AN INVERSE SOURCE PROBLEM FOR A FULLY NONLINEAR ELLIPTIC EQUATION

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Abstract. We study an inverse source problem for fully nonlinear elliptic equations of the form

$$F(D^2u) = f \quad \text{in } \Omega.$$

The question is whether the source term can be recovered from the Dirichlet-to-Neumann map. In two dimensions, the first linearization does not immediately give uniqueness: it leaves a natural conformal ambiguity in the linearized coefficients. For homogeneous nonlinearities F with injective differential DF, we show that this ambiguity has a precise meaning at the level of the equation itself, namely that the source is determined up to an explicit scalar factor.

The main point of the paper is to show how this remaining factor can be removed. We use the second linearization to extract information which is invisible at first order, and combine it with an algebraic nondegeneracy condition on the nonlinearity. Under this condition, the residual ambiguity is forced to be trivial, and the Dirichlet-to-Neumann map uniquely determines the source. The result applies, in particular, to homogeneous admissible Hessian equations of Monge–Ampère type and related examples.

Contents

1. Introduction

In this work, we study the inverse source problem for a class of fully nonlinear elliptic equations. From a physical point of view, the source term f models internal forcing mechanisms of the medium, such as distributed loads, heat generation, or external fields acting inside the domain, while the state variable u describes the resulting equilibrium configuration or potential. By prescribing boundary values of u and measuring the induced boundary flux $\partial_\nu u$, one performs noninvasive boundary

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experiments on the system. The inverse problem is then to determine whether these boundary measurements uniquely recover the internal source f.

The mathematical model is given by the boundary value problem

$$\begin{cases} F(D^2u) = f & \text{in } \Omega, \\ u = \varphi & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^2$ is a bounded domain with C^∞ boundary $\partial\Omega$. In this paper, we work in the admissible setting for Hessian equations as in [CNS85]. More precisely, let $\Gamma \subset \mathbb{R}^2$ be an open convex cone with vertex at the origin, symmetric with respect to permutations of the coordinates, and containing the positive cone

$$\Gamma_+ := \left\{(\lambda_1, \lambda_2) \in \mathbb{R}^2 : \ \lambda_1 > 0, \ \lambda_2 > 0\right\}.$$

We assume that $F(M) = \widetilde{f}(\lambda(M))$, where $\lambda(M) = (\lambda_1(M), \lambda_2(M))$ denotes the eigenvalues of M, and $\widetilde{f} \in C^\infty(\Gamma) \cap C^0(\overline{\Gamma})$ is symmetric. Let S denote the vector space of all 2×2 real symmetric matrices. We write

$$\mathcal{A}_\Gamma := \{M \in S : \ \lambda(M) \in \Gamma\}$$

for the admissible set determined by Γ.

Following [CNS85], we impose the following admissibility assumptions on $\widetilde{f}$.

Assumption 1.1.

(i) $\widetilde{f}$ is elliptic in Γ, namely

$$\frac{\partial \widetilde{f}}{\partial \lambda_i} > 0 \quad \text{in } \Gamma, \quad i = 1, 2;$$

(ii) $\widetilde{f}$ is concave in Γ;

(iii) the boundary-side condition

$$\limsup_{\lambda \to \lambda_0} \widetilde{f}(\lambda) \leq \overline{f}_0 \quad \text{for all } \lambda_0 \in \partial\Gamma \tag{1.2}$$

holds for the prescribed source $f > 0$, where

$$f_0 := \min_{\overline{\Omega}} f \quad \text{and} \quad 0 < \overline{f}_0 < f_0;$$

(iv) for every compact set $K \subset \Gamma$ and every $C > 0$, there exists $R = R(K, C) > 0$ such that

$$\widetilde{f}(\lambda_1, \lambda_2 + R) \geq C \quad \text{for all } (\lambda_1, \lambda_2) \in K;$$

(v) for every compact set $K \subset \Gamma$ and every $C > 0$, there exists $R = R(K, C) > 0$ such that

$$\widetilde{f}(R\lambda) \geq C \quad \text{for all } \lambda \in K.$$

We also assume an admissibility condition on the boundary of the domain: there exists $T > 0$ such that, at every point of $\partial\Omega$, if κ_1 denotes the principal curvature of $\partial\Omega$ with respect to the interior normal, then

$$(\kappa_1, T) \in \Gamma. \tag{1.3}$$

Definition 1.2 (Admissible solutions)**.** A solution $u \in C^2(\overline{\Omega})$ of (1.1) with $u = \varphi$ on $\partial\Omega$ is called *admissible* if $\lambda(D^2u(x)) \in \Gamma$ for all $x \in \overline{\Omega}$.

For an admissible matrix M, the ellipticity condition $\partial_{\lambda_i}\widetilde{f} > 0$ implies that the matrix $\big(F_{ab}(M)\big)$ is positive-definite, where

$$F_{ab}(M) := \frac{\partial F}{\partial M_{ab}}(M), \quad a, b = 1, 2. \tag{1.4}$$

In particular, for every admissible solution u, the first linearized operator $F_{ab}(D^2u)\partial_{ab}$ is elliptic.

Under the above assumptions, the direct problem (1.1) is well posed in the admissible class by [CNS85, Theorem 2]. We shall recall in Section 2 the precise well-posedness statement needed in this paper. Thanks to the well-posedness of (1.1), one can define the associated Dirichlet-to-Neumann (DN) map

$$\Lambda_f : C^\infty(\partial\Omega) \to C^\infty(\partial\Omega), \quad \varphi \mapsto \partial_\nu u_\varphi\big|_{\partial\Omega},$$

where $u_\varphi \in C^\infty(\overline{\Omega})$ is the unique admissible solution to (1.1). The inverse problem considered in this paper is the following:

(IP) Can one uniquely determine the source function f in (1.1) from the knowledge of the DN map?

Equations of the form (1.1) arise, for instance, in nonlinear elasticity, where the unknown u represents the displacement of the material and the operator $F(D^2u)$ describes a nonlinear constitutive law relating strain or curvature to internal stress. In this setting, the source term f corresponds to body forces acting inside the material, and the inverse source problem amounts to recovering hidden internal forces from boundary displacement and traction measurements.

Inverse source problems have been extensively studied for linear elliptic equations, but remain far less understood in the fully nonlinear setting. As a notable example, when $F(M) = \det M$, corresponding to the Monge-Ampère equation, an analogue of the inverse problem **(IP)** was recently resolved in [LL25]. The present work is motivated by the question of whether similar uniqueness results continue to hold for a wider class of admissible fully nonlinear elliptic equations. We first obtain a determination result up to an explicit scalar factor, and then impose an additional non-proportionality condition to remove this remaining freedom and obtain full uniqueness.

To study the inverse problem, we impose the following additional structural assumptions on F. For an admissible matrix $M = (M_{ab})_{1\le a,b\le 2}$, we write

$$F_{ab,k\ell}(M) := \frac{\partial^2 F}{\partial M_{ab}\partial M_{k\ell}}(M), \quad a, b, k, \ell = 1, 2.$$

Assumption 1.3. In addition to Assumption 1.1, we further assume that:

(i) F is positively homogeneous of degree m with $0 < m \neq 1$, that is,
$$F(tM) = t^m F(M) \quad \text{for all admissible } M \text{ and all } t > 0;$$
(ii) DF is injective on the admissible class, namely
$$DF(M_1) = DF(M_2) \implies M_1 = M_2$$
for all admissible matrices M_1, M_2;

(iii) for every admissible matrix M, the second derivative tensor $D^2F(M)$ is not proportional to the rank-one tensor $DF(M) \otimes DF(M)$, namely there does not exist any scalar $\gamma \in \mathbb{R}$ such that
$$F_{ab,k\ell}(M) = \gamma\, F_{ab}(M)F_{k\ell}(M) \quad \text{for all } 1 \le a, b, k, \ell \le 2.$$

We are now ready to state the main result of this paper.

Theorem 1.4. *Let $\Omega \subset \mathbb{R}^2$ be a bounded simply connected domain with C^∞ boundary $\partial\Omega$. Let $F(M) = \widetilde{f}(\lambda(M))$, where $\widetilde{f}$ is a smooth symmetric function defined on an open convex cone $\Gamma \subset \mathbb{R}^2$. Assume that (1.3) holds, and that $\widetilde{f}$ satisfies Assumption 1.1 with respect to each source f_j, $j = 1, 2$. Assume furthermore that F satisfies* Assumption 1.3*.*

Let Λ_{f_j} be the DN map associated with the admissible solution of

$$\begin{cases} F(D^2 u^{(j)}) = f_j & \text{in } \Omega, \\ u^{(j)} = \varphi & \text{on } \partial\Omega, \end{cases}$$

for $j = 1, 2$, where $0 < f_j \in C^\infty(\overline{\Omega})$ satisfy (1.2). Suppose that

$$f_1 = f_2 \text{ on } \partial\Omega. \tag{1.5}$$

Then

$$\Lambda_{f_1}(\varphi) = \Lambda_{f_2}(\varphi), \quad \text{for all } \varphi \in C^\infty(\partial\Omega), \tag{1.6}$$

implies $f_1 = f_2$ in $\overline{\Omega}$.

Example 1.5. Let $\Gamma = \Gamma_+$ be the first quadrant of $\mathbb{R}^2$, that is,

$$\Gamma = \{(\lambda_1, \lambda_2) : \lambda_1 > 0, \ \lambda_2 > 0\}.$$

Such a set Γ is referred to as *type 1* in [CNS85]. Equivalently, the corresponding admissible set is the positive definite cone in the space of symmetric matrices. A basic example is given by

$$F(M) = (\det M)^\theta, \quad \theta \in (0, 1/2).$$

Equivalently, $\widetilde{f}(\lambda_1, \lambda_2) = \lambda_1^\theta \lambda_2^\theta$. It is straightforward to check that $\widetilde{f}$ satisfies Assumption 1.1. Moreover, F is positively homogeneous of degree $2\theta \neq 1$, and

$$DF(M) = \theta(\det M)^\theta M^{-1},$$

which is positive-definite for every $M > 0$. Furthermore, when $\theta \in (0, 1/2)$, the functional $F(M) = (\det M)^\theta$ is strictly concave on Γ. In particular, the gradient map DF is injective on Γ.

