UNIQUE DETERMINATION OF COEFFICIENTS AND KERNEL IN NONLOCAL POROUS MEDIUM EQUATIONS WITH ABSORPTION TERM

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Abstract. The main purpose of this article is the study of an inverse problem for nonlocal porous medium equations (NPMEs) with a linear absorption term. More concretely, we show that under certain assumptions on the time-independent coefficients ρ, q and the time-independent kernel K of the nonlocal operator L_K, the (partial) Dirichlet-to-Neumann map uniquely determines the three quantities (ρ, K, q) in the nonlocal porous medium equation $\rho\partial_t u + L_K(u^m) + qu = 0$, where $m > 1$. In the first part of this work we adapt the Galerkin method to prove existence and uniqueness of nonnegative, bounded solutions to the homogeneous NPME with regular initial and exterior conditions. Additionally, a comparison principle for solutions of the NPME is proved, whenever they can be approximated by sufficiently regular functions like the one constructed for the homogeneous NPME. These results are then used in the second part to prove the unique determination of the coefficients (ρ, K, q) in the inverse problem. Finally, we show that the assumptions on the nonlocal operator L_K in our main theorem are satisfied by the fractional conductivity operator $\mathcal{L}_\gamma$, whose kernel is $\gamma^{1/2}(x)\gamma^{1/2}(y)/|x-y|^{n+2s}$ up to a normalization constant.

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1. Introduction

In recent years inverse problems for a wide class of nonlocal partial differential equations (PDEs) have been studied. The most classical example is the *fractional Calderón problem.* In this problem one considers Dirichlet problem for the *fractional Schrödinger equation*

$$\begin{cases} ((-\Delta)^s + q)u = 0 & \text{in } \Omega, \\ u = \varphi & \text{in } \Omega_e, \end{cases}$$

where $0 < s < 1$, $(-\Delta)^s$ is the fractional Laplacian, q is a given potential and $\Omega_e = \mathbb{R}^n \setminus \overline{\Omega}$ is the exterior of the bounded domain $\Omega \subset \mathbb{R}^n$. The fractional Calderón problem now asks to uniquely determine the potential q from the (exterior) Dirichlet-to-Neumann (DN) map $\varphi \mapsto \Lambda_q(\varphi) = (-\Delta)^s u|_{\Omega_e}$. A first affirmative answer has been established in [GSU20] for bounded potentials.

Later, this work initiated many further developments in the field of nonlocal inverse problems, which includes determination of singular potentials, lower order local perturbations, higher order fractional Laplacians, single measurement results, generalizations to other nonlocal operators in place of the fractional Laplacian and unbounded domains (see [BGU21, CMR21, CMRU22, GLX17, CL19, CLL19, CLR20, FGKU21, HL19, HL20, GRSU20a, GU21, Gho22, Lin22, LL22, LL23, LLR20, LLU22, KLW22, RS20, RS18, RZ22b] and the references therein). Another type of inverse problems for nonlocal operators has been investigated for example in the articles [RZ22b, RZ22a, CRZ22, RZ22c, CRTZ22, Zim23, GU21, Li21], where the authors try to recover coefficients in a nonlocal operator from the DN map instead of determining lower order perturbations of a nonlocal operator. The solvability of the inverse problems in the above works strongly depend on the linear structure of the nonlocal operator, the unique continuation principle (UCP) and the Runge approximation. The last property is a consequence of the UCP, which in turn can be phrased as follows:

Unique continuation principle. *Let X be a Banach space satisfying $C_c^\infty(\mathbb{R}^n) \hookrightarrow X \hookrightarrow \mathscr{D}'(\mathbb{R}^n)$. We say that an operator $L\colon X \to \mathscr{D}'(\mathbb{R}^n)$ has the UCP on X if $Lu = u = 0$ in some nonempty open set Ω implies $u = 0$ in $\mathbb{R}^n$.*

In particular, in the article [GSU20] it has been shown that the fractional Laplacian $(-\Delta)^s$, $0 < s < 1$, has the UCP on $X = H^t(\mathbb{R}^n)$ for any $t \in \mathbb{R}$. The same clearly remains true for local perturbations of fractional Laplacians. In [GLX17], it has been proved that fractional powers of second order operators $L = -\operatorname{div}(A\nabla\cdot)$, where $A\in C_b^\infty(\mathbb{R}^{n\times n};\mathbb{R})$ is symmetric and uniformly elliptic, satisfy the UCP on $X=H^s(\mathbb{R}^n)$. Furthermore, in [Zim23] it has been established by using the fractional Liouville reduction that the fractional conductivity operator $\mathcal{L}_\gamma$ has the UCP as well, at least when the coefficient γ is sufficiently regular. The fractional conductivity operator $\mathcal{L}_\gamma$ belongs to a subclass of integro-differential operators of order $2s$, that is of operators of the form

$$L_K u(x)=\mathrm{P.V.}\int_{\mathbb{R}^n}K(x,y)\frac{u(x)-u(y)}{|x-y|^{n+2s}}\,dy, \tag{1.1}$$

where $K\colon\mathbb{R}^n\times\mathbb{R}^n\to\mathbb{R}$ is a given function and P.V. stands for the Cauchy principal value. In the rest of the article, we abuse the terminology and refer to K as the kernel of L_K. Such an operator is said to belong to the class $\mathcal{L}_0$, whenever the kernel K satisfies the following properties:

(i) K is symmetric in the sense that

$$K(x,y)=K(y,x)\text{ for all }x,y\in\mathbb{R}^n.$$

(ii) K is uniformly elliptic, that is there holds

$$\lambda\leq K(x,y)\leq\Lambda$$

for all $x,y\in\mathbb{R}^n$ and some constants $0<\lambda\leq\Lambda<\infty$.

Now, the fractional conductivity operator $\mathcal{L}_\gamma$ is the integro-differential operator of order $2s$ of the form (1.1), whose kernel is given by

$$K(x,y)=C_{n,s}\gamma^{1/2}(x)\gamma^{1/2}(y)$$

for some uniformly elliptic function $\gamma\colon\mathbb{R}^n\to\mathbb{R}$ and hence $\mathcal{L}_\gamma\in\mathcal{L}_0$. Here $C_{n,s}$ is the constant given by

$$C_{n,s}:=\frac{4^s\Gamma(n/2+s)}{\pi^{n/2}|\Gamma(-s)|}, \tag{1.2}$$

where Γ is the Gamma function. Let us note that in the special case $\gamma=1$ the operator $\mathcal{L}_\gamma$ reduces to the fractional Laplacian $(-\Delta)^s$ and we remark that the constant $C_{n,s}$ is defined as in (1.2) precisely that the Fourier symbol of $(-\Delta)^s$ is $|\xi|^{2s}$.

Recently, the interplay between nonlocality and nonlinearity of an operator and its influence on the solvability of the related inverse problem has been studied in [KRZ23] and [KLZ22]. The underlying models in these articles are closely related, but exhibit dramatically different unique continuation properties. In the first article vector-valued generalizations of weighted fractional p-biharmonic operators

$$(-\Delta)^s_{p,\sigma}u:=(-\Delta)^{s/2}\left(\sigma|(-\Delta)^{s/2}u|^{p-2}(-\Delta)^{s/2}u\right) \tag{1.3}$$

has been investigated, where one wants to determine the uniformly elliptic function σ from the DN map. This has been achieved by monotonicity methods and the observation that the operators in (1.3) have the following unique continuation property on the Bessel potential space $H^{s,p}(\mathbb{R}^n)$: If $(-\Delta)^s_{p,\sigma}u_1=(-\Delta)^s_{p,\sigma}u_2$ and $(-\Delta)^{s/2}u_1=(-\Delta)^{s/2}u_2$ in an open set $\Omega\subset\mathbb{R}^n$, then there holds $u_1=u_2$ in $\mathbb{R}^n$. This property rests on the fact that the fractional Laplacian satisfies the UCP on the Bessel potential spaces $X=H^{t,p}(\mathbb{R}^n)$ with $t\in\mathbb{R}$ and $1\leq p<\infty$. On the other hand, in [KLZ22] the authors studied weighted fractional p-Laplacians, which are integro-differential operators of the form

$$(-\Delta_p)^s_\sigma u(x)=\mathrm{P.V.}\int_{\mathbb{R}^n}\sigma(x,y)|u(x)-u(y)|^{p-2}\frac{u(x)-u(y)}{|x-y|^{n+sp}}\,dxdy, \tag{1.4}$$

where $0<s<1$, $1<p<\infty$ and $\sigma\colon\mathbb{R}^n\times\mathbb{R}^n\to\mathbb{R}$ is a uniformly elliptic function. A disadvantage concerning inverse problems is the fact that it is not known whether operators of the form (1.4) have an appropriate unique continuation property and hence an alternative approach has been established. In this work an exterior determination method in the spirit of the boundary determination result of Kohn and Vogelius [KV84] has been introduced. A similar method was also used for the classical p-Calderón problem by Salo and Zhong [SZ12] and for the fractional conductivity problem in [CRZ22, RZ22c] or the parabolic fractional conductivity problem in [LRZ22].

In this article we follow the same line of research, namely we investigate an inverse problem for a nonlocal and nonlinear PDE. More concretely, we study an inverse problem for a class of *nonlocal porous medium equations* (NPMEs), which are of the form

$$\rho\partial_t u + L_K(\Phi^m(u)) + qu = 0 \text{ in } \Omega_T. \tag{1.5}$$

Here $\Omega \subset \mathbb{R}^n$ is a bounded Lipschitz domain, $T > 0$, $\Omega_T = \Omega \times (0,T)$[1], the function $\Phi^m\colon \mathbb{R} \to \mathbb{R}$ is given by

$$\Phi^m(t) = |t|^{m-1}t \text{ for some } m > 1,$$

$\rho\colon \mathbb{R}^n \to \mathbb{R}$ is uniformly elliptic, L_K an integro-differential operator of order $2s$ in the class $\mathcal{L}_0$ and $q\colon \mathbb{R}^n \to \mathbb{R}$ a nonnegative potential. More concrete assumptions on the coefficients ρ, q and the nonlocal operator L_K will be given below (Section 1.2). To motivate the investigation of the PDE (1.5) and the related inverse problem, we first review in the next section the classical porous medium equation and then discuss natural nonlocal generalizations of it.

1.1. **Local and nonlocal porous medium equations.** The *porous medium equation* (PME) is one of the simplest examples of a nonlinear evolution equation and its study at least dates back to 1957 [Ole57]. One of its basic forms is

$$\partial_t u - \Delta\Phi(u) = 0 \text{ in } \Omega_T, \tag{1.6}$$

where $\Omega \subset \mathbb{R}^n$ can be any domain and Φ is typically assumed to be of the form $\Phi = \Phi^m$ for some $m > 1$ but not restricted to. Let us note that for $m > 1$ the PME is degenerate parabolic, whereas for $m = 1$ one would obtain the classical heat equation and for $m < 1$ a singular PDE or fast diffusion equation. Physically speaking this equation models the gas flow through a porous medium and it is also used to model various phenomena in other fields, such as plasma physics [RK83] and population dynamics [Nam80]. In the literature many generalizations of (1.6) had been studied like adding a forcing term $f(u, \nabla u)$ on the right hand side. For such a term the dependence on ∇u incorporates convection of the medium and the dependence on u reaction or absorption effects. For a detailed account on the mathematical theory for the porous medium equation, we refer the reader to the monographs [Sch60, Váz92].

Furthermore, inverse problems for the PME (1.6) have been investigated in [CGN21, CGU23]. In [CGN21], the authors determined two coefficients (ρ, γ) for the PME in the form

$$\rho\partial_t u - \mathrm{div}(\gamma\nabla\Phi^m(u)) = 0 \tag{1.7}$$

for $m > 1$. In [CGU23] this result has been generalized to porous medium equations with a possible nonlinear absorption term. They prove a unique determination result for the three parameters (ρ, γ, q) in

$$\rho\partial_t u - \mathrm{div}(\gamma\nabla\Phi^m(u)) + q\Phi^r(u) = 0, \tag{1.8}$$

where $m > 1$ and $m^{-1} < r < \sqrt{m}$.

[1]Throughout the article we write A_τ to denote the space-time cylinder $A \times (0,\tau)$, whenever $\tau > 0$ and $A \subset \mathbb{R}^n$.

An interesting, nonlocal generalization of the PME (1.6) is obtained by replacing the Laplacian by the fractional Laplacian, which leads to

$$\partial_t u + (-\Delta)^s(\Phi^m(u)) = 0 \text{ in } \Omega_T \tag{1.9}$$

for $0 < s < 1$. This model describes anomalous diffusion through a porous medium and additional information on this equation as well as generalizations of it can be found for example in the articles [Váz14, BSV15, BV16, BFRO17].

Hence, based on what we said a natural generalization to consider are nonlocal porous medium equations of the form

$$\rho\partial_t u + L_K(\Phi^m(u)) + q\Phi^r(u) = 0 \text{ in } \Omega_T \tag{1.10}$$

(see (1.8) and (1.9)). To simplify the presentation and the technicalities, we will restrict our considerations in this work to the linear case (1.5) instead of (1.10).

1.2. **Inverse problem for nonlocal porous medium equations.** Next, let us consider the initial-exterior value problem for the NPME

$$\begin{cases} \rho\partial_t u + L_K\left(\Phi^m(u)\right) + qu = 0 & \text{in } \Omega_T, \\ u = \varphi & \text{in } (\Omega_e)_T, \\ u(0) = 0 & \text{in } \Omega, \end{cases} \tag{1.11}$$

where $\Omega \subset \mathbb{R}^n$ is a bounded Lipschitz domain, $L_K \in \mathcal{L}_0$, $m > 1$, $\rho\colon \mathbb{R}^n \to \mathbb{R}$ is uniformly elliptic, $q\colon \mathbb{R}^n \to \mathbb{R}$ is a nonnegative potential and $\varphi\colon \Omega_e \times (0,T) \to \mathbb{R}$ is a given exterior condition.

In Section 3, it is established that under suitable assumptions on the coefficients ρ, q and the exterior condition φ the problem (1.11) is well-posed. More concretely, we show in Theorem 3.13 and Theorem 3.17 that there exists a unique, nonnegative, bounded solution of (1.11). Therefore, we can now introduce the DN map $\Lambda_{\rho,K,q}$ related to (1.11), which is strongly given by

$$\Lambda_{\rho,K,q}\varphi = L_K(\Phi^m(u))|_{(\Omega_e)_T}$$

for suitable exterior conditions φ, where $u\colon \mathbb{R}^n \times (0,T) \to \mathbb{R}$ is the unique solution to (1.11). A more detailed account on the DN map is given in Section 5.1. Now, we can formulate our inverse problem.

Question 1. *Can one uniquely determine the coefficients ρ, q in $\overline{\Omega}$ and the kernel K in $\mathbb{R}^{2n}$ by the nonlocal DN map $\Lambda_{\rho,K,q}$?*

Next, let us recall that for any $L_K \in \mathcal{L}_0$ one can show by the Lax–Milgram theorem and the fractional Poincaré inequality (see (2.1)) that for any $\varphi \in H^s(\mathbb{R}^n)$ there is a unique weak solution $u \in H^s(\mathbb{R}^n)$ of

$$\begin{cases} L_K u = 0 & \text{in } \Omega, \\ u = \varphi & \text{in } \Omega_e. \end{cases} \tag{1.12}$$

Furthermore, it is not difficult to prove that the exterior value to solution map of (1.12) is well-defined on the (abstract) trace space $Z = H^s(\mathbb{R}^n)/\widetilde{H}^s(\Omega)$. Hence, we can define the DN map related to (1.12) by

$$\Lambda_K\colon Z \to Z^*,\ \langle \Lambda_K\varphi, \psi\rangle := B_K(u,\psi), \tag{1.13}$$

where $u \in H^s(\mathbb{R}^n)$ is the unique solution to (1.12), $\psi \in Z$ and $B_K\colon H^s(\mathbb{R}^n) \times H^s(\mathbb{R}^n) \to \mathbb{R}$ is the naturally bilinear form related to L_K (see (2.2)).

In general, the determination of the kernel K from the DN map (1.13) is a highly nontrivial inverse problem. Nevertheless, it can be expected that if the linear elliptic nonlocal inverse problem is not uniquely solvable, then the same remains true for the inverse problem of the NPME. To rule out this possibility, we next introduce a suitable class of nonlocal operators, so called measurement equivalent operators. In a similar spirit, the authors of [RZ22b] formulated general conditions under which local perturbations of linear elliptic nonlocal operators can be uniquely determined from the related DN map.

Definition 1.1 (Measurement equivalent operators). *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain and $0 < s < 1$. We say that two uniformly elliptic nonlocal operators $L_{K_1}, L_{K_2} \in \mathcal{L}_0$ of order $2s$ are* measurement equivalent, *written as $L_{K_1} \sim L_{K_2}$, if the condition*

$$\Lambda_{K_1}\varphi|_{W_2} = \Lambda_{K_2}\varphi|_{W_2}\,, \text{ for all } \varphi \in C_c^\infty(W_1) \text{ with } \varphi \geq 0 \tag{1.14}$$

for some nonempty, open sets $W_1, W_2 \subset \Omega_e$ with $W_1 \cap W_2 \neq \emptyset$, implies $K_1 = K_2$.

Remark 1.2. *We assumed that the measurement sets $W_1, W_2 \subset \Omega_e$ in the above definition are non-disjoint due to the counterexamples constructed in* [RZ22a] *and* [RZ22c]. *For example in* [RZ22c], *it is proved that if the measurement sets are disjoint and have a positive distance to Ω, then one can construct two different conductivities γ_1, γ_2 such that the related fractional conductivity operators $\mathcal{L}_{\gamma_j}$, $j = 1, 2$, satisfy* (1.14) *and the regularity assumptions in Proposition 5.4 below.*

In Section 5.2, we provide some examples of nonlocal operators that fulfill Definition 1.1. Now, we are ready to state our main result of this article, which generalizes the recent work [CGU23] on the local porous medium equation to the a nonlocal setting.

Theorem 1.3. *Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain, $T > 0$, $0 < s < \alpha \leq 1$, $m > 1$ and $W_1, W_2 \Subset \Omega_e$ two nonempty open sets with $W_1 \cap W_2 \neq \emptyset$. Assume that for $j = 1, 2$ we have given nonlocal operators $L_{K_j} \in \mathcal{L}_0$ satisfying the UCP on $H^s(\mathbb{R}^n)$, and coefficients $\rho_j, q_j \in C_+^{1,\alpha}(\mathbb{R}^n)$ such that ρ_j is uniformly elliptic. If $L_{K_1} \sim L_{K_2}$ and there holds*

$$\Lambda_{\rho_1,K_1,q_1}\varphi|_{(W_2)_T} = \Lambda_{\rho_2,K_2,q_2}\varphi|_{(W_2)_T}\,,$$

for any $0 \leq \varphi \in C_c([0,T] \times W_1)$ with $\Phi(\varphi) \in L^2(0,T; H^s(\mathbb{R}^n))$, then we have

$$\rho_1 = \rho_2,\ q_1 = q_2 \text{ in } \overline{\Omega} \text{ and } K_1 = K_2 \text{ in } \mathbb{R}^{2n}.$$

Finally, let us observe that if one can show that the fractional powers $(-\operatorname{div}(A\nabla\cdot))^s$ induce measure equivalent operators, then Theorem 1.3 implies that the related inverse problems for the NPME are uniquely solvable (see Remark 5.6).

1.3. **Inverse problem for the porous medium equation.** In this section, we will have a closer look at some aspects of the proof of the unique determination in the inverse problem for the porous medium equation (1.7), since some of these ideas also enter into the proof of Theorem 1.3. For this purpose let us consider the initial-boundary value problem related (1.7) for $r = 1$, namely

$$\begin{cases} \rho\partial_t u - \operatorname{div}(\gamma\nabla\Phi^m(u)) + qu = 0 & \text{in } \Omega_T, \\ u = \varphi & \text{on } \partial\Omega_T, \\ u(0) = 0 & \text{in } \Omega. \end{cases} \tag{1.15}$$

For any regular boundary value $\varphi \geq 0$, one can prove under suitable regularity assumptions on ρ, γ and q the existence of a unique nonnegative solution to (1.15) (see [CGU23]). Hence, the natural measurements related to (1.15) can be encoded in the DN map

$$\varphi \mapsto \Lambda_{\rho,\gamma,q}\varphi := \gamma\partial_\nu\Phi^m(u)|_{\partial\Omega_T}\,.$$

In [CGU23], they showed that if one has

$$\Lambda_{\rho_1,\gamma_1,q_1}\varphi = \Lambda_{\rho_2,\gamma_2,q_2}\varphi$$

for all sufficiently regular boundary data $\varphi \geq 0$, then there holds $\rho_1 = \rho_2$, $\gamma_1 = \gamma_2$ and $q_1 = q_2$ in Ω. Furthermore, under the additional assumption $\gamma_1 = \gamma_2 = 1$ in Ω, the authors of [CGU23] also obtained a partial data uniqueness result. Actually, these results were established for $r > 0$ satisfying $m^{-1} < r < m^{1/2}$.

Next, we describe the main idea of the uniqueness proof for the inverse problem related to (1.15). In the first step, one introduces the new function $v := \Phi^m(u) = u^m$, which solves the problem

$$\begin{cases} \rho\partial_t v^{1/m} - \operatorname{div}(\gamma\nabla v) + qv^{\frac{1}{m}} = 0 & \text{in } \Omega\times(0,T),\\ v = f & \text{in } \partial\Omega\times(0,T),\\ v(0) = 0 & \text{in } \Omega, \end{cases} \tag{1.16}$$

with $f = \varphi^m$, whenever u is a nonnegative solution to (1.15). The DN map $\Lambda^{red.1}_{\rho,\gamma,q}$ related to (1.16) satisfies

$$\Lambda_{\rho,\gamma,q}\varphi = \Lambda^{red.1}_{\rho,\gamma,q}\varphi^m.$$

Next consider large, time homogeneous boundary values of the form

$$f(x,t) = ht^m g(x)$$

with $g \geq 0$ on $\partial\Omega$ and assume $h \gg 1$ is a fixed large parameter. Now the essential observation is that the time-integral transform

$$V(x) = \int_0^T (T-t)^\alpha v(x,t)\,dt,$$

gives a solution to Dirichlet problem

$$\begin{cases} \operatorname{div}(\gamma\nabla V) = \mathcal{N}_t + \mathcal{N}_a & \text{in} \quad \Omega,\\ V = hT^{1+\alpha+m}\frac{\Gamma(1+\alpha)\Gamma(1+m)}{\Gamma(2+\alpha+m)}g & \text{on} \quad \partial\Omega, \end{cases} \tag{1.17}$$

where the parameters $T > 0$ and $\alpha > 0$ are determined later. The DN map of (1.17) is given by

$$\Lambda^{red.2}_{\rho,\gamma,q} g = h^{-1}\gamma\partial_\nu V\big|_{\partial\Omega}$$

and can be computed from the DN map $\Lambda^{red.1}_{\rho,\gamma,q}$. The approach to establish the unique determination result can be briefly summarized as follows:

(i) The Neumann data $\Lambda^{red.2}_{\rho,\gamma,q}g$ converges (up to a constant) to $\Lambda_\gamma g = \gamma\partial_\nu V_0|_{\partial\Omega}$ as $h \to \infty$, where Λ_γ is the DN map of the conductivity equation and V_0 is defined via the expansion

$$V(x) = hT^{1+\alpha+m}\frac{\Gamma(1+\alpha)\Gamma(1+m)}{\Gamma(2+\alpha+m)}V_0(x) + R_1(x).$$

(ii) In addition, making use of the more delicate asymptotic ansatz

$$V(x) = hV_0(x) + h^{1/m}V_t(x) + R_2(x),$$

for suitable functions V_t and R_2, one can see that the Neumann data $\Lambda^{red.2}_{\rho,\gamma,q}g$ has the asymptotic expansion

$$\begin{aligned} \Lambda^{red.2}_{\rho,\gamma,q}g = {} & T^{1+\alpha+m}\frac{\Gamma(1+\alpha)\Gamma(1+m)}{\Gamma(2+\alpha+m)}V_0(x) \\ & + h^{1/m-1}\gamma\partial_\nu V_2\Big|_{\partial\Omega} + \mathcal{O}(h^{1/m^2-1}) \end{aligned}$$

as $h \to \infty$.

(iii) Finally, if $\{(\rho_j,\gamma_j,q_j)\}_{j=1,2}$ is a triple of coefficients such that the related DN maps $\Lambda_{\rho_j,\gamma_j,q_j}$ coincide, then the property (i) yields that $\gamma_1 = \gamma_2$ and (ii) that $\rho_1 = \rho_2$, $q_1 = q_2$ in Ω.

In fact, based on the above ideas, we also prove Theorem 1.3 in Section 6.

1.4. **Structure of the paper.** The paper is organized as follows. In Section 2, we review several function spaces and integro-differential operators. In Section 3, we study the forward problem (1.11). More specifically, we prove the existence of a unique, nonnegative, bounded solution of (1.11) under suitable sign and regularity assumptions on the data. Moreover, we demonstrate a novel comparison principle for NPMEs in Section 4, which will be utilized to establish Theorem 1.3. In Section 5, we define the DN map rigorously, and provide some examples of measurement equivalent operators. Our main result Theorem 1.3 will be proved in Section 6. Finally, some compactness and density properties will be addressed in Appendix A and B.

2. Preliminaries

Let us emphasize that $\Omega \subset \mathbb{R}^n$ always denotes an open set and $\Omega_e := \mathbb{R}^n \setminus \overline{\Omega}$ is called the exterior of Ω throughout this article.

2.1. **Sobolev and Hölder spaces.** The classical Lebesgue and Sobolev spaces are denoted by $L^p(\Omega)$ and $W^{k,p}(\Omega)$, respectively. As usual we will write $H^k(\Omega)$ for $W^{k,p}(\Omega)$, whenever $p = 2$.

Next, we recall the definition of fractional Sobolev spaces. For $1 \leq p < \infty$ and $0 < s < 1$, the *fractional Sobolev space* $W^{s,p}(\Omega)$ is defined by

$$W^{s,p}(\Omega) := \left\{ u \in L^p(\Omega) \, ; \, [u]_{W^{s,p}(\Omega)} < \infty \right\}.$$

which are naturally endowed with the norm

$$\|u\|_{W^{s,p}(\Omega)} := \left(\|u\|_{L^p(\Omega)}^p + [u]_{W^{s,p}(\Omega)}^p \right)^{1/p},$$

where

$$[u]_{W^{s,p}(\Omega)} := \left(\int_\Omega \int_\Omega \frac{|u(x)-u(y)|^p}{|x-y|^{n+sp}} \, dxdy \right)^{1/p}.$$

Next, we introduce a closed subspace of $W^{s,p}(\mathbb{R}^n)$, which can be regarded as consisting of functions with zero exterior value, namely:

$$\widetilde{W}^{s,p}(\Omega) := \text{closure of } C_c^\infty(\Omega) \text{ with respect to } \|\cdot\|_{W^{s,p}(\mathbb{R}^n)}.$$

Let us note that like the classical Sobolev spaces, the spaces $W^{s,p}(\mathbb{R}^n)$ or $\widetilde{W}^{s,p}(\Omega)$ are separable for $1 \leq p < \infty$ and reflexive for $1 < p < \infty$ (see [BH22, Section 7]). We remark that it is known that $\widetilde{W}^{s,p}(\Omega)$ coincides with the set of all functions $u \in W^{s,p}(\mathbb{R}^n)$ such that $u = 0$ a.e. in Ω^c, when $\partial\Omega \in C^0$, and with

$$W_0^{s,p}(\Omega) := \text{closure of } C_c^\infty(\Omega) \text{ with respect to } \|\cdot\|_{W^{s,p}(\Omega)},$$

whenever $\Omega \Subset \mathbb{R}^n$ has a Lipschitz boundary (see [KLL22, Section 2]).

An important advantage of the spaces $\widetilde{W}^{s,p}(\Omega)$ is that they support a *fractional Poincaré inequality*, that is, for any bounded set $\Omega \subset \mathbb{R}^n$, $0 < s < 1$ and $1 < p < \infty$ there holds

$$\|u\|_{L^p(\mathbb{R}^n)} \leq C[u]_{W^{s,p}(\mathbb{R}^n)} \tag{2.1}$$

for all $u \in \widetilde{W}^{s,p}(\Omega)$.

As for the classical Sobolev spaces, we write $H^s(\mathbb{R}^n)$ and $\widetilde{H}^s(\Omega)$ instead of $W^{s,p}(\mathbb{R}^n)$ and $\widetilde{W}^{s,p}(\Omega)$, when $p = 2$. Finally, let us point out that we denote the dual space of $\widetilde{H}^s(\Omega)$ by $H^{-s}(\Omega)$. This notation is justified by the fact that if one defines the latter via Fourier analytic methods, that is as a Bessel potential space, then they are isomorphic. Throughout the article, we will denote the duality pairing between $\widetilde{H}^s(\Omega)$ and $H^{-s}(\Omega)$ by $\langle \cdot, \cdot \rangle$.

Now, we introduce the used notation for Hölder continuous functions. For all $0<\alpha\leq 1$, the space $C^{0,\alpha}(\Omega)$ consists of all continuous functions $u\in C(\Omega)$ such that the norm

$$\|u\|_{C^{0,\alpha}(\Omega)}:=\|u\|_{L^\infty(\Omega)}+[u]_{C^{0,\alpha}(\Omega)}$$

is finite, where

$$[u]_{C^{0,\alpha}(\Omega)}:=\sup_{x\neq y\in\Omega}\frac{|u(x)-u(y)|}{|x-y|^\alpha}.$$

Finally, we recall some standard function spaces for time-dependent PDEs. If X is a given Banach space and $(a,b)\subset\mathbb{R}$, then we write $L^p(a,b\,;X)$ $(1\leq p<\infty)$ for the space of measurable functions $u\colon(a,b)\to X$ such that

$$\|u\|_{L^p(a,b\,;X)}:=\left(\int_a^b\|u(t)\|_X^p\,dt\right)^{1/p}<\infty$$

and $L^\infty(a,b\,;X)$ for the space of measurable functions $u\colon(a,b)\to X$ such that

$$\|u\|_{L^\infty(a,b\,;X)}:=\inf\{\,M\,;\,\|u(t)\|_X\leq M\,\text{a.e.}\,\}<\infty.$$

Additionally to these Bochner–Lebesgue spaces, we will also make use of the Sobolev spaces $H^1(0,T;X)$ and generalized Sobolev spaces

$$W_T(X,Y)=\left\{u\in L^2(0,T;X)\,;\,\partial_t u\in L^2(0,T;Y)\,\right\},$$

where X,Y are Banach spaces such that $X\hookrightarrow Y$, $T>0$ and ∂_t denotes the distributional time derivative. These spaces carry the natural norms

$$\|u\|_{H^1(0,T;X)}=\left(\|u\|^2_{L^2(0,T;X)}+\|\partial_t u\|^2_{L^2(0,T;X)}\right)^{1/2}$$

and

$$\|u\|_{W_T(X,Y)}=\left(\|u\|^2_{L^2(0,T;X)}+\|\partial_t u\|^2_{L^2(0,T;Y)}\right)^{1/2}.$$

For more details, we refer to the monograph [DL92] or the article [Sim87].

