THE CALDERÓN PROBLEM FOR NONLOCAL PARABOLIC OPERATORS

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Abstract. We investigate inverse problems in the determination of leading coefficients for nonlocal parabolic operators, by knowing the corresponding Cauchy data in the exterior space-time domain. The key contribution is that we reduce nonlocal parabolic inverse problems to the corresponding local inverse problems with the lateral boundary Cauchy data. In addition, we derive a new equation and offer a novel proof of the unique continuation property for this new equation. We also build both uniqueness and non-uniqueness results for both nonlocal isotropic and anisotropic parabolic Calderón problems, respectively.

Keywords. Calderón problem, Cauchy data, nonlocal parabolic operators, unique continuation property, global uniqueness, non-uniqueness.
Mathematics Subject Classification (2020): 35B35, 35R11, 35R30

Contents

1. Introduction

In this work, we study a nonlocal analogue of the Calderón problem for nonlocal parabolic operators. The mathematical formulation in this work is given as follows: Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with Lipschitz boundary $\partial\Omega$ for $n \geq 2$, and $T > 0$ be a real number. Consider the parabolic equation

$$\begin{cases} \mathcal{H}v = 0 & \text{in } \Omega_T := (-T,T)\times\Omega, \\ v(t,x) = f(t,x) & \text{on } \Sigma_T := (-T,T)\times\Sigma, \\ v(-T,x) = 0 & \text{for } x\in\Omega, \end{cases} \tag{1.1}$$

where

$$\mathcal{H} := \partial_t - \nabla\cdot(\sigma\nabla) \tag{1.2}$$

denotes the parabolic operators and $\Sigma := \partial\Omega$. Consider the coefficient $\sigma(x) = (\sigma_{ik}(x))_{1\leq i,k\leq n}$ to be a positive definite Lipschitz continuous matrix-valued function satisfying

$$\begin{cases} \sigma_{ik} = \sigma_{ki}, \text{ for all } i,j = 1,2,\ldots,n, \\ c_0|\xi|^2 \leq \sum_{i,k=1}^n \sigma_{ik}(x)\xi_i\xi_k \leq c_0^{-1}|\xi|^2, \text{ for any } x \text{ and } \xi = (\xi_1,\ldots,\xi_n)\in\mathbb{R}^n, \\ |\sigma(x)-\sigma(z)| \leq C_0|x-z|, \text{ for } x,z\in\mathbb{R}^n, \end{cases} \tag{1.3}$$

where $c_0 \in (0,1)$ and $C_0 > 0$ are constants. Meanwhile, we also adapt the notation

$$B_T := (-T,T)\times B,$$

for any $B\subset\mathbb{R}^n$.

It is known the well-posedness of (1.1) always holds whenever f satisfies suitable regularity assumptions (see Section 2). Once the well-posedness holds for certain equations, we can study inverse problems via either the *Cauchy data* or the *Dirichlet-to-Neumann* (DN) map. In this work, we utilize the lateral boundary Cauchy data as our measurements, which is given by

$$\mathcal{C}_{\Sigma_T} \subset L^2(0,T;H^{1/2}(\Sigma_T))\times L^2(0,T;H^{-1/2}(\Sigma_T))$$

with

$$\mathcal{C}_{\Sigma_T} := \left\{ v_f|_{\Sigma_T}, \left.\sum_{i,k=1}^n \sigma_{ik}\partial_{x_k} v_f \nu_i\right|_{\Sigma_T} \right\}, \tag{1.4}$$

where v_f is a solution of (1.1), and $\nu = (\nu_1,\ldots,\nu_n)$ is the unit outer normal on Σ. The classical Calderón problem for the space-time parabolic equation (1.1) is to determine σ by using the information Λ_σ on Σ_T.

As a matter of fact, we are interested in the Calderón problem for nonlocal parabolic equations, which can be formulated as an initial exterior value problem. Throughout this work, we restrict the function $\sigma = (\sigma_{ik})_{1\leq i,k\leq n}$ to be the $n\times n$ identity matrix $\mathbf{I}_n$ outside $\overline{\Omega}$, so that σ still satisfies the condition (1.3) in $\mathbb{R}^n$. Given $s\in(0,1)$, consider

$$\begin{cases} \mathcal{H}^s u = 0 & \text{in } \Omega_T \\ u(t,x) = f(t,x) & \text{in } (\Omega_e)_T, \\ u(t,x) = 0 & \text{for } t\leq -T \text{ and } x\in\mathbb{R}^n, \end{cases} \tag{1.5}$$

where $\mathcal{H}$ is the parabolic operator given by (1.2), and

$$\Omega_e := \mathbb{R}^n \setminus \overline{\Omega}$$

stands for the exterior domain. Due to the definition $\mathcal{H}^s$ (see the rigorous definition of $\mathcal{H}^s$ in Section 2), we cannot only pose the initial condition for (1.5), but we require the whole past time information in order to make the equation (1.5) well-defined. In short, with suitable regularity assumptions for exterior data f, the well-posedness of (1.5) holds (see Section 2).

Furthermore, we can formulate the Calderón problem for nonlocal parabolic equations as follows. Let $W\subset\Omega_e$ be an arbitrarily nonempty open set, and we define the corresponding exterior partial Cauchy data given by

$$\mathcal{C}_{W_T} \subset \left(\widetilde{\mathbf{H}}^s((\Omega_e)_T)\right)\times\left(\widetilde{\mathbf{H}}^s(W_T)\right)^*$$

with

$$\mathcal{C}_{W_T} := \left\{ u|_{(\Omega_e)_T}, \ \mathcal{H}^s u|_{W_T} \right\}, \tag{1.6}$$

where $\widetilde{\mathbf{H}}^s((\Omega_e)_T)$ is a suitable function space which will be introduced in Section 2, and $\left(\widetilde{\mathbf{H}}^s(W_T)\right)^*$ denotes the dual space of $\mathbf{H}^s(W_T)$. Our inverse problem is to ask whether can we determine σ by using the corresponding exterior partial Cauchy data or not. In particular, we propose the following two inverse problems in space-time domain for both local and nonlocal parabolic equations:

- **(1) Local Calderón's problem.** Can one determine the coefficient σ from the local Cauchy data (1.4) of (1.1)?
- **(2) Nonlocal Calderón's problem.** Can one determine the coefficient σ from the nonlocal Cauchy data (1.6) of (1.5)?

In this work, we will answer the above two questions, and describe the relations between nonlocal and local Calderón's problems for both nonlocal and local parabolic equations. We want to show that the above nonlocal Calderón problem **(2)** can be reduced to the local Calderón problem **(1)**, and new unique continuation/determination results are established in this work.

• **Literature review.** The fractional Calderón problem was first proposed and solved in the work [GSU20], where the authors determined the zero order potential for the fractional Schrödinger equation by using the exterior partial Cauchy data. The main tools in the study of fractional inverse problems are based on the *global unique continuation property* and the *Runge approximation property*. Using these methods, many researchers have investigated inverse problems for fractional equations under various settings of mathematical models, such as [BGU21, BKS22, CRZ22, CLL19, CL19, CLR20, CMRU22, FGKU21, GLX17, GRSU20, KLW22, KW22, Gho21, HL19, HL20, LL22a, LL22b, LL19, RS20, LLR20, Lin22, QU22, RZ22a, RZ22b, RS18] and some references therein. In addition, several interesting properties for nonlocal parabolic operators have been studied in [ABDG22, BG18, BGMN21, BG22].

Meanwhile, the Calderón problem to determine the lower order coefficient for a fractional space-time parabolic equation has been considered by [LLR20] for constant coefficients and [BKS22] for variable coefficients. More precisely, given $0 < s < 1$, consider the following fractional parabolic equation

$$\begin{cases} \mathcal{H}^s u + qu = 0 & \text{in } \Omega_T, \\ u = f & \text{in } (\Omega_e)_T, \\ u(t,x) = 0 & \text{for } t \le -T \text{ and } x \in \mathbb{R}^n, \end{cases}$$

where $q = q(t,x) \in L^\infty(\Omega_T)$. It has been shown that one can determine zero order potential q by using the exterior DN map.

Before stating our main results, let us characterize our mathematical setups in the following.

- **(S)** For $n \ge 2$, Let $\Omega \subset \mathbb{R}^n$ be a bounded open set with Lipschitz boundary $\partial\Omega$, and $T > 0$ be a number. Let $\sigma^{(j)} = \left(\sigma^{(j)}_{ik}\right)_{1 \le i,k \le n}$ satisfy (1.3) in $\overline{\Omega}$, and further assume that $\sigma^{(j)}_{ik}(x) = \delta_{ik}$ to be the Kronecker delta, for $x \in \Omega_e$ and $j = 1, 2$. Consider $\mathcal{H}_j$ to be of the form (1.2), for $j = 1, 2$. For $0 < s < 1$, let $W \subset \Omega_e$ be arbitrarily nonempty open subsets, and define the exterior partial Cauchy data by

$$\mathcal{C}^{(j)}_{W_T} = \left\{ u_j|_{(\Omega_e)_T}, \ \left(\mathcal{H}_j\right)^s u_j|_{W_T} \right\},$$

where $u_j \in \mathbf{H}^s(\mathbb{R}^{n+1})$ is the solution of

$$\begin{cases} (\mathcal{H}_j)^s u_j = 0 & \text{in } \Omega_T \\ u_j = f & \text{in } (\Omega_e)_T, \\ u_j(t,x) = 0 & \text{for } t \leq -T \quad \text{and} \quad x \in \mathbb{R}^n, \end{cases}$$

for $j = 1, 2$. Throughout this paper, we always assume the exterior Dirichlet data $f \in C_c^\infty((\Omega_e)_T)$ for the sake of convenience. Moreover, we define the local Cauchy data as usual to be

$$\mathcal{C}^{(j)}_{\Sigma_T} := \left\{ v_j|_{\Sigma_T},\ \sigma_j \partial_\nu v_j|_{\Sigma_T} \right\},$$

where v_j is a solution of

$$\begin{cases} \mathcal{H}_j v_j = 0 & \text{in } \Omega_T, \\ v_j = f & \text{on } \Sigma_T, \\ v_j(-T,x) = 0 & \text{for } x \in \Omega, \end{cases}$$

and we use the following notation

$$\sigma_j\, \partial_\nu v_j|_{\Sigma_T} := \sum_{i,k=1}^n \sigma^{(j)}_{ik} \partial_{x_k} v_j \nu_i \bigg|_{\Sigma_T} \tag{1.7}$$

to denote the Neumann data, for $j = 1, 2$. Here $\nu = (\nu_1, \ldots, \nu_n)$ denotes the unit outer normal on Σ.

Then we are ready to state the first main theorem.

Theorem 1.1. *Adopting all statements and notations given in* (**S**)*, suppose that the exterior partial Cauchy data*

$$\mathcal{C}^{(1)}_{W_T} = \mathcal{C}^{(2)}_{W_T}, \tag{1.8}$$

then the lateral boundary Cauchy data are the same that

$$\mathcal{C}^{(1)}_{\Sigma_T} = \mathcal{C}^{(2)}_{\Sigma_T}.$$

Via the result of Theorem 1.1, we are able to reduce the Calderón problem for nonlocal parabolic equations to the Calderón problem for local parabolic equations. Based on Theorem 1.1, one can immediately obtain the following result.

Corollary 1.1 (Global uniqueness)**.** *Adopting all statements and notations given in* (**S**)*, let σ_j be positive Lipschitz continuous scalar functions defined in $\mathbb{R}^n$ with $\sigma_j = 1$ in Ω_e. Suppose that the nonlocal Cauchy data*

$$\mathcal{C}^{(1)}_{W_T} = \mathcal{C}^{(2)}_{W_T},$$

then

$$\sigma_1 = \sigma_2 \text{ in } \Omega.$$

Next, we are also interested in the case that the leading coefficient is a matrix-valued function. For the local case (i.e. $s = 1$), the non-uniqueness result has been investigated by [GAV12], and we recall the result as follows. Let $\sigma(x) = (\sigma_{ij}(x))_{1 \leq i,j \leq n}$ be a Lipschitz continuous matrix-valued function satisfying (1.3). Let $\mathbf{F} : \overline{\Omega} \to \overline{\Omega}$ be a C^∞ diffeomorphism with $\mathbf{F}|_{\partial\Omega} = \mathbf{Id}$ (the identity map). It is known that if $v(t,x)$ is a solution to

$$\partial_t v - \nabla \cdot (\sigma \nabla v) = 0 \text{ for } (t,x) \in \Omega_T$$

if and only if $\widetilde{v}(t,y) := v(t, \mathbf{F}^{-1}(y))$ is a solution to

$$\mathbf{F}_* 1(y) \partial_t \widetilde{v} - \nabla \cdot (\mathbf{F}_* \sigma \nabla \widetilde{v}) = 0 \text{ for } (t,y) \in \Omega_T,$$

where $\mathbf{F}_*$ denotes the *push-forward* as

$$\begin{cases} \mathbf{F}_*1(y) = \left.\frac{1}{\det(D\mathbf{F})(x)}\right|_{x=\mathbf{F}^{-1}(y)}, \\ \mathbf{F}_*\sigma(y) = \left.\frac{D\mathbf{F}^T(x)\sigma(x)D\mathbf{F}(x)}{\det(D\mathbf{F})(x)}\right|_{x=\mathbf{F}^{-1}(y)}. \end{cases}$$

Here $D\mathbf{F}$ stands for the (matrix) differential of $\mathbf{F}$ and $D\mathbf{F}^T$ is the transpose of $D\mathbf{F}$. Due to the fact that $\mathbf{F}|_{\partial\Omega} = \mathbf{Id}$, one can see that the (lateral) Cauchy data are the same, i.e.,

$$\mathcal{C}^{\sigma}_{\Sigma_T} := \left\{ v|_{\Sigma_T}, \sigma\partial_\nu v|_{\Sigma_T} \right\} = \left\{ \widetilde{v}|_{\Sigma_T}, \mathbf{F}_*\sigma\partial_\nu \widetilde{v}|_{\Sigma_T} \right\} := \mathcal{C}^{\mathbf{F}_*\sigma}_{\Sigma_T},$$

which implies the non-uniqueness property holds for local parabolic operators.

Similar to the local case, our last main result in this paper is to show that non-uniqueness also holds for the nonlocal parabolic case.

Theorem 1.2 (Non-uniqueness)**.** *Adopting all statements and notations given in* ($\mathbf{S}$)*, let* $W = W_1 = W_2 \Subset \Omega_e$ *be an arbitrary nonempty open subset, and* σ_j *be globally Lipschitz continuous matrix-valued function in* $\mathbb{R}^n$ *satisfying* (1.3)*. Suppose that the exterior Cauchy data*

$$\mathcal{C}^{(1)}_{W_T} = \mathcal{C}^{(2)}_{W_T},$$

then there exists a Lipschitz invertible map $\mathbf{F} : \mathbb{R}^n \to \mathbb{R}^n$ *with* $\mathbf{F} : \overline{\Omega} \to \overline{\Omega}$ *and* $\mathbf{F}|_{\Omega_e} = \mathbf{Id}$ *(the identity map) such that*

$$\sigma_2 = \mathbf{F}_*\sigma_1 \text{ in } \Omega,$$

where $\mathbf{F}_*$ *denotes the push-forward of the map* $\mathbf{F}$ *of the form*

$$\mathbf{F}_*\sigma_1(y) = \left.\frac{D\mathbf{F}^T(x)\sigma_1(x)D\mathbf{F}(x)}{\det(D\mathbf{F})(x)}\right|_{x=\mathbf{F}^{-1}(y)}.$$

The paper is organized as follows. In Section 2, we recall the well-posedness for both local and nonlocal parabolic equations, and review several function spaces which are used in this work. We also provide a rigorous definition for the nonlocal parabolic operator $\mathcal{H}^s$ for $0 < s < 1$, which is defined via the evolutive heat semigroup. In Section 3, we derive a new equation, which plays an essential role in the study of this problem. Meanwhile, we show novel Carleman estimates in order to prove the unique continuation property for the new equation. In Section 4, we prove Theorem 1.1. In particular, we demonstrate a fact that any solution of local parabolic equations can be approximated by solutions of some nonlocal parabolic equations. Finally, we show both global uniqueness and non-uniqueness results for nonlocal parabolic equations in Section 5.

2. Preliminaries

In this section, we provide fundamental properties for both local and nonlocal parabolic equations. We first review the definition of weak solutions and well-posedness for the local linear parabolic equation (1.1), which can be found in [Eva98, Chapter 7].

Consider the local parabolic equation (1.1), and consider a function $\widetilde{f}$ defined on $\overline{\Omega_T}$ such that $\widetilde{f}|_{\Sigma_T} = f$. Let $w := u - f$ in $\overline{\Omega_T}$, then w solves

$$\begin{cases} \mathcal{H}w = F & \text{in } \Omega_T := \Omega \times (-T, T), \\ w(t,x) = 0 & \text{on } \Sigma_T := \Sigma \times (-T,T), \\ w(-T,x) = g(x) & \text{for } x \in \Omega, \end{cases} \tag{2.1}$$

where $F=-\mathcal{H}\widetilde{f}$ and $g=-\widetilde{f}(-T,x)$. Define the bilinear form $B[w,\varphi;t]$ by

$$B[w,\varphi;t]:=\int_{\Omega}\sigma(x)\nabla_x w(x,t)\cdot\nabla_x\varphi(x)\,dx,$$

for any $\varphi\in H_0^1(\Omega)$. Then we are able to define the concept of weak solutions of (2.1).

Definition 2.1 (Weak solutions)**.** *A function $w\in L^2(-T,T;H_0^1(\Omega))$ with $\partial_t w\in L^2(-T,T;H^{-1}(\Omega))$ is called a weak solution of the initial boundary value problem (2.1) if w satisfies the following conditions:*

(a) $\int_\Omega \partial_t w(t,x)\varphi(x)\,dx+B[w,\varphi;t]=\int_\Omega F(t,x)\varphi(x)\,dx$, *for any $\varphi\in H_0^1(\Omega)$ and for almost every (a.e.) time $t\in[-T,T]$,*

(b) $w(-T,x)=g(x)$.

With the definition of weak solutions at hand, we have the following result.

