THE CALDERÓN PROBLEM FOR THE INFINITY LAPLACIAN WITH LARGE AFFINE BOUNDARY DATA

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ABSTRACT. We study the Calderón problem for the equation $-\Delta_\infty u+q(x)u=0$ in a bounded convex domain with C^2 boundary. We prove that a positive potential q is uniquely determined and explicitly reconstructible from the measurements $\Lambda_q\big(t(e\cdot x+b)|_{\partial\Omega}\big)$, where $e\in\mathbb{S}^{n-1}$, the offset $b>\sup_{\overline{\Omega}}|x|$ is fixed, and $t\to\infty$. The first potential-dependent term in the large amplitude asymptotics determines weighted integrals of q over the chords parallel to e. Combining the measurements in the directions e and $-e$ gives the X-ray transform of the zero extension of q, and the Fourier slice identity yields reconstruction and uniqueness.

CONTENTS

1. INTRODUCTION

The infinity Laplacian is the second-order operator

$$\Delta_\infty u=\langle D^2u\,Du,Du\rangle=\sum_{i,j=1}^{n}u_{x_i}u_{x_j}u_{x_ix_j}.$$

For the equation considered in this paper, define

$$F_q(x,r,p,X)=-p^TXp+q(x)r. \tag{1.1}$$

If $X\le Y$ in the order of symmetric matrices, then $F_q(x,r,p,X)\ge F_q(x,r,p,Y)$. Thus, F_q is degenerate elliptic in the viscosity convention. Its principal coefficient matrix is $p\otimes p$. When $p\neq 0$, this matrix has eigenvalues $|p|^2,0,\dots,0$, while it vanishes when $p=0$. The equation is therefore not uniformly elliptic. Since (1.1) is linear in the Hessian variable, with coefficients depending quadratically on the gradient, it is a quasilinear equation of nondivergence form. The zeroth-order term does not affect ellipticity. If $q\ge c_0>0$, then F_q is strictly increasing in the solution variable, which gives the comparison principle used below.

2020 *Mathematics Subject Classification.* 35R30, 35J94, 35D40, 49L25, 44A12.
Key words and phrases. Calderón problem, infinity Laplacian, quasilinear degenerate elliptic equations, viscosity solutions, Dirichlet-to-Neumann map, large boundary data, X-ray transform.

The infinity Laplacian arose in Aronsson's study of absolutely minimizing Lipschitz extensions and supremal variational problems [Aro67, Aro68]. The homogeneous equation $\Delta_\infty u = 0$ is naturally understood in the viscosity sense. Its relation to absolutely minimizing functions, comparison with cones, and limits of p-harmonic equations was developed in [Jen93, CEG01, ACJ04]; see also [PSSW09, Lin14]. At every point where $Du \neq 0$, writing $\widehat{Du} = Du/|Du|$, one has $\Delta_\infty u = |Du|^2 D^2 u[\widehat{Du}, \widehat{Du}]$. Thus, the operator records the second derivative only in the gradient direction. This rank-one structure is the main feature used in the inverse problem.

The infinity Laplacian also appears as a nonlinear interpolation operator. The absolutely minimizing Lipschitz extension selects an extension by controlling its largest local slope, a principle used in image interpolation and restoration [CMS98]. On weighted graphs, the same principle leads to Lipschitz learning for semi-supervised classification, whose continuum limit is governed by infinity Laplace type equations [Cal19]. The tug-of-war representation gives a stochastic dynamic-programming interpretation of the equation [PSSW09]. In the problem considered here, q is a spatially varying lower-order coefficient, and the question is whether it can be determined from boundary inputs and the corresponding normal derivatives.

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain, let q be a zeroth-order coefficient, and let $f \in C(\partial\Omega)$. We consider

$$\begin{cases} -\Delta_\infty u + q(x)u = 0 & \text{in } \Omega, \\ u = f & \text{on } \partial\Omega. \end{cases} \tag{1.2}$$

If $q \in C(\overline{\Omega})$ and $q \geq c_0 > 0$, then (1.2) has a unique viscosity solution $u_f^q \in C(\overline{\Omega}) \cap \mathrm{Lip}_{\mathrm{loc}}(\Omega)$. The boundary condition is attained pointwise, while the local Lipschitz regularity follows by writing the equation as $\Delta_\infty u_f^q = q u_f^q$ and applying the regularity theory for bounded inhomogeneities. The viscosity formulation itself only requires continuity, and the interior Lipschitz regularity does not by itself provide a normal derivative on $\partial\Omega$. A separate boundary regularity result is therefore needed in order to define the boundary measurements.

Suppose that $\partial\Omega$ is of class C^1 and that $f \in C^1(\partial\Omega)$. Since u_f^q satisfies $\Delta_\infty u_f^q = h$ with $h(x) = q(x)u_f^q(x)$ bounded and continuous, the boundary differentiability theorem for inhomogeneous infinity Laplace equations implies that u_f^q is differentiable at every point $x_0 \in \partial\Omega$. More precisely, there exists a vector $Du_f^q(x_0) \in \mathbb{R}^n$ such that

$$u_f^q(x) = u_f^q(x_0) + Du_f^q(x_0) \cdot (x - x_0) + o(|x - x_0|)$$

as $x \to x_0$ within $\overline{\Omega}$. In particular, the outward normal derivative $\partial_\nu u_f^q(x_0) = Du_f^q(x_0) \cdot \nu(x_0)$ is well defined at every boundary point. We may therefore define the DN map by

$$\Lambda_q(f)(x) = \partial_\nu u_f^q(x), \quad x \in \partial\Omega, \tag{1.3}$$

where ν denotes the outward unit normal. The same boundary differentiability will be used later to convert the linear vanishing of the chord-barrier error at the endpoints into the large amplitude asymptotics of the normal derivatives.

The Calderón problem is to determine q from Λ_q. We show that the values of $\Lambda_q(f)$ for arbitrary boundary functions are not needed. Let $R_\Omega = \sup_{x \in \overline{\Omega}} |x|$, fix $b > R_\Omega$, and set $\ell_{e,b}(x) = e \cdot x + b$ for $e \in \mathbb{S}^{n-1}$. Then $\ell_{e,b} > 0$ on $\overline{\Omega}$. The measurements used in this paper are $\Lambda_q\big(t\ell_{e,b}|_{\partial\Omega}\big)$, where $e \in \mathbb{S}^{n-1}$ and t is sufficiently large. For each fixed e, the boundary values therefore lie on the one-parameter ray $\{t\ell_{e,b}|_{\partial\Omega} : t > 0\}$.

Question 1.1. *Do the measurements $\Lambda_q\big(t\ell_{e,b}|_{\partial\Omega}\big)$, with $e \in \mathbb{S}^{n-1}$ and t sufficiently large, uniquely determine q?*

Our main result gives an affirmative answer. The first potential-dependent term in the large amplitude measurements determines the X-ray transform of the zero extension of q, from which both uniqueness and reconstruction follow.

Earlier literature. The classical Calderón problem asks whether an isotropic conductivity can be recovered from its Dirichlet-to-Neumann map [Cal80]. In dimensions $n \geq 3$, global uniqueness was proved by Sylvester and Uhlmann using complex geometric optics solutions [SU87]. Their work initiated a broad development of inverse boundary value problems for linear elliptic equations. For nonlinear equations, the dependence of the boundary measurements on the boundary values provides information that is absent from a single linearized equation.

A standard approach to nonlinear inverse problems is to differentiate the nonlinear Dirichlet-to-Neumann map with respect to the boundary data. Early applications include Isakov's work on semilinear parabolic equations [Isa93], the determination of nonlinear conductivities by Kang and Nakamura [KN02], and the elliptic results of Sun [Sun96, Sun10]. The higher-order linearization method, based on interactions of several linearized solutions, was introduced in a geometric hyperbolic setting in [KLU18] and later developed for semilinear elliptic equations in [FO20, LLLS21a]. Related results include partial data problems and nonlinearities with noninteger powers [LLLS21b, LLST22].

Quasilinear elliptic inverse problems have been studied using first and higher-order linearization, complex geometric optics solutions, and boundary determination; see [CNV19, CFK+21, KKU23] and the references therein. Geometric quasilinear problems include the recovery of Riemannian metrics and coefficients from minimal area or minimal surface data [ABN20, CLLO24, Nur24]. We refer to [Las25] for a broader introduction to nonlinear inverse problems.

The argument is also related to inverse source problems. For a linear equation, a compactly supported perturbation of a solution may change the source without changing the boundary Cauchy data. Nonlinear effects can remove this obstruction, as shown for semilinear elliptic and reaction-diffusion equations in [LL24, KLL24]. More recent results concern inverse source problems for the Monge–Ampère equation and for broader classes of admissible fully nonlinear equations [LL25, LLW26, Lin26b]. Large boundary data give a mechanism different from higher-order linearization. In several fully nonlinear problems, a rank-deficient leading profile forces the first source-dependent correction to satisfy lower-dimensional equations, leading to chord or affine section tomography [CG26, Lin26a]. The Hessian equations in those works are fully nonlinear in the second derivatives. The equation considered here is quasilinear and degenerate elliptic, but its large affine limit produces a related rank-one equation.

The forward theory for inhomogeneous infinity Laplace equations was developed in [LW08, BM12]. The strong degeneracy of the operator makes regularity substantially different from the uniformly elliptic theory; characteristic and adjoint methods were developed in [EY05, ES11b, ES11a]. The boundary differentiability theorem used in (1.3) is due to Hong [Hon14]. Inverse boundary value problems for the finite p-Laplacian have also been studied, for example in [SZ12]. The limit $p \to \infty$ does not directly give the argument used here, since the first correction around an affine infinity harmonic function is governed by a rank-one equation rather than a uniformly elliptic one.

Difficulties and ideas of the proof. The proof has three main difficulties. First, solutions of (1.2) are viscosity solutions, and the equation cannot be differentiated through a formal classical expansion. Second, the infinity Laplacian is not in divergence form, so the usual Alessandrini identity is not available. Third, the equation governing the first large amplitude correction contains second derivatives in only one direction.

The different homogeneities of the two terms provide the starting point. Since $\Delta_\infty(tu) = t^3\Delta_\infty u$, uniqueness for the forward problem gives the exact identity $u_{tf}^q = tu_f^{q/t^2}$. Thus, increasing the amplitude of the boundary value for a fixed potential is equivalent to keeping the boundary value fixed and replacing q by $t^{-2}q$. This reduction is specific to the linear potential $q(x)u$. For the equation with $q(x)u^3$, both terms have degree three under $u \mapsto tu$, so the same normalization leaves the potential unchanged.

We apply this identity to the affine infinity harmonic function

$$\ell_{e,b}(x) = e \cdot x + b. \tag{1.4}$$

Let $\varepsilon = t^{-2}$ and set $v_\varepsilon = t^{-1}u^q_{t\ell_{e,b}}$, then v_ε is the unique viscosity solution of

$$\begin{cases} -\Delta_\infty v_\varepsilon + \varepsilon q(x) v_\varepsilon = 0 & \text{in } \Omega, \\ v_\varepsilon = \ell_{e,b} & \text{on } \partial\Omega. \end{cases} \tag{1.5}$$

Define

$$z_\varepsilon = \frac{v_\varepsilon - \ell_{e,b}}{\varepsilon}, \quad \text{equivalently} \quad \frac{u^q_{t\ell_{e,b}}}{t} = \ell_{e,b} + \varepsilon z_\varepsilon. \tag{1.6}$$

This is an exact decomposition. Since v_ε and $\ell_{e,b}$ have the same boundary trace, $z_\varepsilon = 0$ on $\partial\Omega$.

