, t) = (0, 0)$ gives

$$\partial_\nu w_1 = \partial_\nu w_2 \quad \text{on } \partial\Omega$$

in the original Euclidean coordinates. Since $w_1 = w_2 = 0$ on the boundary, their tangential derivatives also vanish there. Hence, the full first boundary jets of w_1 and w_2 agree in the original

coordinates. Applying the same fixed coordinate change to both equations preserves equality of these first boundary jets in the transformed coordinate.

Finally, since w_2 is a scalar function of the target variable and $J = J(x)$, the chain rule gives $\partial_i(w_2 \circ J) = ((\partial_\alpha w_2) \circ J) \partial_i J^\alpha$. Equivalently, if gradients are written as column vectors, then $\nabla(w_2 \circ J) = DJ^T(\nabla w_2) \circ J$. Using $DJ = I$ on $\partial\tilde{\Omega}$, we obtain

$$\nabla(w_2 \circ J) = (\nabla w_2) \circ J = \nabla w_2 \quad \text{on } \partial\tilde{\Omega},$$

because $J = \text{Id}$ on $\partial\tilde{\Omega}$. Since the full first boundary jets of w_1 and w_2 agree in the transformed coordinate, it follows that $\nabla(w_1 - w_2 \circ J) = 0$ on $\partial\tilde{\Omega}$. Therefore, $w_1 - w_2 \circ J$ has zero trace and zero first boundary jet. The Green identity for L_1 and $L_{1,d\mu}^*$ has boundary terms involving only the trace and first derivatives of this difference, and all these boundary terms vanish.

Testing the equation for $w_1 - w_2 \circ J$ against v^* and integrating by parts with respect to $d\mu$, the interior term involving $L_{1,d\mu}^* v^*$ vanishes. This proves (4.4) for real-valued solutions.

The complex-valued case follows by applying the real identity to real and imaginary parts and polarizing. The pairing is the complex-bilinear distributional pairing $\int v^*(\cdot) d\mu$, not the Hermitian L^2 pairing, so no conjugation is introduced. $\square$

Lemma 4.3 (Full pulled-back quadratic difference). *Let $N = DJ$, $\eta_\alpha = \partial_\alpha v_2^{(1)} \circ J$, and $\xi_\alpha = \partial_\alpha v_2^{(2)} \circ J$. Then $\nabla v^{(1)} = N^T \eta$ and $\nabla v^{(2)} = N^T \xi$. Moreover,*

$$\nabla^2 v_2^{(1)} \circ J = N^{-T} (\nabla^2 v^{(1)} - \eta_\alpha \mathcal{H}^\alpha) N^{-1}, \quad \nabla^2 v_2^{(2)} \circ J = N^{-T} (\nabla^2 v^{(2)} - \xi_\alpha \mathcal{H}^\alpha) N^{-1}.$$

Consequently, the full quadratic difference in (4.4) can be written pointwise as

$$(4.5) \quad \mathcal{Q}_1(\mathcal{J}^2 v^{(1)}, \mathcal{J}^2 v^{(2)}) - \rho(\mathcal{Q}_2(\mathcal{J}^2 v_2^{(1)}, \mathcal{J}^2 v_2^{(2)}) \circ J) = -\gamma^2 (\mathcal{P}_{\text{HH}} + \mathcal{P}_{\nabla\nabla}),$$

where

$$(4.6) \quad \mathcal{P}_{\text{HH}} = \text{tr}(\nabla^2 v^{(1)} \nabla^2 v^{(2)}) - \rho^{-1} \text{tr}((\nabla^2 v^{(1)} - \eta_\alpha \mathcal{H}^\alpha)(\nabla^2 v^{(2)} - \xi_\beta \mathcal{H}^\beta)),$$

and

$$(4.7) \quad \mathcal{P}_{\nabla\nabla} = -\gamma^{-2} [\mathcal{G}_1(\nabla v^{(1)}, \nabla v^{(2)}) - \rho(\mathcal{G}_2 \circ J)(\eta, \xi)].$$

Here,

$$(4.8) \quad \mathcal{G}_j(p, \ell) = -\frac{4}{\sigma_j} \mathbf{g}^{ab} p_a \ell_b + \frac{8}{\sigma_j^2} (\mathbf{g}^{ab} \partial_a u_{j,0} p_b) (\mathbf{g}^{cd} \partial_c u_{j,0} \ell_d).$$

The notation $(\mathcal{G}_2 \circ J)(\eta, \xi)$ means that all coefficient functions appearing in $\mathcal{G}_2$ are evaluated at $J(x)$ and then the covectors $\eta(x)$ and $\xi(x)$ are inserted into the resulting bilinear form.

Proof. We first prove the gradient and Hessian transformation rules. Since $v^{(1)} = v_2^{(1)} \circ J$ and $v^{(2)} = v_2^{(2)} \circ J$, the chain rule gives $\partial_i v^{(1)} = (\partial_\alpha v_2^{(1)} \circ J) \partial_i J^\alpha$ and $\partial_i v^{(2)} = (\partial_\alpha v_2^{(2)} \circ J) \partial_i J^\alpha$. In matrix form, this is $\nabla v^{(1)} = N^T \eta$ and $\nabla v^{(2)} = N^T \xi$.

We next differentiate once more. For the first variation,

$$\partial_{ij} v^{(1)} = (\partial_{\alpha\beta} v_2^{(1)} \circ J) \partial_i J^\alpha \partial_j J^\beta + (\partial_\alpha v_2^{(1)} \circ J) \partial_{ij} J^\alpha.$$

Subtracting the domain Christoffel term and adding the target Christoffel contribution inside the pulled-back Hessian gives

$$(\nabla^2 v^{(1)})_{ij} = (\nabla^2 v_2^{(1)})_{\alpha\beta} \circ J \partial_i J^\alpha \partial_j J^\beta + \eta_\alpha \mathcal{H}_{ij}^\alpha.$$

Equivalently, $\nabla^2 v^{(1)} = N^T(\nabla^2 v_2^{(1)} \circ J)N + \eta_\alpha \mathcal{H}^\alpha$. Solving for the pulled-back Hessian gives $\nabla^2 v_2^{(1)} \circ J = N^{-T}(\nabla^2 v^{(1)} - \eta_\alpha \mathcal{H}^\alpha)N^{-1}$. The same calculation for the second variation gives $\nabla^2 v^{(2)} = N^T(\nabla^2 v_2^{(2)} \circ J)N + \xi_\alpha \mathcal{H}^\alpha$, and hence,

$$\nabla^2 v_2^{(2)} \circ J = N^{-T}(\nabla^2 v^{(2)} - \xi_\alpha \mathcal{H}^\alpha)N^{-1}.$$

For the Hessian–Hessian part, use $A_1 = \gamma I$. Since $A_1 = \gamma I$, the Hessian–Hessian part of $\mathcal{Q}_1$ is $-\gamma^2 \text{tr}(\nabla^2 v^{(1)} \nabla^2 v^{(2)})$. The transformed principal gauge relation gives $A_2 \circ J = \rho^{-1} N A_1 N^T$. Therefore,

$$N^{-1}(A_2 \circ J)N^{-T} = \rho^{-1} A_1 = \rho^{-1} \gamma I.$$

Using the two Hessian transformation formulae above, the Hessian–Hessian part of $\rho(\mathcal{Q}_2 \circ J)$ is

$$-\rho(\rho^{-1} \gamma)^2 \text{tr}((\nabla^2 v^{(1)} - \eta_\alpha \mathcal{H}^\alpha)(\nabla^2 v^{(2)} - \xi_\beta \mathcal{H}^\beta)).$$

Hence the Hessian–Hessian part of $\mathcal{Q}_1 - \rho(\mathcal{Q}_2 \circ J)$ is $-\gamma^2 \mathcal{P}_{\text{HH}}$, with $\mathcal{P}_{\text{HH}}$ defined by (4.6).

It remains to compute the gradient–gradient part. By Lemma 4.1, the gradient-dependent part of $\mathcal{Q}_j$ is the bilinear form $\mathcal{G}_j$ in (4.8). Thus the first equation contributes $\mathcal{G}_1(\nabla v^{(1)}, \nabla v^{(2)})$. For the pulled-back second equation, the gradients of $v_2^{(1)}$ and $v_2^{(2)}$ at the target point $J(x)$ are exactly η and ξ . Therefore, the pulled-back second gradient contribution is $\rho(\mathcal{G}_2 \circ J)(\eta, \xi)$.

Combining the Hessian and gradient contributions gives

$$\mathcal{Q}_1(\mathcal{J}^2 v^{(1)}, \mathcal{J}^2 v^{(2)}) - \rho(\mathcal{Q}_2(\mathcal{J}^2 v_2^{(1)}, \mathcal{J}^2 v_2^{(2)}) \circ J) = -\gamma^2 \mathcal{P}_{\text{HH}} + \mathcal{G}_1(\nabla v^{(1)}, \nabla v^{(2)}) - \rho(\mathcal{G}_2 \circ J)(\eta, \xi).$$

By the definition (4.7), the right-hand side is exactly $-\gamma^2(\mathcal{P}_{\text{HH}} + \mathcal{P}_{\nabla\nabla})$. This proves (4.5). $\square$

Using the equality of the nonlinear DN maps and multiplying the second integral identity by -1 , we obtain

$$(4.9) \quad \int_{\tilde{\Omega}} v^* \gamma^2 (\mathcal{P}_{\text{HH}} + \mathcal{P}_{\nabla\nabla}) d\mu = 0.$$

5. ASYMPTOTIC ANALYSIS OF THE SECOND INTEGRAL IDENTITY

We now test (4.9) with CGO solutions. The purpose of this section is only to extract the residual equation; the gauge is not eliminated here.

Set $\mathbf{r} = 1 - \rho^{-1}$ and $M^\alpha = \rho^{-1} \mathcal{H}^\alpha$. Then $\mathbf{r}$ measures the density defect and M^α is the normalized covariant Hessian of the boundary-fixing gauge. From (4.6),

$$\mathcal{P}_{\text{HH}} = \mathbf{r} \text{tr}(\nabla^2 v^{(1)} \nabla^2 v^{(2)}) + \xi_\beta \text{tr}(\nabla^2 v^{(1)} M^\beta) + \eta_\alpha \text{tr}(M^\alpha \nabla^2 v^{(2)}) - \rho^{-1} \eta_\alpha \xi_\beta \text{tr}(\mathcal{H}^\alpha \mathcal{H}^\beta).$$

We write $I_h = I_{\mathbf{r}} + I_{\xi M} + I_{\eta M} + I_{\eta \xi} + I_{\nabla\nabla}$, with the five terms defined by this decomposition. The identity gives $I_h = 0$. We multiply it by h and pass to the limit.

The calculation is first done with the critical point at the origin. It is then translated to an arbitrary center $a \in \tilde{\Omega}$, uniformly for a in compact subsets. The scalar term produces the block $8\bar{\partial}^2(C_a \mathbf{r})$, and the terms linear in M produce $-2\bar{\partial}(C_a \kappa_\alpha \mathcal{A}(M^\alpha))$ plus lower-order terms. The critical CGO remainder also leaves Cauchy-transform terms. These terms are kept, not estimated away, and are localized only in the unique continuation argument.

5.1. The CGO families and the critical estimates. Throughout the CGO asymptotics, we identify the local coordinate plane with $\mathbb{C}$ by $z = x_1 + ix_2$ and use

$$\partial = \frac{1}{2}(\partial_{x_1} - i\partial_{x_2}), \quad \bar{\partial} = \frac{1}{2}(\partial_{x_1} + i\partial_{x_2}).$$

All translated expressions use the same convention with z replaced by $z - a$.

We use the CGO construction and the critical estimates from [LL25a, Section 5]. The first principal coefficient is $A_1 = \gamma I$ in the global isothermal coordinate. After multiplication by a nonzero scalar factor, the forward equation has the local form $-\Delta v + X \cdot \nabla v + Q_0 v = 0$, with $Q_0 = 0$ for the forward equation. The adjoint equation has the same two-dimensional CGO form, with possibly different lower-order coefficients. The three CGO solutions are therefore constructed separately, and their smooth prefactors need not be the same.

The functions inserted into (4.9) are exact solutions of the forward first-linearized equation and of the adjoint equation in the whole transformed domain. The local cutoffs below are used only in the CGO parametrices, in the stationary phase expansions, and in the localized Cauchy inverse estimates. They are not inserted into the second integral identity. The residual equations are extracted first, and the localization of this residual system is carried out only in the unique continuation argument. The same convention is used for the mirror CGO family in Section 6.

For a model equation in this class, let A be the corresponding complex-valued one-form in the factorization, let F_A be the gauge factor defined by $\bar{\partial} F_A = i A F_A$, and let Q be the zeroth-order coefficient in the associated factorization. We set

$$V' := -|F_A|^2, \quad V := \frac{1}{2} Q |F_A|^{-2}.$$

In the notation of [LL25a, Section 5], the first Neumann source term in the critical remainder is Va , where a is the leading critical amplitude. In the present normalization, this leading amplitude is absorbed into the smooth CGO prefactor F_2 . Equivalently, we use the $a \equiv 1$ convention for the critical remainder and write the first Neumann source term as V .

We now fix the localized Cauchy inverse convention used throughout this section. Let C_∂ denote the properly supported localized Cauchy inverse for ∂ . We also write $\partial^{-1} = C_\partial$, and write ∂^{*-1} for the corresponding localized adjoint inverse. Thus,

$$\partial C_\partial f = f + S f,$$

where S is a smoothing cutoff operator. All such smoothing cutoff errors are included in the smoothing cutoff remainders below. When the oscillatory phase $\psi = x_1 x_2 / 2$ is present, we use the conjugated inverses

$$\partial_\psi^{-1} f := C_\partial(e^{-ix_1 x_2 / h} f), \quad \partial_\psi^{*-1} f := \partial^{*-1}(e^{ix_1 x_2 / h} f).$$

The same convention is used after translating the critical point. In particular, identities involving ∂C_∂ , ∂_ψ^{-1} , and adjoints are always understood modulo the smoothing cutoff errors already included in the smoothing cutoff remainder terms.

By the dilation normalization fixed when the global isothermal coordinate was chosen, after translating any chosen critical center to the origin, the points ± 4 lie outside the translated domain. In particular, the noncritical phases below have no critical points on the support of the amplitudes.

At the fixed center, translated to the origin, we use the phases

$$\Phi_1(z) = z + \frac{z^2}{8}, \quad \Phi_2(z) = -\frac{z^2}{4}, \quad \Phi^*(z) = -z + \frac{z^2}{8}.$$

Then

$$\Phi^* + \Phi_1 + \overline{\Phi_2} = \frac{z^2 - \bar{z}^2}{4} = ix_1 x_2.$$

The phase derivatives used below are

$$\partial\Phi_1 = 1 + \frac{z}{4}, \quad \partial^2\Phi_1 = \frac{1}{4}, \quad \overline{\partial}\Phi_2 = -\frac{\bar{z}}{2}, \quad \overline{\partial}^2\Phi_2 = -\frac{1}{2}, \quad \partial\Phi^* = -1 + \frac{z}{4}, \quad \partial^2\Phi^* = \frac{1}{4}.$$

By the CGO construction in [GT11, LL25a], there are solutions of the first linearized equations and the adjoint equation of the form

$$v^* = F_* e^{\Phi^*/h} (1 + r_*), \quad v^{(1)} = F_1 e^{\Phi_1/h} (1 + r_1), \quad v^{(2)} = F_2 e^{\overline{\Phi}_2/h} (1 + \tilde{r}_2).$$

The amplitudes F_*, F_1, F_2 are smooth and nonvanishing near the critical point. We write $d\mu = \mu(x) dx$ and absorb μ into the compactly supported amplitudes in the local stationary phase calculations.

More precisely, for each fixed critical center, we choose a cutoff $\vartheta \in C_0^\infty$ which is equal to 1 in a small neighborhood of that center. In every use of the stationary phase below, the smooth amplitude is first multiplied by this cutoff. The part multiplied by $1 - \vartheta$ has no critical point for the corresponding quadratic phase. Repeated integration by parts in the nonstationary phase variable shows that this part is $O(h^N)$ for any fixed N , after multiplication by any of the finitely many powers of h^{-1} appearing in the smooth-amplitude local terms below. The same convention is used after translating the critical center, uniformly for centers in compact subsets of $\tilde{\Omega}$.

This cutoff convention is used only for the smooth-amplitude stationary phase terms. The terms containing the critical derivative of the CGO remainder are not discarded by this reduction; they are kept and evaluated separately below.

The two remainders associated with the noncritical phases have complete asymptotic expansions. More precisely, for every N and every $p \geq 2$,

$$r_* = hR_{*,h} + O_{W^{2,p}}(h^N), \quad r_1 = hR_{1,h} + O_{W^{2,p}}(h^N),$$

where $R_{*,h}$ and $R_{1,h}$ are finite asymptotic sums with smooth coefficients.

For the critical remainder, we use the conjugated localized inverses above and set

$$\tilde{r}_2 = -\partial_\psi^{-1}(V'\tilde{s}_2), \quad \tilde{s}_2 = \sum_{k=0}^{\infty} \tilde{T}_h^k \partial_\psi^{*-1}(V), \quad \tilde{T}_h = \partial_\psi^{*-1} V \partial_\psi^{-1} V'.$$

Consequently, modulo the smoothing cutoff errors included in the smoothing cutoff remainder terms,

$$(5.1) \quad \partial \tilde{r}_2 = -e^{-ix_1x_2/h} V' \tilde{s}_2.$$

The CGO construction gives the estimates

$$(5.2) \quad \|\tilde{s}_2\|_{L^2} = O(h^{1/2+\varepsilon}), \quad \|H_c D^2 \tilde{r}_2\|_{L^2} = O(h^{-1/2+\varepsilon})$$

for every fixed cutoff $H_c \in C_0^2$, together with the standard localized Cauchy inverse bounds used in the construction.

We shall repeatedly use the stationary phase expansion

$$(5.3) \quad \frac{1}{2\pi h} \int_{\mathbb{R}^2} e^{ix_1x_2/h} F(x) dx = F(0) + h(\overline{\partial}^2 - \partial^2)F(0) + O(h^2),$$

for compactly supported smooth F (for example, see [LL25a, Section 8]).

The next lemma is the additional oscillatory Cauchy estimate needed to control the $h^{-2}\tilde{r}_2$ terms.

Lemma 5.1 (Oscillatory Cauchy inverse estimate). *Let $a \in C_0^\infty(\mathbb{R}^2)$ be fixed and $E = e^{ix_1x_2/h}$. For the localized Cauchy inverse used in the CGO construction,*

$$(5.4) \quad \|\partial^{*-1}(Ea)\|_{L^2} \leq Ch |\log h|^{1/2}.$$

The same estimate holds uniformly after translating the critical point to centers in compact subsets of $\tilde{\Omega}$.

Proof. The localized Cauchy inverse is a properly supported operator of order -1 , modulo a smoothing cutoff error. Hence, it is bounded from H^{-1} to L^2 . It suffices to prove

$$(5.5) \quad \|Ea\|_{H^{-1}} \leq Ch |\log h|^{1/2}.$$

We write

$$\widehat{Ea}(\xi) = \int e^{ix_1x_2/h - ix \cdot \xi} a(x) dx = \int e^{\frac{i}{h}(x_1x_2 - hx \cdot \xi)} a(x) dx.$$

The phase $x_1x_2 - hx \cdot \xi$ has the Hessian $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, which is nondegenerate. We now justify the stationary phase bound uniformly in ξ . Completing the square gives $x_1x_2 - hx \cdot \xi = (x_1 - h\xi_2)(x_2 - h\xi_1) - h^2\xi_1\xi_2$. After the change of variables $y_1 = x_1 - h\xi_2$, $y_2 = x_2 - h\xi_1$, we obtain

$$\widehat{Ea}(\xi) = e^{-ih\xi_1\xi_2} \int e^{iy_1y_2/h} a(y_1 + h\xi_2, y_2 + h\xi_1) dy.$$

Set $a_{h,\xi}(y) := a(y_1 + h\xi_2, y_2 + h\xi_1)$, then the C^2 norms of $a_{h,\xi}$ are bounded uniformly in h and ξ , and its support is a translate of the fixed compact set $\text{supp}(a)$. We split the integral into the regions $|y_1| \leq h$ and $|y_1| > h$. In the first region, the support in the y_2 variable has uniformly bounded length, and hence

$$\left| \int_{|y_1| \leq h} e^{iy_1y_2/h} a_{h,\xi}(y) dy \right| \leq Ch.$$

On the second region, for each fixed y_1 with $|y_1| > h$, we integrate twice by parts in y_2 . Since $e^{iy_1y_2/h} = \frac{h}{iy_1} \partial_{y_2} e^{iy_1y_2/h}$, and $a_{h,\xi}$ is compactly supported with uniformly bounded derivatives, we have

$$\left| \int e^{iy_1y_2/h} a_{h,\xi}(y_1, y_2) dy_2 \right| \leq C \frac{h^2}{|y_1|^2}.$$

Therefore,

$$\left| \int_{|y_1| > h} e^{iy_1y_2/h} a_{h,\xi}(y) dy \right| \leq Ch^2 \int_{|y_1| > h} \frac{dy_1}{|y_1|^2} \leq Ch.$$

Combining the two regions gives the uniform bound

$$(5.6) \quad |\widehat{Ea}(\xi)| \leq Ch.$$

If $|\xi| \geq Ch^{-1}$ with C sufficiently large depending on $\text{supp}(a)$, then the phase has no critical point on $\text{supp}(a)$. Combining this nonstationary integration by parts with the preceding one-variable oscillatory estimate gives, for every N ,

$$(5.7) \quad |\widehat{Ea}(\xi)| \leq C_N h (1 + h|\xi|)^{-N}.$$

Combining (5.6) and (5.7), we get

$$\begin{aligned} \|Ea\|_{H^{-1}}^2 &= \int (1 + |\xi|^2)^{-1} |\widehat{Ea}(\xi)|^2 d\xi \\ &\leq Ch^2 \int_{|\xi| \leq Ch^{-1}} (1 + |\xi|^2)^{-1} d\xi + C_N h^2 \int_{|\xi| \geq Ch^{-1}} (1 + |\xi|^2)^{-1} (1 + h|\xi|)^{-2N} d\xi \\ &\leq Ch^2 |\log h|. \end{aligned}$$

This proves (5.5), and hence (5.4). The translated estimate is identical after replacing x_1x_2 by the translated quadratic phase. $\square$

We record the critical estimates in the forms needed below.

Lemma 5.2 (Critical remainder estimates). *Let $a \in C_0^\infty(\tilde{\Omega})$ be fixed. Then*

$$(5.8) \quad h \left[\frac{1}{h} \int e^{ix_1 x_2 / h} a(x) \tilde{r}_2(x) dx \right] = o(1),$$

$$(5.9) \quad h \left[\frac{1}{h^2} \int e^{ix_1 x_2 / h} a(x) \tilde{r}_2(x) dx \right] = o(1),$$

and

$$(5.10) \quad h \left[\frac{1}{h^2} \int e^{ix_1 x_2 / h} z a(x) \tilde{r}_2(x) dx \right] = o(1),$$

$$(5.11) \quad h \left[\frac{1}{h^2} \int e^{ix_1 x_2 / h} \bar{z} a(x) \tilde{r}_2(x) dx \right] = o(1).$$

Moreover,

$$(5.12) \quad h \left[\frac{1}{h} \int e^{ix_1 x_2 / h} a(x) \partial \tilde{r}_2(x) dx \right] = o(1),$$

and

$$(5.13) \quad h \left[\frac{1}{h} \int e^{ix_1 x_2 / h} a(x) \bar{\partial} \tilde{r}_2(x) dx \right] = o(1).$$

In addition, for every smooth compactly supported amplitude a ,

$$(5.14) \quad h \left[\frac{1}{h^2} \int e^{ix_1 x_2 / h} a(x) e^{-\bar{\Phi}_2 / h} \bar{\partial} (F_2 e^{\bar{\Phi}_2 / h} \tilde{r}_2) dx \right] = o(1),$$

and

$$(5.15) \quad h \left[\frac{1}{h} \int e^{ix_1 x_2 / h} a(x) e^{-\bar{\Phi}_2 / h} \bar{\partial}^2 (F_2 e^{\bar{\Phi}_2 / h} \tilde{r}_2) dx \right] = o(1).$$

The same conclusions hold uniformly after translating the critical point to centers in compact subsets of $\tilde{\Omega}$.

Proof. Let $E = e^{ix_1 x_2 / h}$. Since $\tilde{r}_2 = -\partial^{-1}(E^{-1} V' \tilde{s}_2)$ in the localized Cauchy convention, adjointness gives

$$\int E a \tilde{r}_2 dx = \int \partial^{*-1}(E a) E^{-1} V' \tilde{s}_2 dx.$$

By Lemma 5.1 and (5.2),

$$\left| \int E a \tilde{r}_2 dx \right| \leq C \|\partial^{*-1}(E a)\|_{L^2} \|\tilde{s}_2\|_{L^2} \leq C h^{3/2+\varepsilon} |\log h|^{1/2}.$$

This proves both (5.8) and (5.9). The same argument with the amplitudes za and $\bar{z}a$ gives

$$\left| \int E z a \tilde{r}_2 dx \right| + \left| \int E \bar{z} a \tilde{r}_2 dx \right| \leq C h^{3/2+\varepsilon} |\log h|^{1/2},$$

which proves (5.10) and (5.11).

For (5.12), use (5.1). Then

$$\frac{1}{h} \int E a \partial \tilde{r}_2 dx = -\frac{1}{h} \int a V' \tilde{s}_2 dx.$$

After multiplication by h , the right-hand side is bounded by $C \|\tilde{s}_2\|_{L^2} = o(1)$.

For (5.13), integrate by parts:

$$\frac{1}{h} \int E a \bar{\partial} \tilde{r}_2 dx = -\frac{1}{h} \int \bar{\partial}(E a) \tilde{r}_2 dx.$$

Since $\bar{\partial}E = -\frac{\bar{z}}{2h}E$, the right-hand side is

$$\frac{1}{2h^2} \int E \bar{z} a \tilde{r}_2 dx - \frac{1}{h} \int E(\bar{\partial}a) \tilde{r}_2 dx.$$

After multiplication by h , the first term is controlled by (5.11), and the second by (5.8). This proves (5.13).

For (5.14), use $Ee^{-\bar{\Phi}_2/h} = e^{(\Phi^* + \Phi_1)/h}$. Then the phase $\Phi^* + \Phi_1$ is holomorphic. Thus, an integration by parts yields

$$\frac{1}{h^2} \int Ea e^{-\bar{\Phi}_2/h} \bar{\partial}(F_2 e^{\bar{\Phi}_2/h} \tilde{r}_2) dx = -\frac{1}{h^2} \int E(\bar{\partial}a) F_2 \tilde{r}_2 dx.$$

After multiplication by h , this is controlled by (5.9).

For (5.15), integrate by parts twice in $\bar{\partial}$ before expanding the total derivative:

$$\frac{1}{h} \int Ea e^{-\bar{\Phi}_2/h} \bar{\partial}^2(F_2 e^{\bar{\Phi}_2/h} \tilde{r}_2) dx = \frac{1}{h} \int E(\bar{\partial}^2 a) F_2 \tilde{r}_2 dx.$$

After multiplication by h , this is controlled by (5.8). The uniformity after translating the critical point follows from the uniform translated version of Lemma 5.1 and from the uniform CGO estimates. This concludes the proof. $\square$

The next lemma records the critical Cauchy contribution. This is the only place where a term with prefactor h^{-2} and one $\partial \tilde{r}_2$ derivative has to be kept rather than estimated away.

Lemma 5.3 (Critical Neumann tail estimate for the Cauchy pairing). *Let $B \in C_0^\infty$ be independent of h . Write*

$$(5.16) \quad \tilde{s}_2 = \partial_\psi^{*-1}(V) + \tilde{s}_{2,\text{rem}}, \quad \tilde{s}_{2,\text{rem}} := \sum_{k=1}^{\infty} \tilde{T}_h^k \partial_\psi^{*-1}(V).$$

Then

$$(5.17) \quad \frac{1}{h} \int BV' \tilde{s}_{2,\text{rem}} dx = o(1).$$

The same estimate holds uniformly after translating the critical point to centers in compact subsets of $\tilde{\Omega}$.