Lemma 2.2 (Well-posedness)**.** *Let $\sigma=(\sigma_{ik})_{1\leq i,k\leq n}$ be a Lipschitz continuous matrix-valued function satisfying (1.3). For any $g\in L^2(\Omega)$, $F\in L^2(-T,T;L^2(\Omega))$, the parabolic equation (2.1) admits a unique weak solution $w\in L^2(-T,T;H_0^1(\Omega))$. Moreover,*

$$\begin{aligned}&\max_{0\leq t\leq T}\|w\|_{L^2(\Omega)}+\|w\|_{L^2(-T,T;H_0^1(\Omega))}+\|\partial_t w\|_{L^2(-T,T;H^{-1}(\Omega))}\\&\quad\leq C\left(\|F\|_{L^2(-T,T;L^2(\Omega))}+\|g\|_{L^2(\Omega)}\right),\end{aligned}$$

for some constant $C>0$ depending on Ω, T and σ.

Notice that given arbitrary lateral Dirichlet data $f\in L^2(-T,T;H^{3/2}(\Sigma))$, there exists $\widetilde{f}\in L^2(-T,T;H^2(\Omega))$ such that $\widetilde{f}=f$ on Σ_T in the trace sense.

2.1. **The nonlocal parabolic operator $\mathcal{H}^s$.** The definition for the nonlocal parabolic operator $\mathcal{H}^s$ can be found in [BDLCS21, BKS22]. In the rest of this paper, we adopt the notation

$$\mathcal{L}:=-\nabla\cdot(\sigma\nabla)$$

to denote the second order elliptic operator of divergence form, where $\sigma=(\sigma_{ik})_{1\leq i,k\leq n}$ is a matrix-valued function given via (1.3) in $\overline{\Omega}$, and we define $\sigma_{ik}=\delta_{ik}$ in Ω_e, for $i,k=1,\ldots,n$. With this positive definite matrix-valued function σ defined in the whole $\mathbb{R}^n$, we assume that the parabolic operator $\partial_t+\mathcal{L}$ in $\mathbb{R}\times\mathbb{R}^n$ possesses a globally defined fundamental solution $p(x,z,\tau)$, which satisfies

$$\mathcal{P}_t 1(t,x)=\int_{\mathbb{R}^n}p(x,z,\tau)\,dz=1,\ \text{for every } x\in\mathbb{R}^n \text{ and } \tau>0,$$

where $\mathcal{P}_t$ stands for the heat semigroup.

Consider the following evolution semigroup

$$\mathcal{P}_\tau^{\mathcal{H}}u(t,x):=\int_{\mathbb{R}^n}p(x,z,\tau)u(t-\tau,z)\,dz,\qquad \text{for } u\in\mathcal{S}(\mathbb{R}^{n+1}),\tag{2.2}$$

where $p(x,z,\tau)$ is the heat kernel corresponding to $\partial_\tau+\mathcal{L}$ and $\mathcal{S}(\mathbb{R}^{n+1})$ denotes the Schwarz space. In addition, the heat kernel $p(x,z,\tau)$ satisfies

$$C_1\left(\frac{1}{4\pi\tau}\right)^{n/2}e^{-\frac{c_1|x-z|^2}{4\tau}}\leq p(x,z,\tau)\leq C_2\left(\frac{1}{4\pi\tau}\right)^{n/2}e^{-\frac{c_2|x-z|^2}{4\tau}},\tag{2.3}$$

for $j=1,2$, for some positive constants c_1,c_2,C_1 and C_2. Noticing that $\left\{\mathcal{P}_\tau^{\mathcal{H}}\right\}_{\tau\geq 0}$ is a strongly continuous contractive semigroup such that $\|\mathcal{P}_\tau^{\mathcal{H}}u-u\|_{L^2(\mathbb{R}^{n+1})}=\mathcal{O}(\tau)$. We are able to give the explicit definition of $\mathcal{H}^s$.

Definition 2.3. *Given $s \in (0,1)$ and $u \in \mathcal{S}(\mathbb{R}^{n+1})$, the nonlocal parabolic operator $\mathcal{H}^s$ can be defined via the Balakrishnan formula (see [BKS22]) as*

$$\mathcal{H}^s u(t,x) := -\frac{s}{\Gamma(1-s)} \int_0^\infty \left(\mathcal{P}_\tau^{\mathcal{H}} u(t,x) - u(t,x) \right) \frac{d\tau}{\tau^{1+s}}. \tag{2.4}$$

One may also use the definition from [BDLCS21] to define the nonlocal parabolic operator $\mathcal{H}^s$. In further, by using the Fourier transform with respect to the time-variable $t \in \mathbb{R}$, one can express $\mathcal{H}^s u$ in terms of the Fourier transform. It is known that the heat semigroup $\{P_t\}_{t\geq 0}$ can be written by spectral measures as an identity of gamma functions:

$$\mathcal{P}_t = \int_0^\infty e^{-\lambda t}\, dE_\lambda \quad \text{and} \quad -\frac{s}{\Gamma(1-s)} \int_0^\infty \frac{e^{-(\lambda + \mathbf{i}\rho)t} - 1}{\tau^{1+s}}\, d\tau = (\lambda + \mathbf{i}\rho)^s, \tag{2.5}$$

for $\lambda > 0$ and $\rho \in \mathbb{R}$, where $\mathbf{i} = \sqrt{-1}$. Consider the Fourier transform $\mathcal{F}_t$ of $\mathcal{P}_\tau^{\mathcal{H}} u$ with respect to the t-variable, then we have

$$\mathcal{F}_t \left(\mathcal{P}_\tau^{\mathcal{H}} u \right)(\rho, \xi) = e^{-\mathbf{i}\rho\tau} \mathcal{P}_\tau \left(\mathcal{F}_t u(\rho, \cdot) \right)(\xi),$$

which infers that the Fourier analogue of the definition (2.4)

$$\begin{aligned} \mathcal{F}_t \left(\mathcal{H}^s u \right)(\rho, \cdot) &= -\frac{s}{\Gamma(1-s)} \int_0^\infty \frac{1}{\tau^{1+s}} \int_0^\infty \left(e^{-(\lambda+\mathbf{i}\rho)\tau} - 1 \right) dE_\lambda \left(\mathcal{F}_t u(\rho, \cdot) \right) d\tau \\ &= \int_0^\infty (\lambda + \mathbf{i}\rho)^s\, dE_\lambda \left(\mathcal{F}_t u(\cdot, \rho) \right). \end{aligned}$$

2.2. **Function spaces.** We next turn to define several function spaces. By using previous discussion, for any $u \in \mathcal{S}(\mathbb{R}^{n+1})$, one can write

$$\left\| \mathcal{F}_t \left(\mathcal{H}^s u \right)(\rho, \cdot) \right\|_{L^2(\mathbb{R}^n)} = \int_0^\infty |\lambda + \mathbf{i}\rho|^{2s}\, d\| E_\lambda \left(\mathcal{F}_t u(\rho, \cdot) \right) \|^2,$$

for $\rho \in \mathbb{R}$. With this relation at hand, we can define the space $\mathbf{H}^{2s}(\mathbb{R}^{n+1})$ to be the completion of $\mathcal{S}(\mathbb{R}^{n+1})$ under the norm

$$\|u\|_{\mathbf{H}^{2s}(\mathbb{R}^{n+1})} = \left(\int_{\mathbb{R}} \int_0^\infty \left(1 + |\lambda + \mathbf{i}\rho|^2 \right)^s d\| E_\lambda \left(\mathcal{F}_t u(\rho, \cdot) \right) \|^2\, d\rho \right)^{1/2}. \tag{2.6}$$

Now, let $r \in \mathbb{R}$ and $\mathcal{O} \subset \mathbb{R}^{n+1}$ be an open set, then one can define

$$\begin{aligned} \mathbf{H}^r(\mathbb{R}^{n+1}) =& \Big\{ \text{Completion of } \mathcal{S}(\mathbb{R}^{n+1}) \text{ with respect to the norm :} \\ & \qquad \int_{\mathbb{R}} \int_0^\infty \left(1 + |\lambda + \mathbf{i}\rho|^2 \right)^{r/2} d\| E_\lambda \left(\mathcal{F}_t u(\rho, \cdot) \right) \|^2\, d\rho \Big\}, \\ \mathbf{H}^r(\mathcal{O}) =& \left\{ u|_{\mathcal{O}} : \ u \in \mathbf{H}^r(\mathbb{R}^{n+1}) \right\}, \\ \widetilde{\mathbf{H}}^r(\mathcal{O}) =& \text{closure of } C_c^\infty(\mathcal{O}) \text{ in } \mathbf{H}^r(\mathbb{R}^{n+1}). \end{aligned}$$

Moreover, we define

$$\|u\|_{\mathbf{H}^r(\mathcal{O})} := \inf \left\{ \|v\|_{\mathbf{H}^r(\mathbb{R}^{n+1})} : \ v|_{\mathcal{O}} = u \right\}. \tag{2.7}$$

We also denote the dual spaces

$$\mathbf{H}^{-r}(\mathcal{O}) = \left(\widetilde{\mathbf{H}}^r(\mathcal{O}) \right)^* \quad \text{and} \quad \widetilde{\mathbf{H}}^{-r}(\mathcal{O}) = \left(\mathbf{H}^{-r}(\mathcal{O}) \right)^*.$$

On the other hand, given $a \in \mathbb{R}$, one may consider the parabolic type fractional Sobolev space

$$\mathbb{H}^a(\mathbb{R}^{n+1}) := \left\{ u \in L^2(\mathbb{R}^{n+1}) : \ \left(|\xi|^2 + \mathbf{i}\rho \right)^{a/2} \widehat{u}(\rho, \xi) \in L^2(\mathbb{R}^{n+1}) \right\},$$

where $\widehat{u}(\xi, \rho) = \int_{\mathbb{R}^{n+1}} e^{-\mathbf{i}(t,x)\cdot(\rho,\xi)} u(x,t)\, dt dx$ denotes the Fourier transform of u with respect to both t and x variables.

Meanwhile, the graph norm of $\mathbb{H}^a$-functions is given by

$$\|u\|_{\mathbb{H}^a(\mathbb{R}^{n+1})}^2 := \int_{\mathbb{R}^{n+1}} \left(1 + \left(|\xi|^4 + |\rho|^2\right)^{1/2}\right)^{a/2} \widehat{u}(\rho,\xi)\, d\rho d\xi. \tag{2.8}$$

In addition, one can express the space

$$\mathbb{H}^a(\mathbb{R}^{n+1}) = \mathbb{H}^{a/2,a}(\mathbb{R}^{n+1}),$$

where the exponents $a/2$ and a denote the (fractional) derivatives of time and space, respectively. In particular, as $a = s \in (0,1)$, via the discussion in [BKS22], it is known that

$$\mathbf{H}^s(\mathbb{R}^{n+1}) = \mathbb{H}^s(\mathbb{R}^{n+1}), \text{ for } s \in (0,1), \tag{2.9}$$

and we denote

$$\mathbb{H}^s_E := \left\{ u \in \mathbb{H}^s(\mathbb{R}^{n+1}) : \operatorname{supp}(u) \subset E \right\},$$

for any closed set $E \subset \mathbb{R}^{n+1}$.

2.3. **Initial exterior value problems.** In this section, let us consider the initial exterior value problem of (1.5). In order to study the well-posedness of the initial exterior value problem (1.5), as shown in [BKS22, LLR20], one can consider the adjoint operator $\mathcal{H}^s_*$ of $\mathcal{H}^s$. More precisely, $\mathcal{H}^s_*$ can be defined in terms of the spectral resolution via

$$\mathcal{F}_t\left(\mathcal{H}^s_* u\right)(\rho,\cdot) = \int_0^\infty (\lambda - \mathbf{i}\rho)^s\, dE_\lambda\left(\mathcal{F}_t u(\rho,\cdot)\right),$$

for $u \in \mathcal{S}(\mathbb{R}^{n+1})$. Furthermore, one has that $\mathcal{H}^s_* = (-\partial_t + \mathcal{L})^s$ for $s \in (0,1)$.

Next, for any $f, g \in \mathcal{S}(\mathbb{R}^{n+1})$, one can derive that

$$\begin{aligned} \langle \mathcal{H}^s f, g\rangle_{\mathbb{R}^{n+1}} &= \left\langle \mathcal{H}^{s/2} f, \mathcal{H}^{s/2}_* g\right\rangle_{\mathbb{R}^{n+1}} = \langle f, \mathcal{H}^s_* g\rangle_{\mathbb{R}^{n+1}} \\ &= \int_{\mathbb{R}} \int_0^\infty (\lambda + \mathbf{i}\rho)^s\, d\langle E_\lambda \mathcal{F}_t f, \overline{\mathcal{F}_t g}\rangle(\rho,\cdot)\, d\rho \\ &\leq C\|f\|_{\mathbb{H}^s(\mathbb{R}^{n+1})}\|g\|_{\mathbb{H}^s(\mathbb{R}^{n+1})}, \end{aligned} \tag{2.10}$$

for some constant $C > 0$ independent of f and g. In view of (2.10), one has the mapping property $\mathcal{H}^s : \mathbb{H}^s(\mathbb{R}^{n+1}) \to \mathbb{H}^{-s}(\mathbb{R}^{n+1})$, where $\mathbb{H}^{-s}(\mathbb{R}^{n+1})$ stands for the dual space of $\mathbb{H}^s(\mathbb{R}^{n+1})$. In the rest of this paper, we adopt the notation $\langle \cdot,\cdot\rangle_D$ to denote the natural pairing between a function and its duality, where $D \subset \mathbb{R}^{n+1}$ is an arbitrary set. For instance, given $g \in \widetilde{H}^s(D)$, we can write

$$\langle \mathcal{H}^s f, g\rangle_D = \langle \mathcal{H}^s f, g\rangle_{\widetilde{H}^s(D)^* \times \widetilde{H}^s(D)},$$

where $\widetilde{H}^s(D)^*$ stands for the dual space of $\widetilde{H}^s(D)$ for some set $D \subset \mathbb{R}^{n+1}$.

With these properties of $\mathcal{H}^s$ and $\mathcal{H}^s_*$ at hand, we can define the bilinear map $\mathbf{B}(\cdot,\cdot)$ on $\mathbb{H}^s(\mathbb{R}^{n+1}) \times \mathbb{H}^s(\mathbb{R}^{n+1})$ via

$$\mathbf{B}(f,g) := \left\langle \mathcal{H}^{s/2} f, \mathcal{H}^{s/2}_* g\right\rangle_{\mathbb{R}^{n+1}}.$$

By (2.10), it is known that

$$|\mathbf{B}(f,g)| \leq C\|f\|_{\mathbb{H}^s(\mathbb{R}^{n+1})}\|g\|_{\mathbb{H}^s(\mathbb{R}^{n+1})},$$

for some constant $C > 0$ independent of f and g, which shows the boundedness of the bilinear form $\mathbf{B}(\cdot,\cdot)$. On the other hand, for the coercive, one can consider the cutoff solution akin to [LLR20, BKS22] by considering

$$u_T(t,x) := u(t,x)\chi_{[-T,T]}(t),$$

where $\chi_{[-T,T]}(t) = \begin{cases} 1 & \text{for } t \in [-T,T] \\ 0 & \text{otherwise} \end{cases}$ is the characteristic function. Moreover, $\chi_{[-T,T]}$ is also a multiplier in the fractional Sobolev space $H^a(\mathbb{R}^n)$, for $|a| \leq \frac{1}{2}$ (for example, see [LM12, Theorem 11.4 in Chapter 1]).

More precisely, the coercivity can be seen via

$$\begin{aligned} \left\langle \mathcal{H}^{s/2} f, \mathcal{H}_*^{s/2} f \right\rangle_{\mathbb{R}^{n+1}} &= \int_{\mathbb{R}} \int_0^\infty (\lambda + \mathbf{i}\rho)^s \, d\|E_\lambda \left(\mathcal{F}_t f\right)(\rho, \cdot)\|^2 \, d\rho \\ &= \int_{\mathbb{R}} \int_0^\infty |\lambda + \mathbf{i}\rho|^s \left(\cos(s\theta) + \mathbf{i}\sin(s\theta)\right) d\|E_\lambda \left(\mathcal{F}_t f\right)(\rho, \cdot)\|^2 \, d\rho \\ &= \int_{\mathbb{R}} \int_0^\infty |\lambda + \mathbf{i}\rho|^s \cos(s\theta) \, d\|E_\lambda \left(\mathcal{F}_t f\right)(\rho, \cdot)\|^2 \, d\rho, \end{aligned}$$

where $\tan\theta = \frac{\rho}{\lambda}$ such that $\theta \in (-\pi/2, \pi/2)$ since $\lambda \geq 0$. Here we used that $\sin(s\theta)$ is an odd function so that the third identity holds in the preceding identities. Moreover, due the range of $\theta \in (-\pi/2, \pi/2)$ and $s \in (0,1)$, one can obtain that

$$\cos(s\theta) \geq \cos(s\pi/2) := C_s > 0,$$

so that

$$\left\langle \mathcal{H}^{s/2} f, \mathcal{H}_*^{s/2} f \right\rangle_{\mathbb{R}^{n+1}} \geq C_s \int_{\mathbb{R}} \int_0^\infty |\lambda + \mathbf{i}\rho|^s \, d\|E_\lambda \left(\mathcal{F}_t u\right)(\rho, \cdot)\|^2 \, d\rho \geq C \|f\|^2_{L^2(\mathbb{R}^{n+1})},$$

for some constant $C > 0$ independent of f. This proves the coercivity for the bilinear form $\mathbf{B}(\cdot,\cdot)$.

Moreover, as shown in [BKS22, LLR20], the information of $u(t,x)|_{t>T}$ will not affect the behavior of the solution $u|_{\Omega_T}$, so we can define the weak solution of (1.5) with the cutoff function.

