AN INVERSE PROBLEM FOR THE MONGE-AMPÈRE EQUATION

TONY LIIMATAINEN AND YI-HSUAN LIN

Abstract. We extend the study of inverse boundary value problems to the setting of fully nonlinear PDEs by considering an inverse source problem for the Monge–Ampère equation

$$\det D^2 u = F.$$

We prove that, on a convex Euclidean domain in the plane, the associated Dirichlet-to-Neumann (DN) map uniquely determines a positive source function F. The proof relies on recovering the Hessian of a solution to the equation, which is interpreted as a Riemannian metric g. Interestingly, although the equation is posed on a Euclidean domain, the inverse problem becomes anisotropic since the metric g appears as a coefficient matrix in the linearized equation.

As an intermediate step, we prove that the DN map of the non-divergence form equation

$$g^{ab}\partial_{ab} v = 0$$

uniquely determines the conformal class of the metric g on a simply connected planar domain, without the usual diffeomorphism invariance. To address the challenges of full nonlinearity, we develop asymptotic expansions for complex geometric optics solutions in the planar setting and solve a resulting nonlocal $\overline{\partial}$-equation by proving a unique continuation principle for it. These techniques are expected to be applicable to a wide range of inverse problems for nonlinear equations.

Contents

2020 *Mathematics Subject Classification.* 35R30, 35J25, 35J60, 35J96.

Key words and phrases. Inverse source problem, Monge-Ampère equation, fully nonlinear, anisotropic Calderón problem, stationary phase method.

1. Introduction

The Monge–Ampère equation is a paradigmatic *fully nonlinear* (possibly degenerate) partial differential equation (PDE), introduced nearly two centuries ago by Monge [Mon84] and Ampère [Amp19]. In general form, it is written as

$$\det D^2 u = f(x, u, \nabla u) \quad \text{in } \Omega, \tag{1.1}$$

where $\Omega \subset \mathbb{R}^n$ is an open domain, $u : \Omega \to \mathbb{R}$ is a solution, $D^2 u$ denotes the Hessian matrix of u, and $f : \Omega \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ is a given source function. Under mild assumptions on the solution, f and Ω, the equation becomes elliptic, enabling the application of classical regularity theory (see Remark 1.3). Notably, Figalli was awarded the Fields Medal in 2018 in part for his contributions to the Monge–Ampère equation: see [Fig17] for a comprehensive introduction to its elliptic theory and applications.

The Monge–Ampère equation is deeply intertwined with geometry, analysis, applied mathematics, and physics.

- An optimal transport ∇u between mass densities ρ_0 and ρ_1 with quadratic cost is governed by the Monge-Ampere equation
$$\det D^2 u(x) = \frac{\rho_0(x)}{\rho_1(\nabla u(x))}.$$
Optimal transport appears in many applications such as economics, meteorology, image processing and computer vision, and fluid dynamics. We refer to the book [Vil09] by Villani for further details about the applications.
- In differential geometry, the graph $(x, u(x))$ of a solution to
$$\det D^2 u = K(x)(1 + |\nabla u|^2)^{\frac{n+2}{2}},$$
has prescribed Gaussian curvature $K(x)$, relating to the classical Minkowski problem [Min97, Min03] of constructing convex hypersurfaces with specified curvature.
- The Calabi–Yau conjecture asserts that a compact Kähler manifold with zero first Chern class admits a Ricci-flat Kähler metric, which reduces to solving the complex Monge-Ampère equation. We refer readers to [Yau78, Aub82] for further studies.

For more comprehensive introduction and studies of the Monge-Ampère equation, see [Cal72, CY86, TW00, TW02, TW05, Fig17].

In this work, we focus on the spatially dependent source case,

$$f(x, u, \nabla u) = F(x),$$

and study an *inverse source problem* of recovering the function F from boundary measurements. In this case, the equation (1.1) reads

$$\det D^2 u = F(x), \tag{1.2}$$

for some positive function F satisfying suitable regularity conditions. Our primary objective is to investigate the recovery of the function F from the DN map corresponding to (1.2).

To the best of our knowledge, this work establishes the first uniqueness result for an inverse source problem governed by a fully nonlinear PDE. A central novelty lies in the recovery of the metric only up to a conformal factor, but without a natural diffeomorphism from the first linearized equation. Moreover, after somewhat involved asymptotic analysis for the integral identity of the second linearized equation, the resulting equation is a second-order $\overline{\partial}$-equation with a nonlocal $\overline{\partial}^{-1}$ lower-order perturbation. We prove a unique continuation property (UCP) for this non-local $\overline{\partial}$ equation, specifically the equation to show that the conformal factor is identically equal to 1. We also anticipate that these methods will have influence well beyond the Monge–Ampère model. The reason for the nonlocal equation seems to be in the full nonlinearity.

Inverse problems of parameter identification, encompassing both coefficients and source functions, in nonlinear partial differential equations have attracted considerable interest in recent decades. Among these, the determination of nonlinear laws presents profound challenges due to inherent nonlinearity and severe ill-posedness. The modern approach to such problems can be traced back to the early 1990s, notably through Isakov's pioneering work [Isa93], which introduced the idea of linearizing the nonlinear Dirichlet-to-Neumann (DN) map $C^\infty(\partial\Omega) \to C^\infty(\partial\Omega)$. This linearization reduces the nonlinear inverse problem to one for a linear PDE, enabling the use of classical techniques. Subsequently, second-order linearizations, involving data depending on two parameters, have further advanced the field [AZ21, CNV19, KN02, Sun96, Sun10, SU97].

More recently, a novel method has emerged in the study of inverse problems for semilinear elliptic equations [FO20, LLLS21a]. These works exploit nonlinearity not as an obstacle, but as a constructive tool, building on the foundational insights of [KLU18], which examined inverse problems for nonlinear equations on Lorentzian manifolds and developed the so-called higher order linearization method. By harnessing nonlinear interactions via higher order linearizations, these approaches have solved inverse problems in contexts where methods for linear equations fail.

Following these breakthroughs, a substantial body of literature has developed using higher order linearization techniques to address inverse problems for various nonlinear PDEs. Let us mention here works that address nonlinear elliptic equations. Key contributions include [LLLS21b, LLST22, KU20b, KU20a, FLL23] on semilinear elliptic equations, often with partial boundary data. Quasilinear elliptic inverse problems have been studied in [KKU23, CFK$^+$21, LW23], while inverse problems for the minimal surface equation (quasilinear) are treated in [ABN20, CLLO24, CLLT23, CLT24, Nur24]. The latter have also led to novel applications in AdS/CFT physics [JLST25]. Other related works, including semilinear elliptic equations under various settings and fractional elliptic inverse problems, can be found in [LL19, LL22, LSX22, HL23, ST23, LL25]. We refer the reader to the recent survey [Las25] for a comprehensive introduction and for further references to inverse problems for semilinear elliptic and hyperbolic equations.

1.1. **Mathematical formulations and main results.** The main contribution of this work is a uniqueness result for an inverse source problem for the Monge-Ampère equation in a convex planar domain. The mathematical formulation is as follows.

Let $\Omega \subset \mathbb{R}^2$ be a bounded, uniformly convex domain with C^∞-smooth boundary $\partial\Omega$. Given a source function $F = F(x) \in C^\infty(\overline{\Omega})$ satisfying $F \geq c_0 > 0$ for some constant $c_0 > 0$, let $u : \Omega \to \mathbb{R}$ be the solution to the Dirichlet boundary value problem:

$$\begin{cases} \det D^2 u = F & \text{in } \Omega, \\ u = \varphi & \text{on } \partial\Omega, \end{cases} \tag{1.3}$$

where D^2u denotes the Hessian matrix of u.

To ensure ellipticity (i.e., convexity of solutions), we assume that the source function satisfies

$$F(x) \geq c_0 > 0 \quad \text{in } \Omega,$$

for some constant $c_0 > 0$. We also assume that the boundary data $\varphi \in C^\infty(\partial\Omega)$. Under these conditions, the boundary value problem (1.3) is (locally) well-posed. Further details will be provided in Section 2.2 (see also [Fig17] for additional discussion). Thanks to this well-posedness, we can define the Dirichlet-to-Neumann (DN) map associated with (1.3) as

$$\Lambda_F : C^\infty(\partial\Omega) \to C^\infty(\partial\Omega), \qquad \varphi \mapsto \partial_\nu u_\varphi|_{\partial\Omega}, \tag{1.4}$$

for any φ sufficiently small in an appropriate sense. Here $u_\varphi \in C^\infty(\overline{\Omega})$ denotes the unique solution to (1.3), and

$$\partial_\nu u_\varphi = \nu \cdot \nabla u_\varphi$$

is the Neumann derivative with respect to the unit outer normal ν on $\partial\Omega$. The inverse problem for the Monge-Ampere equation we address is as follows.

(**IP**) **Inverse Source Problem.** Can we determine the unknown source F in Ω by using the knowledge of the DN map Λ_F?

Remark 1.1. *Before addressing the nonlinear setting, we briefly recall the obstruction to non-uniqueness to an inverse source problem in the linear case. Let $\Omega \subset \mathbb{R}^n$ be a bounded open set with sufficiently regular boundary $\partial\Omega$, where $n \geq 2$. Consider the Poisson equation*

$$\begin{cases} \Delta u = F & \text{in } \Omega, \\ u = \varphi & \text{on } \partial\Omega. \end{cases} \tag{1.5}$$

Given an arbitrary function $\psi \in C^2(\Omega)$ with vanishing Cauchy data on $\partial\Omega$, define $v := u + \psi$ in Ω. Then v satisfies

$$\begin{cases} \Delta v = F + \Delta\psi & \text{in } \Omega, \\ v = \varphi & \text{on } \partial\Omega. \end{cases} \tag{1.6}$$

Then, the following observations can be made:

(i) The Cauchy data of (1.5) and (1.6) coincide. Since $\psi \in C^2(\Omega)$ with vanishing Cauchy data was otherwise arbitrary, the inverse source problem is solvable only up to the gauge symmetry $F \mapsto F + \Delta\psi$. In particular, even if the sources agree on all orders on the boundary, the interior source cannot be uniquely determined.

(ii) Related non-uniqueness phenomena have been investigated in the context of semilinear equations and shown that for some nonlinearities the gauge symmetry breaks, leading to unique recovery: see [LL24] for the semilinear elliptic case and [KLL24] for the semilinear parabolic case. Very recently, the work [LN25] determined the source uniquely for a quasilinear elliptic equation, and [QXYZ25] addresses the inverse problem of simultaneously recovering multiple unknown parameters for semilinear wave equations.

Interestingly, for our inverse problem (**IP**), we can provide an affirmative answer in two dimensions: the source function F can be uniquely determined from the DN map (1.4) associated with the Monge–Ampère equation (1.3). Before presenting the main result, we introduce the following set of admissible boundary data:

$$B_\delta(\partial\Omega) := \left\{ \varphi \in C^\infty(\partial\Omega) : \|\varphi\|_{C^{4,\alpha}(\partial\Omega)} < \delta \right\}, \tag{1.7}$$

for some $\alpha \in (0,1)$ and sufficiently small $\delta > 0$. With these preparations in place, we are now ready to state our main theorem.

Theorem 1.2 (Unique determination)**.** *Let $\Omega \subset \mathbb{R}^2$ be a bounded, uniformly convex domain with C^∞ boundary $\partial\Omega$. Let $F \in C^\infty(\overline{\Omega})$ be a source with $F \geq c_0 > 0$ for some positive constant c_0. Suppose that F is known up to second order on the boundary, then the DN map Λ_F of (1.3) determines the source F in Ω uniquely.*

More specifically, let $F_1, F_2 \in C^\infty(\overline{\Omega})$ be sources, and $F_1, F_2 \geq c_0 > 0$ for some positive constant c_0. Let Λ_{F_j} be the DN map of

$$\begin{cases} \det D^2 u^{(j)} = F_j & \text{in } \Omega, \\ u^{(j)} = \varphi & \text{on } \partial\Omega, \end{cases} \tag{1.8}$$

for $j = 1, 2$. Suppose that F_1 and F_2 agree up to second order on the boundary, then

$$\Lambda_{F_1}(\varphi) = \Lambda_{F_2}(\varphi) \text{ on } \partial\Omega, \quad \text{for any } \varphi \in B_\delta(\partial\Omega), \tag{1.9}$$

for $\delta > 0$ sufficiently small, implies

$$F_1 = F_2 \text{ in } \Omega.$$

Remark 1.3. *We clarify the assumptions in Theorem 1.2:*

(i) ***Uniform convexity of Ω and positivity of F.*** *These ensure global regularity—namely, that a solution u admits a classical second derivative D^2u and remains convex on Ω (see* [Fig17, Remark 1.1]*). Consequently, the Monge–Ampère equation is locally well-posed on $B_\delta(\Omega)$, as its linearization around a convex solution is elliptic. This local well-posedness is essential for applying the first linearization method in the inverse problem.*

(ii) ***F known up to second order on $\partial\Omega$.*** *Knowledge of the DN map along with F and up to its second order derivatives behavior on $\partial\Omega$ allows the recovery of the solution to the Monge-Ampère equation up to at least fourth order on the boundary. This assumption, while convenient for avoiding standard boundary determination arguments in our analysis, can most likely be lifted.*

Condition (i) is essential for the forward problem of (1.3)*, while condition (ii) is only used for the inverse problem.*

The first linearization of the Monge-Ampère equation (1.3) (see Section 2) yields

$$u_0^{ab}\partial_{ab}v = 0 \quad \text{in } \Omega, \tag{1.10}$$

where the coefficient matrix $\left(u_0^{ab}\right) = \left(\partial_{ab}u_0\right)^{-1}$ is defined via the solution u_0 to the original Monge-Ampère equation with zero Dirichlet data ($\varphi = 0$ on $\partial\Omega$). Meanwhile, thanks to the convexity and regularity assumptions, the equation (1.10) is a second-order, anisotropic elliptic equation in non-divergence form, as it has matrix-valued non-constant coefficients given by u_0^{ab}. As an intermediate step in proving Theorem 1.2, we establish a uniqueness result for the associated Calderón problem of this linearized equation in the plane.

Theorem 1.4. *Let $\Omega \subset \mathbb{R}^2$ be a bounded open simply connected domain with C^∞-smooth boundary $\partial\Omega$, and $g = \left(g_{ab}\right)$ is a symmetric, positive definite and C^∞-smooth 2×2 matrix-valued function. Let Λ'_g be the DN map of*

$$g^{ab}\partial_{ab}v = 0 \text{ in } \Omega, \tag{1.11}$$

where $\left(g^{ab}\right) = \left(g_{ab}\right)^{-1}$. Then Λ'_g determines g up to a conformal factor $c = c(x) > 0$ in Ω with $c|_{\partial\Omega} = 1$.

More specifically, let $g = g_j$ be as above, and Λ'_{g_j} be the DN map of the equation

$$\begin{cases} g_j^{ab}\partial_{ab}v_j = 0 & \text{in } \Omega, \\ v_j = \phi & \text{on } \partial\Omega, \end{cases}$$

for $j=1,2$. Suppose that

$$\Lambda'_{g_1}(\phi)=\Lambda'_{g_2}(\phi) \quad \textit{for any} \quad \phi\in C^\infty(\partial\Omega),$$

then there exists a C^∞-smooth conformal factor $c>0$ with $c|_{\partial\Omega}=1$, such that

$$g_1 = cg_2 \ \textit{in}\ \Omega.$$

Note that the determination of the metric in Theorem 1.4 is free from diffeomorphism ambiguity. In this sense, it constitutes a stronger result than the corresponding one for the divergence form anisotropic Calderón problem in two dimensions. It is not immediately clear whether the assumption that the domain Ω is simply connected can be lifted.

Novelty of the methods and outline of the proof. Theorem 1.2 establishes, for the first time to our knowledge, a uniqueness result for an inverse problem governed by a fully nonlinear elliptic PDE. Key novel contributions of our method include:

- *Solving the first linearized problem:* The first linearization leads to an elliptic, second-order PDE in non-divergence form, where the leading coefficient is the Hessian of the Monge–Ampère solution (1.3) with zero Dirichlet data, considered as a Riemannian metric g. By reformulating this non-divergence form equation (1.11) as
$$\big(-\Delta_g + X_g\cdot\nabla\big)u=0,$$
where the drift term X_g is given by the contracted Christoffel symbols of g (see Section 4.2). After applying a known result for the anisotropic Calderón problem to the above equation, we study the transformation properties of Christoffel symbols to eliminate the diffeomorphism gauge on a simply connected domain. This leads to global recovery of the metric tensor only up to a conformal factor $c>0$, normalized on the boundary by $c|_{\partial\Omega}=1$ (see Theorem 1.4).
- *Analysis of the second linearized equation:* To resolve the remaining conformal factor, we employ a class of complex geometric optics (CGO) solutions to the first linearized PDE. The second linearization of the Monge–Ampère equation leads to the integral identity
$$\int_\Omega v^*\ \mathrm{tr}\left\{(D^2u_0)^{-1}D^2v^{(1)}(D^2u_0)^{-1}D^2v^{(2)}\right\}dx, \tag{1.12}$$
where u_0 is the solution to (1.3) with zero Dirichlet data, and $\mathrm{tr}(A)$ denotes the trace of a matrix A. Here $v^{(1)}$ and $v^{(2)}$ are CGO solutions of the first linearized equation of (1.3), and v^* is a CGO solution of the corresponding adjoint equation.

  The number of derivatives versus solutions in the integral (1.12) makes classical CGO constructions and asymptotic arguments insufficient in this context. For this purpose, we refine the earlier CGOs by deriving a polynomial expansion in h for their correction terms when the associated phases do not have critical points. Moreover, the asymptotic analysis ultimately yields a PDE of the form
$$\overline{\partial}(A\overline{\partial}\mathbf{c}(z)+\alpha(z)\mathbf{c}(z))=\beta(z)\overline{\partial}^{-1}(\gamma(z)\mathbf{c}(z))+H(z), \tag{1.13}$$
with a nonlocal lower operator $\overline{\partial}^{-1}$, where $A\neq 0$, α,β,γ are possibly complex-valued functions, and H is a holomorphic function. Here, $\mathbf{c}$ is the unknown, and we want to determine $\mathbf{c}=0$. To address this difficulty, we establish a unique continuation property (UCP) via a Carleman estimate

for the equation (1.13), where the UCP holds only when H is holomorphic (see Section 6). This is a delicate result, and allows us to conclude that the conformal factor is identically 1, thereby guaranteeing the unique recovery of the source F.

1.2. **Organization of the article.** Section 2 presents the preliminaries: basic notations from complex analysis, local well-posedness results for (1.3), and the higher-order linearization framework. In Section 3, we prove a boundary determination result for solutions of (1.3), which allows us to transfer the DN map from (1.3) to its linearized equations. Section 4 shows Theorem 1.4, that is, the diffeomorphism relating the metrics is the identity, and the metric can be uniquely determined only up to a conformal factor for an elliptic equation of non-divergence form. In Section 5, we present CGO solutions for the first linearized equation and carry out a refined asymptotic analysis of the remainder terms. In Section 6, we prove a UCP for a PDE with nonlocal lower order perturbations. This UCP, together with the CGO solutions, is applied in Section 7 to show that the conformal factor is identically one, via the integral identity of the second linearized equation. Finally, Section 8 combines all these results and completes the proof of Theorem 1.2.

2. Preliminaries

In this section, we will prepare several useful notations and tools for the study of the Monge-Ampère equation.

2.1. Notations, function spaces and some fundamental tools.

2.1.1. *Notations in complex analysis.* Let us introduce the following standing notation in this article, which is used to identify $\mathbb{R}^2 = \mathbb{C}$. The differential operators $\nabla = (\partial_{x_1}, \partial_{x_2})$, ∂ and $\overline{\partial}$ on $\mathbb{C}$, which are given by

$$\partial = \partial_z = \frac{\partial}{\partial z} = \frac{1}{2}\left(\partial_{x_1} - \mathsf{i}\partial_{x_2}\right), \quad \overline{\partial} = \partial_{\overline{z}} = \frac{\partial}{\partial \overline{z}} = \frac{1}{2}\left(\partial_{x_1} + \mathsf{i}\partial_{x_2}\right), \tag{2.1}$$

where $z = x_1 + \mathsf{i}x_2 \in \mathbb{C}$, $z = x_1 + \mathsf{i}x_2$ with $x_1, x_2 \in \mathbb{R}$ and $\mathsf{i} = \sqrt{-1}$. In addition, let us use $\partial_k \equiv \partial_k$ to simplify the notation, for $j = 1, 2$, then direct computations yield that

$$\partial_1 = \partial + \overline{\partial}, \quad \partial_2 = \mathsf{i}\big(\partial - \overline{\partial}\big), \tag{2.2}$$

and

$$\overline{\partial}^2 - \partial^2 = \frac{1}{4}\big(\partial_1^2 + 2\mathsf{i}\partial_{x_1}\partial_{x_2} - \partial_{x_2}^2 - \partial_{x_1}^2 + 2\mathsf{i}\partial_{x_1}\partial_{x_2} + \partial_{x_2}^2\big) = \mathsf{i}\partial_{x_1}\partial_{x_2}.$$

Note that the Hessian of any C^2 function $f = f(x) = f(x_1, x_2)$ can be written as

$$D^2 f = \begin{pmatrix} \partial_{11} & \partial_{12} \\ \partial_{12} & \partial_{22} \end{pmatrix} f = \begin{pmatrix} \left(\partial + \overline{\partial}\right)^2 & \mathsf{i}\left(\partial^2 - \overline{\partial}^2\right) \\ \mathsf{i}\left(\partial^2 - \overline{\partial}^2\right) & -\left(\partial - \overline{\partial}\right)^2 \end{pmatrix} f(z), \tag{2.3}$$

where we identify $z = x_1 + \mathsf{i}x_2 \in \mathbb{C}$. In particular, as Φ is holomorphic (i.e., $\overline{\partial}\Phi = 0$), one can obtain

$$D^2\Phi = \begin{pmatrix} \left(\partial + \overline{\partial}\right)^2 & \mathsf{i}\left(\partial^2 - \overline{\partial}^2\right) \\ \mathsf{i}\left(\partial^2 - \overline{\partial}^2\right) & -\left(\partial - \overline{\partial}\right)^2 \end{pmatrix} \Phi = \begin{pmatrix} 1 & \mathsf{i} \\ \mathsf{i} & -1 \end{pmatrix} \partial^2\Phi, \tag{2.4}$$

and Ψ is antiholomorphic (i.e., $\partial\Psi = 0$), we have

$$D^2\Psi = \begin{pmatrix} \left(\partial + \overline{\partial}\right)^2 & \mathsf{i}\left(\partial^2 - \overline{\partial}^2\right) \\ \mathsf{i}\left(\partial^2 - \overline{\partial}^2\right) & -\left(\partial - \overline{\partial}\right)^2 \end{pmatrix} \Psi = \begin{pmatrix} 1 & -\mathsf{i} \\ -\mathsf{i} & -1 \end{pmatrix} \overline{\partial}^2\Psi. \tag{2.5}$$

Therefore, the Hessian (2.3) can be written as

$$D^2 f = \begin{pmatrix} \left(\partial+\overline{\partial}\right)^2 & \mathsf{i}\left(\partial^2-\overline{\partial}^2\right) \\ \mathsf{i}\left(\partial^2-\overline{\partial}^2\right) & -\left(\partial-\overline{\partial}\right)^2 \end{pmatrix} f = A\partial^2 f + B\overline{\partial}^2 f + 2I_{2\times 2}\partial\overline{\partial}f, \tag{2.6}$$

where A, B are matrices given by

$$A := \begin{pmatrix} 1 & \mathsf{i} \\ \mathsf{i} & -1 \end{pmatrix} \quad \text{and} \quad B := \begin{pmatrix} 1 & -\mathsf{i} \\ -\mathsf{i} & -1 \end{pmatrix} \tag{2.7}$$

are derived from (2.4) and (2.5), and we used $4\overline{\partial}\partial = \Delta$. Here, $I_{2\times 2}$ denotes the 2×2 identity matrix. These notations will be used throughout the article. In particular, the formula (2.6) is crucial for the asymptotic analysis of the second integral identity.

2.1.2. *Function spaces.* Let us introduce the notion of function spaces that we use in this article. The notation $C^{k,\alpha}(K)$ denotes the Hölder continuous space, for some compact set $K\subset\mathbb{R}^2$, where $k\in\mathbb{N}\cup\{0\}$ denotes the k-th order differentiability, and the exponent $\alpha\in(0,1)$. It is also known that $C^{k,\alpha}(K)$ is an algebra, in the sense that

$$\|uv\|_{C^{k,\alpha}(K)} \le C\big(\|u\|_{C^{k,\alpha}(K)}\|v\|_{L^\infty(K)} + \|u\|_{L^\infty(K)}\|v\|_{C^{k,\alpha}(K)}\big),$$

for some constant $C>0$ independent of $u,v\in C^{k,\alpha}(K)$. It is known that $C^{k,\alpha}(K)$ is a Banach space.

2.1.3. *Matrices computations.* We also collect several useful properties for matrix computations. Let us first recall the Jacobi formula for a matrix, which is given by

$$\frac{d}{dt}\det A(t) = (\det A(t))\operatorname{tr}\left(A(t)^{-1}\frac{dA(t)}{dt}\right), \tag{2.8}$$

for any differentiable $n\times n$ matrix-valued functions $A(t)$. Moreover, it also holds

$$\frac{d}{dt}A^{-1}(t) = -A(t)A'(t)A^{-1}(t), \tag{2.9}$$

for any differentiable matrix $A(t)$. These formulas will be used in the forthcoming analysis to address our problems.

2.2. **Well-posedness.** Let us establish the (local) well-posedness of (1.3) and prove the continuous dependence of solutions on the Dirichlet data. Following the approach in [Fig17], consider a nonempty, bounded, and uniformly convex domain $\Omega\subset\mathbb{R}^2$. Since the source term F is bounded away from zero, the solution u to (1.3) is convex in Ω. Moreover, under these assumptions, one can obtain improved regularity results for certain classes of solutions.

Define the nonlinear differential operator

$$Q(u) := \det D^2 u.$$

Our goal is to prove the following result regarding the solvability and stability of solutions to the equation involving $Q(u)$.

Proposition 2.1 (Well-posedness)**.** *Let $\Omega\subset\mathbb{R}^2$ be a uniformly convex domain with C^∞-boundary $\partial\Omega$.*

(i) Given $\alpha\in(0,1)$, let $F\in C^{4,\alpha}(\overline{\Omega})$ with $F(x)\ge c_0>0$, for some constant c_0. Then there exists a unique convex solution $u_0\in C^{4,\alpha}(\overline{\Omega})$ of

$$\begin{cases} \det D^2 u_0 = F & \text{in } \Omega, \\ u_0 = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.10}$$

(ii) *There exists constants $\delta, C > 0$ such that for any φ in the set $B_\delta(\partial\Omega)$ given by (1.7), there exists a solution $u \in C^{4,\alpha}(\overline{\Omega})$ of*

$$\begin{cases} \det D^2 u = F & \text{in } \Omega, \\ u = \varphi & \text{on } \partial\Omega, \end{cases} \tag{2.11}$$

which satisfies

$$\|u - u_0\|_{C^{4,\alpha}(\overline{\Omega})} \leq C\|\varphi\|_{C^{4,\alpha}(\partial\Omega)},$$

where $u_0 \in C^{4,\alpha}(\overline{\Omega})$ is the convex solution to (2.10). Furthermore, the solution u is unique in the class $\left\{w \in C^{4,\alpha}(\overline{\Omega}) : \|w - u_0\|_{C^{4,\alpha}(\overline{\Omega})} \leq C\delta\right\}$

(iii) *In particular, if $F \in C^\infty(\overline{\Omega})$ and $\varphi \in B_\delta(\partial\Omega)$, the solution u of (2.11) belongs to $C^\infty(\overline{\Omega})$. In addition, there exist C^∞-Fréchet differentiable maps*

$$\begin{aligned} &S : B_\delta(\partial\Omega) \to C^\infty(\overline{\Omega}), \qquad \varphi \mapsto u_\varphi, \\ &\Lambda : B_\delta(\partial\Omega) \to C^\infty(\partial\Omega), \quad \varphi \mapsto \partial_\nu u_\varphi|_{\partial\Omega} . \end{aligned} \tag{2.12}$$

Proof. For (i), as the Dirichlet boundary value vanishes, since $0 < c_0 \leq F \in C^{2,\alpha}(\overline{\Omega})$ in $\overline{\Omega}$, using the method of continuity in [Fig17, Theorem 3.4] and [Fig17, Remark 1.1], there exists a unique convex solution $u_0 \in C^{4,\alpha}(\overline{\Omega})$ of (2.10). More generally, if $F \in C^{k,\alpha}(\overline{\Omega})$, then there exists a unique solution $u \in C^{k+2,\alpha}(\overline{\Omega})$ solves (2.10), for any integer $k \geq 2$.

For (ii), let us prove the existence of solutions to (2.11) by the implicit function theorem for Banach spaces (see [RR04, Theorem 10.5]). Let

$$X := C^{4,\alpha}(\partial\Omega), \quad Y := C^{4,\alpha}(\overline{\Omega}), \quad Z := C^{2,\alpha}(\overline{\Omega}) \times C^{4,\alpha}(\partial\Omega)$$

be Banach spaces. Consider the map

$$\Phi : X \times Y \to Z, \quad \Phi(\varphi, u) := \left(Q(u), u|_{\partial\Omega} - \varphi\right), \tag{2.13}$$

We want to show that the map Φ enjoys the mapping property (2.13). This can be seen since the map

$$C^{4,\alpha}(\overline{\Omega}) \ni u \mapsto Q(u) = \det D^2 u \in C^{2,\alpha}(\overline{\Omega}),$$

where we used $\det D^2 u$ is a polynomial in $\partial_{ab} u$, for $a, b = 1, 2$ and $C^{2,\alpha}(\overline{\Omega})$ is an algebra. For the same reason, the mapping Φ is C^∞ smooth in the Frechét sense.

Now, using the equation (2.10), we have

$$\Phi(0, u_0) = (Q(u_0), u_0|_{\partial\Omega}) = (0, 0),$$

and the partial differential operator is given by

$$\begin{aligned} &\partial_u \Phi(0, u_0) : Y \to Z, \\ &\partial_u \Phi(0, u_0) v = \Big(\underbrace{(\det D^2 u_0)}_{=F>0 \text{ in } \overline{\Omega}} \operatorname{tr}\left((D^2 u_0)^{-1} D^2 v\right), v|_{\partial\Omega}\Big), \end{aligned}$$

for any $v \in Y$, where we used the Jacobi formula (2.8) for $Q(u)$.

Since u_0 is convex with $\det D^2 u_0 > 0$, it is known that $D^2 u_0$ is a positive definite matrix-valued function. Then $\operatorname{tr}\left((D^2 u_0)^{-1} D^2 \,\cdot\,\right)$ is a second order elliptic operator of non-divergence form. Thanks to $F > 0$ in $\overline{\Omega}$ with $F \in C^{2,\alpha}(\overline{\Omega})$, and the ellipticity of $\operatorname{tr}\left((D^2 u_0)^{-1} D^2 \,\cdot\,\right)$, using the results [GT01, Chapter 6], we want to show the map

$$\partial_u \Phi(0, u_0) : Y \to Z, \quad v \mapsto \left(F \operatorname{tr}\left((D^2 u_0)^{-1} D^2 v\right), v|_{\partial\Omega}\right)$$

is a linear isomorphism. On the one hand, it is easy to see that the function $(\det D^2u_0)\,\mathrm{tr}\left((D^2u_0)^{-1}D^2v\right) \in C^{2,\alpha}(\overline{\Omega})$ for any $v \in Y$. On the other hand, consider the Dirichlet problem

$$\begin{cases} F\,\mathrm{tr}\left((D^2u_0)^{-1}D^2v\right) = G & \text{in } \Omega, \\ v = \phi & \text{on } \partial\Omega, \end{cases} \tag{2.14}$$

which is equivalent to

$$\begin{cases} \mathrm{tr}\left((D^2u_0)^{-1}D^2v\right) = \frac{G}{F} & \text{in } \Omega, \\ v = \phi & \text{on } \partial\Omega, \end{cases} \tag{2.15}$$

since $F > 0$ in $\overline{\Omega}$, where $v|_{\partial\Omega} = \phi \in C^{4,\alpha}(\partial\Omega)$. If $G \in C^{2,\alpha}(\overline{\Omega})$, then $\frac{G}{F} \in C^{2,\alpha}(\overline{\Omega})$ because of $F > 0$ and $F \in C^{2,\alpha}(\overline{\Omega})$. Since the equation (2.15) has no zero order coefficients, by [GT01, Chapter 6], there exists a unique solution $v \in C^{2,\alpha}(\overline{\Omega})$ to (2.15). Moreover, applying the (global) Schauder estimate (see [GT01, Chapter 6]) again, one can see that the solution $v \in C^{4,\alpha}(\overline{\Omega})$ of (2.14).