Proof. The critical Neumann construction in [LL25a, Section 5] is used with the fixed local CGO cutoffs introduced above. Thus, the Cauchy inverses are properly supported local inverses in the chosen coordinate chart, and all smoothing cutoff errors are included in the smoothing cutoff remainder terms. With this convention, there is a number $\varepsilon_0 > 0$ such that

$$\|\partial_\psi^{*-1}(V)\|_{L^2} = O(h^{1/2+\varepsilon_0}), \quad \|\tilde{T}_h\|_{L^2 \rightarrow L^2} = O(h^{1/2-\varepsilon_0/2}).$$

Here $\tilde{T}_h = \partial_\psi^{*-1} V \partial_\psi^{-1} V'$ is the critical Neumann operator in this localized CGO parametrix. Hence, for h sufficiently small,

$$\|\tilde{s}_{2,\text{rem}}\|_{L^2} \leq \sum_{k=1}^{\infty} \|\tilde{T}_h\|_{L^2 \rightarrow L^2}^k \|\partial_\psi^{*-1}(V)\|_{L^2} \leq Ch^{1/2-\varepsilon_0/2} h^{1/2+\varepsilon_0} = Ch^{1+\varepsilon_0/2}.$$

Since B and V' are smooth and compactly supported in the local chart,

$$\left| \frac{1}{h} \int BV' \tilde{s}_{2,\text{rem}} dx \right| \leq Ch^{-1} \|\tilde{s}_{2,\text{rem}}\|_{L^2} \leq Ch^{\varepsilon_0/2} = o(1).$$

This proves (5.17). The translated statement follows from the same estimates, uniformly for translated centers in compact subsets of the coordinate patch. $\square$

Lemma 5.4 (Cauchy transform rule for the critical pairing). *Let $B \in C_0^\infty$ be independent of h . Then*

$$(5.18) \quad h \left[\frac{1}{h^2} \int e^{ix_1 x_2 / h} B(x) \partial \tilde{r}_2(x) dx \right] = 2\pi (\partial^{*-1}(BV'))(0)V(0) + o(1).$$

The same formula holds uniformly after translating the critical point to a center a , with 0 replaced by a and with the translated phase factors.

Proof. We work at the center 0. The translated case is identical after replacing $x_1 x_2$ by the translated quadratic phase and using the uniform translated CGO estimates.

By (5.1), $\partial \tilde{r}_2 = -e^{-ix_1 x_2 / h} V' \tilde{s}_2$. Hence, the left hand side of (5.18) is $-\frac{1}{h} \int BV' \tilde{s}_2 dx$. We split the critical Neumann series as in (5.16). Lemma 5.3 gives

$$\frac{1}{h} \int BV' \tilde{s}_{2,\text{rem}} dx = o(1).$$

It remains to compute the first Neumann term.

With the localized convention fixed above, $\partial_\psi^{*-1}(V) = \partial^{*-1}(e^{ix_1 x_2 / h} V)$. All factors have already been multiplied by the fixed local cutoffs from the CGO parametrix, so the pairing is a compactly supported full-plane integral. Let $K^*(x, y)$ be the properly supported kernel of the localized operator ∂^{*-1} . Our convention for the moved operator is

$$(\partial^{*-1} f)(y) := - \int K^*(x, y) f(x) dx.$$

Equivalently, for compactly supported local factors f and g ,

$$- \int f \partial^{*-1} g dx = \int \partial^{*-1} f g dx.$$

This is just Fubini applied to the fixed localized kernel, and no boundary integration by parts is involved.

Applying this with $f = BV'$ and $g = e^{ix_1 x_2 / h} V$, we obtain

$$-\frac{1}{h} \int BV' \partial^{*-1}(e^{ix_1 x_2 / h} V) dx = \frac{1}{h} \int \partial^{*-1}(BV') e^{ix_1 x_2 / h} V dx.$$

The function $\partial^{*-1}(BV')$ is now a fixed smooth localized coefficient on the chart. Applying (5.3) to $F(x) = \partial^{*-1}(BV')(x)V(x)$ gives

$$\frac{1}{h} \int e^{ix_1 x_2 / h} \partial^{*-1}(BV') V dx = 2\pi (\partial^{*-1}(BV'))(0)V(0) + O(h).$$

Combining this with the Neumann-tail estimate proves (5.18). The uniform translated statement follows from the uniform translated stationary phase expansion and the uniform local form of the critical CGO estimates. $\square$

For reference, we record the derivative expansions of the critical factor. Put $c_2 = \bar{\partial} \bar{\Phi}_2 = -\bar{z}/2$ and $d_2 = \bar{\partial}^2 \bar{\Phi}_2 = -1/2$. Then

$$(5.19) \quad \begin{aligned} e^{-\bar{\Phi}_2 / h} \partial (F_2 e^{\bar{\Phi}_2 / h} \tilde{r}_2) &= (\partial F_2) \tilde{r}_2 + F_2 \partial \tilde{r}_2, \\ e^{-\bar{\Phi}_2 / h} \bar{\partial} (F_2 e^{\bar{\Phi}_2 / h} \tilde{r}_2) &= h^{-1} c_2 F_2 \tilde{r}_2 + (\bar{\partial} F_2) \tilde{r}_2 + F_2 \bar{\partial} \tilde{r}_2. \end{aligned}$$

Also,

$$\begin{aligned}
e^{-\bar{\Phi}_2/h} \partial^2 (F_2 e^{\bar{\Phi}_2/h} \tilde{r}_2) &= (\partial^2 F_2) \tilde{r}_2 + 2(\partial F_2) \partial \tilde{r}_2 + F_2 \partial^2 \tilde{r}_2, \\
e^{-\bar{\Phi}_2/h} \partial \bar{\partial} (F_2 e^{\bar{\Phi}_2/h} \tilde{r}_2) &= h^{-1} c_2 (\partial F_2) \tilde{r}_2 + h^{-1} c_2 F_2 \partial \tilde{r}_2 + (\partial \bar{\partial} F_2) \tilde{r}_2 \\
&\quad + (\bar{\partial} F_2) \partial \tilde{r}_2 + (\partial F_2) \bar{\partial} \tilde{r}_2 + F_2 \partial \bar{\partial} \tilde{r}_2, \\
e^{-\bar{\Phi}_2/h} \bar{\partial}^2 (F_2 e^{\bar{\Phi}_2/h} \tilde{r}_2) &= h^{-2} c_2^2 F_2 \tilde{r}_2 + h^{-1} d_2 F_2 \tilde{r}_2 + 2h^{-1} c_2 (\bar{\partial} F_2) \tilde{r}_2 + 2h^{-1} c_2 F_2 \bar{\partial} \tilde{r}_2 \\
&\quad + (\bar{\partial}^2 F_2) \tilde{r}_2 + 2(\bar{\partial} F_2) \bar{\partial} \tilde{r}_2 + F_2 \bar{\partial}^2 \tilde{r}_2.
\end{aligned}$$

These expansions will be used in the scalar critical normal form below.

5.2. The scalar block. We first record the boundary vanishing needed for integrations by parts.

Lemma 5.5 (Boundary vanishing for the residual quantities). *On $\partial\tilde{\Omega}$, one has*

$$(5.20) \quad \mathbf{r} = 0, \quad D\mathbf{r} = 0, \quad M^\alpha = 0, \quad \alpha = 1, 2.$$

In particular, if U is a boundary coordinate patch and the superscript 0 denotes zero extension from $U \cap \tilde{\Omega}$ to U , then

$$(5.21) \quad D^\beta \mathbf{r}^0 = (D^\beta \mathbf{r})^0 \quad \text{in } \mathcal{D}'(U), \quad |\beta| \leq 2,$$

and

$$(5.22) \quad D_j (M^\alpha)^0 = (D_j M^\alpha)^0 \quad \text{in } \mathcal{D}'(U), \quad j = 1, 2, \quad \alpha = 1, 2.$$

Thus, the scalar integrations by parts involving the coefficient $\mathbf{r}$ create no boundary distribution up to the order used below, while the M^α -terms have vanishing boundary contribution in the single integrations by parts where M^α is the boundary coefficient. No boundary vanishing of DM^α is asserted or used. Moreover, the integrations by parts in the total first- and second-derivative critical classes are performed with the compactly supported local CGO-parametrix amplitudes fixed in Subsection 5.1; hence, those total critical classes require no boundary information on DM^α .

Proof. Recall that $\mathbf{r} = 1 - \rho^{-1}$ and $M^\alpha = \rho^{-1} \mathcal{H}^\alpha$. By the boundary normalization from the first linearized gauge, $J = \text{Id}$, $DJ = I$, $\rho = 1$ on $\partial\tilde{\Omega}$, then we have $\mathbf{r} = 0$ on $\partial\tilde{\Omega}$.

Write

$$W = J - \text{Id}, \quad q = \log \frac{\hat{K}_1}{\hat{K}_2 \circ J}, \quad \theta = \log \det DJ - q.$$

Since $J = \text{Id}$ on $\partial\tilde{\Omega}$ and $\hat{K}_j = (K_j \circ \chi^{-1})/m_\chi$ with the same positive factor m_χ for $j = 1, 2$, the boundary curvature normalization gives $\hat{K}_1 = \hat{K}_2$ on $\partial\tilde{\Omega}$. Thus, $q = 0$ on $\partial\tilde{\Omega}$. Differentiating q gives

$$Dq = D \log \hat{K}_1 - DJ^T (D \log \hat{K}_2) \circ J.$$

Using $J = \text{Id}$, $DJ = I$, and the first-order boundary curvature normalization, the common factor m_χ cancels and gives $D \log \hat{K}_1 = D \log \hat{K}_2$ on $\partial\tilde{\Omega}$, which implies $Dq = 0$ on $\partial\tilde{\Omega}$.

We next prove that $D^2 W = 0$ on $\partial\tilde{\Omega}$. In boundary normal coordinates for the Euclidean metric in the isothermal chart, let τ be the tangential coordinate and let ν be the inward normal coordinate. Since $W = 0$ and $DW = 0$ on $\partial\tilde{\Omega}$, the tangential derivatives of the boundary traces of W and $\partial_\nu W$ vanish, then one has $\partial_{\tau\tau} W = 0 = \partial_{\tau\nu} W = 0$ on $\partial\tilde{\Omega}$. The partial derivative form of the drift equation gives

$$\Delta W^\alpha = \Delta J^\alpha = \partial_i J^\alpha \partial_i q + \mathfrak{R}^\alpha(x, J, DJ).$$

On $\partial\tilde{\Omega}$, we have $Dq = 0$, $J = \text{Id}$, and $DJ = I$. By (3.25), the connection remainder vanishes there. Thus $\Delta W^\alpha = 0$ on $\partial\tilde{\Omega}$. In the same boundary normal coordinates,

$$\Delta W^\alpha = \partial_{\nu\nu} W^\alpha + \partial_{\tau\tau} W^\alpha + \kappa_{\partial\tilde{\Omega}} \partial_\nu W^\alpha \quad \text{on } \partial\tilde{\Omega}.$$

The last two terms vanish, so $\partial_{\nu\nu} W^\alpha = 0$ on $\partial\tilde{\Omega}$. Hence $D^2 W = 0$ on $\partial\tilde{\Omega}$.

By Jacobi's formula,

$$D \log \det DJ = D \log \det(I + DW) = 0 \quad \text{on } \partial\tilde{\Omega},$$

because $DW = 0$ and $D^2 W = 0$ there. Since $Dq = 0$, we get

$$D\theta = D \log \det DJ - Dq = 0 \quad \text{on } \partial\tilde{\Omega}.$$

As $\rho = e^\theta$, this implies $D\rho = 0$ on $\partial\tilde{\Omega}$. Therefore

$$D\mathbf{r} = \rho^{-2} D\rho = 0 \quad \text{on } \partial\tilde{\Omega}.$$

This proves the first two identities in (5.20).

It remains to prove $M^\alpha = 0$. The covariant Hessian $\mathcal{H}^\alpha$ is the covariant second derivative of $J^\alpha = \text{Id}^\alpha + W^\alpha$ with respect to the same pulled-back flat connection in the domain and target. The identity map has zero covariant Hessian. Since $J = \text{Id}$, $DJ = I$, and $D^2 W = 0$ on $\partial\tilde{\Omega}$, we obtain

$$\mathcal{H}^\alpha = 0 \quad \text{on } \partial\tilde{\Omega}.$$

Because $\rho = 1$ on $\partial\tilde{\Omega}$, this gives

$$M^\alpha = \rho^{-1} \mathcal{H}^\alpha = 0 \quad \text{on } \partial\tilde{\Omega}, \quad \alpha = 1, 2.$$

This proves (5.20).

We now justify the distributional statements. In a flattened boundary chart, write f^0 for the zero extension of a smooth one-sided function f . The standard trace formula for derivatives of zero extensions shows that no boundary measure appears in $D^\beta f^0$, $|\beta| \leq m$, provided all traces $D^\gamma f|_{\partial\tilde{\Omega}}$ with $|\gamma| \leq m - 1$ vanish. Applying this with $f = \mathbf{r}$ and $m = 2$, using $\mathbf{r} = 0$ and $D\mathbf{r} = 0$ on $\partial\tilde{\Omega}$, gives (5.21). Applying the same trace formula with $f = M^\alpha$ and $m = 1$, using $M^\alpha = 0$ on $\partial\tilde{\Omega}$, gives (5.22).

Finally, the total critical derivative estimates are applied with the local CGO-parametrix cutoffs from Subsection 5.1. Thus, for each fixed critical center, the coefficients in those total critical classes are compactly supported in the local chart before the total derivative is expanded. The corresponding integrations by parts are, therefore, compactly supported full-plane integrations and do not use any trace of DM^α on $\partial\tilde{\Omega}$. This completes the proof. $\square$

Remark 5.6 (Cutoffs in the critical expansions). *The cutoffs used below are part of the local asymptotic extraction, not part of the second integral identity. For a fixed interior critical center a , the exact CGO solutions are first inserted into (4.9). In each resulting model oscillatory integral, the smooth coefficient is then split as $1 = \vartheta_a + (1 - \vartheta_a)$, where $\vartheta_a = 1$ near a . The $(1 - \vartheta_a)$ -term is nonstationary for the quadratic phase and is $O(h^N)$, for any fixed N , after the finitely many powers of h^{-1} appearing in the expansion are taken into account.*

For terms containing the differentiated critical remainder, the same local cutoffs are part of the properly supported Cauchy parametrices fixed in Subsection 5.1. The corresponding smoothing cutoff errors are included in the remainders in the critical estimates. Near the boundary, the residual identities are obtained for interior centers. Later, in the boundary Carleman argument, only localized one-sided inputs are extended by zero to the reflected disk.

Lemma 5.7 (Finite scalar critical normal forms). *Let $I_{\mathbf{r},\text{crit}}$ be the part of $I_{\mathbf{r}}$ in which the second forward CGO solution is replaced by the critical factor $F_2 e^{\bar{\Phi}_2/h} \tilde{r}_2$. After expanding the scalar critical part of $I_{\mathbf{r}}$ and performing the integrations by parts allowed by Lemma 5.5, every scalar critical term that survives after multiplication by h is a finite sum of terms of the form*

$$(5.23) \quad h \left[\frac{1}{h^2} \int e^{i x_1 x_2 / h} B_\nu(x; \mathbf{r}) \partial \tilde{r}_2(x) dx \right] + o(1),$$

where $B_\nu(x; \mathbf{r}) = a_{\nu,0}(x) \mathbf{r}(x) + a_{\nu,1}(x) \partial \mathbf{r}(x) + a_{\nu,2}(x) \bar{\partial} \mathbf{r}(x)$. The coefficients $a_{\nu,k}$ are smooth, compactly supported, independent of h , and independent of $\mathbf{r}$. The sum over ν is finite. All remaining scalar critical terms are $o(1)$ after multiplication by h .

Proof. We work at the critical center 0. The translated statement is obtained by replacing z by $z - a$ and using the uniform translated estimates in Lemma 5.2. Write $E = e^{i x_1 x_2 / h}$ and $T_2 = F_2 e^{\bar{\Phi}_2/h} \tilde{r}_2$. The scalar critical part is a finite sum of model terms of the form

$$\int E \mathcal{C}(x; \mathbf{r}) \text{tr}(\nabla^2 v^{(1)} \nabla^2 T_2) dx,$$

where $\mathcal{C}$ denotes a smooth compactly supported coefficient containing $\gamma^2 \mu F_* F_1$ and a finite number of smooth coefficients coming from the noncritical expansions of r_* and r_1 . Since the noncritical expansions are finite to the order needed after multiplication by h , it is enough to classify these model terms.

In this proof, D^2 denotes the ordinary Euclidean Hessian in the local complex coordinate. The covariant Hessian satisfies $\nabla^2 f = D^2 f - \Gamma_{ij}^k \partial_k f$. Thus, every term in which at least one Hessian is replaced by the connection correction contains only a first derivative of the corresponding CGO factor. Such a term has one fewer effective phase derivative than the leading Hessian-Hessian term and belongs to the lower classes considered below.

We use

$$(5.24) \quad D^2 f = \mathbb{A} \partial^2 f + \mathbb{B} \bar{\partial}^2 f + 2I \partial \bar{\partial} f,$$

where

$$\mathbb{A} = \begin{pmatrix} 1 & i \\ i & -1 \end{pmatrix}, \quad \mathbb{B} = \begin{pmatrix} 1 & -i \\ -i & -1 \end{pmatrix}.$$

The contractions are $\text{tr}(\mathbb{A}\mathbb{B}) = 4$, $\text{tr}(\mathbb{A}\mathbb{A}) = \text{tr}(\mathbb{B}\mathbb{B}) = 0$, and $\text{tr} \mathbb{A} = \text{tr} \mathbb{B} = 0$. The leading ordinary Hessian of the first forward CGO solution is the $\mathbb{A} \partial^2$ part. Hence the only leading Hessian-Hessian scalar contraction with the second forward solution is the $\mathbb{A}\mathbb{B}$ contraction. For the critical factor this identifies the total ∂T_2 and total $\bar{\partial}^2 T_2$ classes as the only classes which must be treated before expanding T_2 .

We first treat these total $\bar{\partial}$ classes. Whenever the critical factor occurs as a total $\bar{\partial} T_2$ or $\bar{\partial}^2 T_2$, we integrate by parts in $\bar{\partial}$ before expanding T_2 . Since $E e^{-\bar{\Phi}_2/h} = e^{(\Phi^* + \Phi_1)/h}$ and $\Phi^* + \Phi_1$ is holomorphic, these integrations do not differentiate the exponential factor. The derivatives fall only on smooth coefficients, on $\mathbf{r}$, or on the noncritical amplitudes. By Lemma 5.5, the boundary terms vanish, because $\mathbf{r} = 0$ and $\partial_\nu \mathbf{r} = 0$ on $\partial \tilde{\Omega}$. The resulting terms are precisely of the types controlled by (5.14) and (5.15), or else reduce to terms containing $\tilde{r}_2$ without derivatives and are controlled by (5.8), (5.9), (5.10), and (5.11). Thus no term from a total $\bar{\partial} T_2$ or total $\bar{\partial}^2 T_2$ class contributes after multiplication by h .

After these total classes have been removed, every remaining critical model term contains one of

$$\tilde{r}_2, \quad \partial \tilde{r}_2, \quad \bar{\partial} \tilde{r}_2, \quad \partial^2 \tilde{r}_2, \quad \partial \bar{\partial} \tilde{r}_2, \quad \bar{\partial}^2 \tilde{r}_2.$$

Terms containing $\tilde{r}_2$ without derivatives have prefactor at most h^{-2} after the previous integrations by parts, and are $o(1)$ after multiplication by h by (5.8) and (5.9). Terms containing $z\tilde{r}_2$ or $\bar{z}\tilde{r}_2$ with prefactor h^{-2} are controlled by (5.10) and (5.11).

Terms containing $\bar{\partial}\tilde{r}_2$ with prefactor at most h^{-1} are controlled by (5.13). A term with bare prefactor $h^{-2}\bar{\partial}\tilde{r}_2$ can only arise from expanding a total $\bar{\partial}T_2$ or total $\bar{\partial}^2T_2$ class. These classes have already been kept in total form and estimated by (5.14) and (5.15). Hence no $\bar{\partial}\tilde{r}_2$ term survives.

Terms containing $\partial\tilde{r}_2$ with prefactor at most h^{-1} are controlled by (5.12). Thus the only possible nonnegligible first-derivative critical class is the class with prefactor $h^{-2}\partial\tilde{r}_2$. In this class, the coefficient multiplying $\partial\tilde{r}_2$ is obtained after at most one integration by parts from a coefficient containing $\mathbf{r}$ and smooth factors. Hence it is a finite sum of terms

$$a_0\mathbf{r} + a_1\partial\mathbf{r} + a_2\bar{\partial}\mathbf{r}$$

with smooth compactly supported a_0, a_1, a_2 .

It remains to consider the terms with two derivatives on $\tilde{r}_2$. The term $F_2\bar{\partial}^2\tilde{r}_2$ belongs to the $\mathbb{B}$ polarization and is part of the total $\bar{\partial}^2T_2$ class already treated. The term $F_2\partial\bar{\partial}\tilde{r}_2$ lies in the identity polarization. Its contraction with the leading $\mathbb{A}$ part of $\nabla^2v^{(1)}$ vanishes because $\text{tr}\mathbb{A} = 0$. If the first Hessian is not the leading $\mathbb{A}$ part, the term has one fewer effective phase derivative; integrating once reduces it either to a term controlled by Lemma 5.2 or to the already identified $h^{-2}\partial\tilde{r}_2$ class. Finally, $F_2\partial^2\tilde{r}_2$ lies in the $\mathbb{A}$ polarization, and its contraction with the leading $\mathbb{A}$ part of $\nabla^2v^{(1)}$ vanishes because $\text{tr}(\mathbb{A}\mathbb{A}) = 0$. If the first Hessian is lower order, we integrate once in ∂ . The boundary term vanishes because $\mathbf{r} = 0$ on $\partial\tilde{\Omega}$. When ∂ hits E , it gives the factor $z/(2h)$ and produces a term of the form (5.23); when ∂ hits a smooth coefficient or $\mathbf{r}$, the resulting $\bar{\partial}\tilde{r}_2$ term has either prefactor at most h^{-1} and is negligible, or has prefactor h^{-2} and again has coefficient $a_0\mathbf{r} + a_1\partial\mathbf{r} + a_2\bar{\partial}\mathbf{r}$.

Derivatives falling on F_* , F_1 , F_2 , on γ , on μ , on a cutoff, or on a connection coefficient only change the smooth coefficients above or lower the effective phase order. The noncritical remainders r_* and r_1 also do not create a new class, since their finite expansions start with a factor h . Therefore the only scalar critical terms not estimated as $o(1)$ after multiplication by h are finite sums of (5.23). This proves the lemma. $\square$

We compute

$$I_{\mathbf{r}} = \int_{\tilde{\Omega}} v^* \gamma^2 \mathbf{r} \text{tr}(\nabla^2 v^{(1)} \nabla^2 v^{(2)}) d\mu.$$

Write $d\mu = \mu(x) dx$ in the local coordinate and set $\Lambda_0 := \gamma^2 \mu$, then $\Lambda_0(0) \neq 0$. Set $C_0 := \Lambda_0 F_* F_1 F_2 \mathbf{r}$, then the corresponding leading contribution is

$$I_{\mathbf{r},1} = 4 \int C_0 e^{\Phi^*/h} \partial^2(e^{\Phi_1/h}) \bar{\partial}^2(e^{\bar{\Phi}_2/h}) dx.$$

By Lemma 5.5, the boundary terms produced by the following integrations by parts vanish:

$$\begin{aligned} I_{\mathbf{r},1} &= 4 \int C_0 e^{\Phi^*/h} \partial^2(e^{\Phi_1/h}) \bar{\partial}^2(e^{\bar{\Phi}_2/h}) dx \\ &= -4 \int \partial(C_0 e^{\Phi^*/h}) \partial(e^{\Phi_1/h}) \bar{\partial}^2(e^{\bar{\Phi}_2/h}) dx \\ &= 4 \int \bar{\partial}[\partial(C_0 e^{\Phi^*/h}) \partial(e^{\Phi_1/h})] \bar{\partial}(e^{\bar{\Phi}_2/h}) dx. \end{aligned}$$

Since $\bar{\partial}e^{\Phi^*/h} = 0$ and $\bar{\partial}\partial(e^{\Phi_1/h}) = 0$, we have $\bar{\partial}[\partial(C_0 e^{\Phi^*/h}) \partial(e^{\Phi_1/h})] = \partial\bar{\partial}(C_0 e^{\Phi^*/h}) \partial(e^{\Phi_1/h})$. Thus,

$$I_{\mathbf{r},1} = 4 \int \partial\bar{\partial}(C_0 e^{\Phi^*/h}) \partial(e^{\Phi_1/h}) \bar{\partial}(e^{\bar{\Phi}_2/h}) dx.$$

Now, $\partial\bar{\partial}(C_0 e^{\Phi^*/h}) = e^{\Phi^*/h}[\partial\bar{\partial}C_0 + h^{-1}(\partial\Phi^*)\bar{\partial}C_0]$, $\partial(e^{\Phi_1/h}) = h^{-1}(\partial\Phi_1)e^{\Phi_1/h}$ and $\bar{\partial}(e^{\bar{\Phi}_2/h}) = h^{-1}(\bar{\partial}\bar{\Phi}_2)e^{\bar{\Phi}_2/h}$. Therefore,

$$I_{\mathbf{r},1} = \frac{4}{h^3} \int (\partial\Phi^*)\bar{\partial}C_0\partial\Phi_1\bar{\partial}\bar{\Phi}_2 e^{ix_1x_2/h} dx + \frac{4}{h^2} \int (\partial\bar{\partial}C_0)\partial\Phi_1\bar{\partial}\bar{\Phi}_2 e^{ix_1x_2/h} dx.$$

The second integral is $O(1)$ because its amplitude contains $\bar{\partial}\bar{\Phi}_2 = -\bar{z}/2$. Hence, it does not contribute to $\lim_{h \rightarrow 0} hI_{\mathbf{r},1}$.

For the first integral, $4(\partial\Phi^*)\partial\Phi_1\bar{\partial}\bar{\Phi}_2 = 2\bar{z}(1 - \frac{z^2}{16})$. Thus,

$$I_{\mathbf{r},1} = \frac{1}{h^3} \int 2\bar{z}\left(1 - \frac{z^2}{16}\right)\bar{\partial}C_0 e^{ix_1x_2/h} dx + O(1).$$

Apply (5.3) to $F(x) = 2\bar{z}(1 - \frac{z^2}{16})\bar{\partial}C_0(x)$, then $F(0) = 0$, $\partial^2 F(0) = 0$, and $\bar{\partial}^2 F(0) = 4\bar{\partial}^2 C_0(0)$. Therefore,

$$(5.25) \quad \lim_{h \rightarrow 0} hI_{\mathbf{r},1} = 8\pi\bar{\partial}^2 C_0(0) = 8\pi\bar{\partial}^2(\Lambda_0 F_* F_1 F_2 \mathbf{r})(0).$$

Lemma 5.8 (Local scalar lower order terms). *After the leading scalar contribution (5.25) is removed, the remaining local scalar terms satisfy*

$$(5.26) \quad \lim_{h \rightarrow 0} h(I_{\mathbf{r},\text{amp}} + I_{\mathbf{r},\text{cov}} + I_{\mathbf{r},\text{nc}}) = \pi \mathcal{L}_{\mathbf{r}, \leq 1}^{\text{loc}} \mathbf{r}(0),$$

where $\mathcal{L}_{\mathbf{r}, \leq 1}^{\text{loc}}$ is a local scalar differential operator of order at most one.

Proof. We work at the critical center 0. The translated statement is identical. Write $E = e^{ix_1x_2/h}$ and $\Lambda_0 = \gamma^2 \mu$. The local scalar part is obtained from

$$I_{\mathbf{r}} = \int v^* \gamma^2 \mathbf{r} \text{tr}(\nabla^2 v^{(1)} \nabla^2 v^{(2)}) d\mu$$

after replacing v^* , $v^{(1)}$, and the noncritical part of $v^{(2)}$ by their smooth-amplitude CGO expansions. Since r_* and r_1 have finite expansions beginning with an explicit factor h , it is enough to classify finitely many model terms at each fixed order.

We use the decomposition (5.24), then the leading scalar term already extracted in (5.25) is the term in which the $\mathbb{A}\partial^2$ part of $v^{(1)}$ is paired with the $\mathbb{B}\bar{\partial}^2$ part of the noncritical part of $v^{(2)}$, and both second derivatives fall on the exponential phases. This is the only local scalar term with the full leading order and with the precise $\mathbb{A}$ - $\mathbb{B}$ contraction producing the displayed second-order block.

Every other local scalar term belongs to one of the following finite classes. First, at least one derivative falls on F_* , F_1 , F_2 , Λ_0 , a cutoff, or a fixed coefficient. Second, at least one of the second derivatives in a Hessian is replaced by a connection correction from $\nabla^2 = D^2 - \Gamma\partial$. Third, one of the noncritical remainders r_* or r_1 is used. Fourth, the lower phase derivative terms $h^{-1}\partial^2\Phi_1$ or $h^{-1}\bar{\partial}^2\bar{\Phi}_2$ are used instead of the two leading phase derivatives. In all four cases, the term loses at least one effective phase derivative compared with the leading scalar term.