To verify the non-proportionality condition, note that

$$D^2F(M)[H, K] = \theta(\det M)^\theta \big(\theta \operatorname{tr}(M^{-1}H) \operatorname{tr}(M^{-1}K) - \operatorname{tr}(M^{-1}HM^{-1}K)\big).$$

Working in a basis in which

$$M = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}, \quad \lambda_1, \lambda_2 > 0,$$

choose

$$H = \begin{pmatrix} \lambda_1 & 0 \\ 0 & -\lambda_2 \end{pmatrix}.$$

Then $\operatorname{tr}(M^{-1}H) = 1 - 1 = 0$, so $DF(M) : H = 0$. On the other hand, $\operatorname{tr}(M^{-1}HM^{-1}H) = 2$, which yields

$$D^2F(M)[H, H] = -2\theta(\det M)^\theta \neq 0.$$

If $D^2F(M)$ were proportional to $DF(M) \otimes DF(M)$, then there would exist a scalar $\gamma \in \mathbb{R}$ such that

$$D^2F(M)[H, H] = \gamma(DF(M) : H)^2$$

for every symmetric matrix H. Since $DF(M) : H = 0$ for the above choice of H, this would imply $D^2F(M)[H, H] = 0$, which is a contradiction. Therefore, $D^2F(M)$ is not proportional to $DF(M) \otimes DF(M)$.

Example 1.6. We consider the *type 2* example given in [CNS85, p. 264]. Let

$$\widetilde{f}(\lambda_1,\lambda_2) = \big((\lambda_1+\lambda_2)^2 - \tau(\lambda_1-\lambda_2)^2\big)^{\theta}, \quad 0<\tau<1, \quad 0<\theta<1/2.$$

The corresponding cone is

$$\Gamma := \left\{(\lambda_1,\lambda_2)\in\mathbb{R}^2 : \lambda_1+\lambda_2 > \sqrt{\tau}\,|\lambda_1-\lambda_2|\right\}.$$

Then Γ is an open convex cone with vertex at the origin, symmetric with respect to permutations of the coordinates, and it contains Γ_+ as a proper subset.

For $M\in S$, set $A(M) := (1-\tau)(\operatorname{tr} M)^2 + 4\tau\det M$. Then $F(M) := A(M)^{\theta}$ corresponds to $\widetilde{f}(\lambda_1,\lambda_2)$, since $A(M) = (\lambda_1+\lambda_2)^2 - \tau(\lambda_1-\lambda_2)^2$. Equivalently, if $k = 4\tau/(1-\tau)$, then, after factoring out the positive constant $(1-\tau)^{\theta}$, this corresponds to

$$F(D^2u) = \big((\Delta u)^2 + k(u_{xx}u_{yy} - u_{xy}^2)\big)^{\theta}.$$

Let $\mu_1 := \lambda_1+\lambda_2+\sqrt{\tau}(\lambda_1-\lambda_2)$ and $\mu_2 := \lambda_1+\lambda_2-\sqrt{\tau}(\lambda_1-\lambda_2)$. Then Γ is transformed into the positive quadrant $\{\mu_1>0,\ \mu_2>0\}$, and $\widetilde{f}(\lambda_1,\lambda_2) = (\mu_1\mu_2)^{\theta}$. Hence $\widetilde{f}$ is elliptic and concave in Γ, since $0<\theta<1/2$. Moreover, $\widetilde{f}\to 0$ as λ approaches $\partial\Gamma$, and the growth conditions in Assumption 1.1 follow from the positive homogeneity. Thus, $\widetilde{f}$ satisfies Assumption 1.1.

We now verify Assumption 1.3. First, F is positively homogeneous of degree 2θ, and $0<2\theta<1$. Thus, Assumption 1.3 (i) holds. Next, we prove that DF is injective on the admissible class. Since $A(M) = (1-\tau)(\operatorname{tr} M)^2 + 4\tau\det M$ and, in dimension two, $\operatorname{cof} M = (\operatorname{tr} M)I - M$ for symmetric matrices, we have

$$DA(M) = 2(1-\tau)(\operatorname{tr} M)I + 4\tau\operatorname{cof} M = 2(1+\tau)(\operatorname{tr} M)I - 4\tau M.$$

The map $M\mapsto DA(M)$ is a linear isomorphism on S. Indeed, writing $M = aI + M_0$ with $\operatorname{tr} M_0 = 0$, one obtains

$$DA(M) = 4aI - 4\tau M_0,$$

so the scalar and trace free parts are both recovered uniquely from $DA(M)$.

Suppose that $M_1, M_2\in\mathcal{A}_\Gamma$ and $DF(M_1) = DF(M_2)$. Since $DF(M) = \theta A(M)^{\theta-1}DA(M)$ and $A(M)>0$ on $\mathcal{A}_\Gamma$, we obtain $DA(M_1) = \rho\,DA(M_2)$ for some $\rho>0$. Since $M\mapsto DA(M)$ is a linear isomorphism, this implies $M_1 = \rho M_2$. Using the homogeneity of DF, we get $DF(M_1) = DF(\rho M_2) = \rho^{2\theta-1}DF(M_2)$. Since $DF(M_1) = DF(M_2)$ and $2\theta-1\neq 0$, it follows that $\rho = 1$. Hence, $M_1 = M_2$, and Assumption 1.3 (ii) holds.

It remains to verify the non-proportionality condition. Fix $M\in\mathcal{A}_\Gamma$. Since M is symmetric, we may work in an orthonormal basis in which $M = \operatorname{diag}(\lambda_1,\lambda_2)$. Choose

$$H = \begin{pmatrix} 0 & 1\\ 1 & 0\end{pmatrix}.$$

Then $\operatorname{tr} H = 0$, and since $\operatorname{cof} M$ is diagonal, we have $DA(M)[H] = 0$. Consequently, $DF(M)[H] = \theta A(M)^{\theta-1}DA(M)[H] = 0$. On the other hand, $A(M+sH) = A(M) - 4\tau s^2$, so $D^2A(M)[H,H] = -8\tau$. Therefore,

$$D^2F(M)[H,H] = \theta A(M)^{\theta-1}D^2A(M)[H,H] = -8\theta\tau A(M)^{\theta-1}\neq 0.$$

If $D^2F(M)$ were proportional to $DF(M)\otimes DF(M)$, then there would exist a scalar $\gamma\in\mathbb{R}$ such that $D^2F(M)[H,H] = \gamma(DF(M)[H])^2$. For the above choice of H, the right hand side is zero, while the left hand side is nonzero. This is a contradiction. Hence $D^2F(M)$ is not proportional to $DF(M)\otimes DF(M)$, and Assumption 1.3 (iii) holds.

Before proving the uniqueness result, we record the part of the argument that does not use the non-proportionality condition. It shows that the equality of the DN maps already determines the two possible sources up to an explicit scalar factor.

Proposition 1.7 (Determination up to a gauge)**.** *Let the assumptions of* Theorem 1.4 *hold, except possibly the non-proportionality condition in* Assumption 1.3 (iii)*. Let* $u_0^{(j)} \in C^\infty(\overline{\Omega})$*,* $j = 1, 2$*, be the admissible solutions to*

$$\begin{cases} F(D^2 u_0^{(j)}) = f_j & \text{in } \Omega, \\ u_0^{(j)} = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.7}$$

If (1.5) *and* (1.6) *hold, then there exists a positive function* $c_0 = c_0(x)$ *in* Ω *such that*

$$f_1 = c_0^{\frac{m}{m-1}} f_2 \quad \text{in } \Omega. \tag{1.8}$$

Ideas of the proof. The equality of the nonlinear DN maps allows us to compare the first linearizations at suitable reference solutions. In two dimensions, the inverse problem for the resulting non-divergence form linear equations determines the linearized coefficients only up to a conformal factor. Thus the first linearization gives $DF(D^2 u_0^{(1)}) = c_0 DF(D^2 u_0^{(2)})$ for a positive function c_0. This is the source of the factor appearing in Proposition 1.7. The homogeneity of F and the injectivity of DF then convert this relation into the source relation (1.8).

The main point is to show that this remaining factor is trivial under the non-proportionality condition. For this purpose, we use the second linearization. It produces an identity involving the second derivative tensor D^2F and Hessians of solutions to the first linearized equation. A local construction of solutions, together with Runge approximation, gives enough Hessians at each interior point to identify the possible defect in the second derivative tensor. This yields the rank-one defect identity used in the final step. The non-proportionality condition then excludes the possibility that the conformal factor is nontrivial, and hence gives $c_0 = 1$ and $f_1 = f_2$.

This mechanism is the main novelty of the paper. The first linearization alone leaves a natural conformal ambiguity, while the second linearization detects enough additional information to remove this ambiguity under an algebraic condition on F. Thus the argument separates the inverse problem into two parts: determining the source up to an explicit factor, and eliminating that factor by using the structure of the nonlinearity.

Earlier literature. We also recall that inverse problems for nonlinear partial differential equations have attracted increasing attention in recent years. A classical starting point is the linearization of the nonlinear Dirichlet-to-Neumann map, going back to the work of Isakov [Isa93], which reduces the nonlinear inverse problem to a linear one. Later developments showed that higher order linearizations can reveal genuinely nonlinear effects that are invisible at first order; see, for instance, [KN02, Sun96, Sun10]. A major development in this direction was the higher order linearization method introduced in nonlinear geometric settings in [KLU18], and subsequently developed for semilinear elliptic equations in [FO20, LLLS21]. Since then, this method has led to a broad range of uniqueness and parameter recovery results for nonlinear equations.

In the elliptic setting, higher order linearization techniques have been applied to several semilinear inverse problems, including problems with partial data and related nonlinear interactions; see [LLLS20, LLST22, KU20b, KU20a, FLL23]. Quasilinear elliptic inverse problems have also been studied in [KKU23, CFK+21b, LW23]. For inverse problems related to the minimal surface equation and related quasilinear models, we refer for example to [ABN20, CLLO24, Nur24]. We also mention the inverse source problems for semilinear elliptic and parabolic equations in [LL24, KLL24].

Among nonlinear inverse problems, the recovery of source terms is of particular interest. In the linear case, such problems generally suffer from a natural obstruction, and uniqueness fails in general. In contrast, nonlinear equations may break this obstruction. Recent progress in this direction includes uniqueness results for inverse source problems for semilinear elliptic and parabolic equations [LL24, KLL24], and more recently for the fully nonlinear Monge-Ampère equation in two dimensions [LL25]. The present work continues this line of study for a broader class of admissible fully nonlinear elliptic equations.

