THE CALDERÓN PROBLEM FOR NONLOCAL PARABOLIC OPERATORS: A NEW REDUCTION FROM THE NONLOCAL TO THE LOCAL

CHING-LUNG LIN, YI-HSUAN LIN, AND GUNTHER UHLMANN

Abstract. In this article, we investigate the Calderón problem for nonlocal parabolic equations, where we are interested to recover the leading coefficient of nonlocal parabolic operators. The main contribution is that we can relate both (anisotropic) variable coefficients local and nonlocal Calderón problem for parabolic equations. More concretely, we show that the (partial) Dirichlet-to-Neumann map for the nonlocal parabolic equation determines the (full) Dirichlet-to-Neumann map for the local parabolic equation. This article extends our earlier results [LLU22] by introducing completely different methods. Moreover, the results hold for any spatial dimension $n \geq 2$.

Keywords. Calderón problem, nonlocal parabolic operators, Dirichlet-to-Neumann map, Cauchy data
Mathematics Subject Classification (2020): 35B35, 35R11, 35R30

Contents

1. Introduction

In this paper, we study the relation of the Calderón problem between both local and nonlocal parabolic equations. The key tool is to use a known extension problem for nonlocal parabolic operators $(\partial_t - \nabla\cdot\sigma\nabla)^s$, for $s \in (0,1)$, so that one can reduce the exterior measurements for nonlocal equations suitably to the boundary measurements for their local counterparts. Fractional type inverse problems have attracted a lot attention in recent years. The Calderón problem for the fractional Schrödinger equation was first investigated in [GSU20], where the authors demonstrated that the potential in a given region can be determined uniquely by the associated exterior measurement. The essential approach is relied on the *strong unique continuation property* (strong UCP in short) for the fractional Laplacian $(-\Delta)^s$, so that one can deduce the useful *Runge approximation* for the fractional Schrödinger equation. Based on these robust results, there is a huge literature developed in this direction.

Let us briefly summarize several works related to fractional inverse problems. In [CLR20], the authors determined both drift term and potential uniquely, which remains open for the local case. In [CLL19], the researchers used single measurement to determine unknown cavity, which cannot hold in their local counterparts. Meanwhile, in the works [HL19, HL20], the authors derived an if-and-only-if monotonicity relation, which leads to a simple reconstruction algorithm. Later, fractional/nonlocal type inverse problems are widely developed in the field of inverse problems, which consists of determination of singular potentials, lower order local perturbations, higher order fractional Laplacians, single measurement, and generalizations to many other nonlocal operators. We refer readers to those works [BGU21, CMR21, CMRU22, GLX17, CLL19, CLR20, FGKU21, HL19, HL20, GRSU20, GU21, Lin22, LL22, LL23, LLR20, LLU22, KLW22, RS20, RS18, RZ22b, RZ22b, RZ22a, CRZ22, RZ22c, CRTZ22, Zim23, GU21, LRZ22, LZ23] and the references therein. We also point out that the recovery of leading coefficients has been addressed in recent works [Fei21, FGKU21, CO23, Rül23] by using the local source-to-solution map. The main approach is based on the heat kernel representation of nonlocal operators and Kannai transmutation. These materials transfer the elliptic type nonlocal inverse problem to a local hyperbolic problem, which has been studied by utilizing the boundary control method.

As a matter of fact, the solvability for the most of these fractional inverse problems based on the linear structure of nonlocal operators, in particular, the (strong) UCP and the Runge approximation play essential roles in related studies. Most of mentioned works, the authors investigated uniquely recovering problem for lower order coefficients, i.e., the main nonlocal operator is known a priori. In general, the determination of the leading parameter is harder than the recovery of lower coefficients. In this work, we want to give another possible description, which builds a bridge between nonlocal and local problems, and this connection will help us to recover leading coefficients for a nonlocal parabolic operator.