After the same integrations by parts as in the computation of (5.25), every remaining local scalar model term has one of the two forms

$$\frac{1}{h^m} \int E a_\nu(x) \mathbf{r}(x) dx, \quad m \leq 2, \quad \text{or} \quad \frac{1}{h^3} \int E \ell_\nu(z, \bar{z}) a_\nu(x) \mathbf{r}(x) dx,$$

where a_ν is smooth and compactly supported, and ℓ_ν vanishes to first order at the critical point. The finite number of such terms follows from the finite expansion of the two Hessians and from the fact that only two derivatives are present on each forward CGO factor.

For the first type, the stationary phase formula gives $\int E a_\nu \mathbf{r} dx = 2\pi h(a_\nu \mathbf{r})(0) + O(h^2)$. Thus, after multiplication by h , only the case $m = 2$ can survive, and it gives a local zeroth-order expression in $\mathbf{r}$. The cases $m \leq 1$ vanish after multiplication by h .

For the second type, the leading stationary phase coefficient is zero because $\ell_\nu(0) = 0$. Hence, the only possible surviving contribution after multiplication by h comes from the next stationary phase coefficient $h(\bar{\partial}^2 - \partial^2)(\ell_\nu a_\nu \mathbf{r})(0)$. Since ℓ_ν has one explicit linear factor z or $\bar{z}$, applying two derivatives to $\ell_\nu a_\nu \mathbf{r}$ leaves at most one derivative on $\mathbf{r}$. Therefore, this class contributes only local terms involving $\mathbf{r}$, $\partial \mathbf{r}$, and $\bar{\partial} \mathbf{r}$.

The noncritical remainders r_* and r_1 do not change this conclusion. Their expansions start with a factor h , so their model terms have one fewer effective power of h^{-1} than the corresponding leading exponential term. If a differentiated remainder appears, the noncritical expansion above gives a finite smooth expansion in the same form, again with the same loss of one effective phase derivative.

The covariant Hessian corrections also do not change the conclusion. Each connection correction contains only one derivative of the corresponding CGO factor, so it loses one effective phase derivative. If a derivative falls on a connection coefficient, the term loses another effective derivative or becomes a smooth coefficient multiplying one of the model terms above.

Combining the finite list of surviving local terms gives a local differential expression of order at most one in $\mathbf{r}$. This expression is denoted by $\pi \mathcal{L}_{\mathbf{r}, \leq 1}^{\text{loc}} \mathbf{r}(0)$, which proves (5.26). $\square$

It remains to isolate the part containing the critical remainder of the second forward solution. Write $v^{(2)} = F_2 e^{\bar{\Phi}_2/h} + F_2 e^{\bar{\Phi}_2/h} \tilde{r}_2$. The first term gives the leading scalar block and the local lower-order terms. The second term is denoted by $I_{\mathbf{r}, \text{crit}}$.

In the estimates below we use a finite decomposition according to where the phase derivatives, the derivatives of $\tilde{r}_2$, and the derivatives of the residual coefficient fall. Total $\bar{\partial}$ derivatives on the critical factor are integrated by parts before the critical factor is expanded. Smoothing cutoff errors from the localized Cauchy inverses are included in the smoothing remainders. Thus, only finitely many model terms remain, and they are handled by Lemma 5.2 or by Lemma 5.4. We write

$$I_{\mathbf{r}, \text{crit}} = I_{\mathbf{r}, \tilde{r}_2} + I_{\mathbf{r}, \partial \tilde{r}_2} + I_{\mathbf{r}, \bar{\partial} \tilde{r}_2} + I_{\mathbf{r}, D^2 \tilde{r}_2}.$$

Lemma 5.9 (Order of the scalar critical contribution). *After the integration by parts allowed by Lemma 5.5, and after multiplication by h , the scalar critical contribution has the form*

$$(5.27) \quad h I_{\mathbf{r}, \text{crit}} = \mathcal{N}_{\mathbf{r}}(\mathbf{r})(0) + o(1).$$

Here, $\mathcal{N}_{\mathbf{r}}$ is a finite sum of localized Cauchy transform terms. More precisely, after absorbing universal constants and the fixed factor $V(0)$ into the coefficients, every summand has the form

$$(5.28) \quad b_\nu(0) (\partial^{*-1} [V'(a_{\nu,0} \mathbf{r} + a_{\nu,1} \partial \mathbf{r} + a_{\nu,2} \bar{\partial} \mathbf{r})])(0).$$

The coefficients $a_{\nu,k}$ and b_ν are smooth and compactly supported in the local chart, are independent of $\mathbf{r}$ and h , and the sum over ν is finite.

Consequently, after the substitution $\mathbf{r} = 1 - e^{-\theta}$, the linearization at $\theta = 0$ of $\mathcal{N}_{\mathbf{r}}(1 - e^{-\theta})$ is of order at most one in θ , possibly followed by a localized Cauchy transform. In particular, the scalar critical contribution does not modify the displayed second-order scalar principal block $8\bar{\partial}^2(C_a \mathbf{r})$.

Proof. We work at the center 0. The translated case is identical, with z replaced by $z - a$. By Lemma 5.7, after multiplication by h , the only scalar critical terms are finite sums of

$$h \left[\frac{1}{h^2} \int e^{ix_1 x_2/h} B_\nu(x; \mathbf{r}) \partial \tilde{r}_2(x) dx \right],$$

where $B_\nu(x; \mathbf{r}) = a_{\nu,0}(x) \mathbf{r}(x) + a_{\nu,1}(x) \partial \mathbf{r}(x) + a_{\nu,2}(x) \bar{\partial} \mathbf{r}(x)$. All other scalar critical terms are $o(1)$ after multiplication by h .

Applying Lemma 5.4 to each such term gives

$$h \left[\frac{1}{h^2} \int e^{ix_1 x_2/h} B_\nu(x; \mathbf{r}) \partial \tilde{r}_2(x) dx \right] = 2\pi (\partial^{*-1} [V' B_\nu(\cdot; \mathbf{r})]) (0) V(0) + o(1).$$

After absorbing the fixed factor $2\pi V(0)$ and fixed smooth coefficient factors into the coefficients, every such summand has the form $b_\nu(0) (\partial^{*-1} [V'(a_{\nu,0} \mathbf{r} + a_{\nu,1} \partial \mathbf{r} + a_{\nu,2} \bar{\partial} \mathbf{r})]) (0)$. This proves (5.27) and the structure (5.28).

Finally, for $R(\theta) = 1 - e^{-\theta}$, the linearization at $\theta = 0$ is $R'(0)\vartheta = \vartheta$. Hence, the linear part of each such summand contains at most one derivative of ϑ , possibly followed by the localized Cauchy inverse. Therefore, the scalar critical contribution does not change the displayed second-order scalar principal block. $\square$

We keep the notation $\mathcal{N}_{\mathbf{r}}$ from Lemma 5.9. Thus, $\mathcal{N}_{\mathbf{r}}(\mathbf{r})(0)$ denotes the finite Cauchy transform part of $hI_{\mathbf{r},\text{crit}}$ at the center 0. When no confusion is possible, we write this term simply as $\mathcal{N}_{\mathbf{r}}(0)$. Combining the scalar pieces gives

$$(5.29) \quad hI_{\mathbf{r}} = 8\pi \bar{\partial}^2 (\Lambda_0 F_* F_1 F_2 \mathbf{r})(0) + \pi \mathcal{L}_{\mathbf{r}, \leq 1}^{\text{loc}} \mathbf{r}(0) + \mathcal{N}_{\mathbf{r}}(\mathbf{r})(0) + o(1).$$

5.3. The gauge Hessian block and the nonlocal Cauchy term. We next compute the two terms linear in M^α :

$$I_{\xi M} = \int_{\bar{\Omega}} v^* \gamma^2 \xi_\beta \text{tr}(\nabla^2 v^{(1)} M^\beta) d\mu, \quad \text{and} \quad I_{\eta M} = \int_{\bar{\Omega}} v^* \gamma^2 \eta_\alpha \text{tr}(M^\alpha \nabla^2 v^{(2)}) d\mu.$$

Recall $\eta_\alpha = (N^{-T})_\alpha^j \partial_j v^{(1)}$, $\xi_\beta = (N^{-T})_\beta^j \partial_j v^{(2)}$, and $N = DJ$, then define

$$\lambda_\alpha = (N^{-T})_\alpha^1 + i(N^{-T})_\alpha^2, \quad \kappa_\alpha = (N^{-T})_\alpha^1 - i(N^{-T})_\alpha^2.$$

Then the leading first derivative factors are $\eta_\alpha^{\text{lead}} = \frac{1}{h} F_1 e^{\Phi_1/h} (\partial \Phi_1) \lambda_\alpha$ and $\xi_\beta^{\text{lead}} = \frac{1}{h} F_2 e^{\bar{\Phi}_2/h} (\bar{\partial} \Phi_2) \kappa_\beta$. Let

$$\mathcal{A}(T) := \text{tr}(\mathbb{A}T), \quad \mathcal{B}(T) := \text{tr}(\mathbb{B}T).$$

With the conventions above, we have $\mathcal{A}(D^2 f) = 4\bar{\partial}^2 f$ and $\mathcal{B}(D^2 f) = 4\partial^2 f$.

For $I_{\xi M}$, the term that can produce a differentiated M -contribution comes from the leading $\mathbb{A}\partial^2$ component of $\nabla^2 v^{(1)}$ and the leading phase derivative in ξ_β . Set $B_\xi := \Lambda_0 F_* F_1 F_2 \kappa_\beta \mathcal{A}(M^\beta)$. Since $\partial^2(e^{\Phi_1/h}) = [h^{-2}(\partial \Phi_1)^2 + h^{-1}\partial^2 \Phi_1] e^{\Phi_1/h}$, and the leading part of ξ_β contains $h^{-1}(\bar{\partial} \Phi_2) \kappa_\beta F_2 e^{\bar{\Phi}_2/h}$, the local term in $I_{\xi M}$ which carries one exterior derivative of M in the limiting equation is

$$I_{\xi M,1} = -\frac{1}{2h^3} \int \bar{z} \left(1 + \frac{z}{4}\right)^2 B_\xi e^{ix_1 x_2/h} dx + O(1).$$

Here we used $\bar{\partial} \Phi_2 = -\bar{z}/2$ and $\partial \Phi_1 = 1 + z/4$. The remaining local terms either have one fewer effective phase derivative, contain the vanishing factor $\bar{\partial} \Phi_2$ in a lower order position, or give only zeroth order contributions in M . Applying stationary phase expansion (5.3) to $F_\xi = -\frac{1}{2}\bar{z}(1 + \frac{z}{4})^2 B_\xi$ gives $F_\xi(0) = 0$, $\partial^2 F_\xi(0) = 0$, and $\bar{\partial}^2 F_\xi(0) = -\bar{\partial} B_\xi(0)$. Hence,

$$(5.30) \quad \lim_{h \rightarrow 0} hI_{\xi M,1} = -2\pi \bar{\partial} B_\xi(0) = -2\pi \bar{\partial} (\Lambda_0 F_* F_1 F_2 \kappa_\beta \mathcal{A}(M^\beta))(0).$$

There is one local zeroth-order contribution in M , which is not included in (5.30). It comes from the amplitude-derivative component of ξ_β when the second forward solution is replaced by its noncritical factor $F_2 e^{\bar{\Phi}_2/h}$. Indeed,

$$\xi_{\beta,\text{nc}} = (N^{-T})_\beta^j \partial_j (F_2 e^{\bar{\Phi}_2/h}) = e^{\bar{\Phi}_2/h} [h^{-1} \kappa_\beta (\bar{\partial} \Phi_2) F_2 + \lambda_\beta \partial F_2 + \kappa_\beta \bar{\partial} F_2].$$

The first term is the leading phase-derivative term already used in (5.30). The remaining amplitude-derivative component is $e^{\bar{\Phi}_2/h}(\lambda_\beta \partial F_2 + \kappa_\beta \bar{\partial} F_2)$. Pairing this component with the leading $\mathbb{A}\partial^2$ component of $\nabla^2 v^{(1)}$ gives

$$\frac{1}{h^2} \int e^{ix_1 x_2/h} \Lambda_0 F_* F_1 \left(1 + \frac{z}{4}\right)^2 (\lambda_\beta \partial F_2 + \kappa_\beta \bar{\partial} F_2) \mathcal{A}(M^\beta) dx.$$

By the stationary phase formula, after multiplication by h this contributes

$$2\pi \Lambda_0(0) F_*(0) F_1(0) (\lambda_\beta(0) \partial F_2(0) + \kappa_\beta(0) \bar{\partial} F_2(0)) \mathcal{A}(M^\beta)(0).$$

Accordingly, define

$$\mathcal{Z}_M(M)(0) = 2\Lambda_0(0) F_*(0) F_1(0) (\lambda_\beta(0) \partial F_2(0) + \kappa_\beta(0) \bar{\partial} F_2(0)) \mathcal{A}(M^\beta)(0).$$

All other noncritical local terms in $I_{\xi M}$ either lose one effective power of h^{-1} , contain the vanishing factor $\bar{\partial} \Phi_2 = -\bar{z}/2$, or arise from covariant Hessian corrections. They produce no local derivative of M beyond the already extracted $-2\bar{\partial}(\cdots)$ term. We include them in a local order zero operator in M .

For $I_{\eta M}$, set $B_\eta := \Lambda_0 F_* F_1 F_2 \lambda_\alpha \mathcal{B}(M^\alpha)$, then the leading Hessian of $v^{(2)}$ without $\tilde{r}_2$ is the $\mathbb{B}\bar{\partial}^2$ component. Hence,

$$I_{\eta M,1} = \frac{1}{h^3} \int B_\eta (\partial \Phi_1) (\bar{\partial} \Phi_2)^2 e^{ix_1 x_2/h} dx + \frac{1}{h^2} \int B_\eta (\partial \Phi_1) \bar{\partial}^2 \Phi_2 e^{ix_1 x_2/h} dx.$$

Using the phase derivatives,

$$I_{\eta M,1} = \frac{1}{4h^3} \int \bar{z}^2 \left(1 + \frac{z}{4}\right) B_\eta e^{ix_1 x_2/h} dx - \frac{1}{2h^2} \int \left(1 + \frac{z}{4}\right) B_\eta e^{ix_1 x_2/h} dx.$$

For the first integral, set $F_{\eta,1} = \frac{1}{4} \bar{z}^2 \left(1 + \frac{z}{4}\right) B_\eta$, then $F_{\eta,1}(0) = 0$, $\partial^2 F_{\eta,1}(0) = 0$, and $\bar{\partial}^2 F_{\eta,1}(0) = B_\eta(0)/2$. Thus, the first integral contributes $\pi B_\eta(0)$ to $\lim h I_{\eta M,1}$. For the second integral, set $F_{\eta,2} = -\frac{1}{2} \left(1 + \frac{z}{4}\right) B_\eta$, then the leading stationary phase coefficient is $2\pi F_{\eta,2}(0) = -\pi B_\eta(0)$. Thus,

$$(5.31) \quad \lim_{h \rightarrow 0} h I_{\eta M,1} = 0.$$

The remaining noncritical local terms in $I_{\eta M}$ either lose another effective power of h^{-1} or contain a vanishing factor. They do not produce an additional local derivative of M . Together with the lower local terms from $I_{\xi M}$, they are denoted by $\pi \mathcal{L}_{M,0}^{\text{loc}}(M)(0)$, where $\mathcal{L}_{M,0}^{\text{loc}}$ is local of order zero in M .

Lemma 5.10 (Finite M -critical normal form). *After expanding the critical part of $I_{\xi M} + I_{\eta M}$ and performing the integrations by parts allowed by Lemma 5.5, every M -critical term is $o(1)$ after multiplication by h , except the following nonnegligible contribution*

$$(5.32) \quad h \left[\frac{1}{h^2} \int e^{ix_1 x_2/h} Q_A(x; M) \left(1 - \frac{z^2}{16}\right) \partial \tilde{r}_2 dx \right] + o(1),$$

where $Q_A(x; M) := \Lambda_0 F_* F_1 F_2 \lambda_\alpha \mathcal{A}(M^\alpha)$.

Proof. We work at the critical center 0. The translated statement is identical, with z replaced by $z - a$. Put $E = e^{ix_1 x_2/h}$ and $T_2 = F_2 e^{\bar{\Phi}_2/h} \tilde{r}_2$. We use the convention of Remark 5.6. A single integration by parts with boundary coefficient M^α gives no boundary term, since $M^\alpha = 0$ on $\partial \tilde{\Omega}$. The total critical classes containing $\bar{\partial} T_2$ or $\bar{\partial}^2 T_2$ are kept in total form until the compactly supported local critical amplitude has been inserted; these integrations by parts are then full-plane integrations in the local chart.

If a derivative falls on M^α after such an integration by parts, the term has at most the size of an $h^{-1} \partial \tilde{r}_2$ or $h^{-1} \bar{\partial} \tilde{r}_2$ pairing, or contains $\tilde{r}_2$ without a critical derivative. These terms are covered

by Lemma 5.2. Thus, the only retained $h^{-2}\partial\tilde{r}_2$ terms are those where the derivative falls on the oscillatory factor.

The M -critical part is the part of $I_{\xi M} + I_{\eta M}$ in which the second forward CGO solution is replaced by T_2 . Since the noncritical factors v^* and $v^{(1)}$ have finite asymptotic expansions to the required order, and since the covariant Hessian corrections contain only first derivatives of the CGO factors, it is enough to classify the finite family of model terms obtained by expanding the leading exponential factors, the smooth CGO amplitude factors, the density, and the connection coefficients.

We use the Hessian decomposition (5.24) with

$$\mathcal{A}(T) = \text{tr}(\mathbb{A}T), \quad \mathcal{B}(T) = \text{tr}(\mathbb{B}T).$$

The covariant Hessian differs from the ordinary Hessian by a fixed connection term of order one. Hence, every term containing a connection correction loses one effective phase derivative. Such terms are either local order zero in M , already included in $\mathcal{L}_{M,0}^{\text{loc}}(M)$, or critical terms controlled by Lemma 5.2. They do not change the $h^{-2}\partial\tilde{r}_2$ class.

We first consider $I_{\xi M}$. The critical part of ξ_β is

$$\xi_{\beta,\text{crit}} = \kappa_\beta e^{-\bar{\Phi}_2/h} \bar{\partial} T_2 + \lambda_\beta e^{-\bar{\Phi}_2/h} \partial T_2.$$

The component containing the total $\bar{\partial} T_2$ is kept in total form and integrated by parts in $\bar{\partial}$ before T_2 is expanded. Since $E e^{-\bar{\Phi}_2/h} = e^{(\Phi^* + \Phi_1)/h}$ and $\Phi^* + \Phi_1$ is holomorphic, this integration does not differentiate the exponential factor. The resulting terms contain $\tilde{r}_2$ without derivatives, or contain $\bar{z}\tilde{r}_2$ after differentiating an already expanded oscillatory expression. After multiplication by h , they are controlled by (5.8), (5.9), (5.11), and (5.14).

It remains in $I_{\xi M}$ to examine the $\lambda_\beta e^{-\bar{\Phi}_2/h} \partial T_2$ component. Since $\partial \bar{\Phi}_2 = 0$, one has $e^{-\bar{\Phi}_2/h} \partial T_2 = (\partial F_2)\tilde{r}_2 + F_2 \partial \tilde{r}_2$. The first term gives an integral with prefactor at most h^{-2} and with $\tilde{r}_2$ without derivatives, hence it is $o(1)$ after multiplication by h by (5.9). The second term is paired with the leading $\mathbb{A}\partial^2$ component of $\nabla^2 v^{(1)}$. Since $\partial^2(e^{\Phi_1/h}) = h^{-2}(\partial\Phi_1)^2 e^{\Phi_1/h} + h^{-1}(\partial^2\Phi_1)e^{\Phi_1/h}$ and $\partial\Phi_1 = 1 + z/4$, the only nonnegligible part of $I_{\xi M}$ is

$$(5.33) \quad I_{\xi M, \partial\tilde{r}_2}^{\text{lead}} = \frac{1}{h^2} \int E Q_A(x; M) \left(1 + \frac{z}{4}\right)^2 \partial\tilde{r}_2 dx + o(h^{-1}).$$

The term with $h^{-1}\partial^2\Phi_1$ has prefactor at most h^{-1} in front of $\partial\tilde{r}_2$ and is $o(h^{-1})$ before multiplication by h by (5.12). Derivatives falling on F_* , F_1 , Λ_0 , κ_β , λ_β , a cutoff, or a connection coefficient are handled in the same way.

We now consider $I_{\eta M}$. The leading part of η_α is $h^{-1}F_1 e^{\Phi_1/h}(\partial\Phi_1)\lambda_\alpha$. The critical Hessian of T_2 is decomposed by the same formula for $D^2 T_2$. The $\mathbb{B}\bar{\partial}^2 T_2$ part is a total $\bar{\partial}^2$ critical class. With the compactly supported local critical amplitude fixed above, it is kept in total form and controlled after multiplication by h by (5.15). The $2I\partial\bar{\partial} T_2$ part is a total $\bar{\partial}$ critical class after the exterior ∂ is treated as part of the smooth differentiated amplitude. Integrating once in $\bar{\partial}$ before expanding T_2 reduces it to terms containing $\tilde{r}_2$ or $\partial\tilde{r}_2$ with prefactor at most h^{-1} , and these are controlled by (5.8) and (5.12). Thus, neither the $\mathbb{B}$ part nor the I part gives a nonnegligible contribution. Since $\partial\bar{\Phi}_2 = 0$,

$$e^{-\bar{\Phi}_2/h} \partial^2 T_2 = (\partial^2 F_2)\tilde{r}_2 + 2(\partial F_2)\partial\tilde{r}_2 + F_2 \partial^2 \tilde{r}_2.$$

The first two terms have a prefactor at most h^{-1} after being multiplied by the leading η_α factor, and are controlled by Lemma 5.2. The last term gives

$$\frac{1}{h} \int E Q_A(x; M) \left(1 + \frac{z}{4}\right) \partial^2 \tilde{r}_2 dx.$$

We integrate once in ∂ . The boundary term vanishes because $M^\alpha = 0$ on $\partial\tilde{\Omega}$ by Lemma 5.5. Since $\partial E = \frac{z}{2h}E$, the differentiated exponential gives the nonnegligible part

$$(5.34) \quad I_{\eta M, A}^{\partial^2 \tilde{r}_2} = -\frac{1}{h^2} \int E \frac{z}{2} \left(1 + \frac{z}{4}\right) Q_A(x; M) \partial \tilde{r}_2 dx + o(h^{-1}).$$

When ∂ falls on $Q_A(x; M)(1 + z/4)$ instead of E , the resulting prefactor is only h^{-1} in front of $\partial \tilde{r}_2$, so it is $o(h^{-1})$ before multiplication by h by (5.12).

Combining (5.33) and (5.34), the coefficient of the nonnegligible $h^{-2} \partial \tilde{r}_2$ term is $(1 + \frac{z}{4})^2 - \frac{z}{2}(1 + \frac{z}{4}) = 1 - \frac{z^2}{16}$. All other critical terms are $o(1)$ after multiplication by h . Therefore, the only nonnegligible M -critical contribution is exactly (5.32). $\square$

By Lemma 5.10 and Lemma 5.4, the M -critical Cauchy contribution is $2\pi(\partial^{*-1}[Q_A(\cdot; M)(1 - z^2/16)V']) (0)V(0)$. Therefore,

$$(5.35) \quad \lim_{h \rightarrow 0} h(I_{\xi M} + I_{\eta M}) = -2\pi \bar{\partial}(\Lambda_0 F_* F_1 F_2 \kappa_\beta \mathcal{A}(M^\beta))(0) + \pi \mathcal{Z}_M(M)(0) + \pi \mathcal{L}_{M,0}^{\text{loc}}(M)(0) \\ + 2\pi(\partial^{*-1}[Q_A(\cdot; M)(1 - z^2/16)V']) (0)V(0).$$

5.4. The residual equation.

Lemma 5.11 (First-derivative terms). *One has*

$$(5.36) \quad \lim_{h \rightarrow 0} h I_{\eta \xi} = 0,$$

and

$$(5.37) \quad \lim_{h \rightarrow 0} h I_{\nabla \nabla} = 0.$$

Proof. Both $I_{\eta \xi}$ and $I_{\nabla \nabla}$ contain only first derivatives of the two forward CGO solutions. We first consider the purely local terms, namely the terms in which the second forward solution is replaced by its noncritical part $F_2 e^{\bar{\Phi}_2/h}$ and the noncritical remainders are expanded to finite order. The leading local model has the form

$$\frac{1}{h^2} \int e^{ix_1 x_2/h} a(x) (\partial \Phi_1) (\bar{\partial} \bar{\Phi}_2) dx,$$

where a is smooth and compactly supported. Since $\bar{\partial} \bar{\Phi}_2 = -\bar{z}/2$, the amplitude vanishes at the critical point. The stationary phase formula then gives this integral as $O(1)$ before multiplication by h . Hence it contributes $o(1)$ to $h I_{\eta \xi}$ and $h I_{\nabla \nabla}$.

If one derivative falls on F_* , F_1 , F_2 , the density, a cutoff, N^{-T} , ρ^{-1} , $\mathcal{H}^\alpha$, or any fixed coefficient in $\mathcal{P}_{\nabla \nabla}$, the term loses one power of h^{-1} . Thus the corresponding model has prefactor at most h^{-1} , and stationary phase gives $o(h^{-1})$ before multiplication by h . If one of the noncritical remainders r_* or r_1 is used, its expansion starts with a factor h , and the same conclusion holds.

It remains to check the critical remainder from the second forward solution. Put $T_2 = F_2 e^{\bar{\Phi}_2/h} \tilde{r}_2$ and $E = e^{ix_1 x_2/h}$. Since only first derivatives of the second forward solution occur in $I_{\eta \xi}$ and $I_{\nabla \nabla}$, the critical factor can only appear through $e^{-\bar{\Phi}_2/h} T_2$, $e^{-\bar{\Phi}_2/h} \partial T_2$, and $e^{-\bar{\Phi}_2/h} \bar{\partial} T_2$. By (5.19),

$$e^{-\bar{\Phi}_2/h} \partial T_2 = (\partial F_2) \tilde{r}_2 + F_2 \partial \tilde{r}_2,$$

and

$$e^{-\bar{\Phi}_2/h} \bar{\partial} T_2 = h^{-1} c_2 F_2 \tilde{r}_2 + (\bar{\partial} F_2) \tilde{r}_2 + F_2 \bar{\partial} \tilde{r}_2, \quad c_2 = -\frac{\bar{z}}{2}.$$

Multiplying by the leading first derivative of $v^{(1)}$ gives only the following critical model classes:

$$\frac{1}{h} \int E a(x) \tilde{r}_2 dx, \quad \frac{1}{h^2} \int E \bar{z} a(x) \tilde{r}_2 dx,$$

and

$$\frac{1}{h} \int E a(x) \partial \tilde{r}_2 dx, \quad \frac{1}{h} \int E a(x) \bar{\partial} \tilde{r}_2 dx.$$

Here a denotes a smooth compactly supported amplitude, and the finite number of such amplitudes comes from the finite number of first derivative factors and coefficient factors in $I_{\eta\xi}$ and $I_{\nabla\nabla}$.

After multiplication by h , the first model is controlled by (5.8), the second by (5.11), the third by (5.12), and the fourth by (5.13). Hence, no critical first-derivative term survives.