Organization of the article. The remainder of this article is organized as follows. In Section 2, we establish the well-posedness of the admissible Dirichlet problem and develop the higher-order linearization scheme. Section 3 is devoted to the analysis of the first linearization, including the boundary determination and the recovery of the linearized coefficients up to a conformal factor. In Section 4, we study the second linearization and derive the key tensorial identities needed for the inverse problem. Finally, in Section 5, we combine these ingredients to prove Theorem 1.4.

2. Preliminaries

2.1. **The well-posedness.** Let us demonstrate the well-posedness of (1.1). Given $\phi \in C^\infty(\partial\Omega)$, we introduce the following set of boundary data:

$$B_{\phi,\delta}(\partial\Omega) := \left\{ \varphi \in C^\infty(\partial\Omega) : \|\varphi - \phi\|_{C^{4,\alpha}(\partial\Omega)} < \delta \right\}, \tag{2.1}$$

for some $\alpha \in (0,1)$ and sufficiently small $\delta > 0$. To solve the inverse problem **(IP)**, we employ the higher-order linearization technique, which requires the following well-posedness result describing the smooth dependence on the boundary data in the admissible class.

Proposition 2.1. *Let $\Omega \subset \mathbb{R}^2$ be a bounded open set with C^∞-smooth boundary $\partial\Omega$. Let $F(M) = \widetilde{f}(\lambda(M))$, where $\widetilde{f}$ is a smooth symmetric function defined on an open convex cone $\Gamma \subset \mathbb{R}^2$ satisfying* Assumption 1.1, *and suppose that Ω satisfies* (1.3). *Let $f \in C^\infty(\overline{\Omega})$ be positive and satisfy* (1.2). *Given $\varphi \in C^\infty(\partial\Omega)$, the following statements hold:*

(i) *There exists a unique admissible solution $u = u_\varphi \in C^\infty(\overline{\Omega})$ to the Dirichlet problem* (1.1).

(ii) *Given $\phi \in C^\infty(\partial\Omega)$, there exist $\delta, C > 0$ such that, for any $\varphi \in B_{\phi,\delta}(\partial\Omega)$ defined by* (2.1), *there exists a unique admissible solution $u_\varphi \in C^\infty(\overline{\Omega})$ of* (1.1) *satisfying*

$$\|u_\varphi - u_\phi\|_{C^{4,\alpha}(\overline{\Omega})} \le C\|\varphi - \phi\|_{C^{4,\alpha}(\partial\Omega)},$$

where $u_\phi \in C^\infty(\overline{\Omega})$ is the solution to

$$\begin{cases} F(D^2 u_\phi) = f & \text{in } \Omega, \\ u_\phi = \phi & \text{on } \partial\Omega. \end{cases} \tag{2.2}$$

(iii) *Moreover, for $\varphi \in B_{\phi,\delta}(\partial\Omega)$, there exist C^∞-Fréchet differentiable maps*

$$\begin{aligned} S &: B_{\phi,\delta}(\partial\Omega) \to C^\infty(\overline{\Omega}), \quad \varphi \mapsto u_\varphi, \\ \Lambda &: B_{\phi,\delta}(\partial\Omega) \to C^\infty(\partial\Omega), \quad \varphi \mapsto \partial_\nu u_\varphi|_{\partial\Omega}. \end{aligned}$$

For the well-posedness, we only use Assumption 1.1, not the additional assumptions in Assumption 1.3 needed for the inverse problem.

Proof of Proposition 2.1. For (i), the existence and uniqueness of an admissible solution follow from [CNS85, Theorem 2]. The assumptions imposed above correspond to the hypotheses required there: the cone condition, symmetry, ellipticity, concavity, the boundary-side condition (1.2), the growth conditions, and the boundary admissibility condition (1.3). Therefore, (1.1) admits a unique admissible classical solution. Since f, φ, and $\partial\Omega$ are smooth, standard elliptic regularity and bootstrap imply that this admissible solution is smooth up to the boundary. Hence $u_\varphi \in C^\infty(\overline{\Omega})$.

For (ii), we first note that the solution $u_\phi \in C^\infty(\overline{\Omega})$ is guaranteed by (i). We now study the local dependence of the admissible solution on the boundary data by the implicit function theorem for Banach spaces; see, for example, [RR04, Theorem 10.6]. Let

$$X := C^{4,\alpha}(\overline{\Omega}), \quad Y := C^{2,\alpha}(\overline{\Omega}) \times C^{4,\alpha}(\partial\Omega).$$

Since u_ϕ is admissible and the cone Γ is open, there exists an open neighborhood $\mathcal{U} \subset X$ of u_ϕ such that $\lambda(D^2u(x)) \in \Gamma$ for all $u \in \mathcal{U}$ and all $x \in \overline{\Omega}$. Thus every $u \in \mathcal{U}$ remains admissible. Consider the map

$$\Psi : C^{4,\alpha}(\partial\Omega) \times \mathcal{U} \to Y, \quad \Psi(\varphi, u) := \left(F(D^2u), u|_{\partial\Omega} - \varphi\right).$$

Since F is smooth on the admissible class, the map Ψ is well defined and C^∞-smooth in the Fréchet sense.

From (2.2), we have

$$\Psi(\phi, u_\phi) = (F(D^2u_\phi), u_\phi|_{\partial\Omega} - \phi) = (f, 0),$$

and the partial differential with respect to u is given by

$$\begin{aligned} &\partial_u\Psi(\phi, u_\phi) : X \to Y, \\ &\partial_u\Psi(\phi, u_\phi)v = \left(F_{ab}(D^2u_\phi)\partial_{ab}v, v|_{\partial\Omega}\right), \end{aligned}$$

for any $v \in X$. Since u_ϕ is admissible and smooth up to the boundary, the image $\{D^2u_\phi(x) : x \in \overline{\Omega}\}$ is a compact subset of the admissible class. By the ellipticity condition in Assumption 1.1(i), the matrix $\left(F_{ab}(D^2u_\phi(x))\right)$ is positive-definite for every $x \in \overline{\Omega}$, and hence uniformly positive-definite on $\overline{\Omega}$. Therefore $F_{ab}(D^2u_\phi)\partial_{ab}$ is a uniformly elliptic operator in Ω.

We now claim that

$$\partial_u\Psi(\phi, u_\phi) : X \to Y, \quad v \mapsto \left(F_{ab}(D^2u_\phi)\partial_{ab}v, v|_{\partial\Omega}\right)$$

is a linear isomorphism. Indeed, consider the Dirichlet problem

$$\begin{cases} F_{ab}(D^2u_\phi)\partial_{ab}v = G & \text{in } \Omega, \\ v = \psi & \text{on } \partial\Omega, \end{cases} \tag{2.3}$$

where $G \in C^{2,\alpha}(\overline{\Omega})$ and $\psi \in C^{4,\alpha}(\partial\Omega)$. As in [LL25, Section 2], and using the standard solvability theory for linear uniformly elliptic equations in non-divergence form together with the Schauder estimates in [GT01, Chapter 6], the Dirichlet problem (2.3) has a unique solution $v \in C^{2,\alpha}(\overline{\Omega})$. Moreover, the global Schauder estimates imply that $v \in C^{4,\alpha}(\overline{\Omega})$. Hence $\partial_u\Psi(\phi, u_\phi)$ is a linear isomorphism from X onto Y.

Therefore, by the implicit function theorem, there exist $\delta > 0$ and a unique C^∞-Fréchet differentiable map

$$S : B_{\phi,\delta}(\partial\Omega) \to C^\infty(\overline{\Omega}), \quad \varphi \mapsto S(\varphi),$$

such that $S(\phi) = u_\phi$ and $\Psi(\varphi, S(\varphi)) = (f, 0)$ for all $\varphi \in B_{\phi,\delta}(\partial\Omega)$, provided that $\delta > 0$ is sufficiently small. Writing $u_\varphi := S(\varphi)$, we obtain a unique admissible solution near u_ϕ for all nearby boundary data. Since S is locally Lipschitz and $S(\phi) = u_\phi$, there exists a constant $C > 0$ such that

$$\|u_\varphi - u_\phi\|_{C^{4,\alpha}(\overline{\Omega})} \le C\|\varphi - \phi\|_{C^{4,\alpha}(\partial\Omega)}.$$

This proves (ii).

Finally, (iii) follows from the C^∞-Fréchet smoothness of the solution map S and the fact that the normal trace map $u \mapsto \partial_\nu u|_{\partial\Omega}$ is continuous and linear from $C^{4,\alpha}(\overline{\Omega})$ to $C^{3,\alpha}(\partial\Omega)$. Hence it defines a smooth map into $C^\infty(\partial\Omega)$ on the smooth solutions under consideration. This completes the proof. $\square$

2.2. **The higher order linearization.** Inspired by [LLLS21, FO20], let us apply the higher order linearization technique to (1.1), with the Dirichlet data

$$\varphi = \phi + \varepsilon_1\phi_1 + \varepsilon_2\phi_2 \text{ on } \partial\Omega, \tag{2.4}$$

where $\phi, \phi_1, \phi_2 \in C^\infty(\partial\Omega)$ are given functions, and $\varepsilon_k \in \mathbb{R}$ are parameters with $|\varepsilon_k|$ sufficiently small for $k = 1, 2$. By Proposition 2.1, for sufficiently small $\varepsilon = (\varepsilon_1, \varepsilon_2)$, there exists a unique admissible solution $u_\varepsilon \in C^\infty(\overline{\Omega})$ to

$$\begin{cases} F(D^2 u_\varepsilon) = f & \text{in } \Omega, \\ u_\varepsilon = \phi + \varepsilon_1\phi_1 + \varepsilon_2\phi_2 & \text{on } \partial\Omega. \end{cases} \tag{2.5}$$

2.2.1. *The first linearization.* Let u_ϕ be the admissible solution to

$$\begin{cases} F(D^2 u_\phi) = f & \text{in } \Omega, \\ u_\phi = \phi & \text{on } \partial\Omega. \end{cases} \tag{2.6}$$

Differentiating (2.5) with respect to ε_k at $\varepsilon = 0$, we obtain the first linearized equation

$$\begin{cases} F_{ab}(D^2 u_\phi)\partial_{ab} v_k = 0 & \text{in } \Omega, \\ v_k = \phi_k & \text{on } \partial\Omega, \end{cases} \tag{2.7}$$

where

$$v_k = \partial_{\varepsilon_k}\big|_{\varepsilon=0} u_\varepsilon, \quad k = 1, 2.$$

Since u_ϕ is admissible and smooth up to the boundary, the set $\{D^2 u_\phi(x) : x \in \overline{\Omega}\}$ is a compact subset of the admissible class. By the ellipticity condition in Assumption 1.1 (i), the matrix $\big(F_{ab}(D^2 u_\phi(x))\big)$ is positive definite for every $x \in \overline{\Omega}$, and hence uniformly positive definite on $\overline{\Omega}$. Therefore the operator $F_{ab}(D^2 u_\phi)\partial_{ab}$ is uniformly elliptic in Ω.