1.1. **Mathematical model and main results.** Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with Lipschitz boundary for $n \geq 2$, and $T \in (0,\infty)$. Consider the fractional Calderón problem for the nonlocal parabolic equation

$$\begin{cases} \left(\partial_t - \nabla\cdot\sigma(x)\nabla\right)^s u(t,x) = 0 & \text{in } \Omega_T, \\ u(t,x) = f(t,x) & \text{in } (\Omega_e)_T, \\ u(t,x) = 0 & \text{for } x\in\mathbb{R}^n \text{ and } t\leq -T, \end{cases} \tag{1.1}$$

where $\Omega_e := \mathbb{R}^n \setminus \overline{\Omega}$. Throughout this work, we always assume the set

$$A_T := (-T,T)\times A,$$

for any subset $A \subseteq \mathbb{R}^n$. Here $\sigma = (\sigma_{ik}(x))_{1\leq i,k\leq n} \in C^2(\mathbb{R}^n;\mathbb{R}^{n\times n})$ satisfies the symmetry and ellipticity conditions

$$\begin{cases} \sigma_{ik} = \sigma_{ki}, \text{ for all } i,j = 1,2,\dots,n, \\ \lambda|\xi|^2 \leq \displaystyle\sum_{i,k=1}^n \sigma_{ik}(x)\xi_i\xi_k \leq \lambda^{-1}|\xi|^2, \end{cases} \tag{1.2}$$

for any $x \in \mathbb{R}^n$ and $\xi = (\xi_1, \dots, \xi_n) \in \mathbb{R}^n$, where $\lambda \in (0,1)$ is a positive constant. The well-posedness of (1.1) has been studied by [BKS22] with respect to suitable function spaces. Let $W \subset \Omega_e$ be a nonempty open subset , we are able to define the corresponding (partial) Dirichlet-to-Neumann (DN) map

$$\begin{aligned} \Lambda_\sigma^s :\ &\mathbf{H}^s(W_T) \to \mathbf{H}^{-s}(W_T), \\ &f \mapsto \left(\partial_t - \nabla \cdot \sigma \nabla\right)^s u_f|_{W_T}, \end{aligned} \tag{1.3}$$

where $u_f \in \mathbb{H}^s(\mathbb{R}^{n+1})$ is the unique solution to (1.1). These function spaces that we are using will be introduced in Section 2.

(IP1) **Nonlocal inverse problem.** Can we determine σ by using the nonlocal DN map Λ_σ^s given by (1.3)?

In fact, it is a nontrivial problem to determine the leading coefficient σ for nonlocal models.

On the other hand, let $\Omega \subset \mathbb{R}^n$ be a bounded set as before, and we consider the classical Calderón problem for (local) parabolic equations. One tries to recover an unknown, possibly anisotropic leading coefficient $\sigma = \sigma(x) : \overline{\Omega} \to \mathbb{R}^{n\times n}$, where σ could be a sufficiently regular anisotropic symmetric matrix-valued function. More concretely, consider the local parabolic problem

$$\begin{cases} \left(\partial_t - \nabla \cdot \sigma(x) \nabla\right) v(t,x) = 0 & \text{in } \Omega_T, \\ v(t,x) = g(t,x) & \text{on } (\partial\Omega)_T, \\ v(-T,x) = 0 & \text{for } x \in \Omega. \end{cases} \tag{1.4}$$

It is known that the initial-boundary value problem (1.4) is well-posed (for example, see [DL92, Chapter XVIII]), so that we can define the corresponding local (full) DN map

$$\begin{aligned} \Lambda_\sigma :\ &L^2(-T,T;H^{1/2}(\partial\Omega)) \to L^2(-T,T;H^{-1/2}(\partial\Omega)), \\ &g \mapsto \sigma \nabla v_g \cdot \nu|_{(\partial\Omega)_T}, \end{aligned} \tag{1.5}$$

where $v_g \in L^2(0,T;H^1(\Omega))$ is the unique solution to (1.4).

(IP2) **Local inverse problem.** Can we determine σ by using the local DN map Λ_σ given by (1.5)?

In fact, the answer of (IP2) is resolved for some cases. More precisely, in [CK01], the author investigated that if $\sigma = \sigma(x)$ is a scalar function, then Λ_σ determines σ in Ω. In this article, we want to use similar ideas as in our earlier work [LLU22], where we want to show that the nonlocal DN map Λ_σ^s determines the local DN map Λ_σ. In this work, we introduce an alternative approach, which is motivated by the very recent work [CGRU23]. This new method is mainly based on the Caffarelli-Silvestre type extension problem for the nonlocal parabolic operator $\left(\partial_t - \nabla \cdot \sigma \nabla\right)^s$.

