THE CALDERÓN PROBLEM FOR THE FRACTIONAL WAVE EQUATION: UNIQUENESS AND OPTIMAL STABILITY

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Abstract. We study an inverse problem for the fractional wave equation with a potential by the measurement taking on arbitrary subsets of the exterior in the space-time domain. We are interested in the issues of uniqueness and stability estimate in the determination of the potential by the exterior Dirichlet-to-Neumann map. The main tools are the qualitative and quantitative unique continuation properties for the fractional Laplacian. For the stability, we also prove that the log type stability estimate is optimal. The log type estimate shows the striking difference between the inverse problems for the fractional and classical wave equations in the stability issue. The results hold for any spatial dimension $n \in \mathbb{N}$.

Keywords. Calderón problem, peridynamic, fractional Laplacian, nonlocal, fractional wave equation, strong uniqueness, Runge approximation, logarithmic stability.
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Contents

1. Introduction

In this paper, we study an inverse problem for the fractional wave equation with a potential. The mathematical model for the fractional wave equation is formulated as follows. Let $\Omega \subset \mathbb{R}^n$ be a nonempty bounded Lipschitz domain, for $n \in \mathbb{N}$. Given $T > 0$, $s \in (0,1)$ and $q = q(x) \in L^\infty(\Omega)$, consider the initial exterior value problem for the wave equation with the fractional Laplacian,

$$\begin{cases} \left(\partial_t^2 + (-\Delta)^s + q\right) u = 0 & \text{in } \Omega_T := \Omega \times (0,T), \\ u = f & \text{in } (\Omega_e)_T := \Omega_e \times (0,T), \\ u = \partial_t u = 0 & \text{in } \mathbb{R}^n \times \{0\}, \end{cases} \tag{1.1}$$

where $(-\Delta)^s$ is the standard fractional Laplacian[1], and

$$\Omega_e := \mathbb{R}^n \setminus \overline{\Omega}$$

denotes the exterior domain. The fractional wave equation can be regarded as a special case of the peridynamics which models the nonlocal elasticity theory, see e.g. [Sil16].

In recent years, inverse problems involving the fractional Laplacian have received a lot of attention. Ghosh-Salo-Uhlmann [GSU20] first proposed the Calderón problem for the fractional Schrödinger equation, and the proof relies on the *strong uniqueness* of the fractional Laplacian ([GSU20, Theorem 1.2]) and the *Runge approximation* ([GSU20, Theorem 1.3]). Based on these two useful tools, there are many related works appeared in past few years, such as [BGU21, CLL19, CL19, CLR20, GLX17, GRSU20, HL19, HL20, LL20, LL19, RS20, LLR20, Lin20] and the references therein.

Throughout this work, we assume that the (lateral) exterior data f is compactly supported in the set $W_T := W \times (0,T) \subset (\Omega_e)_T$, where $W \subset \Omega_e$ with $\overline{W} \cap \overline{\Omega} = \emptyset$ can be any nonempty open subset with Lipschitz boundary, and, to simplify our notations, we assume that both q and f are real-valued functions. Note that the initial boundary value problem (1.1) is a mixed *local-nonlocal* type equation. In order to study the inverse problem of (1.1), we will use the strong approximation property of (1.1), which is due to the *nonlocality* of the fractional Laplacian $(-\Delta)^s$, for $0 < s < 1$. Hence, by the well-posedness of (1.1) (see Theorem 2.1), one can formally define the associated *Dirichlet-to-Neumann* (DN) map Λ_q

$$\Lambda_q : C_c^\infty((\Omega_e)_T) \to L^2(0,T;H^{-s}(\Omega_e)), \quad \Lambda_q : f \mapsto (-\Delta)^s u|_{(\Omega_e)_T}, \tag{1.2}$$

where u is the unique solution to (1.1). The precise definitions of the Sobolev spaces will be given in Section 2.1. Let us state the first main result of our work.

Theorem 1.1 (Global uniqueness)**.** *Consider $T > 0$, $s \in (0,1)$, and $q_j = q_j(x) \in L^\infty(\Omega)$, for $j = 1,2$. Assume that $W_1, W_2 \subset \Omega_e$ are arbitrary nonempty open sets with Lipschitz boundary such that $\overline{W_1} \cap \overline{\Omega} = \overline{W_2} \cap \overline{\Omega} = \emptyset$. Let Λ_{q_j} be the DN map of*

$$\begin{cases} \left(\partial_t^2 + (-\Delta)^s + q_j\right) u = 0 & \text{in } \Omega_T, \\ u = f & \text{in } (\Omega_e)_T, \\ u(x,0) = \partial_t u(x,0) = 0 & \text{in } \mathbb{R}^n \times \{0\}, \end{cases} \tag{1.3}$$

for $j = 1,2$. If

$$\Lambda_{q_1}(f)|_{(W_2)_T} = \Lambda_{q_2}(f)|_{(W_2)_T}, \text{ for any } f \in C_c^\infty((W_1)_T), \tag{1.4}$$

then $q_1 = q_2$ in Ω_T[2].

[1] A rigorous definition is given in Section 2.

[2] Throughout this paper, we adapt the notation $A_T := A \times (0,T)$, for any set $A \subset \mathbb{R}^n$.

The proof of Theorem 1.1 is based on the qualitative form of the *Runge approximation* for the fractional wave equation: For any $g \in L^2(\Omega_T)$, there exists a sequence of functions $\{f_k\}_{k\in\mathbb{N}} \in C_c^\infty((W_1)_T)$ such that $u_k \to g$ in $L^2(\Omega_T)$ as $k \to \infty$, where u_k is the solution to (1.1) with $u_k = f_k$ in $(\Omega_e)_T$, for all $k \in \mathbb{N}$. The preceding characterization can be regarded as an *exterior control* approach, in the sense that one can always control the solution by choosing appropriate exterior data.

The second main result of the paper is a quantitative version of Theorem 1.1, which provides a stability estimate for our fractional Calderón problem. Before we state the stability result, we introduce some notations.

Definition 1.1. *Let $H^2(0,T;\widetilde{H}^\alpha(\Omega))$ be the Sobolev space equipped with the norm*

$$\|u\|_{H^2(0,T;\widetilde{H}^\alpha(\Omega))} = \|u\|_{L^2(0,T;\widetilde{H}^\alpha(\Omega))} + \|\partial_t u\|_{L^2(0,T;\widetilde{H}^\alpha(\Omega))} + \|\partial_t^2 u\|_{L^2(0,T;\widetilde{H}^\alpha(\Omega))}.$$

We also denote

$$H_0^2(0,T;\widetilde{H}^\alpha(\Omega)) := \left\{ u \in H^2(0,T;\widetilde{H}^\alpha(\Omega)) : \ u = \partial_t u = 0 \text{ in } \mathbb{R}^n \times \{0\} \right\},$$

and let $H^{-2}(0,T;H^{-\alpha}(\Omega))$ be the dual space of $H_0^2(0,T;\widetilde{H}^\alpha(\Omega))$. We shall explain the space $\widetilde{H}^\alpha(\Omega)$ in more detail later in Section 2.

Definition 1.2. *For each $\alpha > 0$ and $T > 0$, we define*

$$\|q\|_{Z^{-\alpha}(\Omega;T)} := \sup \left\{ \left| \int_{\Omega_T} q(x,t)\phi_1(x,t)\phi_2(x,t)\, dx\, dt \right| \right\},$$

where the supremum is taken over all functions $\phi_1, \phi_2 \in C_c^\infty(\Omega_T)$ with

$$\|\phi_j\|_{H^2(0,T;\widetilde{H}^\alpha(\Omega))} = 1 \quad (j = 1,2),$$

and let $Z^{-\alpha}(\Omega;T)$ be the Banach space equipped with this norm.

Remark 1.3. *Since $\|\phi_j\|_{L^2(\Omega_T)} \le \|\phi_j\|_{H^2(0,T;\widetilde{H}^\alpha(\Omega))} = 1$ for all $\phi \in H_0^2(0,T;\widetilde{H}^\alpha(\Omega))$, $\alpha > 0$ and $T > 0$, it is easy to see that*

$$\|q\|_{Z^{-\alpha}(\Omega;T)} \le \|q\|_{L^\infty(\Omega_T)}$$

for all $q = q(x,t)$, which implies $L^\infty(\Omega_T) \subset Z^{-\alpha}(\Omega;T)$.

To shorten our notations, we denote the operator norm as

$$\|\cdot\|_* = \|\cdot\|_{L^2(0,T;H^{2s}_{\overline{W}}) \to L^2(0,T;H^{-2s}(W))},$$

where the Sobolev space $H^{2s}_{\overline{W}}$ will be described in Section 2.1. We are now ready to state the second main result of our work.

Theorem 1.2 (Logarithmic stability)**.** *Let $T > 0$, $s \in (0,1)$, and $q_j = q_j(x) \in L^\infty(\Omega)$, for $j = 1,2$. Assume that $W_1, W_2 \subset \Omega_e$ be arbitrary nonempty open sets with Lipschitz boundary such that $\overline{W_1} \cap \overline{\Omega} = \overline{W_2} \cap \overline{\Omega} = \emptyset$. Let Λ_{q_j} be the DN map of (1.3) for $j = 1,2$. We also fix a regularizing parameter $\gamma > 0$. If q_1 and q_2 both satisfy the apriori bound*

$$\|q_j\|_{L^\infty(\Omega)} \le M \qquad \text{for } j = 1,2,$$

then

$$\|q_1 - q_2\|_{Z^{-s-\gamma}(\Omega;T)} \le \omega\left(\|\Lambda_{q_1} - \Lambda_{q_2}\|_*\right)$$

where ω satisfies

$$\omega(t) \le C|\log t|^{-\sigma}, \quad 0 \le t \le 1,$$

for some constants $C > 0$ and $\sigma > 0$ depending only on $n, s, \Omega, W_1, W_2, \gamma, T, M$.

Inspired by Theorem 1.1, we will prove Theorem 1.2 by using a quantitative version of Runge approximation, which involves the well-known *Caffarelli-Silvestre extension* for the fractional Laplacian and the *propagation of smallness*. Moreover, Theorem 1.1 and Theorem 1.2 are satisfied for any spatial dimension $n \in \mathbb{N}$.

The third main result of this work studies the *exponential instability* of the Calderón problem for the fractional wave equation. In other words, the stability result in Theorem 1.2 is optimal. For brevity, we denote the operator norm

$$\|\mathcal{A}\|_*' = \sup_{0\not\equiv\chi\in C_c^\infty((0,T))} \frac{\sup_{t\in(0,T)} \|\chi\mathcal{A}\chi\|_{L^2(B_3\setminus\overline{B_2})\to L^2(B_3\setminus\overline{B_2})}(t)}{\|\chi\|_{W^{2,\infty}(0,T)}^2},$$

where B_r with $r>0$ stands for the ball of radius r centered at the origin.

Theorem 1.3 (Exponential instability I)**.** *Let $\Omega = B_1 \subset \mathbb{R}^n$, for $n \geq 2$, $n \in \mathbb{N}$. Given any $T>0$, $s\in(0,1)$, $\alpha>0$ and $R>0$. There exists a positive constant $c_{R,T,n,s}$ such that: Given any $0<\epsilon<c_{R,T,n,s}$, there exist potentials $q_1, q_2 \in C^\alpha(\Omega)$ such that $\|q_j\|_{L^\infty(\Omega)} \leq R$, $j=1,2$, satisfying*

$$\|q_1 - q_2\|_{L^\infty(\Omega)} \geq \epsilon, \tag{1.5}$$

but

$$\|\Lambda_{q_1} - \Lambda_{q_2}\|_*' \leq K_{R,T,n,s} \exp\left(-\epsilon^{-\frac{n}{(2n+1)\alpha}}\right) \tag{1.6}$$

for some positive constant $K_{R,T,n,s}$.

For 1-dimensional case ($n=1$), we can also establish the same estimate.

Theorem 1.4 (Exponential instability II)**.** *For $n=1$, Theorem 1.3 is also valid with the norm $\|\cdot\|_*'$ being replaced by the following norm:*

$$\|\mathcal{A}\|_*'' := \sup_{0\not\equiv\chi\in C_c^\infty((0,T))} \frac{\sup_{t\in(0,T)} \|\chi\mathcal{A}\chi\|_{L^2((2,3))\to L^2((2,3))}(t)}{\|\chi\|_{W^{2,\infty}(0,T)}^2}.$$

For the local counterpart, let us consider the following initial boundary value problem for the local wave equation:

$$\begin{cases} \left(\partial_t^2 - \Delta + q(x)\right) u = 0 & \text{in } \Omega_T, \\ \partial_\nu u(x,t) = g(x,t) & \text{in } (\partial\Omega)_T, \\ u = \partial_t u = 0 & \text{in } \Omega \times \{0\}, \end{cases} \tag{1.7}$$

where $q = q(x) \in L^\infty(\Omega)$. It is known that (1.7) is well-posed (for example, see [Eva98]) with suitable compatibility conditions. Assuming the well-posedness of (1.7), the corresponding (hyperbolic) *Neumann-to-Dirichlet map* of (1.7) is defined by

$$\widetilde{\Lambda}_q g := u|_{\partial\Omega\times[0,T]} \quad \text{for all } g \in C_c^\infty((\partial\Omega)_T).$$

In fact, $\widetilde{\Lambda}_q : L^2(\partial\Omega\times(0,T)) \to H^1(0,T;L^2(\partial\Omega))$ is a bounded linear operator, which can be proved by the energy estimate of (1.7), see e.g. [CP82, Section 6.7.5]. Now we assume

$$T > \operatorname{diam}(\Omega). \tag{1.8}$$

Under assumption (1.8), in [RW88], they showed the global uniqueness result for time-independent potentials:

$$\widetilde{\Lambda}_{q_1} = \widetilde{\Lambda}_{q_2} \quad \text{implies} \quad q_1 = q_2 \text{ in } \Omega.$$

In [Sun90], the author showed that, if (1.8) holds, under some apriori assumptions, the following estimate hold:

$$\|q_1 - q_2\|_{L^2(\Omega)} \leq C \left\|\widetilde{\Lambda}_{q_1} - \widetilde{\Lambda}_{q_2}\right\|_{\mathcal{L}}^\alpha \tag{1.9}$$

for some constants C and α, where $\|\cdot\|_{\mathcal{L}}$ stands for the operator norm for the Neumann-to-Dirichlet map. A similar estimate also holds for the hyperbolic Dirichlet-to-Neumann map [AS90]. In other words, the stability of the inverse problem for the local wave equation is of *Hölder-type*. We also mention other related results of inverse problems for the local wave equation with potentials [Esk06, Esk07, Isa91, Kia17, RS91, Sal13].

Similar to the local version, we can prove the global uniqueness result for time-independent potentials for the fractional wave equation (see Theorem 1.1). However, in the nonlocal counterpart of (1.9), we show that the stability of the inverse problem for the fractional wave equation is of (optimal) *logarithmic-type* in view of Theorem 1.2 and Theorem 1.3. We also want to point out that we do not need to assume the *large influence time condition* (1.8). One possible explanation is that while the *speed of propagation* of the local wave equation is *finite*, the speed of propagation of the fractional wave operator is *infinite* due the nonlocal nature of the fractional Laplacian $(-\Delta)^s$, for $0 < s < 1$. We will offer some detailed arguments in Section 2.

Before ending this section, we would like to discuss some interesting results for the time-harmonic wave equation. Consider the time-harmonic wave equation with a potential (a.k.a. Schrödinger equation):

$$\left(-\Delta + q(x) - \kappa^2\right) v = 0 \quad \text{in } \Omega. \tag{1.10}$$

Ignoring the effect of the frequency $\kappa > 0$, Alessandrini [Ale88] proved the well-known logarithmic stability estimate for the inverse boundary value problem of (1.10), and Mandache [Man01] established that this logarithmic estimate is optimal by showing that the inverse problem is exponentially unstable. Nonetheless, by taking the frequency into account, it was shown in [INUW14] that

$$\|q_1 - q_2\|_{H^{-\alpha}(\mathbb{R}^n)} \leq C\left(\kappa + \log \frac{1}{\operatorname{dist}\left(\mathcal{C}_{q_1}, \mathcal{C}_{q_2}\right)}\right)^{-2\alpha - n} + C\kappa^4 \operatorname{dist}\left(\mathcal{C}_{q_1}, \mathcal{C}_{q_2}\right), \tag{1.11}$$

where $\mathcal{C}_{q_1}, \mathcal{C}_{q_2}$ are the Cauchy data of the Schrödinger equation (1.10) corresponding to q_1, q_2, and $\operatorname{dist}\left(\mathcal{C}_{q_1}, \mathcal{C}_{q_2}\right)$ is the Hausdorff distance between $\mathcal{C}_{q_1}$ and $\mathcal{C}_{q_2}$. Isakov [Isa11] proved a similar estimate in terms of the DN maps.

The estimate (1.11) is shown to be optimal in the recent paper [KUW21]. The estimate (1.11) clearly indicates that the logarithmic part decreases as the frequency $\kappa > 0$ increases and the estimate changes from a logarithmic type to a Hölder type. This phenomena is termed as the *increasing stability*. It is interesting to compare the stability estimate (1.11) of the time-harmonic wave equation (1.10) with the stability estimate (1.9) of the local wave equation (1.7).

Similarly to the local wave equation, we consider the following time-harmonic fractional wave equation

$$\left((-\Delta)^s + q(x) - \kappa^2\right) v = 0 \quad \text{in } \Omega, \tag{1.12}$$

which is a fractional Schrödinger equation. Without considering the effect of the frequency $\kappa > 0$, Rüland and Salo [RS20] obtained a logarithmic type stability estimate for the inverse boundary value problem of the time-harmonic fractional wave equation (1.12) and, in [RS18], they proved that such logarithmic estimate is optimal by showing the exponential instability phenomenon. These results give rise to a natural question: in the inverse boundary value problem for (1.12), if we take the frequency κ into account, does the increasing stability estimate similar to (1.11) hold? In view of the optimal logarithmic stability results in Theorem 1.2 and Theorem 1.3, we have a strong reason to believe that the answer to this question is negative.

The paper is organized as follows. We discuss and prove the well-posedness of the fractional wave equation in Section 2 and in Appendix A, respectively. We then prove Theorem 1.1 in Section 3, and prove Theorem 1.2 in Section 4. The approach is mainly based on the qualitative and quantitative Runge approximation properties for the fractional wave equation. Finally, we prove Theorem 1.3 and Theorem 1.4 in Section 5.

2. The forward problems for the fractional wave equation

In this section, we provide all preliminaries that we need in the rest of the paper. Let us first recall (fractional) Sobolev spaces and prove the well-posedness of the fractional wave equation (1.1).

2.1. **Sobolev spaces.** Let $\mathcal{F}$, $\mathcal{F}^{-1}$ be Fourier transform and its inverse, respectively. For $s \in (0,1)$, the fractional Laplacian is defined via

$$(-\Delta)^s u := \mathcal{F}^{-1}\left(|\xi|^{2s}\mathcal{F}(u)\right), \text{ for } u \in H^{2s}(\mathbb{R}^n),$$

where $H^s(\mathbb{R}^n)$ stands for the L^2-based fractional Sobolev space (see [DNPV12, Kwa17, Ste16]). The space $H^a(\mathbb{R}^n) = W^{a,2}(\mathbb{R}^n)$ denotes the (fractional) Sobolev space equipped with the norm

$$\|u\|_{H^a(\mathbb{R}^n)} := \left\|\mathcal{F}^{-1}\left\{\langle\xi\rangle^a \mathcal{F}u\right\}\right\|_{L^2(\mathbb{R}^n)},$$

for any $a \in \mathbb{R}$, where $\langle\xi\rangle = (1+|\xi|^2)^{\frac{1}{2}}$. It is known that for $s \in (0,1)$, $\|\cdot\|_{H^s(\mathbb{R}^n)}$ has the following equivalent representation

$$\|u\|_{H^s(\mathbb{R}^n)} := \|u\|_{L^2(\mathbb{R}^n)} + [u]_{H^s(\mathbb{R}^n)}$$

where

$$[u]^2_{H^s(\mathcal{O})} := \int_{\mathcal{O}\times\mathcal{O}} \frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}}\,dxdy,$$

for any open set $\mathcal{O} \subset \mathbb{R}^n$.