2.2.2. *The second linearization.* Using the Dirichlet data (2.4) and applying the second derivative $\partial_{\epsilon_1\epsilon_2}\big|_{\epsilon=0}$ on the equation (2.5), then one has the following second linearized equation

$$\begin{cases} g^{ab}\partial_{ab} w + g^{ab,k\ell}\partial_{ab} v_1 \partial_{k\ell} v_2 = 0 & \text{in } \Omega, \\ w = 0 & \text{on } \partial\Omega, \end{cases}$$

where v_k is the solution to (2.7) for $k = 1, 2$, and $w = \partial_{\epsilon_1\epsilon_2}\big|_{\epsilon=0} u_\epsilon$. Here,

$$g^{ab} := F_{ab}(D^2 u_\phi) \text{ and } g^{ab,k\ell} := F_{ab,k\ell}(D^2 u_\phi) \text{ for } x \in \overline{\Omega},$$

for all $a, b, k, \ell = 1, 2$, where u_ϕ is the admissible solution to (2.6). In the rest of the paper, we will adopt these notations to simplify the analysis.

3. The first linearization

We first extract useful information from the first derivative of F at suitable admissible solutions.

3.1. **Boundary determination.** Let us first show the boundary determination of F on $\partial\Omega$. To this end, given $\phi \in C^\infty(\partial\Omega)$, let us consider the following admissible Dirichlet problem

$$\begin{cases} F(D^2 u_\phi^{(j)}) = f_j & \text{in } \Omega, \\ u_\phi^{(j)} = \phi & \text{on } \partial\Omega, \end{cases} \tag{3.1}$$

for $j = 1, 2$.

Lemma 3.1 (Boundary determination). *Adopting the assumptions in* Theorem 1.4, *the relations* (1.5) *and* (1.6) *imply*

$$F_{ab}(D^2u_\phi^{(1)})\big|_{\partial\Omega} = F_{ab}(D^2u_\phi^{(2)})\big|_{\partial\Omega},$$

where F_{ab} is given by (1.4) *for $1 \le a, b \le 2$, and $u_\phi^{(j)} \in C^\infty(\overline{\Omega})$ is the admissible solution to* (3.1), *for $j = 1, 2$.*

Proof. Recall that $\mathcal{A}_\Gamma$ denotes the admissible class. Fix a boundary point $x_0 \in \partial\Omega$. Without loss of generality, we may assume that $x_0 = (0, 0)$, and the boundary near x_0 is parameterized by $x_2 = \eta(x_1)$ for $x_1 \in (-\varepsilon, \varepsilon)$, where $\eta \in C^\infty((-\varepsilon, \varepsilon))$ and $\eta(0) = \eta'(0) = 0$.

Since $u_\phi^{(1)} = u_\phi^{(2)} = \phi$ on $\partial\Omega$ and the DN maps agree, we have $\partial_\nu u_\phi^{(1)} = \partial_\nu u_\phi^{(2)}$ on $\partial\Omega$. Hence the tangential and normal derivatives of $u_\phi^{(1)}$ and $u_\phi^{(2)}$ coincide on $\partial\Omega$, and therefore $\nabla u_\phi^{(1)} = \nabla u_\phi^{(2)}$ on $\partial\Omega$. In particular,

$$\partial_1 u_\phi^{(1)}(x_1, \eta(x_1)) = \partial_1 u_\phi^{(2)}(x_1, \eta(x_1))$$

and

$$\partial_2 u_\phi^{(1)}(x_1, \eta(x_1)) = \partial_2 u_\phi^{(2)}(x_1, \eta(x_1))$$

for x_1 near 0. Differentiating these identities with respect to x_1 and evaluating at $x_1 = 0$, we obtain

$$\partial_{11} u_\phi^{(1)}(0) + \eta'(0)\partial_{12} u_\phi^{(1)}(0) = \partial_{11} u_\phi^{(2)}(0) + \eta'(0)\partial_{12} u_\phi^{(2)}(0)$$

and

$$\partial_{12} u_\phi^{(1)}(0) + \eta'(0)\partial_{22} u_\phi^{(1)}(0) = \partial_{12} u_\phi^{(2)}(0) + \eta'(0)\partial_{22} u_\phi^{(2)}(0).$$

Since $\eta'(0) = 0$, it follows that

$$\partial_{11} u_\phi^{(1)}(0) = \partial_{11} u_\phi^{(2)}(0) \quad \text{and} \quad \partial_{12} u_\phi^{(1)}(0) = \partial_{12} u_\phi^{(2)}(0). \tag{3.2}$$

It remains to show that $\partial_{22} u_\phi^{(1)}(0) = \partial_{22} u_\phi^{(2)}(0)$.

Write $t_j := \partial_{22} u_\phi^{(j)}(0)$ for $j = 1, 2$, and define

$$M_0 = \begin{pmatrix} m_0^{(11)} & m_0^{(12)} \\ m_0^{(12)} & 0 \end{pmatrix},$$

where $m_0^{(11)} = \partial_{11} u_\phi^{(1)}(0) = \partial_{11} u_\phi^{(2)}(0)$ and $m_0^{(12)} = \partial_{12} u_\phi^{(1)}(0) = \partial_{12} u_\phi^{(2)}(0)$. Then $D^2 u_\phi^{(j)}(0) = M_0 + t_j\, e_2 \otimes e_2$ for $j = 1, 2$, where $e_2 = (0, 1)$. Since $u_\phi^{(j)}$ is admissible, both matrices $D^2 u_\phi^{(j)}(0)$ belong to $\mathcal{A}_\Gamma$. Consider

$$I := \{t \in \mathbb{R} : \ M_0 + t\, e_2 \otimes e_2 \in \mathcal{A}_\Gamma\}.$$

Since $\mathcal{A}_\Gamma$ is an open convex subset of S, the set I is an open interval containing t_1 and t_2.

Define $g(t) := F(M_0 + t\, e_2 \otimes e_2)$ for $t \in I$. Then g is well defined on I, and by the chain rule,

$$g'(t) = F_{22}(M_0 + t\, e_2 \otimes e_2), \quad t \in I.$$

By ellipticity in the admissible class, the matrix $\big(F_{ab}(M)\big)$ is positive definite for every $M \in \mathcal{A}_\Gamma$. In particular, $g'(t) = F_{22}(M_0 + t\, e_2 \otimes e_2) > 0$ for all $t \in I$. Hence g is strictly increasing on I. On the other hand, by construction,

$$g(t_j) = F(D^2 u_\phi^{(j)}(0)) = f_j(0), \quad j = 1, 2.$$

Since $x_0 = 0 \in \partial\Omega$ and $f_1 = f_2$ on $\partial\Omega$, we have $g(t_1) = f_1(0) = f_2(0) = g(t_2)$. Because g is strictly increasing on I, it follows that $t_1 = t_2$. Therefore $\partial_{22} u_\phi^{(1)}(0) = \partial_{22} u_\phi^{(2)}(0)$.

Together with (3.2), this implies

$$D^2u_\phi^{(1)}(0) = D^2u_\phi^{(2)}(0). \tag{3.3}$$

Since $x_0 \in \partial\Omega$ was arbitrary, we conclude that $D^2u_\phi^{(1)}\big|_{\partial\Omega} = D^2u_\phi^{(2)}\big|_{\partial\Omega}$. Because F is known a priori, this yields

$$F_{ab}(D^2u_\phi^{(1)})\big|_{\partial\Omega} = F_{ab}(D^2u_\phi^{(2)})\big|_{\partial\Omega},$$

for $1 \le a, b \le 2$. This proves the lemma. □

Remark 3.2. *Note that* (3.3) *already implies not only*

$$F_{ab}(D^2u_\phi^{(1)})\big|_{\partial\Omega} = F_{ab}(D^2u_\phi^{(2)})\big|_{\partial\Omega},$$

but also

$$D^kF_{ab}(D^2u_\phi^{(1)})\big|_{\partial\Omega} = D^kF_{ab}(D^2u_\phi^{(2)})\big|_{\partial\Omega}$$

for all $k \in \mathbb{N}$, since the nonlinear functional F is known a priori. In addition, the concavity of F is not used in the derivation of the boundary determination.

3.2. **Interior recovery up to a conformal factor.** First, since we know the DN maps of (1.1), we obtain the boundary information of $\nabla u|_{\partial\Omega}$ for any admissible solution u to (1.1). Second, thanks to the boundary determination, we also know the matrix $\big(F_{ab}(D^2u_\phi(x))\big)_{1\le a,b\le 2}$ for $x \in \partial\Omega$. Thus, we can obtain the following alternative DN map for the linearized equation

$$\Lambda'_g : C^\infty(\partial\Omega) \to C^\infty(\partial\Omega), \quad \phi \mapsto g^{ab}\partial_b v\nu_a\big|_{\partial\Omega},$$

where $v \in C^\infty(\overline{\Omega})$ is the solution to (2.7), $\nu = (\nu_1, \nu_2)$ denotes the unit outer normal on $\partial\Omega$.

Now, we reduce our original inverse problem **(IP)** to the inverse problem for the first linearized equation (2.7). More concretely, we use the information in (1.6) to determine the metric g whenever $\Omega \subset \mathbb{R}^2$ is a bounded simply connected domain. Therefore, we can apply [LL25, Theorem 1.4] to recover the metric up to a conformal factor $c > 0$ in Ω, with $c|_{\partial\Omega} = 1$. For the reader's convenience, we record the following result, which gives the recovery of the metric g in the non-divergence elliptic equation.