There is no nonnegligible $h^{-2} \partial \tilde{r}_2$ pairing in $I_{\eta\xi}$ or $I_{\nabla\nabla}$. Such a pairing would require one additional effective phase derivative, but these two terms contain only first derivatives of the forward CGO factors. Therefore, all local, noncritical-remainder, and critical-remainder contributions vanish after multiplication by h , proving (5.36) and (5.37). $\square$

Since $I_h = 0$, multiplying by h and passing to the limit gives, by (5.29), (5.35), (5.36), and (5.37),

$$\begin{aligned} 0 &= 8\pi \bar{\partial}^2 (\Lambda_0 F_* F_1 F_2 \mathbf{r})(0) + \pi \mathcal{L}_{\mathbf{r}, \leq 1}^{\text{loc}} \mathbf{r}(0) - 2\pi \bar{\partial} (\Lambda_0 F_* F_1 F_2 \kappa_\alpha \mathcal{A}(M^\alpha))(0) \\ &\quad + \pi \mathcal{Z}_M(M)(0) + \pi \mathcal{L}_{M,0}^{\text{loc}}(M)(0) + \mathcal{N}_{\mathbf{r}}(0) + 2\pi (\partial^{*-1} [Q_A(\cdot; M)(1 - z^2/16)V'])(0)V(0). \end{aligned}$$

Now translate the critical point to $a \in \tilde{\Omega}$ by using

$$\Phi_{1,a}(z) = (z - a) + \frac{(z - a)^2}{8}, \quad \Phi_{2,a}(z) = -\frac{(z - a)^2}{4}, \quad \Phi_a^*(z) = -(z - a) + \frac{(z - a)^2}{8}.$$

The smooth CGO prefactors and the fixed coefficient factors are translated in the same way. The same calculation gives

$$\begin{aligned} (5.38) \quad 0 &= 8\pi \bar{\partial}^2 (C_a \mathbf{r})(a) + \pi \mathcal{L}_{\mathbf{r}, \leq 1,a}^{\text{loc}} \mathbf{r}(a) - 2\pi \bar{\partial} (C_a \kappa_\alpha \mathcal{A}(M^\alpha))(a) + \pi \mathcal{Z}_{M,a}(M)(a) \\ &\quad + \pi \mathcal{L}_{M,0,a}^{\text{loc}}(M)(a) + \mathcal{N}_{\mathbf{r},a}(a) + 2\pi (\partial^{*-1} P_a(\cdot; M))(a)V(a), \end{aligned}$$

where $C_a := \Lambda_{0,a} F_{*,a} F_{1,a} F_{2,a}$. The translated nonlocal amplitude is

$$\begin{aligned} P_a(x; M) &= Q_a(x; M) \left(1 - \frac{(z - a)^2}{16} \right) V'(x), \\ Q_a(x; M) &= C_a(x) \lambda_\alpha(x) \mathcal{A}(M^\alpha)(x). \end{aligned}$$

Moreover,

$$(5.39) \quad \mathcal{Z}_{M,a}(M)(a) = 2\Lambda_{0,a}(a) F_{*,a}(a) F_{1,a}(a) (\lambda_\beta(a) \partial F_{2,a}(a) + \kappa_\beta(a) \bar{\partial} F_{2,a}(a)) \mathcal{A}(M^\beta)(a).$$

Dividing (5.38) by π and absorbing fixed numerical factors into the notation for the scalar critical term and the nonlocal Cauchy term, we define

$$\mathcal{K}_{\text{nl},a}(M)(a) = 2(\partial^{*-1} P_a(\cdot; M))(a)V(a).$$

The residual equation is therefore,

$$(5.40) \quad 8\bar{\partial}^2 (C_a \mathbf{r}) - 2\bar{\partial} (C_a \kappa_\alpha \mathcal{A}(M^\alpha)) + \mathcal{Z}_M(M) + \mathcal{L}_{\mathbf{r}, \leq 1}^{\text{loc}} \mathbf{r} + \mathcal{L}_{M,0}^{\text{loc}}(M) + \mathcal{N}_{\mathbf{r}} + \mathcal{K}_{\text{nl}}(M) = 0 \quad \text{in } \tilde{\Omega}.$$

The displayed local second order block in (5.40) is $8\bar{\partial}^2 (C_a \mathbf{r}) - 2\bar{\partial} (C_a \kappa_\alpha \mathcal{A}(M^\alpha))$. The terms $\mathcal{L}_{\mathbf{r}, \leq 1}^{\text{loc}} \mathbf{r}$, $\mathcal{L}_{M,0}^{\text{loc}}(M)$, and $\mathcal{Z}_M(M)$ are local lower order terms. The Cauchy transform terms are kept in operator form and will be localized in the unique continuation argument.

6. ASYMPTOTIC ANALYSIS OF THE MIRROR SECOND INTEGRAL IDENTITY

We next use the mirror CGO family, obtained by interchanging the holomorphic and antiholomorphic roles in the preceding construction. This gives the second residual equation paired with (5.40). The equation is derived directly from the same second integral identity; it is not inserted as a complex conjugate of the first residual equation.

The residual variables are the same: $\mathbf{r} = 1 - \rho^{-1}$ and $M^\alpha = \rho^{-1}\mathcal{H}^\alpha$. We use the same decomposition of I_h , but with the mirror CGO solutions below, and denote the resulting terms by $I_{\mathbf{r}}^m$, $I_{\xi M}^m$, $I_{\eta M}^m$, $I_{\eta \xi}^m$, and $I_{\nabla \nabla}^m$. Again, the computation is first made at the origin and then translated to a center $a \in \Omega$.

The Cauchy transform terms coming from the mirror critical remainder are kept in the residual equation. As before, no extra cutoff is inserted into the second integral identity. The localization of these terms belongs only to the unique continuation argument.

6.1. The mirror CGO family. At the fixed center, translated to the origin, set

$$\Psi^*(z) = -\bar{z} + \frac{\bar{z}^2}{8}, \quad \Psi_1(z) = \bar{z} + \frac{\bar{z}^2}{8}, \quad \Psi_2(z) = -\frac{z^2}{4}.$$

Then

$$\Psi^* + \Psi_1 + \Psi_2 = \frac{\bar{z}^2 - z^2}{4} = -ix_1x_2.$$

The derivatives used below are

$$\bar{\partial}\Psi_1 = 1 + \frac{\bar{z}}{4}, \quad \bar{\partial}^2\Psi_1 = \frac{1}{4}, \quad \partial\Psi_2 = -\frac{z}{2}, \quad \partial^2\Psi_2 = -\frac{1}{2}, \quad \bar{\partial}\Psi^* = -1 + \frac{\bar{z}}{4}, \quad \bar{\partial}^2\Psi^* = \frac{1}{4}.$$

By the scaling normalization fixed in Section 5.1, the noncritical mirror phases Ψ_1 and Ψ^* have no critical points on the translated domain.

We insert mirror CGO solutions of the form

$$v^* = \tilde{F}_* e^{\Psi^*/h} (1 + \tilde{r}_*), \quad v^{(1)} = \tilde{F}_1 e^{\Psi_1/h} (1 + \tilde{r}_1), \quad v^{(2)} = \tilde{F}_2 e^{\Psi_2/h} (1 + r_2).$$

The factors $\tilde{F}_*$, $\tilde{F}_1$, and $\tilde{F}_2$ are the corresponding smooth nonvanishing CGO prefactors near the critical point. The noncritical remainders $\tilde{r}_*$ and $\tilde{r}_1$ satisfy complete asymptotic expansions of the same type as r_* and r_1 in Section 5. The critical remainder is now r_2 .

The localized mirror inverses $\bar{\partial}_\psi^{-1}$ and $\bar{\partial}_\psi^{*-1}$ are understood with the reflected convention corresponding to the localized $\bar{\partial}$ Cauchy inverse, modulo the same type of smoothing cutoff errors as in Section 5. The mirror critical Neumann series has the form

$$r_2 = -\bar{\partial}_\psi^{-1}(\tilde{V}' \tilde{s}_2^m), \quad \tilde{s}_2^m = \sum_{k=0}^{\infty} \tilde{T}_{h,m}^k \bar{\partial}_\psi^{*-1}(\tilde{V}),$$

where $\tilde{T}_{h,m} = \bar{\partial}_\psi^{*-1} \tilde{V} \bar{\partial}_\psi^{-1} \tilde{V}'$. Thus,

$$(6.1) \quad \bar{\partial} r_2 = -e^{ix_1x_2/h} \tilde{V}' \tilde{s}_2^m.$$

The basic bounds are

$$\|\tilde{s}_2^m\|_{L^2} = O(h^{1/2+\epsilon}), \quad \|H_c D^2 r_2\|_{L^2} = O(h^{-1/2+\epsilon})$$

for every fixed cutoff $H_c \in C_0^2$.

The estimates used for this family are obtained from the previous ones by the reflection $w = \bar{z}$. This interchanges ∂ and $\bar{\partial}$ and changes the oscillatory factor from $e^{ix_1x_2/h}$ to $e^{-ix_1x_2/h}$. The residual identity itself is still derived by inserting the mirror CGO family into the second integral identity.

Under this reflection, the critical pairing with $h^{-2}\bar{\partial}\tilde{r}_2$ becomes the mirror pairing with $h^{-2}\bar{\partial}r_2$, and the total $\bar{\partial}^2$ critical class becomes the total ∂^2 critical class. The polarizations are interchanged

as $\mathcal{A}(D^2 f) = 4\bar{\partial}^2 f$ and $\mathcal{B}(D^2 f) = 4\partial^2 f$. Thus the plus block is paired with the mirror block, and the plus critical polarization $\lambda_\alpha \mathcal{A}(M^\alpha)$ is paired with $\kappa_\alpha \mathcal{B}(M^\alpha)$.

Lemma 6.1 (Mirror estimates by reflection). *The mirror CGO family satisfies the same noncritical expansion, critical remainder estimates, oscillatory Cauchy inverse estimates, and Cauchy transform rule for the critical pairing as the CGO family in Section 5, with ∂ and $\bar{\partial}$ interchanged and with the phase $e^{-ix_1 x_2/h}$ in place of $e^{ix_1 x_2/h}$. In particular, for fixed compactly supported amplitudes a and B ,*

$$\begin{aligned} h \left[\frac{1}{h} \int e^{-ix_1 x_2/h} a r_2 dx \right] &= o(1), & h \left[\frac{1}{h^2} \int e^{-ix_1 x_2/h} a r_2 dx \right] &= o(1), \\ h \left[\frac{1}{h^2} \int e^{-ix_1 x_2/h} z a r_2 dx \right] &= o(1), & h \left[\frac{1}{h} \int e^{-ix_1 x_2/h} a \bar{\partial} r_2 dx \right] &= o(1), \end{aligned}$$

and

$$h \left[\frac{1}{h} \int e^{-ix_1 x_2/h} a \partial r_2 dx \right] = o(1).$$

Moreover,

$$h \left[\frac{1}{h^2} \int e^{-ix_1 x_2/h} a e^{-\Psi_2/h} \partial(\tilde{F}_2 e^{\Psi_2/h} r_2) dx \right] = o(1),$$

and

$$h \left[\frac{1}{h} \int e^{-ix_1 x_2/h} a e^{-\Psi_2/h} \partial^2(\tilde{F}_2 e^{\Psi_2/h} r_2) dx \right] = o(1).$$

The only nonnegligible mirror critical pairing is

$$(6.2) \quad h \left[\frac{1}{h^2} \int e^{-ix_1 x_2/h} B(x) \bar{\partial} r_2(x) dx \right] = 2\pi(\bar{\partial}^{*-1}(B\tilde{V}'))(0)\tilde{V}(0) + o(1).$$

All these estimates hold uniformly after translating the critical point to centers in compact subsets of $\tilde{\Omega}$.

Proof. Let $\mathcal{R}f(z) = f(\bar{z})$. Then $\mathcal{R}\partial\mathcal{R} = \bar{\partial}$, $\mathcal{R}\bar{\partial}\mathcal{R} = \partial$, and the phase $e^{ix_1 x_2/h}$ is changed into $e^{-ix_1 x_2/h}$. Applying this reflection to the localized model parametrices used in Section 5 gives the same noncritical expansions, the same critical Neumann bounds, and the same oscillatory Cauchy inverse estimates with ∂ and $\bar{\partial}$ interchanged.

The reflection is used only to transfer the local estimates. The residual identity itself is still obtained by inserting the mirror CGO family into the same second integral identity. Since the principal part is conformally Euclidean in the chosen isothermal coordinate and all lower-order coefficients are smooth, the local CGO construction applies with the antiholomorphic critical phase in the same way as in Section 5.

For the critical pairing, (6.1) gives

$$h \left[\frac{1}{h^2} \int e^{-ix_1 x_2/h} B \bar{\partial} r_2 dx \right] = -\frac{1}{h} \int B\tilde{V}' \tilde{s}_2^m dx.$$

The part of $\tilde{s}_2^m$ after the first Neumann term is $o(h)$ in this pairing, exactly as in Lemma 5.3. Moving the localized mirror Cauchy inverse from the oscillatory input to the fixed coefficient by the same kernel convention as in Lemma 5.4, and then applying (6.3), gives (6.2). The translated estimates are obtained by replacing z by $z - a$, uniformly for a in compact subsets. $\square$

The mirror stationary phase formula is

$$(6.3) \quad \frac{1}{2\pi h} \int_{\mathbb{R}^2} e^{-ix_1 x_2/h} F(x) dx = F(0) + h(\partial^2 - \bar{\partial}^2)F(0) + O(h^2),$$

for compactly supported smooth F .

For reference, we also record the only critical derivative expansions used below: $e^{-\Psi_2/h}\bar{\partial}(\tilde{F}_2e^{\Psi_2/h}r_2) = (\bar{\partial}\tilde{F}_2)r_2 + \tilde{F}_2\bar{\partial}r_2$ and $e^{-\Psi_2/h}\partial(\tilde{F}_2e^{\Psi_2/h}r_2) = h^{-1}\left(-\frac{z}{2}\right)\tilde{F}_2r_2 + (\partial\tilde{F}_2)r_2 + \tilde{F}_2\partial r_2$. The second derivative expansions are obtained by differentiating these identities once more. All resulting terms are either controlled by Lemma 6.1 or are of the form (6.2).

6.2. Extraction of the mirror local terms. Set

$$\tilde{\Lambda}_0 := \gamma^2\mu, \quad \tilde{C}_0 := \tilde{\Lambda}_0\tilde{F}_*\tilde{F}_1\tilde{F}_2\mathbf{r}.$$

The leading scalar term pairs the $\mathbb{B}\bar{\partial}^2$ component of $v^{(1)}$ with the $\mathbb{A}\partial^2$ component of $v^{(2)}$:

$$I_{\mathbf{r},1}^m = 4 \int \tilde{C}_0 e^{\Psi^*/h} \bar{\partial}^2(e^{\Psi_1/h}) \partial^2(e^{\Psi_2/h}) dx.$$

As in the scalar computation in Section 5, the boundary terms vanish by Lemma 5.5. Two integrations by parts give

$$I_{\mathbf{r},1}^m = 4 \int \partial\bar{\partial}(\tilde{C}_0 e^{\Psi^*/h}) \bar{\partial}(e^{\Psi_1/h}) \partial(e^{\Psi_2/h}) dx.$$

Since $\partial e^{\Psi^*/h} = 0$, this equals

$$\frac{4}{h^3} \int (\bar{\partial}\Psi^*) \partial\tilde{C}_0 (\bar{\partial}\Psi_1) (\partial\Psi_2) e^{-ix_1x_2/h} dx + O(1).$$

Using the phase derivatives, $4(\bar{\partial}\Psi^*)(\bar{\partial}\Psi_1)(\partial\Psi_2) = 2z(1 - \frac{\bar{z}^2}{16})$.

Thus,

$$I_{\mathbf{r},1}^m = \frac{1}{h^3} \int 2z \left(1 - \frac{\bar{z}^2}{16}\right) \partial\tilde{C}_0 e^{-ix_1x_2/h} dx + O(1).$$

Applying (6.3) gives

$$(6.4) \quad \lim_{h \rightarrow 0} h I_{\mathbf{r},1}^m = 8\pi \partial^2 \tilde{C}_0(0).$$

All remaining mirror scalar local terms are lower order after the explicit contribution (6.4) is removed. They place at most one derivative on $\mathbf{r}$ at the limiting order, and are denoted by $\pi \mathcal{L}_{\mathbf{r}, \leq 1}^{m, \text{loc}} \mathbf{r}(0)$. The mirror critical terms are organized by the same finite decomposition convention as in Lemma 5.9, with ∂ and $\bar{\partial}$ interchanged. In particular, total ∂ derivatives falling on the critical factor are integrated by parts before the critical factor is expanded, and the only nonnegligible pairings are the $h^{-2}\bar{\partial}r_2$ terms.

Lemma 6.2 (Order of the mirror scalar critical contribution). *After the allowed integrations by parts, every mirror scalar critical term that survives after multiplication by h is a finite sum of localized mirror Cauchy transform contributions whose inputs are linear in $\mathbf{r}$ and in at most one derivative of $\mathbf{r}$. More precisely, every such summand comes from a term of the form*

$$h \left[\frac{1}{h^2} \int e^{-ix_1x_2/h} B_\nu^m(x; \mathbf{r}) \bar{\partial}r_2(x) dx \right] + o(1),$$

where $B_\nu^m(x; \mathbf{r}) = a_{\nu,0}^m(x)\mathbf{r}(x) + a_{\nu,1}^m(x)\partial\mathbf{r}(x) + a_{\nu,2}^m(x)\bar{\partial}\mathbf{r}(x)$. Consequently, after the substitution $\mathbf{r} = 1 - e^{-\theta}$, the linear part of the mirror scalar critical contribution is of order at most one in θ , possibly followed by a localized mirror Cauchy transform. In particular, it does not modify the displayed mirror second-order scalar principal block $8\partial^2(\tilde{C}_a\mathbf{r})$.

Proof. We work at the center 0. The translated case is identical, with z replaced by $z - a$. The mirror scalar critical part is obtained by replacing the second forward solution in $I_{\mathbf{r}}^m$ by $\tilde{F}_2e^{\Psi_2/h}r_2$. We keep the differentiated critical factor in this total form until the integrations by parts have been made.

The leading Hessian of $v^{(1)}$ is now the $\mathbb{B}\bar{\partial}^2$ part. The critical Hessian of $\tilde{F}_2 e^{\Psi_2/h} r_2$ is decomposed into the $\mathbb{A}\partial^2$ part, the $\mathbb{B}\bar{\partial}^2$ part, and the identity part $2I\partial\bar{\partial}$. Terms in which the critical factor appears as a total ∂ or ∂^2 derivative are integrated by parts before expanding the derivative. Since $e^{-ix_1 x_2/h} e^{-\Psi_2/h} = e^{(\Psi^* + \Psi_1)/h}$ and $\Psi^* + \Psi_1$ is antiholomorphic, these integrations do not differentiate the exponential factor. The resulting terms contain either r_2 without derivatives or the factor $z r_2$, and are controlled by Lemma 6.1 after multiplication by h .

Terms in which ∂r_2 appears after the total ∂ and total ∂^2 classes have been removed have a prefactor at most h^{-1} . These terms are integrated once in ∂ . Differentiating the oscillatory factor then produces terms containing either r_2 or $z r_2$ with prefactors covered by Lemma 6.1. No bare $h^{-2}\partial r_2$ term is treated in this way. Any term which would have such a prefactor is kept as a total ∂ or total ∂^2 derivative of $\tilde{F}_2 e^{\Psi_2/h} r_2$ before expansion, and is controlled by the two total-derivative estimates in Lemma 6.1. Terms containing $\bar{\partial} r_2$ with prefactor at most h^{-1} are controlled directly by Lemma 6.1. Hence, the only possible nonnegligible mirror class is the class with prefactor h^{-2} and one $\bar{\partial} r_2$ derivative.

It remains to check the terms with two derivatives on r_2 . The term $\tilde{F}_2 \bar{\partial}^2 r_2$ belongs to the $\mathbb{B}$ polarization, and its contraction with the leading $\mathbb{B}$ part of $v^{(1)}$ vanishes because $\text{tr}(\mathbb{B}\mathbb{B}) = 0$. The term $\tilde{F}_2 \partial\bar{\partial} r_2$ belongs to the identity polarization, and its contraction with the leading $\mathbb{B}$ part vanishes because $\text{tr} \mathbb{B} = 0$. The term $\tilde{F}_2 \partial^2 r_2$ belongs to the $\mathbb{A}$ polarization, but it is part of the total ∂^2 class already treated above. Thus, no two-derivative mirror critical term gives a new nonnegligible contribution.

The covariant Hessian corrections contain only first derivatives of the CGO factors. Therefore, every term involving such a correction has one fewer effective phase derivative, or has a first derivative of r_2 with prefactor at most h^{-1} . These terms are negligible after multiplication by h , except for the already identified $h^{-2}\bar{\partial} r_2$ class. The same conclusion holds when derivatives fall on $\tilde{F}_*$, $\tilde{F}_1$, $\tilde{F}_2$, on the density factor, on a cutoff, or on a fixed connection coefficient.

We have therefore reduced every nonnegligible mirror scalar critical term to

$$h \left[\frac{1}{h^2} \int e^{-ix_1 x_2/h} B_\nu^m(x; \mathbf{r}) \bar{\partial} r_2(x) dx \right] + o(1),$$

with B_ν^m as displayed. Applying the mirror Cauchy transform rule (6.2) gives a finite sum of localized mirror Cauchy transform terms whose inputs contain $\mathbf{r}$, $\partial \mathbf{r}$, and $\bar{\partial} \mathbf{r}$, but no second derivative of $\mathbf{r}$. Substituting $\mathbf{r} = 1 - e^{-\theta}$ shows that the linear part is of order at most one in θ , possibly followed by a localized mirror Cauchy transform. This proves the lemma. $\square$

We denote the total mirror scalar critical contribution by $\mathcal{N}_{\mathbf{r}}^m$. Combining the mirror scalar pieces gives

$$(6.5) \quad hI_{\mathbf{r}}^m = 8\pi\partial^2(\tilde{\Lambda}_0 \tilde{F}_* \tilde{F}_1 \tilde{F}_2 \mathbf{r})(0) + \pi \mathcal{L}_{\mathbf{r}, \leq 1}^{m, \text{loc}} \mathbf{r}(0) + \mathcal{N}_{\mathbf{r}}^m(0) + o(1).$$

We next record the mirror terms linear in M^α . For the mirror family, the leading derivative of the second forward solution is

$$\xi_\beta^{\text{lead}, m} = \frac{1}{h} \tilde{F}_2 e^{\Psi_2/h} (\partial \Psi_2) \lambda_\beta.$$

The leading Hessian of $v^{(1)}$ is the $\mathbb{B}\bar{\partial}^2$ component. Set $B_\xi^m := \tilde{\Lambda}_0 \tilde{F}_* \tilde{F}_1 \tilde{F}_2 \lambda_\beta \mathcal{B}(M^\beta)$, then the same stationary phase computation as above gives

$$(6.6) \quad \lim_{h \rightarrow 0} hI_{\xi M, 1}^m = -2\pi\partial B_\xi^m(0) = -2\pi\partial(\tilde{\Lambda}_0 \tilde{F}_* \tilde{F}_1 \tilde{F}_2 \lambda_\beta \mathcal{B}(M^\beta))(0).$$

There is one local zeroth-order contribution in M which is not included in (6.6). It comes from the amplitude-derivative component of ξ_β when the second forward solution is replaced by its noncritical

factor $\tilde{F}_2 e^{\Psi_2/h}$. Indeed,

$$\xi_{\beta, \text{nc}}^m = (N^{-T})_\beta^j \partial_j (\tilde{F}_2 e^{\Psi_2/h}) = e^{\Psi_2/h} [h^{-1} \lambda_\beta (\partial \Psi_2) \tilde{F}_2 + \lambda_\beta \partial \tilde{F}_2 + \kappa_\beta \bar{\partial} \tilde{F}_2].$$

The first term is the leading phase-derivative term already used in (6.6). The remaining amplitude-derivative component gives the following local zeroth-order contribution:

$$\mathcal{Z}_M^m(M)(0) = 2\tilde{\Lambda}_0(0)\tilde{F}_*(0)\tilde{F}_1(0)(\lambda_\beta(0)\partial\tilde{F}_2(0) + \kappa_\beta(0)\bar{\partial}\tilde{F}_2(0))\mathcal{B}(M^\beta)(0).$$

All other noncritical local mirror M -linear terms either lose one effective power of h^{-1} , contain the vanishing factor $\partial\Psi_2 = -z/2$, or come from covariant Hessian corrections. They are denoted by $\pi\mathcal{L}_{M,0}^{m,\text{loc}}(M)(0)$, where $\mathcal{L}_{M,0}^{m,\text{loc}}$ is local of order zero in M .

For $I_{\eta M}^m$, the leading derivative of the first forward solution is $\eta_\alpha^{\text{lead},m} = \frac{1}{h}\tilde{F}_1 e^{\Psi_1/h}(\bar{\partial}\Psi_1)\kappa_\alpha$. The leading local term from the noncritical Hessian of $v^{(2)}$ is the sum of two stationary phase contributions which cancel exactly, as in (5.31). Thus, it produces no additional local derivative of M .

It remains to record the mirror M -critical Cauchy contribution. Define

$$Q_B^m(x; M) := \tilde{\Lambda}_0 \tilde{F}_* \tilde{F}_1 \tilde{F}_2 \kappa_\alpha \mathcal{B}(M^\alpha).$$

Lemma 6.3 (Finite mirror M -critical normal form). *After expanding the mirror critical part of $I_{\xi M}^m + I_{\eta M}^m$ and performing the integrations by parts allowed by Lemma 5.5, every mirror M -critical term is $o(1)$ after multiplication by h , except the following nonnegligible contribution*

$$(6.7) \quad h \left[\frac{1}{h^2} \int e^{-ix_1 x_2/h} Q_B^m(x; M) \left(1 - \frac{\bar{z}^2}{16}\right) \bar{\partial} r_2 dx \right] + o(1).$$

Proof. We work at the critical center 0. The translated statement is identical, with z replaced by $z - a$. Put $E_m = e^{-ix_1 x_2/h}$ and $T_2^m = \tilde{F}_2 e^{\Psi_2/h} r_2$. We use the mirror version of Remark 5.6. A single integration by parts with boundary coefficient M^α gives no boundary term, since $M^\alpha = 0$ on $\partial\tilde{\Omega}$. The total mirror critical classes containing ∂T_2^m or $\partial^2 T_2^m$ are kept in total form until the compactly supported local mirror critical amplitude has been inserted; these integrations by parts are then full-plane integrations in the local chart.

If a derivative falls on M^α after such an integration by parts, the term has at most the size of an $h^{-1}\bar{\partial}r_2$ or $h^{-1}\partial r_2$ pairing, or contains r_2 without a critical derivative. These terms are covered by Lemma 6.1. Thus the only retained $h^{-2}\bar{\partial}r_2$ terms are those where the derivative falls on the oscillatory factor.

The mirror M -critical part is the part of $I_{\xi M}^m + I_{\eta M}^m$ in which the second forward CGO solution is replaced by T_2^m . The noncritical factors v^* and $v^{(1)}$ have finite expansions to the needed order, and all connection corrections contain only first derivatives of the CGO factors. Hence, it is enough to classify the finite family of model terms obtained from the leading exponential factors and the fixed smooth coefficients.

In $I_{\xi M}^m$, the critical part of ξ_β is

$$\xi_{\beta, \text{crit}}^m = \lambda_\beta e^{-\Psi_2/h} \partial T_2^m + \kappa_\beta e^{-\Psi_2/h} \bar{\partial} T_2^m.$$

The part containing the total ∂T_2^m is kept in total form and integrated by parts in ∂ before expanding T_2^m . Since $E_m e^{-\Psi_2/h} = e^{(\Psi^* + \Psi_1)/h}$ and $\Psi^* + \Psi_1$ is antiholomorphic, this integration does not differentiate the exponential factor. The resulting terms contain r_2 without derivatives, or contain $z r_2$ after differentiating an already expanded oscillatory expression. After multiplication by h , they are controlled by Lemma 6.1.

It remains in $I_{\xi M}^m$ to examine the $\kappa_\beta e^{-\Psi_2/h} \bar{\partial} T_2^m$ component. Since $\bar{\partial}\Psi_2 = 0$, one has $e^{-\Psi_2/h} \bar{\partial} T_2^m = (\bar{\partial}\tilde{F}_2)r_2 + \tilde{F}_2 \bar{\partial} r_2$. The first term gives an integral with r_2 without derivatives and is controlled

by Lemma 6.1. The second term is paired with the leading $\mathbb{B}\bar{\partial}^2$ component of $\nabla^2 v^{(1)}$. Since $\bar{\partial}^2(e^{\Psi_1/h}) = h^{-2}(\bar{\partial}\Psi_1)^2 e^{\Psi_1/h} + h^{-1}(\bar{\partial}^2\Psi_1)e^{\Psi_1/h}$ and $\bar{\partial}\Psi_1 = 1 + \bar{z}/4$, the only possible nonnegligible term is

$$(6.8) \quad I_{\xi M, \bar{\partial}r_2}^{m, \text{lead}} = \frac{1}{h^2} \int E_m Q_B^m(x; M) \left(1 + \frac{\bar{z}}{4}\right)^2 \bar{\partial}r_2 dx + o(h^{-1}).$$

The term containing $h^{-1}\bar{\partial}^2\Psi_1$ has only an h^{-1} prefactor in front of $\bar{\partial}r_2$ and is negligible after multiplication by h by Lemma 6.1. The same applies when derivatives fall on $\tilde{F}_*$, $\tilde{F}_1$, $\tilde{\Lambda}_0$, λ_β , κ_β , a cutoff, or a connection coefficient.