For $s \in (0,1)$, the extension formula for $\left(\partial_t - \nabla \cdot \sigma \nabla\right)^s$ is characterized as follows. Let $u \in \mathbf{H}^s(\mathbb{R}^{n+1})$, let $\widetilde{u} = \widetilde{u}(t,x,y)$ be the solution of the Dirichlet problem for $(t,x,y) \in \mathbb{R}^{n+2}_+ := \mathbb{R}^{n+1} \times (0,\infty)$ with $(t,x) \in \mathbb{R}^{n+1}$ and $y \in (0,\infty)$ that

$$\begin{cases} y^{1-2s}\partial_t \widetilde{u} - \nabla_{x,y} \cdot \left(y^{1-2s}\widetilde{\sigma}(x)\nabla_{x,y}\widetilde{u}\right) = 0 & \text{in } \mathbb{R}^{n+2}_+, \\ \widetilde{u}(t,x,0) = u(t,x) & \text{on } \mathbb{R}^{n+1}, \\ \widetilde{u}(t,x,y) = 0 & \text{on } t \le -T, \end{cases} \tag{1.6}$$

where $\widetilde{\sigma}$ is of the form

$$\widetilde{\sigma}(x) = \begin{pmatrix} \sigma(x) & 0 \\ 0 & 1 \end{pmatrix}, \tag{1.7}$$

where σ always satisfies the condition (1.2) throughout this paper. Meanwhile, we use the notation $\nabla \equiv \nabla_x$, $\nabla_{x,y} \equiv (\nabla_x, \partial_y)$ and Id stands for the identity in this article. Then the nonlocal parabolic operator $(\partial_t - \nabla\cdot\sigma\nabla)^s$ is realized in terms of the Dirichlet-to-Neumann relation

$$(\partial_t - \nabla\cdot\sigma\nabla)^s u(t,x) = d_s \lim_{y\to 0} y^{1-2s}\partial_y \widetilde{u}(t,x,y), \tag{1.8}$$

where d_s is a constant depending only on $s\in(0,1)$. In order to study (IP1), we will use (IP2), more specifically, we will show the following theorem, which states that the nonlocal DN map (1.3) determines the local DN map (1.5).

Theorem 1.1. *Let $\Omega, W\subset\mathbb{R}^n$ be bounded sets with Lipschitz boundaries with $\overline{\Omega}\cap\overline{W}=\emptyset$, for $n\geq 2$, $T\in(0,\infty)$ and $s\in(0,1)$. Let $\widetilde{\sigma}\in C^2(\mathbb{R}^n;\mathbb{R}^{(n+1)\times(n+1)})$ be of the form (1.7), where $\sigma(x)\in C^2(\mathbb{R}^n;\mathbb{R}^{n\times n})$ satisfies (1.2) with $|\sigma|+|\nabla\sigma|\leq M<\infty$ in $\mathbb{R}^n$, for some constant $M>0$. Let $\widetilde{u}$ be a weak solution to (1.6), then we have*

$$v(t,x) := \int_0^\infty y^{1-2s}\widetilde{u}(t,x,y)\,dy \in L^2(0,T;H^1(\Omega)), \tag{1.9}$$

such that the function v is a weak solution to the parabolic equation (1.4). Moreover, for any $f\in\widetilde{\mathbf{H}}^s(W_T)$, the nonlocal (partial) DN map

$$\Lambda^s_\sigma : \widetilde{\mathbf{H}}^s(W_T)\to\mathbf{H}^{-s}(W_T), \quad f\mapsto d_s\lim_{y\to 0} y^{1-2s}\partial_y\widetilde{u}\big|_{W_T}$$

determines the local (full) DN map

$$\Lambda_\sigma : L^2(-T,T;H^{1/2}(\partial\Omega))\to L^2(-T,T;H^{-1/2}(\partial\Omega)), \quad g\mapsto \sigma\nabla v\cdot\nu\big|_{(\partial\Omega)_T}.$$

Remark 1.2. *Let us point out that we do not need to assume $\sigma=\mathrm{Id}$ in the exterior domain Ω_e in our parabolic case, which is unlike the assumption for the elliptic setting demonstrated in [CGRU23]. We refer readers to page 35 of this article for more detailed explanations.*

We can reformulate Theorem 1.1 in terms of the next result.