Definition 2.4 (Weak solutions)**.** *Let $\Omega \subset \mathbb{R}^n$ and $T > 0$ be given as before. Given $F \in \left(\mathbb{H}^s_{\overline{\Omega_T}}\right)^*$ and $f \in \mathbf{H}^s((\Omega_e)_T)$. A function $u \in \mathbb{H}^s(\mathbb{R}^{n+1})$ is called a weak solution of*

$$\begin{cases} \mathcal{H}^s u = F & \text{in } \Omega_T \\ u(t,x) = f(t,x) & \text{in } (\Omega_e)_T, \\ u(t,x) = 0 & \text{for } t \leq -T \quad \text{and} \quad x \in \mathbb{R}^n, \end{cases} \tag{2.11}$$

if $v := (u-f)_T \in \mathbb{H}^s_{\overline{\Omega_T}}$ and

$$\mathbf{B}(u,\phi) = \langle F, \phi \rangle_{\mathbb{R}^{n+1}}, \text{ for any } \phi \in \mathbb{H}^s_{\overline{\Omega_T}},$$

or

$$\mathbf{B}(v,\phi) = \langle F - \mathcal{H}^s f, \phi \rangle_{\mathbb{R}^{n+1}}, \text{ for any } \phi \in \mathbb{H}^s_{\overline{\Omega_T}}.$$

Now, the well-posedness of (2.11) can be stated as follows.

Proposition 2.5 (Well-posedness)**.** *Let $\Omega \subset \mathbb{R}^n$ and $T > 0$ be given as before. Given $F \in \left(\mathbb{H}^s_{\overline{\Omega_T}}\right)^*$ and $f \in \mathbf{H}^s((\Omega_e)_T)$. Then there exists a unique $u_T \in \mathbb{H}^s(\mathbb{R}^{n+1})$ with $(u-f)_T \in \mathbb{H}^s_{\overline{\Omega_T}}$ satisfying $\mathcal{H}^s u_T = F$ in Ω_T, and*

$$\|u_T\|_{\mathbb{H}^s(\mathbb{R}^{n+1})} \leq C \left(\|F\|_{(\mathbb{H}^s_{\overline{\Omega_T}})^*} + \|f\|_{\mathbf{H}^s((\Omega_e)_T)} \right),$$

for some constant $C > 0$ independent of u, f and F.

With boundedness and coercivity of $\mathbf{B}(\cdot,\cdot)$ at hand, the proof of Proposition 2.5 is based on the Lax-Milgram theorem for the bilinear map $\mathbf{B}(\cdot,\cdot)$ (a similar trick as in [BKS22, LLR20]), so we skip the detailed proof.

In addition, once we obtain the well-posedness of (1.5), we are able to define the corresponding exterior Cauchy data (or Dirichlet-to-Neumann map) via the bilinear form $\mathbf{B}(\cdot,\cdot)$ (same relation has been investigated in the works [BKS22, LLR20]). In fact, given arbitrarily open sets $W_1, W_2 \subset \Omega_e$, the nonlocal Cauchy data is given by

$$\mathcal{C}_{(W_1)_T,(W_2)_T} := \left\{ u|_{(W_1)_T},\ \mathcal{H}^s u|_{(W_2)_T} \right\} \subset (\mathbf{H}^s((W_1)_T)) \times (\mathbf{H}^s((W_2)_T))^*,$$

where the adjoint space $(\mathbf{H}^s((W_2)_T))^*$ can be verified by $\mathbf{B}(\cdot,\cdot)$.

2.4. **The extension problem for $\mathcal{H}^s$.** We now review the extension problem for $\mathcal{H}^s$, which is a degenerate parabolic equation. Given $u \in \mathbf{H}^s(\mathbb{R}^{n+1})$, let $U = U(t,x,y)$ be the solution of the Dirichlet problem in $\mathbb{R}^{n+2}_+ := \mathbb{R}^{n+1} \times (0,\infty)$

$$\begin{cases} \mathcal{L}_s U = y^{1-2s}\partial_t U - \nabla_{x,y}\cdot\left(y^{1-2s}\widetilde{\sigma}(x)\nabla_{x,y}U\right) = 0 & \text{in } \mathbb{R}^{n+2}_+, \\ U(t,x,0) = u(t,x) & \text{on } \mathbb{R}^{n+1}, \end{cases} \tag{2.12}$$

where

$$\widetilde{\sigma}(x) = \begin{pmatrix} \sigma(x) & 0 \\ 0 & 1 \end{pmatrix}$$

denotes $(n+1)\times(n+1)$ matrix. It is known that (2.12) can be viewed as the parabolic counterpart as the famous Caffarelli-Silvestre extension problem of the fractional Laplacian (see [CS07]) for $\mathcal{H}^s$.

For any open set $\mathcal{D} \subset \mathbb{R}^{n+1}\times(0,\infty)$, we define the weighted Sobolev space

$$H^1(\mathcal{D}; y^{1-2s}dtdxdy) := \left\{ U:\ \|U\|_{H^1(\mathcal{D};y^{1-2s}dtdxdy)} < \infty \right\},$$

where

$$\|U\|^2_{H^1(\mathcal{D};y^{1-2s}dtdxdy)} := \int_{\mathcal{D}} y^{1-2s}\left(|U|^2 + |\nabla_x U|^2 + |\partial_y U|^2\right) dtdxdy.$$

Then we have the following result.

Proposition 2.6 (Extension problem)**.** *Let $s\in(0,1)$ and $u \in \mathbb{H}^s(\mathbb{R}^{n+1})$, then there is a solution $U = U(t,x,y)$ of (2.12) such that*

1. $\lim_{y\to 0+} U(\cdot,\cdot,y) = u(\cdot,\cdot)$ *in* $\mathbb{H}^s(\mathbb{R}^{n+1})$,
2. $\lim_{y\to 0+} \dfrac{2^{1-2s}\Gamma(s)}{\Gamma(1-s)} y^{1-2s}\partial_y U(\cdot,\cdot,y) = \mathcal{H}^s u$ *in* $\mathbb{H}^{-s}(\mathbb{R}^{n+1})$,
3. $\|U\|_{H^1(\mathbb{R}^{n+1}\times(0,M);y^{1-2s}dxdtdy)} \le C_M \|u\|_{\mathbb{H}^s(\mathbb{R}^{n+1})}$, *where $C_M>0$ is a constant depending on M, which is independent of u and U.*

The proof of the above proposition was shown in [BKS22, Theorem 3.1], so we omit the proof.

3. The new equation and its properties

Recall that $\mathcal{H}_j := \partial_t + \mathcal{L}_j$ is a parabolic operator, where $\mathcal{L}_j = -\nabla\cdot(\sigma_j\nabla)$, and σ_j is a matrix-valued function satisfying (1.3) in $\mathbb{R}^n$, such that $\sigma_j = \mathbf{I}_n$ in Ω_e, for $j=1,2$. Due to the definition of $\mathcal{H}_j$, it is not hard to see that $\mathcal{H}_1|_{(\Omega_e)_T} = \mathcal{H}_2|_{(\Omega_e)_T} = (\partial_t - \Delta)|_{(\Omega_e)_T}$ is the heat operator.

3.1. **Basic properties of the new equation.** Given arbitrarily nonempty open sets W_1, W_2 in Ω_e and $0 < s < 1$, let $f \in \mathbf{H}^s((W_1)_T)$, and by utilizing the well-posedness of the initial exterior value problem, there exists a unique solution $u_j \in \mathbb{H}^s(\mathbb{R}^{n+1})$ of

$$\begin{cases} (\mathcal{H}_j)^s u_j = 0 & \text{in } \Omega_T \\ u_j = f & \text{in } (\Omega_e)_T, \\ u_j(t,x) = 0 & \text{for } t \leq -T \quad \text{and} \quad x \in \mathbb{R}^n, \end{cases} \tag{3.1}$$

for $j = 1, 2$. With the condition (1.8) at hand, one can always assume that

$$(\mathcal{H}_1)^s u_1 = (\mathcal{H}_2)^s u_2 \text{ in } (W_2)_T. \tag{3.2}$$

Notice that the global unique continuation property for nonlocal parabolic equation has been studied by [LLR20, Theorem 1.3] and [BKS22, Theorem 1.3], for constant coefficients and variable coefficients nonlocal parabolic operators, respectively. However, even given $u_1 = u_2 = f$ in $(\Omega_e)_T$, with the condition (3.2), one cannot apply the global unique continuation property directly in this work. Thus, we need to analyze the relation (3.2) in a more detailed way.

Let $p_j(x, z, \tau)$ be the heat kernel corresponding to $\partial_\tau + \mathcal{L}_j$ in $\mathbb{R}^n \times \mathbb{R}$ for $j = 1, 2$, which was introduced in Section 2. By using the notation (2.2), we consider the function

$$\mathbf{U}_j(t, \tau, x) := \mathcal{P}_\tau^{\mathcal{H}_j} u_j(t, x) = \int_{\mathbb{R}^n} p_j(x, z, \tau) u_j(t - \tau, z)\, dz, \tag{3.3}$$

for $j = 1, 2$. Unlike the (nonlocal) elliptic case as in [GU21], the function $\mathbf{U}_j$ defined by (3.3) is no longer a solution to any parabolic equation. As a matter of fact, the next lemma plays an essential role in our study.

Lemma 3.1. *Let $u_j \in \mathbb{H}^s(\mathbb{R}^{n+1})$ be the solution of (3.1) and $\mathbf{U}_j$ be the function defined by (3.3), then $\mathbf{U}_j$ solves*

(3.4)
$$\begin{cases} (\partial_t + \partial_\tau)\, \mathbf{U}_j(t, \tau, x) + \mathcal{L}_j \mathbf{U}_j(t, \tau, x) = 0, & \textit{for } (t, \tau, x) \in (-T, T) \times (0, \infty) \times \mathbb{R}^n, \\ \mathbf{U}_j(t, 0, x) = u_j(t, x) & \textit{for } (t, x) \in (-T, T) \times \mathbb{R}^n, \\ \mathbf{U}_j(-T, \tau, x) = 0 & \textit{for } (\tau, x) \in (0, \infty) \times \mathbb{R}^n. \end{cases}$$

Proof. The following arguments hold for $j = 1, 2$. With the definition (3.3) of $\mathbf{U}_j(x, t, \tau)$ at hand, a direct computation yields that

$$\begin{aligned} &(\partial_\tau + \mathcal{L}_j)\, \mathbf{U}_j \\ =&\int_{\mathbb{R}^n} \left[(\partial_\tau + \mathcal{L}_j)\, p_j(x, z, \tau) \right] u_j(t - \tau, z)\, dz \\ &+ \int_{\mathbb{R}^n} p_j(x, z, \tau) \partial_\tau \left(u_j(t - \tau, z) \right) dz \\ =&\int_{\mathbb{R}^n} p_j(x, z, \tau) \partial_\tau \left(u_j(t - \tau, z) \right) dz, \end{aligned} \tag{3.5}$$

for $(t, \tau, x) \in (-T, T) \times (0, \infty) \times \mathbb{R}$, where we used that $p_j(x, z, \tau)$ is the heat kernel of $\partial_\tau + \mathcal{L}_j$, for $j = 1, 2$. By interchanging the derivatives of τ and t, the right hand side of (3.5) can be rewritten as

$$\begin{aligned} (\partial_\tau + \mathcal{L}_j)\, \mathbf{U}_j =& -\partial_t \left(\int_{\mathbb{R}^n} p_j(x, z, \tau) u_j(t - \tau, z)\, dz \right) \\ =& -\partial_t \mathbf{U}_j \quad \text{for } (t, \tau, x) \in (-T, T) \times (0, \infty) \times \mathbb{R}^n, \end{aligned}$$

which shows the first equation of (3.4) holds. Meanwhile, it is not hard to see that

$$
\begin{aligned}
\mathbf{U}_j(t,0,x) &= \lim_{\tau\to 0}\mathcal{P}^{\mathcal{H}_j}_\tau u_j(t,x)\\
&= \lim_{\tau\to 0}\int_{\mathbb{R}^n} p_j(x,z,\tau)u_j(t-\tau,z)\,dz = u_j(t,x), \text{ for } (t,x)\in(-T,T)\times\mathbb{R}^n.
\end{aligned}
$$

Finally, since the parameter $\tau\in(0,\infty)$, one can directly find that

$$
\mathbf{U}_j(-T,\tau,x)=\int_{\mathbb{R}^n} p_j(x,z,\tau)u_j(-T-\tau,z)\,dz=0, \text{ for } (\tau,x)\in(0,\infty)\times\mathbb{R}^n,
$$

where we used $u_j(-T-\tau,x)=0$ for $\tau>0$ and $j=1,2$. This proves the assertion. □

Lemma 3.2. *Adopting all notations in Lemma 3.1, we have*

(3.6)

$$
\max_{-T\le t\le T}\int_0^\infty\int_{\mathbb{R}^n}|\mathbf{U}_j|^2\,dxd\tau+\int_{-T}^T\int_0^\infty\int_{\mathbb{R}^n}|\nabla\mathbf{U}_j|^2\,dxd\tau dt\le C\|u_j\|_{\mathbb{H}^s(\mathbb{R}^{n+1})},
$$

for some constant $C>0$ independent of $\mathbf{U}_j$ and u_j, for $j=1,2$.

Proof. Multiplying (3.4) by $\mathbf{U}_j$, an integration by parts with respect to the x-variable yields that

$$
\frac{\partial_t+\partial_\tau}{2}\left(\int_{\mathbb{R}^n}|\mathbf{U}_j|^2\,dx\right)+\int_{\mathbb{R}^n}\sigma_j\nabla\mathbf{U}_j\cdot\nabla\mathbf{U}_j\,dx=0. \tag{3.7}
$$

We next integrate (3.7) with respect to both t and τ variables, which gives rise to

$$
\begin{aligned}
0 &= \int_0^\infty\int_{\mathbb{R}^n}|\mathbf{U}_j|^2(\widetilde{t},\tau,x)\,dxd\tau-\int_0^\infty\int_{\mathbb{R}^n}|\mathbf{U}_j|^2(-T,\tau,x)\,dxd\tau\\
&\quad+\left[\int_{-T}^{\widetilde{t}}\int_{\mathbb{R}^n}|\mathbf{U}_j|^2(t,\tau,x)\,dxdt\right]_{\tau=0}^{\tau=\infty}\\
&\quad+2\int_{-T}^{\widetilde{t}}\int_0^\infty\int_{\mathbb{R}^n}\sigma_j\nabla\mathbf{U}_j\cdot\nabla\mathbf{U}_j\,dxd\tau dt\\
&=\int_0^\infty\int_{\mathbb{R}^n}|\mathbf{U}_j|^2(\widetilde{t},\tau,x)\,dxd\tau-\int_{-T}^{\widetilde{t}}\int_{\mathbb{R}^n}|u_j|^2(t,x)\,dxdt\\
&\quad+2\int_{-T}^{\widetilde{t}}\int_0^\infty\int_{\mathbb{R}^n}\sigma_j\nabla\mathbf{U}_j\cdot\nabla\mathbf{U}_j\,dxd\tau dt
\end{aligned} \tag{3.8}
$$

for any $\widetilde{t}\in(-T,T)$, where we used

$$
\lim_{\tau\to\infty}\mathbf{U}_j(t,\tau,x)=0
$$

from theheat kernel estimate (2.3). By rewriting (3.8), we have

(3.9)

$$
\int_0^\infty\int_{\mathbb{R}^n}|\mathbf{U}_j|^2(\widetilde{t},\tau,x)\,dxd\tau+2\int_{-T}^{\widetilde{t}}\int_0^\infty\int_{\mathbb{R}^n}\sigma_j\nabla\mathbf{U}_j\cdot\nabla\mathbf{U}_j\,dxd\tau dt\le\|u\|_{\mathbb{H}^s(\mathbb{R}^{n+1})}.
$$

Combined with the ellipticity of σ_j, the inequality (3.6) holds. □

With Lemma 3.2 at hand, we immediately obtain the following result.

Corollary 3.3. *The equation (3.4) possesses a unique solution.*

Proof. If there are two solutions with the same initials $\mathbf{U}_j(t,0,x)$ and $\mathbf{U}_j(-T,\tau,x)$, then the right hand side of (3.6) is zero. Therefore, the solution is unique. □

3.2. **The Carleman estimate.** The proof of main theorem are based on suitable Carleman estimates. In the rest of this section, we will derive the needed Carleman estimates. In fact, our aim is to derive Carleman estimates with the weight

$$\varphi_\beta = \varphi_\beta(x) = \exp(\psi(y)),$$

where $\beta > 0$, $y = -\log|x|$ and $\psi(y) = \beta y + \frac{1}{16}\beta e^{-y/2}$. From [KLW16, Appendix], $\psi(y)$ is a convex function satisfying

$$\begin{cases} \frac{1}{2}\beta \leq \psi' \leq \beta, \\ \operatorname{dist}(2\psi', \mathbb{Z}) + \psi'' \geq \frac{1}{32}. \end{cases} \tag{3.10}$$

Further, h satisfies that for any $C > 0$ there exists $R_0 > 0$ such that

$$\frac{1}{16}|x|\beta \leq (1 + \psi''(-\ln|x|)) \tag{3.11}$$

for all β and $|x| \leq R_0$.

We will modify the arguments of [LW22, Lemma 2.1]. First, let us introduce polar coordinates in $\mathbb{R}^n\backslash\{0\}$ by writing $x = r\omega$, with $r = |x|$, $\omega = (\omega_1, \cdots, \omega_n) \in \mathbb{S}^{n-1}$. Using new coordinate $y = -\log r$, we obtain that

$$\frac{\partial}{\partial x_j} = e^y\left(-\omega_j\partial_y + \Omega_j\right), \quad 1 \leq j \leq n,$$

where Ω_j is a vector field in $\mathbb{S}^{n-1}$. We could check that the vector fields Ω_j satisfy

$$\sum_{j=1}^n \omega_j\Omega_j = 0 \quad \text{and} \quad \sum_{j=1}^n \Omega_j\omega_j = n-1.$$

Since $r \to 0$ if and only if $y \to \infty$, we are interested in values of y near ∞.