2.2. **Nonlocal operators.** As explained in Section 1, in this article we consider uniformly elliptic integro-differential operators of order $2s$. That is, we assume our operators are given by

$$L_K u(x)=\text{P.V.}\int_{\mathbb{R}^n}K(x,y)\frac{u(x)-u(y)}{|x-y|^{n+2s}}\,dxdy,$$

whenever $u\colon\mathbb{R}^n\to\mathbb{R}$ is sufficiently regular and the function $K\colon\mathbb{R}^n\times\mathbb{R}^n\to\mathbb{R}$ satisfies the following properties

(i) $K(x,y)=K(y,x)$ for all $x,y\in\mathbb{R}^n$,
(ii) $\lambda\leq K(x,y)\leq\Lambda$ for some $0<\lambda\leq\Lambda<\infty$.

Recall that if the kernel K satisfies the above conditions, then we say L_K belongs to the class $\mathcal{L}_0$. The natural bilinear form related to L_K on $H^s(\mathbb{R}^n)$, will be denoted by B_K and is given by

$$B_K(u,v)=\int_{\mathbb{R}^{2n}}K(x,y)\frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{n+2s}}\,dxdy \tag{2.2}$$

for all $u,v\in H^s(\mathbb{R}^n)$. One easily sees that a nonlocal operator L_K in the ellipticity class $\mathcal{L}_0$ defines via (2.2) a bounded linear operator from $\widetilde{H}^s(\Omega)$ to $H^{-s}(\Omega)$.

Finally, let us point out that the following operators belong to the class $\mathcal{L}_0$ (see Section 1 and [GLX17, eq: (2.15)-(2.16)]):

(i) *Fractional Laplacians* $(-\Delta)^s$,
(ii) *Fractional conductivity operators* $\mathcal{L}_\gamma$,

(iii) *Fractional powers of second order divergence form operators* $L = -\operatorname{div}(A\nabla\cdot)$, where $A \in C_b^\infty(\mathbb{R}^n;\mathbb{R}^{n\times n})$ is symmetric and uniformly elliptic.

3. The forward problem

In this section, we study the forward problem of the NPME. We begin with an auxiliary lemma, in order to prove the well-posedness of (1.11).

3.1. Auxiliary lemma.

Definition 3.1. *Let $m > 0$ be a fixed number, then we define the function $\Phi^m\colon \mathbb{R} \to \mathbb{R}$ by $\Phi^m(t) = |t|^{m-1}t$ for all $t \in \mathbb{R}$.*

Lemma 3.2. *Let $m \geq 1$ be an arbitrary number, then there exists a sequence $\Phi^m_\varepsilon\colon \mathbb{R} \to \mathbb{R}$, $\varepsilon > 0$ sufficiently small, satisfying the following conditions*

(i) $\Phi^m_\varepsilon \in C^1(\mathbb{R})$,
(ii) $\Phi_\varepsilon(t) = -\Phi_\varepsilon(-t)$,
(iii) $c_\varepsilon \leq (\Phi^m_\varepsilon)' \leq C_\varepsilon$ *for some constants* $0 < c_\varepsilon \leq C_\varepsilon < \infty$,
(iv) $\Phi^m_\varepsilon \to \Phi^m$ *as* $\varepsilon \to 0$ *uniformly on any compact set.*

Proof. To simplify the notation, let us replace m by $p-1$ and write Φ instead of Φ^{p-1}. Note that in the simple case $m = 1$ or equivalently $p = 2$, one can take $\Phi_\varepsilon = \Phi$, which satisfies all conditions (i)-(iv). Hence, we can assume without loss of generality that $p > 2$. Now choose $M \in 2\mathbb{N}$ satisfying $M > p \geq 2$. Next, we introduce the strictly positive coefficients

$$a_\varepsilon = \frac{p-2}{M-2}\varepsilon^{p-M}, \quad b_\varepsilon = \frac{M-p}{M-2}\varepsilon^{p-2}$$

and define the family $\Phi_\varepsilon\colon \mathbb{R} \to \mathbb{R}$ by

$$\Phi_\varepsilon(t) = \begin{cases} \Phi(t), & \text{for} \quad \varepsilon \leq |t| \leq 1/\varepsilon, \\ a_\varepsilon t^{M-1} + b_\varepsilon t, & \text{for} \quad -\varepsilon \leq |t| \leq \varepsilon, \\ (p-1)\varepsilon^{2-p}t + (2-p)\varepsilon^{1-p}, & \text{for} \quad t \geq 1/\varepsilon, \\ (p-1)\varepsilon^{2-p}t - (2-p)\varepsilon^{1-p}, & \text{for} \quad t \leq -1/\varepsilon \end{cases}$$

for $t \in \mathbb{R}$. We directly see that these functions satisfy (ii).

Next, we show the condition (i). Clearly, Φ_ε is a continuous function. Note that

$$\Phi'(t) = (p-1)|t|^{p-2} \quad \text{for} \quad t \neq 0 \tag{3.1}$$

and hence Φ'_ε exists and is continuous for $|t| \geq \varepsilon$. Since $\Phi_\varepsilon|_{[-\varepsilon,\varepsilon]}$ is clearly C^1 it remains to check that its derivative coincides with Φ'_ε at the endpoints. Let us define $\Psi\colon [-\varepsilon,\varepsilon] \to \mathbb{R}$ by

$$\Psi(t) = \frac{d}{dt}\left(a_\varepsilon t^{M-1} + b_\varepsilon t\right) = (M-1)a_\varepsilon t^{M-2} + b_\varepsilon, \text{ for } |t| \leq \varepsilon. \tag{3.2}$$

Since Ψ is symmetric it is enough to show that $\Psi(\varepsilon)=\Phi'(\varepsilon)$. We have

$$\begin{aligned}\Psi(\varepsilon)&=(M-1)a_\varepsilon\varepsilon^{M-2}+b_\varepsilon\\&=(M-1)\frac{p-2}{M-2}\varepsilon^{p-M}\varepsilon^{M-2}+\frac{M-p}{M-2}\varepsilon^{p-2}\\&=\frac{(Mp-2M-p+2)+(M-p)}{M-2}\varepsilon^{p-2}\\&=\frac{Mp-M-2p+2}{M-2}\varepsilon^{p-2}\\&=\frac{(M-2)p-(M-2)}{M-2}\varepsilon^{p-2}\\&=(p-1)\varepsilon^{p-2}\\&=\Phi'(\varepsilon).\end{aligned}$$

This establishes the condition (i). The condition (iii) is easily seen from (3.1), (3.2) and the fact that Φ_ε is for $|t|\geq 1/\varepsilon$ an affine function with strictly positive slope.

Finally, let us prove the assertion (iv). By the symmetry condition (ii), which is also satisfied by Φ, it is enough to prove that for any $K>0$ we have

$$\|\Phi_\varepsilon-\Phi\|_{L^\infty([0,K])}\to 0\quad\text{as}\quad\varepsilon\to 0.$$

For this purpose, let us fix $K>0$. Without loss of generality we can assume that $\varepsilon>0$ is sufficiently small that $K\leq 1/\varepsilon$. For any $0\leq t\leq\varepsilon$, we have

$$\begin{aligned}|\Phi_\varepsilon(t)-\Phi(t)|&=\left|\frac{p-2}{M-2}\varepsilon^{p-M}t^{M-1}+\frac{M-p}{M-2}\varepsilon^{p-2}t-|t|^{p-2}t\right|\\&=\left|\left(\frac{p-2}{M-2}\varepsilon^{p-M}t^{M-2}-\frac{p-2}{M-2}\varepsilon^{p-2}\right)t+(\varepsilon^{p-2}-|t|^{p-2})t\right|\\&\leq\left(\frac{p-2}{M-2}(\varepsilon^{p-2}-\varepsilon^{p-M}t^{M-2})+(\varepsilon^{p-2}-t^{p-2})\right)\varepsilon\\&\overset{(*)}{\leq}\left(\frac{p-2}{M-2}\varepsilon^{p-2}+\varepsilon^{p-2}\right)\varepsilon\\&=\frac{M+p-4}{M-2}\varepsilon^{p-1}.\end{aligned}$$

In $(*)$ we used that both terms are decreasing in t on $[0,\varepsilon]$. Hence, taking $\Phi_\varepsilon|_{[\varepsilon,1/\varepsilon]}=\Phi$ into account, we have

$$\|\Phi_\varepsilon-\Phi\|_{L^\infty([0,K])}\leq\frac{M+p-4}{M-2}\varepsilon^{p-1},$$

which shows the claimed convergence. □

3.2. **Existence of solutions to the forward problem.** Let us begin with the definition of weak solutions.

Definition 3.3 (Weak solutions)**.** *Let $\Omega\subset\mathbb{R}^n$ be a bounded Lipschitz domain, $T>0$, $0<s<1$, $m\geq 1$ and $L_K\in\mathcal{L}_0$. For given $u_0\in L^\infty(\Omega)$, $\varphi\in C_c([0,T]\times\Omega_e)$ with $\Phi^m(\varphi)\in L^2(0,T;H^s(\mathbb{R}^n))$ and $f\in L^2(0,T;H^{-s}(\Omega))$, we say that $u\colon\mathbb{R}^n_T\to\mathbb{R}$ is a* (weak) solution *of*

$$\begin{cases}\partial_t u+L_K(\Phi^m(u))=f & \text{in } \Omega_T,\\ u=\varphi & \text{in } (\Omega_e)_T,\\ u(0)=u_0 & \text{in } \Omega,\end{cases}\tag{3.3}$$

provided that

(i) $\Phi^m(u)\in L^2(0,T;H^s(\mathbb{R}^n))$,

(ii) $\Phi^m(u-\varphi)\in L^2(0,T;\widetilde{H}^s(\Omega))$,

(iii) u *satisfies* (3.3) *in the sense of distributions, that is there holds*

$$-\int_{\Omega_T} u\partial_t\psi\,dxdt+\int_0^T B_K(\Phi^m(u),\psi)\,dt=\int_0^T\langle f,\psi\rangle\,dxdt+\int_\Omega u_0\psi(0)\,dx$$

for all $\psi\in C_c^\infty([0,T)\times\Omega)$.

If a (weak) solutions u *of* (3.3) *additionally satisfies* $u\in L^\infty(\Omega_T)$, *then it is called* bounded solution*.*

Remark 3.4. *Let* u *be a (weak) solution of* (3.3). *Observe that the preceding definition directly implies* $\partial_t u\in L^2(0,T;H^{-s}(\Omega))$ *and thus* u *belongs to the same space. If we have an additional nonlinear absorption term this does not need to hold.*

Theorem 3.5 (Basic existence result)**.** *Let* $\Omega\subset\mathbb{R}^n$ *be a bounded Lipschitz domain,* $T>0$, $0<s<1$, $m>1$ *and* $L_K\in\mathcal{L}_0$. *For any* $0\le u_0\in L^\infty(\Omega)\cap\widetilde{H}^s(\Omega)$ *and* $0\le\varphi\in C_c([0,T]\times\Omega_e)$ *with* $\Phi^m(\varphi)\in L^2(0,T;H^s(\mathbb{R}^n))$, *there exists a non-negative, bounded solution of*

$$\begin{cases}\partial_t u+L_K(\Phi^m(u))=0 & \text{in } \Omega_T,\\ u=\varphi & \text{in } (\Omega_e)_T,\\ u(0)=u_0 & \text{in } \Omega.\end{cases}\tag{3.4}$$

Moreover, there holds

$$\|u\|_{L^\infty(\Omega_T)}\le\sup_{x\in\Omega}u_0+\sup_{(x,t)\in[0,T]\times\Omega_e}\varphi.\tag{3.5}$$

Proof. Throughout the proof we will write $\Phi,\Phi_\varepsilon,L,B$ instead of $\Phi^m,\Phi^m_\varepsilon,L_K,B_K$, where Φ^m_ε is the family of functions constructed in Lemma 3.2.

We prove existence of a solution u by the Galerkin method. For this purpose, let us first observe that since φ is compactly supported in $[0,T]\times\Omega_e$, u is a solution of (3.4) if and only if $v=u-\varphi$ solves

$$\begin{cases}\partial_t v+L(\Phi(v))=f & \text{in } \Omega_T,\\ v=0 & \text{in } (\Omega_e)_T,\\ v(0)=u_0 & \text{in } \Omega,\end{cases}\tag{3.6}$$

with $f=\partial_t\varphi-L(\Phi(\varphi))=-L(\Phi(\varphi))$ in Ω_T by using $\varphi\equiv 0$ in Ω_T. Note that for any $(x,t)\in\Omega_T$, we have

$$f(x,t)=-\int_{\mathbb{R}^n}K(x,y)\frac{\Phi(\varphi)(x,t)-\Phi(\varphi)(y,t)}{|x-y|^{n+2s}}dy=\int_{\Omega_e}K(x,y)\frac{\Phi(\varphi)(y,t)}{|x-y|^{n+2s}}\,dy,$$

where we used that $\Phi(0)=0$ and $\varphi|_{\Omega_T}=0$. The compact support of φ in $[0,T]\times\Omega_e$, $\Phi(0)=0$, the uniform ellipticity of K, $\Phi(t)\ge 0$ for $t\ge 0$ and $\varphi\ge 0$ clearly implies that f has the following properties:

(a) $0\le f(x,t)\le C\int_{\Omega_e}\Phi(\varphi)(y,t)\,dy<\infty$ for $(x,t)\in\Omega_T$,
(b) $f\in C([0,T];L^2(\Omega))$.

Next recall that $\widetilde{H}^s(\Omega)$ is a separable Hilbert space and hence the finite dimensional subspaces

$$E_m=\operatorname{span}\{w_1,\dots,w_m\},\quad m\in\mathbb{N},$$

form a Galerkin approximation for $\widetilde{H}^s(\Omega)$. Here $(w_j)_{j\in\mathbb{N}}\subset\widetilde{H}^s(\Omega)$ is a priori any orthogonal basis of $\widetilde{H}^s(\Omega)$. Since Ω is bounded, we will take in the following $(w_j)_{j\in\mathbb{N}}$ to be the eigenfunctions to the positive discrete eigenvalues $(\lambda_j)_{j\in\mathbb{N}}$ of the positive, symmetric, nonlocal operator L. This is possible as the embedding $\widetilde{H}^s(\Omega)\hookrightarrow$

$L^2(\Omega)$ is by the Rellich–Kondrachov theorem compact (see [Eva10, Section 6.5, Theorem 1], [DL92, Chapter XVIII, Section 2.1-2.2] and [KRZ23, Theorem 3.2]). Observe by density of $\widetilde{H}^s(\Omega)$ in $L^2(\Omega)$ the family $(E_m)_{m\in\mathbb{N}}$ is also a Galerkin approximation for $L^2(\Omega)$. Moreover, we will assume in the following that $(w_j)_{j\in\mathbb{N}}$ are normalized in $L^2(\Omega)$.

Let us note that if

$$\psi_N = \sum_{j=1}^{N} \langle \psi, w_j \rangle_{L^2} w_j, \tag{3.7}$$

for some $\psi \in \widetilde{H}^s(\Omega)$, then we have

$$[\psi_N]_{H^s(\mathbb{R}^n)} \leq C[\psi]_{H^s(\mathbb{R}^n)}, \tag{3.8}$$

for some constant $C > 0$ independent of ψ and $N \in \mathbb{N}$. Here and in the following $\langle \cdot, \cdot \rangle_{L^2}$ always denotes the inner product in $L^2(\Omega)$. In fact, by assumption we have

$$B(w_j, \psi) = \lambda_j \langle w_j, \psi \rangle_{L^2} \tag{3.9}$$

for all $j \in \mathbb{N}$. Thus, (3.7) and (3.9) yield that

$$\psi_N = \sum_{j=1}^{N} \langle \psi, w_j \rangle_{L^2} w_j = \sum_{j=1}^{N} B(\psi, w_j/\sqrt{\lambda_j})(w_j/\sqrt{\lambda_j})$$

and $w_j/\sqrt{\lambda_j}$ for $j \in \mathbb{N}$, is an orthonormal basis of $\widetilde{H}^s(\Omega)$ with equivalent inner product induced by B. The last part follows from the symmetry and uniform ellipticity of K and the fractional Poincaré inequality. But then by Parseval's identity we have

$$B(\psi, \psi) = \sum_{j\in\mathbb{N}} \left| B(\psi, w_j/\sqrt{\lambda_j}) \right|^2 \geq \sum_{j=1}^{N} \left| B(\psi, w_j/\sqrt{\lambda_j}) \right|^2 = B(\psi_N, \psi_N).$$

Now, the uniform ellipticity of K gives (3.8). The rest of the proof is divided into three steps.

Step 1: Construction of approximate solutions.

Let $\varepsilon > 0$, $v_0 \in L^2(\Omega)$, $F \in L^2(0, T; H^{-s}(\Omega))$ and consider the initial-exterior value problem

$$\begin{cases} \partial_t v + L(\Phi_\varepsilon(v)) = F & \text{in } \Omega_T, \\ v = 0 & \text{in } (\Omega_e)_T, \\ v(0) = v_0 & \text{in } \Omega. \end{cases}$$

Next, choose a sequence $F_N \in C([0, T]; H^{-s}(\Omega))$, $N \in \mathbb{N}$, converging to F in $L^2(0, T; H^{-s}(\Omega))$. This can be achieved by setting $F(t) = 0$ for $t \notin [0, T]$ and then mollify in time. As usual in the Galerkin approximation, in a first step we will be looking for functions $v_{\varepsilon,N} \in C^1([0, T], E_N)$, that is they can be written as

$$v_{\varepsilon,N}(x, t) = \sum_{j=1}^{N} c^j_{N,\varepsilon}(t) w_j(x)$$

for $N \in \mathbb{N}$, and solve in E_N the problem

$$\begin{cases} \partial_t v + L(\Phi_\varepsilon(v)) = F_N & \text{in } \Omega_T, \\ v = 0 & \text{in } (\Omega_e)_T, \\ v(0) = v_{0,N} & \text{in } \Omega, \end{cases} \tag{3.10}$$

where $v_{0,N} = \sum_{j=1}^N \langle v_0, w_j\rangle_{L^2} w_j$. If $v_{\varepsilon,N} \in C^1([0,T];E_N)$ solves (3.10), then we have

$$\langle \partial_t v_{\varepsilon,N}, \psi\rangle_{L^2} + B(\Phi_\varepsilon(v_{\varepsilon,N}),\psi) = \langle F_N, \psi\rangle \tag{3.11}$$

for a.e. $t \in (0,T)$ and $\psi \in E_N$. In particular, choosing $\psi = w_j$ for $1 \leq j \leq N$ as a test function, we get

$$\langle \partial_t v_{\varepsilon,N}, w_j\rangle_{L^2} + B(\Phi_\varepsilon(v_{\varepsilon,N}),w_j) = \langle F_N, w_j\rangle \tag{3.12}$$

for all $1 \leq j \leq N$ and $0 < t < T$. Expanding everything, we see that

$$\hat{c}_{\varepsilon,N} := \left(c^1_{\varepsilon,N}, \dots, c^N_{\varepsilon,N}\right) \in C^1([0,T];\mathbb{R}^N)$$

is a solution of the ordinary differential equation (ODE)

$$\begin{cases} \partial_t \hat{c}_{\varepsilon,N} + b(\hat{c}_{\varepsilon,N}) = G_N(\hat{c}_{\varepsilon,N}), & \text{for} \quad 0<t<T \\ \hat{c}_{\varepsilon,N}(0) = \hat{c}_{0,N}, \end{cases} \tag{3.13}$$

where

$$\hat{c}_{0,N} = \left(\langle v_0, w_1\rangle_{L^2}, \dots, \langle v_0, w_N\rangle_{L^2}\right), \quad G_N(\hat{c}_{\varepsilon,N}) = \left\langle F_N, \sum_{j=1}^N c^j_{\varepsilon,N} w_j \right\rangle$$

and for $1 \leq j \leq N$ the components $b_j \colon \mathbb{R}^N \to \mathbb{R}$ of $b = (b_1, \dots, b_N)$ are defined by

$$b_j(c) = B\left(\Phi_\varepsilon\left(\sum_{k=1}^N c_k w_k\right), w_j\right)$$

for $c = (c_1, \dots, c_N) \in \mathbb{R}^N$.

Claim 3.6. *The functions $G_N \colon \mathbb{R}^N \to \mathbb{R}$ and $b \colon \mathbb{R}^N \to \mathbb{R}^N$ are continuous.*

Proof of Claim 3.6. The continuity of G_N is immediate since $F_N \in C([0,T], H^{-s}(\Omega))$. To see the continuity of b, consider a sequence $(\hat{c}_k)_{k\in\mathbb{N}} = \left(\hat{c}^1_k, \dots, \hat{c}^N_k\right)_{k\in\mathbb{N}} \subset \mathbb{R}^N$, which converges to $\hat{c} = \left(\hat{c}^1, \dots, \hat{c}^N\right) \in \mathbb{R}^N$ as $k \to \infty$. Note that by uniform ellipticity of K and Hölder's inequality we have

$$\begin{aligned} |b(\hat{c}_k) - b(\hat{c})| &\leq \max_{1\leq j\leq N} \left| B\left(\Phi_\varepsilon\left(\sum_{\ell=1}^N \hat{c}^\ell_k w_\ell\right) - \Phi_\varepsilon\left(\sum_{\ell=1}^N \hat{c}^\ell w_\ell\right), w_j\right)\right| \\ &\leq C\left[\Phi_\varepsilon\left(\sum_{\ell=1}^N \hat{c}^\ell_k w_\ell\right) - \Phi_\varepsilon\left(\sum_{\ell=1}^N \hat{c}^\ell w_\ell\right)\right]_{H^s(\mathbb{R}^n)} \end{aligned} \tag{3.14}$$

Using the Lipschitz continuity of Φ_ε and Lebesgue's dominated convergence theorem, one easily sees that the last expression in (3.14) goes to zero as $k \to \infty$ and hence showing the continuity of b. This proves the claim. $\square$

Hence, by Peano's existence theorem for ODEs there exist a solution $\hat{c}_{N,\varepsilon} \in C^1([0,\delta];\mathbb{R}^N)$ of (3.13) for possibly a small $\delta > 0$. Later, we show that the solution actually extends to $[0,T]$. Let us denote the corresponding approximate solution of (3.10) by $v_{\varepsilon,N} \in C^1([0,\delta],E_N)$. Multiplying (3.12) by $c^j_{N,\varepsilon}$ and summing from $j=1$ to N gives

$$\langle \partial_t v_{\varepsilon,N}, v_{\varepsilon,N}\rangle_{L^2} + B(\Phi_\varepsilon(v_{\varepsilon,N}),v_{\varepsilon,N}) = \langle F_N, v_{\varepsilon,N}\rangle. \tag{3.15}$$

We have

$$\begin{aligned} \langle \partial_t v_{\varepsilon,N}, v_{\varepsilon,N}\rangle_{L^2} &= \frac{d}{dt}\frac{\|v_{\varepsilon,N}\|^2_{L^2}}{2}, \\ |\langle F_N, v_{\varepsilon,N}\rangle| &\leq \|F_N\|_{H^{-s}(\Omega)}\|v_{\varepsilon,N}\|_{H^s(\mathbb{R}^n)}. \end{aligned} \tag{3.16}$$

Moreover, using the fundamental theorem of calculus, the uniform ellipticity of Φ_ε' and K the second term in (3.15) can be lower bounded as (suppressing the t dependence)

$$
\begin{aligned}
&B(\Phi_\varepsilon(v_{\varepsilon,N}), v_{\varepsilon,N})\\
&= \int_{\mathbb{R}^{2n}} K(x,y) \frac{(\Phi_\varepsilon(v_{\varepsilon,N})(x) - \Phi_\varepsilon(v_{\varepsilon,N})(y))(v_{\varepsilon,N}(x) - v_{\varepsilon,N}(y))}{|x-y|^{n+2s}}\, dxdy\\
&= \int_{\mathbb{R}^{2n}} K(x,y) \frac{\int_0^1 \frac{d}{d\tau}\left[\Phi_\varepsilon(v_{\varepsilon,N}(y) + \tau(v_{\varepsilon,N}(x) - v_{\varepsilon,N}(y)))\right] d\tau}{|x-y|^{n+2s}}\\
&\qquad \cdot (v_{\varepsilon,N}(x) - v_{\varepsilon,N}(y))\, dxdy\\
&= \int_{\mathbb{R}^{2n}} K(x,y) \frac{\int_0^1 \Phi_\varepsilon'(v_{\varepsilon,N}(y) + \tau(v_{\varepsilon,N}(x) - v_{\varepsilon,N}(y)))\, d\tau}{|x-y|^{n+2s}}\\
&\qquad \cdot (v_{\varepsilon,N}(x) - v_{\varepsilon,N}(y))^2\, dxdy\\
&\geq C\, [v_{\varepsilon,N}]^2_{H^s(\mathbb{R}^n)},
\end{aligned} \tag{3.17}
$$

where the constant $C > 0$ is independent of N and t but depends generally on ε. Thus, by the fractional Poincaré inequality we have

$$
B(\Phi_\varepsilon(v_{\varepsilon,N}), v_{\varepsilon,N}) \geq C \left\|v_{\varepsilon,N}\right\|^2_{H^s(\mathbb{R}^n)}. \tag{3.18}
$$

Thus, an integration of (3.15) over $[0, T_0]$ with $0 < T_0 \leq \delta$ and using (3.16) and (3.18), we obtain

$$
\begin{aligned}
&\left\|v_{\varepsilon,N}(T_0)\right\|^2_{L^2(\Omega)} + \left\|v_{\varepsilon,N}\right\|^2_{L^2(0,T_0;H^s(\mathbb{R}^n))}\\
&\leq C\left(\left\|F_N\right\|_{L^2(0,T_0;H^{-s}(\Omega))} \left\|v_{\varepsilon,N}\right\|_{L^2(0,T_0;H^s(\mathbb{R}^n))} + \left\|v_{0,N}\right\|^2_{L^2(\Omega)}\right),
\end{aligned} \tag{3.19}
$$

for some constant $C > 0$ independent of N and $0 < T_0 \leq \delta$. By applying Young's inequality we can absorb the factor $\left\|v_{\varepsilon,N}\right\|_{L^2(0,T_0;H^s(\mathbb{R}^n))}$, appearing on the right of (3.19), on the left hand side to obtain

$$
\begin{aligned}
&\left\|v_{\varepsilon,N}(T_0)\right\|^2_{L^2(\Omega)} + \left\|v_{\varepsilon,N}\right\|^2_{L^2(0,T_0;H^s(\mathbb{R}^n))}\\
&\leq C\left(\left\|F_N\right\|^2_{L^2(0,T_0;H^{-s}(\Omega))} + \left\|v_{0,N}\right\|^2_{L^2(\Omega)}\right),
\end{aligned} \tag{3.20}
$$

for some constant $C > 0$ independent of N and $0 < T_0 \leq \delta$.

Notice that by the convergence $F_N \to F$ in $L^2(0,T;H^{-s}(\Omega))$, then for $N \in \mathbb{N}$, each term $\|F_N\|_{L^2(0,T_0;H^{-s}(\Omega))}$ is uniformly bounded in $N \in \mathbb{N}$ and T_0 and additionally by Parseval's identity, one has

$$
\left\|v_{0,N}\right\|_{L^2(\Omega)} \leq \left\|v_0\right\|_{L^2(\Omega)}. \tag{3.21}
$$

Thus, after taking the supremum in $T_0 \in [0, \delta]$ we get

$$
\left\|v_{\varepsilon,N}\right\|_{L^\infty(0,\delta;L^2(\Omega))} + \left\|v_{\varepsilon,N}\right\|_{L^2(0,\delta;H^s(\mathbb{R}^n))} \leq C
$$

uniformly in $N \in \mathbb{N}$, for some constant $C > 0$ independent of N. But this means $v_{\varepsilon,N}$ remains in $L^2(\Omega)$ as $t \to \delta$ and hence we can repeat our local existence result finitely many times to conclude that $v_{\varepsilon,N} \in C^1([0,T]; E_N)$ solves (3.10) on $[0,T]$.

Next observe that (3.20) and (3.21) gives us by standard arguments a useful energy estimate for $v_{\varepsilon,N}$, namely

$$
\begin{aligned}
&\left\|v_{\varepsilon,N}\right\|_{L^\infty(0,T;L^2(\Omega))} + \left\|v_{\varepsilon,N}\right\|_{L^2(0,T;H^s(\mathbb{R}^n))}\\
&\leq C\left(\left\|F_N\right\|_{L^2(0,T;H^{-s}(\Omega))} + \left\|v_0\right\|_{L^2(\Omega)}\right)\\
&\leq C\left(1 + \left\|F\right\|_{L^2(0,T;H^{-s}(\Omega))} + \left\|v_0\right\|_{L^2(\Omega)}\right)
\end{aligned} \tag{3.22}
$$

uniformly in $N \in \mathbb{N}$. In addition, we want to control $\partial_t v_{\varepsilon,N}$ in $L^2(0,T;H^{-s}(\Omega))$. Note that by (3.10), Hölder's inequality, the uniform ellipticity of Φ'_ε and K, (3.22), the convergence $F_N \to F$ in $L^2(0,T;H^{-s}(\Omega))$ and (3.8), for any $\psi \in L^2(0,T;\widetilde{H}^s(\Omega))$ we have

$$
\begin{aligned}
&\left|\int_0^T \langle \partial_t v_{\varepsilon,N}, \psi\rangle_{L^2}\, dt\right| \\
&= \left|\int_0^T \langle \partial_t v_{\varepsilon,N}, \psi_N\rangle_{L^2}\, dt\right| \\
&= \left|-\int_0^T B(\Phi_\varepsilon(v_{\varepsilon,N}),\psi_N) + \langle F_N, v_N\rangle\, dt\right| \\
&\leq C\,[\Phi_\varepsilon(v_{\varepsilon,N})]_{L^2(0,T;H^s(\mathbb{R}^n))}\,[\psi_N]_{L^2(0,T;H^s(\mathbb{R}^n))} \\
&\quad + \|F_N\|_{L^2(0,T;H^{-s}(\Omega))}\|v_N\|_{L^2(0,T;H^s(\mathbb{R}^n))} \\
&\leq C\left([v_{\varepsilon,N}]_{L^2(0,T;H^s(\mathbb{R}^n))} + \|F_N\|_{L^2(0,T;H^{-s}(\Omega))}\right)\|\psi_N\|_{L^2(0,T;H^s(\mathbb{R}^n))} \\
&\leq C\,\|\psi_N\|_{L^2(0,T;H^s(\mathbb{R}^n))} \\
&\leq C\,\|\psi\|_{L^2(0,T;H^s(\mathbb{R}^n))},
\end{aligned}
\tag{3.23}
$$

for some constant C independent of N, where we set $\psi_N = \sum_{j=1}^N \langle \psi, w_j\rangle_{L^2} w_j$. Finally, using $\langle U,V\rangle_{L^2} = \langle U,V\rangle$ whenever $U,V \in \widetilde{H}^s(\Omega)$, we see that

$$
\partial_t v_{\varepsilon,N} \in L^2(0,T;H^{-s}(\Omega)) \quad \text{with} \quad \|\partial_t v_{\varepsilon,N}\|_{L^2(0,T;H^{-s}(\Omega))} \leq C, \tag{3.24}
$$

for some $C > 0$ independent of N.