Given any nonempty open set $\mathcal{O}$ of $\mathbb{R}^n$, the collection $C_c^\infty(\mathcal{O})$ consists all functions in $C^\infty(\mathbb{R}^n)$ supported in $\mathcal{O}$. Given any $a \in \mathbb{R}$, let us define the following Sobolev spaces,

$$\begin{aligned} H^a(\mathcal{O}) &:= \{u|_{\mathcal{O}};\, u \in H^a(\mathbb{R}^n)\},\\ \widetilde{H}^a(\mathcal{O}) &:= \text{closure of } C_c^\infty(\mathcal{O}) \text{ in } H^a(\mathbb{R}^n),\\ H_0^a(\mathcal{O}) &:= \text{closure of } C_c^\infty(\mathcal{O}) \text{ in } H^a(\mathcal{O}), \end{aligned}$$

and

$$H^a_{\overline{\mathcal{O}}} := \{u \in H^a(\mathbb{R}^n);\, \mathrm{supp}(u) \subset \overline{\mathcal{O}}\}.$$

In addition, the Sobolev space $H^a(\mathcal{O})$ is complete under the norm

$$\|u\|_{H^a(\mathcal{O})} := \inf\left\{\|v\|_{H^a(\mathbb{R}^n)};\, v \in H^a(\mathbb{R}^n) \text{ and } v|_{\mathcal{O}} = u\right\}.$$

It is not hard to see that $\widetilde{H}^a(\mathcal{O}) \subseteq H_0^a(\mathcal{O})$, and that $H^a_{\overline{\mathcal{O}}}$ is a closed subspace of $H^a(\mathbb{R}^n)$. We also denote $H^{-s}(\mathcal{O})$ to be the dual space of $\widetilde{H}^s(\mathcal{O})$. In fact, $H^{-s}(\mathcal{O})$ has the following characterization:

$$H^{-s}(\mathcal{O}) = \left\{u|_{\mathcal{O}} : u \in H^{-s}(\mathbb{R}^n)\right\} \quad \text{with} \quad \inf_{w\in H^s(\mathbb{R}^n),\, w|_{\mathcal{O}}=u} \|w\|_{H^s(\mathbb{R}^n)},$$

see e.g. [GSU20, Section 2.1], [McL00, Chapter 3], or [Tri02] for more details about the fractional Sobolev spaces. Moreover, we will use

$$(f,g)_{L^2(A)} := \int_A fg\,dx, \quad (F,G)_{L^2(A_T)} := \int_0^T\int_A FG\,dxdt,$$

in the rest of this paper, for any set $A \subset \mathbb{R}^n$.

2.2. **The forward problem.** We first state the well-posedness of the fractional wave equation. As above, let $\Omega \subset \mathbb{R}^n$ be a nonempty bounded Lipschitz domain with $n \in \mathbb{N}$. Given $T > 0$, $s \in (0,1)$, and $q = q(x) \in L^\infty(\Omega)$, consider the initial exterior value problem for the fractional wave equation

$$\begin{cases} \left(\partial_t^2 + (-\Delta)^s + q\right) u = F & \text{in } \Omega_T, \\ u = f & \text{in } (\Omega_e)_T, \\ u = \varphi, \quad \partial_t u = \psi & \text{in } \mathbb{R}^n \times \{0\}, \end{cases} \tag{2.1}$$

where $f \in C_c^\infty(W_T)$ for some open set with Lipschitz boundary $W \subset \Omega_e$ satisfying $\overline{W} \cap \overline{\Omega} = \emptyset$, $\varphi \in \widetilde{H}^s(\Omega)$, and $\psi \in L^2(\mathbb{R}^n)$ with $\operatorname{supp}(\psi) \subset \Omega$. We want to show the well-posedness of (2.1). Setting $v := u - f$, we then consider the fractional wave equation with zero exterior data

$$\begin{cases} \left(\partial_t^2 + (-\Delta)^s + q\right) v = \widetilde{F} & \text{in } \Omega_T, \\ v = 0 & \text{in } (\Omega_e)_T, \\ v = \widetilde{\varphi}, \quad \partial_t v = \widetilde{\psi} & \text{in } \mathbb{R}^n \times \{0\}, \end{cases} \tag{2.2}$$

where $\widetilde{F} := F - (-\Delta)^s f$, $\widetilde{\varphi}(x) = \varphi(x) - f(x,0) = \varphi(x)$ and $\widetilde{\psi}(x) = \psi(x) - \partial_t f(x,0) = \psi(x)$. Hence, we simply denote the initial data as (φ, ψ) in the rest of the paper. Now it suffices to study the well-posedness of (2.1).

Let us introduce the following notations. Define

$$\boldsymbol{u} : [0,T] \to \widetilde{H}^s(\Omega)$$

by

$$[\boldsymbol{u}(t)](x) := u(x,t), \text{ for } x \in \mathbb{R}^n, t \in [0,T].$$

Similarly, the function $\widetilde{\boldsymbol{F}} : [0,T] \to L^2(\Omega)$ can be defined analogously by

$$[\widetilde{\boldsymbol{F}}(t)](x) := \widetilde{F}(x,t), \text{ for } x \in \mathbb{R}^n, \ t \in [0,T].$$

With these notations at hand, we can define the weak formulation for the fractional wave equation. Let $\phi \in \widetilde{H}^s(\Omega)$ be any test function, multiplying (2.2) with ϕ gives

$$(\boldsymbol{v}'', \phi)_{L^2(\Omega)} + B[\boldsymbol{v}, \phi; t] = (\widetilde{\boldsymbol{F}}, \phi)_{L^2(\Omega)}, \text{ for } 0 \le t \le T,$$

where $B[\boldsymbol{v}, \phi; t]$ is the bilinear form defined via

$$B[\boldsymbol{v}, \phi; t] := \int_{\mathbb{R}^n} (-\Delta)^{s/2} \boldsymbol{v} (-\Delta)^{s/2} \phi \, dx + \int_\Omega q \boldsymbol{v} \phi \, dx.$$

Definition 2.1 (Weak solutions)**.** *A function*

$$\boldsymbol{v} \in L^2(0,T; \widetilde{H}^s(\Omega)), \text{ with } \boldsymbol{v}' \in L^2(0,T; L^2(\Omega)) \text{ and } \boldsymbol{v}'' \in L^2(0,T; H^{-s}(\Omega))$$

is a weak solution *of the initial exterior value problem* (2.2) *if*

(1) $(\boldsymbol{v}''(t), \phi)_{L^2(\Omega)} + B[\boldsymbol{v}, \phi; t] = \left(\widetilde{\boldsymbol{F}}, \phi\right)_{L^2(\Omega)}$, *for all* $\phi \in \widetilde{H}^s(\Omega)$, *and for* $0 \le t \le T$ *a.e.*

(2) $\boldsymbol{v}(0) = \widetilde{\varphi}$ *and* $\boldsymbol{v}'(0) = \widetilde{\psi}$.

Theorem 2.1 (Well-posedness)**.** *For any* $\widetilde{\boldsymbol{F}} \in L^2(0,T; L^2(\Omega))$, $\widetilde{\varphi} \in \widetilde{H}^s(\Omega)$, *and* $\widetilde{\psi} \in L^2(\mathbb{R}^n)$ *with* $\operatorname{supp}(\widetilde{\psi}) \subset \Omega$, *there exists a unique weak solution* $\boldsymbol{v}$ *to* (2.2). *Moreover, the following estimate holds:*

$$\begin{aligned} &\|\boldsymbol{v}\|_{L^\infty(0,T;\widetilde{H}^s(\Omega))} + \|\partial_t \boldsymbol{v}\|_{L^\infty(0,T;L^2(\Omega))} \\ &\quad \le C \left(\|\widetilde{\boldsymbol{F}}\|_{L^2(0,T;L^2(\Omega))} + \|\widetilde{\varphi}\|_{\widetilde{H}^s(\Omega)} + \|\widetilde{\psi}\|_{L^2(\Omega)} \right). \end{aligned} \tag{2.3}$$

Corollary 2.2. *Let $\Omega \subset \mathbb{R}^n$ be a nonempty bounded Lipschitz domain for $n \in \mathbb{N}$, and $W \subset \Omega_e$ be any nonempty open set with Lipschitz boundary satisfying $\overline{W} \cap \overline{\Omega} = \emptyset$. Then for any $F = F(x,t) \in L^2(0,T;L^2(\Omega))$, $f = f(x,t) \in C_c^\infty(W_T)$, $\varphi \in \widetilde{H}^s(\Omega)$, and $\psi \in L^2(\mathbb{R}^n)$ with $\mathrm{supp}\,(\psi) \subset \Omega$, there exists a unique weak solution $u = v + f$ of (2.1), where $v \in L^2(0,T;\widetilde{H}^s(\Omega)) \cap H^1(0,T;L^2(\Omega))$ is the unique weak solution of (2.2). Furthermore, we have the following estimate*

$$
\begin{aligned}
&\|u-f\|_{L^\infty(0,T;\widetilde{H}^s(\Omega))} + \|\partial_t(u-f)\|_{L^\infty(0,T;L^2(\Omega))} \\
&\quad \le C\left(\|F-(-\Delta)^s f\|_{L^2(0,T;L^2(\Omega))} + \|\varphi\|_{\widetilde{H}^s(\Omega)} + \|\psi\|_{L^2(\Omega)}\right).
\end{aligned} \tag{2.4}
$$

The proof of Theorem 2.1 is similar to the well-posedness of the classical wave equation (i.e., $s=1$) and, for the sake of completeness, we will give a comprehensive proof in Appendix A. In this article, we only consider the time-independent potential $q = q(x) \in L^\infty(\Omega)$. In fact, the well-posedness for a space-time dependent potential $q = q(x,t) \in L^\infty(\Omega_T)$ has been studied. We refer to [Bre11, Theorem 10.14] for the well-posedness of the abstract wave equations, and to [DFA20] for the well-posedness result for non-local semi-linear integro-differential wave equations which involve both the fractional Laplacian (in space) and the Caputo fractional derivative operator (in time).

2.3. **The DN map and its duality.** With the well-posedness at hand, one can define the corresponding DN map (1.2) for the fractional wave equation (1.1). Let us define the solution operator

$$
\mathcal{P}_q : C_c^\infty(W_T) \to L^2(0,T;H^s(\Omega)), \quad f \mapsto u|_{\Omega_T}, \tag{2.5}
$$

where $W \subset \Omega_e$ is a Lipschitz set with $\overline{W} \cap \overline{\Omega} = \emptyset$, and u is the solution of (1.1). Given any $\varphi(x,t)$ defined in $(\Omega_{\mathrm{e}})_T$, we define

$$
\varphi^*(x,t) := \varphi(x, T-t) \quad \text{for all} \quad (x,t) \in (\Omega_{\mathrm{e}})_T,
$$

and we define the following backward DN-map:

$$
\Lambda_q^*(f) := (\Lambda_q(f))^* \quad \text{for any} \quad f \in C_c^\infty((\Omega_{\mathrm{e}})_T).
$$

Lemma 2.3. *Given any $q \in L^\infty(\Omega)$, Λ_q^* is self-adjoint, that is,*

$$
\int_{(\Omega_e)_T} \Lambda_q^*(f_1) f_2 \, dxdt = \int_{(\Omega_e)_T} f_1 \Lambda_q^*(f_2)\, dxdt, \quad \textit{for all } f_1, f_2 \in C_c^\infty((\Omega_{\mathrm{e}})_T).
$$

Proof. Let $u_1 = \mathcal{P}_q f_1$ and $u_2 = \mathcal{P}_q f_2$. Using integration by parts, we have

$$
\int_{\Omega_T} \left[u_1(\partial_t^2 u_2^*) - (\partial_t^2 u_1) u_2^*\right] dxdt = 0. \tag{2.6}
$$

Therefore,

$$
\begin{aligned}
0 &= \int_{\Omega_T} \left[u_1\left(\partial_t^2 u_2^* + (-\Delta)^s u_2^* + q u_2^*\right) - \left(\partial_t^2 u_1 + (-\Delta)^s u_1 + q(x) u_1\right) u_2^*\right] dxdt \\
&= \int_{\Omega_T} [u_1((-\Delta)^s u_2^*) - ((-\Delta)^s u_1) u_2^*]\, dxdt \\
&= \left(\int_{(\mathbb{R}^n)_T} - \int_{(\Omega_{\mathrm{e}})_T}\right) [u_1((-\Delta)^s u_2^*) - ((-\Delta)^s u_1) u_2^*]\, dxdt \\
&= -\int_{(\Omega_{\mathrm{e}})_T} [u_1((-\Delta)^s u_2^*) - ((-\Delta)^s u_1) u_2^*]\, dxdt \\
&= -\int_{(\Omega_{\mathrm{e}})_T} \left[f_1 \Lambda_q^*(f_2) - \Lambda_q(f_1) f_2^*\right] dxdt.
\end{aligned}
$$

Finally, changing the variable $t \mapsto T-t$, we have

$$\int_{(\Omega_e)_T} f_1 \Lambda_q^*(f_2)\,dxdt = \int_{(\Omega_e)_T} \Lambda_q(f_1) f_2^*\,dxdt = \int_{(\Omega_e)_T} \Lambda_q^*(f_1) f_2\,dxdt, \tag{2.7}$$

which is our desired lemma. □

Since Λ_q^* is self-adjoint, we can derive the following identity immediately.

Lemma 2.4 (Integral identity)**.** *Let $q_1, q_2 \in L^\infty(\Omega)$, and given any $f_1, f_2 \in C_c^\infty((\Omega_e)_T)$. Let $u_1 := \mathcal{P}_{q_1} f_1$ and $u_2 := \mathcal{P}_{q_2} f_2$, where the operator $\mathcal{P}_q$ is given in (2.5), for $q = q_1$ and $q = q_2$, respectively. Then*

$$\int_{\Omega_T} (q_1 - q_2) u_1 u_2^*\,dxdt = \int_{(\Omega_e)_T} ((\Lambda_{q_1} - \Lambda_{q_2}) f_1) f_2^*\,dxdt. \tag{2.8}$$

Proof. Using (2.6), we have

$$\begin{aligned}
&\int_{\Omega_T} (q_1 - q_2) u_1 u_2^*\,dxdt \\
&= \int_{\Omega_T} [q_1 u_1 u_2^* - u_1(q_2 u_2^*)]\,dxdt \\
&= -\int_{\Omega_T} \left[(\partial_t^2 u_1 + (-\Delta)^s u_1) u_2^* - u_1(\partial_t^2 u_2^* + (-\Delta)^s u_2^*)\right] dxdt \\
&= -\int_{\Omega_T} [((-\Delta)^s u_1) u_2^* - u_1((-\Delta)^s u_2^*)]\,dxdt \\
&= \left(\int_{(\Omega_e)_T} - \int_{(\mathbb{R}^n)_T}\right) [((-\Delta)^s u_1) u_2^* - u_1((-\Delta)^s u_2^*)]\,dxdt \\
&= \int_{(\Omega_e)_T} [((-\Delta)^s u_1) u_2^* - u_1((-\Delta)^s u_2^*)]\,dxdt \\
&= \int_{(\Omega_e)_T} \left[\Lambda_{q_1}(f_1) f_2^* - f_1 \Lambda_{q_2}^*(f_2)\right] dxdt.
\end{aligned}$$

Combining with (2.7), we obtain

$$\int_{\Omega_T} (q_1 - q_2) u_1 u_2^*\,dxdt = \int_{(\Omega_e)_T} [(\Lambda_{q_1} f_1) f_2^* - (\Lambda_{q_2} f_1) f_2^*]\,dxdt,$$

which is our desired lemma. □

3. Global uniqueness for the fractional wave equation

In this section, let us state and prove a qualitative Runge type approximation for the fractional wave equation, and then prove Theorem 1.1. Before further discussion, let us comment on the speeds of propagation of the local and nonlocal wave equations. Given $V = V(x) \in L^\infty(\mathbb{R}^n)$, let u be a solution of

$$\left(\partial_t^2 - \Delta + V\right) u = 0 \text{ in } \mathbb{R}^n \times (0, \infty).$$

It is known that if $u(x,0) = \phi(x)$, for $x \in \mathbb{R}^n$, such that $\phi \not\equiv 0$ and ϕ is compactly supported, then for every $t > 0$, the solution $u(\cdot, t)$ has compact support.

On the other hand, the speed of propagation for the fractional wave equation is infinite due to the nonlocal nature of the fractional Laplacian. To prove this rigorously, let us recall the strong uniqueness property for the fractional Laplacian. Given $0 < s < 1$, $r \in \mathbb{R}$, if $u \in H^r(\mathbb{R}^n)$ satisfies $u = (-\Delta)^s u = 0$ in any nonempty open subset of $\mathbb{R}^n$, then $u \equiv 0$ in $\mathbb{R}^n$. By this property, we can prove the following lemma.

Lemma 3.1. *Given $V = V(x) \in L^\infty(\mathbb{R}^n)$, let u be a solution of*

$$\left(\partial_t^2 + (-\Delta)^s + V\right) u = 0 \quad \text{in } \mathbb{R}^n \times (0,T), \tag{3.1}$$

then u does not have a finite speed of propagation.

Proof. Suppose the contrary, that the speed of propagation of (3.1) is finite. If we choose $u(x,0) = \phi(x)$ for some $0 \not\equiv \phi \in C_c^\infty(\mathbb{R}^n)$, given any $T > 0$, there exists a bounded set Ω such that

$$u = 0 \quad \text{in } (\Omega_e)_T,$$

therefore, $\partial_t^2 u = 0$ in $(\Omega_e)_T$. Using (3.1), we also have

$$(-\Delta)^s u = 0 \quad \text{in } (\Omega_e)_T.$$

Using the strong uniqueness for the fractional Laplacian, we conclude that $u \equiv 0$, which implies $\phi \equiv 0$, this is a contradiction. □

3.1. **Qualitative Runge approximation.** The qualitative approximation property is based on the strong uniqueness for the fractional Laplacian ([GSU20, Theorem 1.2]).

Theorem 3.1 (Qualitative Runge approximation)**.** *Let $\Omega \subset \mathbb{R}^n$ be a nonempty bounded Lipschitz domain for $n \in \mathbb{N}$, and $W \subset \Omega_e$ be a nonempty open set with Lipschitz boundary satisfying $\overline{W} \cap \overline{\Omega} = \emptyset$. For $s \in (0,1)$, let $\mathcal{P}_q$ be the solution operator given by (2.5), and define*

$$\mathcal{D} := \left\{ u|_{\Omega_T} : \ u = \mathcal{P}_q f, \ f \in C_c^\infty(W_T) \right\}.$$

Then $\mathcal{D}$ is dense in $L^2(\Omega_T)$.