Substituting $v_\varepsilon = \ell_{e,b} + \varepsilon z_\varepsilon$ into (1.5) and dividing by ε gives

$$-(e + \varepsilon D z_\varepsilon)^T D^2 z_\varepsilon (e + \varepsilon D z_\varepsilon) + q(x)(\ell_{e,b} + \varepsilon z_\varepsilon) = 0 \quad \text{in } \Omega,$$

where the equation is understood in the viscosity sense. Lemma 4.1 proves this change of variables directly with test functions. No differentiability of the solution with respect to ε is assumed.

The limiting equation is

$$-\partial_e^2 w + q(x)(e \cdot x + b) = 0, \tag{1.7}$$

where $\partial_e^2 w = D^2 w[e, e]$. On each line parallel to e, this is a one-dimensional equation. We define w by solving (1.7) on each chord of Ω with zero endpoint values and prove, by explicit barriers, that z_ε converges to w on every fixed positive length chord.

The exact identity (1.6) can also be written as $u^q_{t\ell_{e,b}} = t\ell_{e,b} + t^{-1} z_{t^{-2}}$. Taking the normal derivative gives

$$\Lambda_q\big(t\ell_{e,b}|_{\partial\Omega}\big) = t\, e \cdot \nu + t^{-1} \partial_\nu z_{t^{-2}}.$$

Thus, the first potential-dependent term in the DN measurements is determined by the boundary behavior of z_ε, and it is necessary to control the normal derivatives of z_ε, rather than only its values in the interior.

To obtain this boundary information, we use the chord notation introduced below. For $y \in Y_e$, the line $y + \mathbb{R}e$ meets Ω in the interval $\{y + se : \alpha_e(y) < s < \beta_e(y)\}$. A global slab barrier first gives an ε-independent bound for z_ε; see Proposition 4.2. For a fixed $y_0 \in Y_e$, Lemma 5.1 constructs local barriers in a small tube around the chord through y_0 and proves the central chord estimate (5.2). The factor on the right-hand side of that estimate vanishes linearly at both endpoints of the chord.

Proposition 5.3 applies one-sided difference quotients to (5.2) and obtains an $O(\varepsilon)$ estimate for the directional derivatives $\partial_e z_\varepsilon - \partial_e w_{q,e,b}$ at the two endpoints. Since both functions vanish on the corresponding boundary faces, their gradients are normal to $\partial\Omega$ there. At a non-glancing endpoint one has $e \cdot \nu \neq 0$, so the identity $\partial_e = (e \cdot \nu)\partial_\nu$ converts the directional derivative estimate into an estimate for $\partial_\nu z_\varepsilon - \partial_\nu w_{q,e,b}$. Proposition 5.4 makes this estimate uniform on compact subsets of Γ_e. The argument does not require an interior C^1 estimate or a nonvanishing condition on the gradient.

Finally, integrating (1.7) along a chord expresses a weighted integral of q in terms of the two endpoint values of $\partial_\nu w$. Repeating the construction in the direction $-e$ removes the affine weight. The resulting quantities give the X-ray transform of the zero extension of q, and the Fourier slice identity yields reconstruction and uniqueness.

Mathematical formulation and main results. Throughout the paper, $n \geq 2$, $\Omega \subset \mathbb{R}^n$ is a bounded convex domain with C^2 boundary, and

$$q \in C^2(\overline{\Omega}), \quad q(x) \geq c_0 > 0. \tag{1.8}$$

The positivity of q gives comparison and uniqueness for the forward problem. The C^2 regularity is used in the local chord construction, while the forward theory requires less.

For $e \in \mathbb{S}^{n-1}$, let $P_e = \mathrm{Id} - e \otimes e$ and $Y_e = P_e\Omega \subset e^{\perp}$. For each $y \in Y_e$, convexity gives numbers $\alpha_e(y) < \beta_e(y)$ such that

$$\Omega \cap (y + \mathbb{R}e) = \{y + se : \alpha_e(y) < s < \beta_e(y)\}.$$

We write $x_e^-(y) = y + \alpha_e(y)e$ and $x_e^+(y) = y + \beta_e(y)e$. These endpoints are non-glancing: $e \cdot \nu(x_e^+(y)) > 0$ and $e \cdot \nu(x_e^-(y)) < 0$. We also set

$$\Gamma_e = \{x \in \partial\Omega : e \cdot \nu(x) \neq 0\}.$$

The first theorem identifies the first potential-dependent term in the large amplitude measurements.

Theorem 1.1 (large amplitude DN asymptotics)**.** *Assume (1.8) and fix $b > R_\Omega$. For $e \in \mathbb{S}^{n-1}$ and $t > 0$, let $u^q_{t,e,b} \in C(\overline{\Omega})$ be the unique viscosity solution of*

$$\begin{cases} -\Delta_\infty u^q_{t,e,b} + q(x)u^q_{t,e,b} = 0 & \text{in } \Omega, \\ u^q_{t,e,b} = t\ell_{e,b} & \text{on } \partial\Omega, \end{cases} \tag{1.9}$$

where $\ell_{e,b}$ is the affine function defined by (1.4). Then, for every $x \in \Gamma_e$, the limit

$$H_q(e, b; x) = \lim_{t\to\infty} t\left[\Lambda_q\big(t\ell_{e,b}|_{\partial\Omega}\big)(x) - t\, e \cdot \nu(x)\right] \tag{1.10}$$

exists, locally uniformly on Γ_e.

For each $y \in Y_e$, let $w_{q,e,b}$ be the unique solution of

$$\begin{cases} \partial_s^2 w_{q,e,b}(y, s) = q(y + se)(s + b) & \text{for } \alpha_e(y) < s < \beta_e(y), \\ w_{q,e,b}(y, s) = 0 & \text{for } s = \alpha_e(y) \text{ and } s = \beta_e(y). \end{cases} \tag{1.11}$$

The function $w_{q,e,b}$ is of class C^2 on every closed chord tube whose transverse base is compactly contained in Y_e. In particular, it is C^2 up to every point of Γ_e, and

$$H_q(e, b; x) = \partial_\nu w_{q,e,b}(x), \quad x \in \Gamma_e. \tag{1.12}$$

Moreover, for every compact set $K \Subset \Gamma_e$, there are constants $C_K > 0$ and $t_K > 0$ such that

$$\sup_{x\in K} \left|\Lambda_q\big(t\ell_{e,b}|_{\partial\Omega}\big)(x) - t\, e \cdot \nu(x) - t^{-1}\partial_\nu w_{q,e,b}(x)\right| \leq C_K t^{-3} \tag{1.13}$$

for $t \geq t_K$.

Theorem 1.2 (Restricted data uniqueness)**.** *Let $q_1, q_2 \in C^2(\overline{\Omega})$ satisfy $q_j \geq c_0 > 0$, and fix $b > R_\Omega$. Suppose that, for every $e \in \mathbb{S}^{n-1}$, there is $t_e > 0$ such that*

$$\Lambda_{q_1}\big(t\ell_{e,b}|_{\partial\Omega}\big) = \Lambda_{q_2}\big(t\ell_{e,b}|_{\partial\Omega}\big) \quad \text{on } \partial\Omega$$

for all $t \geq t_e$. Then $q_1 = q_2$ in Ω.

We next describe the reconstruction formula provided by the proof. Let $Q = q\mathbf{1}_\Omega$ be the zero extension of q to $\mathbb{R}^n$. For $e \in \mathbb{S}^{n-1}$ and $y \in e^\perp$, define the *X-ray transform* of Q by

$$\mathcal{X}Q(e, y) := \int_{\mathbb{R}} Q(y + se)\, ds.$$

If $y \in Y_e$, this is the integral of q over the chord $\Omega \cap (y + \mathbb{R}e)$. If $y \notin Y_e$, the line $y + \mathbb{R}e$ does not meet the open set Ω, and hence, $\mathcal{X}Q(e, y) = 0$.

We define

$$I_q(e,b;y) := \begin{cases} (e\cdot\nu(x_e^+(y)))H_q(e,b;x_e^+(y)) - (e\cdot\nu(x_e^-(y)))H_q(e,b;x_e^-(y)) & \text{for } y\in Y_e,\\ 0 & \text{for } y\notin Y_e. \end{cases}$$

To see how the endpoint data determine a chord integral, fix $y\in Y_e$ and write $\alpha=\alpha_e(y)$, $\beta=\beta_e(y)$, $x_-=y+\alpha e$, and $x_+=y+\beta e$. By (1.11), the function $W_y(s)=w_{q,e,b}(y+se)$ satisfies

$$W_y''(s) = q(y+se)(s+b), \quad \alpha<s<\beta.$$

Integrating this equation gives

$$\int_\alpha^\beta q(y+se)(s+b)\,ds = W_y'(\beta)-W_y'(\alpha) = \partial_e w_{q,e,b}(x_+) - \partial_e w_{q,e,b}(x_-).$$

Since $w_{q,e,b}$ vanishes on $\partial\Omega$ near both endpoints, its tangential derivatives vanish there, then we have $\partial_e w_{q,e,b}(x_\pm) = (e\cdot\nu(x_\pm))\partial_\nu w_{q,e,b}(x_\pm)$. By (1.12) in Theorem 1.1, the definition of I_q gives $I_q(e,b;y) = \int_{\alpha_e(y)}^{\beta_e(y)} q(y+se)(s+b)\,ds$. The same chord can be parametrized in the opposite direction $-e$. Since $P_{-e}=P_e$, the transverse parameter y remains unchanged, while $\alpha_{-e}(y)=-\beta_e(y)$ and $\beta_{-e}(y)=-\alpha_e(y)$. Applying the preceding identity in the direction $-e$ and reversing the chord parameter gives $I_q(-e,b;y)=\int_{\alpha_e(y)}^{\beta_e(y)} q(y+se)(b-s)\,ds$. Adding this identity to the corresponding formula for $I_q(e,b;y)$ cancels the term linear in s and gives

$$I_q(e,b;y)+I_q(-e,b;y) = 2b\int_{\alpha_e(y)}^{\beta_e(y)} q(y+se)\,ds.$$

The integral on the right-hand side is the X-ray transform of the zero extension $Q=q\mathbf{1}_\Omega$ along the line $y+\mathbb{R}e$. Thus, the large amplitude DN measurements determine $\mathcal{X}Q(e,y)$ for every $e\in\mathbb{S}^{n-1}$ and $y\in e^\perp$.

The Fourier slice identity then recovers the Fourier transform of Q. More precisely, if $\xi\in e^\perp$, then

$$\widehat{Q}(\xi) = \int_{e^\perp} e^{-\mathrm{i}y\cdot\xi}\mathcal{X}Q(e,y)\,dy.$$

For every $\xi\neq 0$, one may choose a direction $e_\xi\in\mathbb{S}^{n-1}$ satisfying $e_\xi\cdot\xi=0$. Substitution of the recovered X-ray transform into the Fourier slice identity therefore determines $\widehat{Q}(\xi)$. We write $\mathcal{F}^{-1}$ for the inverse Fourier transform.

Corollary 1.3 (Reconstruction formula)**.** *Under the assumptions of Theorem 1.1, one has*

$$\mathcal{X}Q(e,y) = \frac{I_q(e,b;y)+I_q(-e,b;y)}{2b} \tag{1.14}$$

for every $e\in\mathbb{S}^{n-1}$ and $y\in e^\perp$.