Proposition 3.3 ([LL25, Theorem 1.4])**.** *Let $\Omega \subset \mathbb{R}^2$ be a bounded open simply connected domain with C^∞-smooth boundary $\partial\Omega$, and let $g = \big(g_{ab}\big)$ be a symmetric, positive definite, and C^∞-smooth 2×2 matrix-valued function and $(g^{ab}) = (g_{ab})^{-1}$. Let $g = g_j$ be two metrics, and let*

$$\Lambda'_{g_j} : C^\infty(\partial\Omega) \to C^\infty(\partial\Omega), \quad h \mapsto g_j^{ab}\partial_b v^{(j)}\nu_a\big|_{\partial\Omega},$$

be the DN map of the equation

$$\begin{cases} g_j^{ab}\partial_{ab}v^{(j)} = 0 & \text{in } \Omega, \\ v^{(j)} = h & \text{on } \partial\Omega, \end{cases}$$

for $j = 1, 2$. Suppose that

$$\Lambda'_{g_1}(h) = \Lambda'_{g_2}(h) \quad \text{for all } h \in C^\infty(\partial\Omega),$$

then there exists a C^∞-smooth conformal factor $c = c(x) > 0$ with $c|_{\partial\Omega} = 1$ such that

$$g_1^{ab} = cg_2^{ab} \text{ in } \Omega, \quad 1 \le a, b \le 2.$$

The above proposition applies since the coefficient matrix (g^{ab}) is smooth and positive definite on $\overline{\Omega}$, hence uniformly elliptic on $\overline{\Omega}$.

Lemma 3.4. *Assume the hypotheses of Theorem 1.4, except possibly the non-proportionality condition in Assumption 1.3 (iii). Let $v_k^{(j)}\in C^\infty(\overline{\Omega})$ be the solution to the first linearized equation*

$$\begin{cases} F_{ab}(D^2u_\phi^{(j)})\partial_{ab}v_k^{(j)}=0 & \text{in } \Omega,\\ v_k^{(j)}=\phi_k & \text{on } \partial\Omega, \end{cases} \tag{3.4}$$

for $k=1,2$, where $u_\phi^{(j)}\in C^\infty(\overline{\Omega})$ is the admissible solution to

$$\begin{cases} F(D^2u_\phi^{(j)})=f_j & \text{in } \Omega,\\ u_\phi^{(j)}=\phi & \text{on } \partial\Omega. \end{cases} \tag{3.5}$$

for $j=1,2$. Then condition (1.6) implies $v_k^{(1)}=v_k^{(2)}$ in Ω, for $k=1,2$.

Proof. First, thanks to the boundary determination in Lemma 3.1, we have $g_1^{ab}=g_2^{ab}$ on $\partial\Omega$, where

$$g_j^{ab}:=F_{ab}(D^2u_\phi^{(j)}),\quad a,b=1,2. \tag{3.6}$$

Second, since we know (1.6), we obtain the boundary information

$$\nabla u_\phi^{(1)}\big|_{\partial\Omega}=\nabla u_\phi^{(2)}\big|_{\partial\Omega},$$

where $u_\phi^{(j)}\in C^\infty(\overline{\Omega})$ is the admissible solution to (3.5), for $j=1,2$. In view of (3.6), it is clear that g_j^{ab} depends on ϕ, for $a,b,j=1,2$.

Since we know $g_1^{ab}=g_2^{ab}$ on $\partial\Omega$, for $a,b=1,2$, and the Dirichlet data ϕ_k can be chosen arbitrarily in $C^\infty(\partial\Omega)$, the first linearization of (1.6) gives

$$\Lambda'_{g_1}(h)=\Lambda'_{g_2}(h)\quad \text{for all } h\in C^\infty(\partial\Omega).$$

Applying Proposition 3.3, we can recover the leading coefficients for the first linearized equation (3.4) up to a conformal factor. More precisely, there exists a conformal factor $c_\phi=c_\phi(x)>0$ such that $c_\phi|_{\partial\Omega}=1$ and

$$F_{ab}(D^2u_\phi^{(1)})=c_\phi F_{ab}(D^2u_\phi^{(2)}) \text{ in } \Omega. \tag{3.7}$$

On the one hand, for each $k=1,2$, we have

$$\begin{cases} g_1^{ab}\partial_{ab}v_k^{(1)}=c_\phi g_2^{ab}\partial_{ab}v_k^{(1)}=0 & \text{in } \Omega,\\ v_k^{(1)}=\phi_k & \text{on } \partial\Omega, \end{cases}$$

and hence we may rewrite the equation as

$$\begin{cases} g_2^{ab}\partial_{ab}v_k^{(1)}=0 & \text{in } \Omega,\\ v_k^{(1)}=\phi_k & \text{on } \partial\Omega, \end{cases}$$

since $c_\phi>0$ in $\overline{\Omega}$. On the other hand, $v_k^{(2)}$ also solves the Dirichlet problem

$$\begin{cases} g_2^{ab}\partial_{ab}v_k^{(2)}=0 & \text{in } \Omega,\\ v_k^{(2)}=\phi_k & \text{on } \partial\Omega, \end{cases}$$

for $k=1,2$. Therefore, $v_k^{(1)}$ and $v_k^{(2)}$ solve the same equation with the same Dirichlet data, which implies that $v_k^{(1)}=v_k^{(2)}$ in $\overline{\Omega}$, for $k=1,2$. This proves the assertion. $\square$

For the forthcoming analysis, let us adopt the notation

$$v_k := v_k^{(1)} = v_k^{(2)} \text{ in } \overline{\Omega}, \text{ for } k = 1, 2,$$

which will always denote the solution to the first linearized equation (3.4).

3.3. Determination up to a gauge.

Thanks to the first linearization, we can prove Proposition 1.7.

Proof of Proposition 1.7. Let $u_0^{(j)} \in C^\infty(\overline{\Omega})$, $j = 1, 2$, be the admissible solutions to (1.7). By Proposition 3.3 and Lemma 3.4, there exists a positive function $c_0 = c_0(x)$ such that

$$F_{ab}(D^2 u_0^{(1)}) = c_0 F_{ab}(D^2 u_0^{(2)}) \quad \text{in } \Omega. \tag{3.8}$$

Fix $x \in \Omega$ and set

$$M_1 := D^2 u_0^{(1)}(x), \quad M_2 := D^2 u_0^{(2)}(x).$$

Since $u_0^{(1)}$ and $u_0^{(2)}$ are admissible, both M_1 and M_2 belong to the admissible class. From (3.8), we have

$$DF(M_1) = c_0(x) DF(M_2).$$

Since F is positively homogeneous of degree $m \neq 1$, its first derivative is homogeneous of degree $m - 1$. Thus

$$DF(tM_2) = t^{m-1} DF(M_2)$$

for every $t > 0$. Define $t(x) := c_0(x)^{1/(m-1)}$. Then

$$DF(M_1) = DF(t(x)M_2).$$

Since the admissible class is a cone, $t(x)M_2$ is admissible. By Assumption 1.3 (ii), we obtain

$$M_1 = t(x)M_2.$$

Using the homogeneity of F again, we get

$$f_1(x) = F(M_1) = F(t(x)M_2) = t(x)^m F(M_2) = c_0(x)^{\frac{m}{m-1}} f_2(x).$$

Since $x \in \Omega$ was arbitrary, this proves (1.8). $\square$

4. The second linearization

For the second linearization, let $\zeta \in C^\infty(\partial\Omega)$ be given, and consider the one-parameter family of boundary data

$$\phi = \delta\zeta \text{ on } \partial\Omega$$

in (2.4), where $\delta \in (-\varepsilon, \varepsilon)$ and $\varepsilon > 0$ is sufficiently small. For $j = 1, 2$, let $u_\delta^{(j)}$ denote the admissible solution to

$$\begin{cases} F(D^2 u_\delta^{(j)}) = f_j & \text{in } \Omega, \\ u_\delta^{(j)} = \delta\zeta & \text{on } \partial\Omega. \end{cases}$$

Then the identity (3.7) can be rewritten as

$$F_{ab}(D^2 u_\delta^{(1)}) = c_\delta \, F_{ab}(D^2 u_\delta^{(2)}) \quad \text{in } \Omega. \tag{4.1}$$

Furthermore, since the solution maps depend smoothly on δ and the conformal factor in (4.1) is uniquely determined, c_δ depends smoothly on δ. We now differentiate (4.1) with respect to δ at $\delta = 0$. This yields

$$F_{ab,k\ell}(D^2 u_0^{(1)}) \, \partial_{k\ell} v^{(1)} = c_0 \, F_{ab,k\ell}(D^2 u_0^{(2)}) \, \partial_{k\ell} v^{(2)} + \dot{c} \, F_{ab}(D^2 u_0^{(2)}) \quad \text{in } \Omega, \tag{4.2}$$

where

$$c_0 := c_\delta|_{\delta=0}, \quad \dot{c} := \partial_\delta c_\delta|_{\delta=0}.$$

Here $u_0^{(j)}$ is the admissible solution of

$$\begin{cases} F(D^2 u_0^{(j)}) = f_j & \text{in } \Omega, \\ u_0^{(j)} = 0 & \text{on } \partial\Omega, \end{cases}$$

and $v^{(j)}$ denotes the solution to the first linearized equation

$$\begin{cases} F_{ab}(D^2 u_0^{(j)})\partial_{ab} v^{(j)} = 0 & \text{in } \Omega, \\ v^{(j)} = \zeta & \text{on } \partial\Omega, \end{cases} \tag{4.3}$$

for $j = 1, 2$.

By Lemma 3.4, there exists a conformal factor $\tilde{c} > 0$ in Ω such that

$$F_{ab}(D^2 u_0^{(1)}) = \tilde{c} F_{ab}(D^2 u_0^{(2)}) \quad \text{in } \Omega.$$

Since the two equations in (4.3) have the same boundary value ζ, uniqueness for second order elliptic equations implies

$$v := v^{(1)} = v^{(2)} \quad \text{in } \Omega.$$

Substituting this into (4.2), we obtain

$$\big(F_{ab,k\ell}(D^2 u_0^{(1)}) - c_0 F_{ab,k\ell}(D^2 u_0^{(2)})\big)\partial_{k\ell} v = \dot{c} F_{ab}(D^2 u_0^{(2)}) \quad \text{in } \Omega, \tag{4.4}$$

for $a, b = 1, 2$.