Remark 3.2. *The Runge approximation plays an essential role in the study of fractional inverse problems, for example, see [GSU20, GLX17, RS20, CLR20] and references therein.*

Proof of Theorem 3.1. By using the Hahn-Banach theorem and the duality arguments, it suffices to show that if $v \in L^2(\Omega_T)$, which satisfies

$$\left(\mathcal{P}_q f, v\right)_{L^2(\Omega_T)} = 0, \quad \text{for any } f \in C_c^\infty(W_T), \tag{3.2}$$

then $v \equiv 0$ in Ω_T. Now, consider the adjoint wave equation

$$\begin{cases} \left(\partial_t^2 + (-\Delta)^s + q\right) w = v & \text{in } \Omega_T, \\ w = 0 & \text{in } (\Omega_e)_T, \\ w = \partial_t w = 0 & \text{in } \mathbb{R}^n \times \{T\}. \end{cases} \tag{3.3}$$

Similar to the proof of Theorem 2.1, it is easy to see that (3.3) is well-posed.

For $f \in C_c^\infty(W_T)$, let u and w be the solutions of (1.1) and (3.3), respectively. Note that $u - f$ is only supported in $\overline{\Omega}_T$, then we have

$$\begin{aligned} \left(\mathcal{P}_q f, v\right)_{L^2(\Omega_T)} &= \left(u - f, (-\partial_t^2 + (-\Delta)^s + q) w\right)_{L^2(\Omega_T)} \\ &= -\left(f, (-\Delta)^s w\right)_{L^2(W_T)}, \end{aligned} \tag{3.4}$$

where we have used u is the solution of (1.1), $u(x,0) = \partial_t u(x,0) = 0$ and $w(x,T) = \partial_t w(x,T) = 0$ for $x \in \mathbb{R}^n$ in last equality of (3.4) . By using the conditions (3.2) and (3.4), one must have $(f, (-\Delta)^s w)_{L^2(W_T)} = 0$, for any $f \in C_c^\infty(W_T)$, which implies that

$$w = (-\Delta)^s w = 0 \text{ in } W_T.$$

Fix any fixed $t \in (0,T)$, the strong uniqueness for the fractional Laplacian (see [GSU20, Theorem 1.2]) yields that $w(\cdot,t) = 0$ in $\mathbb{R}^n \times \{t\}$, for all $t \in (0,T)$. Therefore, we derive $v = 0$ as desired, and the Hahn-Banach theorem infers the density property. This proves the assertion. □

Remark 3.3. *By using similar arguments, one can also consider the well-posedness (Theorem 2.1) and the Runge approximation (Theorem 3.1) also hold for the case $q = q(x,t) \in L^\infty(\Omega_T)$. In this work, we are only interested in time-independent potentials $q = q(x)$.*

Remark 3.4. *For other unique continuation property for the fractional elliptic operators, we refer the reader to* [FF14, GFR19, Rül15, Yu17] *and references therein.*

3.2. **Proof of Theorem 1.1.** With the help of Lemma 2.4 and Theorem 3.1, we can prove the global uniqueness of the inverse problem for the fractional wave equation.

Proof of Theorerm 1.1. Given any $g \in L^2(\Omega_T)$, using Theorem 3.1, there exists a sequence $f_{1,k} \in C_c^\infty((W_1)_T)$ such that

$$\lim_{k\to\infty} \|u_{1,k} - g\|_{L^2((0,T)\times\Omega)} = 0, \quad \text{where} \quad u_{1,k} = \mathcal{P}_q f_{1,k}.$$

Since $1 \in L^2(\Omega_T)$, similarly, we can choose a sequence $f_{2,k} \in C_c^\infty((W_2)_T)$ such that

$$\lim_{k\to\infty} \left\|u_{2,k}^* - 1\right\|_{L^2((0,T)\times\Omega)} = 0, \quad \text{where} \quad u_{2,k} = \mathcal{P}_q f_{2,k}.$$

Combining (1.4) and (2.8), we know that

$$\int_{\Omega_T} (q_1 - q_2) u_{1,k} u_{2,k}^* \, dxdt = 0.$$

Taking the limit $k \to \infty$, we obtain

$$\int_{\Omega_T} (q_1 - q_2) g \, dxdt = 0.$$

Finally, by the arbitrariness of $g \in L^2(\Omega_T)$, we conclude that $q_1 = q_2$ in Ω_T. □

4. Stability for the fractional wave equation

In order to understand the stability estimate for the fractional wave equation, let us recall the famous *Caffarelli-Silvestre extension* [CS07] for the fractional Laplacian. For each $x' \in \mathbb{R}^n$ and $x_{n+1} \in \mathbb{R}_+$, we denote $x = (x', x_{n+1}) \in \mathbb{R}^n \times \mathbb{R}_+ = \mathbb{R}^{n+1}_+$. Fixing any $0 < s < 1$ and $t \in (0,T)$. If there exists $\gamma \in \mathbb{R}$ such that $\boldsymbol{v}(t) = v(x',t) \in H^\gamma(\mathbb{R}^n)$, using [RS20, Lemma 4.1], there exists a *Caffarelli-Silvestre extension* $\boldsymbol{v}^{\mathrm{cs}}(t) = v^{\mathrm{cs}}(x', x_{n+1}, t) \in C^\infty(\mathbb{R}^{n+1}_+)$ of v satisfies

$$\begin{aligned} \nabla \cdot x_{n+1}^{1-2s} \nabla v^{\mathrm{cs}} &= 0 \quad \text{in } \mathbb{R}^{n+1}_+, \\ v^{\mathrm{cs}} &= v \quad \text{on } \mathbb{R}^n \times \{0\}, \\ \lim_{x_{n+1}\to 0} x_{n+1}^{1-2s} \partial_{n+1} v^{\mathrm{cs}} &= -a_{n,s}(-\Delta)^s v, \end{aligned}$$

where $a_{n,s} := 2^{1-2s}\frac{\Gamma(1-s)}{\Gamma(s)}$ and $\nabla = (\nabla_{x'}, \partial_{x_{n+1}}) = (\nabla', \partial_{n+1})$.

4.1. **Logarithmic stability of the Caffarelli-Silvestre extension.** We now define

$$\hat{\Omega} := \left\{ x' \in \mathbb{R}^n : \operatorname{dist}(x', \Omega) < \frac{1}{2}\operatorname{dist}(\Omega, W) \right\}.$$

We now prove a lemma, which concerns the *propagation of smallness* for the Caffarelli-Silvestre extension. By using similar ideas as in [RS20, Section 5], we can derive the following boundary logarithmic stability estimate.

Lemma 4.1. *Let $W \subset \Omega_e$ be a nonempty open bounded Lipschitz set such that $\overline{W} \cap \overline{\Omega} = \emptyset$. Let $v^{\rm cs}(x', x_{n+1}, t)$ be the Caffarelli-Silvestre extension of $v(x', t)$. Define*

$$\eta(t) := \left\| \lim_{x_{n+1}\to 0} x_{n+1}^{1-2s} \partial_{n+1} \boldsymbol{v}^{\rm cs}(t) \right\|_{H^{-s}(W)} = a_s \|(-\Delta)^s \boldsymbol{v}(t)\|_{H^{-s}(W)}$$

Suppose that there exist constants $C_1 > 1$ and $E > 0$ such that $\eta(t) \le E$ and

$$\left\| x_{n+1}^{\frac{1-2s}{2}} v^{\rm cs} \right\|_{L^\infty(0,T;L^2(\mathbb{R}^n \times [0,C_1]))} + \left\| x_{n+1}^{\frac{1-2s}{2}} \nabla v^{\rm cs} \right\|_{L^\infty(0,T;L^2(\mathbb{R}^{n+1}_+))} \le E, \tag{4.2}$$

then

$$\left\| x_{n+1}^{\frac{1-2s}{2}} \boldsymbol{v}^{\rm cs}(t) \right\|_{L^2(\hat{\Omega}\times[0,1])} \le CE \log^{-\mu} \left(\frac{CE}{\eta(t)} \right) \tag{4.3}$$

for some constants $C > 1$ and $\mu > 0$, both depending only on n, s, C_1, Ω, W. Moreover, given any $\gamma > 0$, we have

$$\left\| x_{n+1}^{\frac{1-2s}{2}+\gamma} \nabla \boldsymbol{v}^{\rm cs}(t) \right\|_{L^2(\hat{\Omega}\times[0,1])} \le CE \log^{-\mu} \left(\frac{CE}{\eta(t)} \right), \tag{4.4}$$

for some constants $C > 1$ and $\mu > 0$, both depending only on n, s, C_1, Ω, W, as well as γ.

Proof. Estimate (4.3) is an immediate consequence of [RS20, (5.3) of Theorem 5.1]. Replacing [RS20, (5.67) of Theorem 5.1] by the following inequality:

$$\begin{aligned} &\left\| x_{n+1}^{\frac{1-2s}{2}+\gamma} \nabla' \boldsymbol{v}^{\rm cs}(t) \right\|_{L^2(\hat{\Omega}\times[0,h])} \\ \le& C \left\| x_{n+1}^{\frac{1-2s}{2}} \nabla' \boldsymbol{v}^{\rm cs}(t) \right\|_{L^2(\hat{\Omega}\times[0,h])} \left\| x_{n+1}^{\gamma} \right\|_{L^\infty(\Omega\times[0,h])} \\ \le& C h^\gamma E, \end{aligned}$$

and

$$\begin{aligned} &\left\| x_{n+1}^{\frac{1-2s}{2}+\gamma} \partial_{n+1} \boldsymbol{v}^{\rm cs}(t) \right\|_{L^2(\hat{\Omega}\times[0,h])} \\ \le& C \left\| x_{n+1}^{\frac{1-2s}{2}} \partial_{n+1} \boldsymbol{v}^{\rm cs}(t) \right\|_{L^2(\hat{\Omega}\times[0,h])} \left\| x_{n+1}^{\gamma} \right\|_{L^\infty(\Omega\times[0,h])} \\ \le& C h^\gamma E, \end{aligned}$$

we can prove (4.4) using the similar argument as in the proof of [RS20, (5.5), Theorem 5.1], with a slight modification as indicated above. □

Remark 4.2. *In view of* [RS20, Lemma 4.2]*, we have*

$$\begin{aligned} &\left\| x_{n+1}^{\frac{1-2s}{2}} v^{\rm cs} \right\|_{L^\infty(0,T;L^2(\mathbb{R}^n \times [0,C_1]))} + \left\| x_{n+1}^{\frac{1-2s}{2}} \nabla v^{\rm cs} \right\|_{L^\infty(0,T;L^2(\mathbb{R}^{n+1}_+))} \\ \le& C \|v\|_{L^\infty(0,T;H^s(\mathbb{R}^n))}, \end{aligned}$$

therefore, (4.2) can be achieved by the following sufficient condition:

$$\|v\|_{L^\infty(0,T;H^s(\mathbb{R}^n))} \le E.$$

We now define

$$\omega_1(z) := \log^{-\mu} \left(\frac{C}{z} \right), \quad \text{for } 0 < z < 1.$$

Note that $\omega_1^2(z)$ is concave on $z \in (0, z_0)$ for some sufficiently small $z_0 = z_0(\mu) > 0$. From (4.3) and (4.4), together with [RS20, Lemma 4.4], we obtain

$$\begin{aligned}
&\mathbb{1}_{\{\eta(t)<z_0E\}}\|\boldsymbol{v}(t)\|_{H^{s-\gamma}(\Omega)} \\
\leq &\mathbb{1}_{\{\eta(t)<z_0E\}}C\left(\left\|x_{n+1}^{\frac{1-2s}{2}}\boldsymbol{v}^{\mathrm{cs}}(t)\right\|_{L^2(\hat{\Omega}\times[0,1])}+\left\|x_{n+1}^{\frac{1-2s}{2}+\gamma}\nabla\boldsymbol{v}^{\mathrm{cs}}(t)\right\|_{L^2(\hat{\Omega}\times[0,1])}\right) \\
\leq &\mathbb{1}_{\{\eta(t)<z_0E\}}CE\omega_1\left(\frac{\eta(t)}{E}\right) \\
= &CE\omega_1\left(\mathbb{1}_{\{\frac{\eta(t)}{E}<z_0\}}\frac{\eta(t)}{E}\right).
\end{aligned} \tag{4.5}$$

Using Jensen's inequality for concave functions, we have

$$\frac{1}{T}\int_0^T \omega_1^2\left(\mathbb{1}_{\{\frac{\eta(t)}{E}<z_0\}}\frac{\eta(t)}{E}\right)dt \leq \omega_1^2\left(\frac{1}{T}\int_0^T \mathbb{1}_{\{\frac{\eta(t)}{E}<z_0\}}\frac{\eta(t)}{E}\,dt\right). \tag{4.6}$$

Combining (4.5) and (4.6) implies

$$\int_0^T \mathbb{1}_{\{\eta(t)<z_0E\}}\|\boldsymbol{v}(t)\|^2_{H^{s-\gamma}(\Omega)}\,dt \leq C^2E^2T\omega_1^2\left(\frac{1}{TE}\int_0^T \mathbb{1}_{\{\frac{\eta(t)}{E}<z_0\}}\eta(t)\,dt\right). \tag{4.7}$$

We extend ω_1 so that it is continuous and monotone increasing on $(0,\infty)$. Therefore, (4.7) gives

$$\begin{aligned}
\int_0^T \mathbb{1}_{\{\eta(t)<z_0E\}}\|\boldsymbol{v}(t)\|^2_{H^{s-\gamma}(\Omega)}\,dt \leq &C^2E^2T\omega_1^2\left(\frac{1}{TE}\int_0^T \eta(t)\,dt\right) \\
\leq &C^2E^2T\omega_1^2\left(\frac{\|\eta\|_{L^2(0,T)}}{T^{\frac{1}{2}}E}\right) \\
= &C^2E^2T\omega_1^2\left(\frac{a_s\|(-\Delta)^s v\|_{L^2(0,T;H^{-s}(W))}}{T^{\frac{1}{2}}E}\right).
\end{aligned} \tag{4.8}$$

On the other hand, from [RS20, Lemma 4.4], it follows that

$$\begin{aligned}
&\mathbb{1}_{\{\eta(t)\geq z_0E\}}\|\boldsymbol{v}(t)\|_{H^{s-\gamma}(\Omega)} \\
\leq &\mathbb{1}_{\{\eta(t)\geq z_0E\}}C\left(\left\|x_{n+1}^{\frac{1-2s}{2}}\boldsymbol{v}^{\mathrm{cs}}(t)\right\|_{L^2(\hat{\Omega}\times[0,1])}+\left\|x_{n+1}^{\frac{1-2s}{2}+\gamma}\nabla\boldsymbol{v}^{\mathrm{cs}}(t)\right\|_{L^2(\hat{\Omega}\times[0,1])}\right) \\
\leq &\mathbb{1}_{\{\eta(t)\geq z_0E\}}C\left(\left\|x_{n+1}^{\frac{1-2s}{2}}v^{\mathrm{cs}}\right\|_{L^\infty(0,T;L^2(\mathbb{R}^n\times[0,C_1]))}+\left\|x_{n+1}^{\frac{1-2s}{2}}\nabla v^{\mathrm{cs}}\right\|_{L^\infty(0,T;L^2(\mathbb{R}^{n+1}_+))}\right) \\
\leq &\mathbb{1}_{\{\eta(t)\geq z_0E\}}CE \\
\leq &Cz_0^{-1}\eta(t) \\
= &Cz_0^{-1}a_s\|(-\Delta)^s\boldsymbol{v}(t)\|_{H^{-s}(W)}.
\end{aligned} \tag{4.9}$$

Squaring both sides of (4.9) and subsequently integrating it, we obtain

$$\begin{aligned}
&\int_0^T \mathbb{1}_{\{\eta(t)\geq z_0E\}}\|\boldsymbol{v}(t)\|^2_{H^{s-\gamma}(\Omega)}\,dt \\
\leq &C^2z_0^{-2}a_s^2\|(-\Delta)^s v\|^2_{L^2(0,T;H^{-s}(W))} \\
= &C^2z_0^{-2}E^2a_s^2\left(\frac{\|(-\Delta)^s v\|_{L^2(0,T;H^{-s}(W))}}{E}\right)^2.
\end{aligned} \tag{4.10}$$

Summing (4.8) and (4.10) yields

$$
\begin{aligned}
\|v\|^2_{L^2(0,T;H^{s-\gamma}(\Omega))} =& \int_0^T \|\boldsymbol{v}(t)\|^2_{H^{s-\gamma}(\Omega)}\,dt \\
\leq& C^2E^2\left[T\omega_1^2\left(\frac{a_s\|(-\Delta)^s v\|_{L^2(0,T;H^{-s}(W))}}{T^{\frac{1}{2}}E}\right)\right. \\
&\left. + z_0^{-2}a_s^2\left(\frac{\|(-\Delta)^s v\|_{L^2(0,T;H^{-s}(W))}}{E}\right)^2\right].
\end{aligned} \tag{4.11}
$$

We now define

$$\omega(z) := \left(T\omega_1^2(T^{-\frac{1}{2}}a_s z) + z_0^{-2}a_s^2 z^2\right)^{\frac{1}{2}}.$$

Note that $\omega(z)$ is of logarithmic type when z is small. Therefore, (4.11) can be written as

$$\|v\|_{L^2(0,T;H^{s-\gamma}(\Omega))} \leq CE\omega\left(\frac{\|(-\Delta)^s v\|_{L^2(0,T;H^{-s}(W))}}{E}\right).$$

We summarize the above discussions in the following corollary.