For $\xi\neq 0$, choose any $e_\xi\in\mathbb{S}^{n-1}$ satisfying $e_\xi\cdot\xi=0$. With the Fourier transform convention $\widehat{Q}(\xi)=\int_{\mathbb{R}^n} e^{-\mathrm{i}x\cdot\xi}Q(x)\,dx$, one has

$$\widehat{Q}(\xi) = \frac{1}{2b}\int_{e_\xi^\perp} e^{-\mathrm{i}y\cdot\xi}\left[I_q(e_\xi,b;y)+I_q(-e_\xi,b;y)\right]dy.$$

The value $\widehat{Q}(0)$ is determined by continuity of the Fourier transform of the compactly supported function $Q\in L^1(\mathbb{R}^n)$. Consequently, $Q=\mathcal{F}^{-1}\widehat{Q}$ in $\mathcal{S}'(\mathbb{R}^n)$, and $q=Q|_\Omega$.

Two features of the result are worth noting. For each fixed direction e, the offset b is fixed and only the amplitude t varies. Thus, the boundary values belong to the one-parameter family $t\ell_{e,b}|_{\partial\Omega}$. The transverse variable $y \in e^{\perp}$ is not an additional boundary input; it specifies the chord whose two endpoints are used to evaluate the pointwise DN measurements. The proof only uses the large amplitude coefficient H_q, and hence equality of the measurements along an unbounded sequence of amplitudes is sufficient; see Remark 6.5.

Convexity is used to ensure that each intersection $\Omega \cap (y + \mathbb{R}e)$ is a single interval and that the endpoints of every positive length chord are non-glancing. These properties allow the correction equation to be solved chordwise and the endpoint directional derivatives to be converted into normal derivatives. No positive lower bound for the curvature of $\partial\Omega$ is needed. In particular, the domain need not be strictly or uniformly convex.

Organization of the paper. Section 2 establishes the forward well-posedness and defines the DN map. Section 3 develops the geometry of parallel chords and constructs the rank-one correction. Section 4 derives the exact equation for the difference quotient and proves the global slab estimate. Section 5 establishes the local chord-tube estimate and the resulting DN asymptotics. Section 6 recovers the X-ray transform and proves the reconstruction and uniqueness results.

2. The forward problem and the Dirichlet-to-Neumann map

For $q \in C(\overline{\Omega})$, recall the operator F_q is given by (1.1).

Proposition 2.1 (Degenerate ellipticity and quasilinear structure)**.** *For every (x, r, p), the map $X \mapsto F_q(x, r, p, X)$ is degenerate elliptic. More precisely, if $X \le Y$, then*

$$F_q(x, r, p, X) - F_q(x, r, p, Y) = p^T(Y - X)p \ge 0. \tag{2.1}$$

Written as $-a^{ij}(Du)u_{ij} + q(x)u = 0$, the principal coefficient matrix is $a(Du) = Du \otimes Du$. If $Du \neq 0$, this matrix has rank one and eigenvalues $|Du|^2, 0, \dots, 0$; if $Du = 0$, it vanishes. Hence, the equation is degenerate elliptic, but not uniformly elliptic. It is quasilinear and of nondivergence form. If $q \ge c_0 > 0$ and $r_2 \ge r_1$, then $F_q(x, r_2, p, X) - F_q(x, r_1, p, X) \ge c_0(r_2 - r_1)$.

Proof. Identity (2.1) follows from $Y - X \ge 0$. The spectral statement follows from $(p \otimes p)p = |p|^2 p$ and $(p \otimes p)\xi = 0$ whenever $\xi \perp p$. The remaining claims follow from $\Delta_\infty u = (Du \otimes Du) : D^2u$ and the lower bound for q. $\square$

Definition 2.2 (Viscosity solution)**.** *A function $u \in C(\Omega)$ is called a viscosity subsolution of $F_q(x, u, Du, D^2u) = 0$ in Ω if, whenever $\phi \in C^2(\Omega)$ and $u - \phi$ has a local maximum at $x_0 \in \Omega$, one has $F_q(x_0, u(x_0), D\phi(x_0), D^2\phi(x_0)) \le 0$. It is a viscosity supersolution if, whenever $\phi \in C^2(\Omega)$ and $u - \phi$ has a local minimum at $x_0 \in \Omega$, one has $F_q(x_0, u(x_0), D\phi(x_0), D^2\phi(x_0)) \ge 0$. A viscosity solution is both a viscosity subsolution and a viscosity supersolution.*

Given $f \in C(\partial\Omega)$, a function $u \in C(\overline{\Omega})$ is a viscosity solution of

$$\begin{cases} -\Delta_\infty u + q(x)u = 0 & \text{in } \Omega, \\ u = f & \text{on } \partial\Omega \end{cases} \tag{2.2}$$

if it is a viscosity solution of $F_q(x, u, Du, D^2u) = 0$ in Ω and satisfies $u = f$ pointwise on $\partial\Omega$.

Since the unnormalized infinity Laplacian is continuous at $p = 0$, no separate convention is needed at critical points. Definition 2.2 is the usual viscosity definition for a continuous degenerate elliptic operator [CIL92].

Proposition 2.3 (Comparison principle). *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set, and let $q \in C(\overline{\Omega})$ satisfy $q \geq c_0 > 0$. Suppose that $u, v \in C(\overline{\Omega})$, where u is a viscosity subsolution and v is a viscosity supersolution of $F_q = 0$ in Ω. If $u \leq v$ on $\partial\Omega$, then $u \leq v$ in $\overline{\Omega}$.*

Proof. Assume that $m = \max_{\overline{\Omega}}(u - v) > 0$. Since $u - v \in C(\overline{\Omega})$ and $\overline{\Omega}$ is compact, there exists $\bar{x} \in \overline{\Omega}$ such that $u(\bar{x}) - v(\bar{x}) = m$. Since $u - v \leq 0$ on $\partial\Omega$ and $m > 0$, the point $\bar{x}$ cannot lie on $\partial\Omega$. Hence, $\bar{x} \in \Omega$. For $\delta > 0$, define

$$\Phi_\delta(x, y) = u(x) - v(y) - \frac{|x - y|^2}{2\delta}$$

on $\overline{\Omega} \times \overline{\Omega}$. Since this set is compact and Φ_δ is continuous, there exists $(x_\delta, y_\delta) \in \overline{\Omega} \times \overline{\Omega}$ at which Φ_δ attains its maximum. Comparing with the admissible point $(\bar{x}, \bar{x})$ gives $\Phi_\delta(x_\delta, y_\delta) \geq \Phi_\delta(\bar{x}, \bar{x}) = u(\bar{x}) - v(\bar{x}) = m$. Hence,

$$u(x_\delta) - v(y_\delta) - \frac{|x_\delta - y_\delta|^2}{2\delta} \geq m. \tag{2.3}$$

Since u and v are bounded, (2.3) implies that $|x_\delta - y_\delta|^2/\delta$ remains bounded. In particular, $|x_\delta - y_\delta| \to 0$ as $\delta \downarrow 0$. Passing to a subsequence, we may assume that $x_\delta \to x_0$ and $y_\delta \to x_0$ as $\delta \downarrow 0$, for some $x_0 \in \overline{\Omega}$. By (2.3) and continuity,

$$m \leq u(x_0) - v(x_0) \leq \max_{\overline{\Omega}}(u - v) = m.$$

Thus, $u(x_0) - v(x_0) = m$. Since $u \leq v$ on $\partial\Omega$, it follows that $x_0 \in \Omega$. We also have $x_\delta, y_\delta \in \Omega$ for all sufficiently small δ.

Set $p_\delta = (x_\delta - y_\delta)/\delta$. The same vector appears in the two viscosity inequalities because the derivatives of the penalization satisfy

$$D_x\Big(\frac{|x - y|^2}{2\delta}\Big)\Big|_{(x_\delta, y_\delta)} = p_\delta, \quad -D_y\Big(\frac{|x - y|^2}{2\delta}\Big)\Big|_{(x_\delta, y_\delta)} = p_\delta.$$

By the theorem of sums [CIL92, Theorem 3.2], there are matrices $X_\delta, Y_\delta \in \mathrm{Sym}(n)$ such that (p_δ, X_δ) belongs to the closed second-order superjet of u at x_δ, while (p_δ, Y_δ) belongs to the closed second-order subjet of v at y_δ. These closed jets are obtained as limits of the first and second derivatives of C^2 test functions touching u from above and v from below at nearby points. Since F_q is continuous, the viscosity inequalities remain valid for these limiting jets. Hence,

$$F_q(x_\delta, u(x_\delta), p_\delta, X_\delta) = -p_\delta^T X_\delta p_\delta + q(x_\delta)u(x_\delta) \leq 0,$$
$$F_q(y_\delta, v(y_\delta), p_\delta, Y_\delta) = -p_\delta^T Y_\delta p_\delta + q(y_\delta)v(y_\delta) \geq 0.$$

The theorem of sums also gives the matrix inequality

$$\begin{pmatrix} X_\delta & 0 \\ 0 & -Y_\delta \end{pmatrix} \leq \frac{3}{\delta}\begin{pmatrix} \mathrm{Id} & -\mathrm{Id} \\ -\mathrm{Id} & \mathrm{Id} \end{pmatrix}. \tag{2.4}$$

Testing (2.4) against the vector $(\xi, \xi) \in \mathbb{R}^n \times \mathbb{R}^n$ gives $\xi^T X_\delta \xi - \xi^T Y_\delta \xi \leq 0$ for every $\xi \in \mathbb{R}^n$, since $\begin{pmatrix} \xi \\ \xi \end{pmatrix}^T \begin{pmatrix} \mathrm{Id} & -\mathrm{Id} \\ -\mathrm{Id} & \mathrm{Id} \end{pmatrix} \begin{pmatrix} \xi \\ \xi \end{pmatrix} = 0$. It follows that $X_\delta \leq Y_\delta$. Taking $\xi = p_\delta$ and using the preceding viscosity inequalities, we obtain

$$q(x_\delta)u(x_\delta) \leq p_\delta^T X_\delta p_\delta \leq p_\delta^T Y_\delta p_\delta \leq q(y_\delta)v(y_\delta). \tag{2.5}$$

As $\delta \downarrow 0$, one has $x_\delta \to x_0$, $y_\delta \to x_0$, $u(x_\delta) \to u(x_0)$, and $v(y_\delta) \to v(x_0)$. Since q is continuous, letting $\delta \downarrow 0$ in (2.5) gives $q(x_0)u(x_0) \leq q(x_0)v(x_0)$. The assumption $q(x_0) \geq c_0 > 0$ then implies $u(x_0) \leq v(x_0)$, which contradicts $u(x_0) - v(x_0) = m > 0$. Therefore, $u \leq v$ in $\overline{\Omega}$. $\square$

Proposition 2.4 (The well-posedness). *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set, $q \in C(\overline{\Omega})$ that satisfies $q \geq c_0 > 0$, and $f \in C(\partial\Omega)$. Then (2.2) has a unique viscosity solution $u_f^q \in C(\overline{\Omega}) \cap \mathrm{Lip}_{\mathrm{loc}}(\Omega)$. Moreover,*

$$\|u_f^q\|_{L^\infty(\Omega)} \leq \|f\|_{L^\infty(\partial\Omega)}.$$

If $f \geq 0$ on $\partial\Omega$, then $u_f^q \geq 0$ in $\overline{\Omega}$.

Proof. Set $C = \|f\|_{L^\infty(\partial\Omega)}$ and write the equation as $\Delta_\infty u = G(x, u)$, where $G(x, r) = q(x)r$. The constants $-C$ and C are respectively a viscosity subsolution and a viscosity supersolution, since $0 \geq G(x, -C)$ and $0 \leq G(x, C)$. They are ordered in $\overline{\Omega}$ and satisfy $-C \leq f \leq C$ on $\partial\Omega$.