Proposition 1.3. *Adopting all assumptions of Theorem 1.1. Define the nonlocal (partial) Cauchy data $\mathcal{C}^s_{\sigma,W_T}$*

$$\mathcal{C}^s_{\sigma,W_T} := \left(f|_{W_T}, (\partial_t-\nabla\cdot\sigma\nabla)^s u_f|_{W_T}\right)\subset\widetilde{\mathbf{H}}^s(W_T)\times\mathbf{H}^{-s}(W_T),$$

where $u_f\in\mathbf{H}^s(\mathbb{R}^{n+1})$ is the solution to (1.1). Define the local (full) Cauchy data

$$\begin{aligned}\mathcal{C}_{\sigma,(\partial\Omega)_T} &:= \left(g|_{(\partial\Omega)_T}, \sigma\nabla v_g|_{(\partial\Omega)_T}\right)\\ &\subset L^2(-T,T;H^{1/2}(\partial\Omega))\times L^2(-T,T;H^{-1/2}(\partial\Omega)),\end{aligned}$$

where $v_g\in L^2(-T,T;H^1(\Omega))$ is the solution to (1.4). Then there exists a bounded linear map

$$\begin{aligned}&\mathrm{T}:\mathcal{C}^s_{\sigma,W_T}\to\mathcal{C}_{\sigma,(\partial\Omega)_T},\\ &(f,\Lambda^s_\sigma f)\mapsto\left(v|_{(\partial\Omega)_T}, \sigma\nabla v\cdot\nu|_{(\partial\Omega)_T}\right)\end{aligned}$$

such that

$$\overline{\mathrm{T}\left(\mathcal{C}^s_{\sigma,W_T}\right)}^{L^2(-T,T;H^{1/2}(\partial\Omega))\times L^2(-T,T;H^{-1/2}(\partial\Omega))} = \mathcal{C}_{\sigma,(\partial\Omega)_T}.$$

Here ν is the unit outer normal to $\partial\Omega$ and $v(t,x)$ is defined by (1.9).

As shown in [LLU22], Theorem 1.1 stands for the reduction of the Calderón problem for nonlocal parabolic equations to the local ones. The main difference is that we need to take the exterior data f is taken from $\mathbf{H}^s((\Omega_e)_T)$, but not $\mathbf{H}^s(W_T)$, for a given open subset $W\subset\Omega_e$.

Remark 1.4. *By using the conclusion of Theorem 1.1, we knows that the determination of leading coefficients for nonlocal parabolic operators depend on their local counterparts. It is natural since that the nonlocal DN map contains more data than the local DN map.*

Corollary 1.5. *Adopting all assumptions in Theorem 1.1. Assume that* $\sigma = \sigma \mathrm{I_n}$ *is an isotropic* $n\times n$ *matrix satisfying* (1.2)*. Then the nonlocal DN map* Λ^s_σ *determines* σ *in (IP1) uniquely.*

Next, we are want to know the case when the leading coefficient is a matrix-valued function. It is known that the non-uniqueness result has been investigated by [GAV12] for the local case (i.e. $s = 1$). Let $\sigma(x) = (\sigma_{ij}(x))_{1\leq i,j\leq n} \in (\sigma_{ik}(x))_{1\leq i,k\leq n} \in C^2(\mathbb{R}^n;\mathbb{R}^{n\times n})$ be a matrix-valued function satisfying (1.2). Consider $\Phi : \overline{\Omega} \to \overline{\Omega}$ as a C^∞ diffeomorphism such that $\Phi|_{\partial\Omega} = \mathrm{Id}$ (the identity map), then $v(t,x)$ is a solution to the parabolic equation

$$\partial_t v - \nabla\cdot(\sigma\nabla v) = 0 \text{ for } (t,x)\in\Omega_T \tag{1.10}$$

if and only if $\widetilde{v}(t,y) := v(t,\Phi^{-1}(y))$ is a solution to

$$\partial_t\left(\Phi_* 1(y)\widetilde{v}\right) - \nabla\cdot(\Phi_*\sigma\nabla\widetilde{v}) = 0 \text{ for } (t,y)\in\Omega_T, \tag{1.11}$$

where Φ_* denotes the *push-forward*

$$\begin{cases} \Phi_* 1(y) = \left.\frac{1}{\det(D\Phi)(x)}\right|_{x=\Phi^{-1}(y)}, \\ \Phi_*\sigma(y) = \left.\frac{D\Phi^T(x)\sigma(x)D\Phi(x)}{\det(D\Phi)(x)}\right|_{x=\Phi^{-1}(y)}. \end{cases}$$