Step 2: Passing to the limit $N \to \infty$.

By (3.22) and (3.24), we see that $(v_{\varepsilon,N})_{N\in\mathbb{N}} \subset X$ is uniformly bounded, where

$$
X := L^2(0,T;\widetilde{H}^s(\Omega)) \cap H^1(0,T;H^{-s}(\Omega)).
$$

Since X is a reflexive Banach space, there exists $v_\varepsilon \in X$ such that

$$
v_{\varepsilon,N} \rightharpoonup v_\varepsilon \text{ in } X \text{ as } N \to \infty \tag{3.25}
$$

(up to extracting a subsequence). Next recall that by the Rellich–Kondrachov theorem $\widetilde{H}^s(\Omega) \hookrightarrow L^2(\Omega)$ is compact and thus Theorem A.1 implies that $(v_{\varepsilon,N})_{N\in\mathbb{N}}$ is precompact in $L^2(\Omega_T)$ and hence up to extracting a subsequence we have

$$
v_{\varepsilon,N} \to v_\varepsilon \text{ in } L^2(\Omega_T) \text{ and a.e. in } \Omega_T \text{ as } N \to \infty. \tag{3.26}
$$

Next, we claim that:

Claim 3.7. *There holds*

$$
\int_0^T B(\Phi_\varepsilon(v_{\varepsilon,N}),\psi)\, dt \to \int_0^T B(\Phi_\varepsilon(v_\varepsilon),\psi)\, dt \ \textit{as } N \to \infty,
$$

for all $\psi \in L^2(0,T;\widetilde{H}^s(\Omega))$.

Proof of Claim 3.7. Using a similar method as in (3.17), the fundamental theorem of calculus yields that

$$
\begin{aligned}
&\left|\int_0^T B(\Phi_\varepsilon(v_{\varepsilon,N}),\psi)\,dt-\int_0^T B(\Phi_\varepsilon(v_\varepsilon),\psi)\,dt\right| \\
&=\Bigg|\int_{\mathbb{R}^{2n}_T} K(x,y)\left(\int_0^1 \Phi_\varepsilon'(v_{\varepsilon,N}(y)+\tau(v_{\varepsilon,N}(x)-v_{\varepsilon,N}(y)))\,d\tau\right) \\
&\qquad\cdot\frac{(v_{\varepsilon,N}(x)-v_{\varepsilon,N}(y))(\psi(x)-\psi(y))}{|x-y|^{n+2s}}\,dxdydt \\
&\qquad-\int_{\mathbb{R}^{2n}_T} K(x,y)\left(\int_0^1 \Phi_\varepsilon'(v_\varepsilon(y)+\tau(v_\varepsilon(x)-v_\varepsilon(y)))\,d\tau\right) \\
&\qquad\cdot\frac{(v_\varepsilon(x)-v_\varepsilon(y))(\psi(x)-\psi(y))}{|x-y|^{n+2s}}\,dxdydt\Bigg| \\
&\leq\left|\int_{\mathbb{R}^{2n}_T} K(x,y)\Psi_{\varepsilon,N}(x,y)\frac{(v_{\varepsilon,N}(x)-v_{\varepsilon,N}(y))(\psi(x)-\psi(y))}{|x-y|^{n+2s}}\,dxdydt\right| \\
&\quad+\Bigg|\int_{\mathbb{R}^{2n}_T} K(x,y)\left(\int_0^1 \Phi_\varepsilon'(v_\varepsilon(y)+\tau(v_\varepsilon(x)-v_\varepsilon(y)))\,d\tau\right) \\
&\qquad\cdot\frac{((v_{\varepsilon,N}-v_\varepsilon)(x)-(v_{\varepsilon,N}-v_\varepsilon)(y))(\psi(x)-\psi(y))}{|x-y|^{n+2s}}\,dxdydt\Bigg| \\
&:=I_1^N+I_2^N,
\end{aligned}
$$

where we have set

$$
\begin{aligned}
&\Psi_{\varepsilon,N}(x,y) \\
&:=\int_0^1 \left(\Phi_\varepsilon'(v_{\varepsilon,N}(y)+\tau(v_{\varepsilon,N}(x)-v_{\varepsilon,N}(y)))-\Phi_\varepsilon'(v_\varepsilon(y)+\tau(v_\varepsilon(x)-v_\varepsilon(y)))\right)d\tau.
\end{aligned}
$$

It is clear that the operator in I_2^N defines an element of $L^2(0,T;H^{-s}(\Omega))$ for fixed ψ and v_ε. Thus, the weak convergence $v_{\varepsilon,N}\rightharpoonup v_\varepsilon$ in $L^2(0,T;\widetilde{H}^s(\Omega))$ (see (3.25)), implies that $I_2^N\to 0$ as $N\to\infty$. Hence, we need only to show that $I_1^N\to 0$ as $N\to\infty$.

As $\Phi_\varepsilon\in C^1(\mathbb{R})$, the convergence (3.26) implies

$$
\Phi_\varepsilon'(v_{\varepsilon,N}(y)+\tau(v_{\varepsilon,N}(x)-v_{\varepsilon,N}(y)))-\Phi_\varepsilon'(v_\varepsilon(y)+\tau(v_\varepsilon(x)-v_\varepsilon(y)))\to 0
$$

for a.e. $x,y\in\mathbb{R}^n$ and $t\in(0,T)$ as $N\to\infty$. Hence, the uniform ellipticity of Φ_ε' and Lebesgue's dominated convergence theorem implies

$$
\Psi_N(x,y)\to 0 \text{ as } N\to\infty, \tag{3.27}
$$

for a.e. $x,y\in\mathbb{R}^n$. Now, let us define

$$
W_{\varepsilon,N}(x,y):=K(x,y)\Psi_{\varepsilon,N}(x,y)\frac{v_{\varepsilon,N}(x)-v_{\varepsilon,N}(y)}{|x-y|^{n/2+s}}\in L^2(\mathbb{R}^{2n}_T).
$$

By (3.22) and the uniform ellipticity of K, this sequence is uniformly bounded in $L^2(\mathbb{R}^{2n}_T)$. Thus up to subsequences, we know that $W_{\varepsilon,N}\rightharpoonup W_\varepsilon$ in $L^2(\mathbb{R}^{2n}_T)$ for some $W_\varepsilon\in L^2(\mathbb{R}^{2n}_T)$. Additionally, we know that $W_{\varepsilon,N}\to 0$ a.e. in $\mathbb{R}^{2n}_T$ by (3.27). As a consequence of Egoroff's theorem (see [EG15, Theorem 1.16]), we deduce that $W_\varepsilon=0$ a.e. in $\mathbb{R}^{2n}_T$. Now, testing the weak convergence $W_{\varepsilon,N}\rightharpoonup 0$ in $L^2(\mathbb{R}^{2n}_T)$ with $\frac{\psi(x)-\psi(y)}{|x-y|^{n/2+s}}\in L^2(\mathbb{R}^{2n}_T)$ establishes $I_1^N\to 0$ as $N\to\infty$ and we can conclude the proof of the claim. ☐

Now, we return back to our proof of Theorem 3.5. Let $N \geq M$, use $\psi \in C^1([0,T];E_M)$ satisfying $\psi(T)=0$ in (3.11) as a test function and integrate by parts in time the resulting equation to obtain

(3.28)
$$-\int_{\Omega_T} v_{\varepsilon,N}\partial_t\psi\,dxdt+\int_0^T B(\Phi_\varepsilon(v_{\varepsilon,N}),\psi)\,dt=\int_0^T\langle F_N,\psi\rangle\,dt+\int_\Omega v_{0,N}\psi(0)\,dx.$$

By (3.25), (3.26), Claim 3.7, $F_N\to F$ in $L^2(0,T;H^{-s}(\Omega))$ and $v_{0,N}\to v_0$ in $L^2(\Omega)$, we can pass to the limit in the above equation (3.28) and get

$$-\int_{\Omega_T} v_\varepsilon\partial_t\psi\,dxdt+\int_0^T B(\Phi_\varepsilon(v_\varepsilon),\psi)\,dt=\int_0^T\langle F,\psi\rangle\,dt+\int_\Omega v_0\psi(0)\,dx$$

for all $\psi\in C^1([0,T];E_M)$ with $\psi(T)=0$ and $M\geq 0$. As $\widetilde{H}^s(\Omega)=\overline{\bigcup_{M\in\mathbb{N}}E_M}$, this actually holds for all $\psi\in C^1([0,T];\widetilde{H}^s(\Omega))$ with $\psi(T)=0$. Hence, in particular it holds for all $\psi\in C_c^\infty([0,T)\times\Omega)$. Thus, for any $\varepsilon>0$, $v_0\in L^2(\Omega)$, $F\in L^2(0,T;H^{-s}(\Omega))$ we have found a solution $v_\varepsilon\in L^2(0,T;\widetilde{H}^s(\Omega))\cap H^1(0,T;H^{-s}(\Omega))$ of

$$\begin{cases}\partial_t v+L(\Phi_\varepsilon(v))=F & \text{in } \Omega_T,\\ v=0 & \text{in } (\Omega_e)_T,\\ v(0)=v_0 & \text{in } \Omega.\end{cases}$$

Step 3: Passing to the limit $\varepsilon\to 0$.

Now, let $0\leq u_0\in L^\infty(\Omega)\cap\widetilde{H}^s(\Omega)$ and $\varphi\in C_c^\infty((\Omega_e)_T)$ or $\varphi\in C_c([0,T]\times\Omega_e)$ with $\Phi(\varphi)\in L^2(0,T;H^s(\mathbb{R}^n))$ as in the assumptions of Theorem 3.5. The same we said at the beginning of the proof for the solvability of (3.4) is true when we replace Φ by Φ_ε. More precisely, u_ε solves

$$\begin{cases}\partial_t u+L(\Phi_\varepsilon(u))=0 & \text{in } \Omega_T,\\ u=\varphi & \text{in } (\Omega_e)_T,\\ u(0)=u_0 & \text{in } \Omega\end{cases}$$

if and only if $v_\varepsilon=u_\varepsilon-\varphi$ solves

(3.29)
$$\begin{cases}\partial_t v+L(\Phi_\varepsilon(v))=f_\varepsilon & \text{in } \Omega_T,\\ v=0 & \text{in } (\Omega_e)_T,\\ v(0)=u_0 & \text{in } \Omega,\end{cases}$$

where $f_\varepsilon=-L(\Phi_\varepsilon(\varphi))$. The functions f_ε satisfy

(A) $0\leq f_\varepsilon(x,t)\leq C\int_{\Omega_e}\Phi_\varepsilon(\varphi)(y,t)\,dy$ for $(x,t)\in\Omega_T$,
(B) $f_\varepsilon\in C([0,T];L^2(\Omega))$.

By *Step 1* and *Step 2*, we know that a solution v_ε of (3.29) exists such that $v_\varepsilon\in L^2(0,T;\widetilde{H}^s(\Omega))\cap H^1(0,T;H^{-s}(\Omega))$. Note that the additional approximation of the inhomogenity is here not needed by (B). Next, we need to establish the following claim:

Claim 3.8. *For any $\varepsilon>0$, there holds*

(i) $\partial_t v_\varepsilon\in L^2(\Omega_T)$,
(ii) $\Phi_\varepsilon(v_\varepsilon)\in L^2(0,T;\widetilde{H}^s(\Omega))$,
(iii) $\partial_t(\Phi_\varepsilon(v_\varepsilon))=\Phi_\varepsilon'(v_\varepsilon)\partial_t v_\varepsilon$ *a.e. in* Ω_T.

Proof of Claim 3.8. Since $\Phi_\varepsilon \in C^1(\mathbb{R})$ and $v_{\varepsilon,N} \in C^1([0,T];E_N)$, we may compute

$$\partial_t\left(\Phi_\varepsilon(v_{\varepsilon,N})\right) = \Phi_\varepsilon'(v_{\varepsilon,N})\partial_t v_{\varepsilon,N}.$$

Hence, $\partial_t\Phi_\varepsilon(v_{\varepsilon,N}) \in E_N$ for a.e. $t \in (0,T)$ and therefore using this function in (3.10), we obtain

$$\int_\Omega \Phi_\varepsilon'(v_{\varepsilon,N})\left|\partial_t v_{\varepsilon,N}\right|^2 dx + \frac{1}{2}\partial_t B(\Phi_\varepsilon(v_{\varepsilon,N}),\Phi_\varepsilon(v_{\varepsilon,N})) = \left\langle f_\varepsilon, \partial_t\Phi_\varepsilon(v_{\varepsilon,N})\right\rangle.$$

An integration over $[0,t] \subset [0,T]$, using the uniform ellipticity of Φ_ε' and K and the fractional Poincaré inequality, we get

$$\begin{aligned}&\int_0^t\int_\Omega \left|\partial_t v_{\varepsilon,N}\right|^2 dxds + \|\Phi_\varepsilon(v_{\varepsilon,N}(t))\|^2_{H^s(\mathbb{R}^n)}\\ &\leq C\left(B(\Phi_\varepsilon(v_{0,N}),\Phi_\varepsilon(v_{0,N})) + \|f_\varepsilon\|_{L^2(\Omega_T)}\|\partial_t\Phi_\varepsilon(v_{\varepsilon,N})\|_{L^2(\Omega_T)}\right).\end{aligned}$$

By standard arguments this gives

$$\begin{aligned}&\|\partial_t v_{\varepsilon,N}\|_{L^2(\Omega_T)} + \|\Phi_\varepsilon(v_{\varepsilon,N})\|_{L^\infty(0,T;H^s(\mathbb{R}^n))}\\ &\leq C\left(B(\Phi_\varepsilon(u_{0,N}),\Phi_\varepsilon(u_{0,N})) + \|f_\varepsilon\|_{L^2(\Omega_T)}\right).\end{aligned}$$

Finally, using the uniform ellipticity of K and Φ_ε, $u_0 \in \widetilde{H}^s(\Omega)$ and (3.8), we get

$$\begin{aligned}&\|\partial_t v_{\varepsilon,N}\|_{L^2(\Omega_T)} + \|\Phi_\varepsilon(v_{\varepsilon,N})\|_{L^\infty(0,T;H^s(\mathbb{R}^n))}\\ &\leq C\left([\Phi_\varepsilon(u_{0,N})]_{H^s(\mathbb{R}^n)} + \|f_\varepsilon\|_{L^2(\Omega_T)}\right)\\ &\leq C\left([u_{0,N}]_{H^s(\mathbb{R}^n)} + \|f_\varepsilon\|_{L^2(\Omega_T)}\right)\\ &\leq C\left([u_0]_{H^s(\mathbb{R}^n)} + \|f_\varepsilon\|_{L^2(\Omega_T)}\right).\end{aligned} \tag{3.30}$$

Hence, $\partial_t\Phi_\varepsilon(v_{\varepsilon,N})$ is uniformly bounded in $L^2(\Omega_T)$ in N. Therefore, we conclude that

$$\partial_t v_{\varepsilon,N} \rightharpoonup \partial_t v_\varepsilon \text{ in } L^2(\Omega_T) \text{ as } N\to\infty \tag{3.31}$$

as we wish (see (3.26)). Hence, we have established (i).

Additionally, we see that $\Phi_\varepsilon(v_{\varepsilon,N})$ is uniformly bounded in $L^\infty(0,T;H^s(\mathbb{R}^n))$ and hence $\Phi_\varepsilon(v_{\varepsilon,N}) \rightharpoonup w_\varepsilon$ in $L^2(0,T;\widetilde{H}^s(\Omega))$ as $N\to\infty$ for some $w_\varepsilon \in L^2(0,T;\widetilde{H}^s(\Omega))$. Again by the Aubin–Lions lemma (Theorem A.1), we have $\Phi_\varepsilon(v_{\varepsilon,N}) \to w_\varepsilon$ in $L^2(\Omega_T)$ as $N\to\infty$. Furthermore, the Lipschitz continuity of Φ_ε and (3.26) imply $w_\varepsilon = \Phi_\varepsilon(v_\varepsilon)$. Hence, we can conclude

$$\Phi_\varepsilon(v_{\varepsilon,N}) \rightharpoonup \Phi_\varepsilon(v_\varepsilon) \text{ in } L^2(0,T;\widetilde{H}^s(\Omega)) \text{ as } N\to\infty. \tag{3.32}$$

This shows (ii). By (3.26) and $\Phi_\varepsilon' \in C^0$, we know

$$\Phi_\varepsilon'(v_{\varepsilon,N}) \to \Phi_\varepsilon'(v_\varepsilon) \text{ a.e. in } \Omega_T \text{ as } N\to\infty,$$

but the uniform boundedness of Φ_ε' and Lebesgue's dominated convergence theorem give rise to

$$\Phi_\varepsilon'(v_{\varepsilon,N}) \to \Phi_\varepsilon'(v_\varepsilon) \text{ in } L^2(\Omega_T) \text{ as } N\to\infty. \tag{3.33}$$

Therefore, we can deduce

$$\partial_t\Phi_\varepsilon(v_{\varepsilon,N}) = \Phi_{\varepsilon,N}'(v_{\varepsilon,N})\partial_t v_{\varepsilon,N} \to \Phi_\varepsilon'(v_\varepsilon)\partial_t v_\varepsilon \text{ in } L^2(\Omega_T) \text{ as } N\to\infty,$$

since it is a product of a weakly and strongly converging sequence in $L^2(\Omega_T)$ (see (3.31) and (3.33)). On the other hand, by (3.32) we know

$$\partial_t\Phi_\varepsilon(v_{\varepsilon,N}) \to \partial_t\Phi_\varepsilon(v_\varepsilon) \text{ in } \mathscr{D}'(\Omega_T) \text{ as } N\to\infty,$$

and hence

$$\partial_t\Phi_\varepsilon(v_\varepsilon) = \Phi_\varepsilon'(v_\varepsilon)\partial_t v_\varepsilon \text{ a.e. in } \Omega_T.$$

This finally shows (iii) and we can conclude the proof of Claim 3.8. □

In order to complete the proof of Theorem 3.5, we next prove a maximum principle for problem (3.29).

Claim 3.9. *There exists $M > 0$ such that*

$$0 \leq v_\varepsilon(x,t) \leq M \text{ a.e. in } \Omega_T. \tag{3.34}$$

Proof of Claim 3.9. First note that by construction the function $u_\varepsilon := v_\varepsilon + \varphi \in L^2(0,T;H^s(\mathbb{R}^n))$ solves

$$\begin{cases} \partial_t u + L(\Phi_\varepsilon(u)) = 0 & \text{in } \Omega_T, \\ u = \varphi & \text{in } (\Omega_e)_T, \\ u(0) = u_0 & \text{in } \Omega. \end{cases}$$

Here, we used that $\mathrm{supp}\,(v_\varepsilon) \subset \Omega_T$, $\mathrm{supp}\,(\varphi) \subset [0,T] \times \Omega_e$ are disjoint, and hence

$$\Phi_\varepsilon(u_\varepsilon) = \Phi_\varepsilon(v_\varepsilon) + \Phi_\varepsilon(\varphi).$$

Moreover, observe by Claim 3.8, there holds

$$\int_{\Omega_T} (\partial_t u_\varepsilon)\, \psi \, dxdt + \int_0^T B(\Phi_\varepsilon(u_\varepsilon), \psi)\, dt = 0$$

for all $\psi \in L^2(0,T;\widetilde{H}^s(\Omega))$ and thus

$$\int_\Omega (\partial_t u_\varepsilon)\, \psi \, dx + B(\Phi_\varepsilon(u_\varepsilon), \psi) = 0 \tag{3.35}$$

for a.e. $0 < t < T$ and all $\psi \in \widetilde{H}^s(\Omega)$.

Next, let us define $w_\varepsilon \in L^2(\mathbb{R}^n_T)$ by

$$w_\varepsilon = (u_\varepsilon - M)_+ \quad \text{with} \quad M := \sup_{x\in\Omega} u_0 + \sup_{(x,t)\in[0,T]\times\Omega_e} \varphi. \tag{3.36}$$

Here and later, we write a subscript $+$ for the positive part and $-$ for the negative part of a given function. We assert that $w_\varepsilon \in H^1(0,T;L^2(\Omega)) \cap L^2(0,T;\widetilde{H}^s(\Omega))$. In fact, by (3.36) we known $\partial_t u_\varepsilon \in L^2(\Omega_T)$ and hence $\partial_t w_\varepsilon \in L^2(\Omega_T)$, then it is known that

$$\partial_t w_\varepsilon = (\partial_t u_\varepsilon)\, \chi_{\{u_\varepsilon \geq M\}} \in L^2(\Omega_T), \tag{3.37}$$

where χ_A denotes the characteristic function of a given set A. On the other hand, we may estimate (suppressing the t dependence)

$$[w_\varepsilon]_{H^s(\mathbb{R}^n)} \leq [u_\varepsilon]_{H^s(\mathbb{R}^n)}\,.$$

where we used that $t \mapsto t_+ = \max(t,0)$ is Lipschitz with Lipschitz constant 1. Taking into account

$$\|w_\varepsilon\|_{L^2(\mathbb{R}^n)} = \|w_\varepsilon\|_{L^2(\Omega)} \leq \|u_\varepsilon\|_{L^2(\Omega)} + M|\Omega|^{1/2},$$

where $|\Omega|$ denotes the Lebesgue measure of Ω, and this implies $w_\varepsilon \in L^2(0,T;H^s(\mathbb{R}^n))$. Moreover, by assumption there holds $w_\varepsilon = 0$ in $(\Omega_e)_T$ and thus the Lipschitz continuity gives $w_\varepsilon \in L^2(0,T;\widetilde{H}^s(\Omega))$. This completes the proof of the fact $w_\varepsilon \in H^1(0,T;L^2(\Omega)) \cap L^2(0,T;\widetilde{H}^s(\Omega))$.

Using $\psi = w_\varepsilon$ as a test function in (3.35) and (3.37), we have

$$\begin{aligned} 0 &= \int_\Omega (\partial_t u_\varepsilon) w_\varepsilon \, dx + B(\Phi_\varepsilon(u_\varepsilon), w_\varepsilon) \\ &= \int_\Omega (\partial_t w_\varepsilon) w_\varepsilon \, dx + B(\Phi_\varepsilon(u_\varepsilon), w_\varepsilon) \\ &= \partial_t \int_\Omega \frac{|w_\varepsilon|^2}{2}\, dx + B(\Phi_\varepsilon(u_\varepsilon), w_\varepsilon). \end{aligned} \tag{3.38}$$

Next, we write the bilinear form as

$$B(\Phi_\varepsilon(u_\varepsilon), w_\varepsilon) = \int_{\mathbb{R}^{2n}} K(x,y) \left(\int_0^1 \Phi_\varepsilon'(u_\varepsilon(y) + \tau(u_\varepsilon(x) - u_\varepsilon(y))\, d\tau \right) \cdot \frac{(u_\varepsilon(x) - u_\varepsilon(y))(w_\varepsilon(x) - w_\varepsilon(y))}{|x-y|^{n+2s}}\, dxdy \tag{3.39}$$

and observe

$$\begin{aligned}
&(u_\varepsilon(x) - u_\varepsilon(y))\,(w_\varepsilon(x) - w_\varepsilon(y)) \\
&= |w_\varepsilon(x) - w_\varepsilon(y)|^2 - ((u_\varepsilon - M)_-(x) - (u_\varepsilon - M)_-(y))\,(w_\varepsilon(x) - w_\varepsilon(y)) \\
&= |w_\varepsilon(x) - w_\varepsilon(y)|^2 + (u_\varepsilon - M)_-\,(x) w_\varepsilon(y) + (u_\varepsilon - M)_-\,(y) w_\varepsilon(x) \\
&\geq 0.
\end{aligned}$$

This guarantees

$$\int_0^t B(\Phi_\varepsilon(u_\varepsilon), w_\varepsilon)\, d\tau \geq 0 \tag{3.40}$$

for all $0 \leq t \leq T$. Integration of (3.38) over $[0,t] \subset [0,T]$ gives

$$\int_\Omega \frac{|w_\varepsilon(t)|^2}{2}\, dx + \int_0^t B(\Phi_\varepsilon(u_\varepsilon(\tau)), w_\varepsilon(\tau)) = \int_\Omega \frac{|w_\varepsilon(0)|^2}{2}\, dx.$$

By the fact that $w_\varepsilon(0) = 0$ and (3.40), we deduce $w_\varepsilon = 0$ a.e. in Ω_T and hence $u_\varepsilon \leq M$ a.e. in Ω_T. This establishes the upper bound.

On the other hand, we want to show that $u_\varepsilon \geq 0$ a.e. in Ω_T. The proof is exactly the same but this time we take

$$w_\varepsilon = (u_\varepsilon)_- \in L^2(0,T; \widetilde{H}^s(\Omega))$$

as a test function in (3.35). This regularity follows precisely the same lines as before. Taking into account

$$\partial_t w_\varepsilon = (-\partial_t u_\varepsilon)\, \chi_{\{u_\varepsilon \leq 0\}},$$

we deduce

$$-\partial_t \int_\Omega \frac{|w_\varepsilon|^2}{2}\, dx + B(\Phi_\varepsilon(u_\varepsilon), w_\varepsilon) = 0 \tag{3.41}$$

for a.e. $0 < t < T$. Next, note that we have

$$\begin{aligned}
&(u_\varepsilon(x) - u_\varepsilon(y))(w_\varepsilon(x) - w_\varepsilon(y)) \\
&= -(w_\varepsilon(x) - w_\varepsilon(y))^2 - (u_\varepsilon)_+(x) w_\varepsilon(y) - (u_\varepsilon)_+(y) w_\varepsilon(x) \\
&\leq 0.
\end{aligned} \tag{3.42}$$

Hence, combining (3.42) with (3.39) we get

$$\int_0^t B(\Phi_\varepsilon(u_\varepsilon), w_\varepsilon)\, d\tau \leq 0$$

for $0 < t < T$. Hence, integrating (3.41) over $[0,t] \subset [0,T]$, we obtain

$$\int_\Omega \frac{|w_\varepsilon(t)|^2}{2}\, dx - \int_0^t B(\Phi_\varepsilon(u_\varepsilon), w_\varepsilon)\, d\tau = \int_\Omega \frac{|w_\varepsilon(0)|^2}{2}\, dx. \tag{3.43}$$

By the fact that $w_\varepsilon(0) = 0$ as $u_0 \geq 0$, the right hand side vanishes. From (3.42), we deduce that terms on the left hand side of (3.43) are nonnegative and thus it follows that $u_\varepsilon \geq 0$ a.e. in Ω_T. Since $\varphi = 0$ in Ω_T, this shows the desired inequality (3.34) for v_ε, which establishes Claim 3.9. □

As an immediate consequence of Claim 3.9, we see that there exists $\mathcal{M}>0$ such that

$$0\leq \mathcal{V}_\varepsilon:=\Phi_\varepsilon(v_\varepsilon)\leq \mathcal{M} \text{ a.e. in } \Omega_T$$

for all sufficiently small $\varepsilon>0$. The lower bound follows from the fact that $v_\varepsilon\geq 0$ and $\Phi_\varepsilon(t)\geq 0$ for $t\geq 0$. On the other hand, $0\leq v_\varepsilon\leq M$ and (iv) of Lemma 3.2, establishes

$$\begin{aligned}\mathcal{V}_\varepsilon&=\Phi_\varepsilon(v_\varepsilon)\\&\leq|\Phi_\varepsilon(v_\varepsilon)-\Phi(v_\varepsilon)|+\Phi(v_\varepsilon)\\&\leq\|\Phi_\varepsilon-\Phi\|_{L^\infty([0,M])}+\Phi(v_\varepsilon)\\&\leq 2\Phi(M),\end{aligned}$$

where we used that $\Phi(M)<\infty$, $M>0$. Hence, $v_\varepsilon,\mathcal{V}_\varepsilon\in L^\infty(\Omega_T)$ are uniformly bounded in ε in this space and thus there exist $\overline{v},\overline{\mathcal{V}}\in L^\infty(\Omega_T)$ such that

$$v_\varepsilon\overset{*}{\rightharpoonup}\overline{v}\quad \mathcal{V}_\varepsilon\overset{*}{\rightharpoonup}\overline{\mathcal{V}}\quad\text{in}\quad L^\infty(\Omega_T),\text{ as }\varepsilon\to 0, \tag{3.44}$$

and by lower semi-continuity of $\|\cdot\|_{L^\infty(\Omega_T)}$ under weak-$*$ convergence in $L^\infty(\Omega_T)$ we have

$$\|\overline{v}\|_{L^\infty(\Omega_T)}\leq M,\quad\text{and}\quad \left\|\overline{\mathcal{V}}\right\|_{L^\infty(\Omega_T)}\leq\mathcal{M}. \tag{3.45}$$

a.e. in Ω_T. By boundedness of Ω_T and (3.44), we have

$$v_\varepsilon\rightharpoonup\overline{v},\ \mathcal{V}_\varepsilon\rightharpoonup\overline{\mathcal{V}}\text{ in }L^p(\Omega_T) \tag{3.46}$$

for any $1\leq p<\infty$.