It is easy to see that

$$\frac{\partial^2}{\partial x_j \partial x_\ell} = e^{2y}\left(-\omega_j\partial_y - \omega_j + \Omega_j\right)\left(-\omega_\ell\partial_y + \Omega_\ell\right), \quad 1 \leq j, \ell \leq n.$$

then the Laplacian becomes

$$e^{-2y}\Delta = \partial_y^2 - (n-2)\partial_y + \Delta_\omega,$$

where $\Delta_\omega = \sum_{j=1}^n \Omega_j^2$ denotes the Laplace-Beltrami operator on $\mathbb{S}^{n-1}$. Let us recall that the eigenvalues of $-\Delta_\omega$ are $k(k+n-2)$, $k \in \mathbb{N}$, and denote the corresponding eigenspaces are E_k, where E_k is the space of spherical harmonics of degree k. We note that

$$\sum_j \iint |\Omega_j v|^2 dy d\omega = \sum_{k \geq 0} k(k+n-2)\int |v_k|^2 dy, \tag{3.12}$$

where v_k is the projection of v onto E_k. Let

$$\Lambda = \sqrt{\frac{(n-2)^2}{4} - \Delta_\omega},$$

then Λ is an elliptic first-order positive pseudodifferential operator in $L^2(\mathbb{S}^{n-1})$. The eigenvalues of Λ are $k + \frac{n-2}{2}$ and the corresponding eigenspaces are E_k which represents the space of spherical harmonics of degree k. Hence

$$\Lambda = \sum_{k \geq 0}\left(k + \frac{n-2}{2}\right)\pi_k, \tag{3.13}$$

where π_k is the orthogonal projector on E_k. Let

$$L^{\pm} = \partial_y - \frac{n-2}{2} \pm \Lambda,$$

then it follows that

$$e^{-2y}\Delta = L^+L^- = L^-L^+.$$

Denote $L_\beta^{\pm} = \partial_y - \frac{n-2}{2} \pm \Lambda - \psi'(y)$. Then we have that $L_\beta^{\pm} v = e^{\psi(y)} L^{\pm}\left(e^{-\psi(y)}v\right)$ and $e^{-2y}e^{\psi(y)}\Delta\left(e^{-\psi(y)}v\right) = L_\beta^+ L_\beta^- v = L_\beta^- L_\beta^+ v$.

Lemma 3.4. *Let $\chi(t), \zeta(\tau) \in C_0^2(\mathbb{R})$. There are sufficiently large constants β_1, depending on n, such that for all $v(t,\tau,y,\omega) \in C^1(\mathbb{R}^2; C^\infty(\mathbb{R}\times\mathbb{S}^{n-1}))$ and $\beta \geq \beta_1$ with $\beta \in \mathbb{N} + \frac{1}{4}$, we have that*

$$\begin{aligned}
&\int |\chi\zeta(L_\beta^+L_\beta^- v - e^{-2y}\partial_t v - e^{-2y}\partial_\tau v)|^2 + \int |\chi'\zeta e^{-2y} v|^2 + \int |\chi\zeta' e^{-2y} v|^2 \\
&\gtrsim \sum_{j+|\alpha|\leq 1} \beta^{2-2(j+|\alpha|)} \int (1+\psi'')|\partial_y^j \Omega^\alpha (\chi\zeta v)|^2,
\end{aligned} \tag{3.14}$$

where $\operatorname{supp} v(t,\tau,y,\omega) \subset \mathbb{R}^2 \times (0,\infty) \times \mathbb{S}^{n-1}$.

Proof. By diagonalizing $v = \sum_k v_k$ and $L_\beta^+ L_\beta^- v = (\partial_y - \psi' + k)(\partial_y - \psi' - k - n + 2)v_k$, it is enough to prove that

$$\begin{aligned}
&\sum_{j\leq 1} \int (1+\psi'')|(\beta^{2-2j} + k^{2-2j})\partial_y^j(\chi\zeta v)|^2 \\
&\lesssim \int |\chi\zeta(L_\beta^+L_\beta^- v - e^{-2y}\partial_t v - e^{-2y}\partial_\tau v)|^2 + \int |\chi'\zeta e^{-2y} v|^2 + \int |\chi\zeta' e^{-2y} v|^2,
\end{aligned}$$

where we abuse the notation $v = v_k$. By direct computations, we can have that

$$\chi\zeta(\partial_y - \psi' + k)(\partial_y - \psi' - k - n + 2)v = \chi\zeta(\partial_y^2 v - \tilde{b}\partial_y v + \tilde{a} v), \tag{3.15}$$

where

$$\begin{cases} \tilde{a} = (\psi' - k)(\psi' + k + n - 2) - \psi'' \\ \tilde{b} = 2\psi' + n - 2. \end{cases}$$

It is helpful to note that

$$\psi' = \beta - \frac{\beta}{32}e^{-y/2}, \quad \psi'' = \frac{\beta}{64}e^{-y/2}, \quad \psi'' = -\frac{\beta}{128}e^{-y/2}.$$

We obtain from (3.15) that

$$\begin{aligned}
&4|\chi\zeta(L_\beta^+L_\beta^- v - e^{-2y}\partial_t v - e^{-2y}\partial_\tau v)|^2 + 4|\chi'\zeta e^{-2y} v|^2 + 4|\chi\zeta' e^{-2y} v|^2 \\
\geq& |\chi\zeta(\partial_y^2 v - \tilde{b}\partial_y v + \tilde{a} v) - e^{-2y}\partial_t(\chi\zeta v) - e^{-2y}\partial_\tau(\chi\zeta v)|^2 \\
=& |H(v)|^2 - 2\tilde{b}\partial_y(\chi\zeta v)H(v) - 2e^{-2y}\partial_t(\chi\zeta v)H(v) - 2e^{-2y}\partial_\tau(\chi\zeta v)H(v) \\
&+ |\tilde{b}\partial_y(\chi\zeta v) + e^{-2y}\partial_t(\chi\zeta v) + e^{-2y}\partial_\tau(\chi\zeta v)|^2,
\end{aligned} \tag{3.16}$$

where $H(v) := \chi\zeta\left(\partial_y^2 v + \tilde{a} v\right)$. Now we write

$$\begin{cases}
-2\int \tilde{b}\partial_y(\chi\zeta v)H(v) = -2\int \tilde{b}\partial_y(\chi\zeta v)\partial_y^2(\chi\zeta v) - 2\int \tilde{a}\tilde{b}\chi\zeta v\partial_y(\chi\zeta v) \\
-2\int e^{-2y}\partial_t(\chi\zeta v)H(v) = -2\int e^{-2y}\partial_t(\chi\zeta v)\partial_y^2(\chi\zeta v) - 2\int \tilde{a}\chi\zeta v e^{-2y}\partial_t(\chi\zeta v) \\
-2\int e^{-2y}\partial_\tau(\chi\zeta v)H(v) = -2\int e^{-2y}\partial_\tau(\chi\zeta v)\partial_y^2(\chi\zeta v) - 2\int \tilde{a}\chi\zeta v e^{-2y}\partial_\tau(\chi\zeta v).
\end{cases} \tag{3.17}$$

Direct computations imply that

$$\begin{cases} -2\int \tilde{b}\partial_y(\chi\zeta v)\partial_y^2(\chi\zeta v) = 2\int \psi''|\partial_y(\chi\zeta v)|^2, \\ -2\int \tilde{a}\tilde{b}\chi\zeta v\partial_y(\chi\zeta v) = \int \partial_y(\tilde{a}\tilde{b})|\chi\zeta v|^2, \end{cases} \tag{3.18}$$

$$-2\int e^{-2y}\partial_t(\chi\zeta v)\partial_y^2(\chi\zeta v) = -4\int e^{-2y}\partial_t(\chi\zeta v)\partial_y(\chi\zeta v), \tag{3.19}$$

$$-2\int \tilde{a}\chi\zeta v e^{-2y}\partial_t(\chi\zeta v) = 0, \tag{3.20}$$

$$-2\int e^{-2y}\partial_\tau(\chi\zeta v)\partial_y^2(\zeta v) = -4\int e^{-2y}\partial_t(\chi\zeta v)\partial_y(\chi\zeta v), \tag{3.21}$$

$$-2\int \tilde{a}\chi\zeta v e^{-2y}\partial_\tau(\chi\zeta v) = 0. \tag{3.22}$$

Note that here $\tilde{a}$ is independent of t, τ. Combining (3.16) to (3.22) yields

$$\begin{aligned} &\int |\chi\zeta(\partial_y^2 v - \tilde{b}\partial_y v + \tilde{a}v) - e^{-2y}\partial_t(\chi\zeta v) - e^{-2y}\partial_\tau(\chi\zeta v)|^2 \\ \geq &\int \left(|H(v)|^2 + |\tilde{b}\partial_y(\chi\zeta v) + e^{-2y}\partial_t(\chi\zeta v) + e^{-2y}\partial_\tau(\chi\zeta v)|^2\right) \\ &+2\int \psi''|\partial_y(\chi\zeta v)|^2 - 4\int e^{-2y}\partial_t(\chi\zeta v)\partial_y(\chi\zeta v) - 4\int e^{-2y}\partial_\tau(\chi\zeta v)\partial_y(\chi\zeta v) \\ &+\frac{17}{3}\int (\psi')^2\psi''|\chi\zeta v|^2 - 2\int (k^2+nk-2k)\psi''|\chi\zeta v|^2 \end{aligned} \tag{3.23}$$

for $\beta \geq \beta_1$. It is helpful to remark that $\psi'' > 0$.

Likewise, we write

$$\begin{cases} |\tilde{b}\partial_y(\chi\zeta v) + e^{-2y}\partial_t(\chi\zeta v) + e^{-2y}\partial_\tau(\chi\zeta v)|^2 \\ = |(\tilde{b}-2)\partial_y(\chi\zeta v) + e^{-2y}\partial_t(\chi\zeta v) + e^{-2y}\partial_\tau(\chi\zeta v) + 2\partial_y(\chi\zeta v)|^2 \\ = |(2\psi'+n-4)\partial_y(\chi\zeta v) + e^{-2y}\partial_t(\chi\zeta v) + e^{-2y}\partial_\tau(\chi\zeta v)|^2 \\ \quad +4(2\psi'+n-3)|\partial_y(\chi\zeta v)|^2 + 4e^{-2y}\partial_t(\chi\zeta v)\partial_y(\chi\zeta v) + 4e^{-2y}\partial_\tau(\chi\zeta v)\partial_y(\chi\zeta v). \\ \frac{1}{2}|H(v)|^2 = \frac{1}{2}|H(v) + 3\psi''\chi\zeta v|^2 - 3\psi''\chi\zeta v H(v) - \frac{9}{2}(\psi'')^2|\chi\zeta v|^2. \end{cases} \tag{3.24}$$

It is easy to check that

$$\begin{aligned} &-3\int \psi''\chi\zeta v H(v) \\ =& -3\int \psi''\chi^2\zeta^2 v(\partial_y^2 v + \tilde{a}v) \\ \geq& \, 3\int \psi''|\partial_y(\chi\zeta v)|^2 - \frac{10}{3}\int (\psi')^2\psi''|\chi\zeta v|^2 + 3\int (k^2+nk-2k)\psi''|(\chi\zeta v)|^2 \end{aligned} \tag{3.25}$$

for all $\beta \geq \beta_1$. Moreover, via (3.23)-(3.25), we have that for $\beta \geq \beta_1$

$$\begin{aligned} &\int |\chi\zeta(\partial_y^2 v - \tilde{b}\partial_y v + \tilde{a}v) - e^{-2y}\partial_t(\chi\zeta v) - e^{-2y}\partial_\tau(\chi\zeta v)|^2 \\ &\geq 8\int \psi'|\partial_y(\chi\zeta v)|^2 + 2\int (\psi')^2\psi''|\chi\zeta v|^2 + \int k^2\psi''|(\chi\zeta v)|^2 \\ &+ \frac{1}{2}\int |H(v)|^2. \end{aligned} \tag{3.26}$$

Now, we write that

$$\begin{aligned} \frac{1}{4}\int |H(v)|^2 =& \frac{1}{4}\int \left| H(v) - \frac{\beta(\psi'-k)\chi\zeta v}{10|\beta-k|}\right|^2 + \int \frac{\beta(\psi'-k)}{20|\beta-k|}\chi\zeta v H(v) \\ &- \int \frac{\beta^2(\psi'-k)^2}{400|\beta-k|}|\chi\zeta v|^2 \\ \geq& \int \frac{\beta(\psi'-k)}{20|\beta-k|}\chi\zeta v H(v) - \int \frac{\beta^2(\psi'-k)^2}{400|\beta-k|}|\chi\zeta v|^2 \end{aligned}$$

and note

$$\begin{aligned} \int \frac{\beta(\psi'-k)}{20|\beta-k|}\chi^2\zeta^2 v\partial_y^2 v =& -\int \frac{\beta(\psi'-k)}{20|\beta-k|}|\partial_y(\chi\zeta v)|^2 + \int \frac{\beta\psi'''}{40|\beta-k|}|\chi\zeta v|^2 \\ =& -\int \frac{\beta(\beta-k)}{20|\beta-k|}|\partial_y(\chi\zeta v)|^2 + \int \frac{\beta^2 e^{-y/2}}{640|\beta-k|}|\partial_y(\chi\zeta v)|^2 \\ &+ \int \frac{\beta\psi'''}{40|\beta-k|}|\chi\zeta v|^2 \end{aligned}$$

with

$$\begin{aligned} &\int \frac{\beta(\psi'-k)}{20|\beta-k|}\chi^2\zeta^2 v a v \\ =& \int \frac{\beta(\psi'-k)^2(\psi'+k+n-2)}{20|\beta-k|}|\chi\zeta v|^2 + \int \frac{\beta(\psi'-k)\psi''}{20|\beta-k|}|\chi\zeta v|^2. \end{aligned}$$

Combining (3.26), we have that

$$\begin{aligned} &\int \left|\chi\zeta(\partial_y^2 v - \tilde{b}\partial_y v + \tilde{a}v) - e^{-2y}\partial_t(\chi\zeta v) - e^{-2y}\partial_\tau(\chi\zeta v)\right|^2 \\ &\geq 7\int \psi'|\partial_y(\chi\zeta v)|^2 + \frac{3}{2}\int (\psi')^2\psi''|\chi\zeta v|^2 + \frac{1}{2}\int k^2\psi''|(\chi\zeta v)|^2 \\ &+ \int \frac{\beta(\psi'-k)^2(\psi'+k+n-2)}{20|\beta-k|}|\chi\zeta v|^2 + \frac{1}{4}\int |H(v)|^2. \end{aligned} \tag{3.27}$$

Thus, we can get the desire estimate if $\beta \geq \beta_1$. □

By Lemma 3.4, we have our main Carleman estimate.

Lemma 3.5 (Carleman estimate). *Let $\chi(t), \zeta(\tau) \in C_0^2(\mathbb{R})$. There is a sufficiently large number β_2 depending on n such that for all $w(t,\tau,x) \in C^1(\mathbb{R}^2; C^\infty(\mathbb{R}^n))$ and $\beta \geq \beta_2$ with $\beta \in \mathbb{N} + \frac{1}{4}$, we have that*

$$\begin{aligned} &\iiint \varphi_\beta^2(1+\psi'')\chi^2\zeta^2\left(|x|^{-n+2}|\nabla(w)|^2 + \beta^2|x|^{-n}|w|^2\right) dx d\tau dt \\ \lesssim& \iiint \varphi_\beta^2|x|^{-n+4}\chi^2\zeta^2\left(\Delta w - \partial_t w - \partial_\tau w\right)^2 dx d\tau dt \\ &+ \iiint \varphi_\beta^2|x|^{-n+4}|\chi'\zeta w|^2\, dx d\tau dt + \iiint \varphi_\beta^2|x|^{-n+4}|\chi\zeta' w|^2\, dx d\tau dt, \end{aligned} \tag{3.28}$$

where $\operatorname{supp}(w(t,\tau,x)) \subset \mathbb{R}\times(0,\infty)\times\{x : |x| < e\}$.

3.3. **Unique continuation property.** This section is devoted to proving the unique continuation property of solutions to

$$\partial_t u + \partial_\tau u - \Delta u = 0. \tag{3.29}$$

The arguments are motivated by the proof of [Ves03, Theorem 15].

Theorem 3.1. *Let $u \in H^1(\mathbb{R}; H^1((0,\infty)); H^2(\mathbb{R}^n))$ be a nontrivial solution of (3.29). Given $t_0, \tau_0, \hat{\tau} > 0$ such that $t < T$ and $\tau_0 < \hat{\tau}/2$. Assume that $u(t,\tau,x) = 0$ in $\{(t,\tau,x) : \|x\| < R_1, 0 < \tau < \hat{\tau}, |t| < T\}$. Then $u(t,\tau,x) = 0$ in $\{(t,\tau,x) : x \in \mathbb{R}^n, 0 < \tau < \hat{\tau}, |t| < T\}$.*

Proof. Let χ be defined as

$$\chi(t) = \begin{cases} 1, & |t| \le T_2, \\ 0, & |t| \ge T_1, \\ \exp\left(-(\frac{T}{T_1-|t|})^3(\frac{|t|-T_2}{T_1-T_2})^4\right), & T_2 < |t| < T_1, \end{cases} \tag{3.30}$$

where $T_1 = T - \frac{t_0}{2}$, $T_2 = T - t_0$.

Similarly, we define ζ as

$$\zeta(\tau) = \begin{cases} 1, & |\tau - \hat{\tau}/2| \le \tau_2, \\ 0, & |\tau - \hat{\tau}/2| \ge \tau_1, \\ \exp\left(-(\frac{\hat{\tau}}{8(\tau_1-|\hat{\tau}/2|)})^3(\frac{|\hat{\tau}/2|-\tau_2}{\tau_1-\tau_2})^4\right), & \tau_2 < |\tau - \hat{\tau}/2| < \tau_1, \end{cases} \tag{3.31}$$

where $\tau_1 = \frac{\hat{\tau}}{2} - \frac{\tau_0}{2}$, $\tau_2 = \frac{\hat{\tau}}{2} - \tau_0$.

Moreover, we let $\theta(x) \in C_0^\infty(\mathbb{R}^n)$ satisfy $0 \le \theta(x) \le 1$ and

$$\theta(x) = \begin{cases} 1, & |x| < R_2, \\ 0, & |x| > 2R_2, \end{cases}$$

where $R_1 < R_2 < R_0/2$. It is easy to see that for any multiindex α

$$|D^\alpha \theta| = O(|x|^{|\alpha|}) \quad \text{if} \quad R_2 < |x| < 2R_2. \tag{3.32}$$

Applying (3.28) to θu gives

$$\begin{aligned} &\int \varphi_\beta^2 (1+\psi'')\chi^2\zeta^2 |x|^{-n}(|x|^2|\nabla(\theta u)|^2 + \beta^2|\theta u|^2) \\ &\lesssim \int \varphi_\beta^2 |x|^{-n+4}\chi^2\zeta^2(\Delta(\theta u) - \partial_t(\theta u) - \partial_\tau(\theta u))^2 \\ &\quad + \int \varphi_\beta^2 |x|^{-n+4}\theta^2\zeta^2|\chi' u|^2 + \int \varphi_\beta^2 |x|^{-n+4}\chi^2\theta^2|\zeta' u|^2, \end{aligned} \tag{3.33}$$

Here and after, C and $\tilde{C}$ denote general constants whose value may vary from line to line. The dependence of C and $\tilde{C}$ will be specified whenever necessary.