The function G is continuous on $\overline{\Omega} \times \mathbb{R}$, nondecreasing in its second variable, and bounded on $\overline{\Omega} \times I$ for every bounded interval $I \subset \mathbb{R}$. The existence theorem for the inhomogeneous infinity Laplace Dirichlet problem can be found in [BM12, Corollary 3.3]. In other words, the ordered constant subsolution and supersolution satisfy the hypotheses of the Perron construction in [BM12, Theorem 3.1]. We obtain a viscosity solution $u_f^q \in C(\overline{\Omega})$ satisfying $-C \leq u_f^q \leq C$ in $\overline{\Omega}$.

The uniqueness is ensured by Proposition 2.3, and the comparison with the two constant barriers also gives

$$\|u_f^q\|_{L^\infty(\Omega)} \leq \|f\|_{L^\infty(\partial\Omega)}.$$

Set $h(x) = q(x)u_f^q(x)$. Since q and u_f^q are continuous on $\overline{\Omega}$, one has $h \in C(\overline{\Omega}) \cap L^\infty(\Omega)$, and u_f^q satisfies $\Delta_\infty u_f^q = h$ in the viscosity sense. The local regularity theorem [BM12, Corollary 2.5] yields $u_f^q \in \mathrm{Lip}_{\mathrm{loc}}(\Omega)$.

If $f \geq 0$ on $\partial\Omega$, then the zero function is a solution of the equation and lies below the boundary data. Proposition 2.3 yields $u_f^q \geq 0$ in $\overline{\Omega}$. □

Remark 2.5. *The assumption $q \geq c_0 > 0$ is not an ellipticity condition. The equation is degenerate elliptic independently of the sign of q. The lower bound is used in the comparison argument, hence, in the uniqueness of the forward problem. If q vanishes or changes sign, Proposition 2.3 does not apply in its present form.*

We next record the boundary differentiability needed to define the DN map. A function $u \in C(\overline{\Omega})$ is differentiable at $x_0 \in \partial\Omega$ if there is a vector $Du(x_0) \in \mathbb{R}^n$ such that

$$u(x) = u(x_0) + Du(x_0) \cdot (x - x_0) + o(|x - x_0|) \tag{2.6}$$

as $x \to x_0$ within $\overline{\Omega}$. A boundary function $f \in C(\partial\Omega)$ is differentiable at x_0 in the tangential sense if its composition with a C^1 local parametrization of $\partial\Omega$ is differentiable at the parameter corresponding to x_0. This definition is independent of the choice of local parametrization.

Proposition 2.6 (Boundary differentiability). *Assume the hypotheses of Proposition 2.4. Let $x_0 \in \partial\Omega$, suppose that $\partial\Omega$ is differentiable at x_0, and assume that f is differentiable at x_0 in the tangential sense. Then u_f^q is differentiable at x_0 in the sense of (2.6).*

Proof. The solution satisfies $\Delta_\infty u_f^q = h(x)$, where $h(x) = q(x)u_f^q(x)$. Proposition 2.4 gives $u_f^q \in C(\overline{\Omega})$, and hence $h \in C(\Omega) \cap L^\infty(\Omega)$. The conclusion follows from the boundary differentiability theorem for inhomogeneous infinity Laplace equations [Hon14, Theorem 1.4]. □

If $\partial\Omega$ is of class C^1 and $f \in C^1(\partial\Omega)$, Proposition 2.6 applies at every boundary point. We define

$$\Lambda_q(f)(x) = Du_f^q(x) \cdot \nu(x) = \partial_\nu u_f^q(x), \quad x \in \partial\Omega.$$

If φ is defined on $\overline{\Omega}$, we use $\Lambda_q(\varphi)$ as shorthand for $\Lambda_q(\varphi|_{\partial\Omega})$.

Lemma 2.7 (Amplitude scaling)**.** *Let $t > 0$ and $f \in C(\partial\Omega)$. Then*

$$u^q_{tf} = t u^{q/t^2}_f. \tag{2.7}$$

If $\partial\Omega$ is of class C^1 and $f \in C^1(\partial\Omega)$, then

$$\Lambda_q(tf) = t\Lambda_{q/t^2}(f). \tag{2.8}$$

Proof. Let $v = t^{-1}u^q_{tf}$. Since $\Delta_\infty(tv) = t^3\Delta_\infty v$, the function v satisfies $-\Delta_\infty v + t^{-2}q(x)v = 0$ in Ω and $v = f$ on $\partial\Omega$. Uniqueness gives $v = u^{q/t^2}_f$ and proves (2.7). Under the additional boundary regularity, taking the normal derivative gives (2.8). □

3. Convex chord geometry and the rank-one correction

Fix $e \in \mathbb{S}^{n-1}$ and let $P_e = \mathrm{Id} - e \otimes e$ be the orthogonal projection onto $e^\perp$. Every $x \in \mathbb{R}^n$ has a unique representation $x = y + se$, where $y = P_e x \in e^\perp$ and $s = e \cdot x$. We write $B^\perp_\delta(y_0)$ for the ball of radius δ in $e^\perp$.

Lemma 3.1 (Geometry of parallel chords)**.** *Let $\Omega \subset \mathbb{R}^n$ be a bounded convex domain with C^2 boundary and set $Y_e = P_e\Omega$.*

(i) *The set Y_e is bounded, open, and convex. For every $y \in Y_e$, there are finite numbers $\alpha_e(y) < \beta_e(y)$ such that*

$$\Omega \cap (y + \mathbb{R}e) = \{y + se : \alpha_e(y) < s < \beta_e(y)\}. \tag{3.1}$$

(ii) *If $x^-_e(y) = y + \alpha_e(y)e$ and $x^+_e(y) = y + \beta_e(y)e$, then*

$$e \cdot \nu(x^+_e(y)) > 0, \quad e \cdot \nu(x^-_e(y)) < 0. \tag{3.2}$$

(iii) *The functions α_e, β_e belong to $C^2(Y_e)$. For every $y_0 \in Y_e$, there is $\delta > 0$ such that*

$$\Omega \cap \{y + se : y \in B^\perp_\delta(y_0)\} = \{y + se : y \in B^\perp_\delta(y_0),\ \alpha_e(y) < s < \beta_e(y)\}, \tag{3.3}$$

and $\beta_e - \alpha_e$ has a positive lower bound on $\overline{B^\perp_\delta(y_0)}$.

(iv) *On the upper and lower faces in (3.3), the outward normals are*

$$\nu_+(y) = \frac{e - D_y\beta_e(y)}{\sqrt{1 + |D_y\beta_e(y)|^2}}, \quad \nu_-(y) = \frac{D_y\alpha_e(y) - e}{\sqrt{1 + |D_y\alpha_e(y)|^2}}. \tag{3.4}$$

There also hold that $e \cdot \nu_+(y) = (1 + |D_y\beta_e(y)|^2)^{-1/2}$ and $e \cdot \nu_-(y) = -(1 + |D_y\alpha_e(y)|^2)^{-1/2}$.

Proof. The projection $P_e : \mathbb{R}^n \to e^\perp$ is a surjective linear map and is open. Hence, $Y_e = P_e\Omega$ is open. Its boundedness and convexity follow from the corresponding properties of Ω.

Fix $y \in Y_e$. The intersection $\Omega \cap (y + \mathbb{R}e)$ is a nonempty, bounded, open, and convex subset of the affine line $y + \mathbb{R}e$, then it is an open interval. This gives uniquely determined finite numbers $\alpha_e(y) < \beta_e(y)$ and proves (3.1).

Let $x_+ = x^+_e(y)$. For every sufficiently small $h > 0$, one has $x_+ - he \in \Omega$. Since Ω is convex and $\partial\Omega$ is of class C^1, the tangent hyperplane at x_+ supports Ω, and hence, $(z - x_+) \cdot \nu(x_+) \le 0$, for every $z \in \overline{\Omega}$. The inequality is strict when $z \in \Omega$. Indeed, if an interior point belonged to the supporting hyperplane, a sufficiently small ball centered at that point would meet the complementary half-space. Taking $z = x_+ - he$ gives $-he \cdot \nu(x_+) < 0$, then $e \cdot \nu(x_+) > 0$. At the lower endpoint, the point $x^-_e(y) + he$ lies in Ω for small $h > 0$, and the same argument gives $e \cdot \nu(x^-_e(y)) < 0$.

Fix $y_0 \in Y_e$ and choose s_0 such that $y_0 + s_0 e \in \Omega$. By openness, $y + s_0 e \in \Omega$ for every y in a sufficiently small neighborhood of y_0. Let ρ_+ and ρ_- be C^2 defining functions in neighborhoods of $x^+_e(y_0)$ and $x^-_e(y_0)$, respectively, with $\Omega = \{\rho_\pm < 0\}$ in the corresponding neighborhoods. Set $G_\pm(y, s) = \rho_\pm(y + se)$. By (3.2), $\partial_s G_+(y_0, \beta_e(y_0)) \ne 0$ and $\partial_s G_-(y_0, \alpha_e(y_0)) \ne 0$. The implicit function theorem gives C^2 functions $\widetilde{\beta}$ and $\widetilde{\alpha}$ whose graphs describe the upper and lower boundary near the two endpoints. After shrinking the transverse neighborhood, we may assume that $\widetilde{\alpha}(y) < s_0 < \widetilde{\beta}(y)$ for every y in that neighborhood. Since $\Omega \cap (y + \mathbb{R}e)$ is an interval containing $y + s_0 e$, the two graph points are precisely the lower and upper endpoints of this interval. It follows that $\widetilde{\alpha} = \alpha_e$ and $\widetilde{\beta} = \beta_e$ locally. These local representations agree on overlaps because the two endpoints of each chord are unique. Consequently, $\alpha_e, \beta_e \in C^2(Y_e)$. Since $\beta_e(y_0) - \alpha_e(y_0) > 0$, continuity gives a positive lower bound for the chord length on a sufficiently small closed transverse ball.

The domain lies below the upper graph $s = \beta_e(y)$ and above the lower graph $s = \alpha_e(y)$. The outward normals are therefore the normalized gradients of $s - \beta_e(y)$ and $\alpha_e(y) - s$, respectively. This gives (3.4) and the stated formulas for their scalar products with e. $\square$

Lemma 3.2. *Let $\Gamma_e = \{x \in \partial\Omega : e \cdot \nu(x) \neq 0\}$. If $x \in \Gamma_e$ and $y = P_e x$, then $y \in Y_e$ and x is either $x_e^-(y)$ or $x_e^+(y)$. Moreover, every endpoint of a positive length chord belongs to Γ_e.*

Proof. We first prove that every endpoint of a positive length chord belongs to Γ_e. Let $y \in Y_e$. By (3.2), we know that $e \cdot \nu$ does not vanish at either endpoint, and therefore $x_e^{\pm}(y) \in \Gamma_e$.

Conversely, let $x \in \Gamma_e$ and set $y = P_e x$. Write $s_0 = e \cdot x$, so that $x = y + s_0 e$. Let ρ be a C^1 defining function for Ω in a neighborhood of x, chosen so that $\Omega = \{\rho < 0\}$ and $D\rho(x) = |D\rho(x)|\nu(x)$. Since $x \in \Gamma_e$,

$$\frac{d}{dh}\rho(x + he)\Big|_{h=0} = D\rho(x) \cdot e = |D\rho(x)|\, e \cdot \nu(x) \neq 0.$$

It follows that, for all sufficiently small $h > 0$, the values $\rho(x + he)$ and $\rho(x - he)$ have opposite signs. Thus, exactly one of the points $x + he$ and $x - he$ lies in Ω. In particular, the line $y + \mathbb{R}e$ contains an interior point of Ω, and hence $y \in P_e\Omega = Y_e$.