Next, let us denote by $\Psi_\varepsilon\colon\mathbb{R}\to\mathbb{R}$ to be the antiderivative of Φ_ε satisfying $\Psi_\varepsilon(0)=0$. Since $\mathcal{V}_\varepsilon\in L^2(0,T;\widetilde{H}^s(\Omega))$ and $\partial_t v_\varepsilon\in L^2(\Omega_T)$ (see Claim 3.8), we can test the equation for v_ε with $\mathcal{V}_\varepsilon$ to obtain

$$\int_{\Omega_t}(\partial_t v_\varepsilon)\,\mathcal{V}_\varepsilon\,dxd\tau+\int_0^t B(\Phi_\varepsilon(v_\varepsilon),\mathcal{V}_\varepsilon)\,d\tau=\int_{\Omega_t}f_\varepsilon\mathcal{V}_\varepsilon\,dxd\tau \tag{3.47}$$

for $0<t\leq T$. Clearly, there holds

$$\partial_t\left(\Psi_\varepsilon(v_\varepsilon)\right)=\Psi_\varepsilon'(v_\varepsilon)\partial_t v_\varepsilon=\mathcal{V}_\varepsilon\partial_t v_\varepsilon.$$

Hence, using the uniform ellipticity of K, the fractional Poincaré inequality and Hölder's inequality, we obtain

$$\begin{aligned}&\int_\Omega\Psi_\varepsilon(v_\varepsilon)(t)\,dx+\int_0^t\|\Phi_\varepsilon(v_\varepsilon)\|^2_{H^s(\mathbb{R}^n)}\,d\tau\\&\leq C\left(\|f_\varepsilon\|_{L^2(\Omega_t)}\|\Phi_\varepsilon\|_{L^2(\Omega_t)}+\int_\Omega\Psi_\varepsilon(u_0)\,dx\right)\end{aligned}$$

By the usual argument this gives

(3.48)

$$\sup_{0\leq t\leq T}\int_\Omega\Psi_\varepsilon(v_\varepsilon)(t)\,dx+\|\Phi_\varepsilon(v_\varepsilon)\|^2_{L^2(0,T;H^s(\mathbb{R}^n))}\leq C\left(\|f_\varepsilon\|^2_{L^2(\Omega_T)}+\int_\Omega\Psi_\varepsilon(u_0)\,dx\right).$$

Assuming ε is sufficient small (i.e. $\|u_0\|_{L^\infty(\Omega)}<1/\varepsilon$), then

$$\int_\Omega\Psi_\varepsilon(u_0)\,dx\leq C \tag{3.49}$$

for all $\varepsilon>0$ sufficiently small.

On the other hand, let $\mathcal{K}\subset[0,T]\times\Omega_e$ be a compact set such that $\mathrm{supp}(\varphi)\subset\mathcal{K}$ and denote its projection onto $\mathbb{R}^n$ by $\mathcal{K}'\subset\Omega_e$, then using (a), $\Phi_\varepsilon(0)=0$, the monotonicity of Φ_ε and $\Phi_\varepsilon|_{[\varepsilon,1/\varepsilon]}=\Phi$ (see Lemma 3.2), we deduce

(3.50)
$$\begin{aligned}\|f_\varepsilon\|^2_{L^2(\Omega_T)}&\leq C\int_{\Omega_T}\left(\int_{\mathcal{K}'}\frac{\Phi_\varepsilon(\varphi(y,t))}{|x-y|^{n+2s}}\,dy\right)^2dxdt\\&\leq\frac{C}{\mathrm{dist}(\partial\Omega,\mathcal{K}')^{2n+4s}}|\Omega|\int_0^T\left(\int_{\mathcal{K}'}\Phi(\max(1,\|\varphi\|_{L^\infty((\Omega_e)_T)}))\,dy\right)^2\\&\leq C,\end{aligned}$$

for all $\varepsilon>0$ sufficiently small, where $C>0$ is a constant independent of $\varepsilon>0$. Therefore, (3.48), (3.49) and (3.50), show that $\mathcal{V}_\varepsilon$ is uniformly bounded in $L^2(0,T;\widetilde{H}^s(\Omega))$. Hence, we have

$$\overline{\mathcal{V}}\in L^2(0,T;\widetilde{H}^s(\Omega))\quad\text{and}\quad\mathcal{V}_\varepsilon\rightharpoonup\overline{\mathcal{V}}\text{ in }L^2(0,T;\widetilde{H}^s(\Omega))\text{ as }\varepsilon\to0.\tag{3.51}$$

Furthermore, we know that $L(\Phi_\varepsilon(\varphi))\to L(\Phi(\varphi))$ a.e. in Ω_T as $\varepsilon\to0$. The convergence $\Phi_\varepsilon(\varphi)\to\Phi(\varphi)$ a.e. in $\mathbb{R}^n_T$ follows from the fact that $\varphi\in C_c([0,T]\times\Omega_e)$ and $\Phi_\varepsilon\to\Phi$ uniformly on compact sets. This pointwise convergence, the uniform ellipticity of K and $\varphi\in C_c([0,T]\times\Omega)$, then imply the asserted convergence. But then (3.50) and Lebesgue's dominated convergence theorem justifies

$$f_\varepsilon\to f:=-L(\Phi(\varphi))\text{ in }L^2(\Omega_T)\text{ as }\varepsilon\to0.\tag{3.52}$$

Now, recall that v_ε satisfies

$$-\int_{\Omega_T}v_\varepsilon\partial_t\psi\,dxdt+\int_0^TB(\Phi_\varepsilon(v_\varepsilon),\psi)\,dt=\int_{\Omega_T}f_\varepsilon\psi\,dxdt+\int_\Omega u_0\psi(0)\,dx,$$

for all $\psi\in C_c^\infty([0,T)\times\Omega)$. Using (3.46), (3.51) and (3.52), we can pass to the limit $\varepsilon\to0$ and obtain

$$-\int_{\Omega_T}\overline{v}\partial_t\psi\,dxdt+\int_0^TB(\overline{\mathcal{V}},\psi)\,dt=\int_{\Omega_T}f\psi\,dxdt+\int_\Omega u_0\psi(0)\,dx,$$

for all $\psi\in C_c^\infty([0,T)\times\Omega)$. Next, we show:

Claim 3.10. *We have $\overline{\mathcal{V}}=\Phi(\overline{v})$.*

Proof of Claim 3.10. We first want to show that ∂_tv_ε is uniformly bounded in $L^2(0,T;H^{-s}(\Omega))$. For this observe that we have

$$\partial_tv_\varepsilon=-L(\mathcal{V}_\varepsilon)+f_\varepsilon$$

as distributions on Ω_T. Now, as $\mathcal{V}_\varepsilon$ is uniformly bounded in $L^2(0,T;\widetilde{H}^s(\Omega))$ and f_ε in $L^2(\Omega_T)$, we see that ∂_tv_ε is uniformly bounded in $L^2(0,T;H^{-s}(\Omega))$. Additionally by Claim 3.9, v_ε is uniformly bounded in $L^2(\Omega_T)$. Since the compact embedding $\widetilde{H}^s(\Omega)\hookrightarrow L^2(\Omega)$ and Schauder's theorem tells us that $L^2(\Omega)\hookrightarrow H^{-s}(\Omega)$ is compact, we see that all conditions of Theorem A.2 with

$$X=L^2(\Omega),\ B=Y=H^{-s}(\Omega),\text{ for }p=r=2,\text{ and }s=1$$

are satisfied. Hence, we can conclude that up to extracting a subsequence we have

$$v_\varepsilon\to\overline{v}\text{ in }L^2(0,T;H^{-s}(\Omega))\text{ as }\varepsilon\to0.$$

But then taking into account (3.51), we get

$$\int_{\Omega_T}v_\varepsilon\mathcal{V}_\varepsilon\,dxdt\to\int_{\Omega_T}\overline{v}\overline{\mathcal{V}}\,dxdt\text{ as }\varepsilon\to0,\tag{3.53}$$

since its the product of a strongly converging sequence in $L^2(0,T;H^{-s}(\Omega))$ and a weakly converging sequence in $L^2(0,T;H^{-s}(\Omega))$.

Next choose $w \in C_c^\infty(\Omega_T)$ and observe that the monotonicity of Φ_ε implies

$$\begin{aligned} 0 &\leq \int_{\Omega_T} (\mathcal{V}_\varepsilon - \Phi_\varepsilon(w))(v_\varepsilon - w)\, dxdt \\ &= \int_{\Omega_T} (\mathcal{V}_\varepsilon v_\varepsilon - \mathcal{V}_\varepsilon w - \Phi_\varepsilon(w) v_\varepsilon + \Phi_\varepsilon(w) w)\, dxdt. \end{aligned}$$

As $v_\varepsilon \rightharpoonup \overline{v}$ in $L^2(\Omega_T)$, $\mathcal{V}_\varepsilon \rightharpoonup \overline{\mathcal{V}}$ in $L^2(0,T;\widetilde{H}^s(\Omega))$, $\Phi_\varepsilon \to \Phi$ uniformly on compact sets and (3.53), we obtain in the limit $\varepsilon \to 0$:

$$\begin{aligned} 0 &\leq \int_{\Omega_T} \left(\overline{\mathcal{V}}\overline{v} - \overline{\mathcal{V}} w - \Phi(w)\overline{v} + \Phi(w) w\right) dxdt \\ &= \int_{\Omega_T} \left(\overline{\mathcal{V}} - \Phi(w)\right)(\overline{v} - w)\, dxdt, \end{aligned} \tag{3.54}$$

for all $w \in C_c^\infty(\Omega_T)$. Now, since $v_\varepsilon, \overline{v} \in L^\infty(\Omega_T)$, we can change Φ (and accordingly Φ_ε) outside a compact set without affecting our established convergence results. In fact, if we make Φ outside a sufficiently large compact set linear, then we see by an approximation argument that (3.54) is true for all $L^2(\Omega_T)$.

But then we have for all $\eta \in L^2(\Omega_T)$ and $0 \leq \lambda \leq 1$ the estimate

$$\left\langle \overline{\mathcal{V}} - \widetilde{\Phi}((1-\lambda)\overline{v} + \lambda\eta), \overline{v} - \eta \right\rangle_{L^2} \geq 0. \tag{3.55}$$

Next, we observe that $L^2(\Omega_T) \ni v \mapsto \langle \widetilde{\Phi}(v), u\rangle_{L^2}$ is continuous for every fixed $u \in L^2(\Omega_T)$. In fact, if $v_k \to v$ in $L^2(\Omega_T)$, then up to extracting a subsequence we have $\widetilde{\Phi}(v_k) \to \widetilde{\Phi}(v)$ a.e. in Ω_T. But then the linearity assumption outside a compact set guarantees that

$$\left|\widetilde{\Phi}(v_k) u\right| \leq \max\left(C, c|v_k|\right)|u| \in L^1(\Omega_T).$$

By [Bre11, Theorem 4.9], $|v_k| \leq |h|$ for some $h \in L^2(\Omega_T)$ and thus $|\widetilde{\Phi}(v_k)u| \leq g$ for some $g \in L^1(\Omega_T)$. But then the dominated convergence theorem guarantees

$$\left\langle \widetilde{\Phi}(v_k), u \right\rangle_{L^2} \to \left\langle \widetilde{\Phi}(v), u \right\rangle_{L^2} \quad \text{as } k \to \infty.$$

Hence, we can pass to the limit $\lambda \to 0$ in (3.55) and get

$$\left\langle \overline{\mathcal{V}} - \widetilde{\Phi}(\overline{v}), \overline{v} - \eta \right\rangle_{L^2(\Omega_T)} \geq 0.$$

This can only hold if $\overline{\mathcal{V}} = \widetilde{\Phi}(\overline{v})$ and we can conclude the proof of Claim 3.10. □

Next, we may observe by the compactnes of the embedding $\widetilde{H}^s(\Omega) \hookrightarrow L^2(\Omega)$ that $\mathcal{V}_\varepsilon \rightharpoonup \overline{\mathcal{V}}$ in $L^2(0,T;\widetilde{H}^s(\Omega))$ implies $\mathcal{V}_\varepsilon \to \mathcal{V}$ in $L^2(\Omega_T)$ and a.e. in Ω_T as $\varepsilon \to 0$, but then $\mathcal{V}_\varepsilon \geq 0$ ensures $\overline{\mathcal{V}} \geq 0$. Since by Claim 3.10 we have $\overline{\mathcal{V}} = \Phi(\overline{v})$, this guarantees $\Phi(\overline{v}) \geq 0$ and thus $\overline{v} \geq 0$ as $\Phi(t) < 0$ for $t < 0$. Therefore, we have found a solution of (3.6). Using the observation at the beginning of the proof, we see that $u_\varepsilon = v_\varepsilon + \varphi$ is a solution of the original problem (3.4) and the estimate (3.5) follows from (3.45). Let us note that $\Phi(u) = \Phi(u-\varphi) + \Phi(\varphi)$ belongs to $L^2(0,T;H^s(\mathbb{R}^n))$, as $\Phi(\varphi) \in L^2(0,T;H^s(\mathbb{R}^n))$. This proves the assertion of Theorem 3.5. □

We next define the used notion of weak solutions to NPMEs with a linear absorption term.

Definition 3.11 (Weak solutions with absorption term)**.** *Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain, $T > 0$, $0 < s < \alpha \leq 1$, $m \geq 1$ and $L_K \in \mathcal{L}_0$. Assume additionally that we have given $\rho, q \in C_+^{1,\alpha}(\mathbb{R}^n)$ with ρ uniformly elliptic. For given $u_0 \in L^\infty(\Omega)$,*

$\varphi \in C_c([0,T]\times\Omega_e)$ with $\Phi^m(\varphi)\in L^2(0,T;H^s(\mathbb{R}^n))$ and $f\in L^2(0,T;H^{-s}(\Omega))$, we say that $u\colon \mathbb{R}^n_T\to\mathbb{R}$ is a (weak) solution *of*

$$\begin{cases} \rho\partial_t u + L_K(\Phi^m(u)) + qu = f & \text{in } \Omega_T,\\ u=\varphi & \text{in } (\Omega_e)_T,\\ u(0)=u_0 & \text{in } \Omega, \end{cases} \tag{3.56}$$

provided that

(i) $\Phi^m(u)\in L^2(0,T;H^s(\mathbb{R}^n))$,

(ii) $\Phi^m(u-\varphi)\in L^2(0,T;\widetilde{H}^s(\Omega))$,

(iii) *u satisfies (3.56) in the sense of distributions, that is there holds*

$$\begin{split} &-\int_{\Omega_T}\rho u\partial_t\psi\,dxdt + \int_0^T B_K(\Phi^m(u),\psi)\,dt + \int_{\Omega_T} qu\psi\,dxdt\\ &=\int_0^T\langle f,\psi\rangle\,dxdt + \int_\Omega \rho u_0\psi(0)\,dx \end{split}$$

for all $\psi\in C_c^\infty([0,T)\times\Omega)$.

If it additionally satisfies $u\in L^\infty(\Omega_T)$, then it is called bounded solution.

Remark 3.12. *Note that by* [CRTZ22, Lemma 3.1] *we have*

$$\|Qw\|_{H^s(\mathbb{R}^n)}\le C\|Q\|_{C^{0,\alpha}}\|w\|_{H^s(\mathbb{R}^n)}$$

for $Q\in C^{1,\alpha}(\mathbb{R}^n)$ and $u\in H^s(\mathbb{R}^n)$, where $0<s<\alpha\le 1$ and $C>0$ only depends on s,α and n. In particular, if $w\in\widetilde{H}^s(\Omega)$ and $\partial\Omega\in C^{0,1}$, then $Qw\in\widetilde{H}^s(\Omega)$. This in turn guarantees $Qv\in L^2(0,T;H^{-s}(\Omega))$, whenever $v\in L^2(0,T;H^{-s}(\Omega))$.

Theorem 3.13 (Existence result with linear absorption term)**.** *Let $\Omega\subset\mathbb{R}^n$ be a bounded Lipschitz domain, $T>0$, $0<s<\alpha\le 1$, $m>1$ and $L_K\in\mathcal{L}_0$. Assume additionally that we have given $\rho,q\in C^{1,\alpha}_+(\mathbb{R}^n)$ with ρ uniformly elliptic. For any $0\le u_0\in L^\infty(\Omega)\cap\widetilde{H}^s(\Omega)$ and $0\le\varphi\in C_c([0,T]\times\Omega_e)$ with $\Phi^m(\varphi)\in L^2(0,T;H^s(\mathbb{R}^n))$ there exists a non-negative, bounded solution of*

$$\begin{cases} \rho\partial_t u + L_K(\Phi^m(u)) + qu = 0 & \text{in } \Omega_T,\\ u=\varphi & \text{in } (\Omega_e)_T,\\ u(0)=u_0 & \text{in } \Omega. \end{cases} \tag{3.57}$$

Moreover, there holds

$$\|u\|_{L^\infty(\Omega_T)}\le \sup_{x\in\Omega} u_0 + \sup_{(x,t)\in[0,T]\times\Omega_e}\varphi.$$

Proof. We use the same conventions as in the proof of Theorem 3.5. Instead of repeating the whole proof, we will only highlight the main differences.

Modifications in Step 1. As in Theorem 3.5 we look for solutions of the form

$$v_{\varepsilon,N}(x,t)=\sum_{j=1}^N c^j_{N,\varepsilon}(t)w_j(x)$$

for $N\in\mathbb{N}$. Next note that in the current situation the ODE to solve in the Galerkin approximation are:

$$\langle\rho\partial_t v_{\varepsilon,N},w_j\rangle_{L^2} + B(\Phi_\varepsilon(v_{\varepsilon,N}),w_j) + \langle qv_{\varepsilon,N},w_j\rangle_{L^2} = \langle F_N,w_j\rangle$$

for all $1 \leq j \leq N$ and $0 < t < T$. Expanding everything, we see that $\hat{c}_{\varepsilon,N} = (c^1_{\varepsilon,N}, \dots, c^N_{\varepsilon,N}) \in C^1([0,T];\mathbb{R}^N)$ solves

$$\begin{cases} A\partial_t \hat{c}_{\varepsilon,N} + b(\hat{c}_{\varepsilon,N}) + Q\hat{c}_{\varepsilon,N} = G_N(\hat{c}_{\varepsilon,N}), & \text{for} \quad 0 < t < T \\ \hat{c}_{\varepsilon,N}(0) = \hat{c}_{0,N}, \end{cases} \tag{3.58}$$

where

$$\begin{cases} A = (A_{i,j})_{1\leq i,j\leq N} \quad \text{with} \quad A_{i,j} = \int_\Omega \rho w_i w_j \, dx, \\ Q = (Q_{i,j})_{1\leq i,j\leq N} \quad \text{with} \quad Q_{i,j} = \langle q w_i, w_j \rangle_{L^2}, \\ \hat{c}_{0,N} = (\langle v_0, w_1\rangle_{L^2}, \dots, \langle v_0, w_N\rangle_{L^2}), \\ G_N(\hat{c}_{\varepsilon,N}) = \left\langle F_N, \sum_{j=1}^N c^j_{\varepsilon,N} w_j \right\rangle, \end{cases}$$

and for $1 \leq j \leq N$ the components $b_j \colon \mathbb{R}^N \to \mathbb{R}$ of $b = (b_1, \dots, b_N)$ are defined by

$$b_j(c) = B\left(\Phi_\varepsilon\left(\sum_{k=1}^N c_k w_k\right), w_j\right)$$

for $c = (c_1, \dots, c_N) \in \mathbb{R}^N$.

We observe that the matrix A is invertible. First note that for all $\xi \in \mathbb{R}^N$, we have

$$\begin{aligned} \xi \cdot A\xi =& \xi_i A_{ij} \xi_j = \int_\Omega \rho(\xi_i w_i)(\xi_j w_j)\, dx \\ &= \langle \sqrt{\rho}\xi_i w_i, \sqrt{\rho}\xi_j w_j\rangle_{L^2} = \|\sqrt{\rho}\xi_i w_i\|^2_{L^2} \geq 0, \end{aligned}$$

where we used the Einstein summation convention. Now, if $\xi \cdot A\xi = 0$ for some $\xi \in \mathbb{R}^N$, then the above calculation shows

$$\sum_{i=1}^N \xi_i \sqrt{\rho} w_i = 0 \text{ a.e. in } \Omega.$$

Since $\rho > 0$, this implies

$$\sum_{i=1}^N \xi_i w_i = 0 \text{ a.e. in } \Omega$$

and hence the linear independence of w_i, $1 \leq i \leq N$, shows $\xi = 0$. Therefore, A is positive definite. As A is additionally symmetric, we see that A is invertible. Therefore, finding a solution of (3.58) is equivalent to find a solution of

$$\begin{cases} \partial_t \hat{c}_{\varepsilon,N} + \widetilde{b}(\hat{c}_{\varepsilon,N}) + \widetilde{Q}\hat{c}_{\varepsilon,N} = \widetilde{G}_N(\hat{c}_{\varepsilon,N}), & \text{for} \quad 0 < t < T \\ \hat{c}_{\varepsilon,N}(0) = \hat{c}_{0,N}, \end{cases}$$

where $\widetilde{b} = A^{-1}b$, $\widetilde{Q} = A^{-1}Q$ and $\widetilde{G}_N = A^{-1}G_N$. Clearly, for this new problem we can again find by Peano's existence theorem for ODEs a solution on possibly a small interval $[0,\delta]$, $\delta > 0$. Due to the fact that ρ is uniformly elliptic and $q \geq 0$, we again arrive at equation (3.22). Hence, also in this setting we can extend $v_{\varepsilon,N}$ to a solution on $[0,T]$. Doing the computation in (3.23) with ψ/ρ instead of ψ and using Remark 3.12, we again see that $\partial_t v_{\varepsilon,N} \in L^2(0,T;H^{-s}(\Omega))$ with

$$\|\partial_t v_{\varepsilon,N}\|_{L^2(0,T;H^{-s}(\Omega))} \leq C$$

for some $C > 0$ independent of N.

Modifications in Step 2. All computations in this step hold without any further insights.

Modifications in Step 3. First note that the uniform ellipticity of ρ, $q \in L^\infty(\Omega)$ and (3.22) guarantee that (3.30) still holds (after usings Young's inequality). Hence, we can again deduce property (i) of Claim 3.8. The proof of the other two properties of Claim 3.8 remain the same. Also the proof of Claim 3.9 is a simple modification after noting

$$\begin{aligned} &q u_\varepsilon \left(u_\varepsilon - M\right)_+ = q\left(u_\varepsilon - M\right)_+^2 + qM\left(u_\varepsilon - M\right)_+ \geq 0, \\ &q u_\varepsilon (u_\varepsilon)_- = -q(u_\varepsilon)_-^2 \leq 0. \end{aligned}$$

Next, recall that to uniformly bound $\mathcal{V}_\varepsilon = \Phi_\varepsilon(v_\varepsilon)$ independently of ε, we computed (3.47). We see that the additional absorption term does not cause any problems since we have

$$q v_\varepsilon \mathcal{V}_\varepsilon \geq 0,$$

by the monotonicity of Φ_ε and $\Phi_\varepsilon(0) = 0$. Using the $v_\varepsilon \geq 0$, $\Psi_\varepsilon(t) \geq 0$ for $t \geq 0$ and the uniform ellipticity of ρ, we have again an estimate of the form (3.48). Then arguing the same way as above, we again see that we can pass to the limit $\varepsilon \to 0$ to obtain

$$\begin{aligned} &-\int_{\Omega_T} \rho \overline{v} \partial_t \psi \, dxdt + \int_0^T B(\overline{\mathcal{V}}, \psi)\, dt + \int_{\Omega_T} q \overline{v} \psi \, dxdt \\ &= \int_{\Omega_T} f \psi \, dxdt + \int_\Omega \rho u_0 \psi(0)\, dx, \end{aligned}$$

for all $\psi \in C_c^\infty([0,T) \times \Omega)$.

Hence, it remains only to check that $\mathcal{V} = \Phi(\overline{v})$. Similarly as in the beginning of the proof of Claim 3.10, we see that $\partial_t v_\varepsilon$ is uniformly bounded in $L^2(0,T; H^{-s}(\Omega))$. The only difference one has to notice is that taking into account the uniform bound of v_ε in $L^2(\Omega_T)$ one gets from the PDE that $\rho \partial_t v_\varepsilon$ is uniformly bounded in $L^2(0,T;H^{-s}(\Omega))$, but then using Remark 3.12 one obtains the uniform boundedness of $\partial_t v_\varepsilon$ in $L^2(0,T;H^{-s}(\Omega))$. The rest of the proof of Claim 3.10 is unaffected. Now, we can finish the proof of Theorem 3.13 completely analogous to the proof of Theorem 3.5. □

From the proof, we directly obtain the following corollary.

Corollary 3.14. *Assume the conditions of Theorem 3.13 are satisfied and let u be the constructed solution of (3.57). Then there exist a sequence $(\varepsilon_k)_{k\in\mathbb{N}} \subset (0,\infty)$ with $\varepsilon_k \to 0$ as $k \to \infty$, and a sequence of solutions $(u_k)_{k\in\mathbb{N}}$ of*

$$\begin{cases} \rho \partial_t u + L_K(\Phi^m_{\varepsilon_k}(u)) + qu = 0 & \text{in } \Omega_T, \\ u = \varphi & \text{in } (\Omega_e)_T, \\ u(0) = u_0 & \text{in } \Omega. \end{cases}$$

Moreover, there holds

(i) $\Phi_{\varepsilon_k}(u_k - \varphi) \in H^1(0,T;L^2(\Omega)) \cap L^2(0,T;\widetilde{H}^s(\Omega))$,
(ii) $u_k \in L^\infty(\Omega_T)$,
(iii) $u_k \in L^2(0,T;H^s(\mathbb{R}^n))$ *with* $\partial_t u_k \in L^2(\Omega_T)$,
(iv) $u_k \rightharpoonup u$ *in* $L^2(\Omega_T)$ *as* $k \to \infty$.

3.3. **Uniqueness of solutions to the forward problem.** In the end of this section, we show the uniqueness of solutions to NPMEs with absorption term.

Theorem 3.15 (Basic uniqueness result)**.** *Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain, $T > 0$, $0 < s < 1$, $m > 1$ and $L_K \in \mathcal{L}_0$. If for $j = 1,2$, we have given* *initial data* $u_{0,j} \in L^\infty(\Omega)\cap \widetilde{H}^s(\Omega)$ *and exterior conditions* $\varphi_j \in C_c([0,T]\times\Omega_e)$ *with* $\Phi^m(\varphi_j)\in L^2(0,T;H^s(\mathbb{R}^n))$ *and* u_j *is a solution of*

$$\begin{cases} \partial_t u + L_K(\Phi^m(u)) = 0 & \text{in } \Omega_T,\\ u = \varphi_j & \text{in } (\Omega_e)_T,\\ u(0) = u_{0,j} & \text{in } \Omega, \end{cases} \tag{3.59}$$

then there holds

$$\begin{split} &\|u_1-u_2\|_{L^\infty(0,T;H^{-s}(\Omega))} + \|\mathcal{U}_1-\mathcal{U}_2\|_{L^\infty(0,T;H^s(\mathbb{R}^n))}\\ &\leq C\left(\|u_{0,1}-u_{0,2}\|_{H^{-s}(\Omega)} + \|\Phi^m(\varphi_1)-\Phi^m(\varphi_2)\|_{L^1(0,T;H^s(\mathbb{R}^n))}\right), \end{split} \tag{3.60}$$

where $\mathcal{U}_j(x,t) = \int_0^t u_j(x,\tau)\,d\tau$. *In particular, the constructed solutions in Theorem 3.5 are unique.*

Remark 3.16. *Note that the same proof works if we have additional sources* $F_j \in L^2(0,T;H^{-s}(\Omega))$ *in (3.59), but then we would have an additional contribution* $\|F_1 - F_2\|_{L^1(0,T;H^{-s}(\Omega))}$ *in the continuity estimate (3.60).*

Proof of Theorem 3.15. As in the previous proofs, we write $\Phi, \Phi_\varepsilon, L, B$ instead of $\Phi^m, \Phi^m_\varepsilon, L_K, B_K$. By assumption the functions $v_j = u_j - \varphi_j$ solve

$$\begin{cases} \partial_t v + L(\Phi(v)) = f_j & \text{in } \Omega_T,\\ u = 0 & \text{in } (\Omega_e)_T,\\ u(0) = u_{0,j} & \text{in } \Omega, \end{cases} \tag{3.61}$$

where $f_j := -L(\Phi(\varphi_j))$, for $j=1,2$. Hence, by subtracting the weak formulations that there holds

$$\begin{split} &-\int_{\Omega_T}(v_1-v_2)\partial_t\psi\,dxdt + \int_0^T B(\Phi(v_1)-\Phi(v_2),\psi)\,dt\\ &=\int_{\Omega_T}(f_1-f_2)\psi\,dxdt + \int_\Omega (u_{0,1}-u_{0,2})\psi(0)\,dx \end{split} \tag{3.62}$$

for all $\psi\in C_c^\infty([0,T)\times\Omega)$. Next, we define for fixed $0\leq t_0\leq T$ the function

$$\Psi(x,t) := \begin{cases} \int_t^{t_0}(\Phi(v_1)-\Phi(v_2))(x,\tau)\,d\tau & \text{if } 0\leq t\leq t_0,\\ 0 & \text{otherwise.}\end{cases} \tag{3.63}$$

Using $\Phi(v_j)\in L^2(0,T;\widetilde{H}^s(\Omega))$ and the fundamental theorem of calculus, one can check that $\Psi\in H^1(0,T;\widetilde{H}^s(\Omega))$ with $\Psi(T)=0$. Recalling Proposition B.1, we see that Ψ can be used as a test function in (3.62). This gives

$$\begin{split} &\int_{\Omega_{t_0}}(v_1-v_2)(\Phi(v_1)-\Phi(v_2))\,dxdt\\ &+\int_0^{t_0} B\left(\Phi(v_1)-\Phi(v_2),\int_t^{t_0}(\Phi(v_1)-\Phi(v_2))(x,\tau)\,d\tau\right)dt\\ &=\int_{\Omega_T}(f_1-f_2)\int_t^{t_0}(\Phi(v_1)-\Phi(v_2))(x,\tau)\,d\tau\,dxdt\\ &+\int_\Omega(u_{0,1}-u_{0,2})\int_0^{t_0}(\Phi(v_1)-\Phi(v_2))(x,\tau)\,d\tau\,dx. \end{split} \tag{3.64}$$