We now consider $I_{\eta M}^m$. The leading part of η_α is $h^{-1}\tilde{F}_1 e^{\Psi_1/h}(\bar{\partial}\Psi_1)\kappa_\alpha$. The critical Hessian of T_2^m is decomposed as

$$D^2T_2^m = \mathbb{A}\partial^2T_2^m + \mathbb{B}\bar{\partial}^2T_2^m + 2I\partial\bar{\partial}T_2^m.$$

The $\mathbb{A}\partial^2T_2^m$ part is a total ∂^2 critical class. It is integrated twice in ∂ before expanding T_2^m , and it is controlled by the total ∂^2 estimate in Lemma 6.1. The identity part is a total ∂ critical class after one derivative is treated as part of the smooth differentiated amplitude, and it is controlled by the total ∂ estimate in the same lemma. Thus, neither the $\mathbb{A}$ part nor the I part gives a nonnegligible contribution.

The only possible nonnegligible term in $I_{\eta M}^m$ comes from the $\mathbb{B}\bar{\partial}^2T_2^m$ part. Since $\bar{\partial}\Psi_2 = 0$,

$$e^{-\Psi_2/h}\bar{\partial}^2T_2^m = (\bar{\partial}^2\tilde{F}_2)r_2 + 2(\bar{\partial}\tilde{F}_2)\bar{\partial}r_2 + \tilde{F}_2\bar{\partial}^2r_2.$$

The first two terms have a prefactor at most h^{-1} after being multiplied by the leading η_α factor, and are controlled by Lemma 6.1. The last term gives

$$\frac{1}{h} \int E_m Q_B^m(x; M) \left(1 + \frac{\bar{z}}{4}\right) \bar{\partial}^2r_2 dx.$$

We integrate once in $\bar{\partial}$. The boundary term vanishes because $M^\alpha = 0$ on $\partial\tilde{\Omega}$ by Lemma 5.5. Since $\bar{\partial}E_m = \frac{\bar{z}}{2h}E_m$, the differentiated exponential gives the nonnegligible part

$$(6.9) \quad I_{\eta M, B}^{m, \bar{\partial}^2r_2} = -\frac{1}{h^2} \int E_m \frac{\bar{z}}{2} \left(1 + \frac{\bar{z}}{4}\right) Q_B^m(x; M) \bar{\partial}r_2 dx + o(h^{-1}).$$

When $\bar{\partial}$ falls on $Q_B^m(x; M)(1 + \bar{z}/4)$ instead of E_m , the resulting prefactor is only h^{-1} in front of $\bar{\partial}r_2$, hence it is negligible after multiplication by h .

Combining (6.8) and (6.9), the coefficient of the nonnegligible $h^{-2}\bar{\partial}r_2$ term is $(1 + \frac{\bar{z}}{4})^2 - \frac{\bar{z}}{2}(1 + \frac{\bar{z}}{4}) = 1 - \frac{\bar{z}^2}{16}$. All other mirror M -critical terms are $o(1)$ after multiplication by h . This proves (6.7). $\square$

By Lemma 6.3 and (6.2), the mirror M -critical Cauchy contribution is

$$(6.10) \quad 2\pi(\bar{\partial}^{*-1}[Q_B^m(\cdot; M)(1 - \bar{z}^2/16)\tilde{V}'])(0)\tilde{V}(0).$$

Combining (6.6), the local zeroth order terms in M , and (6.10), we obtain

$$(6.11) \quad \begin{aligned} \lim_{h \rightarrow 0} h(I_{\xi M}^m + I_{\eta M}^m) &= -2\pi\partial(\tilde{\Lambda}_0\tilde{F}_*\tilde{F}_1\tilde{F}_2\lambda_\beta\mathcal{B}(M^\beta))(0) + \pi\mathcal{Z}_M^m(M)(0) + \pi\mathcal{L}_{M,0}^{m, \text{loc}}(M)(0) \\ &\quad + 2\pi(\bar{\partial}^{*-1}[Q_B^m(\cdot; M)(1 - \bar{z}^2/16)\tilde{V}'])(0)\tilde{V}(0). \end{aligned}$$

6.3. The mirror residual equation. The remaining terms $I_{\eta\xi}^m$ and $I_{\nabla\nabla}^m$ contain only first derivatives of the two forward CGO solutions. Their leading local contribution contains the factor $\partial\Psi_2 = -z/2$, and hence the leading stationary phase coefficient vanishes. If a derivative falls on one of the smooth CGO prefactors $\tilde{F}_*$, $\tilde{F}_1$, $\tilde{F}_2$, on the density factor $\tilde{\Lambda}_0$, on a cutoff, or on a fixed tensorial coefficient, rather than on the exponential phase, then the term loses one power of h^{-1} . Terms containing $\tilde{r}_*$ or $\tilde{r}_1$ gain a factor h . The critical remainder enters only through r_2 , $\bar{\partial}r_2$, and ∂r_2 , never through an unreduced D^2r_2 . The estimates in Lemma 6.1, therefore, give

$$(6.12) \quad \lim_{h \rightarrow 0} h I_{\eta\xi}^m = 0,$$

and

$$(6.13) \quad \lim_{h \rightarrow 0} h I_{\nabla\nabla}^m = 0.$$

Since $I_h^m = 0$, multiplying by h and passing to the limit gives, by (6.5), (6.11), (6.12), and (6.13),

$$\begin{aligned} 0 &= 8\pi\partial^2(\tilde{\Lambda}_0\tilde{F}_*\tilde{F}_1\tilde{F}_2\mathbf{r})(0) + \pi\mathcal{L}_{\mathbf{r},\leq 1}^{m,\text{loc}}\mathbf{r}(0) - 2\pi\partial(\tilde{\Lambda}_0\tilde{F}_*\tilde{F}_1\tilde{F}_2\lambda_\alpha\mathcal{B}(M^\alpha))(0) + \pi\mathcal{Z}_M^m(0) \\ &\quad + \pi\mathcal{L}_{M,0}^{m,\text{loc}}(M)(0) + \mathcal{N}_{\mathbf{r}}^m(0) + 2\pi(\bar{\partial}^{*-1}[Q_B^m(\cdot; M)(1 - \bar{z}^2/16)\tilde{V}'])(0)\tilde{V}(0). \end{aligned}$$

Now translate the critical point to $a \in \tilde{\Omega}$ by using

$$\Psi_{1,a}(z) = (\bar{z} - \bar{a}) + \frac{(\bar{z} - \bar{a})^2}{8}, \quad \Psi_{2,a}(z) = -\frac{(z - a)^2}{4}, \quad \Psi_a^*(z) = -(\bar{z} - \bar{a}) + \frac{(\bar{z} - \bar{a})^2}{8}.$$

The smooth mirror CGO prefactors and the fixed coefficient factors are translated in the same way. The same calculation gives

$$(6.14) \quad \begin{aligned} 0 &= 8\pi\partial^2(\tilde{C}_a\mathbf{r})(a) + \pi\mathcal{L}_{\mathbf{r},\leq 1,a}^{m,\text{loc}}\mathbf{r}(a) - 2\pi\partial(\tilde{C}_a\lambda_\alpha\mathcal{B}(M^\alpha))(a) + \pi\mathcal{Z}_{M,a}^m(M)(a) \\ &\quad + \pi\mathcal{L}_{M,0,a}^{m,\text{loc}}(M)(a) + \mathcal{N}_{\mathbf{r},a}^m(a) + 2\pi(\bar{\partial}^{*-1}P_a^m(\cdot; M))(a)\tilde{V}(a), \end{aligned}$$

where $\tilde{C}_a := \tilde{\Lambda}_{0,a}\tilde{F}_{*,a}\tilde{F}_{1,a}\tilde{F}_{2,a}$. The translated mirror nonlocal amplitude is

$$P_a^m(x; M) = Q_a^m(x; M)\left(1 - \frac{(\bar{z} - \bar{a})^2}{16}\right)\tilde{V}'(x), \quad \text{and} \quad Q_a^m(x; M) = \tilde{C}_a(x)\kappa_\alpha(x)\mathcal{B}(M^\alpha)(x).$$

Moreover,

$$(6.15) \quad \mathcal{Z}_{M,a}^m(M)(a) = 2\tilde{\Lambda}_{0,a}(a)\tilde{F}_{*,a}(a)\tilde{F}_{1,a}(a)(\lambda_\beta(a)\partial\tilde{F}_{2,a}(a) + \kappa_\beta(a)\bar{\partial}\tilde{F}_{2,a}(a))\mathcal{B}(M^\beta)(a).$$

Dividing (6.14) by π and absorbing fixed numerical factors into the notation for the mirror scalar critical term and the mirror nonlocal Cauchy term, we define

$$\mathcal{K}_{\text{nl},a}^m(M)(a) = 2(\bar{\partial}^{*-1}P_a^m(\cdot; M))(a)\tilde{V}(a),$$

and the mirror residual equation is

$$(6.16) \quad 8\partial^2(\tilde{C}_a\mathbf{r}) - 2\partial(\tilde{C}_a\lambda_\alpha\mathcal{B}(M^\alpha)) + \mathcal{Z}_M^m(M) + \mathcal{L}_{\mathbf{r},\leq 1}^{m,\text{loc}}\mathbf{r} + \mathcal{L}_{M,0}^{m,\text{loc}}(M) + \mathcal{N}_{\mathbf{r}}^m + \mathcal{K}_{\text{nl}}^m(M) = 0 \quad \text{in } \tilde{\Omega}.$$

Together with (5.40), this gives the paired residual system extracted from the second variation.

7. REDUCTION TO THE AUGMENTED RESIDUAL SYSTEM

From this point to the end of Section 8, we write Ω for the transformed domain $\tilde{\Omega}$. We now rewrite the plus and mirror residual equations in the variables

$$W = J - \text{Id}, \quad s = q - 2\theta, \quad q = s + 2\theta.$$

The cutoffs used from now on are not inserted into the second integral identity. They are used only to localize the residual system already obtained in Sections 5 and 6. In particular, the Cauchy-transform terms produced by the critical CGO remainders remain in operator form until the unique continuation argument.

Throughout Sections 7 and 8,

$$(7.1) \quad q = \log \frac{\widehat{K}_1}{\widehat{K}_2 \circ J}, \quad \widehat{K}_j = \frac{K_j \circ \chi^{-1}}{m_\chi}, \quad N = DJ, \quad d = \det N, \quad \theta = \log d - q.$$

Then $\rho = e^\theta$, $\rho^{-1} = e^{-\theta}$, and $\mathbf{r} = 1 - e^{-\theta}$. Since $q = s + 2\theta$ and $\theta = \log \det DJ - q$,

$$(7.2) \quad s + 3\theta - \log \det(I + DW) = 0.$$

This algebraic relation will be kept as one equation in the augmented system.

Before substituting into the residual equations, we fix the convention for the translated critical center. Let $\Omega' \Subset O \Subset \Omega$, where O is open in the global isothermal coordinate. For $a \in \Omega'$ and $x \in O$, write

$$C_a(x) = C(a, x), \quad \tilde{C}_a(x) = \tilde{C}(a, x).$$

All derivatives below are taken in the x variable with a fixed, and only afterwards are restricted to the diagonal $x = a$.

The critical Cauchy terms from the plus and mirror residual equations have the form

$$\mathcal{K}_{\text{nl},a}(M)(a) = 2V(a)(\partial_x^{*-1}P_a(\cdot; M))(a)$$

and

$$\mathcal{K}_{\text{nl},a}^m(M)(a) = 2\tilde{V}(a)(\bar{\partial}_x^{*-1}P_a^m(\cdot; M))(a),$$

where

$$P_a(x; M) = p(a, x)\lambda_\alpha(x)\mathcal{A}(M^\alpha)(x), \quad P_a^m(x; M) = p^m(a, x)\kappa_\alpha(x)\mathcal{B}(M^\alpha)(x).$$

Here

$$p(a, x) = C(a, x)\left(1 - \frac{(z - a)^2}{16}\right)V'(x), \quad p^m(a, x) = \tilde{C}(a, x)\left(1 - \frac{(\bar{z} - \bar{a})^2}{16}\right)\tilde{V}'(x).$$

Lemma 7.1 (Smooth dependence on the critical center). *The coefficient functions C , $\tilde{C}$, p , and p^m belong to $C^\infty(\Omega' \times O)$. Moreover, $C(a, a) \neq 0$ and $\tilde{C}(a, a) \neq 0$ for every $a \in \Omega'$. After shrinking O if necessary, C and $\tilde{C}$ are nonvanishing in a neighborhood of the diagonal $\{(a, x) : x = a\}$.*

Consequently, after restricting to $x = a$, the translated plus and mirror residual identities are local differential identities on Ω' with smooth coefficients, apart from the explicitly retained localized Cauchy terms. After the cutoff localization in Section 8, the scalar Cauchy inputs have order at most one. The M -polarization inputs are reduced by the localized drift equation and the local Laplace parametrix, and become local lower-order terms, drift-resolvent perturbations, or drift-Cauchy perturbations.

Proof. The translated phases are polynomial in (a, z) and $(a, \bar{z})$. After shrinking O with $\bar{\Omega}' \Subset O$, the noncritical phase derivatives are uniformly bounded away from zero on $\bar{O}$ for all $a \in \Omega'$. The noncritical amplitude coefficients and the finite asymptotic coefficients are obtained from the smooth coefficients of the first linearized operator by algebraic operations, fixed properly supported Cauchy parametrices, and division by these nonvanishing phase derivatives. Hence, they are smooth in (a, x) .

The critical part is reduced to the first Neumann term by estimates uniform for $a \in \Omega'$. The remaining factors are precisely the explicit functions $p(a, x)$ and $p^m(a, x)$ displayed above, and are therefore smooth in (a, x) .

Since the density factors are positive and the CGO amplitude factors are nonzero at the critical point, $C(a, a)$ and $\tilde{C}(a, a)$ are nonzero. Compactness of $\bar{\Omega}'$ gives the asserted nonvanishing near the diagonal after possibly shrinking O .

Finally, for any smooth function $F(a, x)$ and any multi-index β , the quantity $\partial_x^\beta(F(a, x)u(x))|_{x=a}$ is a local differential expression in u with smooth coefficients depending on a . The critical Cauchy terms have fixed properly supported localized kernels and smooth coefficient factors. After the cutoff localization used in Section 8, the scalar Cauchy terms are localized Cauchy transforms with scalar inputs of order at most one. The M -polarization inputs are not reduced through an auxiliary first-order variable. Instead, Section 8 writes $\mathcal{H}^\alpha = D^2W^\alpha + \mathcal{H}_{\text{low}}^\alpha(W, DW)$, uses the drift equation with a localized Laplace parametrix, and treats the remaining nonlocal pieces as localized drift-resolvent and drift-Cauchy perturbations. $\square$

We next record the local form used near the boundary. Let $p_0 \in \partial\Omega$, and choose a flattened boundary chart

$$B_R^+ = B_R \cap \{x_2 > 0\}, \quad \Gamma_R = B_R \cap \{x_2 = 0\},$$

where B_R^+ is the interior side. We use nested cutoffs $\eta_0 \prec \eta \prec \eta_1$ in this collar. Let $\zeta \in C_0^\infty(B_R)$ be equal to 1 on the active diagonal neighborhood determined by $\text{supp } \eta_1$ and the properly supported localized Cauchy kernels.

For $a, x \in B_R^+$, define

$$\mathcal{B}_+(a; \mathbf{r}, M) = 8\bar{\partial}_x^2(C(a, x)\mathbf{r}(x))|_{x=a} - 2\bar{\partial}_x(C(a, x)\kappa_\alpha(x)\mathcal{A}(M^\alpha)(x))|_{x=a},$$

and

$$\mathcal{B}_-(a; \mathbf{r}, M) = 8\partial_x^2(\tilde{C}(a, x)\mathbf{r}(x))|_{x=a} - 2\partial_x(\tilde{C}(a, x)\lambda_\alpha(x)\mathcal{B}(M^\alpha)(x))|_{x=a}.$$

The terms $\mathcal{L}_+$ and $\mathcal{L}_-$ denote the local lower-order parts. The critical Cauchy terms are written as

$$\mathcal{C}_+^\zeta(a; Z) = \sum_\nu b_\nu(a) [\partial_x^{*-1}(\zeta(x)b_{\nu,1}(a, x)B_{\nu,+}(Z)(x))]_{x=a},$$

and

$$\mathcal{C}_-^\zeta(a; Z) = \sum_\nu \tilde{b}_\nu(a) [\bar{\partial}_x^{*-1}(\zeta(x)\tilde{b}_{\nu,1}(a, x)B_{\nu,-}(Z)(x))]_{x=a}.$$

Before the drift reduction, the inputs $B_{\nu,\pm}$ are the scalar critical inputs and the M -polarization inputs obtained in Sections 5 and 6. In Section 8, the scalar inputs become order-one scalar inputs, while the M -polarization inputs are written using $\mathcal{H}^\alpha = D^2W^\alpha + \mathcal{H}_{\text{low}}^\alpha(W, DW)$ and reduced through the localized drift equation.

Lemma 7.2 (Localized boundary-collar residual identities). *After shrinking R , the coefficient families C and $\tilde{C}$ belong to $C^\infty(\overline{B_R^+} \times \overline{B_R^+})$ and satisfy $C(a, a) \neq 0$, $\tilde{C}(a, a) \neq 0$ for $a \in \overline{B_R^+}$. For every interior center $a \in B_R^+$, the plus and mirror residual equations obtained in Sections 5 and 6 have the localized one-sided forms*

$$0 = \eta_0(a) [\mathcal{B}_+(a; \mathbf{r}, M) + \mathcal{L}_+(a; \mathbf{r}, M) + \mathcal{C}_+^\zeta(a; Z)] + R_{\eta_0, \eta_1} Z(a),$$

and

$$0 = \eta_0(a) [\mathcal{B}_-(a; \mathbf{r}, M) + \mathcal{L}_-(a; \mathbf{r}, M) + \mathcal{C}_-^\zeta(a; Z)] + R_{\eta_0, \eta_1} Z(a).$$

Here, the cutoff localization is performed only on the already-derived residual equations. It is not inserted into the second integral identity. In the boundary Carleman argument, the inputs in $\mathcal{C}_\pm^\zeta$ are first cut off in B_R^+ and then extended by zero to the reflected disk. The original uncut one-sided nonlocal expression is never extended across the boundary.

Proof. For a fixed interior center $a \in B_R^+$, the residual identities are exactly the plus and mirror identities derived in Sections 5 and 6. Near the boundary, the stationary phase cutoff used in those sections may be chosen depending on a , with support contained in B_R^+ and equal to 1 near a . This is only part of the extraction of the residual equation for that fixed interior center. The boundary Carleman argument below uses the resulting residual equation and then localizes it; it does not insert a new cutoff into the second integral identity.

The diagonal coefficients left by the stationary phase computation are independent of the shrinking radius of the auxiliary cutoff. They are determined by finite jets at $x = a$ of the translated phases, the background coefficients, and the CGO amplitude factors. If two such cutoffs are both equal to 1 near a , their difference is supported away from the critical point. The corresponding smooth-amplitude oscillatory contribution is nonstationary to infinite order, and the corresponding Cauchy contribution is a properly supported smoothing cutoff remainder once the local Cauchy kernel is fixed. Thus, the diagonal coefficients and the localized operator kernels are not changed.

The translated phases are polynomial in (a, x) , and after shrinking R , the noncritical phase derivatives stay uniformly nonzero on the local supports used in the plus and mirror constructions. The amplitude factors appearing in the local CGO parametrices therefore depend smoothly on (a, x) . This gives $C, \tilde{C} \in C^\infty(B_R^+ \times \overline{B_R^+})$. Since the diagonal factors are products of a positive density and nonvanishing amplitude factors, $C(a, a)$ and $\tilde{C}(a, a)$ remain nonzero after shrinking the collar.

The smoothness statement is understood in the one-sided sense up to the flattened boundary. The residual identities are used only for interior centers $a \in B_R^+$. The extension of C and $\tilde{C}$ to $\overline{B_R^+} \times \overline{B_R^+}$ is used to obtain uniform coefficient bounds in the boundary Carleman argument, not to construct a CGO solution with a boundary critical center. Indeed, the diagonal coefficient obtained by stationary phase depends only on a finite jet at $x = a$ of the translated phases, the background coefficients, and the CGO amplitudes. These jets have one-sided smooth limits as a approaches the flattened boundary. The same argument applies to the mirror coefficient family.

It remains to justify the cutoff insertion in the critical Cauchy terms. Consider a plus Cauchy term before inserting ζ : $b_\nu(a) [\partial_x^{*-1}(b_{\nu,1}(a, x) B_{\nu,+}(Z)(x))]_{x=a}$. After replacing the input by the cut input, the difference is $b_\nu(a) [\partial_x^{*-1}((1 - \zeta(x)) b_{\nu,1}(a, x) B_{\nu,+}(Z)(x))]_{x=a}$. Multiplying by the active output cutoff $\eta_0(a)$, this term is zero if the localized kernel from $\text{supp } \eta_0$ does not meet $\text{supp}(1 - \zeta)$. It is also zero if the source lies in an already known zero region. In the remaining case, the output and source supports are separated, the localized Cauchy kernel is smooth on the relevant product of supports, and the term is a separated-support smoothing cutoff remainder. Hence, it belongs to $R_{\eta_0, \eta_1} Z$. The mirror Cauchy terms are identical, with $\bar{\partial}^{*-1}$ in place of ∂^{*-1} .

Near Γ_R , the critical Cauchy terms are interpreted only after this localization. The one-sided input is first formed in B_R^+ , multiplied by the cutoff ζ , extended by zero to the reflected disk, and then acted on by the corresponding localized adjoint Cauchy inverse. Thus, no derivative falls on the characteristic function of the half-disk, and no boundary-supported distribution is created. $\square$

7.1. Substitution in the two residual equations. We first rewrite the plus residual equation in terms of θ and W . Since $\rho = e^\theta$, one has $\mathbf{r} = 1 - e^{-\theta}$ and $M^\alpha = e^{-\theta} \mathcal{H}^\alpha$. The tensor $\mathcal{H}^\alpha$ is the covariant Hessian of J defined in (4.2). Because the identity map has zero covariant Hessian for the same connection in the domain and target,

$$\mathcal{H}^\alpha = D^2 W^\alpha + \mathcal{H}_{\text{low}}^\alpha(W, DW),$$

where $\mathcal{H}_{\text{low}}^\alpha$ is local of order at most one in W and vanishes when $W = DW = 0$. In a flat affine coordinate for the pulled-back Euclidean connection, this reduces to $\mathcal{H}^\alpha = D^2W^\alpha$.

Recall from Section 5 that $C_a = \Lambda_{0,a}F_{*,a}F_{1,a}F_{2,a}$. The displayed local second order block in (5.40) is $8\bar{\partial}^2(C_a\mathbf{r}) - 2\bar{\partial}(C_a\kappa_\alpha\mathcal{A}(M^\alpha))$. Using $\mathbf{r} = 1 - e^{-\theta}$ and $M^\alpha = e^{-\theta}\mathcal{H}^\alpha$, this becomes $8\bar{\partial}^2[C_a(1 - e^{-\theta})] - 2\bar{\partial}[C_ae^{-\theta}\kappa_\alpha\mathcal{A}(\mathcal{H}^\alpha)]$. The local zeroth order M -linear term extracted in (5.39) becomes

$$\mathcal{Z}_{M,a}(e^{-\theta}\mathcal{H})(a) = 2\Lambda_{0,a}(a)F_{*,a}(a)F_{1,a}(a)(\lambda_\beta(a)\partial F_{2,a}(a) + \kappa_\beta(a)\bar{\partial}F_{2,a}(a))e^{-\theta(a)}\mathcal{A}(\mathcal{H}^\beta)(a).$$

The nonlocal Cauchy term has the form

$$\mathcal{K}_{\text{nl},a}(e^{-\theta}\mathcal{H})(a) = 2(\partial_x^{*-1}P_a(x; e^{-\theta}\mathcal{H}))|_{x=a}V(a),$$

where

$$P_a(x; e^{-\theta}\mathcal{H}) = C_a(x)\lambda_\alpha(x)e^{-\theta(x)}\mathcal{A}(\mathcal{H}^\alpha)(x)\left(1 - \frac{(z-a)^2}{16}\right)V'(x).$$

The localization of this Cauchy operator is not part of the second integral identity. It is performed only after multiplying the reduced equation by local cutoffs in the unique continuation argument.

Therefore (5.40) becomes

$$(7.3) \quad 0 = 8\bar{\partial}^2[C_a(1 - e^{-\theta})] - 2\bar{\partial}[C_ae^{-\theta}\kappa_\alpha\mathcal{A}(\mathcal{H}^\alpha)] + \mathcal{Z}_M(e^{-\theta}\mathcal{H}) + \mathcal{L}_{\mathbf{r},\leq 1}^{\text{loc}}(1 - e^{-\theta}) + \mathcal{L}_{M,0}^{\text{loc}}(e^{-\theta}\mathcal{H}) + \mathcal{N}_{\mathbf{r}}(1 - e^{-\theta}) + \mathcal{K}_{\text{nl}}(e^{-\theta}\mathcal{H}).$$

Here $\mathcal{L}_{\mathbf{r},\leq 1}^{\text{loc}}$ is local of order at most one in $\mathbf{r}$, $\mathcal{L}_{M,0}^{\text{loc}}$ is local of order zero in M , and $\mathcal{N}_{\mathbf{r}}(1 - e^{-\theta})$ denotes the scalar critical contribution after substituting $\mathbf{r} = 1 - e^{-\theta}$. By Lemma 5.9, its linear part is of order at most one in θ , possibly followed by a localized Cauchy transform.

We now rewrite the mirror residual equation (6.16). Recall from Section 6 that $\tilde{C}_a = \tilde{\Lambda}_{0,a}\tilde{F}_{*,a}\tilde{F}_{1,a}\tilde{F}_{2,a}$. The displayed mirror local second order block in (6.16) is $8\partial^2(\tilde{C}_a\mathbf{r}) - 2\partial(\tilde{C}_a\lambda_\alpha\mathcal{B}(M^\alpha))$. Thus, after substituting $\mathbf{r} = 1 - e^{-\theta}$ and $M^\alpha = e^{-\theta}\mathcal{H}^\alpha$, this block becomes $8\partial^2[\tilde{C}_a(1 - e^{-\theta})] - 2\partial[\tilde{C}_ae^{-\theta}\lambda_\alpha\mathcal{B}(\mathcal{H}^\alpha)]$. The mirror local zeroth order M -linear term extracted in (6.15) becomes

$$\mathcal{Z}_{M,a}^m(e^{-\theta}\mathcal{H})(a) = 2\tilde{\Lambda}_{0,a}(a)\tilde{F}_{*,a}(a)\tilde{F}_{1,a}(a)(\lambda_\beta(a)\partial\tilde{F}_{2,a}(a) + \kappa_\beta(a)\bar{\partial}\tilde{F}_{2,a}(a))e^{-\theta(a)}\mathcal{B}(\mathcal{H}^\beta)(a).$$

The mirror nonlocal Cauchy term has the form

$$\mathcal{K}_{\text{nl},a}^m(e^{-\theta}\mathcal{H})(a) = 2(\bar{\partial}_x^{*-1}P_a^m(x; e^{-\theta}\mathcal{H}))|_{x=a}\tilde{V}(a),$$

where

$$P_a^m(x; e^{-\theta}\mathcal{H}) = \tilde{C}_a(x)\kappa_\alpha(x)e^{-\theta(x)}\mathcal{B}(\mathcal{H}^\alpha)(x)\left(1 - \frac{(\bar{z}-\bar{a})^2}{16}\right)\tilde{V}'(x).$$

Again, the localization of this mirror Cauchy operator is performed only in the unique continuation argument.

Therefore (6.16) becomes

$$(7.4) \quad 0 = 8\partial^2[\tilde{C}_a(1 - e^{-\theta})] - 2\partial[\tilde{C}_ae^{-\theta}\lambda_\alpha\mathcal{B}(\mathcal{H}^\alpha)] + \mathcal{Z}_M^m(e^{-\theta}\mathcal{H}) + \mathcal{L}_{\mathbf{r},\leq 1}^{m,\text{loc}}(1 - e^{-\theta}) + \mathcal{L}_{M,0}^{m,\text{loc}}(e^{-\theta}\mathcal{H}) + \mathcal{N}_{\mathbf{r}}^m(1 - e^{-\theta}) + \mathcal{K}_{\text{nl}}^m(e^{-\theta}\mathcal{H}).$$

Here $\mathcal{L}_{\mathbf{r},\leq 1}^{m,\text{loc}}$ is local of order at most one in $\mathbf{r}$, $\mathcal{L}_{M,0}^{m,\text{loc}}$ is local of order zero in M , and $\mathcal{N}_{\mathbf{r}}^m(1 - e^{-\theta})$ denotes the mirror scalar critical contribution after substituting $\mathbf{r} = 1 - e^{-\theta}$. By Lemma 6.2, its linear part is of order at most one in θ , possibly followed by a localized mirror Cauchy transform.