By using (3.32) and (3.29), we obtain that

$$\begin{aligned} &\int_{W_{T,\hat{\tau}}} \varphi_\beta^2 (1+\psi'')\chi^2\zeta^2 |x|^{-n}(|x|^2|\nabla u|^2 + \beta^2|u|^2) \\ &\lesssim \int \varphi_\beta^2 (1+\psi'')\chi^2\zeta^2 (|x|^{-n+2}|\nabla(\theta u)|^2 + \beta^2|x|^{-n}|\theta u|^2) \\ &\lesssim \int \varphi_\beta^2 |x|^{-n+4}\chi^2\zeta^2(\Delta(\theta u) - \partial_t(\theta u) - \partial_\tau(\theta u))^2 \\ &\quad + \int \varphi_\beta^2 |x|^{-n+4}\theta^2\zeta^2|\chi' u|^2 + \int \varphi_\beta^2 |x|^{-n+4}\chi^2\theta^2|\zeta' u|^2 \\ &\le \int_{\tilde{W}} \varphi_\beta^2 |x|^{-n+4}\zeta^2|\chi' u|^2 + \int_{\tilde{W}} \varphi_\beta^2 |x|^{-n+4}\chi^2|\zeta' u|^2 + \int_{\tilde{Y}} \varphi_\beta^2 |x|^{-n}|\tilde{U}|^2, \end{aligned} \tag{3.34}$$

where $W_{T,\hat{\tau}} = \{(t,\tau,x) : |t| < T,\, 0 < \tau < \hat{\tau},\, |x| < R_2\}$, $\tilde{W} = \{(t,\tau,x) : |t| < T,\, 0 < \tau < \hat{\tau},\, |x| < R_2\}$, $\tilde{Y} = \{(t,\tau,x) : |t| < T,\, 0 < \tau < \hat{\tau},\, R_2 < |x| < 2R_2\}$, and $|\tilde{U}(x)|^2 = |x|^4|\chi' u|^2 + |x|^4|\zeta' u|^2 + |x|^{-2}|\nabla u|^2 + |x|^{-4}|u|^2$. Here, the same terms on the right hand side of (3.34) are absorbed by the left hand side of (3.34). With the choices described above, we obtain from (3.34) that

$$\begin{aligned} &\int_{W_{T,\hat{\tau}}} \varphi_\beta^2 (1+\psi'')\chi^2\zeta^2 |x|^{-n}\beta^2|u|^2 \\ \le& \tilde{J}_1 + \tilde{J}_2 + C\int_{\tilde{Y}} \varphi_\beta^2 |x|^{-n}|\tilde{U}|^2, \end{aligned} \tag{3.35}$$

where

$$\begin{aligned} \tilde{J}_1 &= C\int_{\tilde{W}} \varphi_\beta^2 |x|^{-n+4}\zeta^2 \left|\frac{\chi'}{\chi}\right|^2 |\chi u|^2, \\ \tilde{J}_2 &= C\int_{\tilde{W}} \varphi_\beta^2 |x|^{-n+4}\chi^2 \left|\frac{\zeta'}{\zeta}\right|^2 |\zeta u|^2. \end{aligned}$$

Notice that we define $\frac{\chi'}{\chi} = 0$ as $\chi = 0$. The arguments for estimating $\tilde{J}_1$ and $\tilde{J}_2$ are the same, so we only estimate $\tilde{J}_1$.. To do so, one only needs to consider the integral over $\tilde{W}_1 = \{(t,\tau,x) : T_2 < |t| < T_1,\, 0 < \tau < \hat{\tau},\, |x| < R_2\}$. To this end, we consider the following two cases. Firstly,

$$C\left|\frac{\chi'}{\chi}\right|^2 \le \frac{\beta^3}{4}|x|^{-3} \le \frac{(1+\psi'')\beta^2}{4}|x|^{-4}.$$

In this case, $\tilde{J}_1$ will be absorbed by the left hand side. Secondly, we consider

$$C\left|\frac{\chi'}{\chi}\right|^2 \ge \frac{\beta^3}{4}|x|^{-3}.$$

Since

$$\sqrt{C}\left|\frac{\chi'}{\chi}\right| \le C_1 \frac{T^3}{(T_1 - |t|)^4},$$

we can consider a large set

$$C_1 \frac{T^3}{(T_1 - |t|)^4} \ge \left(\frac{\beta^3}{4|x|^3}\right)^{1/2}. \tag{3.36}$$

As a result, taking $\beta \ge \beta_3$ with $\beta_3 = \left(\frac{C_1^2 4^9 T^6}{t_0^8}\right)^{1/3} \ge \left(\frac{C_1^2 4^9 R_2^3 T^6}{t_0^8}\right)^{1/3}$, we can get from (3.36) that

$$T_1 - |t| \le t_0/4$$

which implies that

$$|t| - T_2 \ge \frac{T_1 - T_2}{2}. \tag{3.37}$$

Combining (3.29), (3.36) and (3.37), we get for $(t,x) \in \tilde{W}_1$ that

$$\chi(t) \lesssim \exp\left(-\frac{1}{16}\left(\frac{\beta^3 T^2}{4|x|^3}\right)^{3/8}\right). \tag{3.38}$$

Thus, we have from (3.38) and (3.29) to obtain for $\beta \ge \beta_3$ that

$$\tilde{J}_1 \le C_2 \int_{\tilde{W}} |u|^2, \tag{3.39}$$

where C_2 is a positive constant depending on λ, n, T, t_0.

Combining (3.35) and (3.39), we get that

$$\begin{aligned}
&\beta^2(R_2)^{-n}\varphi_\beta^2(R_2)\int_{W_2}|u|^2 \\
\le&\int_{W_{T,\hat\tau}}\varphi_\beta^2(1+\psi'')\chi^2\zeta^2|x|^{-n}\beta^2|u|^2 \qquad (3.40)\\
\lesssim&\int_{\tilde W}|u|^2+(R_2)^{-n}\varphi_\beta^2(R_2)\int_{\tilde Y}|\tilde U|^2,
\end{aligned}$$

where $W_2=\{(t,\tau,x):\,|t|<T-t_0,\,\tau_0<\tau<\hat\tau-\tau_0,\,|x|<R_2\}$.

Dividing $\beta^2(R_2)^{-n}\varphi_\beta^2(R_2)$ on both sides of (3.39) and if $\beta\ge n$, we have

$$\int_{W_2}|u|^2 \lesssim \beta^{-2}(R_2)^n\varphi_\beta^{-2}(R_2)\int_{\tilde W}|u|^2+\beta^{-2}\int_{\tilde Y}|\tilde U|^2. \qquad (3.41)$$

Let $\beta\to\infty$ on (3.41), we get that $u=0$ on W_2.

Since t_0 and τ_0 are arbitrary, we derive that $u=0$ on $\{(t,\tau,x):|t|<T,\,\tau<\hat\tau,\,|x|<R_2\}$. By the standard argument, we can obtain that $u=0$ on $\{(t,\tau,x):|t|<T,\,0<\tau<\hat\tau,\,x\in\mathbb{R}^n\}$. Finally, since $\hat\tau$ can be arbitrary, we have that $u=0$ on $\{(t,\tau,x):\,|t|<T,\,\tau\in(0,\infty),\,|x|\in\mathbb{R}^n\}$. □

Corollary 3.6. *Given an open set $D\subset\mathbb{R}^n$ and $T>0$. Let $\mathcal{O}\subset D$ be a subset. Let u be a solution of $(\partial_t+\partial_\tau-\Delta)\,u=0$ for $(x,t,\tau)\in D\times(-T,T)\times(0,\infty)$. If $u(x,t,\tau)=0$ in $\mathcal{O}\times(-T,T)\times(0,\infty)$, then $u=0$ in $D\times(-T,T)\times(0,\infty)$.*

4. From the nonlocal to the local

In this section, let us discuss several useful materials and prove Theorem 1.1.

4.1. Auxiliary tools and regularity results.

Lemma 4.1. *Consider the function*

$$\mathbf{V}_j(t,x):=\int_0^\infty \mathbf{U}_j(t,\tau,x)\,d\tau, \qquad (4.1)$$

then $\mathbf{V}_j$ is the solution of

$$\begin{cases}\mathcal{H}_j\mathbf{V}_j=u_j & \text{in } (\mathbb{R}^n)_T\,,\\ \mathbf{V}_j(-T,x)=0 & \text{on } \mathbb{R}^n,\end{cases} \qquad (4.2)$$

where $u_j\in\mathbb{H}^s(\mathbb{R}^{n+1})$ is the solution of (3.1), for $j=1,2$.

Proof. The following arguments hold for $j=1,2$. Integrating (3.4) with respect to the τ-variable, one has

$$\begin{aligned}
(\partial_t+\mathcal{L}_j)\,\mathbf{V}_j(t,x)=&\int_0^\infty(\partial_t+\mathcal{L}_j)\,\mathbf{U}_j(t,\tau,x)\,d\tau\\
=&-\int_0^\infty\partial_\tau\left(\mathbf{U}_j(t,\tau,x)\right)d\tau \qquad (4.3)\\
=&\mathbf{U}_j(t,0,x),\qquad \text{for } (t,x)\in\mathbb{R}^{n+1},
\end{aligned}$$

for $j=1,2$. and plug the above relation into (4.3), so that (4.2) holds. Finally, one can check that

$$\mathbf{V}_j(-T,x)=\int_0^\infty\int_{\mathbb{R}^n}p_j(x,z,\tau)u_j(-T-\tau,z)\,dzd\tau=0,$$

for $t\in(-T,\infty)$, and $\tau\in(0,\infty)$, where we utilized that $u_j(x,t)=0$ for $t\le -T$ and $x\in\mathbb{R}^n$ (or $u_j(z,-T-\tau)=0$ for $\tau\ge0$ and $z\in\mathbb{R}^n$). This proves the assertion. □

From the above derivation, it is not hard to see that

$$\mathbf{V}_j(\zeta, x) = 0, \text{ for all } \zeta \leq -T, \tag{4.4}$$

which will be used in the forthcoming discussion. We next analyze the regularity result of the solution $\mathbf{V}_j$.

Lemma 4.2 (Regularity estimate)**.** *The function* $\mathbf{V}_j$ *given by* (4.1) *satisfies*

$$(\mathcal{H}_j)^{s/2} \mathbf{V}_j \in L^2(-T, T; H^2(\mathbb{R}^n)) \quad \text{ and } \quad \partial_t (\mathcal{H}_j)^{s/2} \mathbf{V}_j \in L^2((\mathbb{R}^n)_T),$$

where $H^a(\mathbb{R}^n)$ *denotes the (fractional) Sobolev space of order* $a \in \mathbb{R}$*, for* $j = 1, 2$*.*

Proof. Since $u_j \in \mathbb{H}^s(\mathbb{R}^{n+1})$ is the solution of (3.1), it is not hard to check that

$$\widetilde{u}_j := (\mathcal{H}_j)^{s/2} u_j \in L^2(\mathbb{R}^{n+1}),$$

for $j = 1, 2$. Here we used the known result that $(\mathcal{H}_j)^{s/2} : \mathbb{H}^s(\mathbb{R}^{n+1}) \to L^2(\mathbb{R}^{n+1})$ by observing the Fourier symbol in the definition (2.8) of the function space $\mathbb{H}^s(\mathbb{R}^{n+1})$.

Let us consider the function

$$\widetilde{\mathbf{V}}_j := (\mathcal{H}_j)^{s/2} \mathbf{V}_j$$

where $\mathbf{V}_j$ satisfies (4.2). Note that

$$\begin{aligned} \widetilde{\mathbf{V}}_j(-T, x) &= \left((\mathcal{H}_j)^{s/2} \mathbf{V}_j \right)(-T, x) \\ &= - \frac{s/2}{\Gamma(1 - s/2)} \int_0^\infty \frac{\mathcal{P}_\tau^{\mathcal{H}_j} \mathbf{V}_j(-T, x) - \mathbf{V}_j(-T, x)}{\tau^{1+s/2}} \, d\tau \\ &= 0, \end{aligned}$$

for $x \in \mathbb{R}^n$, where we used (4.4). Then $\widetilde{\mathbf{V}}_j$ is the solution of

$$\begin{cases} \mathcal{H}_j \widetilde{\mathbf{V}}_j = \widetilde{u}_j & \text{in } \mathbb{R}^n_T, \\ \widetilde{\mathbf{V}}_j(-T, x) = 0 & \text{on } \mathbb{R}^n, \end{cases} \tag{4.5}$$

where we used the interchangeable property between $(\mathcal{H}_j)^{s/2}$ and $\mathcal{H}_j$, for $j = 1, 2$. Meanwhile, we can have the following computations in $-T \leq t \leq T$:

$$\begin{aligned} \int_{\mathbb{R}^n} \widetilde{u}_j^2 \, dx &= \int_{\mathbb{R}^n} \left(\partial_t \widetilde{\mathbf{V}}_j - \mathcal{L}_j \widetilde{\mathbf{V}}_j \right)^2 dx \\ &= \int_{\mathbb{R}^n} \left[\left(\partial_t \widetilde{\mathbf{V}}_j \right)^2 - 2 \mathcal{L}_j \widetilde{\mathbf{V}}_j \cdot \partial_t \widetilde{\mathbf{V}}_j + \left(\mathcal{L}_j \widetilde{\mathbf{V}}_j \right)^2 \right] dx \\ &= \int_{\mathbb{R}^n} \left[\left(\partial_t \widetilde{\mathbf{V}}_j \right)^2 + 2\sigma_j \nabla \widetilde{\mathbf{V}}_j \cdot \partial_t \nabla \widetilde{\mathbf{V}}_j + \left(\mathcal{L}_j \widetilde{\mathbf{V}}_j \right)^2 \right] dx, \end{aligned} \tag{4.6}$$

for $j = 1, 2$, where we have used the integration by parts in the last equality. Moreover, $2\sigma_j \nabla \mathbf{V}_j \cdot \partial_t \nabla \widetilde{\mathbf{V}}_j = \frac{d}{dt} \left(\sigma_j \nabla \widetilde{\mathbf{V}}_j \cdot \nabla \widetilde{\mathbf{V}}_j \right)$, and

$$\int_{-T}^{\zeta} \int_{\mathbb{R}^n} 2\sigma_j \nabla \widetilde{\mathbf{V}}_j \cdot \partial_t \nabla \widetilde{\mathbf{V}}_j \, dx dt = \int_{\mathbb{R}^n} \sigma_j \nabla \widetilde{\mathbf{V}}_j \cdot \nabla \widetilde{\mathbf{V}}_j \, dx \bigg|_{t=-T}^{t=\zeta},$$

for $\zeta \in [-T,T]$ and $j=1,2$. Hence, integrate (4.6) with respect to the time-variable, then one can show that

(4.7)

$$
\begin{aligned}
& c_0 \max_{-T\leq \zeta\leq T}\int_{\mathbb{R}^n} |\nabla \widetilde{\mathbf{V}}_j|^2\, dx + \int_{-T}^T\int_{\mathbb{R}^n} \left(\partial_t \widetilde{\mathbf{V}}_j\right)^2 dxdt + \int_{-T}^T\int_{\mathbb{R}^n} \left(\mathcal{L}_j \widetilde{\mathbf{V}}_j\right)^2 dxdt \\
\leq & \max_{-T\leq \zeta\leq T}\int_{\mathbb{R}^n} \sigma_j \nabla \widetilde{\mathbf{V}}_j\cdot \nabla \widetilde{\mathbf{V}}_j\, dx + \int_{-T}^T\int_{\mathbb{R}^n} \left(\partial_t \widetilde{\mathbf{V}}_j\right)^2 dxdt + \int_{-T}^T\int_{\mathbb{R}^n} \left(\mathcal{L}_j \widetilde{\mathbf{V}}_j\right)^2 dxdt \\
=& 2\int_{-T}^T\int_{\mathbb{R}^n} \widetilde{u}_j^2\, dxdt,
\end{aligned}
$$

for $j=1,2$, where we used the ellipticity condition (1.3) for σ and $\widetilde{u}_j \in L^2(\mathbb{R}^{n+1})$. The inequality (4.7) shows that for a.e. $t\in[-T,T]$,

$$
\nabla \widetilde{\mathbf{V}}_j(t,\cdot), \quad \partial_t \widetilde{\mathbf{V}}_j(t,\cdot) \quad \text{and} \quad \mathcal{L}_j \widetilde{\mathbf{V}}_j(t,\cdot)\in L^2(\mathbb{R}^n). \tag{4.8}
$$

Let us denote $\widetilde{\mathbf{V}}_j' := \frac{\partial}{\partial t}\widetilde{\mathbf{V}}_j$, multiplying (4.5) by $\widetilde{\mathbf{V}}_j'$, then the integration by parts yields that

$$
\int_{\mathbb{R}^n} |\widetilde{\mathbf{V}}_j'|^2\, dx + \frac{1}{2}\frac{\partial}{\partial t}\int_{\mathbb{R}^n} \sigma(x)\nabla \widetilde{\mathbf{V}}_j\cdot\nabla\widetilde{\mathbf{V}}_j\, dx = \int_{\mathbb{R}^n} \widetilde{u}_j \widetilde{\mathbf{V}}_j'\, dx.
$$

Applying the Young's inequality and integrating with respect to the t-variable in the above identity, for any $\varepsilon>0$, we have

$$
\begin{aligned}
&\int_{-T}^T \|\widetilde{\mathbf{V}}_j'\|^2_{L^2(\mathbb{R}^n)}\, dt + \frac{1}{2}\sup_{-T\leq t\leq T}\|\nabla \widetilde{\mathbf{V}}_j\|^2_{L^2(\mathbb{R}^n)} \\
\leq& \varepsilon\int_{-T}^T \|\widetilde{\mathbf{V}}_j'\|^2_{L^2(\mathbb{R}^n)}\, dt + C(\varepsilon)\int_{-T}^T\|\widetilde{u}_j\|^2_{L^2(\mathbb{R}^n)}\, dt.
\end{aligned}
$$

In addition, by choosing $\varepsilon>0$ sufficiently small, one can absorb the first term from the right to the left so that

$$
\int_{-T}^T \|\widetilde{\mathbf{V}}_j'\|^2_{L^2(\mathbb{R}^n)}\, dt + \sup_{-T\leq t\leq T}\|\nabla \widetilde{\mathbf{V}}_j\|^2_{L^2(\mathbb{R}^n)} \leq C\|\widetilde{u}_j\|^2_{L^2(\mathbb{R}^n)},
$$

for some constant $C>0$ independent of $\mathbf{V}_j$ and $\widetilde{u}_j$, for $j=1,2$.