By (3.1), we have $\Omega \cap (y + \mathbb{R}e) = \{y + se : \alpha_e(y) < s < \beta_e(y)\}$. Since $x = y + s_0 e$ belongs to $\partial\Omega$, one cannot have $\alpha_e(y) < s_0 < \beta_e(y)$, for otherwise $x \in \Omega$. On the other hand, the line contains points of Ω arbitrarily close to x, so s_0 belongs to the closure of $(\alpha_e(y), \beta_e(y))$. Therefore, $s_0 = \alpha_e(y)$ or $s_0 = \beta_e(y)$, and hence, $x = x_e^-(y)$ or $x = x_e^+(y)$. $\square$

Fix $b > R_\Omega$. Since $y \in e^\perp$, one has $\ell_{e,b}(y + se) = s + b$. For each $y \in Y_e$, consider the one-dimensional Dirichlet problem

$$\begin{cases} \partial_s^2 w_{q,e,b}(y,s) = q(y + se)(s + b) & \text{for } \alpha_e(y) < s < \beta_e(y), \\ w_{q,e,b}(y,s) = 0 & \text{for } s = \alpha_e(y) \text{ and } s = \beta_e(y). \end{cases} \tag{3.5}$$

The next lemma shows that (3.5) has a unique solution for every $y \in Y_e$. Since every $x \in \Omega$ has a unique representation $x = y + se$ with $y \in Y_e$ and $\alpha_e(y) < s < \beta_e(y)$, these chordwise solutions define a function on Ω by $w_{q,e,b}(y + se) := w_{q,e,b}(y,s)$. We continue to denote this function by $w_{q,e,b}$.

Lemma 3.3 (Formula, uniqueness, and local regularity)**.** *For every $y \in Y_e$, the problem (3.5) has a unique solution. Let $y_0 \in Y_e$, and let $\alpha = \alpha_e$ and $\beta = \beta_e$ on a closed chord tube around y_0. Set $F(y,r) = q(y + re)(r + b)$ and $L(y) = \beta(y) - \alpha(y)$. Then*

$$w_{q,e,b}(y,s) = \int_{\alpha(y)}^{s} (s - r)F(y,r)\,dr - \frac{s - \alpha(y)}{L(y)} \int_{\alpha(y)}^{\beta(y)} (\beta(y) - r)F(y,r)\,dr. \tag{3.6}$$

The function $w_{q,e,b}$ belongs to C^2 on the closed tube, vanishes on its two physical faces, and satisfies

$$-\partial_e^2 w_{q,e,b} + q\ell_{e,b} = 0. \tag{3.7}$$

Moreover, $w_{q,e,b} \leq 0$ in Ω, and

$$\|w_{q,e,b}\|_{L^\infty(\Omega)} \leq \frac{\operatorname{diam}(\Omega)^2}{8}(b + R_\Omega)\|q\|_{L^\infty(\Omega)}. \tag{3.8}$$

Proof. Fix $y \in Y_e$ and abbreviate $\alpha = \alpha_e(y)$ and $\beta = \beta_e(y)$. Twice integrating (3.5) from α gives

$$w(y,s) = \int_\alpha^s (s-r)F(y,r)\,dr + C_0(y) + C_1(y)(s-\alpha).$$

The boundary condition at $s = \alpha$ gives $C_0(y) = 0$. Imposing the boundary condition at $s = \beta$ gives

$$C_1(y) = -\frac{1}{\beta-\alpha}\int_\alpha^\beta (\beta-r)F(y,r)\,dr.$$

This proves (3.6). It also proves uniqueness of the chordwise Dirichlet problem.

On a closed chord tube, the functions q, α, β are of class C^2, and $L(y) = \beta(y) - \alpha(y)$ has a positive lower bound, then the Leibniz rule may be applied to (3.6) with respect to both y and s, including at the two physical faces. It follows that $w_{q,e,b} \in C^2$ on the closed tube. Differentiating twice with respect to s gives

$$\partial_s^2 w_{q,e,b}(y,s) = F(y,s) = q(y+se)(s+b).$$

Substituting $s = \alpha(y)$ and $s = \beta(y)$ into (3.6) gives the two zero boundary values. Thus, (3.7) holds on the tube.

Since $q > 0$ and $s + b > 0$ on every chord, one has $\partial_s^2 w_{q,e,b} \geq 0$. Hence, $s \mapsto w_{q,e,b}(y,s)$ is convex. A convex function that vanishes at both endpoints of an interval is nonpositive in the interval, so that $w_{q,e,b} \leq 0$ in Ω.

For the uniform bound, fix $y \in Y_e$ and continue to write $\alpha = \alpha_e(y)$ and $\beta = \beta_e(y)$. The Green function of $-\frac{d^2}{ds^2}$ on (α,β) with zero endpoint values is

$$G_{\alpha,\beta}(s,r) = \begin{cases} \dfrac{(s-\alpha)(\beta-r)}{\beta-\alpha} & \text{if } \alpha \leq s \leq r \leq \beta, \\ \dfrac{(r-\alpha)(\beta-s)}{\beta-\alpha} & \text{if } \alpha \leq r \leq s \leq \beta. \end{cases}$$

Indeed, for each fixed $r \in (\alpha,\beta)$, the function $s \mapsto G_{\alpha,\beta}(s,r)$ is affine on (α,r) and (r,β), vanishes at $s = \alpha$ and $s = \beta$, and is continuous at $s = r$. Its first derivative has the jump

$$\partial_s G_{\alpha,\beta}(r^+,r) - \partial_s G_{\alpha,\beta}(r^-,r) = -\frac{r-\alpha}{\beta-\alpha} - \frac{\beta-r}{\beta-\alpha} = -1,$$

and hence, $-\partial_s^2 G_{\alpha,\beta}(s,r) = \delta_r(s)$ in the distributional sense. Since $w_{q,e,b}$ satisfies $\partial_s^2 w_{q,e,b}(y,s) = q(y+se)(s+b)$ with zero endpoint values, its Green representation is

$$w_{q,e,b}(y,s) = -\int_\alpha^\beta G_{\alpha,\beta}(s,r)q(y+re)(r+b)\,dr.$$

For fixed $s \in (\alpha,\beta)$, splitting the integral at $r = s$ gives

$$\int_\alpha^\beta G_{\alpha,\beta}(s,r)\,dr = \frac{\beta-s}{\beta-\alpha}\int_\alpha^s (r-\alpha)\,dr + \frac{s-\alpha}{\beta-\alpha}\int_s^\beta (\beta-r)\,dr = \frac{(s-\alpha)(\beta-s)}{2}.$$

The last expression is maximized at $s = (\alpha+\beta)/2$, and this implies

$$\sup_{\alpha<s<\beta}\int_\alpha^\beta G_{\alpha,\beta}(s,r)\,dr = \frac{(\beta-\alpha)^2}{8}.$$

Moreover, $y + re \in \Omega$ implies $|r| = |e \cdot (y+re)| \leq R_\Omega$, so $|r+b| \leq b + R_\Omega$. Since $\beta - \alpha$ is the length of the chord $\Omega \cap (y + \mathbb{R}e)$, one also has $\beta - \alpha \leq \operatorname{diam}(\Omega)$. Consequently,

$$|w_{q,e,b}(y,s)| \leq \|q\|_{L^\infty(\Omega)}(b+R_\Omega)\int_\alpha^\beta G_{\alpha,\beta}(s,r)\,dr \leq \frac{\operatorname{diam}(\Omega)^2}{8}(b+R_\Omega)\|q\|_{L^\infty(\Omega)}.$$

Taking the supremum over all chords and all points on each chord proves (3.8). ∎

At every non-glancing endpoint, $w_{q,e,b} = 0$ on the corresponding boundary face. Thus, its tangential derivatives vanish and

$$\partial_e w_{q,e,b} = (e \cdot \nu)\partial_\nu w_{q,e,b}. \tag{3.9}$$

4. Exact large amplitude scaling and global estimates

Fix $q \in C^2(\overline{\Omega})$ with $q \geq c_0 > 0$, $e \in \mathbb{S}^{n-1}$, and $b > R_\Omega$. Let $u^q_{t,e,b}$ solve (1.9). For a function φ defined on $\overline{\Omega}$, we write $\Lambda_q(\varphi)$ for $\Lambda_q(\varphi|_{\partial\Omega})$.

Set $\varepsilon = t^{-2}$ and define

$$v_\varepsilon = t^{-1} u^q_{t,e,b}. \tag{4.1}$$

By Lemma 2.7, v_ε is the unique solution of

$$\begin{cases} -\Delta_\infty v_\varepsilon + \varepsilon q(x) v_\varepsilon = 0 & \text{in } \Omega, \\ v_\varepsilon = \ell_{e,b} & \text{on } \partial\Omega. \end{cases} \tag{4.2}$$

Define

$$z_\varepsilon = \frac{v_\varepsilon - \ell_{e,b}}{\varepsilon}. \tag{4.3}$$

Then it is clear that $z_\varepsilon \in C(\overline{\Omega})$ and $z_\varepsilon = 0$ on $\partial\Omega$.

Lemma 4.1 (Exact equation for the difference quotient)**.** *The function z_ε is a viscosity solution of*

$$\mathcal{F}_\varepsilon[z_\varepsilon] = 0 \quad \textit{in } \Omega, \tag{4.4}$$

where

$$\mathcal{F}_\varepsilon[\phi] = -(e + \varepsilon D\phi)^T D^2\phi(e + \varepsilon D\phi) + q(x)(\ell_{e,b} + \varepsilon\phi), \qquad \phi \in C^2(\Omega). \tag{4.5}$$

Proof. Suppose that $z_\varepsilon - \phi$ has a local maximum at x_0. After adding a constant to the test function and continuing to denote the shifted function by ϕ, we may assume that $z_\varepsilon(x_0) = \phi(x_0)$. Since $v_\varepsilon = \ell_{e,b} + \varepsilon z_\varepsilon$, the function $v_\varepsilon - (\ell_{e,b} + \varepsilon\phi)$ has a local maximum equal to zero at x_0. The viscosity subsolution inequality for (4.2) gives

$$-\Delta_\infty(\ell_{e,b} + \varepsilon\phi)(x_0) + \varepsilon q(x_0)(\ell_{e,b}(x_0) + \varepsilon\phi(x_0)) \leq 0.$$

Since $D\ell_{e,b} = e$ and $D^2\ell_{e,b} = 0$, one has

$$\Delta_\infty(\ell_{e,b} + \varepsilon\phi) = \varepsilon(e + \varepsilon D\phi)^T D^2\phi(e + \varepsilon D\phi).$$

Dividing the viscosity inequality by $\varepsilon > 0$ gives $\mathcal{F}_\varepsilon[\phi](x_0) \leq 0$.

If $z_\varepsilon - \phi$ has a local minimum at x_0, then $v_\varepsilon - (\ell_{e,b} + \varepsilon\phi)$ has a local minimum equal to zero there. The viscosity supersolution inequality gives $\mathcal{F}_\varepsilon[\phi](x_0) \geq 0$. Hence, z_ε is both a viscosity subsolution and a viscosity supersolution of (4.4). □

Expanding (4.5) in powers of ε gives

$$\mathcal{F}_\varepsilon[\phi] = -\partial_e^2\phi + q\ell_{e,b} + \varepsilon(q\phi - 2D^2\phi[e, D\phi]) - \varepsilon^2 D^2\phi[D\phi, D\phi]. \tag{4.6}$$

The leading part is the equation in (3.7).