Finally, via (i) and (ii), one can see that if $F \in C^\infty(\overline{\Omega})$ and $\varphi \in B_\delta(\partial\Omega)$, then the corresponding solutions u_0 and u to (2.10) and (2.11) are $C^\infty(\overline{\Omega})$-smooth functions (the integer $k \geq 2$ in (i) can be arbitrary in the argument). Next, using the implicit function theorem for Banach spaces (for instance, see [RR04, Theorem 10.5]), there exists $\delta > 0$ and a unique solution map

$$S : B_\delta(\partial\Omega) \to C^\infty(\overline{\Omega}), \quad \varphi \mapsto S(\varphi),$$

such that $S(0) = u_0$ and $\Phi(\varphi, S(\varphi)) = 0$, for all $\varphi \in B_\delta(\partial\Omega)$, for any sufficiently small $\delta > 0$. Let $u := S(\varphi)$, since S is Lipschitz continuous with $S(0) = u_0$, then there must hold

$$\|u - u_0\|_{C^{4,\alpha}(\overline{\Omega})} \leq C\|\varphi\|_{C^{4,\alpha}(\partial\Omega)}.$$

Moreover, the solution map S is C^∞ in the Frechét sense, and since the normal derivative is a linear map, we have that (iii) holds. This concludes the proof. □

Thanks to (2.12), the (local) well-posedness ensures that one can develop the higher order linearization scheme for the Monge-Ampère equation.

2.3. **Higher order linearization.** With the well-posedness of Proposition 2.1 at hand, it is known that the equation (1.3) admits a unique solution $u \in C^{4,\alpha}(\overline{\Omega})$ provided that $u|_{\partial\Omega} \in B_\delta$ for sufficiently small $\delta > 0$. Consider the boundary data $\varphi = \varphi_\epsilon$ in (1.3) to be of the form

$$\varphi_\epsilon = \epsilon_1\phi_1 + \epsilon_2\phi_2 \text{ on } \partial\Omega, \tag{2.16}$$

where $\epsilon = (\epsilon_1, \epsilon_2)$ with sufficiently small parameters $|\epsilon_k|$, and ϕ_k can be any sufficiently smooth function on $\partial\Omega$, for $k = 1, 2,$. With this parametrization at hand, the corresponding solution u of (1.3) can be expressed as $u_\epsilon(x) = u(x;\epsilon)$. In addition, let us write the solution u_ϵ of (1.3) with the Dirichlet data (2.16) of the form

$$u_\epsilon(x) = u_0 + \epsilon_1 v_1 + \epsilon_2 v_2 + \frac{1}{2}\epsilon_1\epsilon_2 w + \mathcal{O}(\epsilon^3) \tag{2.17}$$

as an asymptotic expansion when $\epsilon \to 0$. The notation $\mathcal{O}(\epsilon^3)$ is the Bachmann–Landau notation. Notice that we have the well-known Jacobi formula (2.8) for the determinant of matrices. In the following, we employ this formula to derive the corresponding linearized equations.

2.3.1. *The first linearization.* Since the unknown source φ is independent of ϵ, let us denote u_ϵ of the form (2.17), which is the solution to (1.3) with the Dirichlet data (2.16). By differentiating (1.3) with respect to ϵ_k, and combining with the Jacobi formula (2.8), we have

$$\begin{cases} \left(\det D^2 u_\epsilon\right) \operatorname{tr}\left(\left(D^2 u_\epsilon\right)^{-1} D^2\left(\partial_{\epsilon_k} u_\epsilon\right)\right) = 0 & \text{in } \Omega, \\ \partial_{\epsilon_k} u_\epsilon = \phi & \text{on } \partial\Omega, \end{cases} \tag{2.18}$$

for $j = 1, 2$. In particular, as $\epsilon = 0$, there holds

$$\begin{cases} \left(\det D^2 u_0\right) \operatorname{tr}\left(\left(D^2 u_0\right)^{-1} D^2 v^{(k)}\right) = 0 & \text{in } \Omega, \\ v^{(k)} = \phi_k & \text{on } \partial\Omega. \end{cases} \tag{2.19}$$

where

$$u_0 = u(x;0), \quad \text{and} \quad v^{(k)} = \partial_{\epsilon_k}\big|_{\epsilon=0} u_\epsilon,$$

for $k = 1, 2$. Moreover, it is easy to see that u_0 is the solution to (1.3) with zero boundary data, i.e.,

$$\begin{cases} \det D^2 u_0 = F & \text{in } \Omega, \\ u_0 = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.20}$$

Now, by plugging (2.20) into (2.19) and using $F > 0$ in Ω, we obtain a *linear* second order elliptic equation[1]

$$\begin{cases} u_0^{ab}\partial_{ab} v^{(k)} = 0 & \text{in } \Omega, \\ v^{(k)} = \phi_k & \text{on } \partial\Omega, \end{cases} \tag{2.21}$$

which is of the *non-divergence form* for $k = 1, 2$, where

$$\left(u_0^{ab}\right)_{1\le a,b\le 2} = \left(D^2 u_0\right)^{-1}.$$

By knowing the Cauchy data of (1.3), the Cauchy data $\{v|_{\partial\Omega}, \partial_\nu v|_{\partial\Omega}\}$ is also known. Then we try to solve the inverse boundary value problem for (2.21), and our goal is to recover the matrix u_0^{ab}. By the positivity of φ, even with the boundary data $f = 0$, we still have $\det D^2 u_0 = F > 0$ in Ω, which implies $\left(D^2 u_0\right)$ is also an invertible matrix. If we can recover the inverse matrix $\left(u_0^{ab}\right)_{1\le a,b\le 2}$, then F can be simply recovered by using $F = \det(D^2 u_0)$ in Ω.

In what follows, let us use the notation

$$g^{ab} := u_0^{ab} \text{ in } \Omega, \text{ for } a, b = 1, 2, \tag{2.22}$$

then one can derive

$$\begin{aligned} g^{ab}\partial_{ab} v^{(k)} = 0 &\iff \sqrt{|g|} g^{ab}\partial_{ab} v^{(k)} = 0 \\ &\iff \partial_a\left(\sqrt{|g|} g^{ab}\partial_b v^{(k)}\right) - \partial_a\left(\sqrt{|g|} g^{ab}\right)\partial_b v^{(k)} = 0, \end{aligned}$$

for $k = 1, 2$, where we used $|g| = \det(g) = \det(g_{ab}) > 0$ in Ω. Hence, we can rewrite (2.21) into

$$\begin{cases} \left(-\Delta_g + X_g \cdot \nabla\right) v^{(k)} = 0 & \text{in } \Omega, \\ v^{(k)} = \phi_k & \text{on } \partial\Omega, \end{cases} \tag{2.23}$$

for $k = 1, 2$, where

$$\Delta_g = \frac{1}{\sqrt{|g|}}\partial_a\left(\sqrt{|g|} g^{ab}\partial_b\right) = g^{ab}\partial_{ab} - X_g^b \partial_b \tag{2.24}$$

[1]Throughout this work, we used the Einstein summation convention that $A^{ab}B_{ab} = \sum_{i=1}^2 A_{ab}B_{ab}$ and $A^{ab}C_a = \sum_{a=1}^2 A^{ab}C_a$, for repeating indices. Any repeated indices will be regarded as a summation with respect to a certain index.

stands for the Laplace-Beltrami operator, and

$$
\begin{aligned}
&X_g = \left(X_g^1(x), X_g^2(x)\right) : \overline{\Omega} \to \mathbb{R}^2, \\
&X_g^b := \frac{1}{\sqrt{|g|}} \sum_{a=1}^2 \partial_a \left(\sqrt{|g|} g^{ab}\right), \quad \text{for} \quad b = 1, 2,
\end{aligned} \tag{2.25}
$$

which denotes the vector-valued coefficient of the first order term. We will adapt the above standing notations (2.24) and (2.25) in the rest of this paper.

2.3.2. *The second linearization.* Consider the Dirichlet data of the form (2.16) in (1.3), and let us rewrite (2.18) as

$$
\begin{cases}
\operatorname{tr}\left(\left(D^2 u_\epsilon\right)^{-1} D^2\left(\partial_{\epsilon_k} u_\epsilon\right)\right) = 0 & \text{in } \Omega, \\
\partial_{\epsilon_k} u_\epsilon = \phi & \text{on } \partial\Omega,
\end{cases} \tag{2.26}
$$

for $k = 1, 2$, where we used $\det D^2 u_\epsilon = F > 0$ in Ω. Differentiating (2.26) with respect to ϵ_ℓ (for $k \neq \ell$) again then direct computations yields that

$$
\begin{aligned}
0 &= \partial_{\epsilon_1}\left\{ \operatorname{tr}\left(\left(D^2 u_\epsilon\right)^{-1} D^2\left(\partial_{\epsilon_2} u_\epsilon\right)\right)\right\} \\
&= \left[\operatorname{tr}\left(\partial_{\epsilon_1}\left(D^2 u_\epsilon\right)^{-1}\right) D^2\left(\partial_{\epsilon_2} u_\epsilon\right) + \operatorname{tr}\left(\left(D^2 u_\epsilon\right)^{-1} \partial_{\epsilon_1} D^2\left(\partial_{\epsilon_2} u_\epsilon\right)\right)\right] \\
&= -\operatorname{tr}\left(\left(D^2 u_\epsilon\right)^{-1} D^2\left(\partial_{\epsilon_1} u_\epsilon\right)\left(D^2 u_\epsilon\right)^{-1} D^2\left(\partial_{\epsilon_2} u_\epsilon\right)\right) \\
&\quad + \operatorname{tr}\left(\left(D^2 u_\epsilon\right)^{-1} D^2\left(\partial^2_{\epsilon_1\epsilon_2} u_\epsilon\right)\right),
\end{aligned} \tag{2.27}
$$

where we used the fact (2.9). Inserting $\epsilon = 0$ into (2.27), we can obtain the second linearized equation as

$$
\begin{cases}
\operatorname{tr}\left(\left(D^2 u_0\right)^{-1} D^2 w\right) = \operatorname{tr}\left(\left(D^2 u_0\right)^{-1} D^2 v^{(1)} \left(D^2 u_0\right)^{-1} D^2 v^{(2)}\right) & \text{in } \Omega, \\
w = 0 & \text{on } \partial\Omega,
\end{cases} \tag{2.28}
$$

where $w = \partial^2_{\epsilon_1\epsilon_2}\big|_{\epsilon=0} u_\epsilon$, and we utilized $\det\left(D^2 u_\epsilon\right) = F > 0$ in Ω. Similar to the first linearized equation (2.21), we can rewrite (2.28) as

$$
\begin{cases}
\left(-\Delta_g + X_g \cdot \nabla\right) w = \operatorname{tr}\left(g^{-1}\left(D^2 v^{(1)}\right) g^{-1}\left(D^2 v^{(2)}\right)\right) & \text{in } \Omega, \\
w = 0 & \text{on } \partial\Omega,
\end{cases} \tag{2.29}
$$

where g and X_g are given by (2.22) and (2.25), respectively. Let us emphasize again that v_k is the solution to the first linearized equation (2.21) for $k = 1, 2$.

3. Boundary determination

In this section, we derive the boundary determination for the Hessian $D^2 u_0$ on $\partial\Omega$ from the DN map under the additional assumption that the source F is known on the boundary. Presumably, this assumption can be removed by considering standard-like boundary determination techniques for the first and second linearized equations.

Lemma 3.1 (Boundary determination)**.** *Adopting all assumptions in Theorem 1.2, let $u_0 \in C^\infty(\overline{\Omega})$ be the solution to (2.20). Suppose that F is known up to second order on the boundary, then $D^\beta u_0\big|_{\partial\Omega}$ can be determined by $\Lambda_F(0)$, where $\beta = (\beta_1, \beta_2) \in (\mathbb{N} \cup \{0\})^2$ is a multi-index, with $|\beta| = \beta_1 + \beta_2 \leq 3$.*

In other words, let $F_1, F_2 \in C^\infty(\overline{\Omega})$ be positive sources, and suppose that F_1 and F_2 agree up to second order on the boundary, then

$$
\Lambda_{F_1}(0) = \Lambda_{F_2}(0) \ \text{on } \partial\Omega
$$

implies $D^\beta u_0^{(1)}\big|_{\partial\Omega} = D^\beta u_0^{(2)}\big|_{\partial\Omega}$*, for all* $|\beta| \le 4$*, where* $u_0^{(j)} \in C^\infty(\overline{\Omega})$ *is the solution to*

$$\begin{cases} \det D^2 u_0^{(j)} = F_j & \text{in } \Omega, \\ u_0^{(j)} = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.1}$$

for $j = 1, 2$*.*

Proof. We aim to show that for any point $x_0 \in \partial\Omega$ and any multi-index β with $|\beta| \le 4$, the derivative $D^\beta u_0(x_0)$ can be determined using the data $F|_{\partial\Omega}$ with its boundary derivatives up to order three, and the Dirichlet-to-Neumann map $\Lambda_F(0)$.

Without loss of generality, we assume $x_0 = 0$. Near x_0, we parameterize the boundary $\partial\Omega$ locally as the graph $x_2 = \varphi(x_1)$, where φ is a convex function defined for $x_1 \in (-\delta, \delta)$, for some $\delta > 0$. We further assume $\varphi(0) = \varphi'(0) = 0$, which can be achieved via rotation and translation of coordinates.

Since Ω is uniformly convex and the source function satisfies $F \ge c_0 > 0$, the solution $u_0 \in C^\infty(\overline{\Omega})$ is strictly convex by Proposition 2.1. In particular, the Hessian matrix $D^2 u_0$ is positive definite in Ω, which implies:

$$\partial_{11} u_0 > 0 \text{ in } \overline{\Omega}, \tag{3.2}$$

by the smoothness of u_0. Meanwhile, since ∂_1 is the tangential derivative at 0 on $\partial\Omega$, with the given information of $u_0|_{\partial\Omega}$,

Since $u_0 = 0$ on $\partial\Omega$, we have $u_0(x_1, \varphi(x_1)) = 0$ for all $x_1 \in (-\delta, \delta)$, as well as its tangential derivatives. Thus, differentiating this identity twice with respect to x_1, we can compute

$$\begin{aligned} 0 &= \underbrace{\frac{d^2}{dx_1^2}\Big|_{x_1=0} u_0(x_1, \varphi(x_1))}_{\text{tangential derivative}} \\ &= \big(\partial_{11} u_0\big)(0) + 2\big(\partial_{12} u_0\big)(0)\varphi'(0) + \big(\partial_{22} u_0\big)(0)\big(\varphi'(0)\big)^2 + \big(\partial_2 u_0\big)(0)\varphi''(0). \end{aligned}$$

Using $\varphi'(0) = 0$, this simplifies to

$$\big(\partial_{11} u_0\big)(0) + \big(\partial_2 u_0\big)(0)\varphi''(0) = 0.$$

Hence,

$$\big(\partial_{11} u_0\big)(0) = -\big(\partial_2 u_0\big)(0)\varphi''(0).$$

The right-hand side is known: $\partial_2 u_0(0)$ is obtained from the DN map Λ_F at $x_0 = 0 \in \partial\Omega$, and $\varphi''(0)$ is the curvature of the boundary at $x_0 = 0$, which is computable from the parametrization. Thus, $\big(\partial_{11} u_0\big)(0)$ is determined.

Next, it is known that Λ_F provides $\partial_\nu u_0$ on $\partial\Omega$ and $u_0 = 0$ is known on the boundary, $\partial_2 u_0$ is known on the boundary. Consequently, $\partial_{12} u_0(0)$ is also determined. Moreover, the Monge–Ampère equation (2.20) gives:

$$\big(\partial_{11} u_0\big)(0)\big(\partial_{22} u_0\big)(0) - \left(\partial_{12} u_0\right)^2(0) = F(0),$$

which yields

$$\big(\partial_{22} u_0\big)(0) = \frac{F(0) + \left(\partial_{12} u_0\right)^2(0)}{\big(\partial_{11} u_0\big)(0)}.$$

This is valid due to the positivity of $\big(\partial_{11} u_0\big)(0)$. Therefore, all second-order derivatives $\big(\partial_{ij} u_0\big)(0)$ with $i, j \in \{1, 2\}$ are now determined. Since $x_0 \in \partial\Omega$ was arbitrary, this argument applies uniformly along $\partial\Omega$, and we conclude that:

$$\partial_{11} u_0|_{\partial\Omega}, \quad \partial_{12} u_0|_{\partial\Omega}, \quad \partial_{22} u_0|_{\partial\Omega}$$

are determined.

We now proceed to third-order derivatives. Since $\partial_{22}u_0|_{\partial\Omega}$ is known, we may take a tangential derivative along $\partial\Omega$ to recover $\partial_{122}u_0(0)$. Meanwhile, differentiating the Monge–Ampère equation with respect to x_2 yields:

$$\partial_{11}u_0(0)\partial_{222}u_0(0) + \partial_{211}u_0(0)\partial_{22}u_0(0) - 2\partial_{12}u_0(0)\partial_{122}u_0(0) = \partial_2 F(0).$$

Here, $\partial_2 F(0) = -\partial_\nu F(0)$ is known by assumption, and all terms except $\partial_{222}u_0(0)$ are already determined. Solving for $\partial_{222}u_0(0)$, we obtain:

$$\partial_{222}u_0(0) = \frac{\partial_2 F(0) - \partial_{211}u_0(0)\partial_{22}u_0(0) + 2\partial_{12}u_0(0)\partial_{122}u_0(0)}{\partial_{11}u_0(0)}.$$

Again, this is valid due to (3.2). As a result, all third-order derivatives $\partial_{abc}u_0(0)$, with $a, b, c \in \{1, 2\}$, are now known.

Continuing this process, we can determine u_0 up to fourth order on the boundary by taking more tangential and normal derivatives from the preceding identities. This completes the proof that all fourth-order derivatives of u_0 at any boundary point $x_0 \in \partial\Omega$ can be determined from the data $F|_{\partial\Omega}$ with its higher order derivatives on $\partial\Omega$, and Λ_F. $\square$

4. Unique determination of the metric

Recall that our goal is to recover the Hessian of u_0 in Ω, where u_0 is the solution to (2.20). Once the Hessian D^2u_0 is determined, the source function F is also fully determined. Let us adopt the notation introduced in (2.22).

Thanks to Theorem 3.1, we already know the boundary values of the metric $g = D^2u_0$, i.e., $g|_{\partial\Omega}$, which in turn implies that the conormal derivative $\partial_{\nu_g} v_\phi$ (associated with the operator $-\Delta_g$) is known.

Using the first linearized equation of the Monge-Ampère equation (see Section 2.3), we define the DN map Λ'_g corresponding to the boundary value problem

$$\begin{cases} (-\Delta_g + X_g \cdot \nabla)\, v = 0 & \text{in } \Omega, \\ v = \phi & \text{on } \partial\Omega, \end{cases} \tag{4.1}$$

as

$$\Lambda'_g : C^\infty(\partial\Omega) \to C^\infty(\partial\Omega), \quad \phi \mapsto \partial_{\nu_g} v_\phi, \tag{4.2}$$

where $v_\phi \in C^\infty(\overline{\Omega})$ denotes the solution to (4.1).

To proceed, we prove the following result, which shows that the DN map of the fully nonlinear Monge-Ampère equation determines the DN map of its linearized counterpart (4.1).

Lemma 4.1. *Adopting all assumptions in Theorem 1.2. The DN map Λ_F of (1.3) (see the definition (1.4)) determines the DN map Λ'_g of (4.2), where $g = D^2u_0$ is the Hessian of u_0 (u_0 is the solution to (2.20)).*

Proof. The proof relies on boundary determination. By Lemma 3.1, the DN map Λ_F determines $D^2u_0|_{\partial\Omega}$. Moreover, we observe that

$$\nabla u|_{\partial\Omega} \text{ is determined, where } u \text{ is the solution to (1.3) with } u|_{\partial\Omega} \in B_\delta(\partial\Omega).$$

In addition, Proposition 2.1 implies that $\nabla v|_{\partial\Omega}$ is also determined. Hence, we obtain the complete information of the DN map

$$\Lambda'_g : \phi \mapsto \left.\partial_{\nu_g} v_\phi\right|_{\partial\Omega},$$

which establishes the claim. $\square$

From this point onward, our goal is to solve the inverse problem of recovering the metric g and the vector field X_g from the DN map Λ'_g associated with the linearized equation (4.2). To facilitate this analysis, we introduce the following notation.

Let $J = (J^1, J^2) : \overline{\Omega} \to \mathbb{R}^2$ be a C^1 diffeomorphism. Let $g = (g_{ab})_{1\le a,b\le 2}$ denote a 2×2 matrix-valued function representing a Riemannian metric on Ω (not necessarily the Hessian matrix D^2u_0, and let $X = (X^1, X^2)$ be a smooth vector field. Under the coordinate transformation J, the pullbacks of the metric, vector field, and function v are defined as follows:

$$\begin{aligned} J^*g &= (\nabla J)^T (g\circ J)\nabla J, \\ J^*X &= (J^{-1})_* X = \nabla(J^{-1})(X\circ J), \\ J^*v &= v\circ J, \end{aligned} \tag{4.3}$$

where ∇J denotes the Jacobian matrix of J, $(\nabla J)^T$ its transpose, and $(J^{-1})_*$ the pushforward by the inverse of J. With these notations in place, we now proceed to analyze the first linearized equation.

4.1. Determination up to isometry and conformal factor.

Recalling that the first linearized equation of the Monge-Ampère equation is of the form (2.23), then we have the next result.

Lemma 4.2 (Simultaneous recovery). *Let $\Omega \subset \mathbb{R}^2$ be a bounded open simply connected domain with C^∞-smooth boundary $\partial\Omega$. Let Λ'_{g_j,X_j} be the DN map of*

$$\begin{cases} \big(-\Delta_{g_j} + X_j\cdot\nabla\big)v_j = 0 & \text{in } \Omega, \\ v_j = \phi & \text{on } \partial\Omega, \end{cases} \tag{4.4}$$

where X_j is a vector field, for $j = 1, 2$. Suppose that

$$\Lambda'_{g_1,X_1}(\phi) = \Lambda'_{g_2,X_2}(\phi), \text{ for any } \phi\in C^\infty(\partial\Omega),$$

then there exists a diffeomorphism $J : \overline{\Omega} \to \overline{\Omega}$ with $J|_{\partial\Omega} = \mathrm{Id}$, and a conformal factor $c > 0$ with $c|_{\partial\Omega} = 1$ such that

(i)

$$g_1 = cJ^*g_2 \quad \text{and} \quad X_1 = c^{-1}J^*X_2 \quad \text{in} \quad \Omega. \tag{4.5}$$

(ii) Moreover, if v_j, $j = 1, 2$, are solutions to (4.4), there holds

$$v_1 = J^*v_2.$$

Here, all notations in (4.5) are given in (4.3).

Remark 4.3. *Note that the vector field X_j in the above lemma could be independent of the metric g_j, for $j = 1, 2$. Hence, we do not use the notation X_{g_j} given by (2.25) to denote the vector field in (4.4) for $j = 1, 2$.*

Proof of Lemma 4.2. For (i), by [IUY12, Theorem 1.1], there is a conformal mapping J from Ω to itself with $J|_{\partial\Omega} = \mathrm{Id}$ such that

$$g_1 = cJ^*g_2 \text{ in } \Omega,$$

for some smooth conformal factor $c > 0$ with $c|_{\partial\Omega} = 1$. Inspection of the proof of the theorem, see [IUY12, Eq. (6.6)], also shows that $\nabla J|_{\partial\Omega} = I_{2\times 2}$ (the 2×2 identity matrix). Let v_2 be a solution to

$$-\Delta_{g_2}v_2 + X_2\cdot\nabla v_2 = 0 \text{ in } \Omega.$$

Let us denote

$$\widetilde{v}_2 = v_2\circ J,$$

then $\widetilde{v}_2$ solves

$$\begin{aligned}\Delta_{g_1}\widetilde{v}_2 &= \Delta_{cJ^*g_2}\widetilde{v}_2 = c^{-1}J^*\Delta_{g_2}v_2\\ &= c^{-1}J^*(X_2\cdot\nabla v_2) = c^{-1}(J^*X_2)\nabla(J^*v_2)\\ &= c^{-1}(J^*X_2)\cdot\nabla\widetilde{v}_2 \quad \text{in} \quad \Omega.\end{aligned}$$

Since $J|_{\partial\Omega} = \mathrm{Id}$ and $\nabla J|_{\partial\Omega} = \mathrm{I}_{2\times 2}$, we also have that

$$v_1|_{\partial\Omega} = \widetilde{v}_2|_{\partial\Omega} \quad \text{and} \quad \partial_\nu v_1|_{\partial\Omega} = \partial_\nu\widetilde{v}_2|_{\partial\Omega}.$$

Here we also used $\Lambda_1' = \Lambda_2'$. Thus, the DN maps of the equations

$$-\Delta_{g_1}v_1 + X_1\cdot\nabla v_1 = 0 \text{ in } \Omega$$

and

$$-\Delta_{g_1}\widetilde{v}_2 + c^{-1}(J^*X_2)\cdot\nabla\widetilde{v}_2 = 0 \text{ in } \Omega$$

agree. Since Ω is assumed to be simply connected, by [Nur24, Lemma 4.2] (based on [Tzo17] or alternatively [GT11b]), we have

$$c^{-1}(J^*X_2) = X_1 \text{ in } \Omega.$$

Thus, we have (4.5).

For (ii), since v_1 and $\widetilde{v}_2$ now satisfy the same elliptic equation (without zeroth order term), we have $v_1 = J^*v_2$. □

We mention that a more general version of Lemma 4.2 on Riemannian surfaces, based on the proof in [CLT24], will appear in a work by the first-mentioned author.

4.2. **Determination of the isometry via the Christoffel symbol.** In this section, we want to claim that $J = \mathrm{Id}$ in $\overline{\Omega}$ by using a coupled system of equations. Thanks to Lemma 4.2, we already know that there is an isometry $J:\overline{\Omega}\to\overline{\Omega}$ with $J|_{\partial\Omega} = \mathrm{Id}$, which relates the metrics g_1 and g_2 via (4.5). Therefore, one can apply the assertion (4.5) in Lemma 4.2, which shows that

$$g_1 = cJ^*g_2 \quad \text{and} \quad X_{g_1} = c^{-1}J^*X_{g_2} \quad \text{in} \quad \Omega,$$

where X_{g_j} is the vector field given by (2.25) with components $X^i_{g_j}$, for $i,j = 1,2$. In addition, let $\Gamma(g)^i_{kl}$ be the Christoffel symbol associated with the metric g, which is given by

$$\Gamma(g)^i_{kl} := \frac{1}{2}g^{im}\Big(\frac{\partial g_{mk}}{\partial x^l} + \frac{\partial g_{ml}}{\partial x^k} - \frac{\partial g_{kl}}{\partial x^m}\Big), \quad \text{for } 1\le i,k,l\le 2,$$

for a given metric g. We also note that

$$X^i_g = -g^{kl}\Gamma(g)^i_{kl}, \tag{4.6}$$

by standard formulas in Riemannian geometry, where X_g is given by (2.25).

Notice that the Christoffel symbols transform under conformal scaling

$$g\mapsto \widetilde{g} = e^{2\sigma}g,$$

is

$$\Gamma(\widetilde{g})^i_{kl} = \Gamma(g)^i_{kl} + \delta^i_k\partial_l\sigma + \delta^i_l\partial_k\sigma - g_{kl}\partial^i\sigma.$$

Thus, writing $c = e^{2\sigma}$, or $\sigma = \frac{1}{2}\log c$, where $c>0$ is the conformal factor, we have

$$\begin{aligned}\Gamma^i_{kl}(g_1) &= \Gamma^i_{kl}(cJ^*g_2)\\ &= \Gamma^i_{kl}(J^*g_2) + \frac{1}{2}\big(\delta^i_k\partial_l\log c + \delta^i_l\partial_k\log c - (g_1)_{kl}g_1^{ij}\partial_j\log c\big).\end{aligned}\tag{4.7}$$

Here, $\delta^i_k = \begin{cases}1 & \text{for } i=k\\ 0 & \text{otherwise}\end{cases}$ denotes the Kronecker delta.

Using Lemma 4.2, we can prove Theorem 1.4. For readers' convenience, let us recall Theorem 1.4 as follows.

Theorem 4.4. *Let $\Omega \subset \mathbb{R}^2$ be a bounded open simply connected domain with C^∞-smooth boundary $\partial\Omega$. Let Λ'_{g_j} be the DN map of*

$$\begin{cases} g_j^{ab}\partial_{ab}v_j = 0 & \text{in } \Omega, \\ v_j = \phi & \text{on } \partial\Omega, \end{cases} \tag{4.8}$$

for $j = 1, 2$. Suppose that

$$\Lambda'_{g_1}(\phi) = \Lambda'_{g_2}(\phi) \quad \text{for any} \quad \phi \in C^\infty(\partial\Omega),$$

then there exists a conformal factor $c > 0$ with $c|_{\partial\Omega} = 1$, such that

$$g_1 = cg_2 \text{ in } \Omega. \tag{4.9}$$

Remark 4.5. *In Theorem 4.4, it is not necessary to assume that Ω is uniformly convex. However, it remains unclear whether the assumption that Ω is simply connected can be removed. The difficulty lies in the fact that the proof ultimately relies on the Poincaré lemma, invoked through* [Nur24, Lemma 4.2]*. Thus, the theorem may be viewed as a realization of the anisotropic Calderón problem for elliptic equations in non-divergence form.*

Proof of Theorem 4.4. Let us divide the proof into several steps:

Step 1. Initialization.

First, we rewrite (4.8) in the form

$$\begin{cases} \big(-\Delta_{g_j} + X_{g_j}\big)v_j = 0 & \text{in } \Omega, \\ v_j = \phi & \text{on } \partial\Omega, \end{cases}$$

where X_{g_j} is the vector field given in (2.25) with $g = g_j$, for $j = 1, 2$. By Lemma 4.2 (i), there exists a diffeomorphism $J : \overline{\Omega} \to \overline{\Omega}$ satisfying $J|_{\partial\Omega} = \mathrm{Id}$ and a conformal factor $c > 0$ with $c|_{\partial\Omega} = 1$, such that

$$g_1 = c\, J^* g_2, \quad X_{g_1} = c^{-1} J^* X_{g_2} \quad \text{in } \Omega. \tag{4.10}$$

Step 2. Unique determination of the diffeomorphism.