Here $D\Phi$ denotes the (matrix) differential of Φ and $D\Phi^T$ is the transpose of $D\Phi$. Since $\Phi|_{\partial\Omega} = \mathrm{Id}$, one can see that the (full) Cauchy data (or DN map) of (1.10) and (1.11) are the same, i.e.,

$$\mathcal{C}_{\sigma,(\partial\Omega)_T} := \left(v|_{\sigma,(\partial\Omega)_T}, \sigma\partial_\nu v|_{(\partial\Omega)_T}\right) = \left(\widetilde{v}|_{(\partial\Omega)_T}, \Phi_*\sigma\partial_\nu\widetilde{v}|_{(\partial\Omega)_T}\right) := \mathcal{C}_{\Phi_*\sigma,(\partial\Omega)_T}.$$

This implies the non-uniqueness property for leading coefficients holds for local parabolic operators in the anisotropic case.

Similar to the local case, our final result in this paper is to demonstrate that non-uniqueness also holds for the nonlocal parabolic case.

Corollary 1.6 (Non-uniqueness)**.** *Let* $\Omega, W \subset \mathbb{R}^n$ *be bounded sets with Lipschitz boundaries with* $\overline{\Omega}\cap\overline{W} = \emptyset$*, for* $n\geq 2$*,* $T\in(0,\infty)$ *and* $s\in(0,1)$*. Let* $\widetilde{\sigma}\in C^2(\mathbb{R}^n;\mathbb{R}^{(n+1)\times(n+1)})$ *be a matrix-valued function in* $\mathbb{R}^n$ *satisfying* (1.2)*. Then* Λ^s_{σ,W_T} *determine* σ *up to diffeomorphism, that is, there exists a Lipschitz invertible map* $\Phi:\mathbb{R}^n\to\mathbb{R}^n$ *with* $\Phi|_W = \mathrm{Id}$ *such that*

$$\Lambda^s_{\sigma,W_T}(f) = \Lambda^s_{\Phi_*\sigma,W_T}(f), \text{ for any } f\in C^\infty_c(W_T),$$

where

$$\Phi_*\sigma(y) = \left.\frac{D\Phi^T(x)\sigma(x)D\Phi(x)}{\det(D\Phi)(x)}\right|_{x=\Phi^{-1}(y)}.$$

1.2. **Outline of the argument.** Let us compute the relation between $\widetilde{u}$ and v, where $\widetilde{u}$ is a solution to (1.6) and v is defined by (1.9). Via direct computations, one can see that v is a solution to the following parabolic equation

$$\begin{aligned} 0 &= \int_0^\infty \left\{ y^{1-2s}\partial_t \widetilde{u} - \nabla_{x,y}\cdot \left(y^{1-2s}\widetilde{\sigma}\nabla_{x,y}\widetilde{u}\right)\right\} dy \\ &= \partial_t \left(\int_0^\infty y^{1-2s}\widetilde{u}\, dy\right) - \nabla\cdot\sigma\nabla \left(\int_0^\infty y^{1-2s}\widetilde{u}\, dy\right) \\ &\quad - \int_0^\infty \partial_y \left(y^{1-2s}\partial_y \widetilde{u}(t,x,y)\right) dy \\ &= \left(\partial_t - \nabla\cdot\sigma\nabla\right) v, \end{aligned} \tag{1.12}$$

for $(t,x)\in\Omega_T$. Here we used the fact that

$$\begin{aligned} \int_0^\infty \partial_y \left(y^{1-2s}\partial_y \widetilde{u}(t,x,y)\right) dy &= \lim_{y\to\infty} y^{1-2s}\partial_y \widetilde{u}(t,x,y) - \lim_{y\to 0} y^{1-2s}\partial_y\widetilde{u}(t,x,y) \\ &= \lim_{y\to\infty} y^{1-2s}\partial_y \widetilde{u}(t,x,y) - \underbrace{\frac{1}{d_s}\left(\partial_t - \nabla\cdot\sigma\nabla\right)^s u(t,x)}_{\text{Here we use (1.8)}} \\ &= 0, \end{aligned}$$

where we also used the decay property of $\widetilde{u}$ (and its derivatives) at infinity and u is a solution to (1.1).