Let $m_e = \min_{\overline{\Omega}} e \cdot x$ and $M_e = \max_{\overline{\Omega}} e \cdot x$, and define

$$h_e(x) := (e \cdot x - m_e)(M_e - e \cdot x).$$

Then $h_e \geq 0$, $D^2 h_e = -2e \otimes e$, and $|\partial_e h_e| \leq M_e - m_e$.

Proposition 4.2 (Global bound)**.** *There are constants $C > 0$ and $\varepsilon_0 > 0$, depending on q, b, e, and Ω, such that*

$$\ell_{e,b} - C\varepsilon h_e \le v_\varepsilon \le \ell_{e,b} + C\varepsilon h_e \quad \textit{in } \overline{\Omega}, \tag{4.7}$$

for $0 < \varepsilon \le \varepsilon_0$. Moreover,

$$|z_\varepsilon(x)| \le Ch_e(x) \le \frac{C}{4}(M_e - m_e)^2. \tag{4.8}$$

Proof. Write $s = e\cdot x$ and set $L_e = M_e - m_e$. By the definition of m_e and M_e, one has $m_e \le s \le M_e$ for every $x \in \overline{\Omega}$. Hence, $h_e(x) = (s - m_e)(M_e - s) \ge 0$. A direct calculation gives $\partial_e h_e = M_e + m_e - 2e\cdot x$, $Dh_e = (\partial_e h_e)e$ and $D^2 h_e = -2e \otimes e$. Since $m_e \le e\cdot x \le M_e$, one has

$$|\partial_e h_e| \le M_e - m_e = L_e \quad \text{in } \overline{\Omega}. \tag{4.9}$$

Notice also that $\ell_{e,b}(x) = e\cdot x + b \ge b - |x| \ge b - R_\Omega > 0$ in $\overline{\Omega}$.

Set $Q_0 = \|q\|_{L^\infty(\Omega)}$, $B_0 = \|\ell_{e,b}\|_{L^\infty(\Omega)}$ and choose $C = 2Q_0B_0 + 1$. Since Ω is a nonempty open set, its width in the direction e is positive, so $L_e > 0$. We next choose $\varepsilon_0 = \min\{1, \frac{1}{2CL_e}\}$. For $0 < \varepsilon \le \varepsilon_0$, estimate (4.9) gives

$$C\varepsilon|\partial_e h_e| \le C\varepsilon L_e \le \frac{1}{2}. \tag{4.10}$$

Define $\underline{v}_\varepsilon = \ell_{e,b} - C\varepsilon h_e$ and $\overline{v}_\varepsilon = \ell_{e,b} + C\varepsilon h_e$ to be the lower and upper barriers, respectively. We first consider the lower barrier. Since $D\ell_{e,b} = e$ and $D^2\ell_{e,b} = 0$, one has $D\underline{v}_\varepsilon = e - C\varepsilon Dh_e = (1 - C\varepsilon\partial_e h_e)e$ and $D^2\underline{v}_\varepsilon = -C\varepsilon D^2 h_e = 2C\varepsilon e \otimes e$. Using $|e| = 1$, we obtain

$$\Delta_\infty \underline{v}_\varepsilon = (D\underline{v}_\varepsilon)^T D^2\underline{v}_\varepsilon D\underline{v}_\varepsilon = \big((1 - C\varepsilon\partial_e h_e)e\big)^T(2C\varepsilon e\otimes e)\big((1 - C\varepsilon\partial_e h_e)e\big) = 2C\varepsilon(1 - C\varepsilon\partial_e h_e)^2.$$

Therefore,

$$-\Delta_\infty \underline{v}_\varepsilon + \varepsilon q \underline{v}_\varepsilon = -2C\varepsilon(1 - C\varepsilon\partial_e h_e)^2 + \varepsilon q(\ell_{e,b} - C\varepsilon h_e). \tag{4.11}$$

By (4.10), $1 - C\varepsilon\partial_e h_e \ge 1 - C\varepsilon|\partial_e h_e| \ge \frac{1}{2}$, and hence, $-2C\varepsilon(1 - C\varepsilon\partial_e h_e)^2 \le -2C\varepsilon\big(\frac{1}{2}\big)^2 = -\frac{C}{2}\varepsilon$. Since $q \ge 0$ and $h_e \ge 0$, one also has $q(\ell_{e,b} - C\varepsilon h_e) \le q\ell_{e,b} \le Q_0B_0$. Substitution into (4.11) yields

$$-\Delta_\infty \underline{v}_\varepsilon + \varepsilon q\underline{v}_\varepsilon \le -\frac{C}{2}\varepsilon + Q_0B_0\varepsilon = \Big(-\frac{2Q_0B_0 + 1}{2} + Q_0B_0\Big)\varepsilon = -\frac{\varepsilon}{2} < 0.$$

Thus, $\underline{v}_\varepsilon$ is a classical subsolution, and therefore also a viscosity subsolution, of (4.2).

We next consider the upper barrier. One has

$$D\overline{v}_\varepsilon = e + C\varepsilon Dh_e = (1 + C\varepsilon\partial_e h_e)e \quad \text{and} \quad D^2\overline{v}_\varepsilon = C\varepsilon D^2 h_e = -2C\varepsilon e\otimes e.$$

It follows that $\Delta_\infty\overline{v}_\varepsilon = -2C\varepsilon(1 + C\varepsilon\partial_e h_e)^2$, so

$$-\Delta_\infty\overline{v}_\varepsilon + \varepsilon q\overline{v}_\varepsilon = 2C\varepsilon(1 + C\varepsilon\partial_e h_e)^2 + \varepsilon q(\ell_{e,b} + C\varepsilon h_e). \tag{4.12}$$

The first term on the right-hand side is nonnegative. The second term is strictly positive because $q \ge c_0 > 0$, $\ell_{e,b} > 0$, and $h_e \ge 0$. More precisely, $q(\ell_{e,b} + C\varepsilon h_e) \ge q\ell_{e,b} \ge c_0(b - R_\Omega) > 0$. Thus, (4.12) gives $-\Delta_\infty\overline{v}_\varepsilon + \varepsilon q\overline{v}_\varepsilon \ge \varepsilon c_0(b - R_\Omega) > 0$. It follows that $\overline{v}_\varepsilon$ is a classical supersolution, and therefore also a viscosity supersolution, of (4.2).

On $\partial\Omega$, the boundary condition for v_ε and the inequality $h_e \ge 0$ give

$$\underline{v}_\varepsilon = \ell_{e,b} - C\varepsilon h_e \le \ell_{e,b} = v_\varepsilon \le \ell_{e,b} + C\varepsilon h_e = \overline{v}_\varepsilon.$$

The potential in (4.2) is εq, and it satisfies $\varepsilon q \ge \varepsilon c_0 > 0$. Proposition 2.3 may therefore be applied first to $\underline{v}_\varepsilon$ and v_ε, and then to v_ε and $\overline{v}_\varepsilon$. We obtain $\underline{v}_\varepsilon \le v_\varepsilon \le \overline{v}_\varepsilon$ in $\overline{\Omega}$, which proves (4.7). By the definition $z_\varepsilon = (v_\varepsilon - \ell_{e,b})/\varepsilon$, estimate (4.7) gives $-Ch_e \le z_\varepsilon \le Ch_e$. Hence, $|z_\varepsilon| \le Ch_e$. Finally, completing the square gives $h_e(x) = \frac{(M_e - m_e)^2}{4} - \big(e\cdot x - \frac{M_e + m_e}{2}\big)^2 \le \frac{(M_e - m_e)^2}{4}$. This proves (4.8). $\square$

5. Local chord-tube barriers and Dirichlet-to-Neumann asymptotics

Fix $y_0 \in Y_e$. In a neighborhood of y_0, write $\alpha = \alpha_e$ and $\beta = \beta_e$. Choose $\delta > 0$ as in Lemma 3.1 and set

$$T_\delta = \{y + se : y \in B_\delta^\perp(y_0),\ \alpha(y) < s < \beta(y)\}.$$

The two physical faces are

$$\Gamma_-^\delta = \{y + \alpha(y)e : y \in B_\delta^\perp(y_0)\}, \quad \Gamma_+^\delta = \{y + \beta(y)e : y \in B_\delta^\perp(y_0)\},$$

and the artificial lateral boundary is

$$\Sigma_\delta = \{y + se : y \in \partial B_\delta^\perp(y_0),\ \alpha(y) \le s \le \beta(y)\}.$$

Define

$$\eta(y) = K_0|y - y_0|^2, \quad \chi(y,s) = (s - \alpha(y))(\beta(y) - s),$$

then the function η is independent of the variable s, while direct differentiation gives $\partial_e^2\chi = \partial_s^2\chi = -2$. Moreover, $\chi \ge 0$ in $\overline{T_\delta}$ and $\chi = 0$ on the two physical faces.

Lemma 5.1 (Local chord-tube barrier)**.** *Let z_ε be defined by (4.3) and let $w = w_{q,e,b}$. There are constants $K_0 > 0$, $A > 0$, and $\varepsilon_1 > 0$, independent of ε, such that*

$$w - \eta - A\varepsilon\chi \le z_\varepsilon \le w + \eta + A\varepsilon\chi \tag{5.1}$$

in $\overline{T_\delta}$ for $0 < \varepsilon \le \varepsilon_1$. In particular,

$$|z_\varepsilon(y_0 + se) - w(y_0 + se)| \le A\varepsilon(s - \alpha_0)(\beta_0 - s), \tag{5.2}$$

where $\alpha_0 = \alpha(y_0)$ and $\beta_0 = \beta(y_0)$.

Proof. Let $M > 0$ be the uniform bound for z_ε supplied by Proposition 4.2, so that $|z_\varepsilon| \le M$ in $\overline{\Omega}$ for all sufficiently small ε, and let $W = \|w\|_{L^\infty(T_\delta)}$. We first choose $K_0 > 0$ so that

$$K_0\delta^2 \ge M + W + 1. \tag{5.3}$$

For a constant $A > 0$ to be chosen later, define

$$\Phi_\varepsilon^+ = w + \eta + A\varepsilon\chi, \quad \Phi_\varepsilon^- = w - \eta - A\varepsilon\chi.$$

We will show that Φ_ε^+ is a strict supersolution and Φ_ε^- is a strict subsolution of the exact equation (4.4). Assume temporarily that $A\varepsilon \le 1$. For $j = 0, 1, 2$, set

$$B_j = \|D^j w\|_{L^\infty(T_\delta)} + \|D^j\eta\|_{L^\infty(T_\delta)} + \|D^j\chi\|_{L^\infty(T_\delta)},$$

where D^0 denotes the function itself and the derivatives are taken with respect to the Euclidean variables $x = y + se$. Once K_0 has been fixed, the constants B_j are independent of A and ε. Since $A\varepsilon \le 1$, one has $\|D^j\Phi_\varepsilon^\pm\|_{L^\infty(T_\delta)} \le B_j$, $j = 0, 1, 2$.