We next claim

$$J = \mathrm{Id} \text{ in } \overline{\Omega}, \tag{4.11}$$

or $J(x) = x$ for all $x \in \overline{\Omega}$. To this end, let us write $\widetilde{x} = J(x)$ and use the typical convention to denote by $\frac{\partial x^i}{\partial \widetilde{x}^m}$ the components of the differential of the inverse of J (evaluated at J). We have the standard Christoffel symbols transform as

$$\Gamma^i_{kl}(J^*g_2) = \frac{\partial x^i}{\partial \widetilde{x}^m}\frac{\partial \widetilde{x}^a}{\partial x^k}\frac{\partial \widetilde{x}^b}{\partial x^l}\Gamma^m_{ab}(g_2)\circ J + \frac{\partial^2 \widetilde{x}^m}{\partial x^k \partial x^l}\frac{\partial x^i}{\partial \widetilde{x}^m}. \tag{4.12}$$

Multiplying (4.12) by the matrix $(J^*g_2)^{kl}$, and applying (4.6), we can obtain

$$X^i_{J^*g_2} = \frac{\partial x^i}{\partial \widetilde{x}^m} X^m_{g_2}\circ J + (J^*g_2)^{kl}\frac{\partial^2 \widetilde{x}^m}{\partial x^k \partial x^l}\frac{\partial x^i}{\partial \widetilde{x}^m}. \tag{4.13}$$

By (4.7) and (4.12), one can find

$$\begin{aligned} \Gamma^i_{kl}(g_1) = {} & \frac{\partial x^i}{\partial \widetilde{x}^m}\frac{\partial \widetilde{x}^a}{\partial x^k}\frac{\partial \widetilde{x}^b}{\partial x^l}\Gamma^m_{ab}(g_2)\circ J + \frac{\partial^2 \widetilde{x}^m}{\partial x^k \partial x^l}\frac{\partial x^i}{\partial \widetilde{x}^m} \\ & + \frac{1}{2}\big(\delta^i_k \partial_l \log c + \delta^i_l \partial_k \log c - (g_1)_{kl} g_1^{ij}\partial_j \log c\big). \end{aligned}$$

By (4.6), (4.7) and (4.13), one has

$$
\begin{aligned}
(J^*g_2)^{kl}\Gamma^i_{kl}(g_1) &= \underbrace{(J^*g_2)^{kl}\Gamma^i_{kl}(cJ^*g_2)}_{\text{By } g_1=cJ^*g_2} \\
&= \underbrace{(J^*g_2)^{kl}\Gamma^i_{kl}(J^*g_2)}_{=-X^i_{J^*g_2}} \\
&\quad + \frac{1}{2}(J^*g_2)^{kl}\big(\delta^i_k\partial_l \log c + \delta^i_l\partial_k \log c - (g_1)_{kl}g_1^{ij}\partial_j \log c\big) \\
&= -\Big(\frac{\partial x^i}{\partial \widetilde{x}^m} X^m_{g_2}\circ J + (J^*g_2)^{kl}\frac{\partial^2 \widetilde{x}^m}{\partial x^k \partial x^l}\frac{\partial x^i}{\partial \widetilde{x}^m}\Big) \\
&\quad + \frac{1}{2}(J^*g_2)^{kl}\big(\delta^i_k\partial_l \log c + \delta^i_l\partial_k \log c - (g_1)_{kl}g_1^{ij}\partial_j \log c\big).
\end{aligned} \tag{4.14}
$$

Via (4.5) and (4.6), the left-hand side of (4.14) is

$$
(J^*g_2)^{kl}\Gamma^i_{kl}(g_1) = c g_1^{kl}\Gamma^i_{kl}(g_1) = -cX_{g_1}. \tag{4.15}
$$

On the one hand, plugging (4.15) into (4.14), and using the second relation in (4.10), i.e., $X_{g_1} = c^{-1}J^*X_{g_2}$, we obtain

$$
\begin{aligned}
(J^*X_{g_2})^i &= (cX_{g_1})^i \\
&= \frac{\partial x^i}{\partial \widetilde{x}^m} X^m_{g_2}\circ J + (J^*g_2)^{kl}\frac{\partial^2 \widetilde{x}^m}{\partial x^k \partial x^l}\frac{\partial x^i}{\partial \widetilde{x}^m} \\
&\quad - \frac{1}{2}(J^*g_2)^{kl}\big(\delta^i_k\partial_l \log c + \delta^i_l\partial_k \log c - (g_1)_{kl}g_1^{ij}\partial_j \log c\big).
\end{aligned} \tag{4.16}
$$

On the other hand, via (4.3), we also have

$$
(J^*X_{g_2})^i = \frac{\partial x^i}{\partial \widetilde{x}^m} X^m_{g_2}\circ J, \tag{4.17}
$$

so we can insert (4.17) into the left-hand side of (4.16), which can be canceled by the first term in the right-hand side of (4.16). Thus, one can obtain

$$
(J^*g_2)^{kl}\frac{\partial^2 \widetilde{x}^m}{\partial x^k \partial x^l}\frac{\partial x^i}{\partial \widetilde{x}^m} = \frac{1}{2}(J^*g_2)^{kl}\big(\delta^i_k\partial_l \log c + \delta^i_l\partial_k \log c - (g_1)_{kl}g_1^{ij}\partial_j \log c\big). \tag{4.18}
$$

Finally, using $g_1 = cJ^*g_2$ to the both sides of (4.18), we have

$$
g_1^{kl}\frac{\partial^2 \widetilde{x}^m}{\partial x^k \partial x^l}\frac{\partial x^i}{\partial \widetilde{x}^m} = \frac{1}{2}g_1^{kl}\big(\delta^i_k\partial_l \log c + \delta^i_l\partial_k \log c - (g_1)_{kl}g_1^{ij}\partial_j \log c\big), \tag{4.19}
$$

which can also be written in an equivalent form

$$
g_1^{kl}\frac{\partial^2 \widetilde{x}^m}{\partial x^k \partial x^l} = \frac{1}{2}\frac{\partial \widetilde{x}^m}{\partial x^i}\big(2g_1^{il}\partial_l \log c - 2g_1^{ij}\partial_j \log c\big) = 0, \tag{4.20}
$$

where we used $g_1^{kl}(g_1)_{kl} = \mathrm{tr}(\mathrm{I}_2) = 2$ for the last term in the right-hand side of (4.19).

Thus, for the individual $m = 1, 2$, the equation (4.20) implies that

$$
\begin{cases} g_1^{kl}\frac{\partial^2 \widetilde{x}^m}{\partial x^k \partial x^l} = 0 & \text{in } \Omega, \\ \widetilde{x}^m = x^m & \text{on } \partial\Omega, \end{cases} \tag{4.21}
$$

which is a second order elliptic equation of non-divergence form, where we apply Lemma 4.2 to have $J|_{\partial\Omega} = \mathrm{Id}$, so that $\widetilde{x}^m = x^m$ on $\partial\Omega$ for $m = 1, 2$. Therefore, by the uniqueness of the boundary value problem (4.21) (for example, see [GT01]), this implies that $\widetilde{x}^m \equiv x^m$ in Ω for $m = 1, 2$ (it is easy to see that x^m is a solution to (4.21)). This infers that $J(x) = x$ in Ω as we wish. This proves the claim (4.11).

Step 3. Summary.

Finally, using (4.10), we know that

$$g_1 = \underbrace{cJ^* g_2 = cg_2}_{\text{By } J=\mathrm{Id}} \text{ in } \Omega,$$

which proves the assertion (4.9). □

Remark 4.6.

(i) *From (4.20), the mapping $J(x)$ satisfies the elliptic equation (4.21) without requiring knowledge of the conformal factor $c > 0$.*

(ii) *In Theorem 4.4, convexity of Ω is not needed; however, simple connectedness is essential for determining X_g.*

The remainder of the paper is devoted to recovering the conformal factor $c > 0$ in Ω using suitable CGO solutions for the first linearized equation.

5. Complex geometrical optics solutions

This section is devoted to constructing CGO solutions for the first linearized equation (2.23) and its adjoint equation (7.2). We also derive expansion formulas for the correction terms and provide estimates for the related oscillatory integrals.

5.1. Isothermal coordinates.

Recall that the isothermal coordinates (see, for example, [Ahl66]) correspond to a change of variables $\chi : \Omega \to \widetilde{\Omega} := \chi(\Omega)$, where we denote by χ the associated quasi-conformal mapping. In what follows, we introduce isothermal coordinates so that the metric g_1 takes the form

$$g_1 = \mu \, I_{2\times 2},$$

for some positive scalar function $\mu = \mu(x)$, which will play a key role in our subsequent analysis.

Using Lemma 4.2, we can rewrite the equation (4.4) (in the case $j = 1$) as

$$\begin{cases} -\dfrac{1}{\mu}\Delta \mathbf{v}_1 + \chi^* X_{g_1} \cdot \nabla \mathbf{v}_1 = 0 & \text{in } \widetilde{\Omega}, \\ \mathbf{v}_1 = \phi \circ \chi & \text{on } \partial\widetilde{\Omega}, \end{cases} \tag{5.1}$$

where $\mathbf{v}_1 := v_1 \circ \chi$. This reformulation is also instrumental in the identification of the conformal factor c in Ω.

Moreover, equation (5.1) can be further simplified to the standard form

$$\begin{cases} -\Delta \mathbf{v}_1 + \mathbf{X}_{g_1} \cdot \nabla \mathbf{v}_1 = 0 & \text{in } \widetilde{\Omega}, \\ \mathbf{v}_1 = \phi \circ \chi & \text{on } \partial\widetilde{\Omega}, \end{cases} \tag{5.2}$$

for some vector field $\mathbf{X}_{g_1}$ depending on g_1, χ, and μ.

Since we have already applied the change of variables to transform all indices from 2 to 1, we now consider the adjoint problem corresponding to equation (5.2). Let $\mathbf{v}_1^*$ denote a solution to the adjoint equation:

$$\begin{cases} \Delta \mathbf{v}_1^* + \nabla \cdot (\mathbf{X}_{g_1} \mathbf{v}_1^*) = 0 & \text{in } \widetilde{\Omega}, \\ \mathbf{v}_1^* = \phi \circ \chi & \text{on } \partial\widetilde{\Omega}, \end{cases} \tag{5.3}$$

where ϕ is an arbitrary function.

Using Lemma 4.2, we may also perform the same change of variables for the adjoint equation (7.2) with $j = 2$, while assigning the Dirichlet boundary condition to be identical to that of $\mathbf{v}_1^*\big|_{\partial\widetilde{\Omega}}$. By the uniqueness of solutions to elliptic equations, this yields that the adjoint problem for $j = 2$ also takes the form of equation (5.3).

Hence, we denote the unified form of the adjoint problem in isothermal coordinates as:

$$\begin{cases} \Delta \mathbf{v}^* + \nabla \cdot (\mathbf{X}_g \mathbf{v}^*) = 0 & \text{in } \widetilde{\Omega}, \\ \mathbf{v}^* = \phi \circ \chi & \text{on } \partial\widetilde{\Omega}, \end{cases} \tag{5.4}$$

where $g = g_1$. From this point forward, all notations introduced above are fixed and will be used consistently throughout the remainder of the paper.

5.2. **The construction of CGOs.** For $X \in C_c^\infty(M, T^*M)$, we recall the construction of [GT11b] of CGO solutions to

$$(-\Delta_g + X \cdot \nabla)\, v = 0 \text{ on } \widetilde{M}. \tag{5.5}$$

Here $X \cdot \nabla v = g(X, dv)$.

We want to show that the desired CGO solutions of the linear equation

$$(-\Delta_g + X \cdot \nabla)\, v = 0 \text{ in } \Omega, \tag{5.6}$$

and its adjoint equation

$$-\Delta_g v - \nabla \cdot (Xv) = 0 \text{ in } \Omega, \tag{5.7}$$

are of the form

$$F_A^{-1} e^{\Phi/h}(a + r),$$

for (small) $h > 0$, where $X : \overline{\Omega} \to \mathbb{R}^2$ is a C^∞-smooth vector field. Here, $\Phi = \Phi(z)$ be is holomorphic Morse function, r is the corresponding remainder term, $F_A = e^{i\alpha}$, where α is a function that solves $\overline{\partial}\alpha = A$, and A will be given by X. We have a similar ansatz for CGOs with antiholomorphic phase, see Lemma 5.3.

The construction of solutions is based on the methods developed in [GT11a] and [GT11b]. Although our setting only requires the construction within global holomorphic coordinates, we follow the general framework used in these references, which are formulated on general Riemannian surfaces. This choice facilitates cross-referencing and provides a flexible foundation for future applications.

We begin by introducing the standard notations and definitions adopted in the aforementioned works. Let Σ be a Riemann surface compactly contained in an open surface M. We extend the metric g and the vector field X smoothly to a larger open surface $\widetilde{M} \supset M$ such that $X \in C_c^\infty(\widetilde{M})$.

5.2.1. *Calculus on Riemannian surfaces.* The complexified cotangent bundle $\mathbb{C}T^*M$ has the splitting

$$\mathbb{C}T^*M = T^*_{1,0}M \oplus T^*_{0,1}M$$

determined by the eigenspaces of the Hodge star operator $\star$. In holomorphic coordinates $z = (x_1, x_2)$ the space $T^*_{1,0}M$ is spanned by dz and $T^*_{0,1}M$ is spanned by $d\overline{z}$, where

$$dz = dx_1 + \mathrm{i}dx_2 \quad \text{and} \quad d\overline{z} = dx_1 - \mathrm{i}dx_2.$$

The invariant definitions of ∂ and $\overline{\partial}$ operators are given as

$$\partial := \pi_{1,0} d \text{ and } \overline{\partial} := \pi_{0,1} d.$$

Then $d = \partial + \overline{\partial}$ and in holomorphic coordinates

$$\partial f = \partial_z f\, dz, \quad \overline{\partial} f = \partial_{\overline{z}} f\, d\overline{z},$$

where ∂ and $\overline{\partial}$ are given (as in (2.1)) by

$$\partial_z = \frac{1}{2}\left(\partial_{x_1} - \mathrm{i}\partial_{x_2}\right), \quad \overline{\partial} = \frac{1}{2}\left(\partial_{x_1} + \mathrm{i}\partial_{x_2}\right).$$

By expressing dx_1 and dx_2 in terms of dz and $d\bar{z}$, a 1-form $X = X_1\, dx_1 + X_2\, dx_2$ in holomorphic coordinates can be written as

$$X = \frac{1}{2}(X_1 - \mathrm{i}X_2)\, dz + \frac{1}{2}(X_1 + \mathrm{i}X_2)\, d\bar{z} =: X_{1,0} + X_{0,1} \tag{5.8}$$

so that $\pi_{1,0}X = \frac{1}{2}(X_1 - \mathrm{i}X_2)$ and $\pi_{0,1}X = \frac{1}{2}(X_1 + \mathrm{i}X_2)$. Using the above formula for X, we define

$$\partial X := dX_{0,1}, \quad \overline{\partial} X := dX_{1,0}.$$

In holomorphic coordinates, this is equivalent to

$$\partial(u\, dz + v\, d\bar{z}) = \partial v \wedge d\bar{z}, \quad \overline{\partial}(u\, dz + v\, d\bar{z}) = \overline{\partial} u \wedge dz.$$

The Laplacian is given by

$$-\Delta_g f = -2\mathrm{i} * \overline{\partial}\partial f.$$

(We note that we use the opposite sign for the Laplacian to [GT11a, GT11b].)

By [GT11b, Proposition 2.1], importantly, there is a right inverse $\overline{\partial}^{-1}$ for $\overline{\partial}$ in the sense that

$$\overline{\partial}\,\overline{\partial}^{-1}\omega = \omega \text{ for all } \omega \in C_0^\infty(M, T_{0,1}^*M) \tag{5.9}$$

such that $\overline{\partial}^{-1}$ is bounded from $L^p(T_{1,0}^*M)$ to $W^{1,p}(M)$ for any $p \in (1,\infty)$. We have analogous properties for the Hermitian adjoint of $\overline{\partial}$

$$\overline{\partial}^* = -\mathrm{i} * \partial : W^{1,p}(T_{0,1}^*M) \to L^p(M).$$

In holomorphic coordinates z, the operator $\overline{\partial}^*$ is just ∂. We define

$$\overline{\partial}_\psi^{-1} := \mathcal{R}\overline{\partial}^{-1} e^{-2i\psi/h}\mathcal{E} \quad \text{and} \quad \overline{\partial}_\psi^{*-1} := \mathcal{R}\overline{\partial}^{*-1} e^{2i\psi/h}\mathcal{E},$$

where $\mathcal{E} : W^{l,p}(\Sigma) \to W_c^{l,p}(M)$ an extension operator for some M compactly containing M and $\mathcal{R}$ is the restriction operator onto Σ. By [GT11b, Lemma 2.2 and Lemma 2.3] we have for $p > 2$ and $2 \le q \le p$ the following estimates

$$\begin{aligned} \|\overline{\partial}_\psi^{-1} f\|_{L^q(M)} &\le Ch^{1/q}\|f\|_{W^{1,p}(M,T_{0,1}^*M)} \\ \|\overline{\partial}_\psi^{*-1} f\|_{L^q(M)} &\le Ch^{1/q}\|f\|_{W^{1,p}(M,T_{1,0}^*M)}. \end{aligned} \tag{5.10}$$

Moreover, there is $\epsilon > 0$ such that

$$\begin{aligned} \|\overline{\partial}_\psi^{-1} f\|_{L^2(M)} &\le Ch^{1/2+\epsilon}\|f\|_{W^{1,p}(M,T_{0,1}^*M)} \\ \|\overline{\partial}_\psi^{*-1} f\|_{L^2(M)} &\le Ch^{1/2+\epsilon}\|f\|_{W^{1,p}(M,T_{1,0}^*M)}. \end{aligned} \tag{5.11}$$

If ψ has no critical points on M, we can obtain better estimates than (5.11) and (5.10). Indeed, we have that for all $f \in C^\infty(M; T_{0,1}^*M)$,

$$\overline{\partial}_\psi^{-1} f = \mathcal{R}\overline{\partial}^{-1} e^{-2\mathrm{i}\psi/h}\mathcal{E}f = \frac{\mathrm{i}h}{2}\mathcal{R}\overline{\partial}^{-1}\Big(\Big(\overline{\partial}e^{-2\mathrm{i}\psi/h}\Big)\frac{\mathcal{E}f}{\overline{\partial}\psi}\Big), \tag{5.12}$$

where for all $\omega \in \mathcal{D}'(M; T_{0,1}^*M)$, $\omega/\overline{\partial}\psi$ denotes the unique scalar function such that $\overline{\partial}\psi(\omega/\overline{\partial}\psi) = \omega$. In holomorphic coordinates, $\overline{\partial}^{-1}$ has Schwartz kernel given by $(z - z')^{-1}$. Thus, writing (5.12) in local coordinates and integrating by parts yields

$$\overline{\partial}_\psi^{-1} f = e^{-2\mathrm{i}\psi/h}\frac{\mathrm{i}h}{2}\frac{f}{\overline{\partial}\psi} + \frac{\mathrm{i}h}{2}\mathcal{R}\overline{\partial}^{-1}\left(e^{-2\mathrm{i}\psi/h}\overline{\partial}\left(\frac{\mathcal{E}f}{\overline{\partial}\psi}\right)\right).$$

Consequently, in the case when ψ has no critical points, continuity of $\overline{\partial}^{-1} : L^p \to W^{1,p}$ immediately gives the estimate

$$\|\overline{\partial}_\psi^{-1} f\|_p \le Ch\|f\|_{W^{1,p}}, \tag{5.13}$$

for $p \in (1,\infty)$. Similarly we have

$$\left\|\overline{\partial}_\psi^{*-1} f\right\|_p \le Ch\|f\|_{W^{1,p}}.$$

If the holomorphic function Φ has no critical points, we have by (the proof of) [CLLT23, Lemma 3.5] to arbitrary order $N \in \mathbb{N} \cup \{0\}$ the following expansion

$$\overline{\partial}_\psi^{-1}(f) = e^{-2\mathrm{i}\psi/h} \sum_{j=1}^{N+1} h^j F^j + h^{N+1}\overline{\partial}_\psi^{-1}(\overline{\partial} F^{N+1}), \tag{5.14}$$

where F^j, $j \in \mathbb{N}$, are defined iteratively by

$$F^1 = \frac{\mathrm{i}}{2}\frac{f}{\overline{\partial}\psi} \in C^\infty$$

and

$$F^{j+1} = \frac{\mathrm{i}}{2}\frac{1}{\overline{\partial}\psi}\overline{\partial} F^j \in C^\infty.$$

Note that the functions $F^j \in C^\infty$, $j = 1, \dots, N+1$ in (5.14), are independent of h. The expansion formula holds since $\overline{\partial}\psi \neq 0$ due to

$$2\mathrm{i}\overline{\partial}\psi = \overline{\partial}(\Phi - \overline{\Phi}) = -\overline{\partial\Phi},$$

which must be nonzero because Φ is holomorphic. We have a similar formula for $\partial_\psi^{-1} f$.

5.2.2. *CGOs with holomorphic phase.* The CGOs we next introduce have the same form

$$v = F_A^{-1} e^{\Phi/h}(a + r_h),$$

as in [GT11b]. Here $\Phi = \phi + \mathrm{i}\psi$ is a holomorphic Morse function and a is a holomorphic function defined on M. Moreover, the function F_A is given by

$$F_A = e^{\mathrm{i}\alpha}, \tag{5.15}$$

where α solves

$$\overline{\partial}\alpha = A$$

with $A = \pi_{0,1}\big(-\frac{\mathrm{i}}{2}X\big) = -\frac{\mathrm{i}}{2}\pi_{0,1}X$. Note that α always exists by (5.9).

The Laplace operator with a drift term and potential

$$L := -\Delta_g + X \cdot \nabla + q,$$

factorized as the magnetic Schrödinger operator

$$L = 2F_{\overline{A}}^{-1}\overline{\partial}^*\left[F_{\overline{A}} F_A^{-1} \overline{\partial} F_A\right] + Q, \tag{5.16}$$

see [GT11b, Section 5], where A and F_A are as before,

$$F_{\overline{A}} = e^{\mathrm{i}\overline{\alpha}}, \tag{5.17}$$

and

$$Q = \frac{\mathrm{i}}{2} * dX - \frac{1}{4}|X|^2 + \frac{1}{2}\nabla \cdot X + q. \tag{5.18}$$

Note that while α solves $\overline{\partial}\alpha = A$, $\overline{\alpha}$ solves

$$\partial\overline{\alpha} = \overline{A}.$$

We will apply the above in the case $q = 0$, but we include the general case for future reference.

For concreteness and to illustrate how the computations using complex derivatives work, let us verify (5.16). Let $f \in C^\infty$, then one may compute

$$
\begin{aligned}
&2F_{\overline{A}}^{-1}\overline{\partial}^*\left[F_{\overline{A}}F_A^{-1}\overline{\partial}(F_A f)\right] \\
&= 2F_{\overline{A}}^{-1}\overline{\partial}^*\left[F_{\overline{A}}F_A^{-1}\overline{\partial}(e^{\mathrm{i}\alpha} f)\right] \\
&= 2F_{\overline{A}}^{-1}\overline{\partial}^*\left[F_{\overline{A}}F_A^{-1}e^{\mathrm{i}\alpha}(\mathrm{i}Af + \overline{\partial}f)\right] \\
&= 2F_{\overline{A}}^{-1}\overline{\partial}^*\left[F_{\overline{A}}(\mathrm{i}Af + \overline{\partial}f)\right] \\
&= -2\mathrm{i} * F_{\overline{A}}^{-1}\partial\left[F_{\overline{A}}(\mathrm{i}Af + \overline{\partial}f)\right] \\
&= -2\mathrm{i} * \left[\mathrm{i}\overline{A} \wedge (\mathrm{i}Af + \overline{\partial}f) + \mathrm{i}(\partial A)f + \mathrm{i}A \wedge \partial f + \partial\overline{\partial}f\right] \\
&= -\Delta_g f + 2 * (\overline{A} \wedge \overline{\partial}f + A \wedge \partial f) - 2\mathrm{i} * (\mathrm{i}\partial A - \overline{A} \wedge A)f.
\end{aligned}
$$

Here $\overline{A} \wedge \overline{\partial}f + A \wedge \partial f = 2\mathrm{Re}(A \wedge \partial f)$. Computing using holomorphic coordinates $z = (x_1, x_2)$ we have by (5.8) that $A = -\frac{\mathrm{i}}{4}(X_1 + \mathrm{i}X_2)\, d\overline{z}$ and consequently

$$
\begin{aligned}
A \wedge \partial f &= -\frac{\mathrm{i}}{4}(X_1 + \mathrm{i}X_2)\, d\overline{z} \wedge \frac{1}{2}(\partial_1 f - \mathrm{i}\partial_2 f)\, dz \\
&= \frac{\mathrm{i}}{8}(X_1 + \mathrm{i}X_2)(\partial_1 f - \mathrm{i}\partial_2 f)\, dz \wedge d\overline{z} \\
&= \frac{1}{4}(X_1 + \mathrm{i}X_2)(\partial_1 f - \mathrm{i}\partial_2 f)\, dx_1 \wedge dx_2,
\end{aligned}
$$

where we used $dz \wedge d\overline{z} = -2\mathrm{i}\, dx_1 \wedge dx_2$. Recalling that

$$
*dV_g = 1 \text{ and } dV_g = \sqrt{|g|}\, dx = \sqrt{|g|}\, dx_1 \wedge dx_2, \tag{5.19}
$$

we obtain

$$
\begin{aligned}
2 * (\overline{A} \wedge \overline{\partial}f + A \wedge \partial f) &= 2 * 2\mathrm{Re}(A \wedge \partial f) \\
&= |g|^{-1/2}(X_1\partial_1 f + X_2\partial_2 f) \\
&= g(X, \nabla f) \\
&= X \cdot \nabla f.
\end{aligned}
$$

Hence, by setting

$$
Q = 2\mathrm{i} * (\mathrm{i}\partial A - \overline{A} \wedge A) + q,
$$

we have

$$
-\Delta_g + X \cdot \nabla + q = 2F_{\overline{A}}^{-1}\overline{\partial}^*\left(F_{\overline{A}}F_A^{-1}\overline{\partial}F_A\right) + Q
$$

as claimed. To have the formula (5.18), let us compute

$$
\begin{aligned}
Q - q &= 2\mathrm{i} * (\mathrm{i}\partial A - \overline{A} \wedge A) \\
&= -2 * \partial_z\left[\frac{-\mathrm{i}}{4}(X_1 + \mathrm{i}X_2)dz \wedge d\overline{z} - 2\mathrm{i} * \frac{\mathrm{i}}{4}(X_1 - \mathrm{i}X_2)\left(\frac{-\mathrm{i}}{4}(X_1 + iX_2)\right)\right] dz \wedge d\overline{z} \\
&= \frac{\mathrm{i}}{4}\left[(\partial_1 X_1 + \partial_2 X_2) + \mathrm{i}(\partial_1 X_2 - \partial_2 X_1)\right] * dz \wedge d\overline{z} - \frac{\mathrm{i}}{8}(X_1^2 + X_2^2) * dz \wedge d\overline{z} \\
&= \frac{1}{2}\nabla \cdot X - \frac{1}{4}|X|^2 + \frac{\mathrm{i}}{2} * dX,
\end{aligned}
$$

which shows the identity.

Lemma 5.1 (CGO solutions)**.** *Let*

$$
V' := -|F_A|^2 \quad \textit{and} \quad V := \frac{1}{2}Q\,|F_A|^{-2},
$$

where F_A and Q are given by (5.17) and (5.18), respectively. Then

$$
v = F_A^{-1}e^{\Phi/h}(a + r_h)
$$

solves the linear equation (5.5)*, where* r_h *is the remainder term given by*

$$r_h = -\overline{\partial}_\psi^{-1} V' s_h \tag{5.20}$$

where

$$s_h = \sum_{j=0}^{\infty} T_h^j \overline{\partial}_\psi^{*-1}(Va). \tag{5.21}$$

with

$$T_h := \overline{\partial}_\psi^{*-1} V \overline{\partial}_\psi^{-1} V'. \tag{5.22}$$

Proof. Although it follows from [GT11b] of (5.5) that v is a solution, let us verify that for concreteness. We compute

$$\begin{aligned}
Lv - Qv &= 2F_{\overline{A}}^{-1}\overline{\partial}^* \left[F_{\overline{A}}F_A^{-1}\overline{\partial}\big(e^{\Phi/h}(a+r_h)\big)\right] \\
&= 2F_{\overline{A}}^{-1}\overline{\partial}^* \left(F_{\overline{A}}F_A^{-1}e^{\Phi/h}\overline{\partial} r_h\right) \\
&= -2F_{\overline{A}}^{-1}\overline{\partial}^* \left(F_{\overline{A}}F_A^{-1}e^{\Phi/h}e^{-2i\psi/h}V' s_h\right).
\end{aligned}$$

We have

$$F_{\overline{A}}F_A^{-1} = e^{i\overline{\alpha}}e^{-i\alpha} = |F_A|^{-2}.$$

Thus

$$\begin{aligned}
Lv - Qv &= 2F_{\overline{A}}^{-1}\overline{\partial}^* \left(e^{\Phi/h}e^{-2i\psi/h}s_h\right) \\
&= 2F_{\overline{A}}^{-1}\underbrace{\overline{\partial}^* \left(e^{\overline{\Phi}/h}s_h\right)}_{\overline{\partial}^* e^{\overline{\Phi/h}}=0} \\
&= 2F_{\overline{A}}^{-1}e^{\overline{\Phi}/h}\overline{\partial}^* s_h \\
&= -2F_{\overline{A}}^{-1}e^{\overline{\Phi}/h}\overline{\partial}^* \left(\overline{\partial}_\psi^{*-1}(Va) + \sum_{j=1}^{\infty} T_h^j \overline{\partial}_\psi^{*-1}(Va)\right) \\
&= -2F_{\overline{A}}^{-1}e^{\overline{\Phi}/h}e^{2i\psi/h}\left(Va + V\overline{\partial}_\psi^{-1}V'\sum_{j=1}^{\infty} T_h^{j-1}\overline{\partial}_\psi^{*-1}(Va)\right) \\
&= -2F_{\overline{A}}^{-1}e^{\Phi/h}V\,(a+r_h) - Qv,
\end{aligned}$$

since

$$F_{\overline{A}}^{-1}\,|F_A|^{-2} = e^{-i\overline{\alpha}}e^{-i\alpha}e^{i\overline{\alpha}} = F_A^{-1}.$$

Thus, $Lv = 0$ as claimed. ∎

Let us next recall and derive estimates for the correction term r_h. By [GT11b, Lemma 3.1], we have

$$\|T_h\|_{L^r\to L^r} = \mathcal{O}(h^{1/r}) \quad \text{and} \quad \|T_h\|_{L^2\to L^2} = \mathcal{O}(h^{1/2-\epsilon}), \tag{5.23}$$

for any $0 < \epsilon < 1/2$. We also have

$$\|T_h\|_{W^{1,p}\to W^{1,p}} = \mathcal{O}(h^{1/p}) \quad \text{and} \quad \|T_h\|_{W^{1,2}\to W^{1,2}} = \mathcal{O}(h^{1/2-\epsilon}), \tag{5.24}$$

for any $0 < \epsilon < 1/2$. Indeed, if $f \in W^{1,p}$, we have for $p > 2$

$$\|\overline{\partial}_\psi^{*-1}V\overline{\partial}_\psi^{-1}V'f\|_{W^{1,p}} \lesssim \|V\overline{\partial}_\psi^{-1}V'f\|_{L^p} \lesssim h^{1/p}\|f\|_{W^{1,p}}$$

by continuity of $\overline{\partial}^{-1} : L^p \to W^{1,p}$ and (5.10). For $p = 2$ the norm estimate in (5.24) follows from the fact that T_h is uniformly bounded $W^{1,r}$ to $W^{1,r}$, $r < 2$, and standard interpolation result [BL76, Theorem 6.4.5]. By (the proof of) [GT11b, Lemma 3.2] it then follows that for any $\epsilon > 0$ small enough

$$\|s_h\|_{L^2} + \|r_h\|_{L^2} = \mathcal{O}(h^{1/2+\epsilon}), \tag{5.25}$$

and similar to [CLT24, Section 4.1], we also have

$$\|s_h\|_{L^p} + \|r_h\|_{L^p} = \mathcal{O}(h^{1/p+\epsilon_p}),$$

for all $p \geq 2$, where $\epsilon_p > 0$ depends on p. Moreover, since $\partial\overline{\partial}^{-1}$ is a Calderón-Zygmund operators, so one can use the Calderón-Zygmund estimate to derive

$$\|r_h\|_{L^p}, \quad \|\partial r_h\|_{L^p}, \quad \|\overline{\partial} r_h\|_{L^p} = \mathcal{O}(h^{1/p+\epsilon_p}),$$

for all $p \geq 2$, where $\epsilon_p > 0$ depends on p.