7.2. Boundary residual jets. We record the boundary information, which starts the boundary Carleman propagation from a flattened boundary segment into an interior collar.

Proposition 7.3 (Boundary determination of the residual jets). *On $\partial\Omega$, the residual variables satisfy*

$$(7.5) \quad q|_{\partial\Omega} = 0, \quad Dq|_{\partial\Omega} = 0, \quad J|_{\partial\Omega} = \text{Id}, \quad DJ|_{\partial\Omega} = I.$$

Consequently, with $W = J - \text{Id}$ and $s = q - 2\theta$,

$$(7.6) \quad \theta|_{\partial\Omega} = 0, \quad s|_{\partial\Omega} = 0, \quad D\theta|_{\partial\Omega} = 0, \quad Ds|_{\partial\Omega} = 0.$$

Moreover,

$$(7.7) \quad W|_{\partial\Omega} = 0, \quad DW|_{\partial\Omega} = 0, \quad D^2W|_{\partial\Omega} = 0.$$

In particular,

$$M^\alpha|_{\partial\Omega} = 0, \quad \alpha = 1, 2.$$

Proof. The boundary-fixing normalization gives $J = \text{Id}$ on $\partial\Omega$, and Corollary 3.6 gives $DJ = I$ on $\partial\Omega$. Since

$$q = \log \frac{\widehat{K}_1}{\widehat{K}_2 \circ J}, \quad \widehat{K}_j = \frac{K_j \circ \chi^{-1}}{m_\chi},$$

the boundary curvature normalization, together with $J = \text{Id}$, gives $q = 0$ on $\partial\Omega$. Differentiating,

$$Dq = D \log \widehat{K}_1 - DJ^T (D \log \widehat{K}_2) \circ J.$$

Using $J = \text{Id}$, $DJ = I$, and the first order boundary matching of K_1 and K_2 , the common factor m_χ cancels and gives $Dq = 0$ on $\partial\Omega$. This proves (7.5).

It remains to derive the higher boundary jets. Since $W = J - \text{Id}$, the identities $J = \text{Id}$ and $DJ = I$ give $W = 0$ and $DW = 0$ on $\partial\Omega$. The drift equation (3.23), together with $Dq = 0$ and (3.25), gives $\Delta W^\alpha = 0$ on $\partial\Omega$. In boundary normal coordinates, $W = 0$ and $DW = 0$ imply $\partial_{\tau\tau} W^\alpha = \partial_{\tau\nu} W^\alpha = 0$ on $\partial\Omega$; hence the identity $\Delta W^\alpha = 0$ gives $\partial_{\nu\nu} W^\alpha = 0$. Therefore, $D^2W = 0$ on $\partial\Omega$.

By Jacobi's formula and $DJ = I$, $D^2W = 0$ gives $D \log \det DJ = 0$ on $\partial\Omega$. Since $\theta = \log \det DJ - q$, we get $\theta = 0$ and $D\theta = 0$ on $\partial\Omega$. Then $s = q - 2\theta$ gives $s = 0$ and $Ds = 0$ on $\partial\Omega$.

Finally, the identity map has zero covariant Hessian for the same connection in the domain and target. Since $J = \text{Id}$, $DJ = I$, and $D^2W = 0$ on $\partial\Omega$, we have $\mathcal{H}^\alpha = 0$ there. As $M^\alpha = e^{-\theta} \mathcal{H}^\alpha$, this gives $M^\alpha = 0$ on $\partial\Omega$ for $\alpha = 1, 2$. $\square$

7.3. The scalar equation for θ and the paired augmented system. We record the scalar equation for θ in the isothermal coordinate.

Lemma 7.4 (Scalar equation for θ). *Let $N = DJ$, $P = N^{-1}$, and write the partial-derivative form of the drift equation as*

$$(7.8) \quad \Delta J^\alpha = N_m^\alpha \partial_m q + \mathfrak{R}^\alpha(x, J, N).$$

Then

$$(7.9) \quad \Delta \theta = -P_\alpha^r (\partial_k N_r^\beta) P_\beta^i (\partial_k N_i^\alpha) + P_\alpha^i (\partial_i N_m^\alpha) \partial_m q + P_\alpha^i \partial_i \mathfrak{R}^\alpha(x, J, N).$$

Equivalently, since $q = s + 2\theta$,

$$(7.10) \quad \Delta \theta = -P_\alpha^r (\partial_k N_r^\beta) P_\beta^i (\partial_k N_i^\alpha) + P_\alpha^i (\partial_i N_m^\alpha) \partial_m (s + 2\theta) + P_\alpha^i \partial_i \mathfrak{R}^\alpha(x, J, N).$$

Proof. By definition, $\theta = \log \det N - q$. Jacobi's formula gives $\partial_k \log \det N = P_\alpha^i \partial_k N_i^\alpha$. Hence,

$$\Delta \theta = \partial_k (P_\alpha^i \partial_k N_i^\alpha) - \Delta q.$$

Since $\partial_k P_\alpha^i = -P_\alpha^r (\partial_k N_r^\beta) P_\beta^i$, we obtain

$$\Delta \theta = -P_\alpha^r (\partial_k N_r^\beta) P_\beta^i (\partial_k N_i^\alpha) + P_\alpha^i \partial_k \partial_k N_i^\alpha - \Delta q.$$

As $N_i^\alpha = \partial_i J^\alpha$, the second term is $P_\alpha^i \partial_i \Delta J^\alpha$. Using (7.8), we get

$$P_\alpha^i \partial_i \Delta J^\alpha = P_\alpha^i (\partial_i N_m^\alpha) \partial_m q + P_\alpha^i N_m^\alpha \partial_i \partial_m q + P_\alpha^i \partial_i \mathfrak{R}^\alpha(x, J, N).$$

Since $P_\alpha^i N_m^\alpha = \delta_m^i$, the second term is Δq , which cancels the $-\Delta q$ above. This proves (7.9). Substituting $q = s + 2\theta$ gives (7.10). $\square$

Combining (7.2), (7.10), (3.23), (7.3), and (7.4), we obtain the paired augmented residual system

$$(7.11) \quad \begin{cases} s + 3\theta - \log \det(I + DW) = 0, \\ \Delta \theta = -P_\alpha^r (\partial_k N_r^\beta) P_\beta^i (\partial_k N_i^\alpha) + P_\alpha^i (\partial_i N_m^\alpha) \partial_m (s + 2\theta) + P_\alpha^i \partial_i \mathfrak{R}^\alpha(x, J, N), \\ \Delta W^\alpha = N_i^\alpha \partial_i (s + 2\theta) + \mathfrak{R}^\alpha(x, J, N), \\ \mathcal{E}_+(\theta, W) = 0, \\ \mathcal{E}_-^m(\theta, W) = 0. \end{cases}$$

Here $J = \text{Id} + W$, $q = s + 2\theta$, $N = I + DW$, $P = N^{-1}$, and $\mathcal{E}_+(\theta, W) = 0$ and $\mathcal{E}_-^m(\theta, W) = 0$ denote (7.3) and (7.4), respectively. The boundary unique continuation argument for (7.11) is carried out in Section 8.

This is the system used in the unique continuation argument. In Section 8, the variables are $Z = (\theta, s, W)$. The principal equations are the scalar Laplace equation for θ , the plus and mirror residual equations for s , and the componentwise drift equation for W . The displayed residual blocks are reduced locally by differentiating the drift equation. The remaining terms are treated as local lower-order terms, localized drift-resolvent perturbations, drift-Cauchy perturbations, or localized Cauchy perturbations.

8. UNIQUE CONTINUATION AND ELIMINATION OF THE RESIDUAL GAUGE

We now prove that the residual system forces

$$\theta = s = 0, \quad W = 0 \quad \text{in } \Omega.$$

The variable θ is kept as an independent scalar unknown; we do not replace it using the algebraic relation. The equation for W is kept in componentwise drift form, and the differentiated contractions involving W are reduced only locally by using the drift equation and properly supported Laplace parametrices. We write

$$Z = (\theta, s, W), \quad q = s + 2\theta, \quad J = \text{Id} + W, \quad N = I + DW, \quad P = N^{-1}.$$

The algebraic identity is

$$s + 3\theta - \log \det(I + DW) = 0.$$

It is used to identify the zero set and to conclude $q = 0$ at the end. It is not used to replace θ by derivatives of W in the Carleman system. The drift equation is

$$(8.1) \quad \Delta W^\alpha = N_i^\alpha \partial_i (s + 2\theta) + \mathfrak{R}^\alpha(x, \text{Id} + W, I + DW).$$

The scalar equation for θ is (7.10). In the local estimates, its right-hand side is denoted by $\mathfrak{T}_\theta(x, Z, DZ, D^2W)$; this expression is smooth, vanishes at $Z = 0$, and its linear part is local of order at most one in θ and s and at most two in W .

The Carleman weight is the convexified logarithmic weight

$$(8.2) \quad \varphi(x) = \log|x - z_0|, \quad \varphi_\varepsilon(x) = \varphi(x) - \frac{|x - z_0|^2}{2\varepsilon\tau}, \quad w_\tau = e^{-\tau\varphi_\varepsilon}.$$

The point z_0 is placed in the already known zero region and outside the active target annulus. After shrinking the propagation disk, $|\nabla\varphi_\varepsilon|$ is bounded from below on every active support for all large τ .

8.1. Localized classes, zero extension, and Carleman estimates. Let B_R be either an interior disk or the reflected full disk associated with a flattened boundary half-disk. We use nested cutoffs

$$\eta_0 \prec \eta \prec \eta_1 \prec \eta_2.$$

All equations below are read on $\text{supp } \eta_0$. The residual equations and the augmented system have already been derived; the cutoffs in this section only localize that system for the Carleman argument.

Every local operator, localized Cauchy inverse, localized adjoint Cauchy inverse, localized Laplace parametrix, and smoothing cutoff operator is applied to cut inputs. The kernels are chosen properly supported and close to the diagonal. Commutator terms outside the active cutoff region either vanish by proper support, lie in an already known zero region, or have separated supports with the favorable Carleman weight orientation.

A separated-support estimate is used only in the following form. Let A_{η_0, η_1} have a smooth kernel on $E_x \times E_y$, where the output variable is restricted to E_x , the source variable is restricted to E_y , and $\text{dist}(E_x, E_y) > 0$. Then, for every integer $m \geq 0$,

$$\|A_{\eta_0, \eta_1} F\|_{L^2(E_x)} \leq C_m \sum_{|\alpha| \leq m} \|D^\alpha F\|_{L^2(E_y)}.$$

For the Carleman weight $w_\tau = e^{-\tau\varphi_\varepsilon}$ this gives

$$(8.3) \quad \|w_\tau A_{\eta_0, \eta_1} F\|_{L^2(E_x)} \leq C_m \frac{\sup_{E_x} w_\tau}{\inf_{E_y} w_\tau} \sum_{|\alpha| \leq m} \|w_\tau D^\alpha F\|_{L^2(E_y)}.$$

We use (8.3) only when $\inf_{E_x} \varphi_\varepsilon \geq \sup_{E_y} \varphi_\varepsilon + c$ for some $c > 0$. Then $\sup_{E_x} w_\tau \leq C e^{-c\tau} \inf_{E_y} w_\tau$.

The notation $R_{\eta_0, \eta_1} Z$ denotes a finite sum of local commutators and smoothing cutoff remainders supported away from the active diagonal region. After multiplication by the Carleman weight and restriction to $\text{supp } \eta_0$, these terms are estimated by $\mathcal{E}_{\eta_0, \eta_1}(\tau)$. The quantity $\mathcal{E}_{\eta_0, \eta_1}(\tau)$ may change from line to line, and consists only of terms supported in the already known zero set, direct local outer commutators satisfying the weight separation above, and smoothing cutoff remainders satisfying the same separation.

A localized Cauchy perturbation means a finite sum of terms of the form

$$C_\partial B_1(\eta Z), \quad C_{\bar{\partial}} B_1(\eta Z), \quad \partial^{*-1} B_1(\eta Z), \quad \bar{\partial}^{*-1} B_1(\eta Z).$$

Here B_1 is a properly supported local differential operator of order at most one in $Z = (\theta, s, W)$. In the scalar critical terms, the input is of order at most one in θ . In the lower-order terms coming from $\mathcal{H}_{\text{low}}(W, DW)$, the input is of order at most one in W . No ordinary localized Cauchy perturbation is allowed to contain second derivatives of W as input. The terms with linear second derivatives of W are reduced through the drift-resolvent classes below.

Let

$$G_{\Delta}^{\zeta, \zeta'} f := \zeta \Delta^{-1}(\zeta' f).$$

This denotes a properly supported local Laplace parametrix. On the active region, whenever the input is supported where $\zeta' = 1$, it satisfies

$$\Delta G_{\Delta}^{\zeta, \zeta'} f = f + S_{\Delta}^{\zeta, \zeta'} f.$$

All smoothing cutoff errors are included in $R_{\eta_0, \eta_1} Z$. A localized scalar drift-resolvent perturbation of source order at most one is a finite sum of terms of the form

$$(8.4) \quad a(x) D^\beta G_\Delta^{\zeta, \zeta'} F_{\text{sc}}(x), \quad |\beta| \leq 2.$$

Here a is smooth, and F_{sc} is a properly supported local differential expression of order at most one in $(\eta\theta, \eta s)$. In the actual drift reduction, the scalar source often has a special form

$$F_{\text{sc}} = D_j(b(x) B_{\text{sc},0}(\eta\theta, \eta s)),$$

where b is smooth and $B_{\text{sc},0}$ is local of order at most zero in (θ, s) . The source-order formulation in (8.4) is used to include both these principal drift sources and the lower drift sources produced by localization.

A localized W -drift-resolvent perturbation of source order at most two is a finite sum of terms of the form $a(x) D^\beta G_\Delta^{\zeta, \zeta'} F_W(x)$, $|\beta| \leq 2$. Here, a is smooth, and F_W is a properly supported local differential expression of order at most two in ηW . These terms arise when the localized drift equation is solved and the resulting second derivatives of W are represented through the local Laplace parametrix.

A localized drift-Cauchy perturbation means a finite sum obtained by placing one localized Cauchy inverse or one localized adjoint Cauchy inverse outside one of the drift-resolvent perturbations above. Thus, the model terms are $\mathfrak{C}\mathfrak{D}_{\text{sc},1}(\eta\theta, \eta s)$ and $\mathfrak{C}\mathfrak{D}_{W,2}(\eta W)$, where $\mathfrak{C} \in \{C_\partial, C_{\bar{\partial}}, \partial^{*-1}, \bar{\partial}^{*-1}\}$. Here $\mathfrak{D}_{\text{sc},1}$ is a localized scalar drift-resolvent perturbation of source order at most one, and $\mathfrak{D}_{W,2}$ is a localized W -drift-resolvent perturbation of source order at most two. We use this notation only for the compositions which arise from the localized M -critical terms after the drift reduction.

If $\zeta' \prec \zeta$, we use

$$C_{\bar{\partial}}^{\zeta, \zeta'} f := \zeta \bar{\partial}^{-1}(\zeta' f), \quad C_{\partial}^{\zeta, \zeta'} f := \zeta \partial^{-1}(\zeta' f).$$

Thus, $\bar{\partial} C_{\bar{\partial}}^{\zeta, \zeta'} f = f + S_{\zeta, \zeta'} f$ and $\partial C_{\partial}^{\zeta, \zeta'} f = f + S_{\zeta, \zeta'}^\partial f$ whenever f is supported where $\zeta' = 1$. The remainders are smoothing cutoff terms and are included in $R_{\eta_0, \eta_1} Z$.

Lemma 8.1 (Cutoff decomposition for localized Cauchy and drift-resolvent perturbations). *Let $\mathcal{C}$ be one of $C_\partial, C_{\bar{\partial}}, \partial^{*-1}$ and $\bar{\partial}^{*-1}$, with the localized properly supported kernel fixed above. Let $\mathfrak{D}$ be a localized drift-resolvent perturbation. Let B be a properly supported local differential operator in Z , and let $\eta_0 \prec \eta \prec \eta_1$. Then, on $\text{supp } \eta_0$,*

$$\eta_0 \mathcal{C} B Z = \eta_0 \mathcal{C} B(\eta Z) + R_{\eta_0, \eta_1} Z, \quad \eta_0 \mathfrak{D} B Z = \eta_0 \mathfrak{D} B(\eta Z) + R_{\eta_0, \eta_1} Z.$$

The same statement holds for localized drift-Cauchy perturbations $\mathcal{C}\mathfrak{D} B Z$. In a flattened boundary chart, the one-sided input is first cut off in B_R^+ , then extended by zero to the reflected disk, and only then acted on by the localized Cauchy inverse, localized adjoint Cauchy inverse, localized Laplace parametrix, or their drift-Cauchy composition.

Proof. Write $Z = \eta Z + (1 - \eta)Z$. If the localized kernel from $\text{supp } \eta_0$ does not reach $\text{supp}(1 - \eta)$, the second term is zero by proper support. If it reaches only a region where the residual variables have already been proved to vanish, it is zero there. In the remaining case the output and source supports are separated, so the Cauchy singularity and the Laplace singularity are absent and the corresponding kernels are smooth on the relevant product of supports. This gives (8.3). The same argument applies after composing a localized drift-resolvent perturbation with one localized Cauchy or adjoint Cauchy inverse.

In a boundary chart, the operator is not applied to an uncut one-sided nonlocal expression. The input is first localized in B_R^+ , then extended by zero to the reflected disk, and the localized full-disk operator is applied only after this step. Thus no derivative falls on the characteristic function of the half-disk, and no boundary-supported distribution is created. $\square$

A nonlinear lower order term $\mathcal{N}_{\text{low}}(Z)$ denotes a finite sum of products of residual quantities. After shrinking the propagation disk, it satisfies

$$(8.5) \quad \|w_\tau \eta_0 \mathcal{N}_{\text{low}}(Z)\|_{L^2}^2 \leq \delta \mathcal{C}_\tau(\eta Z) + \mathcal{E}_{\eta_0, \eta_1}(\tau).$$

Here, $\delta > 0$ can be made arbitrarily small at a propagation front. Indeed, one factor in each nonlinear product is small in L^∞ because the residual variables already vanish at the base point; in the boundary case this uses the boundary jets from Proposition 7.3.

Lemma 8.2 (Boundary conformal charts for the local principal blocks). *Let $p \in \partial\Omega$ in the global isothermal coordinate. After shrinking the neighborhood of p , there is a smooth conformal coordinate ζ mapping the one-sided neighborhood of Ω to a half-disk and mapping $\partial\Omega$ to the diameter $\text{Im } \zeta = 0$. In this coordinate,*

$$\bar{\partial}_z^2 = a_+(\zeta) \bar{\partial}_\zeta^2 + P_{+,1}, \quad \partial_z^2 = a_-(\zeta) \partial_\zeta^2 + P_{-,1}, \quad \Delta_z = a_0(\zeta) \Delta_\zeta.$$

Here a_+ , a_- and a_0 are smooth nonvanishing functions, and $P_{+,1}$ and $P_{-,1}$ are local differential operators of order at most one.

Proof. This is the standard boundary conformal flattening for a smooth planar domain. Since the coordinate change is conformal, the first-order Cauchy–Riemann operators transform by multiplication by nonvanishing smooth factors. Squaring gives the two identities for $\bar{\partial}^2$ and ∂^2 , with derivatives of the conformal factors contributing only order-one terms. The Euclidean Laplacian transforms by a nonvanishing conformal factor. $\square$

Lemma 8.3 (Zero extension across a flattened boundary). *Let $B_R^+ = B_R \cap \{x_2 > 0\}$ be a flattened boundary chart and let $\Gamma_R = B_R \cap \{x_2 = 0\}$. Suppose that $\theta = 0$, $D\theta = 0$, $s = 0$, $Ds = 0$, $W = 0$, $DW = 0$, and $D^2W = 0$ on Γ_R . Let θ^0 , s^0 , and W^0 be the zero extensions to the reflected full disk B_R . Then $\theta^0, s^0 \in H_{\text{loc}}^2(B_R)$ and $W^0 \in H_{\text{loc}}^3(B_R)$. Moreover, $D^\alpha \theta^0 = (D^\alpha \theta)^0$ and $D^\alpha s^0 = (D^\alpha s)^0$ for $|\alpha| \leq 2$, and $D^\beta W^0 = (D^\beta W)^0$ for $|\beta| \leq 3$ in the sense of distributions. Therefore, the local second-order operators in θ and s , the componentwise Laplacian in W , and the localized drift reductions create no boundary distribution on Γ_R .*

Proof. Let f be smooth up to Γ_R from the interior side, and let $f^0 = H(x_2)f$ be its zero extension. When one differentiates f^0 in the normal direction, possible boundary terms are supported on Γ_R and are determined by the traces of f and its lower normal derivatives. Thus, if $D^\gamma f = 0$ on Γ_R for all $|\gamma| \leq m-1$, then no boundary measure occurs in $D^\alpha f^0$ for $|\alpha| \leq m$, and $D^\alpha f^0 = (D^\alpha f)^0$ for all such α . Applying this with $m = 2$ to θ and s gives $\theta^0, s^0 \in H_{\text{loc}}^2(B_R)$. Applying it with $m = 3$ to W gives $W^0 \in H_{\text{loc}}^3(B_R)$. $\square$

Lemma 8.4 (Carleman estimates). *There exist $\varepsilon_0 > 0$, $C > 0$, and $\tau_0 > 0$ such that, for $0 < \varepsilon < \varepsilon_0$, $\tau \geq \tau_0$, and $v \in C_0^\infty$ supported in the active annulus,*

$$(8.6) \quad \|w_\tau v\|_{L^2}^2 \leq C\varepsilon \|w_\tau \bar{\partial} v\|_{L^2}^2,$$

and

$$(8.7) \quad \|w_\tau v\|_{L^2}^2 \leq C\varepsilon \|w_\tau \partial v\|_{L^2}^2.$$

The same estimates hold in a flattened boundary chart after zero extension, provided the zero extension creates no boundary distribution for the differentiated quantity to which the estimate is applied.

Proof. We prove (8.6). The estimate (8.7) follows by complex conjugation. Put $h = \tau^{-1}$ and set

$$(8.8) \quad u = w_\tau v = e^{-\tau \varphi_\varepsilon} v = e^{-\varphi_\varepsilon/h} v.$$

Then $v = e^{\varphi_\varepsilon/h}u$. Since φ_ε is real-valued,

$$(8.9) \quad w_\tau \bar{\partial} v = e^{-\varphi_\varepsilon/h} \bar{\partial} (e^{\varphi_\varepsilon/h} u) = (\bar{\partial} + h^{-1} \bar{\partial} \varphi_\varepsilon) u.$$

We use the local first-order Carleman estimate with convexified logarithmic weight, in the form of [GT11, Lemma 3.2]. The same estimate is also the first-order Carleman input used in the nonlocal unique continuation argument in [LL25a, Section 6]. In the present notation, with $h = \tau^{-1}$ and

$$\varphi_\varepsilon = \varphi - \frac{|x - z_0|^2}{2\varepsilon\tau},$$

the local estimate gives, after shrinking the active annulus if necessary,

$$(8.10) \quad \|u\|_{L^2}^2 \leq C\varepsilon \|e^{-\varphi_\varepsilon/h} \bar{\partial} (e^{\varphi_\varepsilon/h} u)\|_{L^2}^2.$$

Here, the factor ε comes from the positive commutator term produced by the convexification of the logarithmic weight. More precisely, in the estimate of [GT11, Lemma 3.2], the convexified contribution gives a positive bulk term of size $\varepsilon^{-1} \|u\|_{L^2}^2$ on the active annulus. Since the singular point z_0 is outside the active support and $|\nabla \varphi_\varepsilon|$ is bounded from below there, the constants are uniform for all sufficiently large τ and all $0 < \varepsilon < \varepsilon_0$.

Combining (8.8), (8.9), and (8.10) gives

$$\|w_\tau v\|_{L^2}^2 = \|u\|_{L^2}^2 \leq C\varepsilon \|w_\tau \bar{\partial} v\|_{L^2}^2.$$

This proves (8.6).

The ∂ estimate follows by applying (8.6) to $\bar{v}$. Since φ_ε and w_τ are real-valued,

$$\|w_\tau v\|_{L^2}^2 = \|w_\tau \bar{v}\|_{L^2}^2 \leq C\varepsilon \|w_\tau \bar{\partial} \bar{v}\|_{L^2}^2 = C\varepsilon \|w_\tau \partial v\|_{L^2}^2.$$

This proves (8.7).

In an interior chart, there are no boundary terms because the functions are compactly supported in the active annulus. In a flattened boundary chart, the estimate is applied to the zero extension on the reflected full disk. The assumption that the zero extension creates no boundary distribution ensures that the conjugated first-order operator acts on the zero-extended function in the distributional sense without producing an additional boundary measure. The estimate is then followed by the usual density argument in the corresponding Sobolev space. $\square$

Lemma 8.5 (Second-order paired scalar Carleman estimate). *Fix $\varepsilon > 0$. Let O be a coordinate patch whose closure is contained in an annulus around the singular point z_0 of the logarithmic weight, and assume that $|\nabla \varphi_\varepsilon|$ is bounded from below on O . If O is chosen sufficiently small, then there are constants $C > 0$ and $\tau_0 > 0$ such that, for all $\tau \geq \tau_0$ and all $v \in C_0^\infty(O)$,*

$$(8.11) \quad \|w_\tau D^2 v\|_{L^2}^2 + \tau^2 \|w_\tau \nabla v\|_{L^2}^2 + \tau^4 \|w_\tau v\|_{L^2}^2 \leq C \|w_\tau \bar{\partial}^2 v\|_{L^2}^2 + C \|w_\tau \partial^2 v\|_{L^2}^2.$$

The same estimate holds in a flattened boundary chart after zero extension, provided the zero extension creates no boundary distribution for the second-order operators.

Proof. Put $h = \tau^{-1}$ and $u = w_\tau v = e^{-\tau\varphi_\varepsilon} v$. For $\sharp \in \{\partial, \bar{\partial}\}$ define

$$P_{\sharp, \tau} = e^{-\tau\varphi_\varepsilon} \sharp e^{\tau\varphi_\varepsilon} = \sharp + \tau \sharp \varphi_\varepsilon.$$

Then $w_\tau \partial^2 v = P_{\partial, \tau}^2 u$ and $w_\tau \bar{\partial}^2 v = P_{\bar{\partial}, \tau}^2 u$. We first prove a local elliptic estimate for the conjugated pair:

$$(8.12) \quad \|D^2 u\|_{L^2}^2 + \tau^2 \|Du\|_{L^2}^2 + \tau^4 \|u\|_{L^2}^2 \leq C \|P_{\partial, \tau}^2 u\|_{L^2}^2 + C \|P_{\bar{\partial}, \tau}^2 u\|_{L^2}^2.$$

At a fixed point x , the first-order frozen symbols are

$$p_\partial(x, \xi, \tau) = \frac{1}{2}(i\xi_1 + \xi_2) + \tau \partial \varphi_\varepsilon(x) \quad \text{and} \quad p_{\bar{\partial}}(x, \xi, \tau) = \frac{1}{2}(i\xi_1 - \xi_2) + \tau \bar{\partial} \varphi_\varepsilon(x).$$

Since φ_ε is real-valued, the two symbols cannot vanish simultaneously on the active patch. Indeed, if both symbols were zero, then $\xi_2 + \tau \partial_{x_1} \varphi_\varepsilon = 0$, $\xi_1 - \tau \partial_{x_2} \varphi_\varepsilon = 0$, $-\xi_2 + \tau \partial_{x_1} \varphi_\varepsilon = 0$, and $\xi_1 + \tau \partial_{x_2} \varphi_\varepsilon = 0$. Hence, $\nabla \varphi_\varepsilon(x) = 0$ and $\xi = 0$. This is impossible on the active patch, where $|\nabla \varphi_\varepsilon|$ is bounded from below. Therefore, by homogeneity in (ξ, τ) and compactness on $|\xi|^2 + \tau^2 = 1$, after shrinking the patch if necessary, there is $c_0 > 0$ such that

$$(8.13) \quad |p_\partial(x, \xi, \tau)|^4 + |p_{\bar{\partial}}(x, \xi, \tau)|^4 \geq c_0(|\xi|^2 + \tau^2)^2$$

for all x in the patch, all $\xi \in \mathbb{R}^2$, and all $\tau \geq 1$.