Corollary 4.3. *Let $W\subset\Omega_e$ be a nonempty open bounded Lipschitz set and $\overline{W}\cap\overline{\Omega}=\emptyset$. If there exists a constant $E>0$ such that*

$$\|v\|_{L^\infty(0,T;H^s(\mathbb{R}^n))} \leq E, \tag{4.12}$$

then there exists a constant $C>1$ and a function of logarithmic type ω, both depending only on n,s,Ω,W,γ,T, such that

$$\|v\|_{L^2(0,T;H^{s-\gamma}(\Omega))} \leq CE\omega\left(\frac{\|(-\Delta)^s v\|_{L^2(0,T;H^{-s}(W))}}{E}\right).$$

4.2. **Quantitative unique continuation.** Given any $F\in L^2(\Omega_T)$, by Corollary 2.2, there exists a unique solution v_F of the backward wave equation

$$\begin{cases} \left(\partial_t^2 + (-\Delta)^s + q\right)v_F = F & \text{in } \Omega_T, \\ v_F = 0 & \text{in } (\Omega_e)_T, \\ v_F = \partial_t v_F = 0 & \text{in } \mathbb{R}^n\times\{T\}, \end{cases} \tag{4.13}$$

such that

$$\|v_F\|_{L^\infty(0,T;\widetilde{H}^s(\Omega))} \leq C_0\|F\|_{L^2(\Omega_T)},$$

for some constant $C_0>0$ independent of v_F and F. Choosing $E=C_0\|F\|_{L^2(\Omega_T)}$, the condition (4.12) satisfies, and then we can employ Corollary 4.3 with $v=v_F$ to obtain

$$\|v_F\|_{L^2(0,T;H^{s-\gamma}(\Omega))} \leq C\|F\|_{L^2(\Omega_T)}\omega\left(\frac{\|(-\Delta)^s v_F\|_{L^2(0,T;H^{-s}(W))}}{\|F\|_{L^2(\Omega_T)}}\right). \tag{4.14}$$

Meanwhile, for any function $u\in H^{-2}(0,T;H^{s-\gamma}(\Omega))$, by the duality argument, one has

$$\|u\|_{H^{-2}(0,T;H^{s-\gamma}(\Omega))} \leq \|u\|_{L^2(0,T;H^{s-\gamma}(\Omega))}. \tag{4.15}$$

Likewise, if $\boldsymbol{u}(T)=\partial_t\boldsymbol{u}(T)=0$, we can see that

$$\left\|\partial_t^2 u\right\|_{H^{-2}(0,T;H^{-s-\gamma}(\Omega))} \leq \|u\|_{L^2(0,T;H^{-s-\gamma}(\Omega))}. \tag{4.16}$$

Now, back to the backward wave equation (4.13), if $\|q\|_{L^\infty(\Omega)} \leq M$, by (4.15), we have

$$
\begin{aligned}
&\|F\|_{H^{-2}(0,T;H^{-s-\gamma}(\Omega))} \\
\leq &\left\|\partial_t^2 v_F\right\|_{H^{-2}(0,T;H^{-s-\gamma}(\Omega))} \\
&+\|(-\Delta)^s v_F\|_{H^{-2}(0,T;H^{-s-\gamma}(\Omega))} + \|qv_F\|_{H^{-2}(0,T;H^{-s-\gamma}(\Omega))} \\
\leq &\left\|\partial_t^2 v_F\right\|_{H^{-2}(0,T;H^{-s-\gamma}(\Omega))} \\
&+\|v_F\|_{H^{-2}(0,T;H^{s-\gamma}(\Omega))} + \|qv_F\|_{L^2(\Omega_T)} \\
\leq &\left\|\partial_t^2 v_F\right\|_{H^{-2}(0,T;H^{-s-\gamma}(\Omega))} + \|v_F\|_{L^2(0,T;H^{s-\gamma}(\Omega))} + M\|v_F\|_{L^2(\Omega_T)}.
\end{aligned} \tag{4.17}
$$

Since v_F satisfies $v_F = \partial_t v_F = 0$ in $\mathbb{R}^n \times \{T\}$, via (4.16), one obtains

$$
\left\|\partial_t^2 v_F\right\|_{H^{-2}(0,T;H^{-s-\gamma}(\Omega))} \leq \|v_F\|_{L^2(0,T;H^{-s-\gamma}(\Omega))}. \tag{4.18}
$$

Therefore, plugging (4.18) into (4.17) yields

$$
\|F\|_{H^{-2}(0,T;H^{-s-\gamma}(\Omega))} \leq C\|v_F\|_{L^2(0,T;H^{s-\gamma}(\Omega))} \tag{4.19}
$$

provided $0 < \gamma < s$. Combining (4.14) and (4.19)gives

$$
\|F\|_{H^{-2}(0,T;H^{-s-\gamma}(\Omega))} \leq C\|F\|_{L^2(\Omega_T)}\omega\left(\frac{\|(-\Delta)^s v_F\|_{L^2(0,T;H^{-s}(W))}}{\|F\|_{L^2(\Omega_T)}}\right). \tag{4.20}
$$

We now investigate the Poisson operator $\mathcal{P}_q$ given in (2.5) in the following lemma.

Lemma 4.4. *Suppose that $q \in L^\infty(\Omega)$. Let $\mathcal{P}_q$ be the Poisson operator given in (2.5). Then*

$$
\mathcal{P}_q - \mathrm{Id} : L^2(0,T;H^{2s}_{W}) \to L^2(\Omega_T) \tag{4.21}
$$

is a compact injective linear operator. Moreover, for each $F \in L^2(\Omega_T)$, the adjoint operator of $\mathcal{P}_q - \mathrm{Id}$ is given by

$$
(\mathcal{P}_q - \mathrm{Id})^* F = -(-\Delta)^s v_F, \tag{4.22}
$$

where v_F is the solution of (4.13).

Proof of Lemma 4.4. It is worth pointing out that the function $(\mathcal{P}_q - \mathrm{Id})f$ is equal to $\mathcal{P}_q f|_{\Omega_T}$, for any $f \in L^2(0,T;H^{2s}_{W})$. We split the proof into several steps.

Step 1. Compactness.

Using (2.4), we can see that

$$
\|(\mathcal{P}_q - \mathrm{Id})f\|_{L^2(0,T;\widetilde{H}^s(\Omega))} + \|\partial_t((\mathcal{P}_q - \mathrm{Id})f)\|_{L^2(0,T;L^2(\Omega))} \leq C_T\|f\|_{L^2(0,T;H^{2s}_{W})}.
$$

In other words, $\mathcal{P}_q - \mathrm{Id} : L^2(0,T;H^{2s}_{W}) \to L^2(0,T;\widetilde{H}^s(\Omega)) \cap H^1(0,T;L^2(\Omega))$ is a bounded linear operator. Since the embedding $\widetilde{H}^s(\Omega) \hookrightarrow L^2(\Omega)$ is compact, using the Aubin-Lions-Simon Theorem in [BF13, Theorem II.5.16], we know that the embedding

$$
L^2(0,T;\widetilde{H}^s(\Omega)) \cap H^1(0,T;L^2(\Omega)) \hookrightarrow L^2(\Omega_T) \quad \text{is compact.}
$$

Therefore, we see that the operator (4.21) is compact.

Step 2. Injectivity.

Suppose that $f \in \ker(\mathcal{P}_q - \mathrm{Id})$, then $\mathcal{P}_q f = 0$ in Ω_T. From the definition of the Poisson operator, $u = \mathcal{P}_q f$ satisfies

$$
\left(\partial_t^2 + (-\Delta)^s + q\right) u = 0 \quad \text{in} \quad \Omega_T. \tag{4.23}
$$

Since $u=0$ in Ω_T, from (4.23), we have that $(-\Delta)^s u = 0$ in Ω_T. Therefore, using the strong uniqueness for the fractional Laplacian again, we know that $u\equiv 0$ throughout $\mathbb{R}^n$, and hence $f\equiv 0$, which concludes that $\mathcal{P}_q$ is injective.

Step 3. Computing the adjoint operator.

Given any $f\in L^2(0,T;H^{2s}_{\overline{W}})$, $F\in L^2(\Omega_T)$, we have

$$\begin{aligned}
&((\mathcal{P}_q-\mathrm{Id})f,F)_{L^2(\Omega_T)}\\
&=(\mathcal{P}_q f,F)_{L^2(\Omega_T)}\\
&=\int_{\Omega_T}\mathcal{P}_q f\left(\partial_t^2 v_F+(-\Delta)^s v_F+qv_F\right)dxdt\\
&=\int_{\Omega_T}\left(\partial_t^2\mathcal{P}_q f+q\mathcal{P}_q f\right)v_F\,dxdt+\left(\int_{(\mathbb{R}^n)_T}-\int_{(\Omega_e)_T}\right)(\mathcal{P}_q f)(-\Delta)^s v_F\,dxdt\\
&=\int_{\Omega_T}\left(\partial_t^2\mathcal{P}_q f+(-\Delta)^s\mathcal{P}_q f+q\mathcal{P}_q f\right)v_F\,dxdt-\int_{(\Omega_e)_T}(\mathcal{P}_q f)(-\Delta)^s v_F\,dxdt\\
&=-\int_{(\mathbb{R}^n)_T}f(-\Delta)^s v_F\,dxdt.
\end{aligned}$$

Consequently, the arbitrariness of f implies (4.22). □

Remark 4.5. Lemma 4.4 implies that $(\mathcal{P}_q-\mathrm{Id})^*:L^2(\Omega_T)\to L^2(0,T;H^{-2s}(W))$. On the other hand, by the Riesz representation theorem, $L^2(0,T;H^{-2s}(W))$ is isomorphic to $L^2(0,T;H^{2s}_{\overline{W}})$. In view of this identification, $(\mathcal{P}_q-\mathrm{Id})^*(\mathcal{P}_q-\mathrm{Id})$ can be regarded as a compact, self-adjoint, positive-definite operator from $L^2(0,T;H^{2s}_{\overline{W}})$ to itself. Furthermore, (4.22) gives $(\mathcal{P}_q-\mathrm{Id})^*F=-(-\Delta)^s v_F$ for $F\in L^2(\Omega_T)$. However, by the energy estimate (2.3), the regularity of $(-\Delta)^s v_F$ can be improved to $L^2(0,T;H^{-s}(W))$.

Combining (4.20) and (4.22), we obtain the following theorem.

Theorem 4.1. *Let $0<s<1$, $q\in L^\infty(\Omega)$ with $\|q\|_{L^\infty(\Omega)}\le M$, and $W\subset\Omega_e$ be nonempty open such that $\overline{W}\cap\overline{\Omega}=\emptyset$. There exist a constant $C>1$ independent of F and a logarithmic function ω, both depending only on n,s,Ω,W,γ,T,M, such that*

$$\|F\|_{H^{-2}(0,T;H^{-s-\gamma}(\Omega))}\le C\|F\|_{L^2(\Omega_T)}\,\omega\left(\frac{\|(\mathcal{P}_q-\mathrm{Id})^*F\|_{L^2(0,T;H^{-s}(W))}}{\|F\|_{L^2(\Omega_T)}}\right) \tag{4.24}$$

for all $F\in L^2(\Omega_T)$.

4.3. **Quantitative Runge approximation.** It follows from Remark 4.5 that there exist eigenvalues $\{\mu_j\}_{j=1}^\infty$ with $\mu_1\ge\mu_2\ge\cdots\to 0$ and orthonormal eigenfunctions $\{\varphi_j\}_{j=1}^\infty$ of $(\mathcal{P}_q-\mathrm{Id})^*(\mathcal{P}_q-\mathrm{Id})$. Note that the orthonormality of $\{\varphi_j\}_{j=1}^\infty$ is with respect to the inner product $(\cdot,\cdot):=(\cdot,\cdot)_{L^2(0,T;H^{2s}_{\overline{W}})}$. For clarity, we denote $\langle\cdot,\cdot\rangle$ the $L^2(0,T;H^{-2s}(W))\times L^2(0,T;H^{2s}_{\overline{W}})$ duality pair. Define

$$\sigma_j:=\sqrt{\mu_j}\quad\text{and}\quad w_j:=\frac{1}{\sigma_j}(\mathcal{P}_q-\mathrm{Id})\,\varphi_j. \tag{4.25}$$

We can easily verify that

$$\begin{aligned}(w_j, w_k)_{L^2(\Omega_T)} &= \frac{1}{\sigma_j\sigma_k}\left((\mathcal{P}_q - \mathrm{Id})\varphi_j, (\mathcal{P}_q - \mathrm{Id})\varphi_k\right)_{L^2(\Omega_T)} \\ &= \frac{1}{\sigma_j\sigma_k}\langle(\mathcal{P}_q - \mathrm{Id})^*(\mathcal{P}_q - \mathrm{Id})\varphi_j, \varphi_k\rangle \\ &= \frac{\sigma_j}{\sigma_k}\left(\varphi_j, \varphi_k\right)_{L^2(0,T;H^{2s}_{\overline{W}})} = \delta_{jk},\end{aligned}$$

that is, $\{w_j\}_{j=1}^{\infty}$ is an orthonormal set in $L^2(\Omega_T)$. Also, we have

$$(\mathcal{P}_q - \mathrm{Id})^* w_j = \frac{1}{\sigma_j}(\mathcal{P}_q - \mathrm{Id})^*(\mathcal{P}_q - \mathrm{Id})\varphi_j = \frac{\mu_j}{\sigma_j}\varphi_j = \sigma_j\varphi_j. \tag{4.26}$$

We can prove the completeness of $\{w_j\}_{j=1}^{\infty}$.

Lemma 4.6. *The set $\{w_j\}_{j=1}^{\infty}$ is complete in $L^2(\Omega_T)$. In other words, it is an orthonormal basis of $L^2(\Omega_T)$.*

Proof. Let $v \in L^2(\Omega_T)$ be such that $(v, w_j)_{L^2(\Omega_T)} = 0$ for all j, then

$$(v, \mathcal{P}_q\varphi_j)_{L^2(\Omega_T)} = (v, (\mathcal{P}_q - \mathrm{Id})\varphi_j)_{L^2(\Omega_T)} = 0, \quad \text{for all } j.$$

Since $\{\varphi_j\}_{j=1}^{\infty}$ forms an orthonormal basis of $L^2(0,T;H^{2s}_{\overline{W}})$, then

$$(v, \mathcal{P}_q f)_{L^2(\Omega_T)} = 0 \quad \text{ for all } f \in C_c^{\infty}(W_T).$$

In view of the Runge approximation (Theorem 3.1), we conclude that $v \equiv 0$. ☐

We now fix a number $\alpha > 0$. We define the operator $R_\alpha : L^2(\Omega_T) \to L^2(0,T;H^{2s}_{\overline{W}})$ by the finite sum

$$R_\alpha\phi := \sum_{\sigma_j > \alpha}\frac{1}{\sigma_j}(\phi, w_j)_{L^2(\Omega_T)}\varphi_j.$$

Since $\{\varphi_j\}_{j=1}^{\infty}$ is an orthonormal basis of $L^2(0,T;H^{2s}_{\overline{W}})$, and $\{\sigma_j\}$ is non-increasing, using Parseval's identity, then it is easy to see that

$$\begin{aligned}\|R_\alpha\phi\|^2_{L^2(0,T;H^{2s}_{\overline{W}})} &= \sum_{\sigma_j > \alpha}\frac{1}{\sigma_j^2}|(\phi, w_j)_{L^2(\Omega_T)}|^2 \\ &\leq \frac{1}{\alpha^2}\|\phi\|^2_{L^2(\Omega_T)},\end{aligned} \tag{4.27}$$

and

(4.28)

$$\begin{aligned}\|(\mathcal{P}_q - \mathrm{Id})(R_\alpha\phi) - \phi\|^2_{L^2(\Omega_T)} &= \left\|\sum_{\sigma_j > \alpha}\frac{1}{\sigma_j}(\phi, w_j)_{L^2(\Omega_T)}(\mathcal{P}_q - \mathrm{Id})\varphi_j - \phi\right\|^2_{L^2(\Omega_T)} \\ &= \left\|\sum_{\sigma_j > \alpha}(\phi, w_j)_{L^2(\Omega_T)}w_j - \phi\right\|^2_{L^2(\Omega_T)} \\ &= \left\|\sum_{\sigma_j \leq \alpha}(\phi, w_j)_{L^2(\Omega_T)}w_j\right\|^2_{L^2(\Omega_T)} \\ &= \sum_{\sigma_j \leq \alpha}\left|(\phi, w_j)_{L^2(\Omega_T)}\right|^2,\end{aligned}$$

where we have used the orthonormality of $\{w_j\}_{j=1}^{\infty}$ in $L^2(\Omega_T)$.

Let us define

$$r_\alpha := \sum_{\sigma_j \leq \alpha} (\phi, w_j)_{L^2(\Omega_T)} w_j. \tag{4.29}$$

In particular, for any $\phi \in H_0^2(0,T;\widetilde{H}^{s+\gamma}(\Omega)) \subset L^2(\Omega_T)$, combining (4.28) and (4.29), we have

$$\begin{aligned}
&\|(\mathcal{P}_q - \mathrm{Id})\,(R_\alpha\phi) - \phi\|_{L^2(\Omega_T)}^2 \\
=&\left|(\phi, r_\alpha)_{L^2(\Omega_T)}\right| \\
\leq&\|\phi\|_{H_0^2(0,T;\widetilde{H}^{s+\gamma}(\Omega))}\,\|r_\alpha\|_{H^{-2}(0,T;H^{-s-\gamma}(\Omega))}\,.
\end{aligned} \tag{4.30}$$

We now choose $F = r_\alpha \in L^2(\Omega_T)$ in (4.24), and we obtain

$$\|r_\alpha\|_{H^{-2}(0,T;H^{-s-\gamma}(\Omega))} \leq C\,\|r_\alpha\|_{L^2(\Omega_T)}\,\omega\left(\frac{\|(\mathcal{P}_q - \mathrm{Id})^* r_\alpha\|_{L^2(0,T;H^{-s}(W))}}{\|r_\alpha\|_{L^2(\Omega_T)}}\right),$$

and thus (4.30) implies

$$\begin{aligned}
&\|(\mathcal{P}_q - \mathrm{Id})(R_\alpha\phi) - \phi\|_{L^2(\Omega_T)}^2 \\
\leq&C\|\phi\|_{H_0^2(0,T;\widetilde{H}^{s+\gamma}(\Omega))}\|r_\alpha\|_{L^2(\Omega_T)}\omega\left(\frac{\|(\mathcal{P}_q - \mathrm{Id})^* r_\alpha\|_{L^2(0,T;H^{-s}(W))}}{\|r_\alpha\|_{L^2(\Omega_T)}}\right).
\end{aligned} \tag{4.31}$$

In view of (4.26), we have

$$(\mathcal{P}_q - \mathrm{Id})^*\, r_\alpha = \sum_{\sigma_j \leq \alpha} (\phi, w_j)_{L^2(\Omega_T)}\,(\mathcal{P}_q - \mathrm{Id})^* w_j = \sum_{\sigma_j \leq \alpha} (\phi, w_j)_{L^2(\Omega_T)}\sigma_j\varphi_j.$$

From the property that $\{\varphi_j\}_{j=1}^\infty$ is an orthonormal basis of $L^2(0,T;H^{2s}_{W})$, it follows

$$\begin{aligned}
&\|(\mathcal{P}_q - \mathrm{Id})^* r_\alpha\|_{L^2(0,T;H^{-s}(W))}^2 \\
\leq&\|(\mathcal{P}_q - \mathrm{Id})^* r_\alpha\|_{L^2(0,T;H^{2s}_{W})}^2 \\
=&\sum_{\sigma_j \leq \alpha} \sigma_j^2\left|(\phi, w_j)_{L^2(\Omega_T)}\right|^2 \\
\leq&\alpha^2\sum_{\sigma_j \leq \alpha}\left|(\phi, w_j)_{L^2(\Omega_T)}\right|^2 \\
=&\alpha^2\|r_\alpha\|_{L^2(\Omega_T)}^2,
\end{aligned}$$

where we have used (4.29) in the last equality. Since ω is monotone non-decreasing, (4.31) gives

$$\|(\mathcal{P}_q - \mathrm{Id})(R_\alpha\phi) - \phi\|_{L^2(\Omega_T)}^2 \leq C\|\phi\|_{H_0^2(0,T;\widetilde{H}^{s+\gamma}(\Omega))}\,\|r_\alpha\|_{L^2(\Omega_T)}\,\omega(\alpha). \tag{4.32}$$

Furthermore, observe that

$$\begin{aligned}
&(\mathcal{P}_q - \mathrm{Id})(R_\alpha\phi) - \phi \\
=&\sum_{\sigma_j > \alpha}\frac{1}{\sigma_j}(\phi, w_j)_{L^2(\Omega_T)}(\mathcal{P}_q - \mathrm{Id})\varphi_j - \phi \\
=&\sum_{\sigma_j > \alpha}(\phi, w_j)_{L^2(\Omega_T)}w_j - \phi \\
=&-\sum_{\sigma_j \leq \alpha}(\phi, w_j)_{L^2(\Omega_T)}w_j \\
=&-r_\alpha.
\end{aligned} \tag{4.33}$$

Combining (4.32) and (4.33) yields

$$\|(\mathcal{P}_q - \mathrm{Id})(R_\alpha \phi) - \phi\|_{L^2(\Omega_T)} \leq C\|\phi\|_{H_0^2(0,T;\widetilde{H}^{s+\gamma}(\Omega))}\omega(\alpha). \tag{4.34}$$

Given any $\epsilon > 0$, there exists a unique $\alpha > 0$ such that $\epsilon = \omega(\alpha)$. Write $f_\epsilon = R_\alpha \phi$, and we know that (4.27) can be rewritten as

$$\|f_\epsilon\|_{L^2(0,T;H^{2s}_{\overline{W}})} \leq \frac{1}{\omega^{-1}(\epsilon)}\|\phi\|_{L^2(\Omega_T)}, \tag{4.35}$$

where ω^{-1} is the inverse function of ω. Now, as in [RS20, Remark 3.4], since

$$\omega(t) = C|\log t|^{-\sigma} \quad \text{for } t \text{ small},$$

we can take $\frac{1}{\omega^{-1}(\epsilon)} \leq \exp(\widetilde{C}\epsilon^{-\mu})$ with $\widetilde{C} \geq C^{1/\sigma}$ and $\mu = 1/\sigma$ for all $\epsilon > 0$. Therefore, restating (4.34) and (4.35) leads to the following theorem.