Lemma 4.1. *For every choice of $\zeta \in C^\infty(\partial\Omega)$, one has $\dot{c} = 0$.*

Proof. Fix $\zeta \in C^\infty(\partial\Omega)$. For each sufficiently small δ, the conformal factor c_δ satisfies

$$DF(D^2 u_\delta^{(1)}) = c_\delta DF(D^2 u_\delta^{(2)}) \quad \text{in } \Omega.$$

Since F is positively homogeneous of degree $m \neq 1$, its first derivative is homogeneous of degree $m - 1$. Setting $t_\delta := c_\delta^{1/(m-1)}$, we have

$$DF(D^2 u_\delta^{(1)}) = DF(t_\delta D^2 u_\delta^{(2)}).$$

By the injectivity of DF on the admissible class,

$$D^2 u_\delta^{(1)} = t_\delta D^2 u_\delta^{(2)} \quad \text{in } \Omega.$$

Using the homogeneity of F, we obtain

$$f_1 = F(D^2 u_\delta^{(1)}) = t_\delta^m F(D^2 u_\delta^{(2)}) = c_\delta^{\frac{m}{m-1}} f_2 \quad \text{in } \Omega.$$

Since f_1 and f_2 are independent of δ, the last identity shows that c_δ is independent of δ, and hence, $\dot{c} = 0$ in Ω. □

4.1. **Spanning Hessians and Runge approximation.** Consequently, (4.4) simplifies to

$$\big(F_{ab,k\ell}(D^2u_0^{(1)}) - c_0\, F_{ab,k\ell}(D^2u_0^{(2)})\big)\partial_{k\ell}v = 0 \quad \text{in } \Omega, \tag{4.5}$$

for every solution v to (4.3) and every $a, b = 1, 2$.

To proceed further from (4.5), one needs to understand how much information about the tensor

$$T^{ab}_{k\ell} := F_{ab,k\ell}(D^2u_0^{(1)}) - c_0 F_{ab,k\ell}(D^2u_0^{(2)})$$

can be extracted by testing against Hessians of solutions to (4.3). At each point $x \in \Omega$, let

$$A(x) := \big(A_{k\ell}(x)\big)_{1\le k,\ell\le 2} := \big(F_{k\ell}(D^2u_0^{(2)}(x))\big)_{1\le k,\ell\le 2}.$$

Since every solution v to (4.3) satisfies

$$A_{k\ell}(x)\partial_{k\ell}v(x) = 0,$$

the Hessian $D^2v(x)$ necessarily belongs to the subspace

$$K_{A(x)} := \big\{H = (H_{k\ell}) \in S :\ A_{k\ell}(x)H_{k\ell} = 0\big\}.$$

The following elementary linear algebra lemma shows that, once sufficiently many such Hessians are available, the tensor $T^{ab}_{k\ell}(x)$ must be proportional to $A_{k\ell}(x)$ in the (k,ℓ)-indices.

Lemma 4.2. *Let $A = (A_{k\ell}(x))$ be a non-degenerate matrix-valued function in Ω with $A(x) \in S$ for every $x \in \Omega$. For each $x \in \Omega$, define $K_{A(x)}$ as above. Suppose that for each fixed $1 \le a, b \le 2$, the symmetric matrix-valued function*

$$T^{ab}(x) := \big(T^{ab}_{k\ell}(x)\big)_{1\le k,\ell\le 2}$$

satisfies

$$T^{ab}_{k\ell}(x)H_{k\ell} = 0 \quad \textit{for all } H \in K_{A(x)},$$

for every $x \in \Omega$. Then there exists a scalar function $\lambda^{ab} = \lambda^{ab}(x)$ such that

$$T^{ab}_{k\ell}(x) = \lambda^{ab}(x)A_{k\ell}(x) \quad \textit{for all } x \in \Omega, \quad 1 \le k, \ell \le 2.$$

Proof. Fix $x \in \Omega$. Since S is a three-dimensional real inner product space with respect to the Frobenius product, i.e.,

$$M : N := M_{k\ell}N_{k\ell},$$

and $A(x) \neq 0$, the set $K_{A(x)}$ is a codimension-one linear subspace of S. Hence, its orthogonal complement $K^{\perp}_{A(x)}$ is one-dimensional. By definition of $K_{A(x)}$, one has

$$A(x) : H = 0 \quad \text{for all } H \in K_{A(x)},$$

so that $A(x) \in K^{\perp}_{A(x)}$. Therefore

$$K^{\perp}_{A(x)} = \operatorname{span}\{A(x)\}.$$

Now fix $1 \le a, b \le 2$. The assumption implies that

$$T^{ab}(x) : H = 0 \quad \text{for all } H \in K_{A(x)},$$

namely $T^{ab}(x) \in K^{\perp}_{A(x)}$. Since $K^{\perp}_{A(x)} = \operatorname{span}\{A(x)\}$, there exists a scalar $\lambda^{ab}(x)$ such that

$$T^{ab}(x) = \lambda^{ab}(x)A(x).$$

In components, this reads

$$T^{ab}_{k\ell}(x) = \lambda^{ab}(x)A_{k\ell}(x),$$

as claimed. ∎

The next few results are stated for a general uniformly elliptic second-order operator with sufficiently regular coefficients. They will later be applied to the first linearized operator $F_{k\ell}(D^2u_0^{(2)})\partial_{k\ell}$, but their validity does not rely on this particular form.

Proposition 4.3. *Let*

$$Lv := A_{k\ell}(x)\partial_{k\ell}v \quad in\ \Omega,$$

where $A = (A_{k\ell})_{1\le k,\ell\le 2} \in C^\alpha(\Omega; S)$ is uniformly elliptic for some $\alpha \in (0,1)$. Fix $x_0 \in \Omega$, and define

$$K_{A(x_0)} := \{H \in S :\ A_{k\ell}(x_0)H_{k\ell} = 0\}.$$

Then there exist a radius $r > 0$ with $\overline{B_r(x_0)} \subset \Omega$ and two functions $v_1, v_2 \in C^{2,\alpha}(B_r(x_0)) \cap C(\overline{B_r(x_0)})$ solving

$$Lv_j = 0 \quad in\ B_r(x_0), \quad j = 1, 2,$$

such that $D^2v_1(x_0)$ and $D^2v_2(x_0)$ are linearly independent elements of $K_{A(x_0)}$. In particular,

$$\operatorname{span}\{D^2v(x_0) :\ Lv = 0\ near\ x_0\} = K_{A(x_0)}.$$

Proof. By translation, we may assume that $x_0 = 0$. Since $A(0)$ is symmetric positive definite by uniform ellipticity, there exists an invertible matrix P such that

$$P^{-1}A(0)P^{-T} = I_2.$$

Equivalently, $A(0) = PP^T$. Introduce the linear change of variables $x = Pz$, and, for a function v in the x-variables, define $\widetilde{v}(z) := v(Pz)$. Then

$$D_z^2\widetilde{v}(z) = P^T D_x^2 v(x)P, \quad x = Pz.$$

Equivalently,

$$D_x^2 v(x) = P^{-T} D_z^2\widetilde{v}(z)P^{-1}.$$

Since the change of variables is linear, no first order terms are produced. Hence the equation $A(x) : D_x^2 v = 0$ is transformed into

$$\widetilde{A}(z) : D_z^2\widetilde{v} = 0,$$

where

$$\widetilde{A}(z) := \big(\widetilde{A}_{k\ell}(z)\big)_{1\le k,\ell\le 2} := P^{-1}A(Pz)P^{-T}.$$

In particular,

$$\widetilde{A}(0) = P^{-1}A(0)P^{-T} = I_2.$$

We next compare the corresponding constraint spaces. The map $H \mapsto P^THP$ is a linear isomorphism from $K_{A(0)}$ onto K_{I_2}. Indeed, if $H \in K_{A(0)}$, then

$$I_2 : (P^THP) = \operatorname{tr}(P^THP) = \operatorname{tr}(PP^TH) = A(0) : H = 0.$$

Conversely, if $\widetilde{H} \in K_{I_2}$, then $H = P^{-T}\widetilde{H}P^{-1}$ satisfies

$$A(0) : H = A(0) : P^{-T}\widetilde{H}P^{-1} = I_2 : \widetilde{H} = 0.$$

Therefore, it suffices to prove the proposition in the transformed coordinates. Renaming the variable z by x, we may assume without loss of generality that $A(0) = I_2$.

In this case,

$$K_{A(0)} = \{H \in S :\ \operatorname{tr}(H) = 0\},$$

which is a two-dimensional subspace of S. Consider the trace-free matrices

$$H_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad H_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

which form a basis of $K_{A(0)}$. Let

$$q_1(x) := \frac{1}{2}(x_1^2 - x_2^2), \quad q_2(x) := x_1 x_2.$$

Then $D^2 q_1 = H_1$, $D^2 q_2 = H_2$, and since $\Delta q_j = 0$, we have

$$A(0) : D^2 q_j = I_2 : D^2 q_j = 0, \quad j = 1, 2.$$

Choose $r_0 > 0$ so that $\overline{B_{r_0}(0)} \subset \Omega$. For each $r \in (0, r_0)$ and each $j = 1, 2$, let $v_{j,r}$ be the unique solution to

$$\begin{cases} A_{ab}(x)\partial_{ab} v_{j,r} = 0 & \text{in } B_r(0), \\ v_{j,r} = q_j & \text{on } \partial B_r(0). \end{cases}$$

By the standard solvability theory for uniformly elliptic non-divergence form equations with C^α coefficients,

$$v_{j,r} \in C^{2,\alpha}(B_r(0)) \cap C(\overline{B_r(0)}).$$

Define the rescaled functions

$$w_{j,r}(y) := r^{-2} v_{j,r}(ry), \quad y \in B_1(0).$$

A direct computation shows that $w_{j,r}$ solves

$$\begin{cases} A_{ab,r}(y)\partial_{ab} w_{j,r} = 0 & \text{in } B_1(0), \\ w_{j,r}(y) = r^{-2} q_j(ry) = q_j(y) & \text{on } \partial B_1(0), \end{cases}$$

where

$$A_{ab,r}(y) := A_{ab}(ry) \quad \text{and} \quad A_r(y) = (A_{ab,r}(y))_{1 \le a,b \le 2}.$$