The frozen constant-coefficient estimate follows from Plancherel and (8.13). For variable coefficients, cover the active patch by smaller coordinate balls on which $\nabla \varphi_\varepsilon$ varies by less than a fixed small amount. Freezing the coefficients on each ball gives the estimate with an error bounded by

$$\delta(\|D^2 u\|_{L^2}^2 + \tau^2 \|Du\|_{L^2}^2 + \tau^4 \|u\|_{L^2}^2) + C_\delta(\|Du\|_{L^2}^2 + \tau^2 \|u\|_{L^2}^2).$$

Choosing the balls so that δ is sufficiently small and then taking τ sufficiently large absorbs the lower-order error. A partition of unity gives (8.12).

It remains to return from u to v . Since $v = e^{\tau \varphi_\varepsilon} u$, each weighted derivative $w_\tau D^k v$ is a finite sum of terms $\tau^{k-\ell} a_{\ell,k}(x) D^\ell u$, $0 \leq \ell \leq k$, with smooth bounded coefficients on the active patch. Conversely, $D^k u$ is a finite sum of the corresponding weighted derivatives of v with the same powers of τ . Hence the left-hand side of (8.12) is equivalent, with constants independent of large τ , to

$$\|w_\tau D^2 v\|_{L^2}^2 + \tau^2 \|w_\tau \nabla v\|_{L^2}^2 + \tau^4 \|w_\tau v\|_{L^2}^2.$$

Using $P_{\partial, \tau}^2 u = w_\tau \partial^2 v$ and $P_{\bar{\partial}, \tau}^2 u = w_\tau \bar{\partial}^2 v$ gives (8.11).

In a flattened boundary chart, we apply the same full-disk estimate to the zero extension. Lemma 8.3 ensures that the second-order zero extensions create no boundary distribution, so the preceding argument applies in the distributional sense and then by density. $\square$

Lemma 8.6 (Laplacian Carleman estimate). *Fix $\varepsilon > 0$. Let O be a coordinate patch whose closure is contained in an active annulus for the weight (8.2). If O is chosen sufficiently small, then there are constants $C > 0$ and $\tau_0 > 0$ such that, for all $\tau \geq \tau_0$ and all compactly supported scalar functions $v \in C_0^\infty(O)$,*

$$\|w_\tau D^2 v\|_{L^2}^2 + \tau^2 \|w_\tau \nabla v\|_{L^2}^2 + \tau^4 \|w_\tau v\|_{L^2}^2 \leq C \|w_\tau \Delta v\|_{L^2}^2.$$

The same estimate holds componentwise for vector-valued functions and also in a flattened boundary chart after zero extension.

Proof. This is the local second-order Carleman estimate for the conjugated Laplacian with the convexified logarithmic weight. It follows from the same first-order convexified estimate used in Lemma 8.4, equivalently from $\Delta = 4\partial\bar{\partial}$ and the standard local elliptic estimate for the conjugated operator $e^{-\tau \varphi_\varepsilon} \Delta e^{\tau \varphi_\varepsilon}$. The principal part has constant coefficients in the local coordinate, and lower-order conjugation terms are absorbed by the convexification and by taking τ large. The boundary version follows by applying the full-disk estimate to the zero extensions in Lemma 8.3. $\square$

Choose large fixed constants $A_\theta > 1$ and $A_W > 1$. For localized variables define

$$\begin{aligned} \mathcal{C}_\tau(\eta Z) &:= A_\theta \|w_\tau D^2(\eta \theta)\|_{L^2}^2 + A_\theta \tau^2 \|w_\tau \nabla(\eta \theta)\|_{L^2}^2 + A_\theta \tau^4 \|w_\tau \eta \theta\|_{L^2}^2 + \|w_\tau D^2(\eta s)\|_{L^2}^2 \\ &\quad + \tau^2 \|w_\tau \nabla(\eta s)\|_{L^2}^2 + \tau^4 \|w_\tau \eta s\|_{L^2}^2 + A_W \|w_\tau D^2(\eta W)\|_{L^2}^2 \\ &\quad + A_W \tau^2 \|w_\tau D(\eta W)\|_{L^2}^2 + A_W \tau^4 \|w_\tau \eta W\|_{L^2}^2. \end{aligned}$$

Lemma 8.7 (Paired Carleman estimate for the reduced variables). *For compactly supported θ , s , and W in the active annulus, one has*

$$\mathcal{C}_\tau(Z) \leq C A_\theta \|w_\tau \Delta \theta\|_{L^2}^2 + C \|w_\tau \bar{\partial}^2 s\|_{L^2}^2 + C \|w_\tau \partial^2 s\|_{L^2}^2 + C A_W \|w_\tau \Delta W\|_{L^2}^2.$$

The same estimate holds after zero extension in a flattened boundary chart.

Proof. Apply Lemma 8.6 to θ , Lemma 8.5 to s , and Lemma 8.6 componentwise to W . Then multiply the estimates for θ and W by A_θ and A_W . $\square$

For the next estimate, set

$$\mathcal{S}_\tau(\eta\theta, \eta s) := \|w_\tau D(\eta\theta)\|_{L^2}^2 + \tau^2 \|w_\tau \eta\theta\|_{L^2}^2 + \|w_\tau D(\eta s)\|_{L^2}^2 + \tau^2 \|w_\tau \eta s\|_{L^2}^2,$$

and

$$\mathcal{W}_\tau(\eta W) := \|w_\tau D^2(\eta W)\|_{L^2}^2 + \tau^2 \|w_\tau D(\eta W)\|_{L^2}^2 + \tau^4 \|w_\tau \eta W\|_{L^2}^2.$$

We use the following shorthand for the localized operator classes introduced above. The symbols P_1 , $P_{W,2}$, $\mathfrak{D}_{\text{sc},1}$, $\mathfrak{D}_{W,2}$, $\mathfrak{CB}_1$, $\mathfrak{CD}_{\text{sc},1}$ and $\mathfrak{CD}_{W,2}$ denote a properly supported local operator of order at most one in $Z = (\theta, s, W)$, a properly supported local operator of order at most two in W , a localized scalar drift-resolvent perturbation of source order at most one, a localized W -drift-resolvent perturbation of source order at most two, a localized Cauchy perturbation with input of order at most one in Z , and the corresponding scalar and W drift-Cauchy perturbations, respectively.

Lemma 8.8 (Perturbative estimates for the localized operator classes). *Let $\eta_0 \prec \eta \prec \eta_1$ be cutoffs in an interior disk, or in a reflected full disk obtained from a flattened boundary chart after zero extension. Then, for all sufficiently large τ , the localized operator classes above satisfy*

$$(8.14) \quad \|w_\tau \eta_0 P_1(\eta Z)\|_{L^2}^2 \leq C \mathcal{S}_\tau(\eta\theta, \eta s) + C \|w_\tau D(\eta W)\|_{L^2}^2 + C \tau^2 \|w_\tau \eta W\|_{L^2}^2 + \mathcal{E}_{\eta_0, \eta_1}(\tau),$$

$$(8.15) \quad \|w_\tau \eta_0 P_{W,2}(\eta W)\|_{L^2}^2 \leq C \mathcal{W}_\tau(\eta W) + \mathcal{E}_{\eta_0, \eta_1}(\tau),$$

$$(8.16) \quad \|w_\tau \eta_0 \mathfrak{D}_{\text{sc},1}(\eta\theta, \eta s)\|_{L^2}^2 \leq C \mathcal{S}_\tau(\eta\theta, \eta s) + \mathcal{E}_{\eta_0, \eta_1}(\tau),$$

$$(8.17) \quad \|w_\tau \eta_0 \mathfrak{D}_{W,2}(\eta W)\|_{L^2}^2 \leq C \mathcal{W}_\tau(\eta W) + \mathcal{E}_{\eta_0, \eta_1}(\tau),$$

and, for the convexified weight in (8.2),

$$(8.18) \quad \begin{aligned} & \|w_\tau \eta_0 \mathfrak{CB}_1(\eta Z)\|_{L^2}^2 + \|w_\tau \eta_0 \mathfrak{CD}_{\text{sc},1}(\eta\theta, \eta s)\|_{L^2}^2 + \|w_\tau \eta_0 \mathfrak{CD}_{W,2}(\eta W)\|_{L^2}^2 \\ & \leq C \varepsilon \mathcal{S}_\tau(\eta\theta, \eta s) + C \varepsilon \mathcal{W}_\tau(\eta W) + \mathcal{E}_{\eta_0, \eta_1}(\tau). \end{aligned}$$

Proof. The local order-one estimate (8.14) follows from a direct weighted commutator calculation. If P_1 is a properly supported local differential operator of order at most one, then

$$w_\tau P_1(\eta Z) = P_1(w_\tau \eta Z) + \tau P_0(w_\tau \eta Z),$$

where P_0 is a properly supported local operator of order zero with smooth coefficients depending on the weight. The contributions of (θ, s) and W are controlled by $\mathcal{S}_\tau(\eta\theta, \eta s)$ and $\|w_\tau D(\eta W)\|_{L^2}^2 + \tau^2 \|w_\tau \eta W\|_{L^2}^2$, respectively. The cutoff errors away from $\text{supp } \eta_0$ are included in $\mathcal{E}_{\eta_0, \eta_1}(\tau)$. This proves (8.14).

The local order-two estimate (8.15) is the same calculation for a properly supported local differential operator of order at most two in W . After commuting the exponential weight through the operator, the resulting terms are bounded by $\|w_\tau D^2(\eta W)\|_{L^2}^2 + \tau^2 \|w_\tau D(\eta W)\|_{L^2}^2 + \tau^4 \|w_\tau \eta W\|_{L^2}^2$, up to the cutoff remainders included in $\mathcal{E}_{\eta_0, \eta_1}(\tau)$. This proves (8.15).

We next estimate the localized drift-resolvent terms. It is enough to consider a model term $u = a D^\beta G_\Delta^{\zeta, \zeta'} F$, $|\beta| \leq 2$, where $F = F_{\text{sc}}$ has source order at most one in $(\eta\theta, \eta s)$ or $F = F_W$ has source order at most two in ηW . The smooth coefficient a is harmless, so we suppress it. Set

$$F_c = \zeta' F, \quad U = G_\Delta^{\zeta, \zeta'} F = \zeta \Delta^{-1} F_c.$$

Choose an auxiliary cutoff $\chi \in C_0^\infty$ such that $\eta_0 \prec \chi$, $\zeta' \prec \chi \prec \zeta$, and such that $\chi = 1$ in a neighborhood of $\text{supp } \eta_0 \cup \text{supp } F_c$. In particular, $\text{supp } \chi$ is contained in the region where $\zeta = 1$.

Hence, $\eta_0 D^\beta U = \eta_0 D^\beta(\chi U)$. Since χU is compactly supported in the coordinate patch under consideration, Lemma 8.6 applied to χU gives

$$\|w_\tau D^2(\chi U)\|_{L^2}^2 + \tau^2 \|w_\tau D(\chi U)\|_{L^2}^2 + \tau^4 \|w_\tau \chi U\|_{L^2}^2 \leq C \|w_\tau \Delta(\chi U)\|_{L^2}^2,$$

for τ sufficiently large. In a boundary chart this is applied after the zero-extension convention in Lemma 8.3 and the localized-input convention fixed before Lemma 8.1; no boundary-supported distribution is created.

We now use the localized parametrix identity. Since $\chi = 1$ on $\text{supp } F_c$ and $\text{supp } \chi \subset \{\zeta = 1\}$, one has $\chi \Delta U = F_c + \chi S_\Delta^{\zeta, \zeta'} F$. Therefore,

$$\Delta(\chi U) = F_c + [\Delta, \chi]U + \chi S_\Delta^{\zeta, \zeta'} F.$$

The commutator $[\Delta, \chi]U$ is supported where χ is not constant. By the choice of χ , this support is separated from $\text{supp } F_c$ and from $\text{supp } \eta_0$. Thus the operator which maps F to $[\Delta, \chi]U$ has a smooth separated-support kernel in the relevant variables. Its weighted norm is included in $\mathcal{E}_{\eta_0, \eta_1}(\tau)$. The term $\chi S_\Delta^{\zeta, \zeta'} F$ is also a smoothing cutoff remainder. Consequently,

$$\|w_\tau \eta_0 D^\beta G_\Delta^{\zeta, \zeta'} F\|_{L^2}^2 \leq C \|w_\tau F_c\|_{L^2}^2 + \mathcal{E}_{\eta_0, \eta_1}(\tau) \leq C \|w_\tau F\|_{L^2}^2 + \mathcal{E}_{\eta_0, \eta_1}(\tau).$$

If $F = F_{\text{sc}}$ has source order at most one in $(\eta\theta, \eta s)$, then

$$\|w_\tau F_{\text{sc}}\|_{L^2}^2 \leq C S_\tau(\eta\theta, \eta s) + \mathcal{E}_{\eta_0, \eta_1}(\tau),$$

which proves (8.16). If $F = F_W$ has source order at most two in ηW , then

$$\|w_\tau F_W\|_{L^2}^2 \leq C W_\tau(\eta W) + \mathcal{E}_{\eta_0, \eta_1}(\tau),$$

which proves (8.17).

It remains to estimate the localized Cauchy and drift-Cauchy perturbations. Let $v = \mathfrak{C}F$, where $\mathfrak{C}$ is one of the localized Cauchy inverses or localized adjoint Cauchy inverses. Write again $F_c = \zeta' F$. Choose a cutoff $\chi \in C_0^\infty$ such that $\eta_0 \prec \chi$, $\zeta' \prec \chi$, and $\chi = 1$ in a neighborhood of $\text{supp } \eta_0 \cup \text{supp } F_c$. We also choose χ supported where the outer cutoff in the localized Cauchy inverse is identically one. By the defining property of the localized Cauchy inverse or localized adjoint Cauchy inverse, there are $D_\mathfrak{C} \in \{\partial, \bar{\partial}\}$ and $\sigma_\mathfrak{C} \in \{\pm 1\}$ such that

$$D_\mathfrak{C} v = \sigma_\mathfrak{C} F_c + S_\mathfrak{C} F$$

on a neighborhood of $\text{supp } \chi$. Here $S_\mathfrak{C}$ is a smoothing cutoff operator. Hence,

$$D_\mathfrak{C}(\chi v) = F_c + [D_\mathfrak{C}, \chi]v + \chi S_\mathfrak{C} F.$$

The commutator $[D_\mathfrak{C}, \chi]v$ is supported where χ is not constant. By the choice of χ , this support is separated from $\text{supp } F_c$ and from $\text{supp } \eta_0$. Thus the operator which maps F to $[D_\mathfrak{C}, \chi]v$ has a smooth separated-support kernel in the relevant variables. Its weighted norm is included in $\mathcal{E}_{\eta_0, \eta_1}(\tau)$. The smoothing term $\chi S_\mathfrak{C} F$ is included in the same remainder.

Since $\eta_0 v = \eta_0 \chi v$, Lemma 8.4 applied to χv gives

$$(8.19) \quad \|w_\tau \eta_0 v\|_{L^2}^2 \leq C \varepsilon \|w_\tau F_c\|_{L^2}^2 + \mathcal{E}_{\eta_0, \eta_1}(\tau) \leq C \varepsilon \|w_\tau F\|_{L^2}^2 + \mathcal{E}_{\eta_0, \eta_1}(\tau).$$

In a boundary chart this is understood after the localized-input zero-extension convention fixed before Lemma 8.1.

For $F = B_1(\eta Z)$, the right-hand side of (8.19) is controlled by (8.14). For $F = \mathfrak{D}_{\text{sc}, 1}(\eta\theta, \eta s)$, it is controlled by (8.16). For $F = \mathfrak{D}_{W, 2}(\eta W)$, it is controlled by (8.17). This proves (8.18). $\square$

8.2. Localized drift reduction and the reduced paired system. The covariant Hessian of the map satisfies

$$\mathcal{H}^\alpha = D^2 W^\alpha + \mathcal{H}_{\text{low}}^\alpha(W, DW),$$

where $\mathcal{H}_{\text{low}}^\alpha$ is local of order at most one in W and vanishes when $W = DW = 0$. Thus $M^\alpha = e^{-\theta} \mathcal{H}^\alpha$.

Lemma 8.9 (Differentiated drift contractions with drift-resolvent remainders). *Let*

$$\mathcal{D}_+ W := \bar{\partial}(\kappa_\alpha \mathcal{A}(\mathcal{H}^\alpha)), \quad \mathcal{D}_- W := \partial(\lambda_\alpha \mathcal{B}(\mathcal{H}^\alpha)).$$

With the adapted cutoff convention,

$$(8.20) \quad \begin{aligned} \eta_0 \mathcal{D}_+ W &= 2\eta_0 \bar{\partial}^2(s + 2\theta) + P_{+,1}^{\text{dr}}(\eta\theta, \eta s) + \mathfrak{D}_{+, \text{sc}, 1}^{\text{dr}}(\eta\theta, \eta s) + P_{+,2}^{\text{dr}}(\eta W) \\ &\quad + \mathfrak{D}_{+, W, 2}(\eta W) + \mathcal{N}_{+, \text{low}}(Z) + R_{\eta_0, \eta_1} Z. \end{aligned}$$

and

$$(8.21) \quad \begin{aligned} \eta_0 \mathcal{D}_- W &= 2\eta_0 \partial^2(s + 2\theta) + P_{-,1}^{\text{dr}}(\eta\theta, \eta s) + \mathfrak{D}_{-, \text{sc}, 1}^{\text{dr}}(\eta\theta, \eta s) + P_{-,2}^{\text{dr}}(\eta W) \\ &\quad + \mathfrak{D}_{-, W, 2}(\eta W) + \mathcal{N}_{-, \text{low}}(Z) + R_{\eta_0, \eta_1} Z. \end{aligned}$$

Here, $P_{\pm,1}^{\text{dr}}$ are properly supported local operators of order at most one in θ and s , $\mathfrak{D}_{\pm, \text{sc}, 1}^{\text{dr}}$ are localized scalar drift-resolvent perturbations of source order at most one in (θ, s) , $P_{\pm,2}^{\text{dr}}$ are properly supported local operators of order at most two in W , and $\mathfrak{D}_{\pm, W, 2}$ are localized W -drift-resolvent perturbations of source order at most two.

Proof. We first compute the principal symbol for the plus contraction; the mirror contraction is its reflected counterpart. For a constant-coefficient differential operator $P(D)$, let $\sigma(P)$ be the Fourier multiplier, so that $\widehat{P(D)f}(\xi_1, \xi_2) = \sigma(P)(\xi_1, \xi_2) \widehat{f}(\xi_1, \xi_2)$. We use the convention $\widehat{\partial_j f} = i\xi_j \widehat{f}$, with $\xi = \xi_1 + i\xi_2$ and $\bar{\xi} = \xi_1 - i\xi_2$. Then $\sigma(\partial) = i\bar{\xi}/2$, $\sigma(\bar{\partial}) = i\xi/2$, and $\sigma(\Delta) = -|\xi|^2 = -\xi\bar{\xi}$. Since $\mathcal{A}(D^2 f) = 4\bar{\partial}^2 f$ and $\mathcal{B}(D^2 f) = 4\partial^2 f$, we have $\sigma(\mathcal{A}(D^2)) = -\xi^2$ and $\sigma(\mathcal{B}(D^2)) = -\bar{\xi}^2$.

We now compute the plus principal symbol. Fix a point x_0 and freeze $N = DJ$ and $\kappa_\alpha = (N^{-T})_\alpha^1 - i(N^{-T})_\alpha^2$ at x_0 . The principal part of the drift equation is

$$\Delta W^\alpha = N_i^\alpha \partial_i q, \quad q = s + 2\theta.$$

Taking the Fourier transform gives $-|\xi|^2 \widehat{W^\alpha} = iN_i^\alpha \xi_i \widehat{q}$. By the identity $(N^{-T})_\alpha^j N_i^\alpha = \delta_i^j$, the definition of κ_α gives $\kappa_\alpha N_i^\alpha \xi_i = \xi_1 - i\xi_2 = \bar{\xi}$. Thus, for $\xi \neq 0$, $\kappa_\alpha \widehat{W^\alpha} = -i \frac{\bar{\xi}}{|\xi|^2} \widehat{q} = -i \frac{1}{\bar{\xi}} \widehat{q}$. Multiplying by the multiplier $-\xi^2$ of $\mathcal{A}(D^2)$, we obtain $(\kappa_\alpha \mathcal{A}(D^2 W^\alpha))^\wedge = i\xi \widehat{q}$. Applying the exterior operator $\bar{\partial}$ gives the extra multiplier $i\xi/2$. Hence, $(\bar{\partial}(\kappa_\alpha \mathcal{A}(D^2 W^\alpha)))^\wedge = -\frac{\xi^2}{2} \widehat{q}$. On the other hand, $(2\bar{\partial}^2 q)^\wedge = 2(\frac{i\xi}{2})^2 \widehat{q} = -\frac{\xi^2}{2} \widehat{q}$. Therefore, the frozen plus principal identity is

$$\bar{\partial}(\kappa_\alpha \mathcal{A}(D^2 W^\alpha)) = 2\bar{\partial}^2(s + 2\theta)$$

at the principal-symbol level. The value at the zero frequency is irrelevant for this homogeneous principal-symbol computation.

We next record the reflected principal calculation. Freeze N and $\lambda_\alpha = (N^{-T})_\alpha^1 + i(N^{-T})_\alpha^2$ at the same point. Then $\lambda_\alpha N_i^\alpha \xi_i = \xi_1 + i\xi_2 = \xi$. The same frozen drift equation gives $\lambda_\alpha \widehat{W^\alpha} = -i \frac{\xi}{|\xi|^2} \widehat{q} = -i \frac{1}{\xi} \widehat{q}$. Since the multiplier of $\mathcal{B}(D^2)$ is $-\bar{\xi}^2$, we get $(\lambda_\alpha \mathcal{B}(D^2 W^\alpha))^\wedge = i\bar{\xi} \widehat{q}$. Applying the exterior operator ∂ gives the multiplier $i\bar{\xi}/2$. Therefore, $(\partial(\lambda_\alpha \mathcal{B}(D^2 W^\alpha)))^\wedge = -\frac{\bar{\xi}^2}{2} \widehat{q} = (2\partial^2 q)^\wedge$. The frozen mirror principal identity is

$$\partial(\lambda_\alpha \mathcal{B}(D^2 W^\alpha)) = 2\partial^2(s + 2\theta)$$

at the principal-symbol level.

We now pass to the variable-coefficient localized statement. The covariant Hessian of the gauge map has the form

$$\mathcal{H}^\alpha = D^2W^\alpha + \mathcal{H}_{\text{low}}^\alpha(x, W, DW),$$

where $\mathcal{H}_{\text{low}}^\alpha$ contains no second derivatives of W and vanishes when $W = DW = 0$. Hence the only possible third-order terms in $\mathcal{D}_+W$ and $\mathcal{D}_-W$ are the ones coming from $\bar{\partial}(\kappa_\alpha \mathcal{A}(D^2W^\alpha))$ and $\partial(\lambda_\alpha \mathcal{B}(D^2W^\alpha))$. The frozen computations above identify these third-order terms after the principal part of the drift equation is used.

It remains to classify the terms left after subtracting the principal contributions. Derivatives falling on κ_α , λ_α , the coefficients of the pulled-back connection, or the coefficients in $\mathcal{H}_{\text{low}}$ reduce the order. They give properly supported local terms of order at most one in (θ, s) , properly supported local terms of order at most two in W , or nonlinear products containing at least two residual factors.

Next, apply the localized drift equation

$$\Delta W^\alpha = N_i^\alpha \partial_i(s + 2\theta) + \mathfrak{R}^\alpha(x, \text{Id} + W, I + DW).$$

The principal part of the source is $\partial_\alpha(s + 2\theta)$. The frozen symbol computation shows that this principal source produces exactly $2\bar{\partial}^2(s + 2\theta)$ in the plus contraction and exactly $2\partial^2(s + 2\theta)$ in the mirror contraction. The remaining scalar source terms have at most one derivative of θ and s , after the local Laplace parametrix is applied and differentiated in the allowed way. These are precisely the localized scalar drift-resolvent perturbations in the statement.

The part of the source involving $\mathfrak{R}^\alpha$ contains no second derivatives of J . Its linear part is local of order at most one in W , while the terms involving $N - I$ multiplying $\nabla(s + 2\theta)$ contain at least two residual factors. After applying the localized Laplace parametrix and then the differentiated contractions, these terms give properly supported local terms of order at most two in W , localized W -drift-resolvent perturbations of order at most two, or nonlinear lower-order terms.

Finally, all pieces produced by the outer cutoffs and by smoothing cutoff remainders are included in $R_{\eta_0, \eta_1}Z$. Combining the plus principal identity with the preceding classification gives (8.20). Combining the reflected principal identity with the same classification gives (8.21). $\square$

Set $C_\Delta(x) := C(x, x)$ and $\tilde{C}_\Delta(x) := \tilde{C}(x, x)$. Define

$$c_+(x) := -4C_\Delta(x)e^{-\theta(x)}, \quad c_-(x) := -4\tilde{C}_\Delta(x)e^{-\theta(x)}.$$

After shrinking the patch, c_+ and c_- are nonvanishing. Write the displayed plus and mirror local blocks as

$$\mathfrak{B}_+ := 8\bar{\partial}^2[C_a(1 - e^{-\theta})] - 2\bar{\partial}[C_a e^{-\theta} \kappa_\alpha \mathcal{A}(\mathcal{H}^\alpha)], \quad \mathfrak{B}_- := 8\partial^2[\tilde{C}_a(1 - e^{-\theta})] - 2\partial[\tilde{C}_a e^{-\theta} \lambda_\alpha \mathcal{B}(\mathcal{H}^\alpha)].$$

Lemma 8.10 (Local reduction of the displayed residual blocks). *With the adapted cutoff convention,*

$$\eta_0 \mathfrak{B}_+ = c_+ \eta_0 \bar{\partial}^2 s + P_{+,1}(\eta\theta, \eta s) + \mathfrak{D}_{+, \text{sc}, 1}(\eta\theta, \eta s) + P_{+,2}(\eta W) + \mathfrak{D}_{+, W, 2}(\eta W) + \mathcal{N}_{+, \text{low}}(Z) + R_{\eta_0, \eta_1}Z.$$

and

$$\eta_0 \mathfrak{B}_- = c_- \eta_0 \partial^2 s + P_{-,1}(\eta\theta, \eta s) + \mathfrak{D}_{-, \text{sc}, 1}(\eta\theta, \eta s) + P_{-,2}(\eta W) + \mathfrak{D}_{-, W, 2}(\eta W) + \mathcal{N}_{-, \text{low}}(Z) + R_{\eta_0, \eta_1}Z.$$

Here, $P_{\pm,1}$ are properly supported local operators of order at most one in θ and s , $\mathfrak{D}_{\pm, \text{sc}, 1}$ are localized scalar drift-resolvent perturbations of source order at most one in (θ, s) , $P_{\pm,2}$ are properly supported local operators of order at most two in W , and $\mathfrak{D}_{\pm, W, 2}$ are localized W -drift-resolvent perturbations of source order at most two.

Proof. Consider the plus block. The order two scalar contribution from $8\bar{\partial}^2[C_a(1 - e^{-\theta})]$ is $8C_\Delta e^{-\theta} \bar{\partial}^2 \theta$. By Lemma 8.9, the order two scalar contribution from the gauge-Hessian block is $-4C_\Delta e^{-\theta} \bar{\partial}^2(s + 2\theta)$. Their sum is $-4C_\Delta e^{-\theta} \bar{\partial}^2 s = c_+ \bar{\partial}^2 s$. All derivatives falling on the diagonal coefficient family, on κ , on

connection coefficients, or on $e^{-\theta}$ give the stated lower local classes, localized scalar drift-resolvent classes, localized W -drift-resolvent classes, or nonlinear lower-order terms. The mirror block is identical with ∂ , λ , and $\mathcal{B}$ in place of $\bar{\partial}$, κ , and $\mathcal{A}$. $\square$

Lemma 8.11 (Reduction of the M -critical Cauchy terms). *Let $\eta_0 \prec \eta \prec \eta_1$ be adapted cutoffs. After substituting*

$$M^\alpha = e^{-\theta} \mathcal{H}^\alpha, \mathcal{H}^\alpha = D^2 W^\alpha + \mathcal{H}_{\text{low}}^\alpha(W, DW),$$

the parts of the plus and mirror M -critical Cauchy terms, which are linear in the second derivatives of W are finite sums of model terms of the form

$$\partial^{*-1}(a_+(x)\lambda_\alpha \mathcal{A}(D^2 W^\alpha)), \bar{\partial}^{*-1}(a_-(x)\kappa_\alpha \mathcal{B}(D^2 W^\alpha)),$$

with smooth compactly supported coefficients $a_\pm$.