On the other hand, one can write the equation (4.5) in terms of the weak formulation so that

$$
\int_{\mathbb{R}^n} \widetilde{\mathbf{V}}_j'\phi\, dx + \int_{\mathbb{R}^n}\sigma_j\nabla\widetilde{\mathbf{V}}_j\cdot\nabla\phi\, dx = \int_{\mathbb{R}^n}\widetilde{u}_j\phi\, dx,
$$

for any $\phi=\phi(x)\in H^1(\mathbb{R}^n)$. The above identity is equivalent to

$$
\int_{\mathbb{R}^n}\sigma\nabla\widetilde{\mathbf{V}}_j\cdot\nabla\phi\, dx = \int_{\mathbb{R}^n} F\phi\, dx,
$$

for any $\phi=\phi(x)\in H^1(\mathbb{R}^n)$, where $F(t,\cdot):=\widetilde{u}_j(t,\cdot)-\widetilde{\mathbf{V}}_j'(t,\cdot)\in L^2(\mathbb{R}^n)$ for a.e. $t\in[-T,T]$. This shows that $\widetilde{\mathbf{V}}_j(t,\cdot)\in H^1(\mathbb{R}^n)$ is a weak solution of $-\nabla\cdot(\sigma_j\nabla\widetilde{\mathbf{V}}_j)=F$ in $\mathbb{R}^n$ for a.e. $t\in[-T,T]$. Moreover, the classical interior estimate shows that $\widetilde{\mathbf{V}}_j\in H^2_{\mathrm{loc}}(\mathbb{R}^n)$. In particular, there exists a ball B_R containing Ω, such that $\widetilde{\mathbf{V}}_j\in H^2(B_R)$. We also observe that

$$
\Delta\widetilde{\mathbf{V}}_j = \Delta\widetilde{\mathbf{V}}_j\Big|_{\Omega} + \Delta\widetilde{\mathbf{V}}_j\Big|_{\Omega_e} = \Delta\widetilde{\mathbf{V}}_j\Big|_{\Omega} + \nabla\cdot(\sigma\nabla\widetilde{\mathbf{V}}_j)\Big|_{\Omega_e}\in L^2(\mathbb{R}^n),
$$

where we used $\sigma=\mathbf{I}_n$ in Ω_e so that the first term in the right hand side of the above identity vanishes in the set Ω_e, and $\widetilde{\mathbf{V}}_j\in H^2(\Omega)\subset H^2(B_R)$. Therefore, due to the fact that $\|\Delta\widetilde{\mathbf{V}}_j\|^2_{L^2(\mathbb{R}^n)}=\|D^2\widetilde{\mathbf{V}}_j\|^2_{L^2(\mathbb{R}^n)}$, we can show that $\widetilde{\mathbf{V}}_j\in H^2(\mathbb{R}^n)$. In addition, the H^2 estimate is independent of $t \in [-T, T]$. Finally, via the definition of $\widetilde{\mathbf{V}}_j$. This proves the assertion. $\square$

Remark 4.3. *Via Lemma 4.2, it is not hard to check that*

$$(\mathcal{H}_j)^{s/2} \mathbf{V}_j \in \mathbb{H}^{1,2}(\mathbb{R}^{n+1}), \tag{4.9}$$

where one can extend $(\mathcal{H}_j)^s \mathbf{V}_j$ *to be zero for* $(t,x) \in \{t > T\} \times \mathbb{R}^n$[1] *without loss of generality. The preceding lemma will give us desired function spaces for our Cauchy data, which will be utilized in the proof of Theorem 1.1.*

Let us state the unique continuation principle for the nonlocal parabolic equation (1.5), which was shown in [LLR20, Proposition 5.6] by using suitable Carleman estimate.

Proposition 4.4. *Given* $s \in (0,1)$, $n \in \mathbb{N}$ *and arbitrarily nonempty open sets* $W_1, W_2 \subset \Omega_e$. *Let* $u_j \in \mathbb{H}^s(\mathbb{R}^{n+1})$ *with* $\mathrm{supp}(u_j) \subset \left(\overline{\Omega} \cup \overline{W_1}\right)_T$, *for* $j = 1, 2$. *Suppose that*

$$u_1 = u_2 \in C_c^\infty((W_1)_T) \quad \text{and} \quad (\mathcal{H}_1)^s u_1 = (\mathcal{H}_2)^s u_2 \text{ in } (W_2)_T. \tag{4.10}$$

Then $\mathbf{U}_1 = \mathbf{U}_2$ *in* $(-T,T) \times (0,\infty) \times \mathbb{R}^n$, *where* $\mathbf{U}_j$ *is defined by (3.3), for* $j = 1, 2$.

Recalling the nonlocal parabolic operator $\mathcal{H}^s$ is defined via (2.4), with the condition (4.10) at hand, one has that

$$\int_0^\infty \frac{\mathbf{U}_1(t,\tau,x) - \mathbf{U}_2(t,\tau,x)}{\tau^{1+s}} \, d\tau = 0, \quad \text{for } (t,x) \in (W_2)_T, \tag{4.11}$$

where $\mathbf{U}_j$ is given by (3.3), for $j = 1, 2$. By utilizing the condition (4.11), we can prove the proposition.

Proof of Proposition 4.4. Inspired by the proof of [GU21, Proposition 3.1], let us consider bounded open set $\mathcal{O}_j \Subset W_j \subset \Omega_e$ $(j = 1, 2)$ such that $\overline{\mathcal{O}_1} \cap \overline{\mathcal{O}_2} = \emptyset$. Without loss of generality, we may assume that $\mathrm{supp}(u_j) \subset \left(\overline{\Omega} \cup \overline{\mathcal{O}_1}\right)_T$, for $j = 1, 2$. Consider

$$\mathbf{U} := \mathbf{U}_1 - \mathbf{U}_2,$$

then one can have

$$\begin{aligned} \mathbf{U}(t,\tau,x) =& \mathbf{U}_1(t,\tau,x) - \mathbf{U}_2(t,\tau,x) \\ =& \int_{\Omega \cup \mathcal{O}_1} p_1(x,z,\tau) u_1(t-\tau,z) \, dz - \int_{\Omega \cup \mathcal{O}_1} p_2(x,z,\tau) u_2(t-\tau,z) \, dz, \end{aligned} \tag{4.12}$$

where we have utilized the condition $\mathrm{supp}(u_j) \subset \left(\overline{\Omega} \cup \overline{\mathcal{O}_1}\right)_T$ and $p_j(x,z,\tau)$ is the corresponding heat kernel of $\partial_\tau + \mathcal{L}_j$, for $j = 1, 2$. Moreover, it is known that heat kernels $p_j(x,z,\tau)$ satisfies

$$C_1 \left(\frac{1}{4\pi\tau}\right)^{n/2} e^{-\frac{c_1|x-z|^2}{4\tau}} \le p_j(x,z,\tau) \le C_2 \left(\frac{1}{4\pi\tau}\right)^{n/2} e^{-\frac{c_2|x-z|^2}{4\tau}}, \tag{4.13}$$

for $j = 1, 2$, for some positive constants c_1, c_2, C_1 and C_2.

Claim 1. $\dfrac{\mathbf{U}(t,\tau,x)}{\tau^{N+s}} \in L^1(0,\infty)$, *for all* $N \in \mathbb{N}$, *and for any given* $(t,x) \in (\mathcal{O}_2)_T$.

[1]The space $\mathbb{H}^{1,2}(\mathbb{R}^{n+1})$ is introduced in Section 2, and the information in the future time domain will not affect the solution in Ω_T.

In order to show the claim, one can examine whether the integral $\int_0^\infty \left|\frac{\mathbf{U}(t,\tau,x)}{\tau^{N+s}}\right| d\tau$ is finite or not. Similar to the arguments as in the proof of [GU21, Proposition 3.1], given $\delta \in (0,1)$, we can divide the integral

$$\int_0^\infty \left|\frac{\mathbf{U}(t,\tau,x)}{\tau^{N+s}}\right| d\tau = I_\delta + II_\delta,$$

where

$$I_\delta := \int_0^\delta \left|\frac{\mathbf{U}(t,\tau,x)}{\tau^{N+s}}\right| d\tau \quad \text{and} \quad II_\delta := \int_\delta^\infty \left|\frac{\mathbf{U}(t,\tau,x)}{\tau^{N+s}}\right| d\tau.$$

For II_δ, by using the Hölder's inequality, one can see that

$$II_\delta \leq C\left(\|u_1\|_{L^2(\mathbb{R}^{n+1})} + \|u_2\|_{L^2(\mathbb{R}^{n+1})}\right)\left(\int_\delta^\infty \frac{1}{\tau^{2N+2s}}\, d\tau\right)^{\frac{1}{2}} < \infty, \tag{4.14}$$

for some constant $C>0$ independent of $\tau>0$. On the other hand, for I_δ, using the Hölder's inequality and the property of the heat kernel estimate (4.13), we have that

$$\begin{aligned} I_\delta \leq& C\left(\|u_1\|_{L^2(\mathbb{R}^{n+1})} + \|u_2\|_{L^2(\mathbb{R}^{n+1})}\right)\left(\int_0^\delta \int_{\Omega\cup\mathcal{O}_1} \frac{e^{-\frac{|x-z|^2}{\tau}}}{\tau^{2N+2s}}\, dz d\tau\right)^{1/2} \\ \leq& \widetilde{C}\left(\int_0^\delta \frac{e^{-\frac{\kappa^2}{\tau}}}{\tau^{2N+2s}}\, d\tau\right)^{1/2} < \infty, \end{aligned} \tag{4.15}$$

for some constants $C, \widetilde{C} > 0$. Here we have used that Ω and $\mathcal{O}_1$ are bounded sets in $\mathbb{R}^n$, and $x \in \mathcal{O}_2$, such that $|x-z| \geq \kappa > 0$, for some $\kappa > 0$ (recalling that $z \in \Omega \cup \mathcal{O}_2$ and $\overline{\Omega \cup \mathcal{O}_2} \cap \overline{\mathcal{O}_1} = \emptyset$). Combining with (4.14) and (4.15), one can conclude that $\frac{\mathbf{U}(t,\tau,x)}{\tau^{N+s}} \in L^1(0,\infty)$ for all $N \in \mathbb{N}$, and for any given $(x,t) \in (\mathcal{O}_2)_T$.

Claim 2. $\displaystyle\int_0^\infty \frac{\mathbf{U}(t,\tau,x)}{\tau^{N+s}}\, d\tau = 0$*, for all* $N \in \mathbb{N}$*, and for any given* $(t,x) \in (\mathcal{O}_2)_T$*.*

With the equation (3.4) at hand, notice that

$$\mathcal{H}_1|_{(\Omega_e)_T} = \mathcal{H}_2|_{(\Omega_e)_T} = (\partial_t - \Delta)|_{(\Omega_e)_T} := \mathcal{H}|_{(\Omega_e)_T},$$

then one can see that $\mathbf{U}$ is a solution of

$$\begin{cases} \partial_\tau \mathbf{U} = -\mathcal{H}\mathbf{U} & \text{in } (-T,T)\times(0,\infty)\times\Omega_e, \\ \mathbf{U}(t,0,x) = 0 & \text{for } (t,x) \in (\Omega_e)_T, \end{cases} \tag{4.16}$$

where we utilized the condition that $\mathbf{U}(t,0,x) = \mathbf{U}_1(t,0,x) - \mathbf{U}_2(t,0,x) = u_1(t,x) - u_2(t,x) = 0$ in $(\Omega_e)_T$. Via the condition (4.11), the function $\mathbf{U}$ satisfies

$$\int_0^\infty \frac{\mathbf{U}(t,\tau,x)}{\tau^{1+s}}\, d\tau = 0 \text{ in } (\mathcal{O}_2)_T, \tag{4.17}$$

which proves the *Claim 2* for the case $N=1$.

Furthermore, since $u_j(t,x)$ are C^∞-smooth for $(t,x) \in (\Omega_e)_T$, and $p_j(x,z,\tau)$ is also smooth, for $j=1,2$, then we get $\mathbf{U}(t,\tau,x)$ is smooth in the (t,x)-variables, for $(t,x) \in (\Omega_e)_T$. Hence, by applying the heat operator $\mathcal{H}^m$ to the equation (4.16) for any $m \in \mathbb{N}\cup\{0\}$, one has that

$$\begin{cases} (\partial_\tau + \mathcal{H})\,\mathcal{H}^m\mathbf{U} = 0 & \text{in } (-T,T)\times(0,\infty)\times\Omega_e, \\ \mathcal{H}^m\mathbf{U}(t,0,x) = 0 & \text{in } (\Omega_e)_T. \end{cases} \tag{4.18}$$

Meanwhile, similar to the arguments as in the *Claim 1*, we can show that

$$\frac{\mathcal{H}^m \mathbf{U}(t,\tau,x)}{\tau^{1+s}} \in L^1(0,\infty), \text{ for } (t,x)\in (\mathcal{O}_2)_T, \text{ and for any } m\in\mathbb{N}\cup\{0\}.$$

For the case $m=N+1$, $N\in\mathbb{N}$, by acting $\mathcal{H}^{N+1}$ on (4.17), we obtain that

$$\int_0^\infty \frac{\mathcal{H}\left(\mathcal{H}^N \mathbf{U}(t,\tau,x)\right)}{\tau^{1+s}}\,d\tau = \mathcal{H}^{N+1}\left(\int_0^\infty \frac{\mathbf{U}(t,\tau,x)}{\tau^{1+s}}\,d\tau\right) = 0 \text{ in } (\mathcal{O}_2)_T,$$

which is equivalent to

$$\int_0^\infty \frac{\partial_\tau\left(\mathcal{H}^N \mathbf{U}(t,\tau,x)\right)}{\tau^{1+s}}\,d\tau = 0 \text{ in } (\mathcal{O}_2)_T, \tag{4.19}$$

where we used the equation (4.18).

Now, an integration by parts for (4.19) yields that

$$\begin{aligned} 0=&\left[\frac{\mathcal{H}^N \mathbf{U}(t,\tau,x)}{\tau^{1+s}}\right]_{\tau=0}^{\tau=\infty} - \int_0^\infty \mathcal{H}^N \mathbf{U}(t,\tau,x)\partial_\tau\left(\frac{1}{\tau^{1+s}}\right)d\tau \\ =&(1+s)\int_0^\infty \frac{\mathcal{H}^N \mathbf{U}(t,\tau,x)}{\tau^{2+s}}\,d\tau \quad \text{ in } (\mathcal{O}_2)_T, \end{aligned} \tag{4.20}$$

where we used that $\left[\frac{\mathcal{H}^N \mathbf{U}(t,\tau,x)}{\tau^{1+s}}\right]_{\tau=0}^{\tau=\infty}=0$. As a result, by repeating the preceding arguments for $m=N-1,N-2,\dots,1$, with (4.20) at hand, we can conclude that

$$\int_0^\infty \frac{\mathbf{U}(t,\tau,x)}{\tau^{N+s}}\,d\tau = 0 \text{ in } (\mathcal{O}_2)_T, \tag{4.21}$$

and this proves the claim.

With (4.21) at hand, for any $\xi\in\mathbb{R}$, since $\frac{\mathbf{U}(t,\tau,x)}{\tau^{1+s}}\in L^1(0,\infty)$ for $(t,x)\in(\mathcal{O}_2)_T$, then $\int_0^\infty \frac{\mathbf{U}(t,\tau,x)}{\tau^{1+s}}e^{\frac{\mathbf{i}\xi}{\tau}}\,d\tau$, for $(t,x)\in(\mathcal{O}_2)_T$ exists. Moreover, by using (4.21) again, one can obtain that

$$\int_0^\infty \frac{\mathbf{U}(t,\tau,x)}{\tau^{1+s}}e^{\frac{\mathbf{i}\xi}{\tau}}\,d\tau = \int_0^\infty \frac{\mathbf{U}(t,\tau,x)}{\tau^{1+s}}\left(\sum_{k=0}^\infty \frac{1}{k!}\frac{(\mathbf{i}\xi)^k}{\tau^k}\right)d\tau = 0, \tag{4.22}$$

for any $\xi\in\mathbb{R}$ and $(t,x)\in(\mathcal{O}_2)_T$. Hence, by using the change of variable $\tau=\alpha^{-1}$, the integral (4.22) is equivalent to

$$\int_0^\infty \frac{\mathbf{U}(t,\alpha^{-1},x)}{\alpha^{1-s}}e^{\mathbf{i}\xi\alpha}\,d\alpha = 0, \quad \text{ for all } \xi\in\mathbb{R}, \tag{4.23}$$

which can be regarded as the one-dimensional Fourier transform with respect to the α-variable (here we can extend the function $\mathbf{U}(t,\alpha^{-1},x)=0$ for $\alpha<0$). Therefore, (4.23) implies that

$$\mathbf{U}(t,\tau,x)=0, \text{ for } (t,\tau,x)\in(-T,T)\times(0,\infty)\times\mathcal{O}_2 \tag{4.24}$$

as we wish. Finally, by using the (weak) unique continuation of (4.16) (Section 3.3), we can show that $\mathbf{U}=0$ in $(\Omega_e)_T\times(0,\infty)$, which is equivalent to

$$\mathbf{U}_1=\mathbf{U}_2 \text{ in } (-T,T)\times(0,\infty)\times\Omega_e. \tag{4.25}$$

This proves the assertion. □

Moreover, we can show the global unique continuation property for $\mathcal{H}^s$.