Next, using Proposition B.1, take a sequence $\psi_k\in C_c^\infty([0,T)\times\Omega)$ such that $\psi_k\to\Psi$ in $H^1(0,T;\widetilde{H}^s(\Omega))$. Since $\Psi(t_0)=0$, we can assume that $\psi_k(t)=0$ for $t_0\leq t\leq T$, then by dominated convergence, we have

$$
\begin{aligned}
&\int_0^{t_0} B\left(\Phi(v_1)-\Phi(v_2), \int_t^{t_0} (\Phi(v_1)-\Phi(v_2))(x,\tau)\,d\tau\right) dt \\
&= -\int_0^{t_0} B(\partial_t \Psi, \Psi)\,dt = -\lim_{k\to\infty}\int_0^{t_0} B(\partial_t\psi_k,\psi_k)\,dt \\
&= -\frac{1}{2}\lim_{k\to\infty}\int_0^{t_0}\partial_t B(\psi_k,\psi_k)\,dt = -\frac{1}{2}\lim_{k\to\infty} B(\psi_k,\psi_k)|_{t=0}^{t=t_0} \\
&= \frac{1}{2}\lim_{k\to\infty} B(\psi_k(0),\psi_k(0)) = \frac{1}{2}B(\Psi(0),\Psi(0)) \\
&= \frac{1}{2}B\left(\int_0^{t_0}(\Phi(v_1)-\Phi(v_2))\,d\tau, \int_0^{t_0}(\Phi(v_1)-\Phi(v_2))\,d\tau\right),
\end{aligned}
\tag{3.65}
$$

where we used the Sobolev embeddding $H^1(0,T;\widetilde{H}^s(\Omega)) \hookrightarrow C([0,T];\widetilde{H}^s(\Omega))$. Additionally, by the monotonicity of Φ we have

$$
\int_{\Omega_{t_0}} (\Phi(v_1)-\Phi(v_2))(v_1-v_2)\,dxdt \geq 0. \tag{3.66}
$$

From now on we use the notation

$$
\mathcal{V}_j(x,t) := \int_0^t \Phi(v_j(x,\tau))\,d\tau,
$$

for $j=1,2$. Hence, using (3.65), (3.66), uniform ellipticity of K and the fractional Poincaré inequality, we deduce from (3.64) the estimate

$$
\begin{aligned}
&\|(\mathcal{V}_1-\mathcal{V}_2)(t_0)\|^2_{H^s(\mathbb{R}^n)} \\
&= \left\|\int_0^{t_0}(\Phi(v_1)-\Phi(v_2))\,d\tau\right\|^2_{H^s(\mathbb{R}^n)} \\
&\leq CB\left(\int_0^{t_0}(\Phi(v_1)-\Phi(v_2))\,d\tau, \int_0^{t_0}(\Phi(v_1)-\Phi(v_2))\,d\tau\right) \\
&\leq C\left\{\int_{\Omega_{t_0}}(v_1-v_2)(\Phi(v_1)-\Phi(v_2))\,dxdt\right. \\
&\qquad \left. + \int_0^{t_0} B\left(\Phi(v_1)-\Phi(v_2), \int_t^{t_0}(\Phi(v_1)-\Phi(v_2))(x,\tau)\,d\tau\right) dt\right\} \\
&= C\left\{\int_{\Omega_T}(f_1-f_2)\left(\int_t^{t_0}(\Phi(v_1)-\Phi(v_2))(x,\tau)\,d\tau\right) dxdt\right. \\
&\qquad \left. + \int_\Omega (u_{0,1}-u_{0,2})\left(\int_0^{t_0}(\Phi(v_1)-\Phi(v_2))(x,\tau)\,d\tau\right) dx\right\} \\
&\leq C\left\{\|f_1-f_2\|_{L^1(0,T;H^{-s}(\Omega))}\left\|\int_t^{t_0}(\Phi(v_1)-\Phi(v_2))\,d\tau\right\|_{L^\infty(0,t_0;H^s(\mathbb{R}^n))}\right. \\
&\qquad \left. + \|u_{0,1}-u_{0,2}\|_{H^{-s}(\Omega))}\left\|\int_0^{t_0}(\Phi(v_1)-\Phi(v_2))\,d\tau\right\|_{H^s(\mathbb{R}^n)}\right\}.
\end{aligned}
\tag{3.67}
$$

Observe that we have

$$\begin{aligned}
&\left\|\int_t^{t_0} (\Phi(v_1)-\Phi(v_2))\,d\tau\right\|_{L^\infty(0,t_0;H^s(\mathbb{R}^n))}\\
&=\|(\mathcal{V}_1-\mathcal{V}_2)(t_0)-(\mathcal{V}_1-\mathcal{V}_2)(t)\|_{L^\infty(0,t_0;H^s(\mathbb{R}^n))}\\
&\leq \|(\mathcal{V}_1-\mathcal{V}_2)(t_0)\|_{H^s(\mathbb{R}^n)}+\|\mathcal{V}_1-\mathcal{V}_2\|_{L^\infty(0,t_0;H^s(\mathbb{R}^n))}\\
&\leq 2\|\mathcal{V}_1-\mathcal{V}_2\|_{L^\infty(0,t_0;H^s(\mathbb{R}^n))}.
\end{aligned} \tag{3.68}$$

Thus, (3.67) simplifies to

$$\begin{aligned}
&\|(\mathcal{V}_1-\mathcal{V}_2)(t_0)\|^2_{H^s(\mathbb{R}^n)}\\
&\leq C\left(\|f_1-f_2\|_{L^1(0,t_0;H^{-s}(\Omega))}+\|u_{0,1}-u_{0,2}\|_{H^{-s}(\Omega)}\right)\|\mathcal{V}_1-\mathcal{V}_2\|_{L^\infty(0,t_0;H^s(\mathbb{R}^n))}\\
&\leq C\left(\|f_1-f_2\|_{L^1(0,T;H^{-s}(\Omega))}+\|u_{0,1}-u_{0,2}\|_{H^{-s}(\Omega)}\right)\|\mathcal{V}_1-\mathcal{V}_2\|_{L^\infty(0,T;H^s(\mathbb{R}^n))}.
\end{aligned}$$

Taking the supremum in $t_0\in(0,T)$ and absorbing the last factor on the left hand side, we obtain

$$\begin{aligned}
&\|\mathcal{V}_1-\mathcal{V}_2\|_{L^\infty(0,T;H^s(\mathbb{R}^n))}\\
&\leq C\left(\|f_1-f_2\|_{L^1(0,T;H^{-s}(\Omega))}+\|u_{0,1}-u_{0,2}\|_{H^{-s}(\Omega)}\right).
\end{aligned} \tag{3.69}$$

Let us point out that the PDE (3.61) guarantees $\partial_t v_j\in L^2(0,T;H^{-s}(\Omega))$ and $v_j\in L^2(0,T;H^{-s}(\Omega))$. The first assertion is immediate.

To see the second one, let $\eta\in C_c^\infty(\Omega_T)$ and insert $\psi(x,t)=-\int_t^T\eta(x,\tau)\,d\tau$ into the weak formulation of (3.61) to obtain

$$\begin{aligned}
\left|\int_{\Omega_T} v_j\eta\,dxdt\right| &= \left|-\int_{\Omega_T} v_j\partial_t\psi\,dxdt\right|\\
&=\left|-\int_0^T B(\Phi(v_j),\psi)\,dt+\int_0^T f_j\psi\,dxdt+\int_\Omega u_{0,j}\psi(0)\,dx\right|\\
&\leq C\big([\Phi(v_j)]_{L^2(0,T;H^s(\mathbb{R}^n))}[\psi]_{L^2(0,T;H^s(\mathbb{R}^n))}\\
&\qquad+\|f_j\|_{L^2(0,T;H^{-s}(\Omega))}\|\psi\|_{L^2(0,T;H^s(\mathbb{R}^n))}\\
&\qquad+\|u_{0,j}\|_{L^2(\Omega)}\|\psi(0)\|_{L^2(\Omega)}\big).
\end{aligned}$$

By Jensen's inequality, we have

$$\|\psi\|_{L^2(\mathbb{R}^n_T)}\leq C\|\eta\|_{L^2(\Omega_T)},\quad \|\psi(0)\|_{L^2(\Omega)}\leq C\|\eta\|_{L^2(\Omega_T)}$$

and

$$[\psi]^2_{H^s(\mathbb{R}^n)}\leq\int_{\mathbb{R}^{2n}}\frac{\left|\int_t^T(\eta(x,\tau)-\eta(y,\tau))\,d\tau\right|^2}{|x-y|^{n+2s}}\,dxdy\leq C[\eta]^2_{L^2(0,T;H^s(\mathbb{R}^n))}.$$

Thus, we get

$$\left|\int_{\Omega_T} v_j\eta\,dxdt\right|\leq C\|\eta\|_{L^2(0,T;H^s(\mathbb{R}^n))},$$

but this means nothing else than $v_j\in L^2(0,T;H^{-s}(\Omega))$ as $C_c^\infty(\Omega_T)$ is dense in $L^2(0,T;\widetilde{H}^s(\Omega))$. By the Sobolev embedding we have $v_j\in C([0,T];H^{-s}(\Omega))$ and hence the fundamental theorem of calculus, (3.61), the uniform ellipticity of K and the estimate (3.69) give

(3.70)
$$\begin{aligned}
&\|(v_1-v_2)(t)\|_{H^{-s}(\Omega)}\\
&\leq \left\|\int_0^t \partial_t(v_1-v_2)(\tau)\,d\tau\right\|_{H^{-s}(\Omega)}+\|(v_1-v_2)(0)\|_{H^{-s}(\Omega)}\\
&\leq \sup_{\psi\in\widetilde{H}^s(\Omega)\colon\|\psi\|_{H^s(\mathbb{R}^n)}\leq 1}\left|\int_0^t\langle\partial_t(v_1-v_2)(\tau),\psi\rangle\,d\tau\right|+\|u_{0,1}-u_{0,2}\|_{H^{-s}(\Omega)}\\
&\leq \sup_{\psi\in\widetilde{H}^s(\Omega)\colon\|\psi\|_{H^s(\mathbb{R}^n)}\leq 1}\left|-\int_0^t B(\Phi(v_1)-\Phi(v_2),\psi)\,d\tau+\int_{\Omega_t}(f_1-f_2)\psi\,dxd\tau\right|\\
&\quad+\|u_{0,1}-u_{0,2}\|_{H^{-s}(\Omega)}\\
&\leq \sup_{\psi\in\widetilde{H}^s(\Omega)\colon\|\psi\|_{H^s(\mathbb{R}^n)}\leq 1}|B((\mathcal{V}_1-\mathcal{V}_2)(t),\psi)|\\
&\quad+\|f_1-f_2\|_{L^1(0,T;H^{-s}(\Omega))}+\|u_{0,1}-u_{0,2}\|_{H^{-s}(\Omega)}\\
&\leq C\left(\|f_1-f_2\|_{L^1(0,T;H^{-s}(\Omega))}+\|u_{0,1}-u_{0,2}\|_{H^{-s}(\Omega))}\right),
\end{aligned}$$

for all $0<t\leq T$. Next, let us observe that by uniform ellipticity of K there holds

(3.71)
$$\begin{aligned}
\|f_1-f_2\|_{L^1(0,T;H^{-s}(\Omega))}&\leq\|L(\Phi(\varphi_1)-\Phi(\varphi_2))\|_{L^1(0,T;H^{-s}(\Omega))}\\
&=\int_0^T\sup_{\psi\in\widetilde{H}^s(\Omega)\colon\ \|\psi\|_{H^s(\mathbb{R}^n)}\leq 1}|B(\Phi(\varphi_1)-\Phi(\varphi_2),\psi)|\,dt\\
&\leq C\|\Phi(\varphi_1)-\Phi(\varphi_2)\|_{L^1(0,T;H^s(\mathbb{R}^n))}.
\end{aligned}$$

Combining (3.71) with (3.70) and (3.69), we have proved

$$\begin{aligned}
&\|v_1-v_2\|_{L^\infty(0,T;H^{-s}(\Omega))}+\|\mathcal{V}_1-\mathcal{V}_2\|_{L^\infty(0,T;H^s(\mathbb{R}^n))}\\
&\leq C\left(\|\Phi(\varphi_1)-\Phi(\varphi_2)\|_{L^1(0,T;H^s(\mathbb{R}^n))}+\|u_{0,1}-u_{0,2}\|_{H^{-s}(\Omega))}\right).
\end{aligned}\tag{3.72}$$

Next, we go back to the variables $u_j=v_j+\varphi_j$. Let us note that

$$\begin{aligned}
\mathcal{U}_1-\mathcal{U}_2&=\int_0^t(\Phi(u_1)-\Phi(u_2))\,d\tau\\
&=\int_0^t(\Phi(v_1+\varphi_1)-\Phi(v_2+\varphi_2))\,d\tau\\
&=\mathcal{V}_1-\mathcal{V}_2+\int_0^t(\Phi(\varphi_1)-\Phi(\varphi_2))\,d\tau,
\end{aligned}\tag{3.73}$$

where we used that v_j and φ_j have disjoint supports. Hence, inserting (3.73) into (3.72) and $\varphi_j|_{\Omega_T}=0$, we get

$$\begin{aligned}
&\|u_1-u_2\|_{L^\infty(0,T;H^{-s}(\Omega))}+\|\mathcal{U}_1-\mathcal{U}_2\|_{L^\infty(0,T;H^s(\mathbb{R}^n))}\\
&\leq C\Bigg(\|\Phi(\varphi_1)-\Phi(\varphi_2)\|_{L^1(0,T;H^s(\mathbb{R}^n))}+\|u_{0,1}-u_{0,2}\|_{H^{-s}(\Omega))}\\
&\qquad+\|\varphi_1-\varphi_2\|_{L^\infty(0,T;H^{-s}(\Omega))}+\left\|\int_0^t(\Phi(\varphi_1)-\Phi(\varphi_2))\,d\tau\right\|_{L^\infty(0,T;H^s(\mathbb{R}^n))}\Bigg)\\
&\leq C\left(\|\Phi(\varphi_1)-\Phi(\varphi_2)\|_{L^1(0,T;H^s(\mathbb{R}^n))}+\|u_{0,1}-u_{0,2}\|_{H^{-s}(\Omega))}\right).
\end{aligned}$$

Hence, we can conclude the proof. □

Theorem 3.17 (Uniqueness with linear absorption term). *Let $\Omega\subset\mathbb{R}^n$ be a bounded Lipschitz domain, $T>0$, $0<s<\alpha\leq 1$, $m>1$ and $L_K\in\mathcal{L}_0$. Assume additionally* *that we have given $\rho, q \in C^{1,\alpha}_+(\mathbb{R}^n)$ with ρ uniformly elliptic. Suppose that $u_0 \in L^\infty(\Omega) \cap \widetilde{H}^s(\Omega)$, $\varphi \in C_c([0,T] \times \Omega_e)$ with $\Phi^m(\varphi) \in L^2(0,T; H^s(\mathbb{R}^n))$ and u_1, u_2 solve*

$$\begin{cases} \rho\partial_t u + L_K(\Phi^m(u)) + qu = 0 & \text{in } \Omega_T, \\ u = \varphi & \text{in } (\Omega_e)_T, \\ u(0) = u_0 & \text{in } \Omega, \end{cases}$$

then there holds $u_1 = u_2$.

Proof. Like in the proofs above we write Φ, B, L for Φ^m, B_K, L_K. As in the proof of Theorem 3.15, we go over to the variables $v_j = u_j - \varphi$. First observe that $v_1 - v_2$ satisfies

$$\begin{cases} \rho\partial_t(v_1 - v_2) + L_K(\Phi(v_1) - \Phi(v_2)) + q(v_1 - v_2) = 0 & \text{in } \Omega_T, \\ v_1 - v_2 = 0 & \text{in } (\Omega_e)_T, \\ (v_1 - v_2)(0) = 0 & \text{in } \Omega \end{cases}$$

and hence there holds

$$-\int_{\Omega_T} (v_1 - v_2)\partial_t\psi \, dxdt + \int_0^T B(\Phi(v_1) - \Phi(v_2), \psi) \, dt + \int_{\Omega_T} q(v_1 - v_2) \, dxdt = 0 \tag{3.74}$$

for all $\psi \in C_c^\infty([0,T) \times \Omega)$. We can again use the function Ψ, as defined in (3.63), as a test function in (3.74). Using (3.65), we obtain

$$\begin{split} &\int_{\Omega_{t_0}} \rho(v_1 - v_2)(\Phi(v_1) - \Phi(v_2)) \, dxdt + \frac{1}{2}B((\mathcal{V}_1 - \mathcal{V}_2)(t_0), (\mathcal{V}_1 - \mathcal{V}_2)(t_0)) \\ &+ \int_{\Omega_{t_0}} q(v_1 - v_2) \int_t^{t_0} \Phi(v_1) - \Phi(v_2) \, d\tau \, dxdt = 0, \end{split}$$

where we used the notation $\mathcal{V}_j(t) = \int_0^t \Phi(v_j) \, d\tau$. Next, note that the monotonicity of Φ and $\rho \geq 0$ implies

$$\int_{\Omega_{t_0}} \rho(\Phi(v_1) - \Phi(v_2))(v_1 - v_2) \, dxdt \geq 0. \tag{3.75}$$

Now by using (3.75), the fractional Poincaré inequality, the uniform ellipticity of K, $q \in C^{0,\alpha}(\mathbb{R}^n)$, (3.68) and Remark 3.12, we deduce that

$$\begin{split} &\|(\mathcal{V}_1 - \mathcal{V}_2)(t_0)\|^2_{H^s(\mathbb{R}^n)} \\ &= \left\| \int_0^{t_0} (\Phi(v_1) - \Phi(v_2)) \, d\tau \right\|^2_{H^s(\mathbb{R}^n)} \\ &\leq CB\left(\int_0^{t_0} (\Phi(v_1) - \Phi(v_2)) \, d\tau, \int_0^{t_0} (\Phi(v_1) - \Phi(v_2)) \, d\tau \right) \\ &\leq C\left(\int_{\Omega_{t_0}} \rho(v_1 - v_2)(\Phi(v_1) - \Phi(v_2)) \, dxdt \right. \\ &\qquad \left. + \frac{1}{2}\int_0^{t_0} B(\Phi(v_1) - \Phi(v_2), \int_t^{t_0} (\Phi(v_1) - \Phi(v_2))(x,\tau) \, d\tau) \, dt \right) \\ &= -C\int_{\Omega_{t_0}} q(v_1 - v_2) \int_t^{t_0} (\Phi(v_1) - \Phi(v_2)) \, d\tau dxdt \\ &\leq C\|q\|_{C^{0,\alpha}(\mathbb{R}^n)} \|v_1 - v_2\|_{L^1(0,t_0;H^{-s}(\Omega))} \left\| \int_t^{t_0} (\Phi(v_1) - \Phi(v_2)) \, d\tau \right\|_{L^\infty(0,t_0;H^s(\mathbb{R}^n))} \\ &\leq C\|q\|_{C^{0,\alpha}(\mathbb{R}^n)} \|v_1 - v_2\|_{L^1(0,t_0;H^{-s}(\Omega))} \|\mathcal{V}_1 - \mathcal{V}_2\|_{L^\infty(0,t_0;H^s(\mathbb{R}^n))}. \end{split}$$

Next, let $0 < \widetilde{T} \leq T$ be a given constant, which we will fix later. The previous estimate then shows

$$\|\mathcal{V}_1 - \mathcal{V}_2\|_{L^\infty(0,\widetilde{T},H^s(\mathbb{R}^n))} \leq C\widetilde{T}\|q\|_{C^{0,\alpha}(\mathbb{R}^n)}\|v_1 - v_2\|_{L^\infty(0,\widetilde{T};H^{-s}(\Omega))}. \tag{3.76}$$

On the other hand, following the computation in (3.70) and use $\rho \in C^{0,\alpha}(\mathbb{R}^n)$, Remark 3.12, [CRTZ22, Lemma 4.1] and the uniform ellipticity of K, we find

$$\begin{aligned}
&\|(v_1 - v_2)(t_0)\|_{H^{-s}(\Omega)} \\
&\leq \left\| \int_0^{t_0} \partial_t(v_1 - v_2)(\tau)\, d\tau \right\|_{H^{-s}(\Omega)} \\
&\leq \sup_{\psi\in\widetilde{H}^s(\Omega):\, \|\psi\|_{H^s(\mathbb{R}^n)}\leq 1} \left| \int_0^{t_0} \langle \partial_t(v_1 - v_2)(\tau), \psi\rangle\, d\tau \right| \\
&\leq \sup_{\psi\in\widetilde{H}^s(\Omega):\, \|\psi\|_{H^s(\mathbb{R}^n)}\leq 1} \left| \int_0^{t_0} \langle \rho\partial_t(v_1 - v_2)(\tau), \psi/\rho\rangle\, d\tau \right| \\
&= \sup_{\psi\in\widetilde{H}^s(\Omega):\, \|\psi\|_{H^s(\mathbb{R}^n)}\leq 1} \left| \int_0^{t_0} B(\Phi(v_1) - \Phi(v_2), \psi/\rho)\, d\tau + \int_{\Omega_{t_0}} q(v_1 - v_2)\psi/\rho\, dxd\tau \right| \\
&\leq C\left(\|(\mathcal{V}_1 - \mathcal{V}_2)(t_0)\|_{H^s(\mathbb{R}^n)} + \|q\|_{C^{0,\alpha}(\mathbb{R}^n)}\|v_1 - v_2\|_{L^1(0,t_0;H^{-s}(\Omega))} \right),
\end{aligned}$$

where the constant $C > 0$ only depends on $\|K\|_{L^\infty(\mathbb{R}^n)}$, the lower bound of ρ and $\|\rho\|_{C^{0,\alpha}(\mathbb{R}^n)}$. Arguing as above this gives

$$\begin{aligned}
\|v_1 - v_2\|_{L^\infty(0,\widetilde{T};H^{-s}(\Omega))} &\leq C\,\|\mathcal{V}_1 - \mathcal{V}_2\|_{L^\infty(0,\widetilde{T},H^s(\mathbb{R}^n))} \\
&\quad + C\widetilde{T}\|q\|_{C^{0,\alpha}(\mathbb{R}^n)}\|v_1 - v_2\|_{L^\infty(0,\widetilde{T};H^{-s}(\Omega))}.
\end{aligned} \tag{3.77}$$

Inserting (3.76) into (3.77), we get

$$\|v_1 - v_2\|_{L^\infty(0,\widetilde{T};H^{-s}(\Omega))} \leq C\widetilde{T}\|q\|_{C^{0,\alpha}(\mathbb{R}^n)}\|v_1 - v_2\|_{L^\infty(0,\widetilde{T};H^{-s}(\Omega))}. \tag{3.78}$$

Hence, by choosing $\widetilde{T}$ sufficiently small such that $C\widetilde{T}\|q\|_{C^{0,\alpha}(\mathbb{R}^n)} \leq 1/2$, (3.78) shows $v_1 = v_2$ for a.e. $0 \leq t \leq \widetilde{T}$. Since $v_1 - v_2 \in C([0,T]; H^{-s}(\Omega))$ is continuous, we know $(v_1 - v_2)(\widetilde{T}) = 0$. Hence, we can repeat our local uniqueness result and finally find $v_1 = v_2$ on $[0,T]$. This completes the proof. □

4. Comparison principle

In this section, we show the comparison principle for NPMEs.

Theorem 4.1 (Basic comparison principle)**.** *Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain, $T > 0$, $0 < s < 1$, $m > 1$ and $L_K \in \mathcal{L}_0$. Assume that for $j = 1,2$ we have given initial data $u_{0,j} \in L^\infty(\Omega) \cap \widetilde{H}^s(\Omega)$, exterior conditions $\varphi_j \in C_c([0,T] \times \Omega_e)$ with $\Phi^m(\varphi_j) \in L^2(0,T; H^s(\mathbb{R}^n))$, sources $F_j \in L^2(\Omega_T)$ and u_j solves*

$$\begin{cases} \partial_t u + L_K(\Phi^m(u)) = F_j & \text{in } \Omega_T, \\ u = \varphi_j & \text{in } (\Omega_e)_T, \\ u(0) = u_{0,j} & \text{in } \Omega. \end{cases}$$

Additionally, suppose that there exist sequences $(F_j^\varepsilon)_{\varepsilon>0} \subset L^2(\Omega_T)$, $(\varphi_j^\varepsilon)_{\varepsilon>0} \subset C_c([0,T] \times \Omega_e)$ with $\Phi_\varepsilon^m(\varphi_j^\varepsilon) \in L^2(0,T; H^s(\mathbb{R}^n))$, $(u_{0,j}^\varepsilon)_{\varepsilon>0} \subset L^\infty(\Omega) \cap \widetilde{H}^s(\Omega)$ and $(u_{j,\varepsilon})_{\varepsilon>0} \subset H^1(0,T; L^2(\Omega)) \cap L^2(0,T; H^s(\mathbb{R}^n))$ satisfying

(i) $\liminf_{\varepsilon\to 0} \int_{\Omega_T} (F_1^\varepsilon - F_2^\varepsilon)_+\, dxdt \leq \int_{\Omega_T} (F_1 - F_2)_+\, dxdt$,

(ii) $\liminf_{\varepsilon\to 0}\int_{\Omega_T}\int_{\Omega_e}\frac{(\Phi_\varepsilon^m(\varphi_1^\varepsilon)-\Phi_\varepsilon^m(\varphi_2^\varepsilon))_+(x,t)}{|x-y|^{n+2s}}\,dxdydt$
$\leq \int_{\Omega_T}\int_{\Omega_e}\frac{(\Phi^m(\varphi_1)-\Phi^m(\varphi_2))_+(x,t)}{|x-y|^{n+2s}}\,dxdydt,$

(iii) $u_{0,j}^\varepsilon \to u_{0,j}$ *in* $L^1(\Omega)$ *as* $\varepsilon\to 0$,

(iv) $u_{j,\varepsilon}\rightharpoonup u_j$ *in* $L^1(\Omega_T)$ *as* $\varepsilon\to 0$

(v) *and* $u_{j,\varepsilon}$ *are solutions of*

$$\begin{cases}\partial_t u + L_K(\Phi_\varepsilon^m(u)) = F_j^\varepsilon & \text{in } \Omega_T,\\ u = \varphi_j^\varepsilon & \text{in } (\Omega_e)_T,\\ u(0) = u_{0,j}^\varepsilon & \text{in } \Omega.\end{cases}$$

Then there exists a constant $C>0$ *independent of these solutions, initial data, boundary data and sources such that*

$$\int_\Omega (u_1-u_2)_+(x,t_0)\,dx \leq C\left(\int_{\Omega_T}(F_1-F_2)_+\,dxdt + \int_\Omega (u_{0,1}-u_{0,2})_+\,dx + \int_{\Omega_T}\int_{\Omega_e}\frac{(\Phi^m(\varphi_1)-\Phi^m(\varphi_2))_+(x,t)}{|x-y|^{n+2s}}\,dxdydt\right). \tag{4.1}$$

Remark 4.2. *If one knows that* $u_{j,\varepsilon}\to u_j$ *strongly in* $L^2(\Omega)$ *as* $\varepsilon\to 0$ *for a.e.* $0<t<T$*, one can replace the left hand side of* (4.1) *by* $\int_\Omega (u_1(x,t)-u_2(x,t))\,dx$.

Proof of Theorem 4.1. As in the existence and uniqueness proofs above, we write $\Phi,\Phi_\varepsilon,L,B$ instead of $\Phi^m,\Phi_\varepsilon^m,L_K,B_K$. Next, let us note that $v_{j,\varepsilon} := u_{j,\varepsilon}-\varphi_j^\varepsilon$ solves

$$\begin{cases}\partial_t v + L(\Phi_\varepsilon(v)) = f_{j,\varepsilon}+F_j^\varepsilon & \text{in } \Omega_T,\\ v = 0 & \text{in } (\Omega_e)_T,\\ v(0) = u_{0,j}^\varepsilon & \text{in } \Omega,\end{cases}$$

where we set $f_{j,\varepsilon} = -L(\Phi_\varepsilon(\varphi_j^\varepsilon))\in L^2(\Omega_T)$. Hence, $v_{1,\varepsilon}-v_{2,\varepsilon}$ solves

$$\begin{cases}\partial_t(v_{1,\varepsilon}-v_{2,\varepsilon}) + L(\Phi_\varepsilon(v_{1,\varepsilon})-\Phi_\varepsilon(v_{2,\varepsilon})) = (f_{1,\varepsilon}-f_{2,\varepsilon})+(F_1^\varepsilon-F_2^\varepsilon) & \text{in } \Omega_T,\\ v_{1,\varepsilon}-v_{2,\varepsilon} = 0 & \text{in } (\Omega_e)_T,\\ (v_{1,\varepsilon}-v_{2,\varepsilon})(0) = u_{0,1}^\varepsilon - u_{0,2}^\varepsilon & \text{in } \Omega.\end{cases}$$

By assumption, the definition of Φ_ε and $\partial\Omega\in C^{0,1}$, we have

$$v_{j,\varepsilon}\in H^1(0,T;L^2(\Omega))\cap L^2(0,T;\widetilde{H}^s(\Omega)).$$

Thus, there holds

$$\int_{\Omega_T}\partial_t(v_{1,\varepsilon}-v_{2,\varepsilon})\,\psi\,dxdt + \int_0^T B(\Phi_\varepsilon(v_{1,\varepsilon})-\Phi_\varepsilon(v_{2,\varepsilon}),\psi)\,dt = \int_{\Omega_T} G_\varepsilon\psi\,dxdt, \tag{4.2}$$

for all $\psi\in L^2(0,T;\widetilde{H}^s(\Omega))$, where we defined $G_\varepsilon := f_{1,\varepsilon}-f_{2,\varepsilon}+F_1^\varepsilon-F_2^\varepsilon$.