Before proceeding, we recall the Calderón–Zygmund estimate for r_2 (see, for instance, [LW23, Section 3]), which yields

$$\|HD^2 r_h\|_{L^2} = \mathcal{O}(h^{-1/2+\varepsilon}), \tag{5.26}$$

for any sufficiently small $\varepsilon > 0$ where $H \in C_0^2(\Omega)$ can be arbitrary. To verify (5.26), we apply the Calderón–Zygmund inequality (for example, see [GT01, Corollary 9.10]) to the product of r_h with H, obtaining

$$\begin{aligned}
\left\|HD^2 r_h\right\|_{L^2} &\leq \left\|D^2(Hr_h)\right\|_{L^2} + 2\left\|\nabla H \otimes \nabla r_h\right\|_{L^2} + \left\|r_h D^2 H\right\|_{L^2} \\
&\lesssim \|\partial\bar{\partial}(Hr_h)\|_{L^2} + \mathcal{O}(h^{1/2+\varepsilon}) \\
&\lesssim \|\partial\big(e^{\mathrm{i}\varphi/h} s_h\big)\|_{L^2} + \mathcal{O}(h^{1/2+\varepsilon}) \\
&\lesssim \frac{1}{h}\|s_h\|_{L^2} + \|\partial s_h\|_{L^2} + \mathcal{O}(h^{1/2+\varepsilon}) \\
&= \mathcal{O}(h^{-1/2+\varepsilon}),
\end{aligned}$$

where we have used (5.25) and (5.21) to conclude that $\|\partial s_h\|_{L^2} = \mathcal{O}(1)$.

5.2.3. *CGOs with antiholomorphic phase.* Next, we construct a CGO solution with an antiholomorphic phase $-\overline{\Phi}$, where $\Phi = \varphi + \mathrm{i}\psi$ is a holomorphic Morse function. Since the coefficients of the linear equation (5.5) are real, we obtain a CGO with an antiholomorphic phase by taking the complex conjugate of the CGO

$$F_A^{-1} e^{-\Phi/h}(a + r_h)$$

given by Lemma 5.1 for the phase $-\Phi$. This gives us a CGO of the form

$$\widetilde{v} = F_{\overline{A}} e^{-\overline{\Phi}/h}(1 + \widetilde{r}_h),$$

where $F_{\overline{A}} = e^{\mathrm{i}\overline{\alpha}}$ and by (5.1)

$$\widetilde{r}_h = -\partial_\psi^{-1} V' \sum_{j=0}^{\infty} \widetilde{T}_h^j \partial_\psi^{*-1}(Va).$$

Here ψ is the imaginary part of Φ as before and

$$\widetilde{T}_h = \partial_\psi^{*-1} V \partial_\psi^{-1} V'$$

with

$$\partial_\psi^{-1} := \mathcal{R}\partial^{-1} e^{-2\mathrm{i}\psi/h}\mathcal{E} \quad \text{and} \quad \partial_\psi^{*-1} := \mathcal{R}\partial^{*-1} e^{2\mathrm{i}\psi/h}\mathcal{E}.$$

(see also [CLT24]). Here $\partial^{*-1} = \overline{\partial}^{-1}$ in holomorphic coordinates. We also write

$$\widetilde{r}_h := -\partial_\psi^{-1} V' \widetilde{s}_h, \quad \widetilde{s}_h := \sum_{j=0}^{\infty} \widetilde{T}_h^j \partial_\psi^{*-1}(Va). \tag{5.27}$$

The remainder $\widetilde{r}_h$ enjoys the same estimates as r_h (corresponding to holomorphic Morse phase), hence, we do not repeat those estimates from the previous subsection. Note that $\widetilde{T}_h$ is not exactly the same as $\overline{T}_h$ since we also changed the sign of Φ. However, $\widetilde{T}_h$ satisfies the same estimates as T_h given by (5.23) and (5.24).

5.2.4. *Estimates and expansions for CGOs in the absence of critical points.* Observe that in the above construction, if Φ has no critical points, we may apply the better estimate (5.13) throughout the construction to get

$$\|r_h\|_p + \|s_h\|_p + \|dr_h\|_p \leq Ch,$$

for all $p \in (1, \infty)$ and for some constant $C > 0$. In fact, we have even the following asymptotic expansion formula for the correction terms associated with a phase function that does not have critical points.

Lemma 5.2. *Let r_h and $\tilde{r}_h$ be as above, and correspond to a holomorphic phase without critical points. Let also $N \in \mathbb{N}$ and $k + l \leq 2$, $p \geq 2$. Then, we can write*

$$r_h = hF_h + \mathcal{O}_{W^{2,p}}(h^N), \quad \tilde{r}_h = h\tilde{F}_h + \mathcal{O}_{W^{2,p}}(h^N), \tag{5.28}$$

where $F_h = F_h(x)$ and $\tilde{F}_h = \tilde{F}_h(x)$ are finite power series in h with C^∞ smooth coefficients depending only on x.

This lemma will be extremely useful when analyzing the integral identity of the second linearized Monge-Ampère equation. The lemma implies that correction terms of CGOs that have phases without critical points can be disregarded as lower order terms in the asymptotic analysis due to the term hF_f (or $h\widetilde{F}_h$).

Proof of Lemma 5.2. Let $N \in \mathbb{N}$. We have

$$T_h : W^{1,p} \to W^{1,p}$$

has an operator norm $\mathcal{O}(h^{1/p})$ for $p > 2$ and $\mathcal{O}(h^{1/2-\epsilon})$ for $p = 2$ by (5.24), where T_h is defined by (5.22). We also note that

$$\overline{\partial}_\psi^{-1} V' : W^{1,p} \to W^{2,p}$$

with

$$\left\|\overline{\partial}_\psi^{-1} V' f\right\|_{W^{2,p}} = \left\|\overline{\partial}^{-1}(e^{-2i\psi/h} V' f)\right\|_{W^{1,p}} \lesssim h^{-1}\|f\|_{W^{1,p}},$$

where we apply [Tzo17, Proposition 2.3] with $\overline{\partial}^{-1} : W^{1,p} \to W^{2,p}$. Recall also that $\overline{\partial}_\psi^{*-1} V$ is uniformly bounded from L^p to $W^{1,p}$, using the above facts, then there is $K = K_N \in \mathbb{N}$ such that

$$\begin{aligned}
r_h &= -\overline{\partial}_\psi^{-1} V' \sum_{k=0}^{\infty} T_h^k \overline{\partial}_\psi^{*-1}(Va) \\
&= -\overline{\partial}_\psi^{-1} V' \Big(\sum_{k=0}^{K} T_h^k \overline{\partial}_\psi^{*-1}(Va) + \mathcal{O}_{W^{1,p}}(h^N) \Big) \\
&= -\overline{\partial}_\psi^{-1} V' \sum_{k=0}^{K} T_h^k \overline{\partial}_\psi^{*-1}(Va) + h^{-1}\mathcal{O}_{W^{2,p}}(h^N).
\end{aligned}$$

Thus, it remains to analyze the finite sum above.

We expand using [CLLT23, Lemma 3.5] as

$$\overline{\partial}_\psi^{*-1}(Va) = e^{2\mathrm{i}\psi/h} V' \sum_{j=1}^{N+1} h^j F^j + h^{N+1} \partial_\psi^{-1}(\partial F^{N+1}),$$

where the functions $F^j \in C^\infty(M)$ are defined recursively by

$$F^1 = \frac{\mathrm{i}}{2} \frac{Va}{\partial \psi}, \quad F^{j+1} = \frac{\mathrm{i}}{2} \frac{1}{\partial \psi} \partial F^j.$$

Since ψ has no critical points, these functions are smooth. In the following, we denote by

$$\check{F}^j, \quad \check{F}^{jk}, \quad \text{etc.}$$

unspecified smooth functions in C^∞, which may vary from line to line.

For each $k \geq 0$, we have:

$$
\begin{aligned}
T_h^k\big(\overline{\partial}_\psi^{*-1}(Va)\big) &= \big(\overline{\partial}_\psi^{*-1} V \overline{\partial}_\psi^{-1} V'\big)^k \big(\overline{\partial}_\psi^{*-1}(Va)\big)\\
&= \big(\overline{\partial}_\psi^{*-1} V \overline{\partial}_\psi^{-1} V'\big)^k \Big(e^{2\mathrm{i}\psi/h}\sum_{j=1}^{N+1} h^j F^j + h^{N+1}\partial_\psi^{-1}(\partial F^{N+1})\Big)\\
&= \big(\overline{\partial}_\psi^{*-1} V \overline{\partial}_\psi^{-1} V'\big)^k \Big(e^{2\mathrm{i}\psi/h}\sum_{j=1}^{N+1} h^j F^j\Big) + h^{N+1}\mathcal{O}_{W^{1,p}}(1).
\end{aligned}
$$

Now

$$
\begin{aligned}
&\big(\overline{\partial}_\psi^{*-1} V \overline{\partial}_\psi^{-1} V'\big)^k \Big(e^{2\mathrm{i}\psi/h}\sum_{j=1}^{N+1} h^j \check{F}^j\Big)\\
&= \sum_{j=1}^{N+1} h^j \big(\overline{\partial}_\psi^{*-1} V \overline{\partial}_\psi^{-1} V'\big)^{k-1}\big(\overline{\partial}_\psi^{*-1} V \overline{\partial}_\psi^{-1} V'\big) e^{2\mathrm{i}\psi/h}\check{F}^j\\
&= \sum_{j=1}^{N+1} h^j \big(\overline{\partial}_\psi^{*-1} V \overline{\partial}_\psi^{-1} V'\big)^{k-1}\overline{\partial}_\psi^{*-1}\check{F}^j\\
&= \sum_{j=1}^{N+1} h^j \big(\overline{\partial}_\psi^{*-1} V \overline{\partial}_\psi^{-1} V'\big)^{k-1}\big(e^{2\mathrm{i}\psi/h}\sum_{l=1}^{N+1} h^l \check{F}^{jl} + h^{N+1}\partial_\psi^{-1}(\partial \check{F}^{jN+1})\big)\\
&= \sum_{j,l=1}^{N+1} h^j h^l \big(\overline{\partial}_\psi^{*-1} V \overline{\partial}_\psi^{-1} V'\big)^{k-1}\big(e^{2\mathrm{i}\psi/h}\check{F}^{jl}\big) + h^{N+1}\mathcal{O}_{W^{1,p}}(1)\\
&= \cdots = e^{2\mathrm{i}\psi/h}\sum_{j_1,\ldots,j_{k+1}=1}^{N+1} h^{j_1+\cdots+j_{k+1}}\check{F}^{j_1\cdots j_{k+1}} + h^{N+1}\mathcal{O}_{W^{1,p}}(1).
\end{aligned}
$$

Thus

$$
\begin{aligned}
r_h &= -\overline{\partial}_\psi^{-1} V' \sum_{k=0}^{K} T_h^k \overline{\partial}_\psi^{*-1}(Va) + h^{-1}\mathcal{O}_{W^{2,p}}(h^N)\\
&= -\sum_{k=0}^{K}\overline{\partial}_\psi^{-1} V'\Big(e^{2\mathrm{i}\psi/h}\sum_{j_1,\ldots,j_{k+1}=1}^{N+1} h^{j_1+\cdots+j_{k+1}}\check{F}^{j_1\cdots j_{k+1}} + h^{N+1}\mathcal{O}_{W^{1,p}}(1)\Big)\\
&\quad + h^{-1}\mathcal{O}_{W^{2,p}}(h^N)\\
&= \sum_{k=0}^{K}\sum_{j_1,\ldots,j_{k+1}=1}^{N+1} h^{j_1+\cdots+j_{k+1}}\check{F}^{j_1\cdots j_{k+1}} + h^N\mathcal{O}_{W^{2,p}}(1) + h^{-1}\mathcal{O}_{W^{2,p}}(h^N).
\end{aligned}
$$

The decrease of the pover of h in the middle term resulted from $\overline{\partial}_\psi^{-1} : W^{1,p} \to W^{2,p}$ with norm $\mathcal{O}(h^{-1})$. Redefining N as $N+1$ yields the first identity in (5.28). The proof of the second identity is similar. □

To conclude this subsection, and for the readers' convenience in the forthcoming analysis, we now summarize all the CGO solutions introduced above.

Lemma 5.3 (CGO solutions)**.**

(i) There exist CGO solutions with holomorphic phases

$$F_{A_1}^{-1} e^{\Phi_1/h}(1+r_1) \quad \textit{and} \quad F_{A^*}^{-1} e^{\Phi^*/h}(1+r_*)$$

to the equations (5.6) *and* (5.7)*, respectively, where* Φ_1 *and* Φ^* *are holomorphic functions without critical points. Here,* $F_{A_1}^{-1}$ *and* $F_{A^*}^{-1}$ *are smooth non-vanishing function independent of* $h > 0$*, and* r_1 *and* r_* *are remainders fulfilling* (5.28)*.*

(ii) There is a CGO solution with an antiholomorphic phase

$$F_{\overline{A_2}} e^{-\overline{\Phi}_2/h}(1 + \widetilde{r}_2)$$

to the equation (5.6)*, where* Φ_2 *is a holomorphic Morse function with critical points. Here,* $F_{\overline{A_2}}$ *is a smooth non-vanishing function independent of* $h > 0$*, and* $\widetilde{r}_2$ *is the remainder fulfilling* (5.27)*.*

In Section 7, we will carefully select these phase functions Φ_1, Φ_2, and Φ_* to recover an unknown conformal factor c uniquely.

6. Carleman estimate and unique continuation

In this section, A is a non-vanishing, possibly complex-valued function. We prove a unique continuation principle (UCP) for solutions of

$$\overline{\partial}(A\overline{\partial}\mathbf{c}(z) + \alpha(z)\mathbf{c}(z)) = \beta(z)\overline{\partial}^{-1}(\gamma(z)\mathbf{c}(z)) + H, \tag{6.1}$$

where $\overline{\partial}^{-1}$ is the standard Cauchy-Riemann integral operator and H is a holomorphic function. We state it as follows

Lemma 6.1 (Unique continuation property)**.** *Let* $U \subset \mathbb{R}^2$ *be a bounded connected open set with* C^∞*-boundary* ∂U*, and* $\mathbf{c}$ *a* C^2*-solution to* (6.1)*. Let* $A \in C^2(U)$ *be a non-vanishing function and* $\alpha, \beta, \gamma \in C^\infty(U)$*. Given a nonempty open subset* $W \subset U$*, then* $\mathbf{c} = 0$ *on* W *implies* $\mathbf{c} = 0$ *in* $\overline{U}$*.*

We prove the above lemma by applying a two-parameter Carleman estimate (see [GT11a, Lemma 3.2]):

$$\|e^{-\tau\phi_\epsilon} v\|^2_{L^2(U)} \le C\epsilon \|e^{-\tau\phi_\epsilon}\overline{\partial} v\|^2_{L^2(U)}, \tag{6.2}$$

for all $v \in C_c^\infty(U)$, where $C > 0$ is a constant independent of ϵ, τ, v, and

$$\phi_\epsilon(z) = \varphi(z) - \frac{1}{2\epsilon\tau}|z|^2, \tag{6.3}$$

with φ a harmonic function (such estimates are often referred to as Carleman estimates with convexified weights).

Let us choose

$$\varphi(z) = \log(|z|^2),$$

which is harmonic away from $z = 0$, blows up at the origin, and hence allows us to apply the Carleman estimate on an annulus. This yields the (weak) UCP for equation (6.1).

Note that $\tau\phi_\epsilon = \tau\varphi - \frac{1}{2\epsilon}|z|^2$, so the weight function involves two independent parameters, ϵ and τ. Concretely, the small parameter ϵ is first used to absorb lower-order terms, after which the large parameter τ is employed to establish the UCP. We want to emphasize that Lemma 6.1 holds only when H is a holomorphic function; otherwise, the result may not hold.

Proof of Lemma 6.1. Without loss of generality, we may assume that the equation (6.1) holds in a ball of radius R and that $\mathbf{c} = 0$ on a ball of radius $r < R$. We show that $\mathbf{c}$ then vanishes on the larger ball $B(0, r + \delta)$, for any $\delta \in (0, R - r)$, and this implies $u = 0$ in $B(0, R)$.

First, we use conjugation for the term $A\overline{\partial}\mathbf{c}(z) + \alpha(z)\mathbf{c}(z)$. For this, let θ solve

$$\overline{\partial}\theta = A^{-1}(\alpha - \overline{\partial}A).$$

Such θ exists due to the existence of $\overline{\partial}^{-1}$ operator (for example, see (5.9)). Consider the function $\widetilde{\mathbf{c}} := e^{\theta} A\mathbf{c}$, then $\widetilde{\mathbf{c}}$ satisfies

$$\overline{\partial}\widetilde{\mathbf{c}} = e^{\theta}\big(A\overline{\partial}\mathbf{c} + \overline{\partial}\theta A\mathbf{c} + \overline{\partial}A\mathbf{c}\big) = e^{\theta}\big(A\overline{\partial}\mathbf{c} + \alpha\mathbf{c}\big),$$

which is equivalent to

$$e^{-\theta}\overline{\partial}\widetilde{\mathbf{c}} = A\overline{\partial}\mathbf{c} + \alpha\mathbf{c}.$$

Using this, we can transform the equation (6.1) into

$$\overline{\partial}\big(e^{-\theta}\overline{\partial}\widetilde{\mathbf{c}}\big) = \beta\overline{\partial}^{-1}(\gamma A^{-1}e^{-\theta}\widetilde{\mathbf{c}}) + H,$$

and we aim to prove unique continuation for the above equation. To simplify the notation, we set

$$u = \widetilde{\mathbf{c}},$$

and redefine γ as $\gamma A^{-1}e^{-\theta}$ in the rest of the proof.

With these notations, we prove UCP for the equation of the form

$$\overline{\partial}(e^{-\theta}\overline{\partial}u) = \beta\overline{\partial}^{-1}(\gamma u) + H. \tag{6.4}$$

Let us then start to estimate. Let $\chi \in C_c^{\infty}(B(0,R))$ and recall that u vanishes on a ball of radius r. Thus χu is supported on an annulus

$$A = B(0,R)\setminus B(0,r)$$

(since $u = 0$ in $B(0,r)$ by assumption), and the Carleman estimate (6.2) holds for the domain A. We will choose χ more precisely later. By using the Carleman estimate consecutively, we have

$$\begin{aligned}
\epsilon^{-1}\|e^{-\tau\phi_\epsilon}\chi u\|^2_{L^2(\mathbb{C})} &= \epsilon^{-1}\|e^{-\tau\phi_\epsilon}\chi u\|^2_{L^2(A)} \\
&\lesssim \underbrace{\|e^{-\tau\phi_\epsilon}\overline{\partial}(\chi u)\|^2_{L^2(A)}}_{\text{By (6.2)}} \\
&\lesssim \|e^{-\tau\phi_\epsilon}(\overline{\partial}\chi)u)\|^2_{L^2(A)} + \|e^{-\tau\phi_\epsilon}\chi\overline{\partial}u)\|^2_{L^2(A)} \\
&= \|e^{-\tau\phi_\epsilon}(\overline{\partial}\chi)u)\|^2_{L^2(A)} + \|e^{-\tau\phi_\epsilon}\chi e^{\theta}(e^{-\theta}\overline{\partial}u)\|^2_{L^2(A)} \\
&\lesssim \|e^{-\tau\phi_\epsilon}(\overline{\partial}\chi)u)\|^2_{L^2(A)} + \|e^{-\tau\phi_\epsilon}(\chi e^{-\theta}\overline{\partial}u)\|^2 \\
&\lesssim \|e^{-\tau\phi_\epsilon}(\overline{\partial}\chi)u)\|^2_{L^2(A)} + \underbrace{\epsilon\|e^{-\tau\phi_\epsilon}\overline{\partial}(\chi e^{-\theta}\overline{\partial}u))\|^2_{L^2(A)}}_{\text{By (6.2)}} \\
&\lesssim \|e^{-\tau\phi_\epsilon}(\overline{\partial}\chi)u\|^2_{L^2(A)} + \epsilon\|e^{-\tau\phi_\epsilon}(\overline{\partial}\chi)e^{-\theta}\overline{\partial}u\|^2_{L^2(A)} \\
&\quad + \epsilon\|e^{-\tau\phi_\epsilon}\chi\overline{\partial}(e^{-\theta}\overline{\partial}u)\|^2_{L^2(A)}.
\end{aligned} \tag{6.5}$$

Here $\lesssim$ refers to an inequality with unspecified constants independent of τ and ϵ.

Next, let us insert the equation (6.4) to the last term in the right-hand side of (6.5). Then we have

$$\begin{aligned}
&\|e^{-\tau\phi_\epsilon}(\overline{\partial}\chi)u\|^2_{L^2(A)} + \epsilon\|e^{-\tau\phi_\epsilon}(\overline{\partial}\chi)e^{-\theta}\overline{\partial}u\|^2_{L^2(A)} + \epsilon\|e^{-\tau\phi_\epsilon}\chi(\beta\overline{\partial}^{-1}(\gamma u) + H)\|^2_{L^2(A)} \\
\lesssim\ &\|e^{-\tau\phi_\epsilon}(\overline{\partial}\chi)u\|^2_{L^2(A)} + \epsilon\|e^{-\tau\phi_\epsilon}(\overline{\partial}\chi)\overline{\partial}u\|^2_{L^2(A)} + \epsilon\|e^{-\tau\phi_\epsilon}\chi(\overline{\partial}^{-1}(\gamma u) + H)\|^2_{L^2(A)} \\
\lesssim\ &\|e^{-\tau\phi_\epsilon}(\overline{\partial}\chi)u\|^2_{L^2(A)} + \epsilon\|e^{-\tau\phi_\epsilon}(\overline{\partial}\chi)\overline{\partial}u\|^2_{L^2(A)} + \underbrace{\epsilon^2\|e^{-\tau\phi_\epsilon}\overline{\partial}(\chi(\overline{\partial}^{-1}(\gamma u) + H))\|^2_{L^2(A)}}_{\text{By (6.2)}} \\
\lesssim\ &\|e^{-\tau\phi_\epsilon}(\overline{\partial}\chi)u\|^2_{L^2(\mathbb{C})} + \epsilon\|e^{-\tau\phi_\epsilon}(\overline{\partial}\chi)\overline{\partial}u\|^2_{L^2(\mathbb{C})} + \epsilon^2\|e^{-\tau\phi_\epsilon}(\overline{\partial}\chi)(\overline{\partial}^{-1}(\gamma u) + H)\|^2_{L^2(\mathbb{C})} \\
&+ \epsilon^2\|e^{-\tau\phi_\epsilon}\chi u\|^2_{L^2(\mathbb{C})},
\end{aligned} \tag{6.6}$$

where we crucially used the fact that $\overline{\partial} H = 0$ since H is holomorphic. We will denote $\|\cdot\|_{L^2(\mathbb{C})}$ by $\|\cdot\|_{L^2}$ from now on. In short, the above estimates (6.5) and (6.6) mean that there is $C > 0$ independent of τ and ϵ, such that

$$
\begin{aligned}
\epsilon^{-1}\|e^{\tau\phi_\epsilon}\chi u\|_{L^2}^2 \leq C\big(&\|e^{-\tau\phi_\epsilon}(\overline{\partial}\chi)u\|_{L^2}^2 + \epsilon\|e^{-\tau\phi_\epsilon}(\overline{\partial}\chi)\overline{\partial}u\|^2 \\
&+ \epsilon^2\|e^{-\tau\phi_\epsilon}(\overline{\partial}\chi)(\overline{\partial}^{-1}(\gamma u) + H)\|_{L^2}^2 + \epsilon^2\|e^{-\tau\phi_\epsilon}\chi u\|_{L^2}^2\big).
\end{aligned} \tag{6.7}
$$

We first absorb the last term on the right-hand side of (6.7), assuming that $\epsilon > 0$ is so small that

$$
C\epsilon^2 \leq \frac{1}{2}\epsilon^{-1} \iff \epsilon \leq (2C)^{-1/3}.
$$

With such values of $\epsilon \in (0, (2C)^{-1/3})$, we have

$$
\begin{aligned}
\epsilon^{-1}\|e^{-\tau\phi_\epsilon}\chi u\|_{L^2}^2 \leq 2C\big(&\|e^{-\tau\phi_\epsilon}(\overline{\partial}\chi)u\|_{L^2}^2 + \epsilon\|e^{-\tau\phi_\epsilon}(\overline{\partial}\chi)e^{-\theta}\overline{\partial}u\|_{L^2}^2 \\
&+ \epsilon^2\|e^{-\tau\phi_\epsilon}(\overline{\partial}\chi)(\overline{\partial}^{-1}(\gamma u) + H)\|_{L^2}^2\big).
\end{aligned} \tag{6.8}
$$

In the forthcoming analysis, we will not change ϵ anymore and it is fixed.

Next, we choose the Carleman weight ϕ_ϵ and the cutoff function χ appropriately, and argue by contradiction. For the harmonic function φ, we take $\varphi = \log(|z|^2)$. Then

$$
e^{-\tau\phi_\epsilon} = |z|^{-2\tau}e^{-\frac{1}{2\epsilon}|z|^2},
$$

where ϕ_ϵ is given by (6.3). Since ϵ is fixed from this point onward, the exponential factor $e^{-\frac{1}{2\epsilon}|z|^2}$ can be regarded as a bounded weight, both above and below, in the forthcoming estimates. On the other hand, the term

$$
|z|^{-2\tau}
$$

decays rapidly as $\tau \to \infty$, because $|z| > 0$ for all $z \neq 0$.

Let $\delta > 0$. We choose the cutoff function χ such that

$$
\chi(z) := \begin{cases} 1 & \text{if } |z| \leq r + \delta, \\ 0 & \text{if } |z| \geq R. \end{cases}
$$

It follows that $\overline{\partial}\chi$ is supported on the annulus $r + \delta \leq |z| \leq R$. With these choices, the right-hand side of (6.8) is bounded by

$$
\begin{aligned}
&2C\|\overline{\partial}\chi\|_{L^\infty}\Big[\|e^{-\tau\phi_\epsilon}u\|_{L^2(B(0,R)\setminus B(0,r+\delta))}^2 + \epsilon\|e^{-\tau\phi_\epsilon}\overline{\partial}u\|_{L^2(B(0,R)\setminus B(0,r+\delta))}^2 \\
&\qquad + \epsilon^2\|e^{-\tau\phi_\epsilon}(\overline{\partial}^{-1}(\gamma u) + H)\|_{L^2(B(0,R)\setminus B(0,r+\delta))}^2\Big] \\
&\leq C'\|e^{-\tau\phi_\epsilon}\|_{L^2(B(0,R)\setminus B(0,r+\delta))}^2,
\end{aligned}
$$

since u is C^1 and $\overline{\partial}^{-1} : L^\infty \to L^\infty$ is bounded. In particular, we have $\|\overline{\partial}^{-1}(\gamma u)\|_{L^\infty} \lesssim \|u\|_{L^\infty}$, which follows directly from the definition of $\overline{\partial}^{-1}$.

Since $|z|^{-2\tau}$ is decreasing in $|z|$, we obtain

$$
\|e^{-\tau\phi_\epsilon}\|_{L^2(B(0,R)\setminus B(0,r+\delta))}^2 = \int_{B(0,R)\setminus B(0,r+\delta)} e^{-2\tau\phi_\epsilon}\, dz \lesssim |r+\delta|^{-2\tau}.
$$

Thus, the right-hand side of (6.8) is bounded by

$$
C''|r+\delta|^{-2\tau},
$$

for some constant C'' independent of τ and ϵ (note that C'' may depend on u and H, but this will not affect the argument).

On the other hand, since $\chi \equiv 1$ on $B(0, r+\delta)$, we have

$$
\|e^{-\tau\phi_\epsilon}u\|_{L^2(B(0,r+\delta))}^2 \;\leq\; \|e^{-\tau\phi_\epsilon}\chi u\|_{L^2(\mathbb{C})}^2.
$$

Consequently,

$$\int_{B(0,r+\delta)} e^{-2\tau\phi_\epsilon}|u|^2\,dz \;\le\; \epsilon C''|r+\delta|^{-2\tau}.$$

Recalling that $e^{-\tau\phi_\epsilon} = |z|^{-2\tau}\, e^{-\frac{1}{2\epsilon}|z|^2}$, this inequality becomes

$$\int_{B(0,r+\delta)} |z|^{-2\tau}\,|u|^2\,dz \;\le\; \epsilon C'''|r+\delta|^{-2\tau},$$

or equivalently,

$$\int_{B(0,r+\delta)} \left(\frac{|z|}{|r+\delta|}\right)^{-2\tau} |u|^2\,dz \;\le\; C''', \tag{6.9}$$

for some constant $C''' > 0$.

Assume then that there is $z_0 \in B(0,r+\delta)$ such that $|u(z_0)| \neq 0$. Thus, by continuity of u there is a neighborhood $\mathcal{N} \subset B(0,r+\delta)$ of z_0 such that

$$|u(z)| \ge \sigma \text{ for } z \in \mathcal{N},$$

for some $\sigma > 0$, where

$$|z| < r+\delta-s, \text{ for all } z \in \mathcal{N}, \tag{6.10}$$

for some $s > 0$. Thus, using (6.9) we have

$$\sigma^2 \int_{\mathcal{N}} \left(\frac{|z|}{r+\delta}\right)^{-2\tau} dz \le \int_{B(0,r+\delta)} \left(\frac{|z|}{r+\delta}\right)^{-2\tau} |u|^2\,dz \le C''' \tag{6.11}$$

Note that for $z \in \mathcal{N}$, we have the condition (6.10), so that

$$\frac{|z|}{r+\delta} \le \frac{r+\delta-s}{r+\delta} = 1 - \frac{s}{r+\delta}$$

which is strictly less than 1. Thus, on the open set $\mathcal{N}$, there holds

$$\left(\frac{|z|}{r+\delta}\right)^{-2\tau} \to \infty \text{ as } \tau \to \infty.$$

As a result, using this to (6.11) leads to a contradiction as the left-hand side blows up with $\tau \to \infty$. We conclude that $u \equiv 0$ on $B(0,R)$, which is larger ball than $B(0,r)$, and u was assumed to be zero in $B(0,r)$. Finally, due to the standard propagation of smallness argument in UCP, one can conclude that $u \equiv 0$ in U, by using $u = 0$ in $B(0,r) \subset U$. This concludes the proof. $\square$

7. Unique determination of the conformal factor

In the previous section, we determined the 2×2 matrix $D^2 u_0$ up to a conformal factor $c > 0$ with $c|_{\partial\Omega} = 1$. We now turn to the problem of recovering this conformal factor inside the domain Ω. To this end, we employ the second linearized equation together with its associated integral identity. The analysis will be somewhat involved due to two reasons: (1) The full nonlinearity leads to complicated asymptotic analysis, (2) The second linearized equation is not coordinate invariant. These complications seem to be unavoidable and lead to a non-local $\overline{\partial}$-equation.

Let $u_0^{(j)} \in C^{4,\alpha}(\overline{\Omega})$ denote the solution to (3.1) for $j = 1,2$. The second linearized Monge–Ampère equation then takes the form

$$\begin{cases} \big(-\Delta_{g_j} + X_{g_j}\cdot\nabla\big) w_j = \operatorname{tr}\big(g_j^{-1}(D^2 v_j^{(1)}) g_j^{-1}(D^2 v_j^{(2)})\big) & \text{in } \Omega, \\ w_j = 0 & \text{on } \partial\Omega, \end{cases} \tag{7.1}$$

where g_j and X_{g_j} are given by (2.22) and (2.25), respectively, with $g = g_j$ and $j = 1,2$. As discussed in Proposition 2.1, the problem (7.1) is well-posed.