Theorem 4.2 (Quantitative Runge approximation)**.** *Let $\|q\|_{L^\infty(\Omega)} \leq M$ and fix a parameter $\gamma > 0$. Given any $\phi \in H_0^2(0,T;\widetilde{H}^{s+\gamma}(\Omega))$, and any $\epsilon > 0$, there exists $f_\epsilon \in L^2(0,T;H^{2s}_{\overline{W}})$ such that*

$$\|\mathcal{P}_q f_\epsilon - \phi\|_{L^2(\Omega_T)} = \|\mathcal{P}_q f_\epsilon - f_\epsilon - \phi\|_{L^2(\Omega_T)} \leq C\|\phi\|_{H_0^2(0,T;\widetilde{H}^{s+\gamma}(\Omega))}\epsilon,$$

and

$$\|f_\epsilon\|_{L^2(0,T;H^{2s}_{\overline{W}})} \leq C\exp(\widetilde{C}\epsilon^{-\mu})\|\phi\|_{L^2(\Omega_T)},$$

for some positive constants μ, C and $\widetilde{C}$, depending only on $n, s, \gamma, \Omega, W, \gamma, T, M$.

4.4. **Proof of Theorem 1.2.** Finally, we can prove our logarithmic stability estimate of the inverse problem for the fractional wave equation.

Proof of Theorem 1.2. Let $\epsilon > 0$ be a parameter to be chosen later. We fix two arbitrary functions $\phi_j \in H_0^2(0,T;\widetilde{H}^{s+\gamma}(\Omega))$ with $\|\phi_j\|_{H_0^2(0,T;\widetilde{H}^{s+\gamma}(\Omega))} = 1$, for $j = 1, 2$. Using Theorem 4.2, there exist functions $f_j \in L^2(0,T;H^{2s}_{\overline{W}})$ such that

$$\|t_j\|_{L^2(\Omega_T)} \leq C\epsilon \quad \text{and} \quad \|f_j\|_{L^2(0,T;H^{2s}_{\overline{W}})} \leq C\exp(\widetilde{C}\epsilon^{-\mu})$$

with

$$t_j = u_j - \phi_j \quad \text{and} \quad u_j = \mathcal{P}_{q_j} f_j,$$

where we have used the fact

$$\|\phi_j\|_{L^2(\Omega_T)} \leq \|\phi_j\|_{H_0^2(0,T;\widetilde{H}^{s+\gamma}(\Omega))} = 1.$$

Inserting u_j into the identity (2.8) in Lemma 2.4, we obtain

$$\begin{aligned}
&\int_{\Omega_T} (q_1 - q_2)\phi_1\phi_2^* \, dxdt \\
=&\int_{(\Omega_e)_T} ((\Lambda_{q_1} - \Lambda_{q_2})f_1)\, f_2^* \, dxdt - \int_{\Omega_T} (q_1 - q_2)\left(\phi_2 t_1^* + \phi_1 t_2^* + t_1 t_2^*\right) dxdt.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\left|\int_{\Omega_T} (q_1 - q_2)\phi_1\phi_2^* \, dxdt\right| \\
\leq& \|\Lambda_{q_1} - \Lambda_{q_2}\|_* \|f_1\|_{L^2(0,T;H^{2s}_{\overline{W}})}\|f_2\|_{L^2(0,T;H^{2s}_{\overline{W}})} \\
&+ 2M\left(\|t_1\|_{L^2(\Omega_T)} + \|t_2\|_{L^2(\Omega_T)} + \|t_1\|_{L^2(\Omega_T)}\|t_2\|_{L^2(\Omega_T)}\right) \\
\leq& C^2 \|\Lambda_{q_1} - \Lambda_{q_2}\|_* \exp\left(2\widetilde{C}\epsilon^{-\mu}\right) + 4CM\epsilon.
\end{aligned}$$

Choosing $\epsilon = |\log(\|\Lambda_{q_1} - \Lambda_{q_2}\|_*)|^{-\frac{1}{\mu}}$, we know that

$$\left|\int_{\Omega_T} (q_1 - q_2)\phi_1\phi_2^* \, dxdt\right| \leq \omega\left(\|\Lambda_{q_1} - \Lambda_{q_2}\|_*\right),$$

for some logarithmic modulus of continuity (which is monotone non-decreasing) ω. Recalling the definition of the function space $Z^{-s}(\Omega, T)$ in Definition 1.2, we finally prove the assertion. □

5. Exponential instability of the inverse problem

In the last section of this paper, we demonstrate that the logarithmic stability in Theorem 1.2 is optimal, by showing the exponential instability phenomenon for the fractional wave equation. The ideas of the construction of the instability are motivated by Mandache's pioneer work [Man01].

5.1. Matrix representation via an orthonormal basis.

For $r > 0$, let B_r be the ball of radius $r > 0$ with center at 0. First of all, we introduce a set of basis of $L^2(B_3 \setminus \overline{B_2})$. The following proposition can be found in [RS18, Lemma 2.1 and Remark 2.2]:

Proposition 5.1. *Let $n \geq 2$. Given any $m \geq 0$, we define*

$$\ell_m := \binom{m+n-1}{n-1} - \binom{m+n-3}{n-1} \leq 2(1+m)^{n-2}.$$

There exists an orthonormal basis $\{Y_{mk\ell} : m \geq 0, k \geq 0, 0 \leq \ell \leq \ell_m\}$ of $L^2(B_3 \setminus \overline{B_2})$ such that

$$\left\|\tilde{Y}_{mk\ell}\right\|_{L^2(B_1)} \leq C'_{n,s} e^{-C''_{n,s}(m+k)} \tag{5.1}$$

for some constants $C'_{n,s}$ and $C''_{n,s}$ (both depending only on n and s), where $\tilde{Y}_{mk\ell} \in H^s(\mathbb{R}^n)$ is the unique solution to

$$\begin{cases} (-\Delta)^s \tilde{Y}_{mk\ell} = 0 & \text{in } B_1, \\ \tilde{Y}_{mk\ell} = \mathbb{1}_{B_3 \setminus \overline{B_2}} Y_{mk\ell} & \text{in } \mathbb{R}^n \setminus \overline{B_1}. \end{cases}$$

Remark 5.2. *For $n = 1$, the "sphere" $\partial B_1 \subset \mathbb{R}$ consists only two end points $\{-1, 1\}$, which is no longer a sphere. Therefore, we need to find another basis for the one-dimensional case. We shall discuss the case of $n = 1$ later.*

Given any bounded linear operator $\mathcal{A} : L^2(B_3 \setminus \overline{B_2}) \to L^2(B_3 \setminus \overline{B_2})$, we define

$$a_{m_1k_1\ell_1}^{m_2k_2\ell_2} := (\mathcal{A}Y_{m_1k_1\ell_1}, Y_{m_2k_2\ell_2})_{L^2(B_3 \setminus \overline{B_2})}.$$

Let $\left(a_{m_1k_1\ell_1}^{m_2k_2\ell_2}\right)$ be the tensor with entries $a_{m_1k_1\ell_1}^{m_2k_2\ell_2}$, and consider the following Banach space:

$$X := \left\{\left(a_{m_1k_1\ell_1}^{m_2k_2\ell_2}\right) : \left\|\left(a_{m_1k_1\ell_1}^{m_2k_2\ell_2}\right)\right\|_X < \infty\right\},$$

where

$$\left\|\left(a_{m_1k_1\ell_1}^{m_2k_2\ell_2}\right)\right\|_X := \sup_{m_ik_i\ell_i} (1 + \max\{m_1 + k_1, m_2 + k_2\})^{n+2} \left|a_{m_1k_1\ell_1}^{m_2k_2\ell_2}\right|, \tag{5.2}$$

see e.g. [RS18, Definition 2.7]. The following lemma can be found in [RS18, (20)], which plays an essential role in our work.

Lemma 5.3. *If $n \geq 2$, then*

$$\|\mathcal{A}\|_{L^2(B_3 \setminus \overline{B_2}) \to L^2(B_3 \setminus \overline{B_2})} \leq 4\left\|\left(a_{m_1k_1\ell_1}^{m_2k_2\ell_2}\right)\right\|_X.$$

Thanks to Lemma 5.3, we can regard the tensor $\left(a_{m_1k_1\ell_1}^{m_2k_2\ell_2}\right)$ as the *matrix representation* of the bounded linear operator $\mathcal{A}$.

5.2. **Special weak solutions.** In view of Proposition 5.1, we need to introduce some special solutions. We begin with the following lemma.

Lemma 5.4. *Let* $\chi = \chi(t) \in C_c^\infty((0,T))$ *and*

$$q \in B_{+,R}^\infty := \{q \textit{ is real-valued}: \ 0 \le q \le R \textit{ a.e.}\}.$$

Given any $f = f(x) \in L^2(B_3 \setminus \overline{B_2})$*, there exists a unique solution* u *to*

$$\begin{cases} \left(\partial_t^2 + (-\Delta)^s + q\right) u = 0 & \textit{in } (B_1)_T, \\ u(x,t) = \chi(t)\mathbb{1}_{B_3\setminus\overline{B_2}}(x)f(x) & \textit{in } (\mathbb{R}^n \setminus \overline{B_1})_T, \\ u = \partial_t u = 0 & \textit{in } \mathbb{R}^n \times \{0\}, \end{cases} \tag{5.3}$$

and

$$\|u\|_{L^\infty(0,T;L^2(B_1))} \le C_{R,T,n,s}\|\chi\|_{W^{2,\infty}(0,T)}\|f\|_{L^2(B_3\setminus\overline{B}_2)} \tag{5.4}$$

for some positive constant $C_{R,T,n,s}$.

Proof of Lemma 5.4. Recall from [RS18, Remark 2.2], there exists a unique solution $\tilde{f}$ to

$$\begin{cases} (-\Delta)^s\tilde{f} = 0 & \text{in } B_1, \\ \tilde{f} = \mathbb{1}_{B_3\setminus\overline{B_2}}f & \text{in } \mathbb{R}^n \setminus \overline{B_1}, \end{cases} \tag{5.5}$$

such that

$$\|\tilde{f}\|_{L^2(\mathbb{R}^n)} \le C_{n,s}\|f\|_{L^2(B_3\setminus\overline{B}_2)} \tag{5.6}$$

for some constant $C_{n,s} > 0$. Let $F(x,t) := -(\chi''(t) + \chi(t)q(x))\tilde{f}(x)$. Since $F \in L^2(B_1 \times (0,T))$, using Theorem 2.1, there exists a unique solution v to

$$\begin{cases} \left(\partial_t^2 + (-\Delta)^s + q\right) v = F & \text{in } (B_1)_T, \\ v = 0 & \text{in } (\mathbb{R}^n \setminus \overline{B_1})_T, \\ v = \partial_t v = 0 & \text{in } \mathbb{R}^n \times \{0\}, \end{cases} \tag{5.7}$$

satisfying

$$\|v\|_{L^\infty(0,T;\tilde{H}^s(B_1))} + \|\partial_t v\|_{L^\infty(0,T;L^2(B_1))} \le C_{R,T,n,s}\|F\|_{L^2(B_1\times(0,T))}. \tag{5.8}$$

In other words,

$$u(x,t) := v(x,t) + \chi(t)\tilde{f}(x)$$

is the unique solution to (5.3). Therefore, we can obtain from (5.8) that

$$\begin{aligned} \|u\|_{L^\infty(0,T;L^2(B_1))} &\le \|v\|_{L^\infty(0,T;L^2(B_1))} + \|\chi\|_{L^\infty(0,T)}\|\tilde{f}\|_{L^2(B_1)} \\ &\le C_{R,T,n,s}\|\chi\|_{W^{2,\infty}(0,T)}\|\tilde{f}\|_{L^2(B_1)}. \end{aligned} \tag{5.9}$$

Combining (5.6) and (5.9) implies (5.4). □

Based on Lemma 5.4, we define the DN map

$$(\Lambda_q\chi)(f) = \Lambda_q(\chi f) := (-\Delta)^s u|_{(B_3\setminus\overline{B}_2)_T} \quad \text{for all } f \in L^2(B_3 \setminus \overline{B_2}), \tag{5.10}$$

where u is given in (5.3). In view of (5.4), we know that

$$\Lambda_q\chi : L^2(B_3 \setminus \overline{B}_2) \to L^\infty(0,T;H^{-2s}(B_3 \setminus \overline{B_2})) \tag{5.11}$$

is a bounded operator. However, the regularity given in (5.11) is insufficient for our purpose. We therefore consider the difference of the DN maps

$$(\Lambda_q - \Lambda_0)(\chi f)$$

for $f \in L^2(B_3 \setminus \overline{B_2})$, where Λ_0 is the DN map (5.10) associated with (5.3) corresponding to $q \equiv 0$. By modifying the idea in the proof of [RS18, Remark 2.5], we can prove the following lemma.

Lemma 5.5. *The operator $(\Lambda_q - \Lambda_0)\chi : L^2(B_3 \setminus \overline{B}_2) \to L^\infty(0,T; L^2(B_3 \setminus \overline{B_2}))$ is bounded. Precisely, the following estimate holds:*

$$\|(\Lambda_q - \Lambda_0)(\chi f)\|_{L^\infty(0,T;L^2(B_3\setminus\overline{B_2}))} \le C_{R,T,n,s}\|\chi\|_{W^{2,\infty}(0,T)}\|f\|_{L^2(B_3\setminus\overline{B}_2)} \tag{5.12}$$

for all $f \in L^2(B_3 \setminus \overline{B}_2)$.

Proof. Let $w(x,t) = u_q(x,t) - u_0(x,t)$, where u_q and u_0 are solutions of (5.3) corresponding to q and 0, respectively. We then have

$$(\Lambda_q - \Lambda_0)(\chi f) = (-\Delta)^s w|_{(B_3\setminus\overline{B_2})_T},$$

where w solves

$$\begin{cases} \left(\partial_t^2 + (-\Delta)^s + q\right) w = -qu_0 & \text{in } (B_1)_T, \\ w = 0 & \text{in } (\mathbb{R}^n \setminus \overline{B_1})_T, \\ w = \partial_t w = 0 & \text{in } \mathbb{R}^n \times \{0\}. \end{cases} \tag{5.13}$$

Using the equivalent definition of $(-\Delta)^s$ via the singular integral, see e.g. [Kwa17, Definition 2.5, Theorem 5.3], we obtain that for each $x \in B_3 \setminus \overline{B_2}$

$$|(-\Delta)^s w(x,t)| = C_{n,s}\left|\int_{\mathbb{R}^n} \frac{w(x,t) - w(y,t)}{|x-y|^{n+2s}}\,dy\right| = C_{n,s}\left|\int_{B_1} \frac{w(y,t)}{|x-y|^{n+2s}}\,dy\right|$$

for some positive constant $C_{n,s}$. Since

$$|x-y|^{n+2s} \ge 1, \quad \text{for all } x \in B_3 \setminus \overline{B_2}, y \in B_1,$$

we immediately observe that

$$|(-\Delta)^s w(x,t)| \le C_{n,s}\left|\int_{B_1} w(y,t)\,dy\right| \le C_{n,s}\|w(\cdot,t)\|_{L^2(B_1)}.$$

Hence, we can estimate

$$\begin{aligned} &\|(\Lambda_q - \Lambda_0)(\chi f)\|_{L^\infty(0,T;L^2(B_3\setminus\overline{B_2}))} \\ =&\|(-\Delta)^s w\|_{L^\infty(0,T;L^2(B_3\setminus\overline{B_2}))} \\ \le& C_{n,s}\|w\|_{L^\infty(0,T;L^2(B_1))} \\ \le& C_{n,s}\|qu_0\|_{L^2(0,T;L^2(B_1))} \quad \text{(by (2.4))} \\ \le& C_{R,T,n,s}\|\chi\|_{W^{2,\infty}(0,T)}\|f\|_{L^2(B_3\setminus\overline{B}_2)} \quad \text{(by applying (5.4) to } u_0\text{)}, \end{aligned}$$

which proves the lemma. $\square$

We now apply Lemma 5.4 with the exterior Dirichlet data $f = Y_{mk\ell}$. Since $\{Y_{mk\ell}\}$ is an orthonormal basis of $L^2(B_3 \setminus \overline{B}_2)$, by Lemma 5.4, there exists a unique solution $u_{mk\ell}$ to

$$\begin{cases} \left(\partial_t^2 + (-\Delta)^s + q\right) u_{mk\ell} = 0 & \text{in } (B_1)_T, \\ u_{mk\ell}(x,t) = \chi(t)\mathbb{1}_{B_3\setminus\overline{B_2}}(x)Y_{mk\ell}(x) & \text{in } (\mathbb{R}^n \setminus \overline{B_1})_T, \\ u_{mk\ell} = \partial_t u_{mk\ell} = 0 & \text{in } \mathbb{R}^n \times \{0\}, \end{cases} . \tag{5.14}$$

Combining Proposition 5.1, (5.9) and (5.1), we obtain

$$\|u_{mk\ell}\|_{L^\infty(0,T;L^2(B_1))} \le C_{T,R,n,s}\|\chi\|_{W^{2,\infty}(0,T)}e^{-c_{n,s}(m+k)} \tag{5.15}$$

Remark 5.6. *Similarly, we can let $\mathring{u}_{mk\ell}$ be the solution to (5.14) with respect to $q \equiv 0$ (i.e. $R = 0$), we have*

$$\|\mathring{u}_{mk\ell}\|_{L^\infty(0,T;L^2(B_1))} \le C_{T,n,s}\|\chi\|_{W^{2,\infty}(0,T)}e^{-c_{n,s}(m+k)}.$$

5.3. **Matrix representation.** We consider the mapping

$$\Gamma(q)(f) := \chi(\Lambda_q - \Lambda_0)(\chi f) \quad \text{for} \quad f \in L^2(B_3 \setminus \overline{B_2}).$$

We define

$$\Gamma^{m_2k_2\ell_2}_{m_1k_1\ell_1}(q)(t) := (\Gamma(q)(Y_{m_1k_1\ell_1}), Y_{m_2k_2\ell_2})_{L^2(B_3\setminus\overline{B_2})}(t). \tag{5.16}$$

Since Λ_q is self-adjoint, then (5.16) infers that

$$\Gamma^{m_2k_2\ell_2}_{m_1k_1\ell_1}(q)(t) = \Gamma^{m_1k_1\ell_1}_{m_2k_2\ell_2}(q)(t).$$

We have the following estimate for the tensor.

Lemma 5.7. *For $n \geq 2$ and given $q \in B^\infty_{+,R}$, there exist constants $C'_{R,T,n,s} > 1$ and $c'_{n,s} > 0$ such that*

$$\sup_{t\in(0,T)} \left|\Gamma^{m_2k_2\ell_2}_{m_1k_1\ell_1}(q)(t)\right| \leq C'_{R,T,n,s}\|\chi\|^2_{W^{2,\infty}(0,T)} e^{-c'_{n,s}\sigma} \tag{5.17}$$

where $\sigma := \max\{m_1 + k_1, m_2 + k_2\}$.