Let $\chi\colon\mathbb{R}\to\mathbb{R}$ be the step function

$$\chi(t) = \begin{cases}1 & \text{if } t>0,\\ 0 & \text{if } t\leq 0.\end{cases}$$

We want to use $\psi = \chi(\Phi_\varepsilon(v_{1,\varepsilon})-\Phi_\varepsilon(v_{2,\varepsilon}))$ as a test function in the weak formulation (4.2). Unfortunately, the test function ψ does not have the right regularity properties in (4.2). Hence, in order to remedy this, let us introduce the auxiliary functions $\chi_\delta \in C^\infty(\mathbb{R})$ satisfying

$$0 \leq \chi_\delta' \leq \frac{2}{\delta} \quad \text{and} \quad \chi_\delta(t) = \begin{cases} 1, & \text{if } t \geq \delta \\ 0, & \text{if } t \leq 0 \end{cases}$$

for $\delta > 0$. One can easily verify that $\chi_\delta \left(\Phi_\varepsilon(v_{1,\varepsilon}) - \Phi_\varepsilon(v_{2,\varepsilon})\right) \in L^2(0,T;\widetilde{H}^s(\Omega))$ by the Lipschitz continuity of χ_δ, then we get

$$\left\|\chi_\delta(\Phi_\varepsilon(v_{1,\varepsilon}) - \Phi_\varepsilon(v_{2,\varepsilon}))\right\|_{L^2(0,T;H^s(\mathbb{R}^n))} \leq C_\delta \left\|\Phi_\varepsilon(v_{1,\varepsilon}) - \Phi_\varepsilon(v_{2,\varepsilon})\right\|_{L^2(0,T;H^s(\mathbb{R}^n))},$$

for some constant $C_\delta > 0$ depending on $\delta > 0$. Let us define

$$\psi_{\delta,k}(x,t) := \chi_\delta \left(\Phi_\varepsilon(v_{1,\varepsilon}) - \Phi_\varepsilon(v_{2,\varepsilon})\right)(x,t)\eta_k(t),$$

where $\eta_k \in C_c^\infty([0,T))$ satisfies

$$0 \leq \eta_k \leq 1, \ \ \eta_k|_{[0,t_0]} = 1, \ \ \eta_k|_{[t_0+1/k,T)} = 0,$$

for $0 < t_0 < T$ and $k \in \mathbb{N}$ sufficiently large. Then $\psi_{\delta,k}$ can be viewed as a test function in (4.2) and we get

$$\begin{aligned} &\int_{\Omega_T} \partial_t(v_{1,\varepsilon} - v_{2,\varepsilon})\chi_\delta \left(\Phi_\varepsilon(v_{1,\varepsilon}) - \Phi_\varepsilon(v_{2,\varepsilon})\right)\eta_k \, dxdt \\ &= -\int_0^T B\left(\Phi_\varepsilon(v_{1,\varepsilon}) - \Phi_\varepsilon(v_{2,\varepsilon}), \chi_\delta(\Phi_\varepsilon(v_{1,\varepsilon}) - \Phi_\varepsilon(v_{2,\varepsilon}))\right)\eta_k \, dt \\ &\quad + \int_{\Omega_T} G_\varepsilon \chi_\delta \left(\Phi_\varepsilon(v_{1,\varepsilon}) - \Phi_\varepsilon(v_{2,\varepsilon})\right)\eta_k \, dxdt. \end{aligned} \tag{4.3}$$

In the rest of the proof, let us define

$$\mathcal{V}_\varepsilon := \Phi_\varepsilon(v_{1,\varepsilon}) - \Phi_\varepsilon(v_{2,\varepsilon})$$

to simplify the notation. We want to show that the term involving the bilinear form in (4.3) is nonnegative. In fact, by the usual trick as used before, we have

$$\begin{aligned} &B\left(\mathcal{V}_\varepsilon, \chi_\delta \mathcal{V}_\varepsilon\right) \\ &= \int_{\mathbb{R}^{2n}} K(x,y)\frac{\left(\mathcal{V}_\varepsilon(x) - \mathcal{V}_\varepsilon(y)\right)\left(\chi_\delta(\mathcal{V}_\varepsilon(x)) - \chi_\delta(\mathcal{V}_\varepsilon(y))\right)}{|x-y|^{n+2s}} \, dxdy \\ &= \int_{\mathbb{R}^{2n}} K(x,y)\left(\int_0^1 \chi_\delta'(\mathcal{V}_\varepsilon(y) + \tau(\mathcal{V}_\varepsilon(x) - \mathcal{V}_\varepsilon(y)))\, d\tau\right)\frac{\left(\mathcal{V}_\varepsilon(x) - \mathcal{V}_\varepsilon(y)\right)^2}{|x-y|^{n+2s}} \, dxdy \\ &\geq 0. \end{aligned} \tag{4.4}$$

Plugging (4.4) into (4.3), we arrive at the estimate

$$\begin{aligned} &\int_{\Omega_T} \partial_t(v_{1,\varepsilon} - v_{2,\varepsilon})\chi_\delta \left(\Phi_\varepsilon(v_{1,\varepsilon}) - \Phi_\varepsilon(v_{2,\varepsilon})\right)\eta_k \, dxdt \\ &\leq \int_{\Omega_T} G_\varepsilon \chi_\delta \left(\Phi_\varepsilon(v_{1,\varepsilon}) - \Phi_\varepsilon(v_{2,\varepsilon})\right)\eta_k \, dxdt. \end{aligned} \tag{4.5}$$

Hence, passing to the limit $\delta \to 0$ by Lebesgue's dominated convergence theorem on (4.5), this shows

$$\begin{aligned} &\int_{\Omega_T} \partial_t(v_{1,\varepsilon} - v_{2,\varepsilon})\chi(\Phi_\varepsilon(v_{1,\varepsilon}) - \Phi_\varepsilon(v_{2,\varepsilon}))\eta_k \, dxdt \\ &\leq \int_{\Omega_T} G_\varepsilon \chi(\Phi_\varepsilon(v_{1,\varepsilon}) - \Phi_\varepsilon(v_{2,\varepsilon}))\eta_k \, dxdt. \end{aligned} \tag{4.6}$$

Note that by monotonicity of Φ_ε and Φ_ε^{-1} there holds

$$\chi\left(\Phi_\varepsilon(v_{1,\varepsilon})-\Phi_\varepsilon(v_{2,\varepsilon})\right)=\chi\left(v_{1,\varepsilon}-v_{2,\varepsilon}\right).$$

Hence, we can deduce from (4.6) the estimate

$$\int_{\Omega_T}\partial_t\left(v_{1,\varepsilon}-v_{2,\varepsilon}\right)\chi\left(v_{1,\varepsilon}-v_{2,\varepsilon}\right)\eta_k\,dxdt\leq\int_{\Omega_T}G_\varepsilon\chi\left(v_{1,\varepsilon}-v_{2,\varepsilon}\right)\eta_k\,dxdt. \tag{4.7}$$

Using $v_{j,\varepsilon}\in H^1(0,T;L^2(\Omega))$ for $j=1,2$ and passing to the limit $k\to\infty$, this ensures $\eta_k\to\chi_{[0,t_0]}$ as $k\to\infty$, and

$$\begin{aligned}\int_{\Omega_{t_0}}\partial_t\left(v_{1,\varepsilon}-v_{2,\varepsilon}\right)_+\,dxdt&\leq\int_{\Omega_{t_0}}G_\varepsilon\chi(v_{1,\varepsilon}-v_{2,\varepsilon})\,dxdt\\&=-\int_{\Omega_{t_0}}L(\Phi_\varepsilon(\varphi_1^\varepsilon)-\Phi_\varepsilon(\varphi_2^\varepsilon))\chi(v_{1,\varepsilon}-v_{2,\varepsilon})\,dxdt\\&\quad+\int_{\Omega_{t_0}}(F_1^\varepsilon-F_2^\varepsilon)\chi(v_{1,\varepsilon}-v_{2,\varepsilon})\,dxdt.\end{aligned}$$

Clearly, we have

$$\int_{\Omega_{t_0}}(F_1^\varepsilon-F_2^\varepsilon)\chi(v_{1,\varepsilon}-v_{2,\varepsilon})\,dxdt\leq\int_{\Omega_{t_0}}(F_1^\varepsilon-F_2^\varepsilon)_+\,dxdt. \tag{4.8}$$

On the other hand, we note that by the uniform ellipticity of K and the support conditions on φ_j and $v_{j,\varepsilon}$, then there holds

$$\begin{aligned}&-\int_{\Omega_{t_0}}L(\Phi_\varepsilon(\varphi_1^\varepsilon)-\Phi_\varepsilon(\varphi_2^\varepsilon))\chi(v_{1,\varepsilon}-v_{2,\varepsilon})\,dxdt\\&=-\int_0^{t_0}\int_{\mathbb{R}^{2n}}K(x,y)\\&\qquad\cdot\frac{(\mathcal{W}_\varepsilon(x)-\mathcal{W}_\varepsilon(y))(\chi(v_{1,\varepsilon}-v_{2,\varepsilon})(x)-\chi(v_{1,\varepsilon}-v_{2,\varepsilon})(y))}{|x-y|^{n+2s}}\,dxdydt\\&=2\int_0^{t_0}\int_{\Omega_e\times\Omega}K(x,y)\frac{\mathcal{W}_\varepsilon(x)\chi(v_{1,\varepsilon}-v_{2,\varepsilon})(y)}{|x-y|^{n+2s}}\,dxdydt\\&\leq2\int_0^{t_0}\int_{\Omega_e\times\Omega}K(x,y)\frac{(\mathcal{W}_\varepsilon)_+(x)\chi(v_{1,\varepsilon}-v_{2,\varepsilon})(y)}{|x-y|^{n+2s}}\,dxdydt\\&\leq C\int_0^{t_0}\int_\Omega\int_{\Omega_e}\frac{(\mathcal{W}_\varepsilon)_+(x)}{|x-y|^{n+2s}}\,dxdydt\\&=C\int_0^{t_0}\int_\Omega\int_{\Omega_e}\frac{(\Phi_\varepsilon(\varphi_1^\varepsilon)-\Phi_\varepsilon(\varphi_2^\varepsilon))_+(x)}{|x-y|^{n+2s}}\,dxdydt,\end{aligned} \tag{4.9}$$

where we set

$$\mathcal{W}_\varepsilon=\Phi_\varepsilon(\varphi_1^\varepsilon)-\Phi_\varepsilon(\varphi_2^\varepsilon).$$

Combining (4.7), (4.8) and (4.9), we obtain

$$\begin{aligned}\int_\Omega\left(v_{1,\varepsilon}-v_{2,\varepsilon}\right)_+(x,t_0)\,dx\leq C\Bigg(&\int_{\Omega_T}(F_1^\varepsilon-F_2^\varepsilon)_+\,dxdt+\int_\Omega\left(u_{0,1}^\varepsilon-u_{0,2}^\varepsilon\right)_+\,dx\\&+\int_{\Omega_T}\int_{\Omega_e}\frac{(\Phi_\varepsilon(\varphi_1^\varepsilon)-\Phi_\varepsilon(\varphi_2^\varepsilon))_+(x)}{|x-y|^{n+2s}}\,dxdydt\Bigg)\end{aligned}$$

for a.e. $0 < t_0 < T$. Going back to $u_{j,\varepsilon}$ and recalling that $\varphi_j^\varepsilon\big|_{\Omega_T} = 0$, shows

(4.10)
$$\int_\Omega (u_{1,\varepsilon} - u_{2,\varepsilon})_+ (x, t_0)\, dx \leq C \left(\int_{\Omega_T} (F_1^\varepsilon - F_2^\varepsilon)_+ \, dxdt + \int_\Omega \left(u_{0,1}^\varepsilon - u_{0,2}^\varepsilon\right)_+ \, dx \right. \\ \left. + \int_{\Omega_T} \int_{\Omega_e} \frac{(\Phi_\varepsilon(\varphi_1^\varepsilon) - \Phi_\varepsilon(\varphi_2^\varepsilon))_+ (x)}{|x-y|^{n+2s}} \, dxdydt \right)$$

for a.e. $0 < t_0 < T$.

Finally, the proof can be accomplished as follows: as $\varepsilon \to 0$. Note that the weak convergence $u_{j,\varepsilon} \rightharpoonup u_j$ in $L^1(\Omega_T)$ and the convexity of $t \mapsto t_+$ guarantees

$$\int_{\Omega_T} (u_1 - u_2)_+ \, dxdt \leq \liminf_{\varepsilon \to 0} \int_{\Omega_T} (u_{1,\varepsilon} - u_{2,\varepsilon})_+ \, dxdt.$$

Thus, the result follows from (4.10) after an integration over $0 \leq t_0 \leq T$, using the convergence assumptions (i)–(iv) and the Lipschitz continuity of $t \mapsto t_+$. This proves the assertion. □

We next prove a similar result for NPME with linear absorption term.

Theorem 4.3 (Comparison principle with absorption term)**.** *Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain, $T > 0$, $0 < s < \alpha \leq 1$, $m > 1$ and $L_K \in \mathcal{L}_0$. Assume additionally that we have given $\rho, q \in C_+^{1,\alpha}(\mathbb{R}^n)$ with ρ uniformly elliptic. Suppose that for $j = 1, 2$ we have given $u_{0,j} \in L^\infty(\Omega) \cap \widetilde{H}^s(\Omega)$, $\varphi_j \in C_c([0,T] \times \Omega_e)$ with $\Phi^m(\varphi_j) \in L^2(0,T; H^s(\mathbb{R}^n))$, $F_j \in L^2(\Omega_T)$ and u_j solves*

$$\begin{cases} \rho \partial_t u + L_K(\Phi^m(u)) + qu = F_j & \text{in } \Omega_T, \\ u = \varphi_j & \text{in } (\Omega_e)_T, \\ u(0) = u_{0,j} & \text{in } \Omega. \end{cases}$$

Additionally, suppose that there exist sequences $\left(F_j^\varepsilon\right)_{\varepsilon>0} \subset L^2(\Omega_T)$, $\left(\varphi_j^\varepsilon\right)_{\varepsilon>0} \subset C_c([0,T] \times \Omega_e)$ with $\Phi_\varepsilon^m(\varphi_j^\varepsilon) \in L^2(0,T; H^s(\mathbb{R}^n))$, $\left(u_{0,j}^\varepsilon\right)_{\varepsilon>0} \subset L^\infty(\Omega) \cap \widetilde{H}^s(\Omega)$ and $(u_{j,\varepsilon})_{\varepsilon>0} \subset H^1(0,T; L^2(\Omega)) \cap L^2(0,T; H^s(\mathbb{R}^n))$ satisfying

(i) $\liminf_{\varepsilon \to 0} \int_{\Omega_T} (F_1^\varepsilon - F_2^\varepsilon)_+ dxdt \leq \int_{\Omega_T} (F_1 - F_2)_+ \, dxdt$,

(ii) $\liminf_{\varepsilon \to 0} \int_{\Omega_T} \int_{\Omega_e} \frac{(\Phi_\varepsilon^m(\varphi_1^\varepsilon) - \Phi_\varepsilon^m(\varphi_2^\varepsilon))_+(x,t)}{|x-y|^{n+2s}} \, dxdydt$
$\leq \int_{\Omega_T} \int_{\Omega_e} \frac{(\Phi^m(\varphi_1) - \Phi^m(\varphi_2))_+(x,t)}{|x-y|^{n+2s}} \, dxdydt$,

(iii) $u_{0,j}^\varepsilon \to u_{0,j}$ *in* $L^1(\Omega)$ *as* $\varepsilon \to 0$,

(iv) $u_{j,\varepsilon} \rightharpoonup u_j$ *in* $L^1(\Omega_T)$ *as* $\varepsilon \to 0$

(v) and $u_{j,\varepsilon}$ are solutions of

$$\begin{cases} \partial_t u + L_K(\Phi_\varepsilon^m(u)) + qu = F_j^\varepsilon & \text{in } \Omega_T, \\ u = \varphi_j^\varepsilon & \text{in } (\Omega_e)_T, \\ u(0) = u_{0,j}^\varepsilon & \text{in } \Omega. \end{cases}$$

Then there exists a constant $C > 0$ independent of these solutions, initial data, boundary data and sources such that

$$\int_\Omega (u_1 - u_2)_+ (x, t_0)\, dx \leq C \left(\int_{\Omega_T} (F_1 - F_2)_+ \, dxdt + \int_\Omega (u_{0,1} - u_{0,2})_+ \, dx \right. \\ \left. + \int_{\Omega_T} \int_{\Omega_e} \frac{(\Phi(\varphi_1) - \Phi(\varphi_2))_+ (x)}{|x-y|^{n+2s}} \, dxdydt \right).$$

Remark 4.4. *Suppose we have given $0 \leq u_0 \in L^\infty(\Omega) \cap \widetilde{H}^s(\Omega)$, $0 \leq \varphi \in C_c([0,T] \times \Omega_e)$ with $\Phi^m(\varphi) \in L^2(0,T;H^s(\mathbb{R}^n))$, then Theorem 3.17 and Corollary 3.14 show that the unique solution u of*

$$
\begin{cases}
\rho\partial_t u + L_K(\Phi^m(u)) + qu = 0 & \text{in } \Omega_T,\\
u = \varphi & \text{in } (\Omega_e)_T,\\
u(0) = 0 & \text{in } \Omega.
\end{cases} \tag{4.11}
$$

can be approximated weakly in $L^1(\Omega_T)$ by a sequence of solutions

$$
(u_\varepsilon)_{\varepsilon>0} \subset H^1(0,T;L^2(\Omega)) \cap L^2(0,T;H^s(\mathbb{R}^n))
$$

to

$$
\begin{cases}
\rho\partial_t v + L_K(\Phi^m_\varepsilon(v)) + qv = 0 & \text{in } \Omega_T,\\
v = \varphi & \text{in } (\Omega_e)_T,\\
v(0) = 0 & \text{in } \Omega.
\end{cases}
$$

Hence, we see that the conditions (iii), (iv), (v) of Theorem 4.3 are satisfied when we take $\varphi^\varepsilon = \varphi$ and $u_0^\varepsilon = u_0 = 0$ and $F^\varepsilon = F = 0$. Additionally, using that $\Phi^m_\varepsilon \to \Phi^m$ uniformly on compact sets and $\varphi \in C_c([0,T] \times \Omega)$, we see that

$$
\int_{\Omega_T}\int_{\Omega_e} \frac{|\Phi^m_\varepsilon(\varphi^\varepsilon) - \Phi^m(\varphi)|(x,t)}{|x-y|^{n+2s}}\, dxdydt \to 0
$$

as $\varepsilon \to 0$. Therefore, we see that in particular Theorem 4.3 can be used to compare solutions to equations of the form (4.11).

Proof of Theorem 4.3. As in the existence and uniqueness proofs above, we write $\Phi, \Phi_\varepsilon, L, B$ instead of $\Phi^m, \Phi^m_\varepsilon, L_K, B_K$. As in the proof of Theorem 4.1 we introduce the new functions $v_{j,\varepsilon} = u_{j,\varepsilon} - \varphi_j^\varepsilon$, which satisfy

$$
\begin{cases}
\rho\partial_t v + L(\Phi_\varepsilon(v)) + qv = f_{j,\varepsilon} + F_j^\varepsilon & \text{in } \Omega_T,\\
v = 0 & \text{in } (\Omega_e)_T,\\
v(0) = u_{0,j}^\varepsilon & \text{in } \Omega,
\end{cases}
$$

where we set $f_{j,\varepsilon} = -L(\Phi_\varepsilon(\varphi_j^\varepsilon)) \in L^2(\Omega_T)$. Therefore, $v_{1,\varepsilon} - v_{2,\varepsilon}$ solves

$$
\begin{cases}
\rho\partial_t(v_{1,\varepsilon} - v_{2,\varepsilon}) + L(\Phi_\varepsilon(v_{1,\varepsilon}) - \Phi_\varepsilon(v_{2,\varepsilon})) + q(v_{1,\varepsilon} - v_{2,\varepsilon}) = G_\varepsilon & \text{in } \Omega_T,\\
v_{1,\varepsilon} - v_{2,\varepsilon} = 0 & \text{in } (\Omega_e)_T,\\
(v_{1,\varepsilon} - v_{2,\varepsilon})(0) = u_{0,1} - u_{0,2} & \text{in } \Omega.
\end{cases}
$$

where $G_\varepsilon := f_{1,\varepsilon} - f_{2,\varepsilon} + F_1^\varepsilon - F_2^\varepsilon$. Arguing as in the proof of Theorem 4.1, one has

$$
\begin{aligned}
&\int_{\Omega_T} \rho\partial_t(v_{1,\varepsilon} - v_{2,\varepsilon})\chi_\delta(\Phi_\varepsilon(v_{1,\varepsilon}) - \Phi_\varepsilon(v_{2,\varepsilon}))\eta_k\, dxdt\\
&= -\int_0^T B(\Phi_\varepsilon(v_{1,\varepsilon}) - \Phi_\varepsilon(v_{2,\varepsilon}), \chi_\delta(\Phi_\varepsilon(v_{1,\varepsilon}) - \Phi_\varepsilon(v_{2,\varepsilon})))\eta_k\, dt\\
&\quad - \int_{\Omega_T} q(v_{1,\varepsilon} - v_{2,\varepsilon})\chi_\delta(\Phi_\varepsilon(v_{1,\varepsilon}) - \Phi_\varepsilon(v_{2,\varepsilon}))\eta_k\, dxdt\\
&\quad + \int_{\Omega_T} G_\varepsilon\chi_\delta(\Phi_\varepsilon(v_{1,\varepsilon}) - \Phi_\varepsilon(v_{2,\varepsilon}))\eta_k\, dxdt.
\end{aligned}
$$

Using (4.4) one obtains

$$\begin{aligned}&\int_{\Omega_T}\rho\partial_t(v_{1,\varepsilon}-v_{2,\varepsilon})\chi_\delta(\Phi_\varepsilon(v_{1,\varepsilon})-\Phi_\varepsilon(v_{2,\varepsilon}))\eta_k\,dxdt\\ \leq &-\int_{\Omega_T}q(v_{1,\varepsilon}-v_{2,\varepsilon})\chi_\delta(\Phi_\varepsilon(v_{1,\varepsilon})-\Phi_\varepsilon(v_{2,\varepsilon}))\eta_k\,dxdt\\ &+\int_{\Omega_T}G_\varepsilon\chi_\delta(\Phi_\varepsilon(v_{1,\varepsilon})-\Phi_\varepsilon(v_{2,\varepsilon}))\eta_k\,dxdt.\end{aligned}$$

Passing to the limit $\delta\to 0$ by Lebesgue's dominated convergence theorem, one can observe by using

$$\chi(\Phi_\varepsilon(v_{1,\varepsilon})-\Phi_\varepsilon(v_{2,\varepsilon}))=\chi(v_{1,\varepsilon}-v_{2,\varepsilon})$$

and $q\geq 0$ that there holds

$$\begin{aligned}&\int_{\Omega_T}\rho\partial_t(v_{1,\varepsilon}-v_{2,\varepsilon})\chi(\Phi_\varepsilon(v_{1,\varepsilon})-\Phi_\varepsilon(v_{2,\varepsilon}))\eta_k\,dxdt\\ \leq&\int_{\Omega_T}G_\varepsilon\chi(\Phi_\varepsilon(v_{1,\varepsilon})-\Phi_\varepsilon(v_{2,\varepsilon}))\eta_k\,dxdt.\end{aligned}$$

Now, we proceed as in the proof of Theorem 4.1 and use the uniform ellipticity of ρ to find

$$\begin{aligned}\int_\Omega(v_{1,\varepsilon}-v_{2,\varepsilon})_+(x,t_0)\,dx\leq C\Bigg(&\int_{\Omega_T}(F_1^\varepsilon-F_2^\varepsilon)_+\,dxdt+\int_\Omega\left(u_{0,1}^\varepsilon-u_{0,2}^\varepsilon\right)_+\,dx\\ &+\int_{\Omega_T}\int_{\Omega_e}\frac{(\Phi_\varepsilon(\varphi_1^\varepsilon)-\Phi_\varepsilon(\varphi_2^\varepsilon))_+(x)}{|x-y|^{n+2s}}\,dxdydt\Bigg).\end{aligned}$$

Again by a limit argument one can conclude the proof. □

5. DN maps and measurement equivalent operators

With the well-posedness of (1.11), we are able to define the DN map rigorously.

5.1. **DN maps.** From now on, since $m>1$ is a fixed number, we write Φ,Φ_ε in place of Φ^m and Φ_ε^m. Moreover, for $0<s<1$ and $W\subset\Omega_e$ we set

$$X_s(W):=\left\{\varphi\in C_c([0,T]\times W)\,;\,\varphi\geq 0\text{ and }\Phi(\varphi)\in L^2(0,T;H^s(\mathbb{R}^n))\,\right\}$$

Additionally, when its notationally convenient we write for a function space X a subscript $+$ to refer to the nonnegative functions in that space. For example,

$$\mathscr{D}_+([0,T]\times\Omega_e):=\{\,\varphi\in C_c^\infty([0,T]\times\Omega_e);\,\varphi\geq 0\,\}\,.$$

Next, we define the DN map for the nonlocal porous medium equation.

Definition 5.1 (The DN map)**.** *Let $\Omega\subset\mathbb{R}^n$ be a bounded Lipschitz domain, $T>0$, $0<s<\alpha\leq 1$, $m>1$ and $L_K\in\mathcal{L}_0$. Assume additionally that we have given $\rho,q\in C_+^{1,\alpha}(\mathbb{R}^n)$ with ρ being uniformly elliptic. Now, we define the DN map $\Lambda_{\rho,K,q}\colon X_s\to L^2(0,T;H^{-s}(\Omega_e))$ by*

$$\langle\Lambda_{\rho,K,q}\varphi,\psi\rangle=\int_0^T B_K(\Phi(u),\psi)\,dt$$

for all $\varphi\in X_s$ and $\psi\in L^2(0,T;\widetilde{H}^s(\Omega_e))$, where u is the unique, nonnegative, bounded solution of

$$\begin{cases}\rho\partial_t u+L_K(\Phi(u))+qu=0, & \text{in }\Omega_T,\\ u=\varphi & \text{in }(\Omega_e)_T,\\ u(0)=0 & \text{in }\Omega,\end{cases}\tag{5.1}$$

(see Theorem 3.13 and 3.17).

Remark 5.2. *We refer the reader to* [LRZ22, Appendix A] *for a discussion on an alternative DN map in nonlocal diffusion models and its relation to the one used in this article.*

For $\varphi \in X_s$, let u be the unique, bounded, nonnegative solution to (5.1). Since Φ is invertible on $\mathbb{R}_+$, the function

$$v(x,t) := \Phi(u)(x,t) \in L^\infty_+(\mathbb{R}^n_T)$$

satisfies

$$\begin{cases} \rho\partial_t \Phi^{-1}(v) + L_K(v) + q\Phi^{-1}(v) = 0 & \text{in } \Omega_T, \\ v = \widetilde{\varphi} & \text{in } (\Omega_e)_T, \\ v(0) = 0 & \text{in } \Omega, \end{cases} \tag{5.2}$$

where $\widetilde{\varphi} = \Phi(\varphi) \in L^2(0,T;H^s(\mathbb{R}^n))$. That v solves the above PDE and initial condition is a direct consequence of Definition 3.11. To see that $v = \widetilde{\varphi}$ observe that $u = \varphi$ if and only if $\Phi(u-\varphi) \in L^2(0,T;\widetilde{H}^s(\Omega))$. Now, since $\partial\Omega \in C^{0,1}$ and $\Phi(t) = 0$ only for $t = 0$ this is equivalent to $u = \varphi$ and therefore $v = \widetilde{\varphi}$ a.e. or in $L^2(0,T;\widetilde{H}^s(\Omega))$ sense.

Next, note that using Theorem 3.13 and 3.17, we deduce that the problem (5.2) is also well-posed for $\widetilde{\varphi} \in \mathscr{D}_+([0,T]\times\Omega_e)$ and $\rho, q \in C^{1,\alpha}_+(\mathbb{R}^n)$ with ρ being uniformly elliptic. In fact, if $\widetilde{\varphi} \in \mathscr{D}_+([0,T]\times\Omega_e)$, then $\varphi := \Phi^{-1}(\widetilde{\varphi}) \in C_c([0,T]\times\Omega_e)$ and $\Phi(\varphi) = \widetilde{\varphi} \in C_c^\infty([0,T]\times\Omega_e) \subset L^2(0,T;H^s(\mathbb{R}^n))$ and hence $\varphi \in X_s$. Now, by Theorem 3.13 and 3.17 there is a unique, nonnegative, bounded solution u of (5.1) and hence $v = \Phi(u)$ is the unique, nonnegative, bounded solution of (5.2).

Hence, we can define the DN map of (5.2), denoted by $\Lambda^\Phi_{\rho,K,q}\colon \mathscr{D}_+([0,T]\times\Omega_e) \to L^2(0,T;H^{-s}(\Omega_e))$, via the formula

$$\left\langle \Lambda^\Phi_{\rho,K,q}\widetilde{\varphi}, \psi \right\rangle = \int_0^T B_K(v,\psi)\,dt \tag{5.3}$$

with $\widetilde{\varphi} \in \mathscr{D}_+([0,T]\times\Omega_e)$ and $\psi \in L^2(0,T;\widetilde{H}^s(\Omega_e))$, where v is the unique, nonnegative, bounded solution of (5.2). It is immediate from our definitions that

$$\langle \Lambda_{\rho,K,q}\varphi, \psi\rangle = \langle \Lambda^\Phi_{\rho,K,q}\Phi(\varphi), \psi\rangle$$

for all $\varphi \in X_s$ with $\Phi(\varphi) \in \mathscr{D}_+([0,T]\times\Omega_e)$ and $\psi \in L^2(0,T;\widetilde{H}^s(\Omega_e))$. Hence, if (ρ_j, K_j, q_j) for $j = 1,2$ are coefficients satisfying the usual assumptions and $W_1, W_2 \subset \Omega_e$ two generic measurement sets, then

$$\Lambda_{\rho_1,K_1,q_1}\varphi\big|_{(W_2)_T} = \Lambda_{\rho_2,K_2,q_2}\varphi\big|_{(W_2)_T}, \quad \text{for all } \varphi \in X_s(W_1)$$

implies

$$\Lambda^\Phi_{\rho_1,K_1,q_1}\widetilde{\varphi}\big|_{(W_2)_T} = \Lambda^\Phi_{\rho_2,K_2,q_2}\widetilde{\varphi}\big|_{(W_2)_T}, \quad \text{for all } \widetilde{\varphi} \in \mathscr{D}_+\left([0,T]\times W_1\right). \tag{5.4}$$

Let us note that these restrictions are meant in the sense that we test against $\psi \in L^2(0,T;\widetilde{H}^s(W_2))$.

Hence, if we can show that the DN maps (5.3) uniquely determine the coefficients ρ, K, q, then this will also give a positive answer to the Question 1.

5.2. **Measurement equivalent nonlocal operators.** Recalling Definition 1.1, the next two propositions provide examples of measurement equivalent nonlocal opeartors. The first example deals with nonlocal operators whose kernel are real analytic and separable.

Proposition 5.3 (Real analytic, separable coefficients)**.** *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set and $0 < s < 1$. Then uniformly elliptic nonlocal operators L_K of order $2s$ such that $K(x,y)$ is of the form*

$$K(x,y) = F(\gamma(x))F(\gamma(y)),$$

for some real analytic functions $\gamma\colon \mathbb{R}^n \to \mathbb{R}$, $F\colon \mathbb{R}_+ \to \mathbb{R}_+$ satisfying

(i) *γ is uniformly elliptic,*
(ii) *F is injective*
(iii) *and for any compact interval $[a,b] \subset \mathbb{R}_+$ there exists $c > 0$ such that $F(\xi) \geq c$ for all $\xi \in [a,b]$*

are measurement equivalent.

Proof. The proof is a consequence of [KLZ22, Proposition 1.4] and the observation that the sequence $(\Phi_N)_{N\in\mathbb{N}}$ in [KLZ22, Theorem 1.1] can be constructed in such a way that $\Phi_N \geq 0$. □

Next, we state another example of measurement equivalent operators.