Lemma 4.5 (Global unique continuation property)**.** *Let $u\in\mathbb{H}^s(\mathbb{R}^{n+1})$, and suppose that $u=\mathcal{H}^s u=0$ in $\mathcal{O}_T$, where $\mathcal{O}\subset\mathbb{R}^n$ is an arbitrarily open set. Then we have $u\equiv 0$ in $(\mathbb{R}^n)_T$.*

Proof. The proof has been demonstrated by [BKS22, Theorem 1.3], whenever the leading coefficient σ is globally Lipschitz continuous on $\mathbb{R}^n$. The arguments are based on suitable unique continuation properties for degenerate parabolic equations (Proposition 2.6), and we refer readers to the detailed explanation in [BKS22]. □

Remark 4.6. *With the preceding lemma at hand, it is not hard to see that the global unique continuation property also holds for the adjoint parabolic operator $\mathcal{H}_*^s$. In other words, given a nonempty open set $\mathcal{O}\subset\mathbb{R}^n$, if $v=\mathcal{H}_*^s v=0$ in $\mathcal{O}_T$, then $v\equiv 0$ in $(\mathbb{R}^n)_T$ as well. The proof can be achieved by repeating the arguments from Proposition 4.4, where one replaces the parabolic operator $\mathcal{H}=\partial_t+\mathcal{L}$ by $\mathcal{H}_*=-\partial_t+\mathcal{L}$.*

4.2. **Proof of Theorem 1.1.** We divide the proof of Theorem 1.1 into two parts.

Proof of Theorem 1.1–Part 1. Recalling that $\mathbf{V}_j=\mathbf{V}_j(t,x)$ is the function defined by (4.1), via Proposition 4.4, one has

$$\mathbf{V}_1=\int_0^\infty \mathbf{U}_1(t,\tau,x)\,d\tau=\int_0^\infty \mathbf{U}_2(t,\tau,x)\,d\tau=\mathbf{V}_2 \text{ in } (\Omega_e)_T, \tag{4.26}$$

and $\mathbf{V}_j$ satisfies (4.2), for $j=1,2$.

Define the function

$$\mathbf{W}_j:=(\mathcal{H}_j)^s\,\mathbf{V}_j \text{ in } (\mathbb{R}^n)_T\,, \text{ for } j=1,2. \tag{4.27}$$

We observe that

$$\mathcal{P}_\tau^{\mathcal{H}_j}\mathbf{V}_j(-T,x)=\int_{\mathbb{R}^n} p_j(x,z,\tau)\mathbf{V}_j(-T-\tau,x)\,dz=0, \tag{4.28}$$

for $j=1,2$ and for all $\tau\in(0,\infty)$, where we have utilized that (4.4). Combining with the definition of $(\mathcal{H}_j)^s$, (4.2) and (4.28), one has that

$$\begin{aligned}\mathbf{W}_j(-T,x)&=\left((\mathcal{H}_j)^s\,\mathbf{V}_j\right)(-T,x)\\&=-\frac{s}{\Gamma(1-s)}\int_0^\infty \frac{\mathcal{P}_\tau^{\mathcal{H}_j}\mathbf{V}_j(-T,x)-\mathbf{V}_j(-T,x)}{\tau^{1+s}}\,d\tau\\&=0,\end{aligned} \tag{4.29}$$

for $x\in\mathbb{R}^n$. Moreover, by interchanging the local and nonlocal parabolic operators, one has that

$$\mathcal{H}_j\left((\mathcal{H}_j)^s\,\mathbf{V}_j\right)=(\mathcal{H}_j)^s\left(\mathcal{H}_j\mathbf{V}_j\right) \text{ in } (\mathbb{R}^n)_T\,. \tag{4.30}$$

Acting $\mathcal{H}_j$ on (4.2), by using (4.29) and (4.30), one obtains that

$$\begin{cases}\mathcal{H}_j\mathbf{W}_j=(\mathcal{H}_j)^s\,u_j & \text{in } (\mathbb{R}^n)_T\,,\\ \mathbf{W}_j(-T,x)=0 & \text{in } \mathbb{R}^n.\end{cases} \tag{4.31}$$

By applying the condition (4.9), we have

$$\mathbf{W}_j\in\mathbb{H}^{1-s/2,2-s}(\mathbb{R}^{n+1})\,(=\mathbb{H}^{2-s}(\mathbb{R}^{n+1})), \tag{4.32}$$

so that

$$\mathbf{W}_j\in L^2(0,T;H^1(\mathbb{R}^n)) \tag{4.33}$$

due to $s\in(0,1)$, for $j=1,2$. On the other hand, recalling that u_j satisfies (3.1), we have

$$\mathcal{H}_j\mathbf{W}_j=0 \text{ in } \Omega_T, \text{ for } j=1,2.$$

We next claim that

$$\mathbf{W}_1=\mathbf{W}_2 \text{ in } (\Omega_e)_T. \tag{4.34}$$

In order to show (4.34), we consider another function

$$\mathbb{U}_j(t,\tau,x) := \int_{\mathbb{R}^n} p_j(x,z,\tau)\mathbf{V}_j(t-\tau,z)\,dz, \tag{4.35}$$

as in Lemma 3.1, $\mathbb{U}_j$ solves

$$\begin{cases} \partial_\tau \mathbb{U}_j(t,\tau,x) + \mathcal{H}_j \mathbb{U}_j(t,\tau,x) = 0, & \text{for } (t,\tau,x) \in \mathbb{R}\times(0,\infty)\times\mathbb{R}^n, \\ \mathbb{U}_j(t,0,x) = \mathbf{V}_j(t,x) & \text{for } (t,x)\in\mathbb{R}^{n+1}, \end{cases} \tag{4.36}$$

for $j=1,2$. Now, by acting the parabolic operator $\mathcal{H}_j$ on both sides of (4.36), we can get

$$\begin{cases} \partial_\tau \widetilde{\mathbb{U}}_j + \mathcal{H}_j \widetilde{\mathbb{U}}_j = 0 & \text{for } (x,t,\tau)\in\mathbb{R}^{n+1}\times(0,\infty), \\ \widetilde{\mathbb{U}}_j(t,0,x) = u_j(t,x) & \text{for } (t,x)\in\mathbb{R}^{n+1}, \end{cases} \tag{4.37}$$

where $\widetilde{\mathbb{U}}_j := \mathcal{H}_j\mathbb{U}_j$ and we used the equations (4.2) and (4.36) in the second equality of (4.37), for $j=1,2$. More precisely, from (4.2) and (4.36), we have that

$$\widetilde{\mathbb{U}}_j(t,0,x) = \mathcal{H}_j\mathbb{U}_j(t,0,x) = \mathcal{H}_j\mathbf{V}_j(t,x) = u_j(t,x),$$

for $j=1,2$. Furthermore, by (4.35), it is known that

$$\mathbb{U}_j(-T,\tau,x) = \int_{\mathbb{R}^n} p_j(x,z,\tau)\mathbf{V}_j(-T-\tau,z)\,dz = 0, \tag{4.38}$$

for all $\tau\in(0,\infty)$, where we used the condition (4.4). Via the definition of $\widetilde{\mathbb{U}}_j$, (4.37), and (4.38), one has that

$$\widetilde{\mathbb{U}}_j(-T,\tau,x) = \mathcal{H}_j\mathbb{U}_j(-T,\tau,x) = -\partial_\tau\mathbb{U}_j(-T,\tau,x) = 0. \tag{4.39}$$

By Corollary 3.3, combining with the condition (4.39), the equation (4.37) possesses a unique solution. Now, comparing the equations (3.4) and (4.37), they both have the same initial condition

$$\mathbf{U}_j(t,0,x) = u_j(x,t) = \widetilde{\mathbb{U}}_j(t,0,x) \text{ for } (t,x)\in\mathbb{R}^{n+1},$$

which yields that

$$\mathbf{U}_j(t,\tau,x) = \widetilde{\mathbb{U}}_j(t,\tau,x) = \mathcal{H}_j\mathbb{U}_j, \text{ for } (t,\tau,x)\in\mathbb{R}\times(0,\infty)\times\mathbb{R}^n, \tag{4.40}$$

for $j=1,2$. Thus, by using (4.36) and (4.40), we have

$$\partial_\tau\mathbb{U}_1 - \partial_\tau\mathbb{U}_2 = -\mathcal{H}_1\mathbb{U}_1 + \mathcal{H}_2\mathbb{U}_2 = -\mathbf{U}_1 + \mathbf{U}_2 = 0 \text{ in } (-T,T)\times(0,\infty)\times\Omega_e, \tag{4.41}$$

where the last equality holds due to the identity (4.25).

In addition, thanks to the identity (4.41), we know that

$$(\mathbb{U}_1 - \mathbb{U}_2)(t,\tau,x) = (\mathbb{U}_1 - \mathbb{U}_2)(t,0,x), \text{ for } (t,\tau,x)\in(-T,T)\times(0,\infty)\times\Omega_e,$$

which is equivalent to

$$\mathbb{U}_1(t,\tau,x) - \mathbb{U}_1(t,0,x) = \mathbb{U}_2(t,\tau,x) - \mathbb{U}_2(t,0,x), \tag{4.42}$$

for $(t,\tau,x)\in(-T,T)\times(0,\infty)\times\Omega_e$. Consequently, (4.42) implies that

$$\int_0^\infty \frac{\mathbb{U}_1(t,\tau,x)-\mathbb{U}_1(t,0,x)}{\tau^{1+s}}\,d\tau = \int_0^\infty \frac{\mathbb{U}_2(t,\tau,x)-\mathbb{U}_2(t,0,x)}{\tau^{1+s}}\,d\tau,$$

for $(t,x)\in(\Omega_e)_T$. Meanwhile, via the definition (2.4) of nonlocal parabolic operators, the above identity gives rise to

$$(\mathcal{H}_1)^s\mathbf{V}_1 = (\mathcal{H}_2)^s\mathbf{V}_2 \text{ in } (\Omega_e)_T. \tag{4.43}$$

Recall that the function $\mathbf{W}_j$ is defined by (4.27) for $j=1,2$, then (4.43) infers that the claim (4.34) holds.

Hence, combining with (4.33), $\mathbf{W}_j \in L^2(-T,T;H^1(\mathbb{R}^n))$ satisfies

$$\begin{cases} \mathcal{H}_j \mathbf{W}_j = 0 & \text{in } \Omega_T, \\ \{\mathbf{W}_1, \sigma_1 \partial_\nu \mathbf{W}_1\} = \{\mathbf{W}_2, \sigma_2 \partial_\nu \mathbf{W}_2\} & \text{on } \Sigma_T, \end{cases}$$

where $\sigma_j \partial_\nu \mathbf{W}_j$ denotes the Neumann data on Σ_T given by (1.7), for $j = 1, 2$. Moreover, by the trace theorem, it is known that

$$\{\mathbf{W}_1, \sigma_1 \partial_\nu \mathbf{W}_1\} \in L^2(0,T;H^{1/2}(\Sigma)) \times L^2(0,T;H^{-1/2}(\Sigma)).$$

Finally, it remains to show that whether we can vary all possible Dirichlet data so that we are able to reduce nonlocal inverse problems to local ones, and the rest of the arguments will be given in next section. □

We want to show that the lateral boundary Cauchy data

$$\mathcal{C}^{(j)}_{\Sigma_T} = \left\{ \mathbf{V}_j|_{\Sigma_T}, \ \sigma_j \partial_\nu \mathbf{V}_j|_{\Sigma_T} \right\},$$

where $\mathbf{V}_j$ is a solution of the initial-boundary value problem

$$\begin{cases} \mathcal{H}_j \mathbf{V}_j = 0 & \text{in } \Omega_T, \\ \mathbf{V}_j = f & \text{on } \Sigma_T, \\ \mathbf{V}_j(-T,x) = 0 & \text{for } x \in \Omega, \end{cases}$$

for $j = 1, 2$. Our aim is to prove

$$\mathcal{C}^{(1)}_{\Sigma_T} = \mathcal{C}^{(2)}_{\Sigma_T}. \tag{4.44}$$

We first demonstrate a connection between local and nonlocal Calderón problems. Adopting all notations in previous sections, we further define two solution spaces that

$$\mathcal{D}_j(\Omega_T) := \left\{ \mathbf{V}_j|_{\Omega_T} : \begin{cases} \mathcal{H}_j \mathbf{V}_j = 0 & \text{in } \Omega_T, \\ \mathbf{V}_j(-T,x) = 0 & \text{for } x \in \Omega, \end{cases} \right\}$$

and

$$\mathcal{E}_j(\Omega_T) := \left\{ \mathbf{W}_j|_{\Omega_T} : \begin{cases} \mathcal{H}_j \mathbf{W}_j = (\mathcal{H}_j)^s u_j & \text{in } \mathbb{R}^n_T \\ \mathbf{W}_j(-T,x) = 0 & \text{for } x \in \mathbb{R}^n \end{cases} \right\}, \tag{4.45}$$

where $u_j \in \mathbb{H}^s(\mathbb{R}^{n+1})$ is the solution of (3.1), for $j = 1, 2$. Then we are able to show:

Lemma 4.7. *$\mathcal{E}_j(\Omega_T)$ is dense in $\mathcal{D}_j(\Omega_T)$ with respect to $L^2(-T,T;H^1(\Omega))$, for $j = 1, 2$.*

We first assume that Lemma 4.7 holds, then we can complete the proof of Theorem 1.1.

Proof of Theorem 1.1–Part 2. Given any $\mathbf{V}_j \in \mathcal{D}_j$ with $\mathbf{V}_1 = \mathbf{V}_2 = f$ on Σ_T for arbitrary $f \in L^2(-T,T;H^{1/2}(\Sigma))$, then there must exist sequences $\left\{ \mathbf{W}^{(k)}_j \right\}_{k \in \mathbb{N}}$ solves (4.31) such that $\mathbf{W}^{(k)}_j \to \mathbf{V}_j$ in $L^2(-T,T;H^1(\Omega))$ as $k \to \infty$, for $j = 1, 2$. Similar as the Part 1 of the proof of Theorem 1.1, $\mathbf{W}^{(k)}_j$ satisfies

$$\begin{cases} \mathcal{H}_j \mathbf{W}^{(k)}_j = 0 & \text{in } \Omega_T, \\ \left\{ \mathbf{W}^{(k)}_1, \sigma_1 \partial_\nu \mathbf{W}^{(k)}_1 \right\} = \left\{ \mathbf{W}^{(k)}_2, \sigma_2 \partial_\nu \mathbf{W}^{(k)}_2 \right\} & \text{on } \Sigma_T, \end{cases} \tag{4.46}$$

for $k \in \mathbb{N}$ and $j = 1, 2$. By taking the limit $k \to \infty$ of (4.46), we can have

$$\begin{cases} \mathcal{H}_j \mathbf{V}_j = 0 & \text{in } \Omega_T, \\ \{\mathbf{V}_1, \sigma_1 \partial_\nu \mathbf{V}_1\} = \{\mathbf{V}_2, \sigma_2 \partial_\nu \mathbf{V}_2\} & \text{on } \Sigma_T, \end{cases} \tag{4.47}$$

for $j = 1, 2$. Hence, we show that (4.44) holds. This shows Theorem 1.1 holds true. □

Proposition 4.8. *Let $\Omega \subset \mathbb{R}^n$, $0 < s < 1$, and $u \in \mathbb{H}^s(\mathbb{R}^{n+1})$ satisfy*

$$\mathcal{H}^s u = 0 \text{ in } \Omega_T. \tag{4.48}$$

Then for any open set $\mathcal{O} \subset \mathbb{R}^n \setminus \overline{(\Omega \cup \mathcal{O}_1)}$, the set

$$\mathcal{X}((\Omega_e)_T) := \left\{ \mathcal{H}^s u|_{(\Omega_e)_T} : u \text{ is a solution to } (4.48) \right\}$$

is dense in $\mathbf{H}^{-s}((\Omega_e)_T)$.

Proof. By the Hahn-Banach theorem, it suffices to show that given $\varphi \in \widetilde{\mathbf{H}}^s((\Omega_e)_T)$ such that

$$\langle \mathcal{H}^s u, \varphi \rangle_{(\Omega_e)_T} \equiv \langle \mathcal{H}^s u, \varphi \rangle_{\mathbf{H}^{-s}((\Omega_e)_T) \times \widetilde{\mathbf{H}}^s((\Omega_e)_T)} = 0, \tag{4.49}$$

for any solutions u of (4.48), then we want to claim $\varphi \equiv 0$.

Consider the adjoint problem and let $v \in \mathbb{H}^s(\mathbb{R}^{n+1})$ be the solution of

$$\begin{cases} \mathcal{H}^s_* v = 0 & \text{in } \Omega_T, \\ v = \varphi & \text{in } (\Omega_e)_T. \end{cases} \tag{4.50}$$

Now, via equations (4.48), (4.50) and (4.49), one has that

$$\begin{aligned} \langle u, \mathcal{H}^s_* v \rangle_{(\Omega_e)_T} &= \langle u, \mathcal{H}^s_* v \rangle_{(\mathbb{R}^n)_T} - \langle u, \mathcal{H}^s_* v \rangle_{\Omega_T} \\ &= \langle \mathcal{H}^s u, v \rangle_{(\mathbb{R}^n)_T} \\ &= \langle \mathcal{H}^s u, \varphi \rangle_{(\Omega_e)_T} = 0. \end{aligned} \tag{4.51}$$

Thus, since $u|_{(\Omega_e)_T}$ can be arbitrary, by varying the value $u|_{(\Omega_e)_T} \in C^\infty_c((\Omega_e)_T)$ and combining with (4.51), one can conclude that $\mathcal{H}^s_* v = 0$ in $(\Omega_e)_T$. Hence, $v = \mathcal{H}^s_* v = 0$ in $(\Omega_e)_T$, by applying Remark 4.6, one obtains $v \equiv 0$ in $(\mathbb{R}^n)_T$. By using the equation (4.50), we have $\varphi = v = 0$ in $(\Omega_e)_T$ as desired. This proves the assertion. □

We are ready to show Lemma 4.7.