Lemma 7.1. *Under the assumptions of Theorem 1.2, the DN map Λ_F associated with (1.3) determines the Neumann derivative $\partial_{\nu_g} w\big|_{\partial\Omega}$. In particular, the condition (1.9) implies $\partial_{\nu_{g_1}} w_1 = \partial_{\nu_{g_2}} w_2$ on $\partial\Omega$.*

Proof. The argument is analogous to Lemma 4.1, but applied to the second linearization of solutions to (1.3). Combining Lemmas 3.1 and 4.1, and differentiating twice with respect to the small parameter ϵ, yields the desired result. ☐

7.1. **The second integral identity.** We now turn to the derivation of the integral identity arising from the second linearization, which will be the key to proving our main result. In particular, we first extract from the first linearization an identity that enables the recovery of the metric g. To this end, we introduce the adjoint problem associated with the first linearized equation (2.23):

$$\begin{cases} \Delta_g v^* + \dfrac{1}{\sqrt{|g|}} \partial_b\big(\sqrt{|g|}\, X_g^b\, v^*\big) = 0 & \text{in } \Omega, \\ v^* = \varphi^* & \text{on } \partial\Omega, \end{cases} \tag{7.2}$$

where $\varphi^* \in C^\infty(\partial\Omega)$ is an arbitrary boundary function. The vector field X_g here is the drift term appearing in the non-divergence-to-divergence recasting of the first linearized equation (see Section 2). Notice that $\sqrt{|g|}\Delta_g = \nabla\cdot\big(\sqrt{|g|}g^{-1}\nabla\big)$, and let w be the solution to the second linearized equation (2.28), then an integration by parts implies

$$\begin{aligned}
&\int_{\partial\Omega} \sqrt{|g|}\varphi^* \partial_{\nu_g} w\, dS \\
&= \int_\Omega v^* \sqrt{|g|}\Delta_g w\, dx + \int_\Omega \sqrt{|g|} g^{-1}\nabla v^* \cdot \nabla w\, dx \\
&= \int_\Omega v^* \sqrt{|g|}\Delta_g w\, dx + \int_{\partial\Omega} \sqrt{|g|}\partial_{\nu_g} v^* w\, dS - \int_\Omega w \sqrt{|g|}\Delta_g v^*\, dx \\
&= \underbrace{\int_\Omega \sqrt{|g|}v^*\big(X_g^b\partial_b w - \mathrm{tr}\,\big(g^{-1}\big(D^2 v^{(1)}\big)g^{-1}\big(D^2 v^{(2)}\big)\big)\big)\, dx}_{\text{By (2.29)}} + \underbrace{\int_\Omega w\partial_b\big(\sqrt{|g|}X_g^b v^*\big)\, dx}_{\text{By (7.2)}} \\
&= \int_\Omega \sqrt{|g|}v^*\big(X_g^b\partial_b w - \mathrm{tr}\,\big(g^{-1}\big(D^2 v^{(1)}\big)g^{-1}\big(D^2 v^{(2)}\big)\big)\big)\, dx - \int_\Omega v^* \sqrt{|g|}X_g^b \partial_b w\, dx \\
&= -\int_\Omega v^*\, \mathrm{tr}\,\big(g^{-1}\big(D^2 v^{(1)}\big)g^{-1}\big(D^2 v^{(2)}\big)\big)\, dV_g,
\end{aligned}$$

where we used $\int_{\partial\Omega} wv^*\sqrt{|g|}X^b\nu_b\, dS = 0$ (since $w|_{\partial\Omega} = 0$). Here $\nu = (\nu_1, \nu_2)$ denotes the unit outer normal to $\partial\Omega$, and $\partial_{\nu_g} w\big|_{\partial\Omega} = \sqrt{|g|}\, g^{ik}\partial_i w\, \nu_k\big|_{\partial\Omega}$ is the conormal derivative. We employ the standard notation from (5.19), which will be used throughout the rest of the work. Finally, recall that v denotes the solution of the first linearized equation (2.23). Combining all of the above computations, we arrive at the following result.

Lemma 7.2 (Integral identity for the second linearization)**.** *The following integral identities hold:*

(i) *Let Λ_F be the DN map of (1.3), then there holds*

$$\int_{\partial\Omega} \sqrt{|g|}\varphi^* \partial_{\nu_g} w\, dS = -\int_\Omega v^*\, \mathrm{tr}\,\big(g^{-1}\big(D^2 v^{(1)}\big)g^{-1}\big(D^2 v^{(2)}\big)\big)\, dV_g,$$

where g is given by (2.22), $v^{(k)}$ is the solution to the first linearized equation (2.23) and w is the solution to (2.28), for $k = 1, 2$.

(ii) Let Λ_{F_j} be the DN map of (1.8) for $j = 1, 2$, and suppose the condition (1.9) holds. Then

$$\begin{aligned}\int_\Omega v_1^* \operatorname{tr}\left(g_1^{-1}(D^2 v_1^{(1)}) g_1^{-1}(D^2 v_1^{(2)})\right) dV_{g_1} \\ - \int_\Omega v_2^* \operatorname{tr}\left(g_2^{-1}(D^2 v_2^{(1)}) g_2^{-1}(D^2 v_2^{(2)})\right) dV_{g_2} = 0,\end{aligned} \tag{7.3}$$

where v_j^* is the solution to the adjoint problem (7.2) with respect to the first linearized equation, for $j = 1, 2$.

Proof. We already proved (i) by the previous computations. Combining Lemma 7.1, one can prove the integral identity (ii) directly. $\square$

Remark 7.3. *To recover the conformal factor c appearing in Lemma 4.2 from the integral identity (7.3), we analyze it using global isothermal coordinates. A complication arises because the Hessian D^2 in the identity is not the invariant Hessian for either metric g_1 or g_2. Consequently, the change to isothermal coordinates introduces additional coordinate artifact terms, which we collectively denote by Y (see (7.6)). These terms Y will ultimately lead to a non-local $\bar\partial$ equation for the conformal factor c, which we then solve.*

7.2. **Change of variables for the Hessian.** Recalling that $D^2 u$ denotes the Hessian matrix of u, let $\chi : \mathbb{R}^2 \to \mathbb{R}^2$ denote a change of coordinates (can be arbitrary), then we have

$$D^2 \widetilde{u} = \nabla\chi^T D^2 u\big|_\chi \nabla\chi + \sum_{k=1}^{2} D^2\chi^k \cdot \partial_k u\big|_\chi,$$

where $\widetilde{u} = u \circ \chi$, and $\chi(x) = \left(\chi^1(x_1, x_2), \chi^2(x_1, x_2)\right)$ denotes the change of variables in the plane. This follows from

$$\begin{aligned}\left(D^2\widetilde{u}\right)_{ab} &= \partial_{ab}(u\circ\chi) = \partial_a\left(\partial_k u|_\chi \partial_b\chi^k\right) \\ &= \partial_{mk} u|_\chi \partial_a\chi^m \partial_b\chi^k + \partial_k u|_\chi \partial_{ab}\chi^k \\ &= (D\chi^T)_a^k (D^2 u)_{km}|_\chi D\chi_b^m + \partial_k u|_\chi \partial_{ab}\chi^k,\end{aligned}$$

for $1 \le a, b \le 2$, and we denote

$$\partial_k u|_\chi \partial_{ab}\chi^k = D^2\chi \cdot \nabla u|_\chi,$$

where we still adopt the Einstein summation convention for repeated indices.

Let us also denote $D^2\chi$ as a three tensor by $(D^2\chi)_{ab}^k$, which is symmetric in the lower indices, i.e., $(D^2\chi)_{ab}^k = (D^2\chi)_{ba}^k$ for all $1 \le a, b, k \le 2$. Recalling that the change of variables for a Riemannian metric g is given by (4.3). Note that the preceding computations hold not only in dimension two but also in any dimension. Thus, we have

$$\begin{aligned}&\operatorname{tr}\left(\widetilde{g}^{-1}(D^2\widetilde{v}^{(1)})\widetilde{g}^{-1}(D^2\widetilde{v}^{(2)})\right) \\ &= \operatorname{tr}\left(g^{-1}(D^2 v^{(1)}) g^{-1}(D^2 v^{(2)})\right)\big|_\chi \\ &\quad + \operatorname{tr}\left(g^{-1}|_\chi (D^2 v^{(1)}) g^{-1}|_\chi (D^2\chi\cdot\nabla v^{(2)})\right) \\ &\quad + \operatorname{tr}\left(g^{-1}|_\chi (D^2\chi\cdot\nabla v^{(1)}) g^{-1}|_\chi (D^2 v^{(2)})\right)\big|_\chi \\ &\quad + \operatorname{tr}\left(g^{-1}|_\chi (D^2\chi\cdot\nabla v^{(1)}|_\chi) g^{-1}|_\chi (D^2\chi\cdot\nabla v^{(2)}|_\chi)\right),\end{aligned} \tag{7.4}$$

where $\widetilde{g} = \chi^* g$ and $\widetilde{v}^{(k)} = v^{(k)} \circ \chi$, for $k = 1, 2$. Let us emphasize again that the mapping χ can be any change of variables at the moment.

Applying the isothermal coordinates, we can transform the Laplace–Beltrami operator into the standard (isotropic) Laplacian, up to a conformal factor. This change of variables is central to the argument that follows. On the one hand in the isothermal coordinates χ, using (7.4), one has

$$\begin{aligned}
&\operatorname{tr}\left(g_1^{-1}D^2v_1^{(1)}g_1^{-1}D^2v_1^{(2)}\right)\\
=&\,\mu^{-2}\operatorname{tr}\left(\left(D^2\mathbf{v}_1^{(1)}\right)\left(D^2\mathbf{v}_1^{(2)}\right)\right)+\mu^{-2}\operatorname{tr}\left(D^2\mathbf{v}_1^{(1)}C\cdot\nabla\mathbf{v}_1^{(2)}\right)\\
&+\mu^{-2}\operatorname{tr}\left(C\cdot\nabla\mathbf{v}_1^{(1)}D^2\mathbf{v}_1^{(2)}\right)+\mu^{-2}\operatorname{tr}\left(C\cdot\nabla\mathbf{v}_1^{(1)}C\cdot\nabla\mathbf{v}_1^{(2)}\right),
\end{aligned}$$

where $C=\left(C^k_{ab}\right)_{1\leq a,b,k\leq 2}$ is some function with 3 indices and depends on the change of variables to the isothermal coordinates.

On the other hand, by Lemma 4.2, one known that the mapping J changes from quantities with index 2 to the index 1 with $v_j^{(1)}=v_j^{(2)}\circ J$, for $j=1,2$. Consider the map $\widetilde{\chi}:=J\circ\chi$, using the same isothermal coordinates mentioned in the previous section, then we can obtain

$$\begin{aligned}
&\operatorname{tr}\left(g_2^{-1}D^2v_2^{(1)}g_2^{-1}D^2v_2^{(2)}\right)\\
=&\,c^{-2}\mu^{-2}\operatorname{tr}\left(\left(D^2\mathbf{v}_1^{(1)}\right)\left(D^2\mathbf{v}_1^{(2)}\right)\right)+c^{-2}\mu^{-2}\operatorname{tr}\left(\left(D^2\mathbf{v}_1^{(1)}\right)\widetilde{C}\cdot\nabla\mathbf{v}_1^{(2)}\right)\\
&+c^{-2}\mu^{-2}\operatorname{tr}\left(\widetilde{C}\cdot\nabla\mathbf{v}_1^{(1)}\left(D^2\mathbf{v}_1^{(2)}\right)\right)+c^{-2}\mu^{-2}\operatorname{tr}\left(\widetilde{C}\cdot\nabla\mathbf{v}_1^{(1)}\widetilde{C}\cdot\nabla\mathbf{v}_1^{(2)}\right),
\end{aligned}$$

where $\widetilde{C}$ is some function with 3 indices $\widetilde{C}^c_{ab}$, and $\widetilde{C}$ is actually depends on the function C and J. Meanwhile, the function C and $\widetilde{C}$ have the same value on the boundary $\partial\widetilde{\Omega}$, since we have utilized only one quasi-conformal mapping χ, and $J|_{\partial\Omega}=\mathrm{Id}$.

Now, adopting all notations introduced in Section 5.1, plugging all the above changes of variables of Hessian into (7.3), we can obtain

$$\begin{aligned}
0=&\int_\Omega v_1^*\operatorname{tr}\left(g_1^{-1}\left(D^2v_1^{(1)}\right)g_1^{-1}\left(D^2v_1^{(2)}\right)\right)dV_{g_1}\\
&-\int_\Omega v_2^*\operatorname{tr}\left(g_2^{-1}\left(D^2v_2^{(1)}\right)g_2^{-1}\left(D^2v_2^{(2)}\right)\right)dV_{g_2}\\
=&\int_\Omega v_1^*\operatorname{tr}\left(g_1^{-1}\left(D^2v_1^{(1)}\right)g_1^{-1}\left(D^2v_1^{(2)}\right)\right)\sqrt{|g_1|}\,dx\\
&-\int_\Omega v_2^*\operatorname{tr}\left(g_2^{-1}\left(D^2v_2^{(1)}\right)g_2^{-1}\left(D^2v_2^{(2)}\right)\right)\sqrt{|g_2|}\,dx\\
=&\int_{\widetilde{\Omega}}\left[G\mathbf{v}^*\operatorname{tr}\left(\left(D^2\mathbf{v}_1^{(1)}\right)\left(D^2\mathbf{v}_1^{(2)}\right)\right)+Y\right]dx,
\end{aligned}$$

where

$$G=\mu^{-1}\left(1-c^{-2}\right)\quad\text{in}\quad\widetilde{\Omega},\tag{7.5}$$

$\sqrt{|g_1|}=\mu$ (after the change of variable), and $\mathbf{v}^*$ is given by (5.4). Here, Y denotes the lower order terms with

$$\begin{aligned}
\mu Y=&\,\mathbf{v}^*\operatorname{tr}\left(D^2\mathbf{v}_1^{(1)}C\cdot\nabla\mathbf{v}_1^{(2)}\right)-c^{-2}\mathbf{v}^*\operatorname{tr}\left(\left(D^2\mathbf{v}_1^{(1)}\right)\widetilde{C}\cdot\nabla\mathbf{v}_1^{(2)}\right)\\
&+\mathbf{v}^*\operatorname{tr}\left(C\cdot\nabla\mathbf{v}_1^{(1)}D^2\mathbf{v}_1^{(2)}\right)-c^{-2}\mathbf{v}^*\operatorname{tr}\left(\widetilde{C}\cdot\nabla\mathbf{v}_1^{(1)}\left(D^2\mathbf{v}_1^{(2)}\right)\right)\\
&+\mathbf{v}^*\operatorname{tr}\left(C\cdot\nabla\mathbf{v}_1^{(1)}C\cdot\nabla\mathbf{v}_1^{(2)}\right)-c^{-2}\mathbf{v}^*\operatorname{tr}\left(\widetilde{C}\cdot\nabla\mathbf{v}_1^{(1)}\widetilde{C}\cdot\nabla\mathbf{v}_1^{(2)}\right)\\
=&\,\mathbf{v}^*\operatorname{tr}\left(D^2\mathbf{v}_1^{(1)}\mathcal{C}\cdot\nabla\mathbf{v}_1^{(2)}\right)+\mathbf{v}^*\operatorname{tr}\left(D^2\mathbf{v}_1^{(2)}\mathcal{C}\cdot\nabla\mathbf{v}_1^{(1)}\right)\\
&+(1-c^{-2})\mathbf{v}^*\operatorname{tr}\left(C\cdot\nabla\mathbf{v}_1^{(1)}C\cdot\nabla\mathbf{v}_1^{(2)}\right),
\end{aligned}\tag{7.6}$$

where we use the notation

$$\mathcal{C}:=C-c^{-2}\widetilde{C}=(1-c^{-2})C,\tag{7.7}$$

for the three tensor function in the rest of the paper, where we use Lemma 4.4 to conclude $\widetilde{C} = C$ as $J = \mathrm{Id}$ in $\overline{\Omega}$.

Now, we have already transformed the metric from g_2 to g_1 and then to the isothermal coordinates. In what follows, we will work on the g_1-domain, and let us denote $g \equiv g_1$ and $\mathbf{v}^{(k)} \equiv \mathbf{v}_1^{(k)}$ for $k = 1, 2$ to simplify our notations. With the above analysis at hand, we can have the next key result, which is used to prove the uniqueness of the conformal factor c in Ω.

Theorem 7.4. *Assume that*

$$\int_\Omega \left[G\mathbf{v}^* \operatorname{tr}\left(\left(D^2\mathbf{v}^{(1)}\right)\left(D^2\mathbf{v}^{(2)}\right)\right) + Y\right] dx = 0, \tag{7.8}$$

for any $\mathbf{v}^{(1)}, \mathbf{v}^{(2)}$ solving (5.2), and $\mathbf{v}^*$ solving (5.3), where Y is given by (7.6). Let G be the function given by (7.5), then $c = 1$ in Ω.

Remark 7.5. *Thanks to the boundary determination of Lemma 3.1, utilizing $c|_{\partial\Omega} = 1$ and $\partial_\nu c|_{\partial\Omega} = 0$, we can rewrite the integral identity (7.8) as*

$$\int_U \left[G\mathbf{v}^* \operatorname{tr}\left(\left(D^2\mathbf{v}^{(1)}\right)\left(D^2\mathbf{v}^{(2)}\right)\right) + Y\right] dx = 0, \tag{7.9}$$

where Ω is compactly contained in U since we have $G = Y = 0$ in $U \setminus \Omega$. Thus, we are going to use the integral identity (7.9) to claim $c = 1$ in U so that $c = 1$ in Ω in the rest of the paper.

Since $\mathbf{v}^{(1)}$, $\mathbf{v}^{(2)}$, and $\mathbf{v}^*$ can be taken as arbitrary solutions to (5.2) and (5.4), respectively, we will employ isothermal coordinates and the associated CGO solutions summarized in Lemma 5.3 for the first linearized equations to prove Theorem 7.4.

7.3. **Asymptotic analysis for the second integral identity.** To prove Theorem 7.4, let us review the known stationary phase formula.

• **Stationary phase formula.** For any $\varphi \in C_0^\infty(\mathbb{R}^2)$, we have the asymptotic expansion of the oscillatory integral

$$\begin{aligned} \frac{1}{2\pi h}\int_{\mathbb{R}^2} e^{\mathrm{i}x_1x_2/h}\varphi(x)\,dx &= \sum_{k=0}^{N-1}\frac{h^k}{k!}\left(\left(\frac{1}{\mathrm{i}}\partial_{x_1}\partial_{x_2}\right)^k\varphi\right)(0,0) + R_N(\varphi;h) \\ &= \sum_{k=0}^{N-1}\frac{h^k}{k!}\left(\left((\overline{\partial}^2 - \partial^2)\right)^k\varphi\right)(0,0) + R_N(\varphi;h), \end{aligned} \tag{7.10}$$

for $N \in \mathbb{N}$ and $h > 0$. Here, $R_N(\varphi;h)$ denotes the error term of the expansion that can be estimated by

$$|R_N(\varphi;h)| \le \frac{Ch^N}{N!}\sum_{\alpha_1+\alpha_2\le N}\left\|\partial_{x_1}^{\alpha_1}\partial_{x_2}^{\alpha_2}\left(\partial_{x_1}\partial_{x_2}\right)^N\varphi\right\|_{L^1(\mathbb{R}^2)},$$

for any $N \in \mathbb{N}$, and $\alpha = (\alpha_1, \alpha_2) \in (\mathbb{N} \cup \{0\})^2$ denotes the multi-indices. Now, we can prove Theorem 7.4.

Proof of Theorem 7.4. The proof relies on the asymptotic behavior of CGO solutions for the first linearized equation. Using the stationary phase method, we extract the principal contributions. Unlike semilinear or quasilinear cases, the full nonlinearity of the Monge–Ampère equation introduces two derivatives of the CGOs in the integral identity, making the asymptotic analysis substantially more delicate and dependent on the estimates from Section 5 and Section 6.

This asymptotic analysis leads to a second-order differential equation with a lower-order nonlocal perturbation

$$\overline{\partial}(A\overline{\partial}\mathbf{c}(z)+\alpha(z)\mathbf{c}(z))=\beta(z)\overline{\partial}^{-1}(\gamma(z)\mathbf{c}(z))+H(z) \quad \text{in } U,$$

for $\mathbf{c}=1-c^{-2}$ (see (7.39)), where the coefficients in the above equation can be explicitly determined (see the last step of the proof). In particular, the leading coefficient A is non-vanishing, and the function H is holomorphic. Therefore, applying UCP of Lemma 6.1, we can deduce $\mathbf{c}\equiv 0$ in U, hence $c=1$ in U. Here U is an open set fulfilling the property $\Omega \Subset U$ that is given in Remark 7.5. Moreover, since G and Y (defined in (7.5) and (7.6)) vanish on $\partial\Omega$ up to higher orders, integration by parts can be performed (at least twice) without any boundary contributions. Meanwhile, we also set $c=1$ in the exterior domain $\mathbb{R}^n\setminus\Omega$, so that $c\in C^2(\mathbb{R}^2)$. The proof is organized into eight steps.

Step 0. Initialization.

Let us consider the holomorphic functions

$$\Phi_1(z)=z+\frac{1}{8}z^2, \quad \text{and} \quad \Phi_2(z)=-\frac{1}{4}z^2$$

in holomorphic coordinates. We may assume by scaling the coordinates z that Φ_1 does not have critical points in U. Let us compute $\operatorname{tr}\left(\left(D^2\mathbf{v}^{(1)}\right)\left(D^2\mathbf{v}^{(2)}\right)\right)$, where

$$\mathbf{v}^{(1)}=F_{A_1}^{-1}e^{\Phi_1/h}(1+r_1), \quad \mathbf{v}^{(2)}=F_{\overline{A_2}}e^{\overline{\Phi_2}/h}(1+\widetilde{r}_2)$$

are CGO solutions for the first linearized equation, for sufficiently small $h>0$, where $r_1,\widetilde{r}_2$ are remainders (see Lemma 5.3 for the formulas). We have

$$\begin{aligned}
&\operatorname{tr}\left(\left(D^2\mathbf{v}^{(1)}\right)\left(D^2\mathbf{v}^{(2)}\right)\right)\\
&=\operatorname{tr}\left(D^2(F_{A_1}^{-1}e^{\Phi_1/h})D^2(F_{\overline{A_2}}e^{\overline{\Phi_2}/h})\right)+\operatorname{tr}\left(D^2(F_{A_1}^{-1}e^{\Phi_1/h}r_1)D^2(F_{\overline{A_2}}e^{\overline{\Phi_2}/h})\right)\\
&\quad+\operatorname{tr}\left(D^2(F_{A_1}^{-1}e^{\Phi_1/h})D^2(F_{\overline{A_2}}e^{\overline{\Phi_2}/h}\widetilde{r}_2)\right)+\operatorname{tr}\left(D^2(F_{A_1}^{-1}e^{\Phi_1/h}r_1)D^2(F_{\overline{A_2}}e^{\overline{\Phi_2}/h}\widetilde{r}_2)\right),
\end{aligned}$$

where we used the matrix representation formula (2.3) for the Hessian to derive the above identities.

Let us begin by analyzing the first term in (7.8), which can be written as the sum

$$\int_U G\mathbf{v}^*\operatorname{tr}\left(\left(D^2\mathbf{v}^{(1)}\right)\left(D^2\mathbf{v}^{(2)}\right)\right)dx:=S_1+S_2+S_3+S_4,$$

where

$$\begin{aligned}
S_1&:=\int_U G\mathbf{v}^*\operatorname{tr}\left(D^2(F_{A_1}^{-1}e^{\Phi_1/h})D^2(F_{\overline{A_2}}e^{\overline{\Phi_2}/h})\right)dx,\\
S_2&:=\int_U G\mathbf{v}^*\operatorname{tr}\left(D^2(F_{A_1}^{-1}e^{\Phi_1/h}r_1)D^2(F_{\overline{A_2}}e^{\overline{\Phi_2}/h})\right)dx,\\
S_3&:=\int_U G\mathbf{v}^*\operatorname{tr}\left(D^2(F_{A_1}^{-1}e^{\Phi_1/h})D^2(F_{\overline{A_2}}e^{\overline{\Phi_2}/h}\widetilde{r}_2)\right)dx,\\
S_4&:=\int_U G\mathbf{v}^*\operatorname{tr}\left(D^2(F_{A_1}^{-1}e^{\Phi_1/h}r_1)D^2(F_{\overline{A_2}}e^{\overline{\Phi_2}/h}\widetilde{r}_2)\right)dx.
\end{aligned} \tag{7.11}$$

Here, r_1 and $\widetilde{r}_2$ are the remainder terms in the CGO solutions satisfying

$$r_1=-\overline{\partial}_\psi^{-1}V's_1 \quad \text{and} \quad \widetilde{r}_2=-\partial_\psi^{-1}V'\widetilde{s}_2 \tag{7.12}$$

with s_1 and $\widetilde{s}_2$ satisfy the decay estimates as constructed in Section 5. In the rest of the article, the function ψ is chosen as

$$\psi = \psi(x_1, x_2) = \frac{x_1 x_2}{2}.$$

For the solution to the adjoint equation, we take

$$\mathbf{v}^* = F_{A^*}^{-1}\, e^{\Phi^*/h}\,(a^* + r^*), \tag{7.13}$$

where $h > 0$ is sufficiently small and Φ^* is the holomorphic phase function

$$\Phi^*(z) := -z + \frac{1}{8}z^2.$$

By scaling the coordinates again, if needed, we may assume that Φ^* neither has critical points in U. Here $F_{A^*}^{-1}$ is a smooth, nowhere-vanishing function as given in Lemma 5.3, and the remainder r^* satisfies the same better decay estimates as r_1. In the following, we will analyze the contributions S_k for $k = 1, 2, 3, 4$ separately.

Step 1. Analysis of S_1.

We define the notation

$$\widetilde{G}_0 := F_{A_1}^{-1} F_{\overline{A_2}} G = \mu^{-1} F_{A_1}^{-1} F_{\overline{A_2}} (1 - c^{-2}). \tag{7.14}$$

Using the expression (2.6), a direct computation yields

$$\begin{aligned} S_1 = &\int_U G\mathbf{v}^* \mathrm{tr}\big\{\big[A\partial^2(F_{A_1}^{-1}e^{\Phi_1/h}) + B\overline{\partial}^2(F_{A_1}^{-1}e^{\Phi_1/h}) + 2I_{2\times 2}\partial\overline{\partial}(F_{A_1}^{-1}e^{\Phi_1/h})\big] \\ &\qquad \cdot \big[A\partial^2(F_{\overline{A_2}}e^{\overline{\Phi_2}/h}) + B\overline{\partial}^2(F_{\overline{A_2}}e^{\overline{\Phi_2}/h}) + 2I_{2\times 2}\partial\overline{\partial}(F_{\overline{A_2}}e^{\overline{\Phi_2}/h})\big]\big\}\, dx \\ = &\int_U G\mathbf{v}^* \mathrm{tr}\big\{\big[A\big(F_{A_1}^{-1}\partial^2 e^{\Phi_1/h} + 2\partial F_{A_1}^{-1}\partial e^{\Phi_1/h} + e^{\Phi_1/h}\partial^2 F_{A_1}^{-1}\big) \\ &\qquad + Be^{\Phi_1/h}\overline{\partial}^2 F_{A_1}^{-1} + 2I_{2\times 2}\big(e^{\Phi_1/h}\partial\overline{\partial}F_{A_1}^{-1} + \overline{\partial}F_{A_1}^{-1}\partial e^{\Phi_1/h}\big)\big] \\ &\qquad \cdot \big[B\big(F_{\overline{A_2}}\overline{\partial}^2 e^{\overline{\Phi_2}/h} + 2\overline{\partial}F_{\overline{A_2}}\overline{\partial}e^{\overline{\Phi_2}/h} + e^{\overline{\Phi_2}/h}\overline{\partial}^2 F_{\overline{A_2}}\big) \\ &\qquad + Ae^{\overline{\Phi_2}/h}\partial^2 F_{\overline{A_2}} + 2I_{2\times 2}\big(e^{\overline{\Phi_2}/h}\partial\overline{\partial}F_{\overline{A_2}} + \partial F_{\overline{A_2}}\overline{\partial}e^{\overline{\Phi_2}/h}\big)\big]\big\}\, dx \end{aligned}$$

where A, B are complex-valued matrices given in (2.7), which satisfy

$$\mathrm{tr}(AB) = 4, \quad \mathrm{tr}(AA) = \mathrm{tr}(BB) = \mathrm{tr}(A) = \mathrm{tr}(B) = 0.$$

The above relations are used to reduce the computations throughout the asymptotic analysis. Next, let us write

$$S_1 := S_{1,1} + S_{1,2},$$

where

$$\begin{aligned} S_{1,1} &= \int_U \widetilde{G}_0\mathbf{v}^* \underbrace{\mathrm{tr}(AB)}_{=4}\, \partial^2(e^{\Phi_1/h})\overline{\partial}^2(e^{\overline{\Phi_2}/h})\, dx \\ &= 4\int_U \partial\overline{\partial}(\widetilde{G}_0\mathbf{v}^*)\partial(e^{\Phi_1/h})\overline{\partial}(e^{\overline{\Phi_2}/h})\, dx \\ &= \frac{1}{h^2}\int_U \Big(\widetilde{G}_0\Delta\mathbf{v}^* + 4\partial\mathbf{v}^*\overline{\partial}\widetilde{G}_0 + 4\overline{\partial}\mathbf{v}^*\partial\widetilde{G}_0 + \mathbf{v}^*\Delta\widetilde{G}_0\Big)\partial\Phi_1\overline{\partial\Phi_2}e^{(\Phi_1+\overline{\Phi_2})/h}\, dx \\ &= \frac{1}{h^2}\int_U \Big(-\widetilde{G}_0\nabla\cdot(\mathbf{X}_g\mathbf{v}^*) + 4\partial\mathbf{v}^*\overline{\partial}\widetilde{G}_0 + 4\overline{\partial}\mathbf{v}^*\partial\widetilde{G}_0 + \mathbf{v}^*\Delta\widetilde{G}_0\Big) \\ &\qquad \cdot \partial\Phi_1\overline{\partial\Phi_2}e^{(\Phi_1+\overline{\Phi_2})/h}\, dx, \end{aligned}$$

and we have employed integration by parts and the adjoint equation (5.4) in the above identities.

Let us emphasize that, throughout the remainder of the proof, integration by parts will produce no boundary contributions due to the boundary determination established in Section 3. Furthermore, we note that the term $S_{1,2} = S_1 - S_{1,1}$ consists of integrals in which at least one derivative falls on either $F_{A_1}^{-1}$ or $F_{\overline{A_2}}$. As we shall demonstrate, these contributions are of lower order and do not affect the leading order behavior. For convenience in future computations, we collect the following identities:

$$\begin{aligned} &\partial\Phi_1 = 1 + \frac{1}{4}z, \quad \overline{\partial}\Phi_2 = -\frac{1}{2}\overline{z}, \quad \partial\Phi^* = -1 + \frac{1}{4}z, \\ &\partial^2\Phi_1 = \partial^2\Phi^* = \frac{1}{4}, \quad \overline{\partial}^2\Phi_2 = -\frac{1}{2}. \end{aligned} \tag{7.15}$$

Step 1-1. Analysis of $S_{1,1}$.

Using the CGO solution of the form (7.13), we can write

$$S_{1,1} := S_{1,1}^m + S_{1,1}^r,$$

where

$$\begin{aligned} S_{1,1}^m := \frac{1}{h^2}\int_U \Big(&- \widetilde{G}_0 \nabla\cdot(\mathbf{X}_g F_{A^*}^{-1} e^{\Phi^*/h}) + 4\partial(F_{A^*}^{-1}e^{\Phi^*/h})\overline{\partial}\widetilde{G}_0 \\ &+ 4\overline{\partial}(F_{A^*}^{-1}e^{\Phi^*/h})\partial\widetilde{G}_0 + F_{A^*}^{-1}e^{\Phi^*/h}\Delta\widetilde{G}_0\Big)(\partial\Phi_1)(\overline{\partial\Phi_2})e^{(\Phi_1+\overline{\Phi_2})/h}\,dx, \end{aligned}$$

and

$$\begin{aligned} S_{1,1}^r := \frac{1}{h^2}\int_U \Big(&- \widetilde{G}_0 \nabla\cdot(\mathbf{X}_g F_{A^*}^{-1} e^{\Phi^*/h}r^*) + 4\partial(F_{A^*}^{-1}e^{\Phi^*/h}r^*)\overline{\partial}\widetilde{G}_0 \\ &+ 4\overline{\partial}(F_{A^*}^{-1}e^{\Phi^*/h}r^*)\partial\widetilde{G}_0 + F_{A^*}^{-1}e^{\Phi^*/h}r^*\Delta\widetilde{G}_0\Big)(\partial\Phi_1)(\overline{\partial\Phi_2})e^{(\Phi_1+\overline{\Phi_2})/h}\,dx. \end{aligned} \tag{7.16}$$

We will show that $S_{1,1}^m$ is a governing term in the asymptotic analysis.