Proposition 5.4 (Fractional conductivity operators)**.** *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set and $0 < s < \min(1, n/2)$. Let $\gamma \in L^\infty(\mathbb{R}^n)$ be uniformly elliptic such that*

(i) *the background deviation $m_\gamma := \gamma^{1/2} - 1$ belongs to the Bessel potential space $H^{s,n/s}(\mathbb{R}^n)$*[2]*,*
(ii) *$\gamma|_{\Omega_e} = \Gamma$ for some fixed $\Gamma \in L^\infty(\mathbb{R}^n)$ with $m_\Gamma \in H^{s,n/s}(\mathbb{R}^n)$,*
(iii) *and $q_\gamma := -(-\Delta)^s m_\gamma/\gamma^{1/2}$ satisfies*

$$q_\gamma \in \begin{cases} C(\overline{\Omega}), & \text{if } 0 < s < 1/4 \\ L^\infty(\Omega), & \text{if } 1/4 \leq s < \min(1, n/2). \end{cases} \tag{5.5}$$

Then the nonlocal operator $L_K \in \mathcal{L}_0$ with

$$K(x,y) = C_{n,s}\gamma^{1/2}(x)\gamma^{1/2}(y),$$

where $C_{n,s} > 0$ is the normalization constant from the fractional Laplacian, is measurement equivalent.

Remark 5.5. *Let us recall that the operators L_K of Proposition 5.4 are called* fractional conductivity operator *as they converge to the conductivity operator* $-\operatorname{div}(\gamma\nabla u)$ *as $s \uparrow 1$ for sufficiently regular functions γ and u.*

Proof of Proposition 5.4. We only give here the needed modifications in the proofs of the results in [RZ22c] and refer for further details to this article. First, let us observe that any nonnegative $\widetilde{H}^s(W)$ with $W \subset \Omega_e$ can approximated by nonnegative test functions $\varphi \in C_c^\infty(W)$. In fact, let $\varphi \in \widetilde{H}^s(W)$ and choose $(\varphi_k)_{k\in\mathbb{N}} \subset C_c^\infty(W)$ such that $\varphi_k \to \varphi$ in $H^s(\mathbb{R}^n)$ as $k \to \infty$. Then we define

$$\psi_k(x) = \sqrt{(\varphi_k(x))^2 + \varepsilon_k^2} - \varepsilon_k$$

for some sequence $\varepsilon_k \downarrow 0$ as $k \to \infty$. We have $\psi_k \in C_c^\infty(W)$ and $\psi_k \geq 0$. Clearly, up to extracting a subsequence we can assume $\varphi_k \to \varphi$ as $k \to \infty$ a.e. in $\mathbb{R}^n$ and hence $\psi_k \to \varphi$ a.e. in $\mathbb{R}^n$ as $k \to \infty$. Since $\varphi_k \to \varphi$ in $L^2(\mathbb{R}^n)$ as $k \to \infty$, there exists $v \in L^2(\mathbb{R}^n)$ such that $|\varphi_k| \leq v$ uniformly in k (see [Bre11, Theorem 4.9]).

[2]Recall that the Bessel potential space $H^{s,p}(\mathbb{R}^n)$ with $1 \leq p < \infty$ and $s \in \mathbb{R}$ is the space of tempered distributions $u \in \mathscr{S}'(\mathbb{R}^n)$ such that $\|u\|_{H^{s,p}(\mathbb{R}^n)} = \|\langle D\rangle^s u\|_{L^p(\mathbb{R}^n)} < \infty$, where $\langle D\rangle^s$ is the Fourier multiplier with symbol $\langle\xi\rangle^s = (1 + |\xi|^2)^{s/2}$.

Therefore, Lebesgue's dominated convergence theorem gives $\psi_k \to \varphi$ in $L^2(\mathbb{R}^n)$ as $k \to \infty$. Additionally one has

$$\frac{\psi_k(x)-\psi_k(y)}{|x-y|^{n/2+s}} \to \frac{\varphi(x)-\varphi(y)}{|x-y|^{n/2+s}} \text{ a.e. in } \mathbb{R}^{2n}, \text{ as } k\to\infty,$$

and using the Lipschitz continuity of the function $\zeta(t) := \sqrt{t^2+\varepsilon^2}-\varepsilon$, we can get $|\zeta'(t)| = \left|\frac{t}{\sqrt{t^2+\varepsilon^2}}\right| \leq 1$, so that

$$\left|\frac{\psi_k(x)-\psi_k(y)}{|x-y|^{n/2+s}}\right| \leq \frac{|\varphi_k(x)-\varphi_k(y)|}{|x-y|^{n/2+s}} \leq V(x,y) \in L^2(\mathbb{R}^{2n}).$$

The existence of such a $V \in L^2(\mathbb{R}^{2n})$ follows from $\varphi_k \to \varphi$ in $H^s(\mathbb{R}^n)$ as $k \to \infty$ and [Bre11, Theorem 4.9]. The Lebesgue's dominated convergence theorem implies

$$\begin{aligned}[\psi_k-\varphi]^2_{H^s(\mathbb{R}^n)} &= \int_{\mathbb{R}^{2n}} \frac{|((\psi_k-\varphi)(x)-(\psi_k-\varphi)(y)|^2}{|x-y|^{n+2s}}\,dxdy \\ &= \int_{\mathbb{R}^{2n}} \left|\frac{(\psi_k(x)-\psi_k(y))-(\varphi(x)-\varphi(y))}{|x-y|^{n/2+s}}\right|^2 dxdy \to 0\end{aligned}$$

as $k\to\infty$. Hence, we can conclude the assertion.

Next, let $\gamma_j \in L^\infty(\mathbb{R}^n)$ be two functions with associated coefficients $K_j(x,y) = C_{n,s}\gamma_j^{1/2}(x)\gamma_j^{1/2}(y)$ for $j=1,2$, and operators L_{K_1}, L_{K_2} such that the conditions in Proposition 5.4 for two non-disjoint, nonempty, open sets $W_1, W_2 \subset \Omega$ hold. Moreover, assume that we have (1.14) for these data. Now one can follow the same arguments as in [RZ22c, Lemma 4.1] or [CRZ22, Lemma 4.1] to deduce that there holds

$$\Lambda_{q_1}\varphi|_{W_2} = \Lambda_{q_2}\varphi|_{W_2}$$

for all nonnegative $\varphi \in C_c^\infty(W_1)$. Here for $j=1,2$ the potentials q_j are given by

$$q_j = -\frac{(-\Delta)^s m_{\gamma_j}}{\gamma_j^{1/2}}$$

and Λ_{q_j} are the DN maps related to the Dirichlet problem

$$\begin{cases} ((-\Delta)^s + q_j)\, u = 0 & \text{in } \Omega \\ u = \varphi & \text{in } \Omega_e, \end{cases}$$

which is well-posed by [RZ22c, Lemma 3.11]. Using [GRSU20b, Theorem 1] (see (5.5)), we deduce that $q_1 = q_2$ in Ω. To apply this theorem we make the measurement sets if needed smaller.

Now, we can follow the proof of [RZ22b, Lemma 8.15] to see that

$$\int_{\Omega_e} (-\Delta)^{s/2}\,(m_{\gamma_1}-m_{\gamma_2})\,(-\Delta)^{s/2}(\varphi\psi)\,dx = 0$$

for all nonnegative functions $\varphi \in C_c^\infty(W_1)$, $\psi \in C_c^\infty(W_2)$. Using a cut-off function this implies

$$\int_{\Omega_e} (-\Delta)^{s/2}\,(m_{\gamma_1}-m_{\gamma_2})\,(-\Delta)^{s/2}\varphi\,dx = 0$$

for all nonnegative functions $\varphi \in C_c^\infty(\omega)$ with $\omega \Subset W_1 \cap W_2$. But now since

$$\varphi\psi = \frac{(\varphi+\psi)^2-\varphi^2-\psi^2}{2},$$

we find

$$\int_{\Omega_e} (-\Delta)^{s/2}\,(m_{\gamma_1}-m_{\gamma_2})\,(-\Delta)^{s/2}(\varphi\psi)\,dx = 0$$

for all $\varphi, \psi \in C_c^\infty(\omega)$. By using again a cut-off function this implies

$$\int_{\Omega_e} (-\Delta)^{s/2} (m_{\gamma_1} - m_{\gamma_2}) (-\Delta)^{s/2} \varphi \, dx = 0$$

for all $\varphi \in C_c^\infty(\omega')$, where $\omega' \Subset \omega$. Therefore, we can conclude that

$$(-\Delta)^s (m_{\gamma_1} - m_{\gamma_2}) = 0 \text{ in } \omega'.$$

Taking into account $\gamma_1|_{\Omega_e} = \gamma_2|_{\Omega_e}$, we deduce from the unique continuation principle for the fractional Laplacian in Bessel potential spaces (see [KRZ23, Theorem 2.2]) that $\gamma_1 = \gamma_2$ in $\mathbb{R}^n$ and therefore $K_1 = K_2$ in $\mathbb{R}^{2n}$ as asserted. □

Remark 5.6. *We have shown that the fractional conductivity operator is a measurement equivalent nonlocal operator. It is also natural to ask a question whether two nonlocal operators of the form $(-\nabla \cdot A\nabla)^s$ are under certain conditions measurement equivalent or not. In fact, this is still an open problem. Very recently, the works [GU21, CGRU23] demonstrated that if the nonlocal DN maps agree for $(-\nabla \cdot A_1\nabla)^s$ and $(-\nabla \cdot A_2\nabla)^s$, under appropriate conditions, then the corresponding DN maps agree for their local counterparts. This connects nonlocal and local inverse problems by using approaches of either heat semigroup or Caffarelli-Silvestre extension. In addition, similar question has been addressed for nonlocal parabolic operators, and we refer readers to [LLU22] for more details.*

6. Uniqueness of the inverse problem

In order to prove Theorem 1.3, we need a simple property about the nonlocal operator $L_K \in \mathcal{L}_0$.

6.1. Dirichlet problem for $L_K \in \mathcal{L}_0$.

Lemma 6.1. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain, $0 < s < 1$, $L_K \in \mathcal{L}_0$, $F \in H^{-s}(\Omega)$ and $f \in H^s(\mathbb{R}^n)$. Then the Dirichlet problem*

$$\begin{cases} L_K u = F & \text{in } \Omega, \\ u = f & \text{in } \Omega_e. \end{cases} \tag{6.1}$$

has a unique solution $u \in H^s(\mathbb{R}^n)$ satisfying

$$\|u\|_{H^s(\mathbb{R}^n)} \leq C \left(\|F\|_{H^{-s}(\Omega)} + \|f\|_{H^s(\mathbb{R}^n)} \right), \tag{6.2}$$

for some constant $C > 0$ independent of V, F and f.

Proof. As noted in the proof of Theorem 3.5 the bilinear form B_K defined via (2.2) induces an equivalent inner product on $\widetilde{H}^s(\Omega)$ and hence the Lax–Milgram theorem guarantees the existence of a unique solution $v \in \widetilde{H}^s(\Omega)$ to

$$\begin{cases} L_K v = G & \text{in } \Omega, \\ v = 0 & \text{in } \Omega_e \end{cases} \tag{6.3}$$

for any $G \in H^{-s}(\Omega)$ and v satisfies the continuity estimate

$$\|v\|_{H^s(\mathbb{R}^n)} \leq C \|G\|_{H^{-s}(\Omega)}$$

for some $C > 0$. Now, the assertions of Lemma 6.1 follow by observing that $u \in H^s(\mathbb{R}^n)$ uniquely solves (6.1) if and only if $v = u - f$ uniquely solves (6.3) with $G = F - L_K f \in H^{-s}(\Omega)$. Here we are using the fact that $L_K \colon \widetilde{H}^s(\Omega) \to H^{-s}(\Omega)$ is a bounded linear operator (see Section 2.2). □

6.2. **An integral-time transformation.** For given $0 < T_0 \leq T$, $\beta > 1$ and $h > 1$, let us consider exterior data $\widetilde{\varphi} \in \mathscr{D}_+([0,T] \times W_1)$ of the form

$$\widetilde{\varphi}(x,t) = t^m h \widetilde{\varphi}_0(x) \quad \text{with} \quad \widetilde{\varphi}_0 \in \mathscr{D}_+(W_1) \setminus \{0\}. \tag{6.4}$$

The parameters $T_0 > 0$ and $\beta > 0$ will be specified later. Now, let v be the unique solution to (5.2) (with $\widetilde{\varphi}$ as in (6.4)) and define $V \in H^s(\mathbb{R}^n)$ by the time-integral

$$V(x) := \int_0^{T_0} (T_0 - t)^\beta v(x,t)\, dt.$$

The regularity of V follows from the fact that $v \in L^2(0,T;H^s(\mathbb{R}^n))$. Note that to define V we only need to observe the nonlinear effects of the porous medium up to a possibly small time T_0. Next, we assert that V solves

$$\begin{cases} L_K V = -(\mathcal{M} + \mathcal{N}) & \text{in } \Omega, \\ V = h\mathcal{B}(\beta+1, m+1) T_0^{\beta+m+1} \widetilde{\varphi}_0 & \text{in } \Omega_e, \end{cases} \tag{6.5}$$

where $\mathcal{M} = \mathcal{M}(x)$ and $\mathcal{N} = \mathcal{N}(x)$ are given by

$$\mathcal{M} = \beta\rho \int_0^{T_0} (T_0 - t)^{\beta-1} \Phi^{-1}(v)\, dt,$$
$$\mathcal{N} = q \int_0^{T_0} (T_0 - t)^{\beta} \Phi^{-1}(v)\, dt$$

and $\mathcal{B}$ is the Euler beta function defined as

$$\mathcal{B}(a,b) = \int_0^1 (1-t)^{a-1} t^{b-1}\, dt,$$

for $a, b > 0$. That $V \in H^s(\mathbb{R}^n)$ attains the prescribed exterior values is easily seen from

$$\int_0^{T_0} (T_0 - t)^\beta t^m\, dt = \mathcal{B}(\beta+1, m+1) T_0^{\beta+m+1}, \tag{6.6}$$

and the fact that the exterior values in (5.2) can be considered in an a.e. sense due to the Lipschitz regularity of $\partial\Omega$. The identity (6.6) follows by a change of variables and the definition of $\mathcal{B}$. In fact, we may calculate

$$\begin{aligned} \int_0^{T_0} (T_0 - t)^\beta t^m\, dt &= T_0^{\beta+m+1} \int_0^1 (1-t)^{(\beta+1)-1} t^{(m+1)-1}\, dt \\ &= T_0^{\beta+m+1} \mathcal{B}(\beta+1, m+1). \end{aligned} \tag{6.7}$$

Next, observe that the condition $\beta > 1$ shows that the function

$$\psi_w = (T_0 - t)^\beta w \text{ with } w \in C_c^\infty(\Omega)$$

satisfies $\psi_w \in H^1(0, T_0; \widetilde{H}^s(\Omega))$ and $\psi_w(T_0) = 0$. Hence, using that v solves (5.2), $\Phi^{-1}(v) = 0$, $\rho \in C_+^{1,\alpha}(\mathbb{R}^n)$ is uniformly elliptic, Remark 3.12, Proposition B.1 and $\Phi^{-1}(v)(0) = 0$, we may compute

$$\begin{aligned} &\left\langle L_K \left(\int_0^{T_0} (T_0 - t)^\beta v\, dt \right), w \right\rangle \\ &= \int_0^{T_0} (T_0 - t)^\beta B_K(v, w)\, dt \\ &= \int_{\Omega_{T_0}} \Phi^{-1}(v) \partial_t(\rho \psi_w)\, dxdt - \int_{\Omega_{T_0}} q\Phi^{-1}(v) \psi_w\, dxdt \\ &= \left\langle -\beta\rho \int_0^{T_0} (T_0 - t)^{\beta-1} \Phi^{-1}(v)\, dt - q \int_0^{T_0} (T_0 - t)^\beta \Phi^{-1}(v)\, dt, w \right\rangle \end{aligned}$$

for all $w \in C_c^\infty(\Omega)$. This establishes that V solves (6.5). Before, proceeding let us remark that since the exterior data $v|_{(\Omega_e)_T}$ depends on h, the quantities V, $\mathcal{M}$ and $\mathcal{N}$ also depend on the parameter $h > 1$.

Next, we derive fundamental estimates for $\mathcal{M}$ and $\mathcal{N}$. Before doing this observe that if $\beta > 1$, $m > 1$ and the Hölder conjugate m' of m satisfies $\beta - m' > -1$, then we have

$$\int_0^{T_0} (T_0 - t)^{\beta - m'}\,dt = T_0^{\beta - m' + 1} \int_0^1 t^{\beta - m'}\,dt = \frac{T_0^{\beta - m' + 1}}{\beta - m' + 1}. \tag{6.8}$$

From now on we assume that

$$\beta > 1 \text{ and } \beta > m' - 1 = \frac{1}{m-1}. \tag{6.9}$$

Hence, by using Hölder's inequality, $v \geq 0$, $\Phi^{-1}(t) = t^{1/m}$ for $t \geq 0$, that ρ is uniformly elliptic and (6.8), we get

$$\begin{aligned} 0 \leq \mathcal{M}(x) &= \beta \rho(x) \int_0^{T_0} (T_0 - t)^{\frac{\beta - m'}{m'}} \left((T_0 - t)^\beta v(x,t) \right)^{\frac{1}{m}} dt \\ &\leq C T_0^{\frac{\beta - m' + 1}{m'}} (V(x))^{\frac{1}{m}} \\ &= C T_0^{\frac{\beta}{m'} - \frac{1}{m}} (V(x))^{\frac{1}{m}}, \end{aligned} \tag{6.10}$$

for some constant $C > 0$ only depending on β, m and $\|\rho\|_{L^\infty(\mathbb{R}^n)}$. Noting that (6.8) remains true for $m' = 0$, we obtain by Hölder's inequality and $0 \leq q \in L^\infty(\mathbb{R}^n)$, the pointwise estimate

$$0 \leq \mathcal{N}(x) \leq C T_0^{\frac{\beta+1}{m'}} (V(x))^{\frac{1}{m}}, \tag{6.11}$$

for some constant $C > 0$ only depending on β and $\|q\|_{L^\infty(\mathbb{R}^n)}$. By applying (6.10), (6.11) and Jensen's inequality, we can obtain the L^2 estimates

$$\begin{aligned} \|\mathcal{M}\|_{L^2(\Omega)} &\leq C T_0^{\frac{\beta}{m'} - \frac{1}{m}} \|V\|_{L^2(\Omega)}^{\frac{1}{m}}, \\ \|\mathcal{N}\|_{L^2(\Omega)} &\leq C T_0^{\frac{\beta+1}{m'}} \|V\|_{L^2(\Omega)}^{\frac{1}{m}} \end{aligned} \tag{6.12}$$

for some constant $C > 0$ only depending on β, m, on the norms $\|\rho\|_{L^\infty(\mathbb{R}^n)}, \|q\|_{L^\infty(\mathbb{R}^n)}$ and $|\Omega|$. On the other hand, applying the estimate (6.2) to the equation (6.5), one has

$$\begin{aligned} \|V\|_{H^s(\mathbb{R}^n)} &\leq C \left(\|\mathcal{M} + \mathcal{N}\|_{H^{-s}(\Omega)} + \mathcal{B}(\beta+1, m+1) T_0^{\beta+m+1} h \|\widetilde{\varphi}_0\|_{H^s(\mathbb{R}^n)} \right) \\ &\leq C \left\{ \left(T_0^{\frac{\beta}{m'} - \frac{1}{m}} + T_0^{(\beta+1)/m'} \right) \|V\|_{L^2(\Omega)}^{\frac{1}{m}} + T_0^{\beta+m+1} h \|\widetilde{\varphi}_0\|_{H^s(\mathbb{R}^n)} \right\}, \end{aligned}$$

for some constant $C > 0$ independent of V, $\widetilde{\varphi}_0$, T_0 and h. Since, $h > 1$ and $m > 1$ this implies

(6.13)

$$\begin{aligned} &\max(\|V\|_{H^s(\mathbb{R}^n)}, h) \\ &\leq C \left\{ \left(T_0^{\frac{\beta}{m'} - \frac{1}{m}} + T_0^{(\beta+1)/m'} \right) (\max(\|V\|_{L^2(\Omega)}, h))^{\frac{1}{m}} + T_0^{\beta+m+1} h \|\widetilde{\varphi}_0\|_{H^s(\mathbb{R}^n)} + h \right\} \\ &\leq C \left\{ \left(T_0^{\frac{\beta}{m'} - \frac{1}{m}} + T_0^{(\beta+1)/m'} \right) \max(\|V\|_{L^2(\Omega)}, h) + T_0^{\beta+m+1} h \|\widetilde{\varphi}_0\|_{H^s(\mathbb{R}^n)} + h \right\} \\ &= C \left\{ \left(T_0^{\frac{\beta}{m'} - \frac{1}{m}} + T_0^{(\beta+1)/m'} \right) \max(\|V\|_{L^2(\Omega)}, h) + h \left(T_0^{\beta+m+1} \|\widetilde{\varphi}_0\|_{H^s(\mathbb{R}^n)} + 1 \right) \right\} \end{aligned}$$

Next, observe that in the estimate (6.13) all exponents of T_0 are strictly positive by our choice of β (see (6.9)) and hence we can absorb the first term in the last line of (6.13) on the left hand side after choosing T_0 sufficiently small. This shows

$$\|V\|_{H^s(\mathbb{R}^n)} \leq Ch\left(T_0^{\beta+m+1}\|\widetilde{\varphi}_0\|_{H^s(\mathbb{R}^n)}+1\right) \tag{6.14}$$

for some $C>0$.

6.3. **Asymptotic analysis.** First, let us consider the ansatz

$$V=V_h=\mathcal{B}(\beta+1,m+1)T_0^{\beta+m+1}hV^{(0)}+R_1, \tag{6.15}$$

where $V^{(0)}$ and R_1 are the solutions to

$$\begin{cases} L_K V^{(0)}=0 & \text{in } \Omega,\\ V^{(0)}=\widetilde{\varphi}_0 & \text{in } \Omega_e \end{cases} \tag{6.16}$$

and

$$\begin{cases} L_K R_1=-(\mathcal{M}+\mathcal{N}) & \text{in } \Omega,\\ R_1=0 & \text{in } \Omega_e, \end{cases}$$

respectively. By using Lemma 6.1, (6.12) and (6.14), we get

$$\begin{aligned} \|R_1\|_{H^s(\mathbb{R}^n)} &\leq C\|\mathcal{M}+\mathcal{N}\|_{H^{-s}(\Omega)}\\ &\leq C\|V\|_{L^2(\Omega)}^{\frac{1}{m}}\\ &\leq Ch^{\frac{1}{m}}(\|\widetilde{\varphi}_0\|_{H^s(\mathbb{R}^n)}+1)^{1/m}, \end{aligned} \tag{6.17}$$

and

$$\|L_K R_1\|_{H^{-s}(\Omega)}=\|\mathcal{M}+\mathcal{N}\|_{H^{-s}(\Omega)}\leq Ch^{\frac{1}{m}}\left(\|\widetilde{\varphi}_0\|_{H^s(\mathbb{R}^n)}+1\right)^{1/m}. \tag{6.18}$$

This implies that

$$h^{-1}L_K R_1=\mathcal{O}\left(h^{\frac{1}{m}-1}\right) \text{ as } h\to\infty,$$

in $H^{-s}(\Omega)$. Thus, we have the asymptotic behavior of

$$L_K\left(h^{-1}V_h\right)=h^{-1}L_K R_1=\mathcal{O}\left(h^{\frac{1}{m}-1}\right) \text{ as } h\to\infty,$$

in $H^{-s}(\Omega)$. For the sake of convenience, let us introduce the following function

$$v_0(x,t):=ht^mV^{(0)}(x)\in H^1(0,T_0;\widetilde{H}^s(\Omega))$$

and observe that

$$0\leq v_0(x,t)\leq M \text{ for a.e. } (x,t)\in\mathbb{R}^n_{T_0} \tag{6.19}$$

for some $M>0$. In fact, by definition $V^{(0)}$ has exterior value $\widetilde{\varphi}_0\geq 0$. Arguing as in the proof of [RO16, Proposition 4.1] one sees that the maximum principle for the equation (6.16) still holds and guarantees the nonnegativity

$$V^{(0)}(x)\geq 0 \text{ for a.e. } x\in\Omega. \tag{6.20}$$

Moreover, that by linearity the maximum principle directly implies the comparison principle for equation (6.16). Furthermore, [RO16, Lemma 5.1] remains true in our setting, since K is uniformly elliptic and the function $w\in C_c^\infty(\mathbb{R}^n)$ constructed in that result has its maximum in Ω. Thus, in our case one can still establish [RO16, Corollary 5.2] and hence conclude that there holds

$$\left\|V^{(0)}\right\|_{L^\infty(\Omega)}\leq C\left\|\widetilde{\varphi}_0\right\|_{L^\infty(\Omega_e)}. \tag{6.21}$$

Therefore, we have shown the estimate (6.19).

Next, let us observe that (6.7) implies

$$\int_0^{T_0}(T_0-t)^\beta v_0(x,t)\,dt = h\mathcal{B}(\beta+1,m+1)T_0^{\beta+m+1}V^{(0)}(x).$$

Motivated by the asymptotic behaviour of the remainder R_1 (see (6.17) and (6.18)), in a next step we refine the ansatz for V as

$$V=\widetilde{V}_h = h\mathcal{B}(\beta+1,m+1)T_0^{\beta+m+1}V^{(0)}+h^{\frac{1}{m}}V^{(1)}+R_2, \tag{6.22}$$

where $V^{(0)}$ is the solution of (6.16) and $V^{(1)}$ is the solution to

$$\begin{cases} L_K V^{(1)} = -h^{-\frac{1}{m}}\left(\mathcal{M}^{(1)}+\mathcal{N}^{(1)}\right) & \text{in } \Omega,\\ V^{(1)}=0 & \text{in } \Omega_e, \end{cases} \tag{6.23}$$

where

$$\begin{aligned} 0\leq \mathcal{M}^{(1)}(x) &= \beta\rho(x)\int_0^{T_0}(T_0-t)^{\beta-1}v_0^{\frac{1}{m}}(x,t)\,dt\\ &= h^{\frac{1}{m}}\rho(x)\frac{2T_0^{\beta+1}}{\beta(\beta+1)}\left(V^{(0)}(x)\right)^{\frac{1}{m}}, \end{aligned} \tag{6.24}$$

and

$$\begin{aligned} 0\leq \mathcal{N}^{(1)}(x) &= q(x)\int_0^{T_0}(T_0-t)^{\beta}v_0^{\frac{1}{m}}(x,t)\,dt\\ &= h^{\frac{1}{m}}q(x)\frac{2T_0^{\beta+2}}{(\beta+1)(\beta+2)}\left(V^{(0)}(x)\right)^{\frac{1}{m}}. \end{aligned} \tag{6.25}$$

The computations in (6.24) and (6.25) are easily justified by using (6.19), (6.7), $\mathcal{B}(a,b)=\frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$, $\Gamma(a+1)=a\Gamma(a)$ and $\Gamma(n)=n!$ for $a,b>0$ and $n\in\mathbb{N}$. Hence, the remainder term R_2 needs to satisfy

$$\begin{cases} L_K R_2 = -\left[(\mathcal{M}-\mathcal{M}_1)+(\mathcal{N}-\mathcal{N}_1)\right] & \text{in } \Omega,\\ R_2=0 & \text{in } \Omega_e. \end{cases} \tag{6.26}$$

Thus, we see that $v_0\in H^1(0,T;\widetilde{H}^s(\Omega))$ solves

$$\begin{cases} \rho\partial_t v_0^{\frac{1}{m}}+L_K v_0 = h^{\frac{1}{m}}\rho\left(V^{(0)}\right)^{\frac{1}{m}}\geq 0 & \text{in } \Omega_{T_0},\\ v_0 = ht^m\widetilde{\varphi}_0 & \text{in } (\Omega_e)_{T_0},\\ v_0(0)=0 & \text{in } \Omega. \end{cases}$$

Next, we want to show that by our comparison principle (Theorem 4.3) there holds

$$v_0(x,t)\geq v(x,t) \text{ for a.e. } (x,t)\in\mathbb{R}^n_{T_0}. \tag{6.27}$$

For this purpose let $(\Phi_\varepsilon)_{\varepsilon>0}$ be the functions constructed in Lemma 3.2. First notice that

$$v_0 = ht^mV^{(0)} = \left(h^{1/m}t(V^{(0)})^{1/m}\right)^m\in H^1(0,T;\widetilde{H}^s(\Omega))$$

and thus the function $u_0=h^{1/m}t(V^{(0)})^{1/m}$ satisfies

$$\begin{cases} \rho\partial_t u_0 + L_K(\Phi(u_0)) = F_0 & \text{in } \Omega_T,\\ u_0=\varphi_0 & \text{in } (\Omega_e)_T,\\ u_0(0)=0 & \text{in } \Omega \end{cases}$$

with $F_0=\Phi^{-1}(h)\rho\Phi^{-1}(V^{(0)})$ and $\varphi_0=\Phi^{-1}(h)t\Phi^{-1}(\widetilde{\varphi}_0)$. Next, we introduce the functions $u_{0,\varepsilon}=\Phi_\varepsilon^{-1}(h)t\Phi_\varepsilon^{-1}(V^{(0)})$. Note that the uniform ellipticity of Φ_ε' implies that Φ_ε is bi-Lipschitz and hence by Remark 3.12 we have $\Phi_\varepsilon^{-1}(V^{(0)})\in H^s(\mathbb{R}^n)$. Thus, $u_{0,\varepsilon}\in H^1(0,T;H^s(\mathbb{R}^n))$.