Since $A \in C^\alpha(\Omega)$ and $A(0) = I_2$, it follows that

$$A_r \to I_2 \quad \text{in } C^\alpha(\overline{B_1(0)}) \quad \text{as } r \to 0.$$

Let $\eta_{j,r} := w_{j,r} - q_j$. Then $\eta_{j,r} = 0$ on $\partial B_1(0)$ and

$$A_{ab,r}(y)\partial_{ab}\eta_{j,r} = -\big(A_{ab,r}(y) - \delta_{ab}\big)\partial_{ab} q_j \quad \text{in } B_1(0).$$

The right hand side tends to zero in $C^\alpha(\overline{B_1(0)})$ as $r \to 0$. By the maximum principle and the interior Schauder estimates, for every $\rho \in (0, 1)$,

$$\eta_{j,r} \to 0 \quad \text{in } C^2(B_\rho(0)) \quad \text{as } r \to 0.$$

In particular, taking $\rho = 1/2$, we obtain

$$w_{j,r} \to q_j \quad \text{in } C^2(B_{1/2}(0)) \quad \text{as } r \to 0.$$

Therefore, as $r \to 0$,

$$D^2 w_{j,r}(0) \to D^2 q_j = H_j, \quad j = 1, 2.$$

On the other hand, by the definition of $w_{j,r}$,

$$D^2 w_{j,r}(0) = D^2 v_{j,r}(0).$$

Hence

$$D^2 v_{j,r}(0) \to H_j, \quad j = 1, 2.$$

Since H_1 and H_2 are linearly independent, it follows that for all sufficiently small $r > 0$, the matrices $D^2 v_{1,r}(0)$ and $D^2 v_{2,r}(0)$ are linearly independent. Moreover, since each $v_{j,r}$ solves $L v_{j,r} = 0$ in $B_r(0)$, evaluating the equation at 0 gives

$$A_{k\ell}(0)\partial_{k\ell} v_{j,r}(0) = 0,$$

so that

$$D^2 v_{j,r}(0) \in K_{A(0)}, \quad j = 1, 2.$$

Thus, for some sufficiently small $r > 0$, the two solutions $v_1 := v_{1,r}$ and $v_2 := v_{2,r}$ satisfy that $D^2 v_1(0)$ and $D^2 v_2(0)$ are linearly independent elements of $K_{A(0)}$. Since $K_{A(0)}$ is two-dimensional, this implies

$$\operatorname{span}\{D^2 v(0) : Lv = 0 \text{ near } 0\} = K_{A(0)}.$$

Finally, returning to the original variables $x = Pz$, if $\widetilde{v}$ is a solution in the transformed variables, then $v(x) := \widetilde{v}(P^{-1}x)$ solves the original equation near 0. Moreover,

$$D_z^2 \widetilde{v}(0) = P^T D_x^2 v(0) P.$$

The map $H \mapsto P^T H P$ is a linear isomorphism from $K_{A(0)}$ onto K_{I_2}, and hence linear independence of the Hessians in the normalized variables is equivalent to linear independence of the corresponding Hessians in the original variables. After restricting to a sufficiently small Euclidean ball contained in the image of the transformed neighborhood, we obtain the conclusion at 0. □

The preceding discussion shows that, to extract more information from (4.5), it is enough to produce sufficiently many global solutions to the first linearized equation whose Hessians at a fixed point span the subspace

$$K_{A(x_0)} := \{H \in S : A_{k\ell}(x_0) H_{k\ell} = 0\},$$

where

$$A(x) := \big(A_{k\ell}(x)\big)_{1 \le k,\ell \le 2} := \big(F_{k\ell}(D^2 u_0^{(2)}(x))\big)_{1 \le k,\ell \le 2}.$$

By Proposition 4.3, such a spanning property holds locally near any fixed point $x_0 \in \Omega$. We next invoke a Runge approximation result to replace these local solutions by global ones.

Proposition 4.4 (Runge approximation)**.** *Let $\Omega_0 \Subset \Omega \subset \mathbb{R}^2$ be a smooth subdomain, and let*

$$L := A_{k\ell}(x)\partial_{k\ell} \quad \text{in } \Omega,$$

where $A = (A_{k\ell})_{1 \le k,\ell \le 2} \in C^\infty(\overline{\Omega}; S)$ is uniformly elliptic. Let $w \in H^2(\Omega_0)$ satisfy $Lw = 0$ in Ω_0. Then for every compact set $K \subset \Omega_0$ and every $\varepsilon > 0$, there exists a global solution $v \in H^2(\Omega)$ of $Lv = 0$ in Ω, such that

$$\|v - w\|_{L^2(K)} < \varepsilon.$$

Proof. The result follows from the standard Hahn-Banach duality argument combined with unique continuation for the adjoint operator. Since $L = A_{k\ell}(x)\partial_{k\ell}$, its formal adjoint is given by

$$L^* \varphi = \partial_{k\ell}(A_{k\ell}\varphi) = \partial_k(A_{k\ell}\partial_\ell \varphi) + \partial_k\big((\partial_\ell A_{k\ell})\varphi\big).$$

Let

$$\mathcal{S}(\Omega) := \{v \in H^2(\Omega) : Lv = 0 \text{ in } \Omega\}, \quad \mathcal{S}(\Omega_0) := \{u \in H^2(\Omega_0) : Lu = 0 \text{ in } \Omega_0\}.$$

We claim that

$$\overline{\{v|_{\Omega_0} : v \in \mathcal{S}(\Omega)\}}^{L^2(\Omega_0)} = \mathcal{S}(\Omega_0).$$

By the Hahn-Banach theorem, it suffices to show the following: if $f \in L^2(\Omega_0)$ satisfies

$$\int_{\Omega_0} f\, v\, dx = 0 \quad \text{for all } v \in \mathcal{S}(\Omega) \implies \int_{\Omega_0} f\, u\, dx = 0 \quad \text{for all } u \in \mathcal{S}(\Omega_0).$$

Extend f by zero outside Ω_0, and denote the extension by $\widetilde{f} \in L^2(\Omega)$. Since the Dirichlet problem for L is uniquely solvable, the corresponding Dirichlet realization of L is invertible. Hence, by the Fredholm alternative for elliptic boundary value problems, the adjoint Dirichlet problem is also uniquely solvable. Therefore, there exists

$$\varphi \in H_0^1(\Omega) \cap H^2(\Omega)$$

such that

$$L^*\varphi = \widetilde{f} \quad \text{in } \Omega.$$

Now let $v \in \mathcal{S}(\Omega)$ be arbitrary, and let $g := v|_{\partial\Omega}$. Using the Green identity for L and L^*, together with the fact that $\varphi = 0$ on $\partial\Omega$, we obtain

$$\int_{\Omega_0} f\, v\, dx = \int_{\Omega} \widetilde{f}\, v\, dx = \int_{\Omega} (L^*\varphi)\, v\, dx - \int_{\Omega} \varphi\, Lv\, dx = \int_{\partial\Omega} \partial_{\nu_A}\varphi\, g\, dS, \tag{4.6}$$

where $\partial_{\nu_A}\varphi := \nu_k A_{k\ell}\partial_\ell\varphi$ is the conormal derivative associated with L^*. Here we used that $\varphi = 0$ on $\partial\Omega$. By assumption,

$$\int_{\Omega_0} f\, v\, dx = 0 \quad \text{for all } v \in \mathcal{S}(\Omega).$$

Since smooth boundary data g are arbitrary and give rise to solutions $v \in \mathcal{S}(\Omega)$, (4.6) implies $\partial_{\nu_A}\varphi = 0$ on $\partial\Omega$. Therefore,

$$\varphi = \partial_{\nu_A}\varphi = 0 \quad \text{on } \partial\Omega.$$

Next, extend φ by zero to a slightly larger smooth domain $\widetilde{\Omega} \supset \Omega$. Since both φ and its conormal derivative vanish on $\partial\Omega$, the zero extension still satisfies

$$L^*\varphi = 0 \quad \text{in } \widetilde{\Omega} \setminus \overline{\Omega_0}$$

in the weak sense. Hence, by the unique continuation principle for second-order elliptic equations, we conclude that

$$\varphi = 0 \quad \text{in } \Omega \setminus \overline{\Omega_0}.$$

In particular, the traces of φ and $\nabla\varphi$ vanish on $\partial\Omega_0$.

Now let $u \in \mathcal{S}(\Omega_0)$ be arbitrary. Using the Green identity again, now in Ω_0, we obtain

$$\int_{\Omega_0} f\, u\, dx = \int_{\Omega_0} (L^*\varphi)\, u\, dx - \int_{\Omega_0} \varphi\, Lu\, dx = \int_{\partial\Omega_0} \big(\partial_{\nu_A}\varphi\, u - \varphi\, \partial_{\nu_A} u\big)\, dS.$$

Since φ and $\nabla\varphi$ vanish on $\partial\Omega_0$, the boundary integral is zero. Hence

$$\int_{\Omega_0} f\, u\, dx = 0 \quad \text{for all } u \in \mathcal{S}(\Omega_0).$$

This proves the L^2-density claim. Therefore, for every compact set $K \subset \Omega_0$ and every $\varepsilon > 0$, there exists $v \in \mathcal{S}(\Omega)$ such that

$$\|v - w\|_{L^2(K)} < \varepsilon,$$

as desired. □

Remark 4.5. Proposition 4.4 *is the weak form of Runge approximation needed below. The passage from L^2-approximation to C^2-approximation on compact subsets follows from interior elliptic regularity for homogeneous solutions.*

Lemma 4.6. *Let $\Omega_1 \Subset \Omega$ be a smooth subdomain. Suppose that $\{u_m\}$ is a sequence of solutions to $Lu_m = 0$ in Ω_1, and that $u_m \to 0$ in $L^2(\Omega_1)$. Then for every compact set $K \subset \Omega_1$, $u_m \to 0$ in $C^2(K)$.*

Proof. Fix compact sets

$$K \Subset \Omega_2 \Subset \Omega_1,$$

where Ω_2 is a smooth subdomain. Since $Lu_m = 0$ in Ω_1, the interior Schauder estimate yields

$$\|u_m\|_{C^{2,\alpha}(K)} \leq C_K \|u_m\|_{L^\infty(\Omega_2)},$$

where $C_K > 0$ is independent of m. On the other hand, by the interior estimate for uniformly elliptic equations, we have

$$\|u_m\|_{L^\infty(\Omega_2)} \le C_{\Omega_2}\|u_m\|_{L^2(\Omega_1)},$$

for some constant $C_{\Omega_2} > 0$ independent of m. Hence

$$\|u_m\|_{C^{2,\alpha}(K)} \le C\|u_m\|_{L^2(\Omega_1)},$$

where $C > 0$ is independent of m. Since $u_m \to 0$ in $L^2(\Omega_1)$, it follows that

$$u_m \to 0 \quad \text{in } C^{2,\alpha}(K),$$

and in particular in $C^2(K)$. □

Combining Proposition 4.3, Proposition 4.4, and Lemma 4.6, we obtain the required richness of global solutions.