To proceed, using $\overline{\partial}e^{\Phi^*/h}=0$, we can compute $S_{1,1}^m$ as

$$\begin{aligned} \frac{h}{2\pi}S_{1,1}^m = \frac{1}{2\pi h}\int_U \Big(&- \widetilde{G}_0\nabla\cdot(F_{A^*}^{-1}\mathbf{X}_g)e^{\Phi^*/h} - \frac{1}{h}\widetilde{G}_0F_{A^*}^{-1}\mathbf{X}_g\cdot\nabla\Phi^* e^{\Phi^*/h} \\ &+ 4\partial(F_{A^*}^{-1}e^{\Phi^*/h})\overline{\partial}\widetilde{G}_0 + 4e^{\Phi^*/h}\overline{\partial}F_{A^*}^{-1}\partial\widetilde{G}_0 + F_{A^*}^{-1}e^{\Phi^*/h}\Delta\widetilde{G}_0\Big) \\ &\cdot(\partial\Phi_1)(\overline{\partial\Phi_2})e^{(\Phi_1+\overline{\Phi_2})/h}\,dx \\ = \frac{1}{2\pi h}\int_U \Big(&- \widetilde{G}_0\nabla\cdot(F_{A^*}^{-1}\mathbf{X}_g) - \frac{1}{h}\widetilde{G}_0F_{A^*}^{-1}\mathbf{X}_g\cdot\nabla\Phi^* \\ &+ 4(\partial F_{A^*}^{-1} + \frac{1}{h}F_{A^*}^{-1}\partial\Phi^*)\overline{\partial}\widetilde{G}_0 + 4\overline{\partial}F_{A^*}^{-1}\partial\widetilde{G}_0 + F_{A^*}^{-1}\Delta\widetilde{G}_0\Big) \\ &\cdot(\partial\Phi_1)(\overline{\partial\Phi_2})e^{\mathrm{i}x_1x_2/h}\,dx \\ = \frac{1}{2\pi h}\int_U \Big\{ &\frac{1}{h}\big[F_{A^*}^{-1}\big(-1+\frac{z}{4}\big)\overline{\partial}\widetilde{G}_0 - F_{A^*}^{-1}(\mathbf{X}_g\cdot\nabla\Phi^*)\widetilde{G}_0\big] \\ &+ \big[-\widetilde{G}_0\nabla\cdot(F_{A^*}^{-1}\mathbf{X}_g) + 4\partial F_{A^*}^{-1}\overline{\partial}\widetilde{G}_0 + 4\overline{\partial}F_{A^*}^{-1}\partial\widetilde{G}_0 \\ &+ F_{A^*}^{-1}\Delta\widetilde{G}_0\big]\Big\}\cdot\underbrace{(1+\frac{1}{4}z)(-\frac{1}{2}\overline{z})}_{\text{By (7.15)}}\,e^{\mathrm{i}x_1x_2/h}\,dx, \end{aligned} \tag{7.17}$$

where we used $\Phi_1 + \overline{\Phi_2} + \Phi^* = \frac{1}{4}\big(z^2 - \overline{z}^2\big) = \mathrm{i}x_1x_2 = 2\mathrm{i}\psi$. Applying the stationary phase expansion (7.10) to the identity (7.17), we can see that

(7.18)
$$
\begin{aligned}
&\frac{1}{2\pi h}S^m_{1,1}\\
&= \big(\overline{\partial}^2 - \partial^2\big)\Big\{\big(F_{A^*}^{-1}\big(-1+\frac{z}{4}\big)\overline{\partial}\widetilde{G}_0 - F_{A^*}^{-1}(\mathbf{X}_g\cdot\nabla\Phi^*)\widetilde{G}_0\big)\\
&\qquad\qquad \cdot (1+\frac{1}{4}z)(-\frac{1}{2}\overline{z})\Big\}\Big|_{z=0} + \mathcal{O}(h)\\
&= \overline{\partial}^2\Big\{\big(F_{A^*}^{-1}\big(-1+\frac{z}{4}\big)\overline{\partial}\widetilde{G}_0 - F_{A^*}^{-1}(\mathbf{X}_g\cdot\nabla\Phi^*)\widetilde{G}_0\big)(1+\frac{1}{4}z)(-\frac{1}{2}\overline{z})\Big\}\Big|_{z=0} + \mathcal{O}(h)\\
&= -\frac{1}{2}\overline{\partial}\Big\{\big(F_{A^*}^{-1}\big(-1+\frac{z}{4}\big)\overline{\partial}\widetilde{G}_0 - F_{A^*}^{-1}(\mathbf{X}_g\cdot\nabla\Phi^*)\widetilde{G}_0\big)(1+\frac{1}{4}z)\Big\}\Big|_{z=0} + \mathcal{O}(h)\\
&= \frac{1}{2}\overline{\partial}\big(F_{A^*}^{-1}\overline{\partial}\widetilde{G}_0 + F_{A^*}^{-1}(\mathbf{X}_g\cdot\nabla\Phi^*)\widetilde{G}_0\big)\big|_{z=0} + \mathcal{O}(h),
\end{aligned}
$$

where we used

$$
\begin{aligned}
&\partial^2\Big\{\big(F_{A^*}^{-1}\big(-1+\frac{z}{4}\big)\overline{\partial}\widetilde{G}_0 - F_{A^*}^{-1}(\mathbf{X}_g\cdot\nabla\Phi^*)\widetilde{G}_0\big)(1+\frac{1}{4}z)(-\frac{1}{2}\overline{z})\Big\}\Big|_{z=0}\\
&= \Big\{\big(-\frac{1}{2}\overline{z}\big)\partial^2\Big[\big(F_{A^*}^{-1}\big(-1+\frac{z}{4}\big)\overline{\partial}\widetilde{G}_0 - F_{A^*}^{-1}(\mathbf{X}_g\cdot\nabla\Phi^*)\widetilde{G}_0\big)(1+\frac{1}{4}z)\Big]\Big\}\Big|_{z=0}\\
&= 0.
\end{aligned}
$$

Furthermore, in view of (7.18), one can conclude that

$$
\lim_{h\to 0}\big(hS^m_{1,1}\big) = \pi\overline{\partial}\big(F_{A^*}^{-1}\overline{\partial}\widetilde{G}_0 + F_{A^*}^{-1}(\mathbf{X}_g\cdot\nabla\Phi^*)\widetilde{G}_0\big)\big|_{z=0}.
$$

Next, we want to show $S^r_{1,1}$ has a faster decay in h than $S^m_{1,1}$. Recalling that $S^r_{1,1}$ is given by (7.16), which has a very similar form as $S^m_{1,1}$. The only difference is that the integral contains an extra error term, r^*, and its first derivative. Thanks to Lemma 5.2 and analysis in Section 5, the remainder term r^* has better decay properties, so that one can apply the stationary phase formula to ensure $\lim_{h\to 0}\big(hS^r_{1,1}\big) = 0$. Hence, we can ensure

$$
\lim_{h\to 0}\big(hS_{1,1}\big) = \pi\overline{\partial}\big(F_{A^*}^{-1}\overline{\partial}\widetilde{G}_0 + F_{A^*}^{-1}(\mathbf{X}_g\cdot\nabla\Phi^*)\widetilde{G}_0\big)\big|_{z=0},
$$

which will play an essential role in the recovery of the conformal factor.

Step 1-2: Analysis of $S_{1,2}$.

Since every term in the integrand of $S_{1,2}$ involves at least one derivative acting on either $F_{A_1}^{-1}$ or $F_{\overline{A_2}}$, it is expected that many of these terms contribute only to lower-order effects and do not influence the leading-order behavior. To illustrate this, let us examine representative terms in $S_{1,2}$ that exhibit the highest possible asymptotic order with respect to the small parameter h. The asymptotic behavior of the remaining terms can be analyzed in a similar manner, and in most cases, they exhibit even faster decay. It is straightforward to verify that there are two terms of order $\mathcal{O}(1/h^3)$, and we compute them term by term below.

We denote that

$$
\widetilde{G}_1 := GF_{A^*}^{-1}F_{A_1}^{-1}\overline{\partial}F_{\overline{A_2}} = \mu^{-1}F_{A^*}^{-1}F_{A_1}^{-1}\overline{\partial}F_{\overline{A_2}}(1-c^{-2}) \tag{7.19}
$$

is a bounded function independent of $h > 0$, then an integration by parts with respect to $\overline{\partial}$ and the stationary phase expansion imply that

$$\begin{aligned}
&2\int_U GF_{A_1}^{-1}\overline{\partial}F_{\overline{A_2}}\mathbf{v}^*\underbrace{\mathrm{tr}(AB)}_{=4}\partial^2(e^{\Phi_1/h})\overline{\partial}(e^{\overline{\Phi_2}/h})\,dx \\
&= -8\int_U (\overline{\partial}\widetilde{G}_1)e^{\Phi^*/h}\Big(\frac{1}{h^2}(1+\frac{z}{4})^2+\frac{1}{4h}\Big)e^{(\Phi_1+\overline{\Phi_2})/h}\,dx + \mathcal{O}(1) \\
&= -\frac{16\pi}{h}\overline{\partial}\widetilde{G}_1\big|_{z=0} + \mathcal{O}(1),
\end{aligned} \tag{7.20}$$

as $h \to 0$, where we utilize $\overline{\partial}e^{\Phi^*/h} = 0$ and

$$\mathbf{v}^* = F_{A^*}^{-1}e^{\Phi^*/h}(1+r^*),$$

where the integral including r^* contribute $\mathcal{O}(1)$ by Lemma 5.2 in the middle line of (7.20). Thus, we obtain

$$\lim_{h\to 0}\left(2h\int_U GF_{A_1}^{-1}\overline{\partial}F_{\overline{A_2}}\mathbf{v}^*\,\mathrm{tr}(AB)\partial^2(e^{\Phi_1/h})\overline{\partial}(e^{\overline{\Phi_2}/h})\,dx\right) = -16\pi\overline{\partial}\widetilde{G}_1\big|_{z=0}.$$

Similarly, we can use the same approach as above for the other term to obtain

$$\begin{aligned}
&2\int_U G\partial F_{A_1}^{-1}F_{\overline{A_2}}\,\mathrm{tr}(AB)\mathbf{v}^*\partial e^{\Phi_1/h}\overline{\partial}^2 e^{\overline{\Phi_2/h}}\,dx \\
&= \frac{4}{h^2}\int_U \underbrace{\overline{\partial}\big(G\partial F_{A_1}^{-1}F_{\overline{A_2}}\mathbf{v}^*\big)}_{=\mathcal{O}(1)}\,(1+\frac{z}{4})\overline{z}e^{(\Phi_1+\overline{\Phi_2})/h}\,dx \\
&= \mathcal{O}(1).
\end{aligned}$$

Therefore, we can obtain

$$\lim_{h\to 0}\left(2h\int_U G\partial F_{A_1}^{-1}F_{\overline{A_2}}\,\mathrm{tr}(AB)\mathbf{v}^*\partial e^{\Phi_1/h}\overline{\partial}^2 e^{\overline{\Phi_2/h}}\,dx\right) = 0.$$

Since the above two terms contribute $\mathcal{O}(1/h)$ and $\mathcal{O}(1)$, and it is not hard to see the rest terms in $S_{1,2}$ have at least one more h factor. This implies

$$\lim_{h\to 0}\big(hS_{1,2}\big) = -16\pi\overline{\partial}\widetilde{G}_1\big|_{z=0}.$$

Therefore, combining all the analyses, we can ensure

$$\lim_{h\to 0}\big(hS_1\big) = \pi\overline{\partial}\big(F_{A^*}^{-1}\overline{\partial}\widetilde{G}_0 + F_{A^*}^{-1}(\mathbf{X}_g\cdot\nabla\Phi^*)\widetilde{G}_0\big)\big|_{z=0} - 16\pi\overline{\partial}\widetilde{G}_1\big|_{z=0}. \tag{7.21}$$

We remark that the terms $\widetilde{G}_0$ and $\widetilde{G}_1$ on the right-hand side each contain $(1-c^{-2})$ as a multiplicative factor. Ultimately, our analysis will lead to a partial differential equation for $(1-c^{-2})$.

Step 2. Analysis of S_2.

Recall that A and B are 2×2 complex-valued matrices (see (2.7)), and D^2 denotes the Hessian operator in $\mathbb{R}^2$. The contributions of S_2, arising from the remainder term r_1, are associated with the linear part of the phase function and can be handled similarly to previous terms. The more challenging components are S_3 and S_4, whose analysis will be presented in the subsequent sections. For S_2, let us compute the matrix BD^2 as

$$BD^2 = \begin{pmatrix} 1 & -\mathsf{i} \\ -\mathsf{i} & -1 \end{pmatrix} \begin{pmatrix} \left(\partial + \overline{\partial}\right)^2 & \mathsf{i}\left(\partial^2 - \overline{\partial}^2\right) \\ \mathsf{i}\left(\partial^2 - \overline{\partial}^2\right) & -\left(\partial - \overline{\partial}\right)^2 \end{pmatrix}$$
$$= \begin{pmatrix} 2\partial^2 + 2\overline{\partial}\partial & \mathsf{i}(2\partial^2 - 2\partial\overline{\partial}) \\ -\mathsf{i}(2\partial^2 + 2\partial\overline{\partial}) & 2\partial^2 - 2\partial\overline{\partial} \end{pmatrix},$$

so that

$$\operatorname{tr}(BD^2) = 4\partial^2. \tag{7.22}$$

With the above computations, applying an integration by parts formula and $\partial(e^{\overline{\Phi_2}/h}) = 0$, we can see

(7.23)
$$\begin{aligned} S_2 &= \int_U G\mathbf{v}^* \operatorname{tr}\left(D^2(F_{A_1}^{-1}e^{\Phi_1/h}r_1)\right)D^2(F_{\overline{A_2}}e^{\overline{\Phi_2}/h})\,dx \\ &= \int_U G\mathbf{v}^* \operatorname{tr}\left\{\left(D^2(F_{A_1}^{-1}e^{\Phi_1/h}r_1)\right)\left[B\left(F_{\overline{A_2}}\overline{\partial}^2 e^{\overline{\Phi_2}/h} + 2\overline{\partial}F_{\overline{A_2}}\overline{\partial}e^{\overline{\Phi_2}/h} + e^{\overline{\Phi_2}/h}\overline{\partial}^2 F_{\overline{A_2}}\right)\right.\right. \\ &\qquad \left.\left. + Ae^{\overline{\Phi_2}/h}\partial^2 F_{\overline{A_2}} + 2I_{2\times 2}\left(e^{\overline{\Phi_2}/h}\partial\overline{\partial}F_{\overline{A_2}} + \partial F_{\overline{A_2}}\overline{\partial}e^{\overline{\Phi_2}/h}\right)\right]\right\}dx \\ &=: S_{2,1} + S_{2,2} \end{aligned}$$

where

$$S_{2,1} := \int_U GF_{\overline{A_2}}\mathbf{v}^* \operatorname{tr}\left(BD^2(F_{A_1}^{-1}e^{\Phi_1/h}r_1)\overline{\partial}^2 e^{\overline{\Phi_2}/h}\right)dx$$

and

$$\begin{aligned} S_{2,2} := \int_U G\mathbf{v}^* \operatorname{tr}&\left\{\left(D^2(F_{A_1}^{-1}e^{\Phi_1/h}r_1)\right)\left[B\left(2\overline{\partial}F_{\overline{A_2}}\overline{\partial}e^{\overline{\Phi_2}/h} + e^{\overline{\Phi_2}/h}\overline{\partial}^2 F_{\overline{A_2}}\right)\right.\right. \\ &\left.\left. + Ae^{\overline{\Phi_2}/h}\partial^2 F_{\overline{A_2}} + 2I_{2\times 2}\left(e^{\overline{\Phi_2}/h}\partial\overline{\partial}F_{\overline{A_2}} + \partial F_{\overline{A_2}}\overline{\partial}e^{\overline{\Phi_2}/h}\right)\right]\right\}dx. \end{aligned}$$

Notice that the integral $S_{2,2}$ contains at least one derivative of $F_{\overline{A_2}}$, and generate one extra h than the first integral in the right-hand side of (7.23). Let us analyze $S_{2,1}$ as follows. Using (7.22), we have

$$\begin{aligned} S_{2,1} &= 4\int_U GF_{\overline{A_2}}\mathbf{v}^*\partial^2(F_{A_1}^{-1}e^{\Phi_1/h}r_1)\overline{\partial}^2 e^{\overline{\Phi_2}/h}\,dx \\ &= 4\int_U \partial\overline{\partial}\big(\widetilde{G}_2\mathbf{v}^*\big)\partial(F_{A_1}^{-1}e^{\Phi_1/h}r_1)\overline{\partial}e^{\overline{\Phi_2}/h}\,dx \\ &\quad + 4\int_U \partial\big(\widetilde{G}_2\mathbf{v}^*\big)\partial\left[\overline{\partial}F_{A_1}^{-1}e^{\Phi_1/h}r_1 + F_{A_1}^{-1}e^{\Phi_1/h}\overline{\partial}r_1\right]\overline{\partial}e^{\overline{\Phi_2}/h}\,dx, \end{aligned}$$

where

$$\widetilde{G}_2 := GF_{\overline{A_2}}.$$

Observing that the analysis of the term $S_{2,1}$ closely parallels that of S_1, the only difference being the appearance of the remainder term r_1, which enjoys better decay properties for $h > 0$ as described in Lemma 5.2, we can proceed analogously. By repeating exactly the same arguments used in the analysis of S_1, and taking into account the improved decay of r_1 and its derivatives, we immediately obtain

$$\lim_{h\to 0}(hS_{2,1}) = 0.$$

For $S_{2,2}$, as explained earlier, there is an extra factor of h compared to $S_{2,1}$, so we omit the detailed derivation. In short, one can ensure that $\lim_{h\to 0}\left(hS_{2,2}\right) = 0$, which in turn yields

$$\lim_{h\to 0}\left(hS_2\right)=0. \tag{7.24}$$

Step 3. Analysis of S_3.

Similar to the analysis for S_2, we have a similar formula for AD^2, such that

$$\mathrm{tr}(AD^2)=4\overline{\partial}^2. \tag{7.25}$$

Different from the analysis of S_2, using (2.6) again, an alternative integration by parts formula yields that

$$\begin{aligned}
S_3&=\int_U G\mathbf{v}^*\,\mathrm{tr}\left(D^2(F_{A_1}^{-1}e^{\Phi_1/h})D^2(F_{\overline{A_2}}e^{\overline{\Phi_2}/h}\widetilde{r}_2)\right)dx\\
&=\int_U G\mathbf{v}^*\mathrm{tr}\big\{\big[A\big(F_{A_1}^{-1}\partial^2e^{\Phi_1/h}+2\partial F_{A_1}^{-1}\partial e^{\Phi_1/h}+e^{\Phi_1/h}\partial^2F_{A_1}^{-1}\big)\\
&\qquad\qquad+Be^{\Phi_1/h}\overline{\partial}^2F_{A_1}^{-1}+2I_{2\times 2}\big(e^{\Phi_1/h}\partial\overline{\partial}F_{A_1}^{-1}+\overline{\partial}F_{A_1}^{-1}\partial e^{\Phi_1/h}\big)\big]\\
&\qquad\qquad\cdot D^2(F_{\overline{A_2}}e^{\overline{\Phi_2}/h}\widetilde{r}_2)\big)\,dx\\
&=:S_{3,1}+S_{3,2},
\end{aligned}$$

where

$$S_{3,1}:=\int_U GF_{A_1}^{-1}\mathbf{v}^*\,\mathrm{tr}\left(AD^2(F_{\overline{A_2}}e^{\overline{\Phi_2}/h}\widetilde{r}_2)\partial^2e^{\Phi_1/h}\right)dx,$$

and

$$\begin{aligned}
S_{3,2}:=\int_U G\mathbf{v}^*\mathrm{tr}\big\{&\big[A\big(2\partial F_{A_1}^{-1}\partial e^{\Phi_1/h}+e^{\Phi_1/h}\partial^2F_{A_1}^{-1}\big)\\
&+Be^{\Phi_1/h}\overline{\partial}^2F_{A_1}^{-1}+2I_{2\times 2}\big(e^{\Phi_1/h}\partial\overline{\partial}F_{A_1}^{-1}+\overline{\partial}F_{A_1}^{-1}\partial e^{\Phi_1/h}\big)\big]\\
&\cdot D^2(F_{\overline{A_2}}e^{\overline{\Phi_2}/h}\widetilde{r}_2)\big)\,dx.
\end{aligned}$$

We also analyze $S_{3,1}$ and $S_{3,2}$ separately.

Let us review an integration by parts as in [LW23, Section 3.1], which yields

$$\int_U\big(\overline{\partial}^{-1}f\big)\varphi\,dx=-\int_U f\big(\overline{\partial}^{-1}\varphi\big)\,dx, \tag{7.26}$$

for $f\in L^1$ and $\varphi\in L^p$ for some $p>2$, such that both f and φ vanish on the boundary ∂U. For $S_{3,1}$, using (7.25) and an integration by parts, we have

$$\begin{aligned}
S_{3,1}&=4\int_U GF_{A_1}^{-1}\mathbf{v}^*\overline{\partial}^2(F_{\overline{A_2}}e^{\overline{\Phi_2}/h}\widetilde{r}_2)\partial^2e^{\Phi_1/h}\,dx\\
&=\underbrace{4\int_U\overline{\partial}^2\big(GF_{A_1}^{-1}\mathbf{v}^*\big)F_{\overline{A_2}}\Big(\frac{1}{h^2}\big(1+\frac{z}{4}\big)^2+\frac{1}{4h}\Big)e^{(\Phi_1+\overline{\Phi_2})/h}\widetilde{r}_2\,dx}_{\text{By }\overline{\partial}e^{\Phi_1/h}=0}\\
&=4\int_U\widetilde{G}_3\Big(\frac{1}{h^2}\big(1+\frac{z}{4}\big)^2+\frac{1}{4h}\Big)e^{\mathrm{i}x_1x_2/h}\widetilde{r}_2\,dx
\end{aligned}$$

where

$$\widetilde{G}_3:=\big[\overline{\partial}^2\big(GF_{A_1}^{-1}\big)(1+r^*)+2\overline{\partial}\big(GF_{A_1}^{-1}\big)\overline{\partial}r^*+GF_{A_1}^{-1}\overline{\partial}^2r^*\big]F_{\overline{A_2}}=\mathcal{O}_{L^2}(1),$$

we used the properties of $\mathbf{v}^*$, r^* again. To proceed, using (7.12), we can rewrite $S_{3,1}$ into

$$
\begin{aligned}
S_{3,1} &= 4\int_U \widetilde{G}_3\Big(\frac{1}{h^2}(1+\frac{z}{4})^2+\frac{1}{4h}\Big)e^{\mathrm{i}x_1x_2/h}\widetilde{r}_2\,dx\\
&= -4\int_U \widetilde{G}_3\Big(\frac{1}{h^2}(1+\frac{z}{4})^2+\frac{1}{4h}\Big)e^{\mathrm{i}x_1x_2/h}\partial_\psi^{-1}V'\widetilde{s}_2\,dx\\
&= 4\int_U \partial^{-1}\Big[\widetilde{G}_3\Big(\frac{1}{h^2}(1+\frac{z}{4})^2+\frac{1}{4h}\Big)e^{\mathrm{i}x_1x_2/h}\Big]V'\widetilde{s}_2e^{\mathrm{i}x_1x_2/h}\,dx\\
&= \mathcal{O}(h^{-1+\epsilon}),\quad \text{as}\quad h\to 0,
\end{aligned}
\tag{7.27}
$$

where we used [GT11b, Lemma 2.2] with

$$
\overline{\partial}_\psi^{-1}f=\mathcal{O}_{L^2}(h^{1/2+\epsilon})\quad\text{and}\quad \partial_\psi^{-1}f=\mathcal{O}_{L^2}(h^{1/2+\epsilon})\quad\text{as}\quad h\to 0,
$$

and $\|s_2\|_{L^2(U)}=\mathcal{O}(h^{1/2+\epsilon})$, for $\epsilon>0$ sufficiently small. The derivation (7.27) ensures that the limit

$$
\lim_{h\to 0}\big(hS_{3,1}\big)=0
$$

holds. Similarly, since $S_{3,2}$ contains at least one addition h factor, similar analysis gives rises to $\lim_{h\to 0}\big(hS_{3,2}\big)=0$, which infers that

$$
\lim_{h\to 0}\big(hS_3\big)=0
\tag{7.28}
$$

as we expect.

Step 4. Analysis of S_4.

Using the Hessian representation (2.6), direct computations imply that

$$
\begin{aligned}
&\operatorname{tr}\big(D^2(F_{A_1}^{-1}e^{\Phi_1/h}r_1)D^2(F_{\overline{A_2}}e^{\overline{\Phi_2}/h}\widetilde{r}_2)\big)\\
={}&\operatorname{tr}\big\{D^2(F_{A_1}^{-1}e^{\Phi_1/h}r_1)\\
&\qquad\cdot\big[A\partial^2(F_{\overline{A_2}}e^{\overline{\Phi_2}/h}\widetilde{r}_2)+B\overline{\partial}^2(F_{\overline{A_2}}e^{\overline{\Phi_2}/h}\widetilde{r}_2)+2I_{2\times 2}\partial\overline{\partial}(F_{\overline{A_2}}e^{\overline{\Phi_2}/h}\widetilde{r}_2)\big]\big\}\\
={}&\operatorname{tr}\big(AD^2(F_{A_1}^{-1}e^{\Phi_1/h}r_1)\big)\partial^2(F_{\overline{A_2}}e^{\overline{\Phi_2}/h}\widetilde{r}_2)\\
&+\operatorname{tr}\big(BD^2(F_{A_1}^{-1}e^{\Phi_1/h}r_1)\big)\overline{\partial}^2(F_{\overline{A_2}}e^{\overline{\Phi_2}/h}\widetilde{r}_2)\\
&+\underbrace{2\operatorname{tr}\big(D^2(F_{A_1}^{-1}e^{\Phi_1/h}r_1)\big)}_{=2\Delta(F_{A_1}^{-1}e^{\Phi_1/h}r_1)}\partial\overline{\partial}(F_{\overline{A_2}}e^{\overline{\Phi_2}/h}\widetilde{r}_2)\\
={}&\underbrace{4\overline{\partial}^2(F_{A_1}^{-1}e^{\Phi_1/h}r_1)\partial^2(F_{\overline{A_2}}e^{\overline{\Phi_2}/h}\widetilde{r}_2)}_{\text{By (7.25)}}+\underbrace{4\partial^2(F_{A_1}^{-1}e^{\Phi_1/h}r_1)\overline{\partial}^2(F_{\overline{A_2}}e^{\overline{\Phi_2}/h}\widetilde{r}_2)}_{\text{By (7.22)}}\\
&+\underbrace{8\partial\overline{\partial}(F_{A_1}^{-1}e^{\Phi_1/h}r_1)\partial\overline{\partial}(F_{\overline{A_2}}e^{\overline{\Phi_2}/h}\widetilde{r}_2)}_{\text{By }\Delta=4\partial\overline{\partial}}.
\end{aligned}
\tag{7.29}
$$

Inserting (7.29) into S_4 given by (7.11), we can write

$$
S_4:=S_{4,1}+S_{4,2}+S_{4,3},
$$

where

$$S_{4,1} := 4\int_U G\mathbf{v}^*\overline{\partial}^2(F_{A_1}^{-1}e^{\Phi_1/h}r_1)\partial^2(F_{\overline{A_2}}e^{\overline{\Phi_2}/h}\widetilde{r}_2)\,dx,$$
$$S_{4,2} := 4\int_U G\mathbf{v}^*\partial^2(F_{A_1}^{-1}e^{\Phi_1/h}r_1)\overline{\partial}^2(F_{\overline{A_2}}e^{\overline{\Phi_2}/h}\widetilde{r}_2)\,dx,$$
$$S_{4,3} := 8\int_U G\mathbf{v}^*\partial\overline{\partial}(F_{A_1}^{-1}e^{\Phi_1/h}r_1)\partial\overline{\partial}(F_{\overline{A_2}}e^{\overline{\Phi_2}/h}\widetilde{r}_2)\,dx.$$

Now, for $S_{4,1}$, direct computations yields that

$$\left|S_{4,1}\right| = 4\left|\int_U G\mathbf{v}^*\overline{\partial}^2(F_{A_1}^{-1}r_1)\partial^2(F_{\overline{A_2}}\widetilde{r}_2)e^{(\Phi_1+\overline{\Phi_2})/h}\,dx\right| = \mathcal{O}(h^{-1/2+\epsilon}),$$

where we use the better estimate for r_1 and (5.26) for $\widetilde{r}_2$. This implies that

$$\lim_{h\to 0}\left(hS_{4,1}\right) = 0.$$

For $S_{4,2}$, we can apply a similar method as in Step 1, then an integration by parts gives

$$\begin{aligned}S_{4,2} = {} & 4\int_U \partial\overline{\partial}\big(G\mathbf{v}^*\big)\partial(F_{A_1}^{-1}e^{\Phi_1/h}r_1)\overline{\partial}(F_{\overline{A_2}}e^{\overline{\Phi_2}/h}\widetilde{r}_2)\,dx\\
&+4\int_U \partial\big(G\mathbf{v}^*\big)\partial\big(e^{\Phi_1/h}\overline{\partial}(F_{A_1}^{-1}r_1)\big)\overline{\partial}(F_{\overline{A_2}}e^{\overline{\Phi_2}/h}\widetilde{r}_2)\,dx\\
&+4\int_U \overline{\partial}\big(G\mathbf{v}^*)\big)\partial(F_{A_1}^{-1}e^{\Phi_1/h}r_1)\overline{\partial}\big(e^{\overline{\Phi_2}/h}\partial(F_{\overline{A_2}}\widetilde{r}_2)\big)\,dx\\
&+4\int_U G\mathbf{v}^*\partial\big(e^{\Phi_1/h}\overline{\partial}(F_{A_1}^{-1}r_1)\big)\overline{\partial}\big(e^{\overline{\Phi_2}/h}\partial(F_{\overline{A_2}}\widetilde{r}_2)\big)\,dx.\end{aligned}$$

Using (5.20) as in the previous step, it is not hard to see the above integral is of $\mathcal{O}(h^{-1/2+\epsilon})$, which implies

$$\lim_{h\to 0}\left(hS_{4,2}\right) = 0.$$

Similarly arguments can be used in the derivation of $S_{4,3}$, and we can conclude that

$$\lim_{h\to 0}\left(hS_4\right) = 0 \tag{7.30}$$

as wanted. Hence, using (7.21), (7.24), (7.28) and (7.30), we can summarize that

$$\begin{aligned}&\lim_{h\to 0}\left(h\int_U G\mathbf{v}^*\operatorname{tr}\left(\big(D^2\mathbf{v}^{(1)}\big)\big(D^2\mathbf{v}^{(2)}\big)\right)dx\right)\\
&=\pi\overline{\partial}\big(F_{A^*}^{-1}\overline{\partial}\widetilde{G}_0 + F_{A^*}^{-1}(\mathbf{X}_g\cdot\nabla\Phi^*)\widetilde{G}_0\big)\big|_{z=0} - 16\pi\overline{\partial}\widetilde{G}_1\big|_{z=0},\end{aligned}$$

where $\mathbf{v}^{(1)}, \mathbf{v}^{(2)}, \mathbf{v}^*$ are the CGO solutions described as before. It remains to analyze the integral of Y.

Step 5. Analysis of the integral of Y.