Formally, the corresponding DN map of (1.12) is given by

$$\begin{aligned} &\Lambda_\sigma : L^2(-T,T;H^{1/2}(\partial\Omega)) \to L^2(-T,T;H^{-1/2}(\partial\Omega)), \\ &\underbrace{\int_0^\infty y^{1-2s}\widetilde{u}(t,x,y)\,dy\Big|_{(\partial\Omega)_T}}_{=v(t,x)|_{(\partial\Omega)_T}} \mapsto \underbrace{\sigma\nabla\left(\int_0^\infty y^{1-2s}\widetilde{u}(t,x,y)\,dy\right)\cdot\nu\Big|_{(\partial\Omega)_T}}_{=\sigma\nabla v\cdot\nu|_{(\partial\Omega)_T}}. \end{aligned}$$

Making the preceding formal computations rigorously plays an essential role in the proof of Theorem 1.1. In fact, we need to prove that the function $v(t,x)$ given by (1.9) belong to suitable function spaces with $v|_{(\partial\Omega)_T}\in L^2(-T,T;H^{1/2}(\partial\Omega))$. Therefore, the local DN map can be determined by the nonlocal DN map as we wish.

1.3. **Organization of the article.** The paper is structured as follows. In Section 2, we define the nonlocal parabolic operator rigorously. We also introduce function spaces and demonstrate the well-posedness for our study. In Section 3, we investigate the Caffarelli-Silvestre type extension problem for the nonlocal parabolic operator, and we offer the useful function to connect the nonlocal and local equations. We provide regularity estimates for solutions to the extension problem in Section 4. With the analysis in Section 4 at hand, we can will demonstrate our key equation of this work in Section 5, and associated density results will be proved in Section 6. Finally, we prove our main results in Section 7.

2. Preliminaries

We review several basic properties and tools, which will be utilized in our work.

2.1. **Nonlocal parabolic operators.** Note that the nonlocal parabolic operator $(\partial_t - \nabla\cdot\sigma\nabla)^s$ is defined in [BDLCS21, BKS22], where $\sigma = (\sigma_{ik})_{1\le i,k\le n}$ is a matrix-valued function given via (1.2) in $\mathbb{R}^n$. Next, it is known that the parabolic operator $\partial_t - \nabla\cdot\sigma\nabla$ in $\mathbb{R}\times\mathbb{R}^n$ possesses a globally defined fundamental solution $p(x,z,\tau)$, which satisfies

$$\mathcal{P}_t 1(t,x) = \int_{\mathbb{R}^n} p(x,z,\tau)\,dz = 1, \text{ for every } x\in\mathbb{R}^n \text{ and } \tau>0,$$

where $\mathcal{P}_t$ stands for the heat semigroup. In addition, the evolutive semigroup $\mathcal{P}_\tau$ is given by

$$\mathcal{P}_\tau u(t,x) := \int_{\mathbb{R}^n} p(x,z,\tau)u(t-\tau,z)\,dz, \qquad \text{for } u\in\mathcal{S}(\mathbb{R}^{n+1}),$$

where $p(x,z,\tau)$ is the heat kernel associated to the elliptic operator $\nabla\cdot\sigma\nabla$ such that

$$\partial_\tau p(x,z,\tau) - \nabla\cdot\sigma\nabla p(x,z,\tau) = 0, \tag{2.1}$$

and $\mathcal{S}(\mathbb{R}^{n+1})$ denotes the Schwarz space. In addition, the heat kernel $p(x,z,\tau)$ satisfies

$$C_1\left(\frac{1}{4\pi\tau}\right)^{n/2} e^{-\frac{c_1|x-z|^2}{4\tau}} \le p(x,z,\tau) \le C_2\left(\frac{1}{4\pi\tau}\right)^{n/2} e^{-\frac{c_2|x-z|^2}{4\tau}},$$

for some positive constants c_1, c_2, C_1 and C_2. Moreover, it is known that the heat kernel possesses the pointwise estimate (see [ST10] for $\ell = 0$ and [CJKS20] for $\ell = 1$)

$$\left|\nabla_x^\ell p(x,z,\tau)\right| \lesssim \tau^{-\frac{n+\ell}{2}} e^{-c\frac{|x-z|^2}{\tau}}, \text{ for } \ell = 0,1. \tag{2.2}$$

Since $\{\mathcal{P}_\tau\}_{\tau\ge 0}$ can be also regarded a strongly continuous contractive semigroup with $\|\mathcal{P}_\tau u - u\|_{L^2(\mathbb{R}^{n+1})} = \mathcal{O}(\tau)$, then the explicit definition of $(\partial_t - \nabla\cdot\sigma\nabla)^s$, for $s\in(0,1)$.