Proof. Since $\|Y_{m_2k_2\ell_2}\|_{L^2(B_3\setminus\overline{B_2})} = 1$, using the equivalent definition of $(-\Delta)^s$ again, we have

$$\begin{aligned}
&\sup_{t\in(0,T)} \left|\Gamma^{m_2k_2\ell_2}_{m_1k_1\ell_1}(q)(t)\right|^2 \\
&\leq \sup_{t\in(0,T)} \|\Gamma(q)(Y_{m_1k_1\ell_1})\|^2_{L^2(B_3\setminus\overline{B_2})}(t) \\
&\leq \|\chi\|^2_{L^\infty(0,T)} \sup_{t\in(0,T)} \|(-\Delta)^s w_{m_1k_1\ell_1}\|^2_{L^2(B_3\setminus\overline{B_2})}(t) \\
&\qquad (w_{m_1k_1\ell_1} := u_{m_1k_1\ell_1} - \mathring{u}_{m_1k_1\ell_1}) \\
&= \|\chi\|^2_{L^\infty(0,T)} C_{n,s} \sup_{t\in(0,T)} \int_{B_3\setminus\overline{B_2}} \left| \int_{\mathbb{R}^d} \frac{w_{m_1k_1\ell_1}(x,t) - w_{m_1k_1\ell_1}(y,t)}{|x-y|^{n+2s}}\, dy \right|^2 dx \\
&\leq \|\chi\|^2_{L^\infty(0,T)} C_{n,s} \sup_{t\in(0,T)} \int_{B_3\setminus\overline{B_2}} \left| \int_{B_1} w_{m_1k_1\ell_1}(y,t)\, dy \right|^2 dx \\
&\qquad (\text{since } \ w_{m_1k_1\ell_1} = 0 \text{ in } (B_3 \setminus \overline{B_2})_T) \\
&= \|\chi\|^2_{L^\infty(0,T)} C_{n,s} \sup_{t\in(0,T)} |B_3 \setminus \overline{B_2}| \left| \int_{B_1} w_{m_1k_1\ell_1}(y,t)\, dy \right|^2 \\
&\leq \|\chi\|^2_{L^\infty(0,T)} C_{n,s} |B_3 \setminus \overline{B_2}| \overbrace{\left(\int_{B_1} 1^2\, dy \right)}^{=|B_1|} \overbrace{\left(\sup_{t\in(0,T)} \int_{B_1} |w_{m_1k_1\ell_1}(y,t)|^2\, dy \right)}^{=\|w_{m_1k_1\ell_1}\|^2_{L^\infty(0,T;L^2(B_1))}} \\
&= \|\chi\|^2_{L^\infty(0,T)} C_{n,s} |B_3 \setminus \overline{B_2}| |B_1| \, \|w_{m_1k_1\ell_1}\|^2_{L^\infty(0,T;L^2(B_1))},
\end{aligned}$$

for some positive constant $C_{n,s}$ depending only on n and $s \in (0,1)$. Combining the estimate above and (5.15) gives

$$\sup_{t\in(0,T)} \left|\Gamma^{m_2k_2\ell_2}_{m_1k_1\ell_1}(q)(t)\right| \leq C_{R,T,n,s}\|\chi\|^2_{W^{2,\infty}(0,T)} e^{-c_{n,s}(m_1+k_1)}. \tag{5.18}$$

Since q is real-valued, $\Gamma(q)$ is self-adjoint. Therefore, we have

$$\begin{aligned}
\sup_{t\in(0,T)} \left|\Gamma^{m_2k_2\ell_2}_{m_1k_1\ell_1}(q)\right| &= \sup_{t\in(0,T)} \left|\Gamma^{m_1k_1\ell_1}_{m_2k_2\ell_2}(q)\right| \\
&\leq C_{R,T,n,s}\|\chi\|^2_{W^{2,\infty}(0,T)} e^{-c_{n,s}(m_2+k_2)}.
\end{aligned} \tag{5.19}$$

The estimate (5.17) then follows from (5.18) and (5.19). ∎

5.4. **Construction of a family of δ-net.** It follows easily from (5.17) that

$$\sup_{t\in(0,T)}(1+\sigma)^{n+2}\left|\Gamma_{m_1k_1\ell_1}^{m_2k_2\ell_2}(q)(t)\right| \le C'_{R,T,n,s}\|\chi\|_{W^{2,\infty}(0,T)}^2(1+\sigma)^{n+2}e^{-c'_{n,s}\sigma}, \tag{5.20}$$

that is,

$$\sup_{t\in(0,T)}\left\|(\Gamma_{m_1k_1\ell_1}^{m_2k_2\ell_2}(q)(t))\right\|_X \le C'_{R,T,n,s}\|\chi\|_{W^{2,\infty}(0,T)}^2\sup_{\sigma\ge 0}(1+\sigma)^{n+2}e^{-c'_{n,s}\sigma} < \infty,$$

and thus $\left(\Gamma_{m_1k_1\ell_1}^{m_2k_2\ell_2}(B_{+,R}^\infty)(t)\right) \subset X$, for all $t\in(0,T)$. Let us define the δ-net as follows.

Definition 5.8 (δ-net)**.** *A set Y of a metric space (M,d) is called a δ-net for $Y_1 \subset M$ if for any $x\in Y_1$, there is a $y\in Y$ such that $\mathsf{d}(x,y)\le\delta$.*

Lemma 5.9. *Let $n\ge 2$. Given any $R>1$ and $0<\delta<\|\chi\|_{W^{2,\infty}(0,T)}^2$. There exists a family $\{Y(t) : t\in(0,T)\}$ such that each $Y(t)$ is a δ-net of*

$$\left((\Gamma_{m_1k_1\ell_1}^{m_2k_2\ell_2}(B_{+,R}^\infty)(t)), \|\cdot\|_X\right)$$

and satisfies

$$\log|Y(t)| \le K_{n,s}\log^{2n+1}\left(\frac{K_{R,T,n,s}\|\chi\|_{W^{2,\infty}(0,T)}^2}{\delta}\right) \tag{5.21}$$

for some positive constants $K_{n,s}$ and $K_{R,T,n,s}$, where $|Y(t)|$ denotes the cardinality of $Y(t)$.

Proof. We first note that it suffices to take $C'_{R,T,n,s}\ge 1$ and $c'_{n,s}>0$ described in Lemma 5.7.

Step 1. Initialization.

Given any $0<\delta<\|\chi\|_{W^{2,\infty}(0,T)}^2$, suggested by (5.20), let $\tilde\sigma>0$ be the unique solution to

$$(1+\tilde\sigma)^{n+2}e^{-c'_{n,s}\tilde\sigma} = \frac{\delta}{C'_{R,T,n,s}\|\chi\|_{W^{2,\infty}(0,T)}^2}.$$

It is easy to see that

$$\frac{\delta}{C'_{R,T,n,s}\|\chi\|_{W^{2,\infty}(0,T)}^2} = (1+\tilde\sigma)^{n+2}e^{-\frac{c'_{n,s}}{2}\tilde\sigma}e^{-\frac{c'_{n,s}}{2}\tilde\sigma} \le C''_{n,s}e^{-\frac{c'_{n,s}}{2}\tilde\sigma}$$

with

$$C''_{n,s} := \sup_{\sigma>0}(1+\sigma)^{n+2}e^{-\frac{c'_{n,s}}{2}\sigma}.$$

Therefore, we have

$$\tilde\sigma \le \frac{2}{c'_{n,s}}\log\left(\frac{C''_{n,s}C'_{R,T,n,s}\|\chi\|_{W^{2,\infty}(0,T)}^2}{\delta}\right).$$

Let $\sigma_* = \lfloor\tilde\sigma\rfloor$, then

$$1+\sigma_* \le 1+\tilde\sigma \le c''_{n,s}\log\left(\frac{C''_{n,s}C'_{R,T,n,s}\|\chi\|_{W^{2,\infty}(0,T)}^2}{\delta}\right) \tag{5.22}$$

for some constant $c''_{n,s}$. Observe that if $\mathbb{Z}\ni\sigma\ge\sigma_*+1$, then

$$(1+\sigma)^{n+2}e^{-c'_{n,s}\sigma} \le \frac{\delta}{C'_{R,T,n,s}\|\chi\|_{W^{2,\infty}(0,T)}^2}, \tag{5.23}$$

where $\mathbb{Z}$ denotes the set of integers.

Step 2. Construction of sets.

Let $\delta' := \frac{\delta}{(1+\sigma_*)^{n+2}}$ and define

$$Y' := \left\{ a \in \delta'\mathbb{Z} : \, |a| \leq C'_{R,T,n,s} \|\chi\|^2_{W^{2,\infty}(0,T)} (1+\sigma_*)^{-(n+2)} \right\},$$

and

$$Y(t) := \left\{ \left(b^{m_2k_2\ell_2}_{m_1k_1\ell_1}(t) \right) : \begin{array}{l} b^{m_2k_2\ell_2}_{m_1k_1\ell_1}(t) \in Y' \text{ if } \mathbb{Z}_+ \ni \sigma \leq \sigma_*, \\ b^{m_2k_2\ell_2}_{m_1k_1\ell_1}(t) = 0, \text{ otherwise,} \end{array} \right\},$$

where $\mathbb{Z}_+$ is the set of non-negative integers.

Step 3. Verifying $Y(t)$ is a δ-net.

Our goal is to construct an appropriate $\left(b^{m_2k_2\ell_2}_{m_1k_1\ell_1}(t) \right) \in Y(t)$ that is an approximation of $\left(\Gamma^{m_2k_2\ell_2}_{m_1k_1\ell_1}(q)(t) \right)$.

- If $\sigma \leq \sigma_*$, we choose $b^{m_2k_2\ell_2}_{m_1k_1\ell_1}(t) \in Y'$ to be the closest element to $\Gamma^{m_2k_2\ell_2}_{m_1k_1\ell_1}(q)(t)$. Then

$$(1+\sigma)^{n+2} \left| b^{m_2k_2\ell_2}_{m_1k_1\ell_1}(t) - \Gamma^{m_2k_2\ell_2}_{m_1k_1\ell_1}(q)(t) \right| \leq (1+\sigma_*)^{n+2}\delta' = \delta.$$

- Otherwise, if $\sigma \geq \sigma_* + 1$, we simply choose $b^{m_2k_2\ell_2}_{m_1k_1\ell_1}(t) = 0$. Consequently, we obtain from (5.20) and (5.23) that

$$\begin{aligned} &(1+\sigma)^{n+2} \left| b^{m_2k_2\ell_2}_{m_1k_1\ell_1}(t) - \Gamma^{m_2k_2\ell_2}_{m_1k_1\ell_1}(q)(t) \right| \\ =&(1+\sigma)^{n+2} \left| \Gamma^{m_2k_2\ell_2}_{m_1k_1\ell_1}(q)(t) \right| \\ \leq& C'_{R,T,n,s} \|\chi\|^2_{W^{2,\infty}(0,T)} (1+\sigma)^{n+2} e^{-c'_{n,s}\sigma} \\ \leq& \delta. \end{aligned}$$

Combining these two cases, and by the definition of X-norm, we have

$$\sup_{t\in(0,T)} \left\| \left(b^{m_2k_2\ell_2}_{m_1k_1\ell_1} - \Gamma^{m_2k_2\ell_2}_{m_1k_1\ell_1}(q) \right) \right\|_X \leq \delta.$$

In other words, we have shown that, $Y(t)$ is a δ-net of $\left((\Gamma^{m_2k_2\ell_2}_{m_1k_1\ell_1}(B^\infty_{+,R})(t)), \|\cdot\|_X \right)$, for each $t \in (0,T)$.

Step 4. Calculating the cardinality of $Y(t)$.

Let N_σ be the number of 6-tuples $(m_1, k_1, \ell_1, m_2, k_2, \ell_2)$ with $\sigma = \max\{m_1 + k_1, m_2 + k_2\}$ as in Lemma 5.7. We now want to estimate N_σ. First of all, we fix any integer $0 \leq \sigma' \leq \sigma$ and compute the number of 6-tuples with $m_1 + k_1 = \sigma$ and $m_2 + k_2 = \sigma'$. It is easy to see that there are

$$\sigma + 1 \text{ choices of } m_1 \quad \text{and} \quad \sigma' + 1 \text{ choices of } m_2.$$

Moreover, the number of choices of ℓ_i is bounded by ℓ_{m_i}, and, from Proposition 5.1, we can see that

$$\ell_{m_i} \leq \ell_\sigma \leq 2(1+\sigma)^{n-2} \qquad \text{for } i = 1, 2.$$

Therefore, the number N_σ of 6-tuples with $m_1+k_1 = \sigma$ and $m_2+k_2 = \sigma'$ is bounded by $4(1+\sigma)^{2n-2}$. Thus, the number of 6-tuples with $m_1 + k_1 = \sigma$ and $m_2 + k_2 \leq \sigma$ is bounded by $4(1+\sigma)^{2n-1}$. Interchanging the role of (m_1, k_1, ℓ_1) with (m_2, k_2, ℓ_2), we obtain a similar bound, and hence

$$N_\sigma \leq 8(1+\sigma)^{2n-1}.$$

Therefore, we derive that

$$N_* := \sum_{\sigma=0}^{\sigma_*} N_\sigma \leq \sum_{\sigma=0}^{\sigma_*} 8(1+\sigma)^{2n-1} \leq 8(1+\sigma_*)^{2n}.$$

From (5.22), it follows

$$N_* \leq 8\left(c''_{n,s}\log\left(\frac{C''_{n,s}C'_{R,T,n,s}\|\chi\|^2_{W^{2,\infty}(0,T)}}{\delta}\right)\right)^{2n}.$$

Since $|Y(t)| = |Y'|^{N_*}$ and

$$\begin{aligned} |Y'| &= 1 + 2\left\lfloor \frac{C'_{R,T,n,s}\|\chi\|^2_{W^{2,\infty}(0,T)}(1+\sigma_*)^{-(n+2)}}{\delta'} \right\rfloor \\ &\leq \frac{3C'_{R,T,n,s}\|\chi\|^2_{W^{2,\infty}(0,T)}}{\delta}, \quad \text{since } 0<\delta<\|\chi\|^2_{W^{2,\infty}(0,T)}, \end{aligned}$$

we can see that

$$\begin{aligned} \log|Y(t)| =& N_*\log|Y'| \\ \leq& 8\left(c''_{n,s}\log\left(\frac{C''_{n,s}C'_{R,T,n,s}\|\chi\|^2_{W^{2,\infty}(0,T)}}{\delta}\right)\right)^{2n} \\ &\times \log\left(\frac{3C'_{R,T,n,s}\|\chi\|^2_{W^{2,\infty}(0,T)}}{\delta}\right). \end{aligned}$$

Setting

$$\begin{aligned} K_{R,T,n,s} &:= C''_{n,s}C'_{R,T,n,s} + 3C'_{R,T,n,s}, \\ \widetilde{C}_{n,s} &:= c''_{n,s} + 1, \end{aligned}$$

we then obtain

$$\begin{aligned} \log|Y(t)| &\leq 8\left(\widetilde{C}_{n,s}\log\left(\frac{K_{R,T,n,s}\|\chi\|^2_{W^{2,\infty}(0,T)}}{\delta}\right)\right)^{2n+1} \\ &= K_{n,s}\log^{2n+1}\left(\frac{K_{R,T,n,s}\|\chi\|^2_{W^{2,\infty}(0,T)}}{\delta}\right) \end{aligned}$$

with $K_{n,s} = 8(\widetilde{C}_{n,s})^{2n+1}$. This proves the assertion. □

Remark 5.10. *Note that*

$$\inf_{0<\delta<\|\chi\|^2_{W^{2,\infty}(0,T)}} \log\left(\frac{K_{R,T,n,s}\|\chi\|^2_{W^{2,\infty}(0,T)}}{\delta}\right) = \log(K_{R,T,n,s}).$$

Therefore, for each $\alpha>0$ *and*

$$0<\epsilon<\log^{-\frac{(2n+1)\alpha}{n}}(K_{R,T,n,s}) =: c_{R,T,n,s}, \tag{5.24}$$

there exists a unique $0<\delta<\|\chi\|^2_{W^{2,\infty}(0,T)}$ *such that*

$$\epsilon^{-\frac{n}{(2n+1)\alpha}} = \log\left(\frac{K_{R,T,n,s}\|\chi\|^2_{W^{2,\infty}(0,T)}}{\delta}\right). \tag{5.25}$$

Therefore, we can rewrite (5.21) *as*

$$\log|Y(t)| \leq K_{n,s}\epsilon^{-\frac{n}{\alpha}}. \tag{5.26}$$

5.5. **Construction of an ϵ-discrete set.** Fixing any $r_0 \in (0,1)$, $\alpha > 0$, $\epsilon > 0$, and $\beta > 0$, we define the following set:

$$\mathcal{N}^{\epsilon}_{\alpha\beta}(B_{r_0}) := \{q \geq 0 : \mathrm{supp}\,(q) \subset B_{r_0},\ \|q\|_{L^\infty} \leq \epsilon,\ \|q\|_{C^\alpha} \leq \beta\}.$$

The following lemma can be found in [KUW21, Proposition 2.1], in [ZZ19, Lemma 5.2], or in [KT59, KT61] in a more abstract form, also see [Man01, Lemma 2] for a direct proof, which is valid for all $n \in \mathbb{N}$. Additionally, we refer to [DCR03b, Proposition 3.1] and [DCR03a, Proposition 2.2], where similar results were derived under different settings.

Definition 5.11 (ϵ-discrete set)**.** *A set Z of a metric space (M, d) is called an ϵ-discrete set if $\mathsf{d}(z_1, z_2) \geq \epsilon$ for all $z_1 \neq z_2 \in Z$.*

Lemma 5.12. *Let $n \in \mathbb{N}$ and $\alpha > 0$. There exists a constant $\mu = \mu(n, \alpha) > 0$ such that the following statement holds for all $\beta > 0$ and for all $\epsilon \in (0, \mu\beta)$. Then there exists a ϵ-discrete (a.k.a. ϵ-distinguishable) subset Z of $\left(\mathcal{N}^{\epsilon}_{\alpha\beta}(B_{r_0}), \|\cdot\|_{L^\infty}\right)$ such that*

$$\log|Z| \geq 2^{-(n+1)}\left(\frac{\mu\beta}{\epsilon}\right)^{\frac{n}{\alpha}},$$

where $|Z|$ denotes the cardinality of Z.

5.6. **Proof of Theorem 1.3.** With Lemma 5.9 and Lemma 5.12 at hand, we can prove the exponential instability of the inverse problem for the fractional wave equation.

Proof of Theorem 1.3. Let μ and $c_{R,T,n,s}$ be the constants given in Lemma 5.12 and in (5.24), respectively. For each $0 < \epsilon < \min\{c_{R,T,n,s}, R, \mu\beta\}$, we can construct an ϵ-discrete set Z as described in Lemma 5.12. Let δ be the constant chosen in (5.25). Next, for each $t \in (0,T)$, we construct a δ-net $Y(t)$ as in Lemma 5.9 and (5.26) holds. Clearly, $Y(t)$ is also a δ-net for $\left(\Gamma^{m_2k_2\ell_2}_{m_1k_1\ell_1}(Z)(t), \|\cdot\|_X\right)$.