Proof of Lemma 4.7. Consider $F \in \left(L^2(-T, T; H^1(\Omega))\right)^*$, which denotes the dual space of $L^2(-T, T; H^1(\Omega))$. Moreover, by using the definition of dual spaces via the natural dual pairing, it is not hard to see that

$$\left(L^2(-T, T; H^1(\Omega))\right)^* = L^2(-T, T; \widetilde{H}^{-1}(\Omega)),$$

where

$$\widetilde{H}^{-1}(\Omega) := \left\{ h \in H^{-1}(\mathbb{R}^n) : \text{supp}(h) \subset \overline{\Omega} \right\}$$

denotes the dual space of $H^1(\Omega)$. In further, we also denote $H^{-1}(\mathbb{R}^n)$ as the dual space of $H^1(\mathbb{R}^n)$. By the Hahn-Banach theorem, it is equivalent to show that

$$\langle F, \mathbf{W}_j \rangle_{L^2(-T,T;\widetilde{H}^{-1}(\Omega)) \times L^2(-T,T;H^1(\Omega))} = 0, \quad \text{for all } \mathbf{W}_j \in \mathcal{E}_j, \tag{4.52}$$

then it follows

$$\langle F, \mathbf{V}_j \rangle_{L^2(-T,T;\widetilde{H}^{-1}(\Omega)) \times L^2(-T,T;H^1(\Omega))} = 0 \quad \text{for all } \mathbf{V}_j \in \mathcal{D}_j, \tag{4.53}$$

for $j = 1, 2$.

For $0 < s < 1$, recalling that $\mathbf{W}_j$ is the solution of (4.31) for $j = 1, 2$. By varying the exterior data $f|_{(\Omega_e)_T} \in C_c^\infty((\Omega_e)_T)$, Proposition 4.8 implies that the set

$$\mathcal{Y}((\Omega_e)_T) := \left\{ \mathcal{H}_j \mathbf{W}_j|_{(\Omega_e)_T} : \mathbf{W}_j \text{ is a solution of } (4.31) \right\} \tag{4.54}$$

is dense in $L^2(-T, T; \widetilde{H}^{-1}(\mathcal{O}))$. With the condition (4.32) at hand, one can directly see that

$$\mathcal{H}_j \mathbf{W}_j \in \mathbb{H}^{-s}(\mathbb{R}^{n+1}) = \mathbf{H}^{-s}(\mathbb{R}^{n+1}), \tag{4.55}$$

for $j = 1, 2$.

Suppose that there exists a function $F \in L^2(-T, T; \widetilde{H}^{-1}(\Omega))$ satisfies (4.52), then we have

$$\begin{aligned} 0 &= \langle F, \mathbf{W}_j \rangle_{L^2(-T,T;\widetilde{H}^{-1}(\Omega)) \times L^2(-T,T;H^1(\Omega))} \\ &= \langle F, \mathbf{W}_j \rangle_{L^2(-T,T;H^{-1}(\mathbb{R}^n)) \times L^2(-T,T;H^1(\mathbb{R}^n))}, \end{aligned} \tag{4.56}$$

where we have utilized that $F \in L^2(-T, T; \widetilde{H}^{-1}(\Omega))$ with $\mathrm{supp}(F) \subset \overline{\Omega_T}$. In addition, there must exists a unique solution $\mathbf{v}_j \in L^2(-T, T; H^1(\mathbb{R}^n))$ of the backward parabolic equation

$$\begin{cases} (\mathcal{H}_j)_* \mathbf{v}_j = F & \text{in } \mathbb{R}^n \times (-T, T), \\ \mathbf{v}_j(x, T) = 0 & \text{in } \mathbb{R}^n, \end{cases} \tag{4.57}$$

where $(\mathcal{H}_j)_* = -\partial_t + \mathcal{L}_j$ denotes the backward parabolic operator, for $j = 1, 2$. We next analyze the regularity of the solution $\mathbf{v}_j$.

Notice that $F \in L^2(-T, T; H^{-1}(\mathbb{R}^n))$, then we can apply the negative fractional Laplacian $(\mathbf{Id} - \Delta)^{-1/2} = (\mathbf{Id} - \Delta_x)^{-1/2}$ to regularize the source term

$$\widetilde{F} := (\mathbf{Id} - \Delta)^{-1/2} F,$$

such that $\widetilde{F} \in L^2(\mathbb{R}^{n+1})$. One can check that $(\mathbf{Id} - \Delta)^{-1/2}$ and $\mathcal{H}_j$ are interchangeable, and apply the result as in Lemma 4.2 and Remark 4.3, then we can obtain $\widetilde{\mathbf{v}}_j \in \mathbb{H}^{1,2}(\mathbb{R}^{n+1})$, where $\widetilde{\mathbf{v}}_j := (\mathbf{Id} - \Delta)^{-1/2} \mathbf{v}_j$ so that

$$\mathbf{v}_j = (\mathbf{Id} - \Delta)^{1/2} \widetilde{\mathbf{v}}_j \in \mathbb{H}^{1,1}(\mathbb{R}^{n+1}) \subset \mathbf{H}^s(\mathbb{R}^{n+1}). \tag{4.58}$$

Next, via (4.56) and (4.57), an integration by parts yields that

$$\begin{aligned} 0 &= \langle F, \mathbf{W}_j \rangle_{L^2(-T,T;H^{-1}(\mathbb{R}^n)) \times L^2(-T,T;H^1(\mathbb{R}^n))} \\ &= \int_{-T}^{T} \int_{\mathbb{R}^n} (\mathcal{H}_j)_* \mathbf{v}_j \cdot \mathbf{W}_j \, dx dt \\ &= \int_{-T}^{T} \int_{\mathbb{R}^n} \mathbf{v}_j \left(\mathcal{H}_j \mathbf{W}_j \right) dx dt \\ &= \langle \mathbf{v}_j, \mathcal{H}_j \mathbf{W}_j \rangle_{\widetilde{\mathbf{H}}^s((\mathbb{R}^n)_T) \times \mathbf{H}^{-s}((\mathbb{R}^n)_T)}, \end{aligned} \tag{4.59}$$

for $j = 1, 2$, where we have utilized (4.55) and (4.58). Via (4.45), one knows that

$$\mathcal{H}_j \mathbf{W}_j = 0 \text{ in } \Omega_T,$$

for $j = 1, 2$. Combining with the preceding equality, the identity (4.59) implies

$$\langle \mathbf{v}_j, \mathcal{H}_j \mathbf{W}_j \rangle_{\widetilde{\mathbf{H}}^s((\Omega_e)_T) \times \mathbf{H}^{-s}((\Omega_e)_T)} = 0 \tag{4.60}$$

for any $\mathbf{W}_j \in \mathbb{H}^{2-s}(\mathbb{R}^{n+1})$ solving (4.31). Moreover, by utilizing the fact that $\mathcal{Y}((\Omega_e)_T)$ is also dense in $\mathbf{H}^{-s}((\Omega_e)_T)$, where $\mathcal{Y}((\Omega_e)_T)$ is defined by (4.54). Thus, (4.60) implies that $\mathbf{v}_j = 0$ in $(\Omega_e)_T$, for $j = 1, 2$.

On the other hand, recalling that $\mathbf{v}_j$ is a solution of (4.57), in particular, $\mathbf{v}_j$ satisfies $(\mathcal{H}_j)_* \mathbf{v}_j = 0$ in $(\Omega_e)_T$. Combining with $\mathbf{v}_j = 0$ in $\mathcal{O}_T$, the unique continuation property for (backward) parabolic equations yields that $\mathbf{v}_j = 0$ in $(\Omega_e)_T$. To summarize, the function $\mathbf{v}_j \in L^2(-T, T; H_0^1(\Omega))$ solves

$$\begin{cases} (\mathcal{H}_j)_* \mathbf{v}_j = F & \text{in } \Omega_T, \\ \mathbf{v}_j = 0 & \text{on } \Sigma_T, \\ \mathbf{v}_j(x, T) = 0 & \text{in } \Omega, \end{cases}$$

and from the well-posedness for the regularity condition (4.58) of $\mathbf{v}_j$, one has that $\mathbf{v}_j \in L^2(-T, T; H_0^1(\Omega))$, such that $\sigma_j \partial_\nu \mathbf{v}_j \in L^2(-T, T; H^{1/2}(\Sigma))$ is well-defined for $j = 1, 2$. Now, since $\mathbf{v}_j = 0$ in $(\Omega_e)_T$, one must have that $\sigma_j \partial_\nu \mathbf{v}_j = 0$ on Σ_T for $j = 1, 2$. Hence, an integration by parts infers that

$$\begin{aligned} &\langle F, \mathbf{V}_j \rangle_{L^2(-T,T;\widetilde{H}^{-1}(\Omega)) \times L^2(-T,T;H^1(\Omega))} \\ =& \int_{-T}^{T} \int_\Omega (\mathcal{H}_j)_* \mathbf{v}_j \cdot \mathbf{V}_j \, dx dt \\ =& \int_{-T}^{T} \int_\Omega \mathbf{v}_j \cdot \mathcal{H}_j \mathbf{V}_j \, dx dt = 0, \end{aligned}$$

where we used that $\mathbf{v}_j(T, x) = \mathbf{V}_j(-T, x) = 0$ in Ω, and $\mathcal{H}_j \mathbf{V}_j = 0$ in Ω_T, which proves (4.53). This completes the proof. □

5. Global uniqueness and non-uniqueness

In the previous section, we have shown that the inverse problem for nonlocal parabolic equations and be reduced to its local counterparts. We first prove Corollary 1.1.

Proof of Corollary 1.1. With Theorem 1.1 at hand, it is known that the information of the nonlocal Cauchy data can be reduced to its local counterpart. Hence, one has that

$$\left\{ v_1|_{\Sigma_T}, \sigma_1 \partial_\nu v_1|_{\Sigma_T} \right\} = \left\{ v_2|_{\Sigma_T}, \sigma_2 \partial_\nu v_2|_{\Sigma_T} \right\},$$

where $v_j \in L^2(0, T; H^1(\Omega))$ is the weak solution of

$$\begin{cases} \mathcal{H}_j v_j = 0 & \text{in } \Omega_T, \\ v_j = f & \text{on } \Sigma_T, \\ v_j(-T, x) = 0 & \text{for } x \in \Omega, \end{cases}$$

for $j = 1, 2$. Moreover, one can apply the completeness of products of solutions to parabolic equations (for example, see [CK01, Theorem 1.3]), then we are able to conclude that $\sigma_1 = \sigma_2$ in Ω as desired. □

Before proving Theorem 1.2, let us analyze the following changing of variables, which can be regarded as the *transformation optics* in the literature. Given $u \in \mathbb{H}^s(\mathbb{R}^{n+1})$, let $\mathbf{U}(t, \tau, x)$ be a solution of

$$\begin{cases} (\partial_t + \partial_\tau) \mathbf{U} - \nabla \cdot (\sigma \nabla \mathbf{U}) = 0 & \text{in } (-T, T) \times (0, \infty) \times \mathbb{R}^n, \\ \mathbf{U}(t, 0, x) = u(t, x) & \text{for } (t, x) \in (-T, T) \times \mathbb{R}^n, \end{cases} \tag{5.1}$$

where σ is a globally Lipschitz continuous matrix-valued function satisfying (1.3).

Let $\mathbf{F} : \mathbb{R}^n \to \mathbb{R}^n$ be a locally Lipschitz invertible map such that the Jacobians satisfy

$$\det(D\mathbf{F})(x), \quad \det(D\mathbf{F}^{-1})(x) \geq C > 0 \text{ for a.e. } x \in \mathbb{R}^n, \tag{5.2}$$

for some positive constant C. By direct computations, one can derive the following proposition known as the *transformation optics* via the standard change of variables technique (for example, see [KSVW08]).

Proposition 5.1. *$\mathbf{U}(t,\tau,x)$ is a solution of (5.1) if and only if $\widetilde{\mathbf{U}}(t,\tau,y) = \mathbf{U}(t,\tau,\mathbf{F}^{-1}(y))$ is a solution of*

$$\begin{cases} \mathbf{F}_*1(y)\left(\partial_t+\partial_\tau\right)\widetilde{\mathbf{U}} - \nabla\cdot(\mathbf{F}_*\sigma(y)\nabla\widetilde{\mathbf{U}}) = 0 & \text{in } (-T,T)\times(0,\infty)\times\mathbb{R}^n, \\ \widetilde{\mathbf{U}}(t,0,y) = \widetilde{u}(t,y) & \text{for } (t,y)\in(-T,T)\times\mathbb{R}^n, \end{cases} \tag{5.3}$$

where $\widetilde{u} = u(t,\mathbf{F}^{-1}(y))$. Here the coefficients are defined by

$$\begin{cases} \mathbf{F}_*1(y) = \left.\frac{1}{\det(D\mathbf{F})(x)}\right|_{x=\mathbf{F}^{-1}(y)}, \\ \mathbf{F}_*\sigma(y) = \left.\frac{D\mathbf{F}^T(x)\sigma(x)D\mathbf{F}(x)}{\det(D\mathbf{F})(x)}\right|_{x=\mathbf{F}^{-1}(y)}. \end{cases}$$

Proof. The result can be seen via the standard change of variables. More precisely, by expressing (5.1) in terms of the weak formulation, one has that

$$\int_{\mathbb{R}^n}\left(\partial_t+\partial_\tau\right)\mathbf{U}\varphi\,dx + \int_{\mathbb{R}^n}\sigma\nabla_x\mathbf{U}\cdot\nabla_x\varphi\,dx = 0, \tag{5.4}$$

for any $\varphi=\varphi(x)\in H^1(\mathbb{R}^n)$. Via the change of variable $y=\mathbf{F}(x)$ (independent of (t,τ)-variables), it is not hard to see that

$$\int_{\mathbb{R}^n}\sum\sigma_{ij}\frac{\partial\mathbf{U}}{\partial x_i}\frac{\partial\varphi}{\partial x_j}\,dx = \int_{\mathbb{R}^n}\sum\sigma_{ij}\frac{\partial\mathbf{U}}{\partial y_k}\frac{\partial y_k}{\partial x_i}\frac{\partial\varphi}{\partial y_\ell}\frac{\partial y_\ell}{\partial x_j}\det\left(\frac{\partial x}{\partial y}\right)dy,$$

where $\det\left(\frac{\partial x}{\partial y}\right)$ denotes the Jacobian of the change of variable $x=\mathbf{F}^{-1}(y)$. Inserting the above identity into (5.4), the assertion is proven. □

Finally, let us prove Theorem 1.2.

Proof of Theorem 1.2. Let $\Omega\subset\mathbb{R}^n$ be a bounded domain and $W\Subset\Omega_e$ be a nonempty open set. Let $\mathbf{F}:\mathbb{R}^n\to\mathbb{R}^n$ be the Lipschitz invertible map described as before, which satisfy $\mathbf{F}:\overline{\Omega}\to\overline{\Omega}$ and (5.2). We also assume that $\mathbf{F}(x)=x$ in W. Let $u\in\mathbb{H}^s(\mathbb{R}^{n+1})$ be a solution of $\left(\mathcal{H}_\sigma\right)^s u=0$ in Ω_T with $u(-T,x)=0$ for $x\in\mathbb{R}^n$, where the nonlocal parabolic operator $\left(\mathcal{H}_\sigma\right)^s$ can be defined by

$$\left(\mathcal{H}_\sigma\right)^s u(t,x) := -\frac{s}{\Gamma(1-s)}\int_0^\infty\frac{\mathbf{U}(t,x,\tau)-u(t,x)}{\tau^{1+s}}\,d\tau.$$

Here $\mathcal{H}_\sigma := \partial_t - \nabla\cdot(\sigma\nabla)$ and $\mathbf{U}$ satisfies (5.1). Thus, adopting all notations in this section,

$$\left(\mathcal{H}_\sigma\right)^s u = 0 \text{ in } \Omega_T \quad\text{and}\quad u(-T,x)=0 \text{ in } \mathbb{R}^n \tag{5.5}$$

imply that

$$\begin{aligned} 0 &= -\frac{s}{\Gamma(1-s)}\int_0^\infty\frac{\mathbf{U}(t,\tau,x)-u(t,x)}{\tau^{1+s}}\,d\tau \\ &= -\frac{s}{\Gamma(1-s)}\int_0^\infty\frac{\widetilde{\mathbf{U}}(t,\tau,y)-\widetilde{u}(t,y)}{\tau^{1+s}}\,d\tau, \qquad \text{for } (t,x),\ (t,y)\in\Omega_T \end{aligned}$$

where $\widetilde{\mathbf{U}}(t,\tau,y)$ is a solution to (5.3). Meanwhile, $\widetilde{u}(-T,y)=0$, which yields that $\widetilde{u}\in\mathbb{H}^s(\mathbb{R}^{n+1})$ is a solution to

$$\left(\mathcal{H}_{\mathbf{F}_*\sigma}\right)^s\widetilde{u} = 0 \text{ in } \Omega_T \quad\text{and}\quad \widetilde{u}(-T,y)=0 \text{ in } \mathbb{R}^n. \tag{5.6}$$

On the other hand, in viewing of the nonlocal Cauchy data, we can derive that $u(t,\cdot)=\widetilde{u}(t,\cdot)$ in W_T and $\mathbf{U}(t,\tau,\cdot)=\widetilde{\mathbf{U}}(t,\tau,\cdot)$ in W, for $(t,\tau)\in(-T,T)\times(0,\infty)$, then

$$\begin{aligned}(\mathcal{H}_\sigma)^s u(t,x) &= -\frac{s}{\Gamma(1-s)}\int_0^\infty \frac{\mathbf{U}(t,\tau,x)-u(t,x)}{\tau^{1+s}}\,d\tau \\ &= -\frac{s}{\Gamma(1-s)}\int_0^\infty \frac{\widetilde{\mathbf{U}}(t,\tau,y)-\widetilde{u}(t,y)}{\tau^{1+s}}\,d\tau \\ &= (\mathcal{H}_{\mathbf{F}_*\sigma})^s\,\widetilde{u} \quad \text{ in } W_T.\end{aligned}$$

The preceding derivation yields that there are two different matrix-valued functions σ and $\mathbf{F}_*\sigma$ can generate the same exterior Cauchy data

$$\left\{u|_{W_T},\, (\mathcal{H}_\sigma)^s u|_{W_T}\right\} = \left\{\widetilde{u}|_{W_T},\, (\mathcal{H}_{\mathbf{F}_*\sigma})^s \widetilde{u}|_{W_T}\right\},$$

where u and $\widetilde{u}$ are solutions to (5.5) and (5.6), respectively. This completes the proof. □

Acknowledgments. Y.-H. Lin is partly supported by MOST 111-2628-M-A49-002. G. Uhlmann was partially supported by NSF, a Walker Professorship at University of Washington and a Si-Yuan Professorship at Institute for Advanced Study, Hong Kong University of Science and Technology.

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