On supp η_0 , these terms admit the decompositions

$$(8.22) \quad \eta_0 \partial^{*-1}(a_+ \lambda_\alpha \mathcal{A}(D^2 W^\alpha)) = \mathfrak{C}_{+, \text{sc}}^{\text{mc}} \mathfrak{D}_{+, \text{sc}, 1}^{\text{mc}}(\eta\theta, \eta s) + \mathfrak{C}_{+, W}^{\text{mc}} \mathfrak{D}_{+, W, 2}^{\text{mc}}(\eta W) + \mathfrak{C}_{+, 1}^{\text{mc}} B_{+, 1}^{\text{mc}}(\eta\theta, \eta s) \\ + P_{+, 1}^{\text{mc}}(\eta Z) + \mathcal{N}_{+, \text{low}}^{\text{mc}}(Z) + R_{\eta_0, \eta_1} Z,$$

and

$$(8.23) \quad \eta_0 \bar{\partial}^{*-1}(a_- \kappa_\alpha \mathcal{B}(D^2 W^\alpha)) = \mathfrak{C}_{-, \text{sc}}^{\text{mc}} \mathfrak{D}_{-, \text{sc}, 1}^{\text{mc}}(\eta\theta, \eta s) + \mathfrak{C}_{-, W}^{\text{mc}} \mathfrak{D}_{-, W, 2}^{\text{mc}}(\eta W) + \mathfrak{C}_{-, 1}^{\text{mc}} B_{-, 1}^{\text{mc}}(\eta\theta, \eta s) \\ + P_{-, 1}^{\text{mc}}(\eta Z) + \mathcal{N}_{-, \text{low}}^{\text{mc}}(Z) + R_{\eta_0, \eta_1} Z.$$

Here, $\mathfrak{D}_{\pm, \text{sc}, 1}^{\text{mc}}$ are localized scalar drift-resolvent perturbations of source order at most one in (θ, s) , and $\mathfrak{D}_{\pm, W, 2}^{\text{mc}}$ are localized W -drift-resolvent perturbations of source order at most two. The operators $\mathfrak{C}_{\pm, \text{sc}}^{\text{mc}}$ and $\mathfrak{C}_{\pm, W}^{\text{mc}}$ denote the outer localized adjoint Cauchy inverses which occur in the critical Cauchy terms. The inputs $B_{\pm, 1}^{\text{mc}}$ are local differential expressions of order at most one in θ and s . Any ordinary localized Cauchy term involving W which comes from $\mathcal{H}_{\text{low}}(W, DW)$ has input of order at most one in W and is included later in the ordinary localized Cauchy class of Lemma 8.12; it is not part of the second-derivative W -polarization reduced in this lemma.

Proof. We prove the plus statement; the mirror statement is identical with ∂^{*-1} , λ_α , and $\mathcal{A}$ replaced by $\bar{\partial}^{*-1}$, κ_α , and $\mathcal{B}$.

The plus M -critical Cauchy term contains the opposite polarization $\lambda_\alpha \mathcal{A}(M^\alpha)$. Since

$$M^\alpha = e^{-\theta} \mathcal{H}^\alpha = e^{-\theta} (D^2 W^\alpha + \mathcal{H}_{\text{low}}^\alpha(W, DW)),$$

the only part which is linear in the second derivatives of W is $\lambda_\alpha \mathcal{A}(D^2 W^\alpha)$. The factor $e^{-\theta} - 1$ gives nonlinear products, and $\mathcal{H}_{\text{low}}(W, DW)$ gives ordinary localized Cauchy inputs of order at most one in W . These terms are not part of the second-derivative W -polarization reduced here.

We first identify the order of the opposite polarization. Freeze $N = DJ$ and $\lambda_\alpha = (N^{-T})_\alpha^1 + i(N^{-T})_\alpha^2$ at a point. With the Fourier convention used in Lemma 8.9, the frozen principal drift equation gives $-|\xi|^2 \widehat{W}^\alpha = iN_i^\alpha \xi_i \widehat{q}$, where $q = s + 2\theta$. Contracting with λ_α gives $\lambda_\alpha \widehat{W}^\alpha = -i\xi|\xi|^{-2} \widehat{q} = -i\bar{\xi}^{-1} \widehat{q}$. Since $\mathcal{A}(D^2)$ has multiplier $-\xi^2$, one obtains

$$(\lambda_\alpha \mathcal{A}(D^2 W^\alpha))^\wedge = i \frac{\xi^2}{\bar{\xi}} \widehat{q}.$$

This is a homogeneous order-one expression on real nonzero frequencies. We therefore first use the localized drift equation, rather than estimating it as a Cauchy term with $D^2 W$ left in the input.

Let $\zeta' \prec \zeta$ be the cutoffs in the local Laplace parametrix and choose them adapted to $\eta_0 \prec \eta \prec \eta_1$. On supp η_0 , the localized drift equation gives

$$W^\alpha = G_\Delta^{\zeta, \zeta'} (\zeta' [N_i^\alpha \partial_i (s + 2\theta) + \mathfrak{R}^\alpha(x, \text{Id} + W, I + DW)]) + R_{\eta_0, \eta_1} Z.$$

Applying the two derivatives occurring in $\lambda_\alpha \mathcal{A}(D^2 W^\alpha)$ gives

$$\eta_0 \lambda_\alpha \mathcal{A}(D^2 W^\alpha) = \eta_0 \lambda_\alpha \mathcal{A}(D^2 G_\Delta^{\zeta, \zeta'} (\zeta' N_i^\alpha \partial_i (s + 2\theta))) + \eta_0 \lambda_\alpha \mathcal{A}(D^2 G_\Delta^{\zeta, \zeta'} (\zeta' \mathfrak{R}^\alpha)) + R_{\eta_0, \eta_1} Z.$$

Now split $N_i^\alpha = \delta_i^\alpha + (N_i^\alpha - \delta_i^\alpha)$. The term with $\delta_i^\alpha \partial_i (s + 2\theta)$ is a localized scalar drift-resolvent perturbation of source order at most one in (θ, s) . The terms containing $(N - I)\nabla(s + 2\theta)$ have at least two residual factors and are included in $\mathcal{N}_{+, \text{low}}^{\text{mc}}(Z)$, after shrinking the propagation patch. Similarly, the coefficients λ_α may be split into their value at the zero gauge and a residual part; the residual part contains DW and therefore gives nonlinear products when multiplied by the scalar source or by $D^2 W$.

The term containing $\mathfrak{R}^\alpha$ is treated as follows. Since $\mathfrak{R}^\alpha(x, \text{Id}, I) = 0$ and $\mathfrak{R}^\alpha$ contains no second derivatives of J , its linear part is local of order at most one in W , and its remaining terms are nonlinear lower-order products. After applying $D^2 G_\Delta^{\zeta, \zeta'}$, the linear part gives a localized W -drift-resolvent perturbation of source order at most two in W , while the nonlinear part is included in $\mathcal{N}_{+, \text{low}}^{\text{mc}}(Z)$. The cutoff commutators produced by applying the parametrix to the localized equation are separated-support or smoothing cutoff remainders, hence belong to $R_{\eta_0, \eta_1} Z$.

We now restore the outer localized adjoint Cauchy inverse. Since the M -critical term already contains ∂^{*-1} , the scalar drift-resolvent contribution becomes a localized scalar drift-Cauchy perturbation, written as $\mathfrak{C}_{+, \text{sc}}^{\text{mc}} \mathfrak{D}_{+, \text{sc}, 1}^{\text{mc}}(\eta\theta, \eta s)$. The W -drift-resolvent contribution becomes $\mathfrak{C}_{+, W}^{\text{mc}} \mathfrak{D}_{+, W, 2}^{\text{mc}}(\eta W)$. Terms in which the scalar source is local of order at most one inside the outer Cauchy inverse are recorded as $\mathfrak{C}_{+, 1}^{\text{mc}} B_{+, 1}^{\text{mc}}(\eta\theta, \eta s)$. Purely local order-one terms are included in $P_{+, 1}^{\text{mc}}(\eta Z)$, nonlinear products in $\mathcal{N}_{+, \text{low}}^{\text{mc}}(Z)$, and all off-diagonal cutoff pieces in $R_{\eta_0, \eta_1} Z$.

Thus,

$$\begin{aligned} \eta_0 \partial^{*-1} (a_+ \lambda_\alpha \mathcal{A}(D^2 W^\alpha)) &= \mathfrak{C}_{+, \text{sc}}^{\text{mc}} \mathfrak{D}_{+, \text{sc}, 1}^{\text{mc}}(\eta\theta, \eta s) + \mathfrak{C}_{+, W}^{\text{mc}} \mathfrak{D}_{+, W, 2}^{\text{mc}}(\eta W) + \mathfrak{C}_{+, 1}^{\text{mc}} B_{+, 1}^{\text{mc}}(\eta\theta, \eta s) \\ &\quad + P_{+, 1}^{\text{mc}}(\eta Z) + \mathcal{N}_{+, \text{low}}^{\text{mc}}(Z) + R_{\eta_0, \eta_1} Z. \end{aligned}$$

This proves (8.22). The mirror reduction is the same, using the opposite polarization $\kappa_\alpha \mathcal{B}(D^2 W^\alpha)$ and the outer operator $\bar{\partial}^{*-1}$, and gives (8.23). $\square$

Lemma 8.12 (Localized lower-order structure of the residual equations). *Let $\eta_0 \prec \eta \prec \eta_1$ be cutoffs in a disk where the adapted convention is valid. Then the nonprincipal parts of the plus and mirror residual equations have the forms*

$$\begin{aligned} \eta_0 \mathfrak{R}_+(\theta, W) &= P_{+, 1}(\eta Z) + P_{+, 2}(\eta W) + \mathfrak{D}_{+, \text{sc}, 1}(\eta\theta, \eta s) + \mathfrak{D}_{+, W, 2}(\eta W) + \sum_{\ell=1}^{N_+} \mathfrak{C}_{+, \ell} B_{+, \ell}(\eta Z) \\ &\quad + \sum_{\ell=1}^{N_+^{\text{sc}}} \mathfrak{C}_{+, \ell}^{\text{sc}} \mathfrak{D}_{+, \ell, \text{sc}, 1}(\eta\theta, \eta s) + \sum_{\ell=1}^{N_+^W} \mathfrak{C}_{+, \ell}^W \mathfrak{D}_{+, \ell, W, 2}(\eta W) + \mathcal{N}_{+, \text{low}}(Z) + R_{\eta_0, \eta_1} Z, \end{aligned}$$

and

$$\begin{aligned} \eta_0 \mathfrak{R}_-^m(\theta, W) &= P_{-, 1}(\eta Z) + P_{-, 2}(\eta W) + \mathfrak{D}_{-, \text{sc}, 1}(\eta\theta, \eta s) + \mathfrak{D}_{-, W, 2}(\eta W) + \sum_{\ell=1}^{N_-} \mathfrak{C}_{-, \ell} B_{-, \ell}(\eta Z) \\ &\quad + \sum_{\ell=1}^{N_-^{\text{sc}}} \mathfrak{C}_{-, \ell}^{\text{sc}} \mathfrak{D}_{-, \ell, \text{sc}, 1}(\eta\theta, \eta s) + \sum_{\ell=1}^{N_-^W} \mathfrak{C}_{-, \ell}^W \mathfrak{D}_{-, \ell, W, 2}(\eta W) + \mathcal{N}_{-, \text{low}}(Z) + R_{\eta_0, \eta_1} Z. \end{aligned}$$

Here $P_{\pm, 1}$ are properly supported local operators of order at most one in $Z = (\theta, s, W)$, and $P_{\pm, 2}$ are properly supported local operators of order at most two in W . The operators $\mathfrak{D}_{\pm, \text{sc}, 1}$ are localized scalar drift-resolvent perturbations of source order at most one in (θ, s) , while $\mathfrak{D}_{\pm, W, 2}$ are localized

W-drift-resolvent perturbations of source order at most two. The ordinary localized Cauchy inputs $B_{\pm,\ell}$ have order at most one in Z ; in particular they contain no second derivatives of W . All Cauchy terms whose linear input contains the second derivatives of W are precisely the M -critical Cauchy terms and have been converted by Lemma 8.11 into localized scalar drift-Cauchy or localized W -drift-Cauchy perturbations.

Proof. The local scalar terms are of order at most one in θ . The local terms which are order zero in M become local order-two terms in W , because $M^\alpha = e^{-\theta}\mathcal{H}^\alpha$ and $\mathcal{H}^\alpha = D^2W^\alpha + \mathcal{H}_{\text{low}}^\alpha(W, DW)$.

The scalar critical terms are localized Cauchy transforms whose inputs contain θ and at most one derivative of θ , modulo nonlinear lower-order products. The M -critical Cauchy terms are handled by Lemma 8.11. Their linear second-derivative W -parts become localized scalar drift-Cauchy terms and localized W -drift-Cauchy terms, while the lower-order $\mathcal{H}_{\text{low}}(W, DW)$ pieces are ordinary localized Cauchy terms with input of order at most one in W . All products with at least two residual factors are included in $\mathcal{N}_{\pm,\text{low}}(Z)$, and all cutoff remainders belong to $R_{\eta_0,\eta_1}Z$. $\square$

Lemma 8.13 (Local reduced system). *Let $\eta_0 \prec \eta \prec \eta_1$ be cutoffs in a disk where the adapted convention is valid. Then the paired augmented residual system implies*

$$(8.24) \quad \begin{aligned} \Delta\theta &= P_{\theta,1}(\eta Z) + P_{\theta,2}(\eta W) + \mathcal{N}_{\theta,\text{low}}(Z) + R_{\eta_0,\eta_1}Z, \\ c_+(x)\bar{\partial}^2 s &= P_{+,1}(\eta Z) + P_{+,2}(\eta W) + \mathfrak{D}_{+,\text{sc},1}(\eta\theta, \eta s) + \mathfrak{D}_{+,W,2}(\eta W) + \sum_{\ell=1}^{N_+} \mathfrak{C}_{+,\ell} B_{+,\ell}(\eta Z) \\ &\quad + \sum_{\ell=1}^{N_+^{\text{sc}}} \mathfrak{C}_{+,\ell}^{\text{sc}} \mathfrak{D}_{+,\ell,\text{sc},1}(\eta\theta, \eta s) + \sum_{\ell=1}^{N_+^W} \mathfrak{C}_{+,\ell}^W \mathfrak{D}_{+,\ell,W,2}(\eta W) + \mathcal{N}_{+,\text{low}}(Z) + R_{\eta_0,\eta_1}Z, \\ c_-(x)\partial^2 s &= P_{-,1}(\eta Z) + P_{-,2}(\eta W) + \mathfrak{D}_{-,\text{sc},1}(\eta\theta, \eta s) + \mathfrak{D}_{-,W,2}(\eta W) + \sum_{\ell=1}^{N_-} \mathfrak{C}_{-,\ell} B_{-,\ell}(\eta Z) \\ &\quad + \sum_{\ell=1}^{N_-^{\text{sc}}} \mathfrak{C}_{-,\ell}^{\text{sc}} \mathfrak{D}_{-,\ell,\text{sc},1}(\eta\theta, \eta s) + \sum_{\ell=1}^{N_-^W} \mathfrak{C}_{-,\ell}^W \mathfrak{D}_{-,\ell,W,2}(\eta W) + \mathcal{N}_{-,\text{low}}(Z) + R_{\eta_0,\eta_1}Z, \end{aligned}$$

and

$$(8.25) \quad \Delta W^\alpha = P_{W,1}^\alpha(\eta Z) + \mathcal{N}_{W,\text{low}}^\alpha(Z) + R_{\eta_0,\eta_1}Z.$$

Here $P_{\theta,1}$, $P_{\pm,1}$, and $P_{W,1}^\alpha$ are local of order at most one in Z ; $P_{\theta,2}$ and $P_{\pm,2}$ are local of order at most two in W . The nonlocal terms are exactly of the localized Cauchy, scalar drift-resolvent, W -drift-resolvent, scalar drift-Cauchy, and W -drift-Cauchy classes described in Subsection 8.1.

Proof. The θ equation is (7.10), localized with the adapted cutoffs. Its linear part contains at most one derivative of θ and s , and at most two derivatives of W ; higher products are nonlinear lower-order terms.

The plus and mirror equations follow from Lemma 8.10 and Lemma 8.12, using the revised operator classes. The W equation is the localized drift equation (8.1). All off-diagonal cutoff terms are included in $R_{\eta_0,\eta_1}Z$. $\square$

Lemma 8.14 (Boundary localized form of the reduced system). *Let $B_R^+ = B_R \cap \{x_2 > 0\}$ be a flattened boundary chart, with $\Gamma_R = B_R \cap \{x_2 = 0\}$ and with B_R^+ corresponding to the interior side. Let (θ, s, W) be a smooth one-sided solution of the paired augmented residual system in B_R^+ . Suppose that $\theta = 0$, $D\theta = 0$, $s = 0$, $Ds = 0$, $W = 0$, $DW = 0$, and $D^2W = 0$ on Γ_R . Let θ^0 , s^0 , and W^0 denote the zero extensions to the reflected full disk B_R . Then the localized reduced system*

(8.24)–(8.25) is available for the zero-extended variables in the reflected disk in the sense needed for the Carleman estimates. The local differential terms create no boundary distributions, the localized Cauchy and drift-resolvent terms are interpreted by the localized-input zero-extension convention, and all cutoff remainders belong to $R_{\eta_0, \eta_1} Z$.

Proof. By Lemma 8.3, the zero extensions of θ , s , and W have exactly the distributional derivatives needed by the local scalar second-order equations and the componentwise Laplacian equation for W . Thus, the local differential terms create no boundary-supported distribution on Γ_R .

All smooth coefficient functions appearing in the localized reduced system are extended smoothly to the reflected disk. The equations are used only after multiplication by the adapted cutoffs, so the particular extension is irrelevant. Unknown-dependent coefficient factors, such as those depending on $e^{-\theta}$, N^{-1} , λ , or κ , are split into their boundary values plus residual factors. The boundary values give fixed smooth coefficients, while the residual parts are included in the nonlinear lower-order class.

The localized Cauchy, drift-resolvent, and drift-Cauchy terms are not obtained by extending an uncut one-sided nonlocal identity. Their coefficient factors and residual inputs are first multiplied by the interior cutoff, then extended by zero, and only then acted on by the localized full-disk operators. In particular, for the scalar critical Cauchy terms, the cut scalar input is extended by zero before the localized adjoint Cauchy inverse is applied. For the retained M -polarization terms, the localized drift reduction is first applied to the cut one-sided input, the resulting localized drift-resolvent input is then extended by zero, and only afterwards the outer adjoint Cauchy inverse is applied.

With this convention, no derivative falls on the characteristic function of the half-disk, and no boundary-supported distribution is produced by the nonlocal terms. Off-diagonal cutoff pieces are exactly the remainders included in $R_{\eta_0, \eta_1} Z$. $\square$

8.3. Local propagation and elimination.

Lemma 8.15 (Local propagation of the full residual zero set). *Let (θ, s, W) be a smooth solution of the localized reduced system in a connected disk. Suppose that the cutoff form (8.24)–(8.25) is available in every smaller disk. If $\theta = s = 0$ and $W = 0$ in a nonempty open subset of the disk, then $\theta = s = 0$ and $W = 0$ in the connected component of the disk.*

Proof. We prove one propagation step. Let B_R be a disk and suppose that $\theta = s = 0$ and $W = 0$ in B_r . Choose $0 < r_0 < r < r_1 < r_2 < R$. The singular point z_0 is placed in B_{r_0} , and the radii are chosen so that the target annulus and the outer commutator annulus have the weight separation described in Subsection 8.1. Choose $\eta \in C_0^\infty(B_R)$ with $\eta = 0$ near $B_{r_0} \cup (\mathbb{R}^2 \setminus B_{r_2})$ and $\eta = 1$ on $B_{r_1} \setminus B_r$. The inner commutators are supported where the solution is already zero. The outer commutators are included in $\mathcal{E}_{\eta_0, \eta_1}(\tau)$. All nonlocal operators are applied only to cut inputs.

Since c_+ and c_- are nonvanishing, division by them only changes the constants and the lower-order operator classes. Applying Lemma 8.7 to ηZ and then substituting the localized reduced system gives

$$\mathcal{C}_\tau(\eta Z) \leq C_0 \mathcal{P}_\tau(\eta Z) + C_0 \mathcal{N}_\tau(Z) + C_0 \mathcal{E}_{\eta_0, \eta_1}(\tau),$$

where $\mathcal{P}_\tau(\eta Z)$ is the sum of the weighted norms of all linear local, drift-resolvent, Cauchy, and drift-Cauchy perturbations, and $\mathcal{N}_\tau(Z)$ is the weighted norm of the nonlinear lower-order terms.

We now estimate $\mathcal{P}_\tau$. The local order-one terms and scalar drift-resolvent terms are bounded by $\mathcal{S}_\tau(\eta\theta, \eta s)$, and

$$\mathcal{S}_\tau(\eta\theta, \eta s) \leq C\tau^{-2} \mathcal{C}_\tau(\eta Z).$$

The local order-two W -terms and the W -drift-resolvent terms are bounded by $\mathcal{W}_\tau(\eta W)$, and

$$\mathcal{W}_\tau(\eta W) \leq C A_W^{-1} \mathcal{C}_\tau(\eta Z).$$

In the θ equation, the order-two W -terms are multiplied by the prefactor A_θ coming from Lemma 8.7, and hence give the contribution $C A_\theta A_W^{-1} \mathcal{C}_\tau(\eta Z)$. In the W equation, the scalar first-order terms

are multiplied by A_W , and hence give $CA_W\tau^{-2}\mathcal{C}_\tau(\eta Z)$. The localized Cauchy, scalar drift-Cauchy, and W -drift-Cauchy terms are bounded by Lemma 8.8; after the preceding two inequalities they contribute

$$C\varepsilon\tau^{-2}\mathcal{C}_\tau(\eta Z) + C\varepsilon A_W^{-1}\mathcal{C}_\tau(\eta Z)$$

up to $\mathcal{E}_{\eta_0, \eta_1}(\tau)$.

Choose A_θ first. Then choose A_W so large that the coefficients containing A_W^{-1} and $A_\theta A_W^{-1}$ are smaller than $1/8$. With A_θ and A_W fixed, choose $\varepsilon > 0$ so small that the Cauchy and drift-Cauchy contributions are smaller than $1/8$. Then choose the propagation disk sufficiently small so that the nonlinear estimate (8.5) gives a contribution smaller than $1/8$. Finally take τ large enough so that all terms containing τ^{-2} , including the $A_W\tau^{-2}$ term from the W equation, are smaller than $1/8$. After these choices all perturbative terms are absorbed into the left hand side, and we obtain

$$\mathcal{C}_\tau(\eta Z) \leq C\mathcal{E}_{\eta_0, \eta_1}(\tau).$$

By construction, every nonzero term in $\mathcal{E}_{\eta_0, \eta_1}(\tau)$ is exponentially smaller than the weighted norm on the target subannulus. Hence, for every compact subannulus $A \Subset B_{r_1} \setminus B_r$,

$$\inf_A w_\tau^2 (\tau^4 \|\theta\|_{L^2(A)}^2 + \tau^4 \|s\|_{L^2(A)}^2 + \tau^4 \|W\|_{L^2(A)}^2) \leq Ce^{-c\tau}.$$

Letting $\tau \rightarrow \infty$ gives $\theta = s = 0$ and $W = 0$ in $B_{r_1} \setminus B_r$. Since the variables already vanish in B_r , we obtain $\theta = s = 0$ and $W = 0$ in B_{r_1} . A finite chain of overlapping disks proves the statement in the connected component. $\square$

Lemma 8.16 (Boundary initiation of the full residual zero set). *Let (θ, s, W) be a smooth one-sided solution of the paired augmented residual system in a flattened boundary chart. If $\theta = 0$, $D\theta = 0$, $s = 0$, $Ds = 0$, $W = 0$, $DW = 0$, and $D^2W = 0$ on a flattened boundary segment, then $\theta = s = 0$ and $W = 0$ in an interior collar of $\partial\Omega$.*

Proof. Work in a flattened boundary chart as in Lemma 8.2. Write $B_R^+ = B_R \cap \{x_2 > 0\}$, $B_R^- = B_R \cap \{x_2 < 0\}$, and $\Gamma_R = B_R \cap \{x_2 = 0\}$, with B_R^+ the interior side. Let θ^0 , s^0 , and W^0 be the zero extensions from B_R^+ to B_R . By Lemma 8.3, the local differential operators create no boundary distribution on Γ_R . By Lemma 8.14, the localized reduced system is available for the zero-extended variables in the reflected disk.

Apply Lemma 8.7 to ηZ^0 . The singular point is chosen on the exterior side, where the zero extensions vanish. On B_R^+ the local differential terms are replaced by the reduced system; on B_R^- they vanish identically.

The nonlocal terms are used only after localization. The scalar critical Cauchy inputs are first cut off on the one-sided interior side, then extended by zero, and only then acted on by the localized adjoint Cauchy inverses. The retained M -polarization terms are first reduced by the localized drift equation with cut one-sided input; the resulting drift-resolvent inputs are then extended by zero before the outer adjoint Cauchy inverse is applied. Thus the reflected disk is used for the operator estimates and the Carleman inequality, not for extending an uncut one-sided nonlocal identity.

The local terms, localized Cauchy perturbations, localized scalar drift-resolvent terms, localized W -drift-resolvent terms, localized scalar drift-Cauchy terms, and localized W -drift-Cauchy terms are estimated exactly as in Lemma 8.15, using the perturbative estimates (8.14)–(8.18). Exterior commutators vanish. The remaining commutators away from the target collar are direct local outer commutators or smoothing cutoff remainders satisfying the weight separation condition.

After choosing A_θ first, choosing A_W large enough to absorb the fixed local order-two W -terms and localized W -drift-resolvent terms, choosing $\varepsilon > 0$ small to absorb the localized Cauchy, scalar drift-Cauchy, and W -drift-Cauchy terms, shrinking the boundary chart so that the nonlinear smallness

parameter δ is small, and taking τ large, we obtain

$$\mathcal{C}_\tau(\eta Z^0) \leq C\mathcal{E}_{\eta_0, \eta_1}(\tau).$$

The logarithmic weight is chosen so that every nonzero term in $\mathcal{E}_{\eta_0, \eta_1}(\tau)$ is exponentially smaller than the weighted norm on the target one-sided collar. Letting $\tau \rightarrow \infty$ gives $\theta = s = 0$ and $W = 0$ in a smaller interior collar of Γ_R . $\square$

Proposition 8.17 (Vanishing of the residual variables and gauge). *One has $\theta = s = 0$, $W = 0$, and hence $J = \text{Id}$ in Ω .*

Proof. Using (7.6) and (7.7) in Proposition 7.3, Lemma 8.16 gives $\theta = s = 0$ and $W = 0$ in an interior collar of $\partial\Omega$. Let $p \in \Omega$. Since Ω is connected, there is a compact path from p to this collar. Cover the path with a finite chain of overlapping disks. Applying Lemma 8.15 successively along the chain propagates $\theta = s = 0$ and $W = 0$ to a neighborhood of p . Since p was arbitrary, $\theta = s = 0$ and $W = 0$ in Ω . Finally, $W = J - \text{Id}$, so $J = \text{Id}$ in Ω . $\square$

Corollary 8.18 (Elimination of the residual gauge). *The paired residual system implies $\theta = 0$, $s = 0$, $q = 0$, and $J = \text{Id}$ in Ω .*

Proof. This follows from Proposition 8.17 and the identity $q = s + 2\theta$. $\square$

Proof of Theorem 1.1. Fix $\varphi \in \mathcal{U}$. The equality of the nonlinear DN maps gives the equality of the first-linearized conormal data. By the planar gauge theorem, the first-linearized coefficients are therefore related by a boundary-fixing gauge pair (J, ρ) . The divergence form of the first linearization gives the density identity used to define q and θ .

The second variation gives the plus and mirror residual equations derived in Sections 5 and 6. The cutoff localizations used after that point belong only to the unique continuation argument for this already-derived residual system. The first-order boundary matching of the curvatures gives the boundary residual jets in Proposition 7.3. Corollary 8.18 then gives $q = 0$ and $J = \text{Id}$ in the isothermal coordinate domain.

By (7.1), $q = 0$ and $J = \text{Id}$ imply $\widehat{K}_1 = \widehat{K}_2$. Since $\widehat{K}_j = (K_j \circ \chi^{-1})/m_\chi$ and the factor m_χ is the same for $j = 1, 2$, we get $K_1 \circ \chi^{-1} = K_2 \circ \chi^{-1}$. Hence $K_1 = K_2$ in the original domain. $\square$

STATEMENTS AND DECLARATIONS

Data availability statement. No datasets were generated or analyzed during the current study.

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