We now choose $\beta = \beta(R, n, s, \alpha)$ sufficiently large (which is independent of ϵ) such that $\mu\beta \geq R$ and

$$\log|Z| \geq 2^{-(n+1)}\left(\frac{\mu\beta}{\epsilon}\right)^{\frac{n}{\alpha}} > K_{n,s}\epsilon^{-\frac{n}{\alpha}} \geq \log|Y(t)|.$$

Therefore, by the pigeonhole principle, for each $t \in (0,T)$, there exist $\left(y^{m_2k_2\ell_2}_{m_1k_1\ell_1}(t)\right) \in Y(t)$, and two different $q_1, q_2 \in Z \subset \mathcal{N}^{\epsilon}_{\alpha\beta}(B_{r_0})$ such that

$$\left\|\left(\Gamma^{m_2k_2\ell_2}_{m_1k_1\ell_1}(q_i)(t) - y^{m_2k_2\ell_2}_{m_1k_1\ell_1}(t)\right)\right\|_X \leq \delta \quad \text{for } i = 1, 2.$$

In view of Lemma 5.3 and by (5.25), we have

$$\begin{aligned}
&\sup_{t\in(0,T)} \|\chi(\Lambda_{q_1} - \Lambda_{q_2})\chi\|_{L^2(B_3\setminus\overline{B_2})\to L^2(B_3\setminus\overline{B_2})}(t) \\
=&\sup_{t\in(0,T)} \|\Gamma(q_1) - \Gamma(q_2)\|_{L^2(B_3\setminus\overline{B_2})\to L^2(B_3\setminus\overline{B_2})}(t) \\
\leq& 4\left\|\Gamma^{m_2k_2\ell_2}_{m_1k_1\ell_1}(q_1)(t) - \Gamma^{m_2k_2\ell_2}_{m_1k_1\ell_1}(q_2)(t)\right\|_X \\
\leq& 8\delta = 8K_{R,T,n,s}\|\chi\|^2_{W^{2,\infty}(0,T)}\exp(-\epsilon^{-\frac{n}{(2n+1)\alpha}}).
\end{aligned}$$

The arbitrariness of $0 \not\equiv \chi \in C^\infty_c((0,T))$ leads to the estimate (1.6), while the estimate (1.5) immediately follows form the definition of Z. Moreover, since $\epsilon < R$, $\|q_i\|_{L^\infty} \leq R$, for $i = 1, 2$. The proof is now completed. ☐

We next prove the exponential instability in the case of $n = 1$, Theorem 1.4. The proof of Theorem 1.4 is very similar to that of Theorem 1.3. The main difference is that when $n = 1$, the boundary ∂B_1 of the interval $B_1 = (-1,1)$ consists only two points $\{-1,1\}$. Therefore, we need to modify the proof of Proposition 5.1.

We first construct an orthonormal basis $\{Y_k\}$ of $L^2((2,3))$ such that the solution $\tilde{Y}_k$ of

$$\begin{cases} (-\Delta)^s \tilde{Y}_k = 0 & \text{in } B_1, \\ \tilde{Y}_k = \mathbb{1}_{(2,3)} Y_k & \text{in } \mathbb{R}^1 \setminus \overline{B_1}, \end{cases} \tag{5.27}$$

satisfies some exponential decay bound. Similar to the proof of [RS18, Lemma 2.1], using the Poisson formula of u_k in [Buc16, Theorem 2.10], there exists a constant $c = c(s) \neq 0$ such that

$$\frac{\tilde{Y}_k(x)}{c(1-x^2)^s} = \int_{\mathbb{R}^1 \setminus \overline{B_1}} \frac{1}{|x-r|} \frac{\mathbb{1}_{(2,3)}(r) Y_k(r)}{(r^2-1)^s}\, dr = \int_2^3 \frac{1}{r-x} \frac{Y_k(r)}{(r^2-1)^s}\, dr \tag{5.28}$$

for all $x \in (-1,1)$. If we choose $\left\{Y_k = e^{2\pi i k(x-2)}\right\}$ to be the usual orthonormal basis of $L^2((2,3))$, it will be difficult to obtain an exponential decay bound for $\tilde{Y}_k$. Therefore, we would like to find another orthonormal basis for $L^2((2,3))$ to meet our goal.

Proposition 5.13. *There exists a* real-valued *orthonormal basis* $\{Y_k\}$ *of* $L^2((2,3))$ *satisfying that*

$$\|\tilde{Y}_k\|_{L^2(B_1)} \le C' e^{-C'' k} \quad \text{for all} \quad k = 0,1,2,\dots,$$

for some positive constants C' *and* C'' *independent of* Y_k *and* $\tilde{Y}_k$*, where* $\tilde{Y}_k \in H^s(\mathbb{R}^1)$ *is the unique solution of* (5.27).

Proof. In view of (5.28), we want to find *real-valued* Y_k of the form

$$Y_k(r) = r(r^2-1)^s g_k(r). \tag{5.29}$$

Plugging the ansatz (5.29) into (5.28), we obtain that for $x \in (-1,1)$

$$\begin{aligned} \tilde{Y}_k(x) &= c(1-x^2)^s \int_2^3 \frac{1}{1-\frac{x}{r}} g_k(r)\, dr \\ &= c(1-x^2)^s \int_2^3 \sum_{j=0}^\infty \left(\frac{x}{r}\right)^j g_k(r)\, dr \\ &= c(1-x^2)^s \sum_{j=0}^\infty x^j \int_2^3 r^{-j} g_k(r)\, dr, \end{aligned} \tag{5.30}$$

provided $g_k \in L^1((2,3))$. In order to derive the desired decaying properties, for each $k \ge 1$, we will choose g_k such that

$$\int_2^3 r^{-j} g_k(r)\, dr = 0 \quad \text{for all} \quad 0 \le j \le k-1. \tag{5.31a}$$

From (5.29), we also require g_k to satisfy

$$\delta_{k\ell} = \int_2^3 Y_k(r) Y_\ell(r)\, dr = \int_2^3 r^2 (r^2-1)^{2s} g_k(r) g_\ell(r)\, dr. \tag{5.31b}$$

Setting

$$h_k(r) := r^2 (r^2-1)^{2s} g_k(r),$$

we can rewrite (5.31a) and (5.31b) as

$$(5.32a)\qquad \left(h_k, r^{-j}\right)_s = 0, \quad \text{for all } 0 \le j \le k-1,$$

$$(5.32b)\qquad (h_k, h_\ell)_s = \delta_{k\ell}, \quad \text{for all non-negative integers } k, \ell,$$

where

$$(h_1, h_2)_s := \int_2^3 r^{-2}(r^2-1)^{-2s} h_1(r) h_2(r)\, dr.$$

Using the Gram-Schmidt process, we can choose

$$h_k(r) \in \operatorname{span}\left(\bigcup_{j=0}^{k} \{r^{-j}\}\right) \qquad \text{for all } k = 0, 1, 2, \cdots$$

which satisfy (5.32a) and (5.32b). In other words,

$$\left\{Y_k(r) = r^{-1}(r^2-1)^{-s} h_k(r) : \ k = 0, 1, 2, \dots\right\}$$

forms an orthonormal basis of $L^2((2,3))$.

We observe that

$$\begin{aligned}
&\left|\int_2^3 r^{-j} g_k(r)\, dr\right| \\
&= \left|\int_2^3 \left(r^{-1}(r^2-1)^{-s} h_k(r)\right)\left(r^{-1}(r^2-1)^{-s} r^{-j}\right) dr\right| \\
&\le \left(\int_2^3 r^{-2}(r^2-1)^{-2s} |h_k(r)|^2\, dr\right)^{\frac{1}{2}} \left(\int_2^3 r^{-2-2j}(r^2-1)^{-2s}\, dr\right)^{\frac{1}{2}} \\
&= \left(\int_2^3 r^{-2-2j}(r^2-1)^{-2s}\, dr\right)^{\frac{1}{2}} \\
&\le 2^{-1-j},
\end{aligned}$$

for all $j \ge k$. Combining this estimate with (5.31a), we have

$$(5.33)\qquad \left|\int_2^3 r^{-j} g_k(r)\, dr\right| \le \mathbb{1}_{\{j \ge k\}} 2^{-1-j}.$$

Plugging (5.33) into (5.30), we obtain that

$$\begin{aligned}
\left|\tilde{Y}_k(x)\right| &\le |c| \left(1-x^2\right)^s \sum_{j=0}^{\infty} |x|^j \left|\int_2^3 r^{-j} g_k(r)\, dr\right| \\
&\le C \sum_{j=k}^{\infty} 2^{-j} = C 2^{-k+1},
\end{aligned}$$

which is our desired result. □

Given any bounded linear operator $\mathcal{A} : L^2((2,3)) \to L^2((2,3))$, we define

$$a_{k_1}^{k_2} := (\mathcal{A} Y_{k_1}, Y_{k_2})_{L^2((2,3))}.$$

Let $\left(a_{k_1}^{k_2}\right)$ be the tensor with entries $a_{k_1}^{k_2}$, and consider the following Banach space:

$$X' := \left\{\left(a_{k_1}^{k_2}\right) : \ \left\|\left(a_{k_1}^{k_2}\right)\right\|_{X'} < \infty\right\},$$

where

$$\left\|\left(a_{k_1}^{k_2}\right)\right\|_{X'} := \sup_{k_1, k_2} \left(1 + \max\{k_1, k_2\}\right)^3 \left|a_{k_1}^{k_2}\right|.$$

Similar to Lemma 5.3, we can prove the following lemma.

Lemma 5.14. *We have*

$$\|\mathcal{A}\|_{L^2(B_3\setminus\overline{B_2})\to L^2(B_3\setminus\overline{B_2})} \le 2\left\|\left(a_{k_1}^{k_2}\right)\right\|_{X'}. \tag{5.34}$$

Proof. In view of the Hilbert-Schmidt norm, we obtain

$$\begin{aligned}\|\mathcal{A}\|^2_{L^2(B_3\setminus\overline{B_2})\to L^2(B_3\setminus\overline{B_2})} &\le \sum_{k_1,k_2}\left|a_{k_1}^{k_2}\right|^2 \\ &\le \sup_{k_1,k_2}\,(1+\max\{k_1,k_2\})^6\left|a_{k_1}^{k_2}\right|^2 \sum_{k_1,k_2\ge 0}(1+\max\{k_1,k_2\})^{-6}.\end{aligned} \tag{5.35}$$

We also note that

$$\sum_{k_1,k_2\ge 0}(1+\max\{k_1,k_2\})^{-6} \le 2\sum_{k=0}^{\infty}(1+k)^{-6} \le 4. \tag{5.36}$$

Combining (5.35) and (5.36) implies (5.34). ☐

Similar to preceding discussions, let us consider the one spatial dimensional case.

- **Special weak solutions.**

Let $\chi = \chi(t) \in C_c^\infty((0,T))$. By the same proof of Lemma 5.4, we can establsih

Lemma 5.15. *If $q \in B^\infty_{+,R}$, then there exists a unique weak solution u_k to*

$$\begin{cases}\left(\partial_t^2 + (-\Delta)^s + q\right)u_k = 0 & \text{in } (B_1)_T,\\ u_k(x,t) = \chi(t)\mathbb{1}_{(2,3)}Y_k & \text{in } (\mathbb{R}^1\setminus\overline{B_1})_T,\\ u_k = \partial_t u_k = 0 & \text{in } \mathbb{R}^1\times\{0\},\end{cases}$$

such that

$$\|u_k\|_{L^\infty(0,T;L^2(B_1))} \le C_{R,T,s}\|\chi\|_{W^{2,\infty}(0,T)}e^{-c_s k}.$$

- **Matrix representation.**

Again, consider the mapping

$$\Gamma(q)(f) := \chi\Lambda_q(\chi f) - \chi\Lambda_0(\chi f) \quad \text{for all } f\in C_c^\infty((\mathbb{R}^1\setminus\overline{B_1})_T).$$

In this case, we define

$$\Gamma_{k_1}^{k_2}(q)(t) := \left(\Gamma(q)Y_{k_1}, Y_{k_2}\right)_{L^2((2,3))}(t).$$

Likewise, Λ_q is self-adjoint, and we have

$$\Gamma_{k_1}^{k_2}(q)(t) = \Gamma_{k_2}^{k_1}(q)(t).$$

Following the same proof of Lemma 5.7, we can prove that

Lemma 5.16. *Given any $q\in B^\infty_{+,R}$, there exist constants $C_{R,T,s}>1$ and $c_s'>0$ such that*

$$\sup_{t\in(0,T)}\left|\Gamma_{k_1}^{k_2}(q)(t)\right| \le C'_{R,T,s}\|\chi\|^2_{W^{2,\infty}(0,T)}e^{-c_s'\sigma},$$

where $\sigma := \max\{k_1,k_2\}$.

- **Construction of a family of δ-net.**

We now construct a δ-net for $(\Gamma_{k_1}^{k_2}(B^\infty_{+,R})(t))$.

Lemma 5.17. *Given any $R > 1$ and $0 < \delta < \|\chi\|^2_{W^{2,\infty}(0,T)}$. There exists a family $\{Y(t) : t \in (0,T)\}$ such that each $Y(t)$ is a δ-net of $\left((\Gamma^{k_2}_{k_1}(B^{\infty}_{+,R})(t)), \|\cdot\|_{X'}\right)$ and satisfies*

$$\log |Y(t)| \leq K_s \log^3 \left(\frac{K_{R,T,s}\|\chi\|^2_{W^{2,\infty}(0,T)}}{\delta} \right)$$

for some positive constants K_s and $K_{R,T,s}$, where $|Y(t)|$ denotes the cardinality of $Y(t)$.

Proof. The proof of Lemma 5.17 is almost identically to the proof of Lemma 5.9 with some minor adjustments in Step 4. Here, let N_σ be the number of 2-tuples (k_1, k_2) with $\max\{k_1, k_2\} = \sigma$. In this case, we can easily obtain

$$N_\sigma \leq 2(1+\sigma) \leq 8(1+\sigma)^{2n-1}$$

with $n = 1$. □

Proof of Theorem 1.4. Finally, we can prove Theorem 1.4 by following the lines in the proof of Theorem 1.3. □

Appendix A. Proofs related to the forward problem

In the end of this work, we prove Theorem 2.1 in details for the self-containedness. The proof of Theorem 2.1 is similar to the proof of the case $s = 1$, i.e., the well-posedness of the classical wave equation. The main difference is that the estimates and results hold in the fractional Sobolev space. Here we will utilize similar ideas shown in [Eva98, Chapter 7]: The *Galerkin approximation*.

We now set up the Galerkin approximation for the fractional wave equation. To this end, let us consider an eigenbasis $\{w_k\}_{k\in\mathbb{N}}$ associated with the Dirichlet fractional Laplacian in a bounded domain Ω, that is,

$$\begin{cases} (-\Delta)^s w_k = \lambda_k w_k & \text{in } \Omega, \\ w_k = 0 & \text{in } \Omega_e. \end{cases}$$

Moreover, we can normalize these eigenfunctions so that

$$\{w_k\}_{k\in\mathbb{N}} \text{ be an orthogonal basis in } \widetilde{H}^s(\Omega), \tag{A.1}$$

and

$$\{w_k\}_{k\in\mathbb{N}} \text{ be an orthonormal basis in } L^2(\Omega). \tag{A.2}$$

Given any fixed integer $m \in \mathbb{N}$, consider the function

$$\boldsymbol{v}_m(t) := \sum_{k=1}^m d^k_m w_k, \tag{A.3}$$

where the coefficients $d^k_m(t)$ ($0 \leq t \leq T$, $k = 1, \dots, m$) satisfy

$$\begin{cases} d^k_m(0) = (\widetilde{\varphi}, w_k), \\ \left(d^k_m\right)'(0) = (\widetilde{\psi}, w_k), \end{cases} \tag{A.4}$$

and, for $0 \leq t \leq T$,

$$(\boldsymbol{v}''_m, w_k)_{L^2(\Omega)} + B[\boldsymbol{v}_m, w_k; t] = \left(\widetilde{\boldsymbol{F}}, w_k\right)_{L^2(\Omega)} \tag{A.5}$$

with $k = 1, \dots, m$. Let us split the proof of Theorem 2.1 into the following lemmas.

Lemma A.1 (Existence of the approximate solution)**.** *For any $m \in \mathbb{N}$, there exists a unique function $\boldsymbol{v}_m$ of the form (A.3) satisfying (A.4) and (A.5).*

Proof. Due to the orthonormality property (A.2) of $\{w_k\}_{k\in\mathbb{N}} \subset L^2(\Omega)$, we have

$$(\boldsymbol{v}_m''(t), w_k)_{L^2(\Omega)} = \left(d_m^k\right)''(t).$$

In addition, we have

$$B[\boldsymbol{v}_m, w_k; t] = \sum_{\ell=1}^m e^{k\ell} d_m^\ell(t),$$

where $e^{k\ell} := B[w_\ell, w_k]$ for $k, \ell = 1, \dots, m$. Let us write $F^k(t) := (\widetilde{\boldsymbol{F}}(t), w_k)$ for $k = 1, \dots, m$. Consequently, (A.5) becomes the following linear system of ordinary differential equation (ODE)

$$\left(d_m^k\right)''(t) + \sum_{\ell=1}^m e^{k\ell} d_m^\ell(t) = F^k(t), \text{ for } 0 \le t \le T, \ k = 1, \dots, m, \tag{A.6}$$

with the initial conditions (A.4). Via the standard ODE theory, there exists a unique C^2 solution $d_m(t) = (d_m^1(t), \dots, d_m^m(t))$ satisfying (A.4), and solving (A.6) for $0 \le t \le T$. □

Our next goal is to take $m \to \infty$, whenever we have a suitable energy estimate, uniform in $m \in \mathbb{N}$.

Lemma A.2 (Energy estimate)**.** *Under the assumptions of Theorem 2.1, there exists a constant $C > 0$, independent of $m \in \mathbb{N}$, such that*

$$\begin{aligned}
&\max_{0\le t\le T}\left(\|\boldsymbol{v}_m(t)\|_{\widetilde{H}^s(\Omega)} + \|\boldsymbol{v}_m'(t)\|_{L^2(\Omega)}\right) + \|\boldsymbol{v}_m''\|_{L^2(0,T;H^{-s}(\Omega))} \\
&\qquad \le C\left(\|\widetilde{F}\|_{L^2(0,T;L^2(\Omega))} + \|\widetilde{\varphi}\|_{\widetilde{H}^s(\Omega)} + \|\psi\|_{L^2(\Omega)}\right)
\end{aligned} \tag{A.7}$$

for all $m \in \mathbb{N}$.

Proof. We divide the proof into several steps.

Step 1. Basic estimates.