Using the relation (7.6), we can write

$$\int_U Y\,dx := S_5 + S_6 + S_7,$$

where

$$
\begin{aligned}
S_5 &:= \int_U \mu^{-1}\mathbf{v}^* \operatorname{tr}\left(D^2\mathbf{v}^{(1)}\mathcal{C}\cdot\nabla\mathbf{v}^{(2)}\right) dx,\\
S_6 &:= \int_U \mu^{-1}\mathbf{v}^* \operatorname{tr}\left(\mathcal{C}\cdot\nabla\mathbf{v}^{(1)}D^2\mathbf{v}^{(2)}\right) dx,\\
S_7 &:= \int_U \mu^{-1}\big(1-c^{-2}\big)\mathbf{v}^* \operatorname{tr}\left(C\cdot\nabla\mathbf{v}^{(1)}C\cdot\nabla\mathbf{v}^{(2)}\right) dx.
\end{aligned}
$$

Similar to the previous analysis, we exploit the structures of the CGO solutions $\mathbf{v}^*$, $\mathbf{v}^{(1)}$, and $\mathbf{v}^{(2)}$. Since S_5 and S_6 share the same structural form, we analyze them jointly.

Step 6. Analysis of S_5 and S_6.

Recall the tensor function $\mathcal{C}=(\mathcal{C}^k_{ab})_{1\le a,b,k\le 2}$, which is independent of $h>0$. Using the CGO solutions $\mathbf{v}^{(k)}$ for $k=1,2$, the leading terms in S_5 and S_6 can be written as

$$
\begin{aligned}
&\operatorname{tr}\left(D^2\mathbf{v}^{(1)}\mathcal{C}\cdot\nabla\mathbf{v}^{(2)}\right)\\
&= \operatorname{tr}\left(D^2\big(F_{A_1}^{-1}e^{\Phi_1/h}\big)\mathcal{C}\cdot\nabla\big(F_{\overline{A_2}}e^{\overline{\Phi_2}/h}\big)\right) + \operatorname{tr}\left(D^2\big(e^{\Phi_1/h}r_1\big)\mathcal{C}\cdot\nabla\big(F_{\overline{A_2}}e^{\overline{\Phi_2}/h}\big)\right)\\
&\quad + \operatorname{tr}\left(D^2\big(F_{A_1}^{-1}e^{\Phi_1/h}\big)\mathcal{C}\cdot\nabla\big(F_{\overline{A_2}}e^{\overline{\Phi_2}/h}\widetilde{r}_2\big)\right)\\
&\quad + \operatorname{tr}\left(D^2\big(F_{A_1}^{-1}e^{\Phi_1/h}r_1\big)\mathcal{C}\cdot\nabla\big(F_{\overline{A_2}}e^{\overline{\Phi_2}/h}\widetilde{r}_2\big)\right)
\end{aligned}
$$

and

$$
\begin{aligned}
&\operatorname{tr}\left(\mathcal{C}\cdot\nabla\mathbf{v}^{(1)}D^2\mathbf{v}^{(2)}\right)\\
&= \operatorname{tr}\left(\mathcal{C}\cdot\nabla\big(F_{A_1}^{-1}e^{\Phi_1/h}\big)D^2\big(F_{\overline{A_2}}e^{\overline{\Phi_2}/h}\big)\right) + \operatorname{tr}\left(\mathcal{C}\cdot\nabla\big(F_{A_1}^{-1}e^{\Phi_1/h}r_1\big)D^2\big(F_{\overline{A_2}}e^{\overline{\Phi_2}/h}\big)\right)\\
&\quad + \operatorname{tr}\left(\mathcal{C}\cdot\nabla\big(F_{A_1}^{-1}e^{\Phi_1/h}\big)\big(D^2\big(F_{\overline{A_2}}e^{\overline{\Phi_2}/h}\widetilde{r}_2\big)\big)\right)\\
&\quad + \operatorname{tr}\left(\mathcal{C}\cdot\nabla\big(F_{A_1}^{-1}e^{\Phi_1/h}r_1\big)D^2\big(F_{\overline{A_2}}e^{\overline{\Phi_2}/h}\widetilde{r}_2\big)\right).
\end{aligned}
$$

Then we can write S_5 and S_6 into

$$
\begin{aligned}
S_5 &:= S_{5,1}+S_{5,2}+S_{5,3}+S_{5,4},\\
S_6 &:= S_{6,1}+S_{6,2}+S_{6,3}+S_{6,4},
\end{aligned}
$$

where

$$
\begin{aligned}
S_{5,1} &:= \int_U \mu^{-1}\mathbf{v}^*\left[\operatorname{tr}\left(D^2\big(F_{A_1}^{-1}e^{\Phi_1/h}\big)\mathcal{C}\cdot\nabla\big(F_{\overline{A_2}}e^{\overline{\Phi_2}/h}\big)\right)\right] dx,\\
S_{5,2} &:= \int_U \mu^{-1}\mathbf{v}^*\left[\operatorname{tr}\left(D^2\big(F_{A_1}^{-1}e^{\Phi_1/h}r_1\big)\mathcal{C}\cdot\nabla\big(F_{\overline{A_2}}e^{\overline{\Phi_2}/h}\big)\right)\right] dx,\\
S_{5,3} &:= \int_U \mu^{-1}\mathbf{v}^*\left[\operatorname{tr}\left(D^2\big(F_{A_1}^{-1}e^{\Phi_1/h}\big)\mathcal{C}\cdot\nabla\big(F_{\overline{A_2}}e^{\overline{\Phi_2}/h}\widetilde{r}_2\big)\right)\right] dx,\\
S_{5,4} &:= \int_U \mu^{-1}\mathbf{v}^*\left[\operatorname{tr}\left(D^2\big(F_{A_1}^{-1}e^{\Phi_1/h}r_1\big)\mathcal{C}\cdot\nabla\big(F_{\overline{A_2}}e^{\overline{\Phi_2}/h}\widetilde{r}_2\big)\right)\right] dx,
\end{aligned}
$$

and

$$
\begin{aligned}
S_{6,1} &:= \int_U \mu^{-1}\mathbf{v}^*\big[\operatorname{tr}\big(\mathcal{C}\cdot\nabla\big(F_{A_1}^{-1}e^{\Phi_1/h}\big)D^2\big(F_{\overline{A_2}}e^{\overline{\Phi_2}/h}\big)\big)\big]\,dx,\\
S_{6,2} &:= \int_U \mu^{-1}\mathbf{v}^*\big[\operatorname{tr}\big(\mathcal{C}\cdot\nabla\big(F_{A_1}^{-1}e^{\Phi_1/h}r_1\big)D^2\big(F_{\overline{A_2}}e^{\overline{\Phi_2}/h}\big)\big)\big]\,dx,\\
S_{6,3} &:= \int_U \mu^{-1}\mathbf{v}^*\big[\operatorname{tr}\big(\mathcal{C}\cdot\nabla\big(F_{A_1}^{-1}e^{\Phi_1/h}\big)D^2\big(F_{\overline{A_2}}e^{\overline{\Phi_2}/h}\widetilde{r}_2\big)\big)\big]\,dx,\\
S_{6,4} &:= \int_U \mu^{-1}\mathbf{v}^*\big[\operatorname{tr}\big(\mathcal{C}\cdot\nabla\big(F_{A_1}^{-1}e^{\Phi_1/h}r_1\big)D^2\big(F_{\overline{A_2}}e^{\overline{\Phi_2}/h}\widetilde{r}_2\big)\big)\big]\,dx.
\end{aligned}
$$

Step 6-1: Analysis of $S_{5,1}$ and $S_{6,1}$.

Let us use the same technique in previous steps, using (2.6) and the stationary phase expansion (7.10), then direct computations give

$$
\begin{aligned}
S_{5,1} &= \sum_{k=1}^2 \int_U \mu^{-1}\mathbf{v}^*\partial^2\big(F_{A_1}^{-1}e^{\Phi_1/h}\big)\partial_{x_k}\big(F_{\overline{A_2}}e^{\overline{\Phi_2}/h}\big)\operatorname{tr}\big(A\mathcal{C}^k\big)\,dx\\
&\quad+\sum_{k=1}^2 \int_U \mu^{-1}\mathbf{v}^*\overline{\partial}^2\big(F_{A_1}^{-1}e^{\Phi_1/h}\big)\partial_{x_k}\big(F_{\overline{A_2}}e^{\overline{\Phi_2}/h}\big)\operatorname{tr}\big(B\mathcal{C}^k\big)\,dx\\
&\quad+2\sum_{k=1}^2 \int_U \mu^{-1}\mathbf{v}^*\partial\overline{\partial}\big(F_{A_1}^{-1}e^{\Phi_1/h}\big)\partial_{x_k}\big(F_{\overline{A_2}}e^{\overline{\Phi_2}/h}\big)\operatorname{tr}\big(\mathcal{C}^k\big)\,dx,
\end{aligned}
$$

where A is the 2×2 complex-valued matrix given in (2.7), and $\mathcal{C}^k = \big(\mathcal{C}^k_{ab}\big)_{1\le a,b\le 2}$ is given by (7.7), for $k=1,2$. Using (2.2), we note that $\partial_{x_1}e^{\overline{\Phi_2}/h} = (\partial+\overline{\partial})e^{\overline{\Phi_2}/h} = -\frac{1}{2h}\overline{z}e^{\overline{\Phi_2}/h}$ and $\partial_{x_2}e^{\overline{\Phi_2}/h} = \mathsf{i}(\partial-\overline{\partial})e^{\overline{\Phi_2}/h} = \frac{\mathsf{i}}{2h}\overline{z}e^{\overline{\Phi_2}/h}$ so that $\partial_{x_k}\big(e^{\overline{\Phi_2}/h}\big)$ will contribute a factor of $\overline{z}$ for $k=1,2$. Moreover, as in the previous steps, the governing terms arise when the derivatives act on $e^{\Phi_1/h}$ and $e^{\overline{\Phi_2}/h}$. Thus, we can write $S_{5,1} := S^m_{5,1} + S^r_{5,1}$, where

$$
S^m_{5,1} := \sum_{k=1}^2 \int_U \mu^{-1}\mathbf{v}^* F_{A_1}^{-1}F_{\overline{A_2}}\partial^2\big(e^{\Phi_1/h}\big)\partial_{x_k}\big(e^{\overline{\Phi_2}/h}\big)\operatorname{tr}\big(A\mathcal{C}^k\big)\,dx,
$$

and $S^r_{5,1} = S_{5,1} - S^m_{5,1}$ is of lower order, as it contains an additional factor of h. Here, we use $\overline{\partial}^2\big(F_{A_1}^{-1}e^{\Phi_1/h}\big) = e^{\Phi_1/h}\overline{\partial}^2F_{A_1}^{-1}$ and $\partial\overline{\partial}\big(F_{A_1}^{-1}e^{\Phi_1/h}\big) = \partial\big(e^{\Phi_1/h}\overline{\partial}F_{A_1}^{-1}\big)$ that produces an extra h factor.

Therefore, the stationary phase expansion (7.10) can be applied to compute $S^m_{5,1}$ such that

(7.31)

$$
\begin{aligned}
S^m_{5,1} &= \sum_{k=1}^2 \int_U \widetilde{\mu}\mathbf{v}^*\partial^2\big(e^{\Phi_1/h}\big)\partial_{x_k}\big(e^{\overline{\Phi_2}/h}\big)\operatorname{tr}\big(A\mathcal{C}^k\big)\,dx\\
&= \frac{1}{h}\int_U \widetilde{\mu}(1+r^*)\Big[\frac{1}{h^2}\big(1+\frac{z}{4}\big)^2+\frac{1}{4h}\Big]\overline{z}\Big[\frac{-1}{2}\operatorname{tr}\big(A\mathcal{C}^1\big)+\frac{\mathsf{i}}{2}\operatorname{tr}\big(A\mathcal{C}^2\big)\Big]e^{\mathsf{i}x_1x_2/h}\,dx\\
&= \frac{2\pi}{h}\Big(\partial^2-\overline{\partial}^2\Big)\Big\{\widetilde{\mu}\big(1+\frac{z}{4}\big)^2\overline{z}\Big[\frac{-1}{2}\operatorname{tr}\big(A\mathcal{C}^1\big)+\frac{\mathsf{i}}{2}\operatorname{tr}\big(A\mathcal{C}^2\big)\Big]\Big\}\Big|_{z=0}+\mathcal{O}(1)\\
&= \frac{2\pi}{h}\overline{\partial}^2\Big\{\widetilde{\mu}\big(1+\frac{z}{4}\big)^2\overline{z}\Big[\frac{1}{2}\operatorname{tr}\big(A\mathcal{C}^1\big)-\frac{\mathsf{i}}{2}\operatorname{tr}\big(A\mathcal{C}^2\big)\Big]\Big\}\Big|_{z=0}+\mathcal{O}(1)\\
&= \frac{2\pi}{h}\overline{\partial}\Big\{\mu^{-1}\big(1+\frac{z}{4}\big)^2\Big[\frac{1}{2}\operatorname{tr}\big(A\mathcal{C}^1\big)-\frac{\mathsf{i}}{2}\operatorname{tr}\big(A\mathcal{C}^2\big)\Big]\Big\}\Big|_{z=0}+\mathcal{O}(1)\\
&= \frac{\pi}{h}\overline{\partial}\big\{\widetilde{\mu}\big[\operatorname{tr}\big(A\mathcal{C}^1\big)-\mathsf{i}\operatorname{tr}\big(A\mathcal{C}^2\big)\big]\big\}\big|_{z=0}+\mathcal{O}(1)
\end{aligned}
$$

as $h \to 0$, where

$$\widetilde{\mu} := \mu^{-1} F_{A_1}^{-1} F_{\overline{A_2}} \tag{7.32}$$

is a bounded function independent of $h > 0$. Meanwhile, it is clear that $S_{5,1}^r = \mathcal{O}(1)$, and we omit the derivation.

Similar analysis can be utilized for $S_{6,1}$. By writing $S_{6,1} := S_{6,1}^m + S_{6,1}^r$, where

$$\begin{aligned} S_{6,1}^m &:= \sum_{k=1}^2 \int_U \widetilde{\mu} \mathbf{v}^* \partial_{x_k} \left(e^{\Phi_1/h}\right) \overline{\partial}^2 \left(e^{\overline{\Phi_2}/h}\right) \operatorname{tr}(B\mathcal{C}^k)\, dx \\ &= \frac{1}{h} \int_U \overline{\partial}^2 \left[\widetilde{\mu}(1+r^*)\left(\operatorname{tr}(B\mathcal{C}^1) + \mathrm{i} \operatorname{tr}(B\mathcal{C}^2)\right)\right] \left(1 + \frac{z}{4}\right) e^{\mathrm{i}x_1x_2/h}\, dx \\ &= \mathcal{O}(1) \quad \text{as} \quad h \to 0, \end{aligned} \tag{7.33}$$

where we have applied twice integration by parts in the above computations, and $\widetilde{\mu}$ is given by (7.32). Here we used the Dirichlet and Neumann data of $\mathcal{C}^k$ are zero, for $k = 1, 2$. To summarize, on the one hand, using (7.31) and $S_{5,1}^r = \mathcal{O}(1)$, one can see that

$$\lim_{h \to 0} \left(hS_{5,1}\right) = \pi \overline{\partial} \left\{ \widetilde{\mu} \left[\operatorname{tr}\left(A\mathcal{C}^1\right) - \mathrm{i} \operatorname{tr}\left(A\mathcal{C}^2\right)\right]\right\}\Big|_{z=0}.$$

On the other hand, with (7.33) at hand, we can ensure

$$\lim_{h \to 0} \left(hS_{6,1}\right) = 0,$$

since $S_{6,1}^r$ has a better decay in h than $S_{6,1}^m$.

Step 6-2. Analysis of $S_{5,2}$ and $S_{6,2}$.

Note that the difference between $S_{5,2}$ and $S_{5,1}$ lies in the presence of an additional remainder term r_1 in the integrand, which enjoys better decay properties and admits a favorable asymptotic expansion. On the one hand, when the derivative does not act on r_1, the analysis proceeds in the same way as the analysis for S_2: applying the stationary phase method (7.10) yields the desired vanishing limit. On the other hand, if the derivative falls on r_1, we can still invoke the stationary phase expansion, as an additional factor of $\overline{z}$ is always present due to the absence of any remainder term $\widetilde{r}_2$ in these computations.

A similar strategy applies to $S_{6,2}$. By integrating by parts in the $\overline{\partial}$ operator and exploiting the holomorphic properties of $\mathbf{v}^*$ and $\mathbf{v}^{(1)}$ from their phase functions, the analysis mirrors that of (7.33). As the arguments follow closely, we omit further details. In summary, we can obtain

$$\lim_{h \to 0} \left(hS_{5,2}\right) = \lim_{h \to 0} \left(hS_{6,2}\right) = 0.$$

It remains to analyze the terms $S_{5,3}$, $S_{5,4}$, $S_{6,3}$, and $S_{6,4}$, which involve the remainder term r_2.

Step 6-3. Analysis of $S_{5,3}$ and $S_{6,3}$.

Let us first analyze $S_{5,3}$ with the same strategy by using (2.6) as before, then we can compute each term in $S_{5,3}$ as

$$S_{5,3} := S_{5,3}^m + S_{5,3}^r,$$

where

$$\begin{aligned} S_{5,3}^m &:= \sum_{k=1}^2 \int_U \widetilde{\mu}\mathbf{v}^* \operatorname{tr}(A\mathcal{C}^k)\partial^2\big(e^{\Phi_1/h}\big)\partial_{x_k}\big(e^{\overline{\Phi_2}/h}\widetilde{r}_2\big)\big)\,dx \\ &= \int_U \widetilde{\mu}\mathbf{v}^* \operatorname{tr}(A\mathcal{C}^1)\partial^2\big(e^{\Phi_1/h}\big)\big(\partial+\overline{\partial}\big)\big(e^{\overline{\Phi_2}/h}\widetilde{r}_2\big)\big)\,dx \\ &\quad + \mathsf{i}\int_U \widetilde{\mu}\mathbf{v}^* \operatorname{tr}(A\mathcal{C}^2)\partial^2\big(e^{\Phi_1/h}\big)\big(\partial-\overline{\partial}\big)\big(e^{\overline{\Phi_2}/h}\widetilde{r}_2\big)\big)\,dx. \end{aligned}$$

Here, the term $S_{5,3}^r$ consists of those contributions in which at least one derivative acts on either $F_{A_1}^{-1}$ or $F_{\overline{A_2}}$, and $\widetilde{\mu}$ is the function defined in (7.32).

By writing $S_{5,3}^m := \sum_{k,\ell=1}^2 S_{5,3}^{k,\ell}$, such that

$$\begin{aligned} S_{5,3}^{1,1} &:= \int_U \widetilde{\mu}\mathbf{v}^* \operatorname{tr}(A\mathcal{C}^1)\partial^2\big(e^{\Phi_1/h}\big)\overline{\partial}\big(e^{\overline{\Phi_2}/h}\widetilde{r}_2\big)\big)\,dx, \\ S_{5,3}^{1,2} &:= -\mathsf{i}\int_U \widetilde{\mu}\mathbf{v}^* \operatorname{tr}(A\mathcal{C}^2)\partial^2\big(e^{\Phi_1/h}\big)\overline{\partial}\big(e^{\overline{\Phi_2}/h}\widetilde{r}_2\big)\big)\,dx, \\ S_{5,3}^{2,1} &:= \int_U \widetilde{\mu}\mathbf{v}^* \operatorname{tr}(A\mathcal{C}^1)\partial^2\big(e^{\Phi_1/h}\big)\partial\big(e^{\overline{\Phi_2}/h}\widetilde{r}_2\big)\big)\,dx, \\ S_{5,3}^{2,2} &:= \mathsf{i}\int_U \widetilde{\mu}\mathbf{v}^* \operatorname{tr}(A\mathcal{C}^2)\partial^2\big(e^{\Phi_1/h}\big)\partial\big(e^{\overline{\Phi_2}/h}\widetilde{r}_2\big)\big)\,dx. \end{aligned}$$

Let us only analyze $S_{5,3}^{1,1}$, and $S_{5,3}^{2,1}$ has a similar structure. Applying the integration by parts, one can obtain

$$\begin{aligned} S_{5,3}^{1,1} &= \int_U \widetilde{\mu}\mathbf{v}^* \operatorname{tr}(A\mathcal{C}^1)\partial^2\big(e^{\Phi_1/h}\big)\overline{\partial}\big(e^{\overline{\Phi_2}/h}\widetilde{r}_2\big)\big)\,dx \\ &= -\int_U \overline{\partial}\big[\widetilde{\mu}\mathbf{v}^* \operatorname{tr}(A\mathcal{C}^1)\big]\partial^2\big(e^{\Phi_1/h}\big)e^{\overline{\Phi_2}/h}\widetilde{r}_2\,dx \\ &= -\int_U \overline{\partial}\big[\widetilde{\mu}\operatorname{tr}(A\mathcal{C}^1)\big](1+r^*)\Big(\frac{1}{h^2}\big(1+\frac{z}{4}\big)^2+\frac{1}{4h}\Big)e^{\mathrm{i}x_1x_2/h}\widetilde{r}_2\,dx \\ &\quad \underbrace{-\int_U \widetilde{\mu}\operatorname{tr}(A\mathcal{C}^1)\overline{\partial}r^*\Big(\frac{1}{h^2}\big(1+\frac{z}{4}\big)^2+\frac{1}{4h}\Big)e^{\mathrm{i}x_1x_2/h}\widetilde{r}_2\,dx}_{\text{By } \overline{\partial}(e^{\Phi^*/h}(1+r^*))=e^{\Phi^*/h}\overline{\partial}r^*}. \end{aligned} \tag{7.34}$$

It is easy to see that the second term in the right-hand side of (7.34) is of order $\mathcal{O}(h^{-1/2+\epsilon})$, and we only need to consider the first term in the right-hand side of (7.34). To this end, we can apply [CLLT23, Proposition 3.9], such that the first term is of order $o(1/h)$. Therefore, we can have

$$S_{5,3}^{1,1} = \mathcal{O}(h^{-1+\epsilon}), \text{ as } h\to 0,$$

which leads

$$\lim_{h\to 0}\big(hS_{5,3}^{1,1}\big)=0.$$

Similar arguments can be applied to $S_{5,3}^{1,2}$, and one can conclude

$$\lim_{h\to 0}\big(hS_{5,3}^{1,1}\big)=\lim_{h\to 0}\big(hS_{5,3}^{1,2}\big)=0.$$

When ∂ hits $e^{\overline{\Phi_2}/h}r_2$, the analysis will become more complicated. Before analyzing $S_{5,3}^{2,1}$ and $S_{5,3}^{2,2}$, let us look into $S_{6,3}$.

Similar to the analysis for $S_{5,3}$, let us write $S_{6,3} = S_{6,3}^m + S_{6,3}^r$, where

$$S_{6,3}^m := \int_U \widetilde{\mu}\mathbf{v}^*\big[\operatorname{tr}\big(\mathcal{C}\cdot\nabla\big(e^{\Phi_1/h}\big)\big(D^2\big(e^{\overline{\Phi_2}/h}\widetilde{r}_2\big)\big)\big)\big]\,dx := S_{6,3}^1+S_{6,3}^2+S_{6,3}^3,$$

and $S^r_{6,3} = S_{6,3} - S^m_{6,3}$ contains those contributions in which at least one derivative acts on either $F^{-1}_{A_1}$ or $F_{\overline{A_2}}$. Thanks to (2.6) again, we can write $S^m_{6,3} := S^1_{6,3} + S^2_{6,3} + S^3_{6,3}$, where

$$\begin{aligned} S^1_{6,3} &:= \sum_{k=1}^{2} \int_U \widetilde{\mu}\mathbf{v}^* \operatorname{tr}\left(B\mathcal{C}^k\right)\partial_{x_k}\left(e^{\Phi_1/h}\right)\overline{\partial}^2\left(e^{\overline{\Phi_2}/h}\widetilde{r}_2\right)dx := S^{1,1}_{6,3} + S^{1,2}_{6,3}, \\ S^2_{6,3} &:= \sum_{k=1}^{2} \int_U \widetilde{\mu}\mathbf{v}^* \operatorname{tr}\left(A\mathcal{C}^k\right)\partial_{x_k}\left(e^{\Phi_1/h}\right)\partial^2\left(e^{\overline{\Phi_2}/h}\widetilde{r}_2\right)dx := S^{2,1}_{6,3} + S^{2,2}_{6,3}, \\ S^3_{6,3} &:= 2\sum_{k=1}^{2} \int_U \widetilde{\mu}\mathbf{v}^* \operatorname{tr}\left(\mathcal{C}^k\right)\partial_{x_k}\left(e^{\Phi_1/h}\right)\partial\overline{\partial}\left(e^{\overline{\Phi_2}/h}\widetilde{r}_2\right)dx := S^{3,1}_{6,3} + S^{3,2}_{6,3}. \end{aligned}$$

Here

$$\begin{aligned} S^{1,1}_{6,3} &:= \int_U \widetilde{\mu}\mathbf{v}^* \operatorname{tr}\left(B\mathcal{C}^1\right)\partial\left(e^{\Phi_1/h}\right)\overline{\partial}^2\left(e^{\overline{\Phi_2}/h}\widetilde{r}_2\right)dx \\ S^{1,2}_{6,3} &:= \mathsf{i}\int_U \widetilde{\mu}\mathbf{v}^* \operatorname{tr}\left(B\mathcal{C}^2\right)\partial\left(e^{\Phi_1/h}\right)\overline{\partial}^2\left(e^{\overline{\Phi_2}/h}\widetilde{r}_2\right)dx, \end{aligned}$$

$$\begin{aligned} S^{2,1}_{6,3} &:= \int_U \widetilde{\mu}\mathbf{v}^* \operatorname{tr}\left(A\mathcal{C}^1\right)\partial\left(e^{\Phi_1/h}\right)\partial^2\left(e^{\overline{\Phi_2}/h}\widetilde{r}_2\right)dx, \\ S^{2,2}_{6,3} &:= \mathsf{i}\int_U \widetilde{\mu}\mathbf{v}^* \operatorname{tr}\left(A\mathcal{C}^2\right)\partial\left(e^{\Phi_1/h}\right)\partial^2\left(e^{\overline{\Phi_2}/h}\widetilde{r}_2\right)dx \end{aligned}$$

and

$$\begin{aligned} S^{3,1}_{6,3} &:= 2\int_U \widetilde{\mu}\mathbf{v}^* \operatorname{tr}\left(\mathcal{C}^1\right)\partial\left(e^{\Phi_1/h}\right)\partial\overline{\partial}\left(e^{\overline{\Phi_2}/h}\widetilde{r}_2\right)dx, \\ S^{3,2}_{6,3} &:= 2\mathsf{i}\int_U \widetilde{\mu}\mathbf{v}^* \operatorname{tr}\left(\mathcal{C}^2\right)\partial\left(e^{\Phi_1/h}\right)\partial\overline{\partial}\left(e^{\overline{\Phi_2}/h}\widetilde{r}_2\right)dx. \end{aligned}$$

For $S^{1,1}_{6,3}$, via twice integration by parts for $\overline{\partial}$, one can obtain

$$S^{1,1}_{6,3} = \frac{1}{h}\int_U \overline{\partial}^2\left(\widetilde{\mu}\mathbf{v}^* \operatorname{tr}(B\mathcal{C}^1)\right)\left(1+\frac{z}{4}\right)e^{(\Phi_1+\overline{\Phi_2})/h}\widetilde{r}_2\,dx = \mathcal{O}(h^{-1/2+\epsilon}),$$

where we used $r_2 = \mathcal{O}_{L^2}(h^{1/2+\epsilon})$ and the term $\overline{\partial}^2\left(\widetilde{\mu}\mathbf{v}^* \operatorname{tr}(B\mathcal{C}^1)\right)$ will not generate extra $1/h$ since its phase is holormorphic. Similar assertion holds for $S^{1,2}_{6,3}$, so we can conclude

$$\lim_{h\to 0}\left(hS^1_{6,3}\right) = 0.$$

Recalling that $S^2_{6,3}$ can be written as $S^2_{6,3} = S^{2,1}_{6,3} + S^{2,2}_{6,3}$, let us first analyze $S^{2,1}_{6,3}$. Applying an integration by parts formula, one has

$$\begin{aligned} S^{2,1}_{6,3} = &-\int_U \partial\left(\widetilde{\mu}\mathbf{v}^* \operatorname{tr}\left(A\mathcal{C}^1\right)\right)\partial\left(e^{\Phi_1/h}\right)\partial\left(e^{\overline{\Phi_2}/h}\widetilde{r}_2\right)dx \\ &\underbrace{-\int_U \widetilde{\mu}\mathbf{v}^* \operatorname{tr}\left(A\mathcal{C}^1\right)\partial^2\left(e^{\Phi_1/h}\right)\partial\left(e^{\overline{\Phi_2}/h}\widetilde{r}_2\right)dx}_{=S^{2,1}_{5,3}}, \end{aligned}$$

which implies

$$S^{2,1}_{5,3} + S^{2,1}_{6,3} = -\int_U \partial\left(\widetilde{\mu}\mathbf{v}^* \operatorname{tr}\left(A\mathcal{C}^1\right)\right)\partial\left(e^{\Phi_1/h}\right)\partial\left(e^{\overline{\Phi_2}/h}\widetilde{r}_2\right)dx.$$

Now, for the right-hand side in the above equation, direct computations imply

$$
\begin{aligned}
&\int_U \partial\big(\widetilde{\mu}\mathbf{v}^* \operatorname{tr}\big(A\mathcal{C}^1\big)\big)\partial\big(e^{\Phi_1/h}\big)\partial\big(e^{\overline{\Phi_2}/h}\widetilde{r}_2\big)\, dx \\
&= \int_U \partial\big(\widetilde{\mu}\operatorname{tr}\big(A\mathcal{C}^1\big)\big)\mathbf{v}^*\partial\big(e^{\Phi_1/h}\big)e^{\overline{\Phi_2}/h}\partial\widetilde{r}_2\, dx \\
&\quad + \int_U \widetilde{\mu}\operatorname{tr}\big(A\mathcal{C}^1\big)\partial\mathbf{v}^*\partial\big(e^{\Phi_1/h}\big)e^{\overline{\Phi_2}/h}\partial\widetilde{r}_2\, dx \\
&= \frac{1}{h}\int_U \partial\big(\widetilde{\mu}\operatorname{tr}\big(A\mathcal{C}^1\big)\big)\mathbf{v}^*\big(1+\frac{z}{4}\big)e^{(\Phi_1+\overline{\Phi_2})/h}\partial\widetilde{r}_2\, dx \\
&\quad + \frac{1}{h}\int_U \widetilde{\mu}\operatorname{tr}\big(A\mathcal{C}^1\big)\partial\big(e^{\Phi^*/h}r^*\big)\big(1+\frac{z}{4}\big)e^{(\Phi_1+\overline{\Phi_2})/h}\partial\widetilde{r}_2\, dx \\
&\quad + \frac{1}{h^2}\int_U \widetilde{\mu}\operatorname{tr}\big(A\mathcal{C}^1\big)\big(-1+\frac{z^2}{16}\big)e^{\mathrm{i}x_1x_2/h}\partial\widetilde{r}_2\, dx.
\end{aligned}
\tag{7.35}
$$

It is easy to see that the first two terms in the right-hand side of (7.35) can be estimated by $\mathcal{O}(h^{-1/2+\epsilon})$, which gives rise to

$$
\begin{aligned}
&\lim_{h\to 0}\int_U \partial\big(\widetilde{\mu}\operatorname{tr}\big(A\mathcal{C}^1\big)\big)\mathbf{v}^*\big(1+\frac{z}{4}\big)e^{(\Phi_1+\overline{\Phi_2})/h}\partial\widetilde{r}_2\, dx \\
&= \lim_{h\to 0}\int_U \widetilde{\mu}\operatorname{tr}\big(A\mathcal{C}^1\big)\partial\big(e^{\Phi^*/h}r^*\big)\big(1+\frac{z}{4}\big)e^{(\Phi_1+\overline{\Phi_2})/h}\partial\widetilde{r}_2\, dx \\
&= 0.
\end{aligned}
$$

Thus, it remains to estimate the last term in the right-hand side of (7.35).