Recall that the function w satisfies $-\partial_e^2 w + q\ell_{e,b} = 0$, while $\partial_e^2\eta = 0$ and $\partial_e^2\chi = -2$. It follows that $-\partial_e^2\Phi_\varepsilon^+ + q\ell_{e,b} = 2A\varepsilon$ and $-\partial_e^2\Phi_\varepsilon^- + q\ell_{e,b} = -2A\varepsilon$. Using the expansion (4.6), together with the bounds $|q\Phi_\varepsilon^\pm| \le \|q\|_{L^\infty(T_\delta)}B_0$, $2|D^2\Phi_\varepsilon^\pm[e, D\Phi_\varepsilon^\pm]| \le 2B_1B_2$, and $|D^2\Phi_\varepsilon^\pm[D\Phi_\varepsilon^\pm, D\Phi_\varepsilon^\pm]| \le B_1^2B_2$, we obtain

$$\mathcal{F}_\varepsilon[\Phi_\varepsilon^+] \ge 2A\varepsilon - C_1\varepsilon - C_2\varepsilon^2, \quad \mathcal{F}_\varepsilon[\Phi_\varepsilon^-] \le -2A\varepsilon + C_1\varepsilon + C_2\varepsilon^2, \tag{5.4}$$

where $C_1 = \|q\|_{L^\infty(T_\delta)}B_0 + 2B_1B_2$ and $C_2 = B_1^2B_2$. Now, we choose $A = C_1 + C_2 + 1$, and then select $0 < \varepsilon_1 \le \min\{\varepsilon_0, 1, A^{-1}\}$, where ε_0 is the constant given in Proposition 4.2. For $0 < \varepsilon \le \varepsilon_1$, one has $A\varepsilon \le 1$ and $2A - C_1 - C_2\varepsilon \ge 2A - C_1 - C_2 = C_1 + C_2 + 2 > 0$. Therefore, with these choices at hand, the inequalities (5.4) read that

$$\mathcal{F}_\varepsilon[\Phi_\varepsilon^+] > 0, \quad \mathcal{F}_\varepsilon[\Phi_\varepsilon^-] < 0 \quad \text{in } T_\delta. \tag{5.5}$$

We next verify the ordering on the boundary of the tube. On the physical faces $\Gamma^\delta_- \cup \Gamma^\delta_+$, one has $z_\varepsilon = 0$, $w = 0$, and $\chi = 0$. Hence,

$$\Phi^-_\varepsilon = -\eta \le 0 = z_\varepsilon \le \eta = \Phi^+_\varepsilon.$$

On the lateral boundary Σ_δ, one has $\eta = K_0\delta^2$. Since $\chi \ge 0$, the choice (5.3) gives

$$\Phi^-_\varepsilon \le W - K_0\delta^2 \le -M - 1 \le z_\varepsilon \le M + 1 \le -W + K_0\delta^2 \le \Phi^+_\varepsilon.$$

Thus, $\Phi^-_\varepsilon \le z_\varepsilon \le \Phi^+_\varepsilon$ on ∂T_δ.

Suppose that $m = \max_{\overline{T_\delta}}(z_\varepsilon - \Phi^+_\varepsilon) > 0$, and let x_* be a point where the maximum is attained. The boundary ordering implies that $x_* \in T_\delta$. By the definition of m, one has $z_\varepsilon - \Phi^+_\varepsilon - m \le 0$ in $\overline{T_\delta}$, with equality at x_*. Hence, $\Phi^+_\varepsilon + m$ touches z_ε from above at x_*. Since z_ε is a viscosity subsolution of (4.4), $\mathcal{F}_\varepsilon[\Phi^+_\varepsilon + m](x_*) \le 0$. Adding the constant m does not change the first or second derivatives, while the zeroth-order term changes by $\varepsilon q(x)m$. Therefore,

$$0 \ge \mathcal{F}_\varepsilon[\Phi^+_\varepsilon + m](x_*) = \mathcal{F}_\varepsilon[\Phi^+_\varepsilon](x_*) + \varepsilon q(x_*)m > 0,$$

where the last inequality follows from (5.5), $q > 0$, and $m > 0$. This contradiction proves that $z_\varepsilon \le \Phi^+_\varepsilon$ in $\overline{T_\delta}$.

Similarly, suppose that $m = \max_{\overline{T_\delta}}(\Phi^-_\varepsilon - z_\varepsilon) > 0$, and let $x^* \in \overline{T_\delta}$ be a point where this maximum is attained. The boundary ordering implies that $x^* \in T_\delta$. Then $\Phi^-_\varepsilon - m$ touches z_ε from below at x^*. Since z_ε is a viscosity supersolution of (4.4),

$$0 \le \mathcal{F}_\varepsilon[\Phi^-_\varepsilon - m](x^*) = \mathcal{F}_\varepsilon[\Phi^-_\varepsilon](x^*) - \varepsilon q(x^*)m < 0,$$

where the last inequality follows from (5.5), $q > 0$, and $m > 0$. This is impossible. Therefore, $\Phi^-_\varepsilon \le z_\varepsilon$ in $\overline{T_\delta}$, and (5.1) follows.

On the central chord $y = y_0$, one has $\eta(y_0) = 0$ and $\chi(y_0, s) = (s - \alpha_0)(\beta_0 - s)$. Restricting (5.1) to this chord gives

$$w(y_0 + se) - A\varepsilon(s - \alpha_0)(\beta_0 - s) \le z_\varepsilon(y_0 + se) \le w(y_0 + se) + A\varepsilon(s - \alpha_0)(\beta_0 - s),$$

which is equivalent to (5.2). □

Remark 5.2 (Structure of the barriers). *The transverse term η does not contribute to the limiting operator $-\partial_e^2$ and separates the upper and lower barriers from z_ε on the artificial lateral boundary. The chord factor χ vanishes on the physical faces and satisfies $\partial_e^2\chi = -2$, which produces the strict signs in (5.5). No positive lower bound for the curvature of $\partial\Omega$ is used.*

Let $L_0 = \beta_0 - \alpha_0$, $x_- = y_0 + \alpha_0 e$, and $x_+ = y_0 + \beta_0 e$.

Proposition 5.3 (Endpoint derivative estimate). *For $0 < \varepsilon \le \varepsilon_1$,*

$$|\partial_e z_\varepsilon(x_\pm) - \partial_e w_{q,e,b}(x_\pm)| \le A\varepsilon L_0. \tag{5.6}$$

Moreover,

$$|\partial_\nu z_\varepsilon(x_\pm) - \partial_\nu w_{q,e,b}(x_\pm)| \le \frac{A\varepsilon L_0}{|e \cdot \nu(x_\pm)|}. \tag{5.7}$$

Proof. The function v_ε satisfies $\Delta_\infty v_\varepsilon = \varepsilon q v_\varepsilon$ with affine boundary values. Its right-hand side is bounded and continuous, and Proposition 2.6 shows that v_ε is differentiable at x_- and x_+. Since $z_\varepsilon = \frac{v_\varepsilon - \ell_{e,b}}{\varepsilon}$, the function z_ε is differentiable at the same points. The correction $w_{q,e,b}$ is C^2 near both endpoints by Lemma 3.3.

Set $g_\varepsilon = z_\varepsilon - w_{q,e,b}$. Both functions vanish at x_+ and x_-, so $g_\varepsilon(x_+) = g_\varepsilon(x_-) = 0$. For $0 < h < L_0$, apply (5.2) at $x_+ - he = y_0 + (\beta_0 - h)e$. Since $\beta_0 - (\beta_0 - h) = h$ and $(\beta_0 - h) - \alpha_0 = L_0 - h$, one obtains $|g_\varepsilon(x_+ - he)| \le A\varepsilon h(L_0 - h)$. Using $g_\varepsilon(x_+) = 0$ and dividing by $h > 0$ gives

$$\left|\frac{g_\varepsilon(x_+) - g_\varepsilon(x_+ - he)}{h}\right| \le A\varepsilon(L_0 - h). \tag{5.8}$$

As $h \downarrow 0$, boundary differentiability implies

$$\frac{g_\varepsilon(x_+) - g_\varepsilon(x_+ - he)}{h} \longrightarrow Dg_\varepsilon(x_+) \cdot e = \partial_e z_\varepsilon(x_+) - \partial_e w_{q,e,b}(x_+).$$

Therefore, together with (5.8), we obtain $|\partial_e z_\varepsilon(x_+) - \partial_e w_{q,e,b}(x_+)| \le A\varepsilon L_0$.

Similarly, at the lower endpoint, apply (5.2) at $x_- + he = y_0 + (\alpha_0 + h)e$. Since $(\alpha_0 + h) - \alpha_0 = h$ and $\beta_0 - (\alpha_0 + h) = L_0 - h$, one obtains $|g_\varepsilon(x_- + he)| \le A\varepsilon h(L_0 - h)$. Using $g_\varepsilon(x_-) = 0$ gives

$$\left|\frac{g_\varepsilon(x_- + he) - g_\varepsilon(x_-)}{h}\right| \le A\varepsilon(L_0 - h).$$

Letting $h \downarrow 0$ yields $|\partial_e z_\varepsilon(x_-) - \partial_e w_{q,e,b}(x_-)| \le A\varepsilon L_0$. This proves (5.6) at both endpoints.

The functions z_ε and $w_{q,e,b}$ vanish on the corresponding physical boundary faces. Their tangential derivatives therefore vanish at $x_\pm$, and their gradients are parallel to the outward unit normal. Hence,

$$\partial_e z_\varepsilon(x_\pm) = (e \cdot \nu(x_\pm))\partial_\nu z_\varepsilon(x_\pm) \quad \text{and} \quad \partial_e w_{q,e,b}(x_\pm) = (e \cdot \nu(x_\pm))\partial_\nu w_{q,e,b}(x_\pm).$$

Subtracting these identities and using $e \cdot \nu(x_\pm) \ne 0$ gives

$$|\partial_\nu z_\varepsilon(x_\pm) - \partial_\nu w_{q,e,b}(x_\pm)| = \frac{|\partial_e z_\varepsilon(x_\pm) - \partial_e w_{q,e,b}(x_\pm)|}{|e \cdot \nu(x_\pm)|}.$$

Estimate (5.7) now follows from (5.6). ∎

Proposition 5.4 (Uniformity away from glancing points). *Let $K \Subset \Gamma_e$. There are $C_K > 0$ and $\varepsilon_K > 0$ such that*

$$\sup_{x \in K} |\partial_\nu z_\varepsilon(x) - \partial_\nu w_{q,e,b}(x)| \le C_K \varepsilon \tag{5.9}$$

for $0 < \varepsilon \le \varepsilon_K$.

Proof. By Lemma 3.2, every point of K is an endpoint of a positive length chord. Hence, $Z = P_e(K)$ is a compact subset of the open set Y_e. Choose open sets $U_1, U_2 \subset e^\perp$ such that $Z \Subset U_1 \Subset U_2 \Subset Y_e$, and set $\delta := \frac{1}{2}\operatorname{dist}(\overline{U}_1, e^\perp \setminus U_2) > 0$, then $\overline{B_\delta^\perp(y_0)} \subset U_2$ for every $y_0 \in Z$. Since $\alpha_e, \beta_e \in C^2(Y_e)$, their C^2 norms are bounded on $\overline{U}_2$. Moreover, $\inf_{y \in \overline{U}_2}\big(\beta_e(y) - \alpha_e(y)\big) > 0$. The explicit formula (3.6), together with $q \in C^2(\overline{\Omega})$, therefore gives a uniform C^2 bound for $w_{q,e,b}$ on the family of chord tubes parametrized by $\overline{U}_2$.

Let M be the global bound for z_ε from Proposition 4.2, and let W_2 be a uniform bound for $w_{q,e,b}$ on these tubes. Choose K_0 so that $K_0\delta^2 \ge M + W_2 + 1$. This choice is independent of the central parameter $y_0 \in Z$. The C^2 norms of $|y - y_0|^2$ on $B_\delta^\perp(y_0)$ are also independent of y_0. In addition, the C^2 norms of $\chi_{y_0}(y, s) = (s - \alpha_e(y))(\beta_e(y) - s)$ are uniformly bounded by the C^2 bounds for α_e and β_e. It follows that the constants B_j, C_1, C_2, and A in the proof of Lemma 5.1, as well as the corresponding threshold ε_1, can be chosen independently of $y_0 \in Z$.