Multiplying the equation (A.5) by $\left(d_m^k\right)'(t)$, and summing over $k = 1, \dots, m$, with the condition (A.2) at hand, we have

$$(\boldsymbol{v}_m'', \boldsymbol{v}_m')_{L^2(\Omega)} + B[\boldsymbol{v}_m, \boldsymbol{v}_m'; t] = \left(\widetilde{\boldsymbol{F}}, \boldsymbol{v}_m'\right)_{L^2(\Omega)} \tag{A.8}$$

for a.e. $0 \le t \le T$. Note that the first term of (A.8) can be written as

$$(\boldsymbol{v}_m'', \boldsymbol{v}_m')_{L^2(\Omega)} = \frac{d}{dt}\left(\frac{1}{2}\|\boldsymbol{v}_m'\|_{L^2(\Omega)}^2\right). \tag{A.9}$$

On the other hand, we can express

$$\begin{aligned}
B[\boldsymbol{v}_m, \boldsymbol{v}_m'; t] &= \int_{\mathbb{R}^n} (-\Delta)^{s/2}\boldsymbol{v}_m (-\Delta)^{s/2}\boldsymbol{v}_m'\, dx + \int_\Omega q\boldsymbol{v}_m\boldsymbol{v}_m'\, dx \\
&= \frac{d}{dt}\left(\frac{1}{2}\int_{\mathbb{R}^n} |(-\Delta)^{s/2}\boldsymbol{v}_m|^2\, dx\right) + \int_\Omega q\boldsymbol{v}_m\boldsymbol{v}_m'\, dx.
\end{aligned} \tag{A.10}$$

Meanwhile, we recall that the Hardy-Littlewood-Sobolev inequality

$$\|\boldsymbol{v}_m\|_{L^2(\Omega)} \le C\|\boldsymbol{v}_m\|_{L^{\frac{2n}{n-s}}(\mathbb{R}^n)} \le C_{n,s}\|(-\Delta)^{s/2}\boldsymbol{v}_m\|_{L^2(\mathbb{R}^n)}, \tag{A.11}$$

holds for $\boldsymbol{v}_m \in \widetilde{H}^s(\Omega)$, see e.g. [Pon16, Proposition 15.5]. Indeed, the Hardy-Littlewood-Sobolev inequality also can be further refined in terms of fractional gradient of order s (a.k.a.) s-gradient, see e.g. [Pon16, Section 15.2] for more details. Putting together (A.8), (A.9), (A.10), and (A.11), we can derive the following inequality

$$\begin{aligned} &\frac{d}{dt}\left(\|\boldsymbol{v}_m'\|^2_{L^2(\Omega)} + \|(-\Delta)^{s/2}\boldsymbol{v}_m\|^2_{L^2(\mathbb{R}^n)}\right) \\ \leq &C\left(\|\boldsymbol{v}_m'\|^2_{L^2(\Omega)} + \|(-\Delta)^{s/2}\boldsymbol{v}_m\|^2_{L^2(\mathbb{R}^n)} + \|\widetilde{F}\|^2_{L^2(\Omega)}\right), \end{aligned} \tag{A.12}$$

for some constant $C > 0$.

Step 2. Gronwall inequality.

We next let

$$\eta(t) := \|\boldsymbol{v}_m'(t)\|^2_{L^2(\Omega)} + \|(-\Delta)^{s/2}\boldsymbol{v}_m(t)\|^2_{L^2(\mathbb{R}^n)}, \tag{A.13}$$

and

$$\zeta(t) := \|\widetilde{F}(t)\|^2_{L^2(\Omega)}, \tag{A.14}$$

for $0 \leq t \leq T$. Then (A.12) yields that

$$\eta'(t) \leq C_1\eta(t) + C_2\zeta(t), \text{ for } 0 \leq t \leq T,$$

for some constants $C_1, C_2 > 0$. Therefore, the Gronwall's inequality implies that

$$\eta(t) \leq e^{C_1 t}\left(\eta(0) + C_2\int_0^t \zeta(s)\, ds\right), \text{ for } 0 \leq t \leq T. \tag{A.15}$$

On the other hand,

$$\begin{aligned} \eta(0) =&\|\boldsymbol{v}_m'(0)\|^2_{L^2(\Omega)} + \|(-\Delta)^{s/2}\boldsymbol{v}_m(0)\|^2_{L^2(\mathbb{R}^n)} \\ \leq &C\left(\|\widetilde{\varphi}\|_{\widetilde{H}^s(\Omega)} + \|\widetilde{\psi}\|_{L^2(\Omega)}\right), \end{aligned}$$

where we have utilized (A.1), (A.2) and $\|(-\Delta)^{s/2}\boldsymbol{v}_m(0)\|_{L^2(\mathbb{R}^n)} \leq C\|\widetilde{\varphi}\|_{\widetilde{H}^s(\Omega)}$, for some constant $C > 0$. Thus, combining (A.13), (A.14), and (A.15), we derive the following bound

$$\begin{aligned} &\|\boldsymbol{v}_m'(t)\|^2_{L^2(\Omega)} + \|(-\Delta)^{s/2}\boldsymbol{v}_m\|^2_{L^2(\mathbb{R}^n)} \\ &\quad \leq C\left(\|\widetilde{\varphi}\|^2_{\widetilde{H}^s(\Omega)} + \|\widetilde{\psi}\|^2_{L^2(\Omega)} + \|\widetilde{F}\|^2_{L^2(0,T;L^2(\Omega))}\right). \end{aligned}$$

Since the above estimate is independent of $t \in [0, T]$, one can conclude that

$$\begin{aligned} &\max_{0\leq t\leq T}\left(\|\boldsymbol{v}_m'(t)\|^2_{L^2(\Omega)} + \|(-\Delta)^{s/2}\boldsymbol{v}_m(t)\|^2_{L^2(\mathbb{R}^n)}\right) \\ &\quad \leq C\left(\|\widetilde{\varphi}\|^2_{\widetilde{H}^s(\Omega)} + \|\widetilde{\psi}\|^2_{L^2(\Omega)} + \|\widetilde{F}\|^2_{L^2(0,T;L^2(\Omega))}\right). \end{aligned}$$

Step 3. Conclusion.

For any $\phi \in \widetilde{H}^s(\Omega)$ with $\|\phi\|_{\widetilde{H}^s(\Omega)} \leq 1$, we write $\phi = \phi_1 + \phi_2$, where $\phi_1 \in \operatorname{span}\{w_k\}_{k=1}^m$ and $(\phi_2, w_k)_{L^2(\Omega)} = 0$, for $k = 1, \dots, m$. It is not hard to see $\|\phi_1\|_{\widetilde{H}^s(\Omega)} \leq 1$. In view of (A.3) and (A.5), we have

$$(\boldsymbol{v}_m'', \phi)_{L^2(\Omega)} = (\boldsymbol{v}_m'', \phi_1)_{L^2(\Omega)} = (\widetilde{\boldsymbol{F}}, \phi_1) - B[\boldsymbol{v}_m, \phi_1; t],$$

so that

$$\left|(\boldsymbol{v}_m'', \phi)_{L^2(\Omega)}\right| \leq C\left(\|\widetilde{\boldsymbol{F}}\|_{L^2(\Omega)} + \|\boldsymbol{v}_m\|_{\widetilde{H}^s(\Omega)}\right),$$

where we used $\|\phi_1\|_{\widetilde{H}^s(\Omega)} \leq 1$. In conclusion,

$$\begin{aligned}\int_0^T \|\boldsymbol{v}_m''\|_{H^{-s}(\Omega)}^2\,dt \leq& C\int_0^T \left(\|\widetilde{\boldsymbol{F}}\|_{L^2(\Omega)}^2 + \|\boldsymbol{v}_m\|_{\widetilde{H}^s(\Omega)}^2\right)dt \\ \leq& C\left(\|\widetilde{\varphi}\|_{\widetilde{H}^s(\Omega)}^2 + \|\widetilde{\psi}\|_{L^2(\Omega)}^2 + \|\widetilde{F}\|_{L^2(0,T;L^2(\Omega))}^2\right).\end{aligned}$$

This proves the assertion. □

Now, we are ready to prove Theorem 2.1.

Proof of Theorem 2.1. Our goal is to pass the limits in the previous Galerkin approximations.

Step 1. Existence of weak solution.

Using the energy estimate (A.7), it is known that the sequence $\{\boldsymbol{v}_m\}_{m\in\mathbb{N}}$, $\{\boldsymbol{v}_m'\}_{m\in\mathbb{N}}$ and $\{\boldsymbol{v}_m''\}_{m\in\mathbb{N}}$ are bounded in $L^2(0,T;\widetilde{H}^s(\Omega))$, $L^2(0,T;L^2(\Omega))$ and $L^2(0,T;H^{-s}(\Omega))$, respectively.

By extracting a subsequence of $\{\boldsymbol{v}_m\}_{m\in\mathbb{N}}$ (still denote the subsequence as $\{\boldsymbol{v}_m\}_{m\in\mathbb{N}}$), there exists $\boldsymbol{v} \in L^2(0,T;\widetilde{H}^s(\Omega))$, with $\boldsymbol{v}' \in L^2(0,T;L^2(\Omega))$ and $\boldsymbol{v}'' \in L^2(0,T;H^{-s}(\Omega))$ such that

$$\begin{cases} \boldsymbol{v}_m \rightharpoonup \boldsymbol{v} & \text{weakly in } L^2(0,T;\widetilde{H}^s(\Omega)), \\ \boldsymbol{v}_m' \rightharpoonup \boldsymbol{v}' & \text{weakly in } L^2(0,T;L^2(\Omega)), \\ \boldsymbol{v}_m'' \rightharpoonup \boldsymbol{v}'' & \text{weakly in } L^2(0,T;H^{-s}(\Omega)). \end{cases}$$

Given a fixed integer N, choose a function $\widetilde{\boldsymbol{v}} \in C^1(0,T;\widetilde{H}^s(\Omega))$ of the form

$$\widetilde{\boldsymbol{v}}(t) := \sum_{k=1}^N d^k(t) w_k, \tag{A.16}$$

where $\{d^k\}_{k=1}^N$ are smooth functions and $\{w_k\}_{k\in\mathbb{N}}$ are the eigenfunctions given by (A.1) and (A.2). By choosing $m \geq N$, multiplying (A.5) by $d^k(t)$, and summing $k = 1,\ldots,N$, we then integrate the resulting identity with respect to t to derive

$$\int_0^T \left((\boldsymbol{v}_m'',\widetilde{\boldsymbol{v}})_{L^2(\Omega)} + B[\boldsymbol{v}_m,\widetilde{\boldsymbol{v}};t]\right)dt = \int_0^T (\widetilde{\boldsymbol{F}},\widetilde{\boldsymbol{v}})_{L^2(\Omega)}\,dt. \tag{A.17}$$

By passing the limit (along a subsequence, if necessary) in (A.17), we have

$$\int_0^T \left((\boldsymbol{v}'',\widetilde{\boldsymbol{v}})_{L^2(\Omega)} + B[\boldsymbol{v},\widetilde{\boldsymbol{v}};t]\right)dt = \int_0^T (\widetilde{\boldsymbol{F}},\widetilde{\boldsymbol{v}})_{L^2(\Omega)}\,dt. \tag{A.18}$$

Note that (A.18) holds for all functions $\widetilde{\boldsymbol{v}}$ of the form (A.16), which are dense in $L^2(0,T;\widetilde{H}^s(\Omega))$. Combining (A.18) and the denseness of $\widetilde{\boldsymbol{v}}$, we obtain

$$(\boldsymbol{v}'',\phi)_{L^2(\Omega)} + B[\boldsymbol{v},\phi;t] = (\widetilde{\boldsymbol{F}},\phi)_{L^2(\Omega)},$$

for any $\phi \in \widetilde{H}^s(\Omega)$ and for a.e. $0 \leq t \leq T$. Moreover, by [Eva98, Theorem 5.9.2], one can show that $\boldsymbol{v} \in C([0,T];L^2(\Omega))$ and $\boldsymbol{v}' \in C([0,T];H^{-s}(\Omega))$.

It remains to show that

$$\boldsymbol{v}(0) = \varphi \quad \text{and} \quad \boldsymbol{v}'(0) = \psi. \tag{A.19}$$

In order to show (A.19), let us select any function $\boldsymbol{w} \in C^2([0,T];\widetilde{H}^s(\Omega))$, with $\boldsymbol{w}(T) = \boldsymbol{w}'(T) = 0$. Integrating by parts twice with respect to t of (A.18) yields that

$$
\begin{aligned}
&\int_0^T \left((\boldsymbol{w}'', \boldsymbol{v})_{L^2(\Omega)} + B[\boldsymbol{v}, \boldsymbol{w}; t]\right) dt \\
&= \int_0^T (\widetilde{\boldsymbol{F}}, \boldsymbol{w})_{L^2(\Omega)}\, dt - (\boldsymbol{v}(0), \boldsymbol{w}'(0))_{L^2(\Omega)} + (\boldsymbol{v}'(0), \boldsymbol{w}(0))_{L^2(\Omega)}.
\end{aligned} \tag{A.20}
$$

Similarly, from (A.17), one also has

$$
\begin{aligned}
&\int_0^T \left((\boldsymbol{w}'', \boldsymbol{v}_m)_{L^2(\Omega)} + B[\boldsymbol{v}_m, \boldsymbol{w}; t]\right) dt \\
&= \int_0^T (\widetilde{\boldsymbol{F}}, \boldsymbol{w})_{L^2(\Omega)}\, dt - (\boldsymbol{v}_m(0), \boldsymbol{w}'(0))_{L^2(\Omega)} + (\boldsymbol{v}_m'(0), \boldsymbol{w}(0))_{L^2(\Omega)}.
\end{aligned} \tag{A.21}
$$

Hence, by taking $m \to \infty$ of (A.21) (along a subsequences as before), we have

$$
\begin{aligned}
&\int_0^T \left((\boldsymbol{w}'', \boldsymbol{v})_{L^2(\Omega)} + B[\boldsymbol{v}, \boldsymbol{w}; t]\right) dt \\
&= \int_0^T (\widetilde{\boldsymbol{F}}, \boldsymbol{w})_{L^2(\Omega)}\, dt - (\widetilde{\varphi}, \boldsymbol{w}'(0))_{L^2(\Omega)} + (\widetilde{\psi}, \boldsymbol{w}(0))_{L^2(\Omega)}.
\end{aligned} \tag{A.22}
$$

Finally, comparing (A.20) and (A.22), by the arbitrariness of $\boldsymbol{w}(0)$, $\boldsymbol{w}'(0)$, we can conclude that $\boldsymbol{v}$ is a weak solution of (2.2).

Step 2. Uniqueness of weak solution.

Let u_1, u_2 be weak solutions of (2.1). Then $u = u_1 - u_2$ satisfies

$$
\begin{cases}
\left(\partial_t^2 + (-\Delta)^s + q\right) u = 0 & \text{in } \Omega_T, \\
u = 0 & \text{in } (\Omega_e)_T, \\
u = \partial_t u = 0 & \text{in } \mathbb{R}^n \times \{0\}.
\end{cases}
$$

For each fix $0 \le r \le T$, we define

$$
\boldsymbol{w}(t) := \begin{cases} \int_t^r \boldsymbol{u}(\tau)\, d\tau & \text{if } 0 \le t \le r, \\ 0 & \text{otherwise.} \end{cases}
$$

Note that $\boldsymbol{w}(t) \in \widetilde{H}^s(\Omega)$ for each $0 \le t \le T$. Therefore, choosing the test function $\phi = \boldsymbol{w}(t)$ in Definition 2.1 yields

$$
(\boldsymbol{u}''(t), \boldsymbol{w}(t))_{L^2(\Omega)} + B[\boldsymbol{u}, \boldsymbol{w}; t] = 0 \quad \text{for a.e. } 0 \le t \le T.
$$

Since $\boldsymbol{u}'(0) = \boldsymbol{w}(r) = 0$ and $\boldsymbol{w}'(t) = -\boldsymbol{u}(t)$, we have

$$
\begin{aligned}
0 &= \int_0^r (\boldsymbol{u}''(t), \boldsymbol{w}(t))_{L^2(\Omega)}\, dt + \int_0^r B[\boldsymbol{u}, \boldsymbol{w}; t]\, dt \\
&= -\int_0^r (\boldsymbol{u}'(t), \boldsymbol{w}'(t))_{L^2(\Omega)}\, dt + \int_0^r B[\boldsymbol{u}, \boldsymbol{w}; t]\, dt \\
&= \int_0^r (\boldsymbol{u}'(t), \boldsymbol{u}(t))_{L^2(\Omega)}\, dt - \int_0^r B[\boldsymbol{w}', \boldsymbol{w}; t]\, dt \\
&= \frac{1}{2}\int_0^r \frac{d}{dt}(\boldsymbol{u}(t), \boldsymbol{u}(t))_{L^2(\Omega)}\, dt - \frac{1}{2}\int_0^r \frac{d}{dt} B[\boldsymbol{w}, \boldsymbol{w}; t]\, dt \\
&= \frac{1}{2}\|\boldsymbol{u}(r)\|_{L^2(\Omega)}^2 - \frac{1}{2}\|\boldsymbol{u}(0)\|_{L^2(\Omega)}^2 - \frac{1}{2}B[\boldsymbol{w}, \boldsymbol{w}; r] + \frac{1}{2}B[\boldsymbol{w}, \boldsymbol{w}; 0] \\
&= \frac{1}{2}\|\boldsymbol{u}(r)\|_{L^2(\Omega)}^2 + \frac{1}{2}B[\boldsymbol{w}, \boldsymbol{w}; 0] \\
&= \frac{1}{2}\|\boldsymbol{u}(r)\|_{L^2(\Omega)}^2 + \frac{1}{2}\|(-\Delta)^{s/2}\boldsymbol{w}(0)\|_{L^2(\mathbb{R}^n)}^2 + \frac{1}{2}\int_\Omega q|\boldsymbol{w}(0)|^2\, dx.
\end{aligned}
$$

That is, we obtain
(A.23)

$$\|\boldsymbol{u}(r)\|_{L^2(\Omega)}^2 + \|(-\Delta)^{s/2}\boldsymbol{w}(0)\|_{L^2(\mathbb{R}^n)}^2 = -\int_\Omega q|\boldsymbol{w}(0)|^2\,dx \le \|q\|_{L^\infty}\int_\Omega |\boldsymbol{w}(0)|^2\,dx$$

Since

$$\int_\Omega |\boldsymbol{w}(0)|^2\,dx = \int_\Omega \left|\int_0^r \boldsymbol{u}(\tau)\,d\tau\right|^2\,dx \le \int_0^r\int_\Omega |\boldsymbol{u}(t)|^2\,dx\,dt = \int_0^r \|\boldsymbol{u}(t)\|_{L^2(\Omega)}^2\,dt,$$

(A.23) implies

$$\|\boldsymbol{u}(r)\|_{L^2(\Omega)}^2 \le \|q\|_{L^\infty}\int_0^r \|\boldsymbol{u}(t)\|_{L^2(\Omega)}^2\,dt. \tag{A.24}$$

Multiplying (A.24) by the integrating factor $e^{-r\|q\|_{L^\infty}}$ yields

$$\begin{aligned}&\frac{d}{dr}\left[e^{-r\|q\|_{L^\infty}}\int_0^r \|\boldsymbol{u}(t)\|_{L^2(\Omega)}^2\,dt\right]\\ =&e^{-r\|q\|_{L^\infty(\Omega)}}\left[-\|q\|_{L^\infty}\int_0^r \|\boldsymbol{u}(t)\|_{L^2(\Omega)}^2\,dt + \|\boldsymbol{u}(r)\|_{L^2(\Omega)}^2\right] \le 0,\end{aligned}$$

that is,

$$e^{-r\|q\|_{L^\infty(\Omega)}}\int_0^r \|\boldsymbol{u}(t)\|_{L^2(\Omega)}^2\,dt = 0 \quad \text{for all } r\in(0,T),$$

and this immediately implies $\boldsymbol{u}\equiv 0$.

Step 3. Energy estimate.

In Step 1 (precisely, (A.7)), we have derived

$$\begin{aligned}&\max_{0\le t\le T}\left(\|\boldsymbol{v}_m(t)\|_{\widetilde{H}^s(\Omega)} + \|\boldsymbol{v}_m'(t)\|_{L^2(\Omega)}\right)\\ \le& C\left(\|\widetilde{\boldsymbol{F}}\|_{L^2(0,T;L^2(\Omega))} + \|\widetilde{\varphi}\|_{\widetilde{H}^s(\Omega)} + \|\psi\|_{L^2(\Omega)}\right).\end{aligned}$$

Passing the limit $m\to\infty$, the estimate (2.3) follows directly from above. □

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