Corollary 4.7. Fix $x_0 \in \Omega$, and let

$$A(x_0) = \big(A_{k\ell}(x_0)\big)_{1\le k,\ell\le 2} := \big(F_{k\ell}(D^2u_0^{(2)}(x_0))\big)_{1\le k,\ell\le 2}.$$

Then there exist two global solutions $v_1, v_2 \in H^2(\Omega)\cap C^{2,\alpha}_{\mathrm{loc}}(\Omega)$ of

$$F_{k\ell}(D^2u_0^{(2)})\partial_{k\ell}v_j = 0 \quad \text{in } \Omega, \quad j = 1,2,$$

such that $D^2v_1(x_0)$ and $D^2v_2(x_0)$ are linearly independent elements of $K_{A(x_0)}$. Consequently,

$$\mathrm{span}\big\{D^2v(x_0) : F_{k\ell}(D^2u_0^{(2)})\partial_{k\ell}v = 0 \text{ in } \Omega\big\} = K_{A(x_0)}.$$

Proof. By Proposition 4.3, there exist $r > 0$ and local solutions $w_1, w_2 \in C^{2,\alpha}(B_r(x_0))$ of

$$F_{k\ell}(D^2u_0^{(2)})\partial_{k\ell}w_j = 0 \quad \text{in } B_r(x_0), \quad j = 1,2,$$

such that $D^2w_1(x_0)$ and $D^2w_2(x_0)$ are linearly independent in $K_{A(x_0)}$.

Choose a smooth domain Ω_1 such that $B_{r/2}(x_0) \Subset \Omega_1 \Subset B_r(x_0)$. By Proposition 4.4, for each $j = 1,2$ there exists a sequence of global solutions $v_{j,m} \in H^2(\Omega)$ of

$$F_{k\ell}(D^2u_0^{(2)})\partial_{k\ell}v_{j,m} = 0 \quad \text{in } \Omega$$

such that $v_{j,m} \to w_j$ in $L^2(\Omega_1)$ as $m \to \infty$. Since each difference $v_{j,m} - w_j$ solves the same homogeneous equation in Ω_1, Lemma 4.6 implies that

$$v_{j,m} \to w_j \quad \text{in } C^2(\overline{B_{r/2}(x_0)})$$

as $m \to \infty$. In particular,

$$D^2v_{j,m}(x_0) \to D^2w_j(x_0) \quad \text{as } m \to \infty.$$

Since linear independence is an open condition, it follows that for all sufficiently large m, the matrices $D^2v_{1,m}(x_0)$ and $D^2v_{2,m}(x_0)$ remain linearly independent. Moreover, since each $v_{j,m}$ satisfies the first linearized equation in Ω, evaluating at x_0 yields

$$A_{k\ell}(x_0)\partial_{k\ell}v_{j,m}(x_0) = 0,$$

and hence

$$D^2v_{j,m}(x_0) \in K_{A(x_0)}, \quad j = 1,2.$$

Therefore, for all sufficiently large m, the two matrices $D^2v_{1,m}(x_0)$ and $D^2v_{2,m}(x_0)$ are linearly independent elements of $K_{A(x_0)}$. Since $K_{A(x_0)}$ is two-dimensional, they form a basis of $K_{A(x_0)}$. Renaming

$$v_j := v_{j,m}, \quad j = 1,2,$$

for such a sufficiently large m, we obtain the desired conclusion. □

We now apply Corollary 4.7 to the identity (4.5). Fix an arbitrary $x_0 \in \Omega$. Since (4.5) holds for every global solution of the first linearized equation, Corollary 4.7 implies that, for each fixed pair $1 \le a, b \le 2$, one has

$$T^{ab}_{k\ell}(x_0) H_{k\ell} = 0 \quad \text{for all } H \in K_{A(x_0)},$$

where

$$T^{ab}_{k\ell} := F_{ab,k\ell}(D^2 u_0^{(1)}) - c_0 F_{ab,k\ell}(D^2 u_0^{(2)}).$$

Therefore, Lemma 4.2 yields a scalar function $\lambda^{ab} = \lambda^{ab}(x)$ such that

$$F_{ab,k\ell}(D^2 u_0^{(1)}(x)) - c_0(x) F_{ab,k\ell}(D^2 u_0^{(2)}(x)) = \lambda^{ab}(x) F_{k\ell}(D^2 u_0^{(2)}(x)) \tag{4.7}$$

for all $x \in \Omega$ and all $1 \le a, b, k, \ell \le 2$.

The next lemma shows that the coefficient λ^{ab} is itself proportional to $A_{ab} = F_{ab}(D^2 u_0^{(2)})$.

Lemma 4.8. *Suppose that*

$$T^{ab}_{k\ell} = \lambda^{ab} A_{k\ell}$$

for some matrix-valued function (λ^{ab}), where $A = (A_{k\ell})$ satisfies the uniform ellipticity. Assume further that $T^{ab}_{k\ell} = T^{k\ell}_{ab}$. Then there exists a scalar function μ such that

$$\lambda^{ab} = \mu A_{ab}.$$

As a result, there holds $T^{ab}_{k\ell} = \mu A_{ab} A_{k\ell}$.

Proof. From the identities

$$T^{ab}_{k\ell} = \lambda^{ab} A_{k\ell} \quad \text{and} \quad T^{ab}_{k\ell} = T^{k\ell}_{ab},$$

we obtain

$$\lambda^{ab} A_{k\ell} = \lambda^{k\ell} A_{ab}$$

for all $1 \le a, b, k, \ell \le 2$. By the ellipticity, $A(x)$ is positive definite. Thus, $A_{11}(x) > 0$ for any $x \in \Omega$. Define

$$\mu(x) := \frac{\lambda^{11}(x)}{A_{11}(x)}.$$

Then, taking $(k, \ell) = (1, 1)$ in the above relation, we obtain

$$\lambda^{ab}(x) A_{11}(x) = \lambda^{11}(x) A_{ab}(x),$$

and hence $\lambda^{ab}(x) = \mu(x) A_{ab}(x)$. Substituting this back into $T^{ab}_{k\ell} = \lambda^{ab} A_{k\ell}$ gives $T^{ab}_{k\ell} = \mu A_{ab} A_{k\ell}$, as claimed. □

Therefore, applying Lemma 4.8 to (4.7) yields a scalar function $\mu = \mu(x)$ such that

$$F_{ab,k\ell}(D^2 u_0^{(1)}(x)) - c_0(x) F_{ab,k\ell}(D^2 u_0^{(2)}(x)) = \mu(x) F_{ab}(D^2 u_0^{(2)}(x)) F_{k\ell}(D^2 u_0^{(2)}(x)) \tag{4.8}$$

for all $x \in \Omega$ and all $1 \le a, b, k, \ell \le 2$.

Remark 4.9. *In this section, the homogeneity of F and the injectivity of DF are used only to show that the conformal factor does not vary with the boundary perturbation. The non-proportionality condition is not used here; it will be used only in Section 5 to remove the remaining scalar factor.*

5. The inverse problem

Finally, we can prove Theorem 1.4.

Proof of Theorem 1.4. By Proposition 1.7, there exists a positive function $c_0 = c_0(x)$ such that $f_1 = c_0^{\frac{m}{m-1}} f_2$ in Ω. It remains to show that $c_0 = 1$.

Let $u_0^{(j)} \in C^\infty(\overline{\Omega})$, $j = 1, 2$, be the admissible solutions to (1.7). From the proof of Proposition 1.7, we have

$$D^2 u_0^{(1)} = t D^2 u_0^{(2)} \quad \text{in } \Omega,$$

where $t = c_0^{1/(m-1)}$. Moreover, by the conclusion of Section 4, in particular (4.8), there exists a scalar function $\mu = \mu(x)$ such that

$$F_{ab,k\ell}(D^2 u_0^{(1)}) - c_0 F_{ab,k\ell}(D^2 u_0^{(2)}) = \mu F_{ab}(D^2 u_0^{(2)}) F_{k\ell}(D^2 u_0^{(2)}) \quad \text{in } \Omega. \tag{5.1}$$

Fix any $x \in \Omega$ and set $M_2 := D^2 u_0^{(2)}(x)$. Since $D^2 u_0^{(1)}(x) = t(x) M_2$, and since the second derivative of F is homogeneous of degree $m-2$, (5.1) gives

$$t(x)^{m-2} F_{ab,k\ell}(M_2) - c_0(x) F_{ab,k\ell}(M_2) = \mu(x) F_{ab}(M_2) F_{k\ell}(M_2).$$

Using $c_0(x) = t(x)^{m-1}$, this becomes

$$t(x)^{m-2}(1 - t(x)) F_{ab,k\ell}(M_2) = \mu(x) F_{ab}(M_2) F_{k\ell}(M_2).$$

If $t(x)^{m-2}(1 - t(x)) \neq 0$, then

$$F_{ab,k\ell}(M_2) = \gamma F_{ab}(M_2) F_{k\ell}(M_2)$$

for some $\gamma \in \mathbb{R}$, which contradicts Assumption 1.3 (iii). Hence $t(x)^{m-2}(1-t(x)) = 0$. Since $t(x) > 0$, we have $t(x) = 1$, and therefore $c_0(x) = 1$. Since $x \in \Omega$ was arbitrary, $c_0 = 1$ in Ω. The gauge relation in Proposition 1.7 then gives

$$f_1 = f_2 \quad \text{in } \Omega.$$

Together with (1.5), this proves $f_1 = f_2$ in $\overline{\Omega}$. ∎

Statements and Declarations

Data availability statement. No datasets were generated or analyzed during the current study.

Conflict of Interests. Hereby, we declare there are no conflicts of interest.

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