The endpoint estimate (5.6) consequently holds with one constant A for every endpoint in K. Since $K \Subset \Gamma_e$ and the outward unit normal is continuous, one has $\kappa_K = \min_{x \in K} |e \cdot \nu(x)| > 0$. Every chord has length at most $\operatorname{diam}(\Omega)$, then Proposition 5.3 gives

$$\sup_{x \in K} |\partial_\nu z_\varepsilon(x) - \partial_\nu w_{q,e,b}(x)| \le \frac{A\operatorname{diam}(\Omega)}{\kappa_K}\varepsilon$$

for all sufficiently small ε. This proves (5.9). □

Proof of Theorem 1.1. Let $\varepsilon = t^{-2}$. Equations (4.1) and (4.3) give the exact identity

$$u^q_{t,e,b} = t\ell_{e,b} + t^{-1} z_{t^{-2}}. \tag{5.10}$$

All terms in (5.10) are differentiable at the boundary. Taking the outward normal derivative gives

$$\Lambda_q(t\ell_{e,b}) = t\, e\cdot\nu + t^{-1}\partial_\nu z_{t^{-2}}. \tag{5.11}$$

For $x \in \Gamma_e$, Proposition 5.3 gives $\partial_\nu z_{t^{-2}}(x) \to \partial_\nu w_{q,e,b}(x)$ as $t \to \infty$. This proves (1.10) and (1.12). If $K \Subset \Gamma_e$, Proposition 5.4 gives $\sup_{x\in K} |\partial_\nu z_{t^{-2}}(x) - \partial_\nu w_{q,e,b}(x)| \le C_K t^{-2}$ for all sufficiently large t. Substitution into (5.11) proves (1.13). □

Corollary 5.5. *For every $e \in \mathbb{S}^{n-1}$,*

$$t\,[\Lambda_q(t\ell_{e,b}) - t\, e\cdot\nu] \longrightarrow \partial_\nu w_{q,e,b} \quad \text{as } t\to\infty$$

locally uniformly on Γ_e.

6. X-ray reconstruction and uniqueness

We now recover the X-ray transform from the coefficient in (1.10).

Lemma 6.1 (Endpoint identity)**.** *For $e \in \mathbb{S}^{n-1}$ and $y \in Y_e$,*

$$I_q(e,b;y) = \int_{\alpha_e(y)}^{\beta_e(y)} q(y+se)(s+b)\,ds. \tag{6.1}$$

Proof. Fix $y \in Y_e$, and write $\alpha = \alpha_e(y)$ and $\beta = \beta_e(y)$. Define the restriction of $w_{q,e,b}$ to the corresponding chord by $W_y(s) = w_{q,e,b}(y+se)$, $\alpha \le s \le \beta$. Since $w_{q,e,b}$ is C^2 up to the two endpoints of the chord, the chain rule gives $W_y'(s) = Dw_{q,e,b}(y+se)\cdot e = \partial_e w_{q,e,b}(y+se)$. The chord equation reads $W_y''(s) = q(y+se)(s+b)$. Integrating it from α to β gives

$$\partial_e w_{q,e,b}(x_e^+(y)) - \partial_e w_{q,e,b}(x_e^-(y)) = \int_{\alpha_e(y)}^{\beta_e(y)} q(y+se)(s+b)\,ds. \tag{6.2}$$

At the two endpoints, (3.9) and Theorem 1.1 give $\partial_e w_{q,e,b}(x_e^\pm(y)) = (e\cdot\nu(x_e^\pm(y)))H_q(e,b;x_e^\pm(y))$. Substituting these identities into (6.2) and using the definition of $I_q(e,b;y)$ proves (6.1). □

Lemma 6.2 (Removal of the affine weight)**.** *For every $e \in \mathbb{S}^{n-1}$ and $y \in Y_e$,*

$$\int_{\alpha_e(y)}^{\beta_e(y)} q(y+se)\,ds = \frac{I_q(e,b;y) + I_q(-e,b;y)}{2b}. \tag{6.3}$$

Proof. Since $P_{-e} = P_e$, the same y parametrizes the chord in both orientations. Moreover, $\alpha_{-e}(y) = -\beta_e(y)$ and $\beta_{-e}(y) = -\alpha_e(y)$. Lemma 6.1 in the direction $-e$ gives

$$I_q(-e,b;y) = \int_{-\beta_e(y)}^{-\alpha_e(y)} q(y-re)(r+b)\,dr.$$

With the change of variables $r = -s$, this becomes

$$I_q(-e,b;y) = \int_{\alpha_e(y)}^{\beta_e(y)} q(y+se)(b-s)\,ds. \tag{6.4}$$

Adding (6.1) and (6.4) proves (6.3). □

Let $Q = q\mathbf{1}_\Omega$ be the zero extension of q to $\mathbb{R}^n$. Its X-ray transform is

$$\mathcal{X}Q(e,y) = \int_{\mathbb{R}} Q(y+se)\,ds, \quad e \in \mathbb{S}^{n-1}, \quad y \in e^\perp.$$

If $y \in Y_e$, then the line $y + \mathbb{R}e$ meets Ω in the chord described by (3.1), and (6.3) gives the value of $\mathcal{X}Q(e,y)$. If $y \notin \overline{Y_e}$, the line does not meet $\overline{\Omega}$, and hence, $\mathcal{X}Q(e,y) = 0$. Finally, if $y \in \partial Y_e$, the line contains no point of the open set Ω. It may meet a flat portion of $\partial\Omega$, but this does not contribute to the integral of the zero extension $Q = q\mathbf{1}_\Omega$. Thus, $\mathcal{X}Q(e,y) = 0$ also for $y \in \partial Y_e$. It follows that (1.14) holds for every $y \in e^\perp$.

Lemma 6.3 (Fourier slice identity)**.** *Let $F \in L^1_c(\mathbb{R}^n)$. For $e \in \mathbb{S}^{n-1}$ and $\xi \in e^\perp$,*

$$\widehat{F}(\xi) = \int_{e^\perp} e^{-\mathrm{i}y\cdot\xi}\,\mathcal{X}F(e,y)\,dy. \tag{6.5}$$

Moreover, if $\mathcal{X}F(e,y) = 0$ for every e and almost every $y \in e^\perp$, then $F = 0$ almost everywhere.

Proof. Write $x = y + se$, where $y \in e^\perp$ and $s \in \mathbb{R}$. This orthogonal change of variables has unit Jacobian, and

$$\int_{e^\perp} |\mathcal{X}F(e,y)|\,dy \le \int_{e^\perp}\int_{\mathbb{R}} |F(y+se)|\,ds\,dy = \|F\|_{L^1(\mathbb{R}^n)}.$$

Thus, Fubini's theorem applies. Since $\xi \cdot e = 0$, one has $(y+se)\cdot\xi = y\cdot\xi$, and (6.5) follows.

If $\mathcal{X}F = 0$, then $\widehat{F}(\xi) = 0$ for every $\xi \neq 0$ by choosing $e \perp \xi$. The continuity of $\widehat{F}$ also gives $\widehat{F}(0) = 0$. The injectivity of the Fourier transform on $L^1(\mathbb{R}^n)$ yields $F = 0$ almost everywhere. ☐

Proposition 6.4 (Recovery of the X-ray transform)**.** *Suppose that $\Lambda_q(t\ell_{e,b})$ is known for every $e \in \mathbb{S}^{n-1}$ and all sufficiently large t. Then the X-ray transform $\mathcal{X}Q(e,y)$ is determined for every $e \in \mathbb{S}^{n-1}$ and $y \in e^\perp$ by (1.14).*

Proof. Theorem 1.1 determines $H_q(e,b;x)$ on Γ_e. If $y \in Y_e$, both endpoints of the corresponding chord belong to Γ_e, and hence, the measurements determine $I_q(e,b;y)$ and $I_q(-e,b;y)$. Lemma 6.2 gives $\mathcal{X}Q(e,y)$. If $y \notin Y_e$, then $\mathcal{X}Q(e,y) = 0$. ☐

Proof of Corollary 1.3. Formula (1.14) follows from Proposition 6.4. Let $\xi \neq 0$ and choose $e_\xi \in \mathbb{S}^{n-1}$ such that $e_\xi \cdot \xi = 0$. Substituting (1.14) into the Fourier slice identity (6.5) gives

$$\widehat{Q}(\xi) = \frac{1}{2b}\int_{e_\xi^\perp} e^{-\mathrm{i}y\cdot\xi}\left[I_q(e_\xi,b;y) + I_q(-e_\xi,b;y)\right]dy.$$

The right-hand side is independent of the choice of e_ξ, since it equals $\widehat{Q}(\xi)$. The value $\widehat{Q}(0)$ is determined by continuity of the Fourier transform of the compactly supported L^1 function Q. Fourier inversion recovers Q in $\mathcal{S}'(\mathbb{R}^n)$. Since $Q = q$ almost everywhere in Ω and q is continuous, this determines q pointwise in Ω. ☐

Proof of Theorem 1.2. Let $Q_j = q_j\mathbf{1}_\Omega$. Fix $e \in \mathbb{S}^{n-1}$ and $x \in \Gamma_e$. The assumed equality of the measurements gives

$$t\left[\Lambda_{q_1}(t\ell_{e,b})(x) - t\,e\cdot\nu(x)\right] = t\left[\Lambda_{q_2}(t\ell_{e,b})(x) - t\,e\cdot\nu(x)\right]$$

for all sufficiently large t. Letting $t \to \infty$ gives $H_{q_1}(e,b;x) = H_{q_2}(e,b;x)$. The same conclusion holds in the direction $-e$. Hence, $I_{q_1}(\pm e,b;y) = I_{q_2}(\pm e,b;y)$ for every $y \in Y_e$, and Lemma 6.2 gives $\mathcal{X}Q_1(e,y) = \mathcal{X}Q_2(e,y)$. Both transforms vanish for $y \notin Y_e$. Thus, $\mathcal{X}(Q_1 - Q_2) = 0$, and Lemma 6.3 gives $Q_1 = Q_2$ almost everywhere. Since q_1 and q_2 are continuous, they agree pointwise in Ω. ☐

Remark 6.5 (A weaker asymptotic hypothesis). *The proof only uses equality of the coefficients H_{q_1} and H_{q_2}. Therefore, it is enough to assume that for every $e \in \mathbb{S}^{n-1}$ and every $x \in \Gamma_e$, there is a sequence $t_k \to \infty$ such that*

$$t_k \left[\Lambda_{q_1}(t_k \ell_{e,b})(x) - \Lambda_{q_2}(t_k \ell_{e,b})(x)\right] \to 0.$$

Remark 6.6 (Reconstruction). *The reconstruction proceeds by extracting H_q from the large amplitude measurements, forming the endpoint quantity I_q, combining the directions e and $-e$, and applying the inverse Fourier transform to the recovered X-ray data.*

Statements and declarations

Data availability statement. No datasets were generated or analyzed during the current study.

Conflict of interest. The author declares no conflict of interest.

Acknowledgments. The author is partially supported by the National Science and Technology Council (NSTC), Taiwan, under the project 113-2115-M-A49-017-MY3. The author also acknowledges financial support from the Alexander von Humboldt Foundation through the Henriette Herz Scouting Programme, hosted by Universität Duisburg-Essen, Germany. The author acknowledge the use of AI tools. All mathematical arguments and proofs in the final manuscript were checked and written by the author.

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