For a certain term, using the relation

$$
\widetilde{r}_2 = -\partial_\psi^{-1}\widetilde{s}_2 = -\partial^{-1}e^{-\mathrm{i}x_1x_2/h}\widetilde{s}_2,
$$

we can conclude that

$$
\begin{aligned}
&\lim_{h\to 0}\left(\frac{1}{h}\int_U \widetilde{\mu}\operatorname{tr}\big(A\mathcal{C}^1\big)\big(-1+\frac{z^2}{16}\big)e^{\mathrm{i}x_1x_2/h}\partial\widetilde{r}_2\, dx\right) \\
&= -\lim_{h\to 0}\left(\frac{1}{h}\int_U \widetilde{\mu}\operatorname{tr}\big(A\mathcal{C}^1\big)\big(-1+\frac{z^2}{16}\big)e^{\mathrm{i}x_1x_2/h}\,\underbrace{\partial\partial^{-1}\big(e^{-\mathrm{i}x_1x_2/h}V'\widetilde{s}_2\big)}_{=e^{-\mathrm{i}x_1x_2/h}V'\widetilde{s}_2}\, dx\right) \\
&= -\lim_{h\to 0}\left(\frac{1}{h}\int_U \widetilde{\mu}\operatorname{tr}\big(A\mathcal{C}^1\big)\big(-1+\frac{z^2}{16}\big)V'\widetilde{s}_2\, dx\right) \\
&= -\lim_{h\to 0}\left[\frac{1}{h}\int_U \widetilde{\mu}\operatorname{tr}(A\mathcal{C}^1)\big(-1+\frac{z^2}{16}\big)V'\Big(\partial^{*-1}\big(e^{\mathrm{i}x_1x_2/h}V\big)\right. \\
&\qquad\qquad \left. + \sum_{k=1}^{\infty}\widetilde{T}_h^k\partial^{*-1}\big(e^{\mathrm{i}x_1x_2/h}V\big)\Big)\right] dx
\end{aligned}
\tag{7.36}
$$

Let us look at the first term in the right-hand side of (7.36). Applying the integration by parts formula (7.26) and the stationary phase formula (7.10), we have

$$
\begin{aligned}
&\frac{1}{h}\int_U \widetilde{\mu}\operatorname{tr}(A\mathcal{C}^1)\big(-1+\frac{z^2}{16}\big)V'\partial^{*-1}\big(e^{\mathrm{i}x_1x_2/h}V\big)\, dx \\
&= -\frac{1}{h}\int_U \partial^{*-1}\Big(\widetilde{\mu}\operatorname{tr}(A\mathcal{C}^1)\big(-1+\frac{z^2}{16}\big)V'\Big)\big(e^{\mathrm{i}x_1x_2/h}V\big)\, dx \\
&\longrightarrow -2\pi\partial^{*-1}\Big(\widetilde{\mu}\operatorname{tr}(A\mathcal{C}^1)\big(-1+\frac{z^2}{16}\big)V'\Big)V\Big|_{z=0} \\
&= 2\pi\mathrm{i}\overline{\partial}^{-1}\Big(\widetilde{\mu}\operatorname{tr}(A\mathcal{C}^1)\big(-1+\frac{z^2}{16}\big)V'\Big)V\Big|_{z=0}
\end{aligned}
$$

as $h \to 0$. Recalling that here we also extended $\mathcal{C}^1$ by zero, outside Ω. This extension is C^2 thanks to the boundary determination (recall that since we are in holomorphic coordinates $\partial^{*-1} = \overline{\partial}^{-1}$). Since the first term in the right-hand side of (7.36) is of $\mathcal{O}(1)$, note that $\widetilde{T}_h$ satisfies (5.23), which ensures that the second term in the right-hand side of (7.36) decays faster than $\mathcal{O}(1)$ as $h \to 0$. Hence,

$$
\begin{aligned}
&\lim_{h\to 0}\left(\frac{1}{h}\int_U \widetilde{\mu}\,\mathrm{tr}\,(A\mathcal{C}^1)\big(-1+\frac{z^2}{16}\big)e^{\mathrm{i}x_1x_2/h}\partial\widetilde{r}_2\,dx\right)\\
&= 2\pi\mathrm{i}\overline{\partial}^{-1}\Big(\widetilde{\mu}\,\mathrm{tr}(A\mathcal{C}^1)\big(-1+\frac{z^2}{16}\big)V'\Big)V\Big|_{z=0}.
\end{aligned}
$$

Similarly, one can use the same derivation for $S_{6,3}^{2,2}$ together with $S_{5,3}^{2,2}$. Therefore, to summarize, we must have

$$
\begin{aligned}
&\lim_{h\to 0}\big[h\big(S_{5,3}^{2,1}+S_{6,3}^{2,1}\big)\big] = 2\pi\mathrm{i}\overline{\partial}^{-1}\Big(\widetilde{\mu}\,\mathrm{tr}(A\mathcal{C}^1)\big(-1+\frac{z^2}{16}\big)V'\Big)V\Big|_{z=0},\\
&\lim_{h\to 0}\big[h\big(S_{5,3}^{2,2}+S_{6,3}^{2,2}\big)\big] = -2\pi\overline{\partial}^{-1}\Big(\widetilde{\mu}\,\mathrm{tr}(A\mathcal{C}^2)\big(-1+\frac{z^2}{16}\big)V'\Big)V\Big|_{z=0}.
\end{aligned}
$$

Similar to previous methods, for $S_{6,3}^{3,1}$, an integration by parts for $\overline{\partial}$ yields

$$
S_{6,3}^{3,1} = -\int_U \underbrace{\overline{\partial}\big[\widetilde{\mu}\mathbf{v}^*\,\mathrm{tr}\,(\mathcal{C}^1)\big]}_{=\mathcal{O}_{L^2}(1)}\big(1+\frac{z}{4}\big)\partial e^{(\Phi_1+\overline{\Phi_2})/h}\partial\widetilde{r}_2\,dx = \mathcal{O}(h^{-1/2+\epsilon}),
$$

and same estimate holds for $S_{6,3}^{3,2}$. Now, we have $\lim_{h\to0}\big(hS_{5,3}^m\big) = \lim_{h\to0}\big(hS_{6,3}^m\big) = 0$, then this implies that the lower terms satisfy $\lim_{h\to0}\big(hS_{5,3}^r\big) = \lim_{h\to0}\big(hS_{6,3}^r\big) = 0$ as well. Hence, one has

$$
\lim_{h\to0}\big(hS_{5,3}^3\big) = \lim_{h\to0}\big(hS_{6,3}^3\big) = 0.
$$

Step 6-4. Analysis of $S_{5,4}$ and $S_{6,4}$.

For $S_{5,4}$, using (2.6) and similar to previous arguments, one can compute

$$
\partial_{x_k}\big(e^{\overline{\Phi_2}/h}r_2\big) = \mathcal{O}_{L^2}(h^{-1/2+\epsilon}),
$$

for $k=1,2$. Thanks to the better estimate for $r_1, \partial r_1, \overline{\partial} r_1 = \mathcal{O}_{L^2}(h)$, from the above computations, one can easily see that

$$
S_{5,4} = \int_U \widetilde{\mu}\mathbf{v}^*\big[\underbrace{\mathrm{tr}\,\big(\big(D^2\big(e^{\Phi_1/h}r_1\big)\big)\mathcal{C}}_{=\mathcal{O}_{L^2}(1)}\cdot\underbrace{\nabla\big(e^{\overline{\Phi_2}/h}\widetilde{r}_2\big)}_{=\mathcal{O}_{L^2}(h^{-1/2+\epsilon})}\big)\big]\,dx = \mathcal{O}(h^{-1/2+\epsilon}),
$$

which implies

$$
\lim_{h\to0}\big(hS_{5,4}\big) = 0.
$$

For $S_{6,4}$, note that $\nabla\big(e^{\Phi_1/h}r_1\big) = \mathcal{O}_{L^2}(1)$, by (2.6), then there holds that

$$
\begin{aligned}
S_{6,4} = \int_U &\mu^{-1}\mathbf{v}^*\big[\mathrm{tr}\,\big(\mathcal{C}\cdot\nabla\big(F_{A_1}^{-1}e^{\Phi_1/h}r_1\big)\\
&\cdot\big[A\partial^2(F_{\overline{A_2}}e^{\overline{\Phi_2}/h}\widetilde{r}_2) + B\overline{\partial}^2(F_{\overline{A_2}}e^{\overline{\Phi_2}/h}\widetilde{r}_2) + 2I_{2\times2}\partial\overline{\partial}(F_{\overline{A_2}}e^{\overline{\Phi_2}/h}\widetilde{r}_2)\big]\big\}\big]\,dx.
\end{aligned}
$$

As in the previous analysis, the term $S_{6,4}$ involves second derivatives acting on $\widetilde{r}_2$. Hence, by applying the Calderón–Zygmund estimate (5.26) to $\widetilde{r}_2$, we obtain

$$
|S_{6,4}| = \mathcal{O}(h^{-1/2+\varepsilon}).
$$

In particular, this implies

$$\lim_{h\to 0}\left(hS_{6,4}\right)=0.$$

Step 7. Analysis of S_7.

In S_7, there is only one derivative on $\mathbf{v}^{(1)}$ and $\mathbf{v}^{(2)}$, from the above analysis, we know that

$$\nabla \mathbf{v}^{(1)}=\frac{1}{h}\begin{pmatrix}1\\ \mathsf{i}\end{pmatrix}\left(1+\frac{z}{4}\right)e^{\Phi_1/h}+\mathcal{O}(1),\quad \nabla \mathbf{v}^{(2)}=-\frac{1}{h}\begin{pmatrix}1\\ -\mathsf{i}\end{pmatrix}\frac{\overline{z}}{2}e^{\overline{\Phi_2}/h}+\mathcal{O}(1).$$

Using the same trick as before, an integration by parts argument between the holomorphic and antiholomorphic functions ensures that

$$\lim_{h\to 0}\left(hS_7\right)=0.$$

Step 8. Finalization.

From Step 1 to Step 7, with the integral identity (7.8) at hand, we can conclude that the only nonzero terms come from S_1 and S_5, which are

$$\begin{aligned}0&=\lim_{h\to 0}\Big(h\sum_{k=1}^{7}S_k\Big)\\ &=\pi\overline{\partial}\big(F_{A^*}^{-1}\overline{\partial}\widetilde{G}_0+F_{A^*}^{-1}(\mathbf{X}_g\cdot\nabla\Phi^*)\widetilde{G}_0\big)\big|_{z=0}-16\pi\overline{\partial}\widetilde{G}_1\big|_{z=0}\\ &\quad+\pi\overline{\partial}\big\{\widetilde{\mu}\big[\operatorname{tr}\left(A\mathcal{C}^1\right)-\mathsf{i}\operatorname{tr}\left(A\mathcal{C}^2\right)\big]\big\}\big|_{z=0}+2\pi\mathsf{i}\overline{\partial}^{-1}(E(w))V\Big|_{z=0},\end{aligned}\tag{7.37}$$

where $E(w):=\widetilde{\mu}(w)\big(\operatorname{tr}(A\mathcal{C}^1(w))+\mathsf{i}\operatorname{tr}(A\mathcal{C}^2(w))\big)\big(-1+\frac{w^2}{16}\big)V'(w)$. We next vary the critical point $z=0$ to the entire domain U by shifting the phases of the CGOs. By doing so, the local terms will be just evaluated at $z\in U$ instead of $z=0$. However, in the nonlocal term $\overline{\partial}^{-1}(E(w))(z)$ in (7.37) the function $\big(-1+\frac{w^2}{16}\big)$ is obtained from phase functions with evaluating point $w=0$. Thus, changing the phase, changes the function $\big(-1+\frac{w^2}{16}\big)$ in the nonlocal term.

Let us give more details to the above observation regarding the nonlocal term. When we choose the critical point to be $a\in\mathbb{C}$, the corresponding phase functions become

$$\Phi_1(z)=(z-a)+\frac{(z-a)^2}{8},\ \Phi_2(z)=-\frac{1}{4}(z-a)^2,\ \text{and}\ \Phi^*(z)=-(z-a)+\frac{(z-a)^2}{8}.$$

Following similar derivations as in Step 6-3, we can see that the nonlocal lower order term will be given by $\overline{\partial}^{-1}\Big(\widetilde{\mu}\big(\operatorname{tr}(A\mathcal{C}^1)+\mathsf{i}\operatorname{tr}(A\mathcal{C}^2)\big)\big(-1+\frac{(w-a)^2}{16}\big)V'\Big)(z)V(z)\Big|_{z=a}$. More concretely, one may compute

$$\begin{aligned}&\overline{\partial}^{-1}\Big(\widetilde{\mu}\big(\operatorname{tr}(A\mathcal{C}^1)+\mathsf{i}\operatorname{tr}(A\mathcal{C}^2)\big)\big(-1+\frac{(w-a)^2}{16}\big)V'\Big)V\Big|_{z=a}\\ &=-\overline{\partial}^{-1}\Big(\widetilde{\mu}\big(\operatorname{tr}(A\mathcal{C}^1)+\mathsf{i}\operatorname{tr}(A\mathcal{C}^2)\big)\Big)V\Big|_{z=a}\\ &\quad+\frac{1}{16}\int\widetilde{\mu}\big(\operatorname{tr}(A\mathcal{C}^1)+\mathsf{i}\operatorname{tr}(A\mathcal{C}^2)\big)\frac{(w-a)^2}{w-z}\,d\overline{w}\wedge dw\Big|_{z=a}\\ &=-\overline{\partial}^{-1}\Big(\widetilde{\mu}\big(\operatorname{tr}(A\mathcal{C}^1)+\mathsf{i}\operatorname{tr}(A\mathcal{C}^2)\big)\Big)V\Big|_{z=a}\\ &\quad+\frac{1}{16}\int\widetilde{\mu}\big(\operatorname{tr}(A\mathcal{C}^1)+\mathsf{i}\operatorname{tr}(A\mathcal{C}^2)\big)(w-a)\,d\overline{w}\wedge dw\\ &=-\overline{\partial}^{-1}\Big(\widetilde{\mu}\big(\operatorname{tr}(A\mathcal{C}^1)+\mathsf{i}\operatorname{tr}(A\mathcal{C}^2)\big)\Big)V\Big|_{z=a}+c_1a+c_2,\end{aligned}$$

where

$$
\begin{aligned}
c_1 &:= -\frac{1}{16}\int \widetilde{\mu}\big(\operatorname{tr}(A\mathcal{C}^1)+\mathsf{i}\operatorname{tr}(A\mathcal{C}^2)\big)\, d\overline{w}\wedge dw \in \mathbb{C},\\
c_2 &:= \frac{1}{16}\int \widetilde{\mu}\big(\operatorname{tr}(A\mathcal{C}^1)+\mathsf{i}\operatorname{tr}(A\mathcal{C}^2)\big) w\, d\overline{w}\wedge dw \in \mathbb{C},
\end{aligned}
$$

are some constants, and we used the function $\widetilde{\mu}\big(\operatorname{tr}(A\mathcal{C}^1)+\mathsf{i}\operatorname{tr}(A\mathcal{C}^2)\big) = \widetilde{\mu}\big(\operatorname{tr}(AC^1)+\mathsf{i}\operatorname{tr}(AC^2)\big)\mathbf{c}$ is compactly supported in $\mathbb{C}$, so that the above integrals must be finite.

Let us write

$$
H(z) = -2\mathsf{i}(c_1 z - c_2),
$$

for the linear function in z, which is thus holomorphic. Using (7.5) and (7.7), by translation, we can vary the critical point $z=0$ (or $z=a$) of the phase functions in CGOs to arbitrary points $z\in U$, then the identity (7.37) yields that

$$
\begin{aligned}
&\overline{\partial}\big(F_{A^*}^{-1}\overline{\partial}\widetilde{G}_0 + F_{A^*}^{-1}(\mathbf{X}_g\cdot\nabla\Phi^*)\widetilde{G}_0\big) - 16\overline{\partial}\widetilde{G}_1\\
&\quad + \overline{\partial}\big\{\widetilde{\mu}\big[\operatorname{tr}\big(A\mathcal{C}^1\big) - \mathsf{i}\operatorname{tr}\big(A\mathcal{C}^2\big)\big]\big\}\\
&= \underbrace{2\mathsf{i}V}_{:=\beta}\,\overline{\partial}^{-1}\Big(\underbrace{\widetilde{\mu}\big(\operatorname{tr}(A\mathcal{C}^1)+\mathsf{i}\operatorname{tr}(A\mathcal{C}^2)\big)}_{:=\gamma\mathbf{c}}\Big) + H \text{ in } U.
\end{aligned}
\tag{7.38}
$$

Using the definitions (7.14), (7.19), (7.32) and (7.7) of $\widetilde{G}_0$, $\widetilde{G}_1$, $\widetilde{\mu}$ and $\mathcal{C}^k$ $(k=1,2)$, respectively, by setting

$$
\mathbf{c} := 1 - c^{-2} \in C^2(\mathbb{R}^2),
$$

where $\mathbf{c}$ has been extended by zero to $\mathbb{R}^2\setminus\Omega$ at the outset of the proof, the equation (7.38) will take the form

$$
\overline{\partial}\big(A\overline{\partial}\mathbf{c} + \alpha\mathbf{c}\big) = \beta(z)\overline{\partial}^{-1}(\gamma\mathbf{c}) + H \text{ in } U,
\tag{7.39}
$$

for some functions α,β,γ independent of $\mathbf{c}$, where the leading coefficient A of the second derivatives in the equation (7.39) is non-vanishing. In particular, we can find A explicitly by

$$
\begin{aligned}
A &:= \mu^{-1}F_{A^*}^{-1}F_{A_1}^{-1}F_{\overline{A_2}},\\
\gamma &:= \mu^{-1}F_{A_1}^{-1}F_{\overline{A_2}}\big(\operatorname{tr}(AC^1)+\mathsf{i}\operatorname{tr}(AC^2)\big),
\end{aligned}
$$

and α,β are some functions that can be computed and are independent of $\mathbf{c}$. Note that by (5.15), $A(z)\neq 0$ for all $z\in U$.

Since $\mathbf{c}=0$ in $U\setminus\Omega$ and recall that H is holomorphic, the UCP of Lemma 6.1 applied to (7.39) yields $\mathbf{c}\equiv 0$ in all of U. As $\mathbf{c} = 1-c^{-2} = 0$ in U, it follows that $c^{-2}=1$ in U, and in particular in Ω. Finally, using $c>0$ in Ω, we conclude

$$
c\equiv 1 \quad \text{in } \Omega,
$$

which completes the proof. □

8. Proof of Theorem 1.2

With all arguments in previous sections, we have proved Theorem 1.2. For the sake of completeness, let us explain the arguments again.

Proof of Theorem 1.2. Let us split the proof into several steps:

- *Step 1.* Using $F_1|_{\partial\Omega} = F_2|_{\partial\Omega}$, the boundary determination (see Lemma 3.1) shows that $D^2u_0^{(1)}|_{\partial\Omega} = D^2u_0^{(2)}|_{\partial\Omega}$.
- *Step 2.* Lemma 4.1 shows that the relation (1.9) determines ,

$$
\Lambda'_{g_1}(\phi) = \Lambda'_{g_2}(\phi), \quad \text{for any } \phi\in C^\infty(\partial\Omega),
\tag{8.1}
$$

  where Λ'_{g_j} denotes the DN map of (4.8), for $j=1,2$.

- *Step 3.* Since Ω is a uniformly convex domain, it must be a simply connected domain. Hence, applying Theorem 4.4, the condition (8.1) implies that there exists $c > 0$ with $c|_{\partial\Omega} = 1$ such that $g_1 = cg_2$ in Ω.
- *Step 4.* Theorem 7.4 yields that $c = 1$ in Ω. This shows that $g_1 = g_2$ in $\overline{\Omega}$. In other words, $D^2 u_0^{(1)} = D^2 u_2^{(0)}$ in $\overline{\Omega}$, which implies

$$F_1 = \det D^2 u_0^{(1)} = \det D^2 u_2^{(0)} = F_2 \text{ in } \Omega,$$

where we used $u_0^{(j)}$ are solutions to (3.1), for $j = 1, 2$.

This concludes the proof. □

Statements and Declarations

Data availability statement. No datasets were generated or analyzed during the current study.

Conflict of Interests. Hereby, we declare there are no conflicts of interest.

Acknowledgments. The authors would like to thank Professor Mikko Salo for fruitful discussions to improve this article. T. L. was partly supported by the Academy of Finland (Centre of Excellence in Inverse Modelling and Imaging and FAME Flagship, grant numbers 312121 and 359208). Y.-H. L. is partially supported by the National Science and Technology Council (NSTC) of Taiwan, under projects 113-2628-M-A49-003 and 113-2115-M-A49-017-MY3. Y.-H. L. is also a Humboldt research fellow (for experienced researchers).

References

[ABN20] Spyros Alexakis, Tracey Balehowsky, and Adrian Nachman. Determining a Riemannian metric from minimal areas. *Adv. Math.*, 366:107025, 71, 2020.

[Ahl66] Lars V. Ahlfors. *Lectures on quasiconformal mappings*, volume No. 10 of *Van Nostrand Mathematical Studies*. D. Van Nostrand Co., Inc., Toronto, Ont.-New York-London, 1966. Manuscript prepared with the assistance of Clifford J. Earle, Jr.

[Amp19] André-Marie Ampère. *Mémoire contenant l'application de la théorie exposée dans le XVII. e Cahier du Journal de l'École polytechnique, à l'intégration des équations aux différentielles partielles du premier et du second ordre*. De l'Imprimerie royale, 1819.

[Aub82] Thierry Aubin. *Nonlinear analysis on manifolds. Monge-Ampère equations*, volume 252 of *Grundlehren der mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, New York, 1982.

[AZ21] Yernat M Assylbekov and Ting Zhou. Direct and inverse problems for the nonlinear time-harmonic Maxwell equations in Kerr-type media. *J. Spectr. Theory*, 11:1–38, 2021.

[BL76] Jöran Bergh and Jörgen Löfström. *Interpolation spaces. An introduction*, volume No. 223 of *Grundlehren der Mathematischen Wissenschaften*. Springer-Verlag, Berlin-New York, 1976.

[Cal72] Eugenio Calabi. Complete affine hyperspheres. I. In *Symposia Mathematica, Vol. X (Convegno di Geometria Differenziale, INDAM, Rome, 1971 & Convegno di Analisi Numerica, INDAM, Rome, 1972)*, pages 19–38. Academic Press, London-New York, 1972.

[CFK+21] Cătălin I. Cârstea, Ali Feizmohammadi, Yavar Kian, Katya Krupchyk, and Gunther Uhlmann. The Calderón inverse problem for isotropic quasilinear conductivities. *Adv. Math.*, 391:Paper No. 107956, 31, 2021.

[CLLO24] Cătălin I. Cârstea, Matti Lassas, Tony Liimatainen, and Lauri Oksanen. An inverse problem for the Riemannian minimal surface equation. *J. Differential Equations*, 379:626–648, 2024.

[CLLT23] Cătălin I Cârstea, Matti Lassas, Tony Liimatainen, and Leo Tzou. An inverse problem for general minimal surfaces. *arXiv preprint arXiv:2310.14268*, 2023.

[CLT24] Cătălin I. Cârstea, Tony Liimatainen, and Leo Tzou. The Calderón problem on Riemannian surfaces and of minimal surfaces. *arXiv preprint arXiv:2406.16944*, 2024.

[CNV19] Cătălin I Cârstea, Gen Nakamura, and Manmohan Vashisth. Reconstruction for the coefficients of a quasilinear elliptic partial differential equation. *Applied Mathematics Letters*, 98:121–127, 2019.

[CY86] Shiu Yuen Cheng and Shing-Tung Yau. Complete affine hypersurfaces. I. The completeness of affine metrics. *Comm. Pure Appl. Math.*, 39(6):839–866, 1986.

[Fig17] Alessio Figalli. *The Monge-Ampère equation and its applications*. Zurich Lectures in Advanced Mathematics. European Mathematical Society (EMS), Zürich, 2017.

[FLL23] Ali Feizmohammadi, Tony Liimatainen, and Yi-Hsuan Lin. An inverse problem for a semilinear elliptic equation on conformally transversally anisotropic manifolds. *Ann. PDE*, 9(2):Paper No. 12, 54, 2023.

[FO20] Ali Feizmohammadi and Lauri Oksanen. An inverse problem for a semi-linear elliptic equation in Riemannian geometries. *J. Differential Equations*, 269(6):4683–4719, 2020.

[GT01] David Gilbarg and Neil S. Trudinger. *Elliptic partial differential equations of second order*. Classics in Mathematics. Springer-Verlag, Berlin, 2001. Reprint of the 1998 edition.

[GT11a] Colin Guillarmou and Leo Tzou. Calderón inverse problem with partial data on Riemann surfaces. *Duke Math. J.*, 158(1):83–120, 2011.

[GT11b] Colin Guillarmou and Leo Tzou. Identification of a connection from Cauchy data on a Riemann surface with boundary. *Geom. Funct. Anal.*, 21(2):393–418, 2011.

[HL23] Bastian Harrach and Yi-Hsuan Lin. Simultaneous recovery of piecewise analytic coefficients in a semilinear elliptic equation. *Nonlinear Anal.*, 228:Paper No. 113188, 14, 2023.

[Isa93] V. Isakov. On uniqueness in inverse problems for semilinear parabolic equations. *Arch. Rational Mech. Anal.*, 124(1):1–12, 1993.

[IUY12] Oleg Imanuvilov, Gunther Uhlmann, and Masahiro Yamamoto. Partial Cauchy data for general second order elliptic operators in two dimensions. *Publ. Res. Inst. Math. Sci.*, 48(4):971–1055, 2012.

[JLST25] Niko Jokela, Tony Liimatainen, Miika Sarkkinen, and Leo Tzou. Bulk metric reconstruction from entanglement data via minimal surface area variations. *Journal of High Energy Physics*, 2025(10):79, 2025.

[KKU23] Yavar Kian, Katya Krupchyk, and Gunther Uhlmann. Partial data inverse problems for quasilinear conductivity equations. *Math. Ann.*, 385(3-4):1611–1638, 2023.

[KLL24] Yavar Kian, Tony Liimatainen, and Yi-Hsuan Lin. On determining and breaking the gauge class in inverse problems for reaction-diffusion equations. *Forum Math. Sigma*, 12:Paper No. e25, 42, 2024.

[KLU18] Yaroslav Kurylev, Matti Lassas, and G. Uhlmann. Inverse problems for Lorentzian manifolds and non-linear hyperbolic equations. *Invent. Math.*, 212(3):781–857, 2018.

[KN02] Hyeonbae Kang and Gen Nakamura. Identification of nonlinearity in a conductivity equation via the Dirichlet-to-Neumann map. *Inverse Problems*, 18:1079–1088, 2002.

[KU20a] Katya Krupchyk and Gunther Uhlmann. Partial data inverse problems for semilinear elliptic equations with gradient nonlinearities. *Mathematical Research Letters*, 27(6):1801–1824, 2020.

[KU20b] Katya Krupchyk and Gunther Uhlmann. A remark on partial data inverse problems for semilinear elliptic equations. *Proc. Amer. Math. Soc.*, 148(2):681–685, 2020.

[Las25] Matti Lassas. Introduction to inverse problems for non-linear partial differential equations. *arXiv preprint arXiv:2503.12448*, 2025.

[LL19] Ru-Yu Lai and Yi-Hsuan Lin. Global uniqueness for the fractional semilinear Schrödinger equation. *Proc. Amer. Math. Soc.*, 147(3):1189–1199, 2019.

[LL22] Ru-Yu Lai and Yi-Hsuan Lin. Inverse problems for fractional semilinear elliptic equations. *Nonlinear Anal.*, 216:Paper No. 112699, 21, 2022.

[LL24] Tony Liimatainen and Yi-Hsuan Lin. Uniqueness results for inverse source problems for semilinear elliptic equations. *Inverse Problems*, 40(4):Paper No. 045030, 32, 2024.

[LL25] Yi-Hsuan Lin and Hongyu Liu. *Inverse Problems for Integro-differential Operators*, volume 222 of *Applied Mathematical Sciences*. Springer, Cham, 2025.

[LLLS21a] Matti Lassas, Tony Liimatainen, Yi-Hsuan Lin, and Mikko Salo. Inverse problems for elliptic equations with power type nonlinearities. *J. Math. Pures Appl. (9)*, 145:44–82, 2021.

[LLLS21b] Matti Lassas, Tony Liimatainen, Yi-Hsuan Lin, and Mikko Salo. Partial data inverse problems and simultaneous recovery of boundary and coefficients for semilinear elliptic equations. *Rev. Mat. Iberoam.*, 37(4):1553–1580, 2021.

[LLST22] Tony Liimatainen, Yi-Hsuan Lin, Mikko Salo, and Teemu Tyni. Inverse problems for elliptic equations with fractional power type nonlinearities. *J. Differential Equations*, 306:189–219, 2022.

[LN25] Tony Liimatainen and Janne Nurminen. An inverse problem for the prescribed mean curvature. *arXiv preprint arXiv:2509.22078*, 2025.

[LSX22] Shuai Lu, Mikko Salo, and Boxi Xu. Increasing stability in the linearized inverse Schrödinger potential problem with power type nonlinearities. *Inverse Problems*, 38(6):Paper No. 065009, 25, 2022.

[LW23] Tony Liimatainen and Ruirui Wu. Calderón problem for the quasilinear conductivity equation in dimension 2. *arXiv preprint arXiv:2309.11047*, 2023.

[Min97] Hermann Minkowski. Allgemeine lehrsätze über die konvexen polyeder. In *Ausgewählte Arbeiten zur Zahlentheorie und zur Geometrie: Mit D. Hilberts Gedächtnisrede auf H. Minkowski, Göttingen 1909*, pages 121–139. Springer, 1897.

[Min03] Hermann Minkowski. Volumen und Oberfläche. *Math. Ann.*, 57(4):447–495, 1903.

[Mon84] Gaspard Monge. *Mémoire sur le calcul intégral des équations aux différences partielles*. Imprimerie royale, 1784.

[Nur24] Janne Nurminen. An inverse problem for the minimal surface equation in the presence of a Riemannian metric. *Nonlinearity*, 37(9):Paper No. 095029, 22, 2024.

[QXYZ25] Dong Qiu, Xiang Xu, Yeqiong Ye, and Ting Zhou. Uniqueness result for semi-linear wave equations with sources. *arXiv preprint arXiv:2510.04810*, 2025.

[RR04] Michael Renardy and Robert C. Rogers. *An introduction to partial differential equations*, volume 13 of *Texts in Applied Mathematics*. Springer-Verlag, New York, second edition, 2004.

[ST23] Mikko Salo and Leo Tzou. Inverse problems for semilinear elliptic PDE with measurements at a single point. *Proc. Amer. Math. Soc.*, 151(5):2023–2030, 2023.

[SU97] Ziqi Sun and Gunther Uhlmann. Inverse problems in quasilinear anisotropic media. *Amer. J. Math.*, 119(4):771–797, 1997.

[Sun96] Ziqi Sun. On a quasilinear inverse boundary value problem. *Math. Z.*, 221(2):293–305, 1996.

[Sun10] Ziqi Sun. An inverse boundary-value problem for semilinear elliptic equations. *Electron. J. Differential Equations*, pages No. 37, 5, 2010.

[TW00] Neil S. Trudinger and Xu-Jia Wang. The Bernstein problem for affine maximal hypersurfaces. *Invent. Math.*, 140(2):399–422, 2000.

[TW02] Neil S. Trudinger and Xu-Jia Wang. Affine complete locally convex hypersurfaces. *Invent. Math.*, 150(1):45–60, 2002.

[TW05] Neil S. Trudinger and Xu-Jia Wang. The affine Plateau problem. *J. Amer. Math. Soc.*, 18(2):253–289, 2005.

[Tzo17] Leo Tzou. The reflection principle and Calderón problems with partial data. *Math. Ann.*, 369(1-2):913–956, 2017.

[Vil09] Cédric Villani. *Optimal transport*, volume 338 of *Grundlehren der mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 2009. Old and new.

[Yau78] Shing Tung Yau. On the Ricci curvature of a compact Kähler manifold and the complex Monge-Ampère equation. I. *Comm. Pure Appl. Math.*, 31(3):339–411, 1978.

Department of Mathematics and Statistics, University of Helsinki, Helsinki, Finland
Email address: tony.liimatainen@helsinki.fi

Department of Applied Mathematics, National Yang Ming Chiao Tung University, Hsinchu, Taiwan & Fakultät für Mathematik, University of Duisburg-Essen, Essen, Germany
Email address: yihsuanlin3@gmail.com