Let us assert that

$$\Phi_\varepsilon^{-1}(t) \to \Phi^{-1}(t) \text{ for all } t \in \mathbb{R} \tag{6.28}$$

as $\varepsilon \to 0$. For completeness we give a proof of this fact. Let $t \in \mathbb{R}$ and $\sigma > 0$. By continuity of Φ^{-1}, there exists $\delta > 0$ such that

$$\left|\Phi^{-1}(\tau) - \Phi^{-1}(t)\right| \leq \sigma$$

for all $\tau \in \overline{B_\delta(t)}$. This in particular implies

$$\left|\Phi^{-1}(t+\delta) - \Phi^{-1}(t)\right| \leq \sigma \quad \text{and} \quad \left|\Phi^{-1}(t-\delta) - \Phi^{-1}(t)\right| \leq \sigma.$$

Using the monotonicity of Φ^{-1}, we obtain

$$\Phi^{-1}(t+\delta) \leq \Phi^{-1}(t) + \sigma \quad \text{and} \quad \Phi^{-1}(t) - \sigma \leq \Phi^{-1}(t-\delta). \tag{6.29}$$

On the other hand, as $\Phi_\varepsilon \to \Phi$ as $\varepsilon \to 0$ on compact sets and Φ^{-1} is continuous, we conclude that there exists $\varepsilon_0 > 0$ such that

$$|\Phi_\varepsilon(z) - \Phi(z)| < \delta, \text{ for all } z \in \Phi^{-1}(\overline{B_\delta(t)}) \text{ and } 0 < \varepsilon < \varepsilon_0,$$

and in particular,

$$\left|\Phi_\varepsilon(\Phi^{-1}(t+\delta)) - (t+\delta)\right| < \delta \quad \text{and} \quad \left|\Phi_\varepsilon(\Phi^{-1}(t-\delta)) - (t-\delta)\right| < \delta$$

for all $\varepsilon < \varepsilon_0$. Now, this implies

$$t \leq \Phi_\varepsilon(\Phi^{-1}(t+\delta)) \quad \text{and} \quad \Phi_\varepsilon(\Phi^{-1}(t-\delta)) \leq t,$$

for all $0 < \varepsilon < \varepsilon_0$. Hence, by the monotonicity of Φ_ε^{-1} we deduce

$$\Phi^{-1}(t-\delta) \leq \Phi_\varepsilon^{-1}(t) \leq \Phi^{-1}(t+\delta),$$

for all $0 < \varepsilon < \varepsilon_0$. Recalling (6.29), we find

$$\Phi^{-1}(t) - \sigma \leq \Phi_\varepsilon^{-1}(t) \leq \Phi^{-1}(t) + \sigma$$

and thus

$$\left|\Phi_\varepsilon^{-1}(t) - \Phi^{-1}(t)\right| \leq \sigma$$

for all $0 < \varepsilon < \varepsilon_0$. This completes the proof of (6.28).

Therefore, using (6.28) we deduce that

$$\Phi_\varepsilon^{-1}(V^{(0)}) \to \Phi^{-1}(V^{(0)}) \text{ as } \varepsilon \to 0, \text{ for a.e. } x \in \Omega.$$

Recalling that by (6.20) and (6.21) there holds $0 \leq V^{(0)}(x,t) \leq M$, we see from the monotonicity of Φ_ε^{-1}, $\Phi_\varepsilon^{-1}(t) \geq 0$ for $t \geq 0$ and $\Phi_\varepsilon^{-1}(t) = \Phi^{-1}(t)$ for $t \in [\Phi(\varepsilon), \Phi(1/\varepsilon)]$ that $\Phi_\varepsilon^{-1}(V^{(0)})$ is uniformly bounded in ε. In fact,

$$0 \leq \Phi_\varepsilon^{-1}(V^{(0)})) \leq \Phi_\varepsilon^{-1}(\max(1,M)) = \Phi^{-1}(\max(1,M))$$

for $\varepsilon > 0$ sufficiently small. Thus, by Lebesgue's dominated convergence theorem we deduce that

$$\Phi_\varepsilon^{-1}(V^{(0)}) \to \Phi^{-1}(V^{(0)}) \text{ in } L^1(\Omega_{T_0}) \text{ as } \varepsilon \to 0.$$

Then clearly the same holds for $u_{0,\varepsilon}$. Furthermore, note that the functions $u_{0,\varepsilon}$ satisfy

$$\begin{cases} \rho\partial_t u_{0,\varepsilon} + L_K(\Phi_\varepsilon(u_{0,\varepsilon})) = F_0^\varepsilon & \text{in } \Omega_T, \\ u_{0,\varepsilon} = \varphi_0^\varepsilon & \text{in } (\Omega_e)_T, \\ u_{0,\varepsilon}(0) = 0 & \text{in } \Omega. \end{cases}$$

with $F_0^\varepsilon = \rho\Phi_\varepsilon^{-1}(h)t\Phi_\varepsilon^{-1}(V^{(0)})$ and $\varphi_0^\varepsilon = \Phi_\varepsilon^{-1}(h)t\Phi_\varepsilon^{-1}(\widetilde{\varphi}_0)$. Meanwhile, let us recall that by construction $u = \Phi^{-1}(v) \geq 0$ solves

$$\begin{cases} \rho\partial_t u + L_K(\Phi(u)) = F_1 & \text{in } \Omega_T, \\ u = \varphi_1 & \text{in } (\Omega_e)_T, \\ u(0) = 0 & \text{in } \Omega \end{cases}$$

with $F_1 = -qu$ and $\varphi_1 = \Phi^{-1}(h)t\Phi^{-1}(\widetilde{\varphi}_0)$. Moreover, by Remark 4.4 there is a sequence $0 \leq u_\varepsilon \in H^1(0,T;L^2(\Omega)) \cap L^2(0,T;H^s(\mathbb{R}^n))$, $\varepsilon > 0$, satisfying

$$\begin{cases} \rho\partial_t u_\varepsilon + L_K(\Phi_\varepsilon(u_\varepsilon)) = F_1^\varepsilon & \text{in } \Omega_T, \\ u_\varepsilon = \varphi_1^\epsilon & \text{in } (\Omega_e)_T, \\ u_\varepsilon(0) = 0 & \text{in } \Omega. \end{cases}$$

with $F_1^\varepsilon = -qu_\varepsilon$ and $\varphi_1^\varepsilon = \Phi^{-1}(h)t\Phi^{-1}(\widetilde{\varphi}_0)$.

As a matter of fact, we can show that the functions $u_0, u_{0,\varepsilon}, u$ and u_ε fulfill all conditions in Theorem 4.3 without absorption term and zero initial condition[3]. First, let us observe that $q \geq 0$, $\rho > 0$, (6.20) and $\Phi^{-1}(t), \Phi_\varepsilon^{-1}(t) \geq 0$ for $t \geq 0$ imply

$$F_1 - F_0 \leq 0 \text{ and } F_1^\varepsilon - F_0^\varepsilon \leq 0.$$

This shows that property (i) of Theorem 4.3 holds. Next, using (6.28) we observe that

$$\varphi_0^\varepsilon = \Phi_\varepsilon^{-1}(h)t\Phi_\varepsilon^{-1}(\widetilde{\varphi}_0) \to \Phi^{-1}(h)t\Phi^{-1}(\widetilde{\varphi}_0) = \varphi_0 \text{ as } \varepsilon \to 0.$$

This pointwise convergence, $\Phi_\varepsilon \to \Phi$ uniformly on compact sets as $\varepsilon \to 0$ and Φ is continuous, implies

$$\Phi_\varepsilon(\varphi_1^\varepsilon) - \Phi_\varepsilon(\varphi_0^\varepsilon) = \Phi_\varepsilon(\Phi^{-1}(h)t\Phi^{-1}(\widetilde{\varphi}_0)) - \Phi_\varepsilon(\Phi_\varepsilon^{-1}(h)t\Phi_\varepsilon^{-1}(\widetilde{\varphi}_0)) \to 0$$

as $\varepsilon \to 0$. Thus, we have in particular

$$(\Phi_\varepsilon(\varphi_1^\varepsilon) - \Phi_\varepsilon(\varphi_0^\varepsilon))_+ \to 0 = (\varphi_1 - \varphi_0)_+ \text{ as } \varepsilon \to 0.$$

Recall that the support of $\Phi_\varepsilon(\varphi_1^\varepsilon) - \Phi_\varepsilon(\varphi_0^\varepsilon)$ is compactly contained in $[0,T] \times W_1$. Hence, by Lebesgue's dominated convergence theorem we can conclude the property (ii) of Theorem 4.3 with equality and right hand side equal to zero. Thus, all assumptions of Theorem 4.3 are satisfied and we can deduce

$$\int_{\Omega_{T_0}} (u - u_0)_+ (x,t)\,dxdt = 0.$$

This gives $u \leq u_0$ and thus by the monotonicity of Φ^{-1} the desired estimate (6.27).

Furthermore, by $\rho \geq 0$, (6.27), the triangle inequality, Hölder's inequality, (6.7), (6.8) and (6.15), we can estimate

$$\begin{aligned} 0 &\leq \mathcal{M}^{(1)} - \mathcal{M} \\ &= \beta\rho \int_0^{T_0} (T_0 - t)^{\beta-1} \left(v_0^{1/m} - v^{1/m} \right) dt \\ &\leq \beta\rho \int_0^{T_0} (T_0 - t)^{\beta-1} (v_0 - v)^{1/m}\, dt \qquad (6.30) \\ &\leq T_0^{\frac{\beta}{m'} - \frac{1}{m}} \frac{\beta\rho}{(\beta - m' + 1)^{\frac{1}{m'}}} \left(\mathcal{B}(\beta+1, m+1) T_0^{\beta+m+1} h V^{(0)} - V \right)^{1/m} \\ &= T_0^{\frac{\beta}{m'} - \frac{1}{m}} \frac{\beta\rho}{(\beta - m' + 1)^{\frac{1}{m'}}} (-R_1)^{1/m}. \end{aligned}$$

[3] Hence, Theorem 4.1 is enough for our purposes.

Similarly, using $q \geq 0$, (6.7) and (6.15), we obtain

$$
\begin{aligned}
0 &\leq \mathcal{N}^{(1)} - \mathcal{N} \\
&= q \int_0^{T_0} (T_0 - t)^\beta \left(v_0^{1/m} - v^{1/m} \right) dt \\
&\leq q \int_0^{T_0} (T_0 - t)^\beta \left(v_0 - v \right)^{1/m} \, dt \\
&\leq q \frac{T_0^{(\beta+1)/m'}}{(\beta+1)^{1/m'}} \left(T_0^{\beta+m+1} \mathcal{B}(\beta+1, m+1) h V^{(0)} - V \right)^{1/m} \\
&\leq q \frac{T_0^{(\beta+1)/m'}}{(\beta+1)^{1/m'}} \left(-R_1 \right)^{1/m}.
\end{aligned} \tag{6.31}
$$

Now, we can deduce the asymptotic behaviour[4]

$$
\left\| \left(\mathcal{M}^{(1)} - \mathcal{M} \right) + \left(\mathcal{N}^{(1)} - \mathcal{N} \right) \right\|_{L^2(\Omega)} = \mathcal{O}\left(h^{\frac{1}{m^2}} \right) \text{ as } h \to \infty.
$$

Here, we used (6.30), (6.31), Jensen's inequality, $\rho, q \in L^\infty(\mathbb{R}^n)$ and (6.17). Combining this with (6.26) and Lemma 6.1, we infer

$$
\|R_2\|_{H^s(\mathbb{R}^n)} = \mathcal{O}\left(h^{\frac{1}{m^2}} \right) \text{ as } h \to \infty. \tag{6.32}
$$

Next, let us denote by $\psi_{w,h} \in L^2(0, T; \widetilde{H}^s(W_2))$, the function

$$
\psi_{w,h}(x,t) = \begin{cases} h^{-1} (T_0 - t)^\beta w(x), & \text{if } 0 \leq t \leq T_0, \\ 0, & \text{otherwise}, \end{cases} \tag{6.33}
$$

where $w \in C_c^\infty(W_2)$. Using (6.4), (6.22) and (6.26), we obtain

(6.34)

$$
\begin{aligned}
\langle \Lambda^\Phi_{\rho,K,q} \widetilde{\varphi}, \psi_{w,h} \rangle &= \int_0^{T_0} (T_0 - t)^\beta h^{-1} B_K(v, w) \, dt \\
&= h^{-1} B_K(V, w) \\
&= h^{-1} B_K(h \mathcal{B}(\beta+1, m+1) T_0^{\beta+m+1} V^{(0)} + h^{\frac{1}{m}} V^{(1)} + R_2, w) \\
&= \mathcal{B}(\beta+1, m+1) T_0^{\beta+m+1} B_K(V^{(0)}, w) + h^{\frac{1}{m}-1} B_K(V^{(1)}, w) + h^{-1} B_K(R_2, w).
\end{aligned}
$$

In the limit $h \to \infty$, we obtain

$$
\lim_{h \to \infty} \langle \Lambda^\Phi_{\rho,K,q} \widetilde{\varphi}, \psi_{w,h} \rangle = \mathcal{B}(\beta+1, m+1) T_0^{\beta+m+1} B_K(V^{(0)}, w) \tag{6.35}
$$

for all $w \in C_c^\infty(W_2)$. Recall that $V^{(0)} \in H^s(\mathbb{R}^n)$ is the unique solution of (6.16) with exterior value $\widetilde{\varphi}_0$. Hence, if we denote the DN map of this equation by $\Lambda_K \colon \widetilde{H}^s(\Omega_e) \to H^{-s}(\Omega_e)$, then (6.35) means nothing else than

$$
\lim_{h \to \infty} \langle \Lambda^\Phi_{\rho,K,q} \widetilde{\varphi}, \psi_{w,h} \rangle = \mathcal{B}(\beta+1, m+1) T_0^{\beta+m+1} \langle \Lambda_K \widetilde{\varphi}_0, w \rangle \tag{6.36}
$$

for all $w \in C_c^\infty(W_2)$ and $\widetilde{\varphi}_0 \in \mathscr{D}_+(W_1)$.

[4] We use the Landau asymptotic notation that $A = \mathcal{O}(B)$ stands for that B is nonnegative, and there is a positive constant C such that $|A| \leq CB$ for large h.

6.4. **Proof of Theorem 1.3.**

Proof of Theorem 1.3. The proof consists of two steps. In the first step we determine the kernel K and then the coefficients ρ and q. Let us start by recalling that the assumptions on the DN maps Λ_{ρ_j,K_j,q_j} for $j=1,2$ guarantee that the transferred DN maps $\Lambda^{\Phi}_{\rho_j,K_j,q_j}$, $j=1,2$, satisfy

$$\langle \Lambda^{\Phi}_{\rho_1,K_1,q_1}\widetilde{\varphi},\psi\rangle = \langle \Lambda^{\Phi}_{\rho_2,K_2,q_2}\widetilde{\varphi},\psi\rangle,$$

for all $\widetilde{\varphi}\in \mathscr{D}_+([0,T]\times W_1)$ and $\psi\in L^2(0,T;\widetilde{H}^s(W_2))$ (see (5.3) and (5.4)). Moreover, let $T_0>0$ be sufficiently small such that the results of Section 6.2 and 6.3 hold, but otherwise be arbitrary.

Step 1: Unique determination of the kernel.

First, let us recall that from the asymptotic expansion of the DN maps (6.36), we deduce

$$\langle \Lambda_{K_1}\widetilde{\varphi}_0,w\rangle = \langle \Lambda_{K_2}\widetilde{\varphi}_0,w\rangle$$

for all $w\in C_c^\infty(W_2)$ and $\widetilde{\varphi}_0\in\mathscr{D}_+(W_1)$. Since L_{K_1} and L_{K_2} are measurement equivalent, we can deduce that there holds

$$K(x,y) := K_1(x,y) = K_2(x,y), \text{ for } x,y\in\mathbb{R}^n$$

as desired (see Definition 1.1). This proves the first step.

Step 2: Unique determination of coefficients.

We now prove the unique determination result for both $\rho_1=\rho_2$ and $q_1=q_2$ in $\overline{\Omega}$. Now, let ψ_{w,h^σ} be the function from (6.33) with h replaced by h^σ and $w\in C_c^\infty(W_1)$. From the results of Section 6.3 and in particular (6.34), we known that for any $\sigma>0$ and $j=1,2$ there holds

(6.37)
$$\begin{aligned}\langle \Lambda^{\Phi}_{\rho_j,K,q_j}\widetilde{\varphi},\psi_{w,h^\sigma}\rangle &= h^{-\sigma}\int_0^{T_0}(T_0-t)^\beta B_K(v,w)\,dt\\ &= B_K(V_j,w)\\ &= h^{-\sigma}B_K(h\mathcal{B}(\beta+1,m+1)T_0^{\beta+m+1}V^{(0)}+h^{\frac{1}{m}}V_j^{(1)}+R_{2,j},w),\end{aligned}$$

where $V_j\in H^s(\mathbb{R}^n)$ is given by

$$V_j=\int_0^{T_0}(T_0-t)^\beta v_j\,dt$$

with v_j denoting the unique solution of

$$\begin{cases}\rho_j\partial_t\Phi^{-1}(v)+L_K(v)+q_j\Phi^{-1}(v)=0 & \text{in } \Omega_T,\\ v=\widetilde{\varphi} & \text{in } (\Omega_e)_T,\\ v(0)=0 & \text{in } \Omega\end{cases}$$

for $\widetilde{\varphi}$ as in (6.4).

Moreover, the asymptotic expansions of V_j are denoted as

$$V_j=V_{h,j}=h\mathcal{B}(\beta+1,m+1)T_0^{\beta+m+1}V^{(0)}+h^{\frac{1}{m}}V_j^{(1)}+R_{2,j},$$

where the $V^{(0)}$ are the same as they are solutions to (6.16) with the same kernel. Subtraction of the expansions (6.37) for $j=1$ and $j=2$ gives

(6.38)
$$\left\langle(\Lambda^{\Phi}_{\rho_1,K,q_1}-\Lambda^{\Phi}_{\rho_2,K,q_2})\widetilde{\varphi},\psi_{w,h^\sigma}\right\rangle = h^{-\sigma}B_K\left(h^{\frac{1}{m}}(V_1^{(1)}-V_2^{(1)})+(R_{2,1}-R_{2,2}),w\right).$$

Now, we take $\sigma=1/m$ and pass to the limit $h\to\infty$ to obtain

$$\lim_{h\to\infty}\left\langle(\Lambda^{\Phi}_{\rho_1,K,q_1}-\Lambda^{\Phi}_{\rho_2,K,q_2})\widetilde{\varphi},\psi_{w,h^\sigma}\right\rangle = B_K(V_1^{(1)}-V_2^{(1)},w) \tag{6.39}$$

for all $w\in C_c^\infty(W_2)$. Here, we are using that by (6.32) for $j=1,2$ we have $R_{2,j}=\mathcal{O}(h^{1/m^2})$ in $H^s(\mathbb{R}^n)$ and hence $h^{-1/m}R_{2,j}\to 0$ in $H^s(\mathbb{R}^n)$ as $h\to\infty$. The last limit vanishes as $1/m^2-1/m<0$. This convergence and the uniform ellipticity of K, now implies that the second term in (6.38) goes to zero in the limit $h\to\infty$. From (6.39) and the definition of $V_j^{(1)}$, $j=1,2$, we deduce that the function $V_1^{(1)}-V_2^{(1)}\in H^s(\mathbb{R}^n)$ satisfies

$$L_K\left(V_1^{(1)}-V_2^{(1)}\right)=V_1^{(1)}-V_2^{(1)}=0 \text{ in } W_2$$

(see (6.23)). As L_K satisfies the UCP on $H^s(\mathbb{R}^n)$ as we assumed, this implies $V_1^{(1)}=V_2^{(1)}$ in $\mathbb{R}^n$.

Now, subtracting the Dirichlet problems (6.23) for $V_1^{(1)}$ and $V_2^{(1)}$, gives

$$\int_\Omega\left[\left(\mathcal{M}_1^{(1)}-\mathcal{M}_2^{(1)}\right)+\left(\mathcal{N}_1^{(1)}-\mathcal{N}_2^{(1)}\right)\right]\psi\,dx=0$$

for any $\psi\in\widetilde{H}^s(\Omega)$, where for $j=1,2$ the quantities $\mathcal{M}_j^{(1)},\mathcal{N}_j^{(1)}$ are given by

$$\mathcal{M}_j^{(1)}=h^{\frac{1}{m}}\rho_j\frac{2T_0^{\beta+1}}{\beta(\beta+1)}\left(V^{(0)}\right)^{\frac{1}{m}},$$
$$\mathcal{N}_j^{(1)}=h^{\frac{1}{m}}q_j\frac{2T_0^{\beta+2}}{(\beta+1)(\beta+2)}\left(V^{(0)}\right)^{\frac{1}{m}}$$

(see (6.24) and (6.25)). This implies

$$\int_\Omega\left[\beta^{-1}(\rho_1-\rho_2)+\frac{T_0}{\beta+2}(q_1-q_2)\right]\left(V^{(0)}\right)^{\frac{1}{m}}\psi\,dx=0 \tag{6.40}$$

for any $\psi\in\widetilde{H}^s(\Omega)$. Passing to the limit $T_0\to 0$ shows

$$\int_\Omega(\rho_1-\rho_2)\left(V^{(0)}\right)^{\frac{1}{m}}\psi\,dx=0$$

for all $\psi\in\widetilde{H}^s(\Omega)$. This implies

$$(\rho_1-\rho_2)\left(V^{(0)}\right)^{\frac{1}{m}}=0 \text{ a.e. in } \Omega. \tag{6.41}$$

Let us assert that this implies $\rho_1=\rho_2$. For contradiction assume that $\rho_1(x_0)\neq\rho_2(x_0)$, then by continuity of ρ_1,ρ_2 implies that there would be an $r>0$ such that $\rho_1\neq\rho_2$ on $B_r(x_0)\subset\Omega$. Now, (6.41) would imply $V^{(0)}=0$ on $B_r(x_0)$. Hence, the UCP on $H^s(\mathbb{R}^n)$ for L_K and (6.16) implies $V^{(0)}=0$ in $\mathbb{R}^n$, which is impossible as $\widetilde{\varphi}_0\neq 0$. Thus, we conclude that $\rho_1=\rho_2$ in Ω. Now, from the continuity of ρ_1,ρ_2, we infer $\rho_1=\rho_2$ in $\overline{\Omega}$. Turning back to equation (6.40), we see that

$$\int_\Omega(q_1-q_2)\left(V^{(0)}\right)^{\frac{1}{m}}\psi\,dx=0$$

for all $\psi\in\widetilde{H}^s(\Omega)$. Now, arguing exactly in the same way as for ρ_1,ρ_2, we can conclude that $q_1=q_2$ in $\overline{\Omega}$. This finishes the proof. $\square$

Appendix A. Some compact embeddings

For the convenience of the reader, we collect here two known compactness results. Here we use the following notation. If $F \subset \mathscr{D}'((0,T);X)$ for a Banach space X and $T>0$, then we set $\partial_t F = \{\partial_t f\,;\, f \in F\}$.

Theorem A.1 (Aubin–Lions lemma, [Sim87, Corollary 4])**.** *Let $X \hookrightarrow B \hookrightarrow Y$ be Banach spaces, where the first embedding is compact, and $1 \leq p < \infty$, $1 < r \leq \infty$. If F is bounded set in $L^p(0,T;X)$ and $\partial_t F$ bounded in $L^1(0,T;Y)$, then F is relatively compact in $L^p(0,T;B)$. If F is bounded in $L^\infty(0,T;X)$ and $\partial_t F$ bounded in $L^r(0,T;Y)$, then F is relatively compact in $C([0,T];B)$.*

Theorem A.2 (Aubin–Lions–Simon lemma, [Sim87, Corollary 5])**.** *Let $X \hookrightarrow B \hookrightarrow Y$ be Banach spaces, where the first embedding is compact, and $1 \leq p \leq \infty$, $1 \leq r \leq \infty$. If F is bounded in $L^p(0,T;X) \cap W^{s,r}(0,T;Y)$ with $s > 0$, $r \geq p$ or $s > 1/r - 1/p$, $r \leq p$. Then F is relatively compact in $L^p(0,T;B)$ if $p < \infty$ and otherwise in $C([0,T];B)$.*

Appendix B. A density results

In some of our proofs the following density result will be important.

Proposition B.1. *Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain, $T > 0$ and $0 < s < 1$. Then the space of test functions $C_c^\infty([0,T) \times \Omega)$ is dense in*

$$\mathcal{W}_T = \{u \in H^1(0,T;\widetilde{H}^s(\Omega))\,;\, u(T) = 0\}.$$

Proof. First of all recall that by standard results we have $H^1(0,T;\widetilde{H}^s(\Omega)) \hookrightarrow C([0,T];\widetilde{H}^s(\Omega))$ and hence $\mathcal{W}_T$ is as closed subspace of $H^1(0,T;\widetilde{H}^s(\Omega))$ itself a Hilbert space. First, we show that any element in $\mathcal{W}_T$ can be approximated by elements in

$$\mathcal{V}_T = \{v \in \mathcal{W}_T \,:\, v(t) = 0 \,\text{in a neighborhood of}\, t = T\}\,.$$

For this purpose choose $u \in \mathcal{W}_T$. Up to time inversion, translation and scaling we can assume that $u \in H^1(0,1;\widetilde{H}^s(\Omega))$ vanishes at $t = 0$. Next, we take any $\eta \in C^\infty(\mathbb{R})$ satisfying

$$0 \leq \eta \leq 1 \quad \text{and} \quad \eta(t) = \begin{cases} 0, & t \leq 1, \\ 1, & t \geq 2. \end{cases}$$

For any $k \in \mathbb{N}$, we now introduce the sequence $\eta_k(t) = \eta(kt)$ and define $u_k(t) = \eta_k(t)u(t)$. Clearly, $u_k \in H^1(0,1;\widetilde{H}^s(\Omega))$ and $u_k(t) = 0$ in a neighborhood of $t = 0$. Hence, it remains to prove that $u_k \to u$ in $H^1(0,1;\widetilde{H}^s(\Omega))$. The convergence $u_k \to u$ in $L^2(0,1;\widetilde{H}^s(\Omega))$ is an immediate consequence of Lebesgue's dominated convergence theorem. Next, observe that by the product rule we have $\partial_t u_k = \eta_k' u + \eta_k \partial_t u$.

Since $\partial_t u \in L^2(0,1;\widetilde{H}^s(\Omega))$, we have again $\eta_k \partial_t u \to \partial_t u$ in $L^2(0,1;\widetilde{H}^s(\Omega))$. Hence, if we can show $\eta_k' u \to 0$ in $L^2(0,1;\widetilde{H}^s(\Omega))$, then we have established $u_k \to u$

in $H^1(0,1;\widetilde{H}^s(\Omega))$. To see this note that

$$\begin{aligned}\|\eta_k' u\|_{L^2(0,1;H^s(\mathbb{R}^n))}^2 &\leq \int_0^1 |\eta_k'(t)|^2 \|u(t)\|_{H^s(\mathbb{R}^n)}^2\,dt\\ &= k^2\int_{1/k}^{2/k} |\eta'(kt)|^2\|u(t)\|_{H^s(\mathbb{R}^n)}^2\,dt\\ &\leq k^2\int_{1/k}^{2/k}\|u(t)\|_{H^s(\mathbb{R}^n)}^2\,dt\\ &= Ck^2\int_{1/k}^{2/k} t^2\frac{\|u(t)\|_{H^s(\mathbb{R}^n)}^2}{t^2}\,dt\\ &\leq C\int_{1/k}^{2/k}\frac{\|u(t)\|_{H^s(\mathbb{R}^n)}^2}{t^2}\,dt.\end{aligned}$$

Thus, if we can show that $\frac{\|u(t)\|_{H^s(\mathbb{R}^n)}^2}{t^2} \in L^1(0,1)$, then it follows from the absolute continuity of the Lebesgue integral that $\eta_k' u \to 0$ in $L^2(0,1;\widetilde{H}^s(\Omega))$. As $u \in H^1(0,1;\widetilde{H}^s(\Omega))$ one has

$$u(t) = u(s) + \int_s^t \partial_t u\,d\tau \tag{B.1}$$

for $0 \leq s \leq t \leq 1$. Moreover, that Hardy's inequality states that for any $1 < p < \infty$ and any measurable function $f\colon \mathbb{R}_+ \to \mathbb{R}_+$ one has

$$\int_0^\infty \left(\frac{1}{t}\int_0^t f(s)\,ds\right)\,dt \leq \left(\frac{p}{p-1}\right)^p \int_0^\infty (f(t))^p\,dt. \tag{B.2}$$

Hence using (B.1), $u(0) = 0$ and (B.2), we get

$$\begin{aligned}\int_0^1 \left(\frac{\|u(t)\|_{H^s(\mathbb{R}^n)}}{t}\right)^2 dt &\leq \int_0^1\left(\frac{1}{t}\int_0^t \|\partial_t u(\tau)\|_{H^s(\mathbb{R}^n)}d\tau\right)^2 dt\\ &\leq 4\int_0^1 \|\partial_t u(t)\|_{H^s(\mathbb{R}^n)}^2\,dt.\end{aligned}$$

This establishes the required integrability of $\|u(t)\|_{H^s(\mathbb{R}^n)}^2/t^2$ and hence $\eta_k' u \to 0$ in $L^2(0,1;\widetilde{H}^s(\Omega))$. Thus, we can conclude that $\mathcal{V}_T$ is dense in $\mathcal{W}_T$.

Next we show that any $u \in \mathcal{V}_T$ can be approximated by elements in

$$\mathcal{D}_T = \left\{\sum_{j=1}^N \varphi_j w_j\,;\, w_j \in C_c^\infty(\Omega),\, \rho_j \in C_c^\infty([0,T)),\, 1 \leq j \leq N,\, N \in \mathbb{N}\right\}.$$

This then establishes that $C_c^\infty([0,T) \times \Omega)$ is dense in $\mathcal{W}_T$. Let $u \in \mathcal{V}_T$. By assumption $\partial_t u \in L^2(0,T;\widetilde{H}^s(\Omega))$ and hence by standard results there exists a sequence $(v_n)_{n\in\mathbb{N}}$ in

$$\mathscr{D}_T = \left\{\sum_{j=1}^N \rho_j\psi_j\,;\, \psi_j \in C_c^\infty(\Omega),\, \rho_j \in C_c^\infty((0,T)),\, 1 \leq j \leq N,\, N \in \mathbb{N}\right\}$$

such that $v_n \to \partial_t v$ in $L^2(0,T;\widetilde{H}^s(\Omega))$. Now define $u_j\colon (0,T) \to \widetilde{H}^s(\Omega)$ by

$$u_j(t) = -\int_t^T v_j(s)\,ds$$

for $j \in \mathbb{N}$. Observe that since u vanishes in a neighborhood of $t = T$, we have $u_j \in \mathcal{D}_T$. Note that there holds

$$\partial_t u_j = v_j \to \partial_t u \,\text{in}\, L^2(0,T\,;\widetilde{H}^s(\Omega)).$$

But since $u(T) = 0$, the fundamental theorem of calculus guarantees

$$u(t) = -\int_t^T \partial_t u\,ds$$

and thus $u_j \to u$ in $L^2(0,T;\widetilde{H}^s(\Omega))$. In fact, first the convergence holds uniformly in t and then by Lebesgue's dominated convergence theorem also in $L^2(0,T;\widetilde{H}^s(\Omega))$. This proves the assertion. □

Statements and Declarations

Data availability statement. No datasets were generated or analyzed during the current study.

Conflict of Interests. Hereby we declare there are no conflict of interests.

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