Definition 2.1. *Let $s\in(0,1)$ and $u\in\mathcal{S}(\mathbb{R}^{n+1})$, then $(\partial_t-\nabla\cdot\sigma\nabla)^s$ is defined by the Balakrishnan formula (see [BKS22]) as*

$$(\partial_t - \nabla\cdot\sigma\nabla)^s u(t,x) := -\frac{s}{\Gamma(1-s)}\int_0^\infty \left(\mathcal{P}_\tau u(t,x) - u(t,x)\right)\frac{d\tau}{\tau^{1+s}}. \tag{2.3}$$

Moreover, via the Fourier transform in the time-variable $t\in\mathbb{R}$, we can write $(\partial_t - \nabla\cdot\sigma\nabla)^s u$ in terms of the Fourier transform. It is known that the heat semigroup $\{\mathcal{P}_t\}_{t\ge0}$ can be written by spectral measures as an identity of gamma functions:

$$\mathcal{P}_t = \int_0^\infty e^{-\lambda t}\,dE_\lambda \quad \text{ and } \quad -\frac{s}{\Gamma(1-s)}\int_0^\infty \frac{e^{-(\lambda+\mathsf{i}\rho)t}-1}{\tau^{1+s}}\,d\tau = (\lambda+\mathsf{i}\rho)^s,$$

for $\lambda>0$ and $\rho\in\mathbb{R}$, where $\mathsf{i}=\sqrt{-1}$. Consider the time Fourier transform $\mathcal{F}_t$ of $\mathcal{P}_\tau u$, then there holds

$$\mathcal{F}_t\left(\mathcal{P}_\tau u\right)(\rho,\xi) = e^{-\mathsf{i}\rho\tau}\mathcal{P}_\tau\left(\mathcal{F}_t u(\rho,\cdot)\right)(\xi),$$

which yields that the Fourier analogue of the definition (2.3)

$$\begin{aligned}\mathcal{F}_t\left(\mathcal{H}^s u\right)(\rho,\cdot) &= -\frac{s}{\Gamma(1-s)}\int_0^\infty \frac{1}{\tau^{1+s}}\int_0^\infty \left(e^{-(\lambda+\mathsf{i}\rho)\tau}-1\right)dE_\lambda\left(\mathcal{F}_t u(\rho,\cdot)\right)d\tau \\ &= \int_0^\infty (\lambda+\mathsf{i}\rho)^s\,dE_\lambda\left(\mathcal{F}_t u(\cdot,\rho)\right).\end{aligned}$$

2.2. **Function spaces.** Given any $u\in\mathcal{S}(\mathbb{R}^{n+1})$, it is known that

$$\|\mathcal{F}_t\left(\mathcal{H}^s u\right)(\rho,\cdot)\|_{L^2(\mathbb{R}^n)} = \int_0^\infty |\lambda+\mathsf{i}\rho|^{2s}\;d\|E_\lambda\left(\mathcal{F}_t u(\rho,\cdot)\right)\|^2,$$

for $\rho\in\mathbb{R}$. With the preceding relation at hand, let us define the space $\mathbf{H}^{2s}(\mathbb{R}^{n+1})$ as the completion of $\mathcal{S}(\mathbb{R}^{n+1})$ with respect to the norm

$$\|u\|_{\mathbf{H}^{2s}(\mathbb{R}^{n+1})} = \left(\int_{\mathbb{R}}\int_0^\infty \left(1+|\lambda+\mathsf{i}\rho|^2\right)^s d\|E_\lambda\left(\mathcal{F}_t u(\rho,\cdot)\right)\|^2\,d\rho\right)^{1/2}.$$

Next, given $a \in \mathbb{R}$ and an open set $\mathcal{O} \subset \mathbb{R}^{n+1}$, we define

$$\begin{aligned}\mathbf{H}^a(\mathbb{R}^{n+1}) &= \Big\{\text{Completion of } \mathcal{S}(\mathbb{R}^{n+1}) \text{ with respect to the norm :} \\ &\qquad \int_{\mathbb{R}}\int_0^\infty \left(1+|\lambda+\mathrm{i}\rho|^2\right)^{a/2} d\|E_\lambda\left(\mathcal{F}_t u(\rho,\cdot)\right)\|^2\, d\rho\Big\}, \\ \mathbf{H}^a(\mathcal{O}) &= \left\{u|_{\mathcal{O}} : u \in \mathbb{H}^a(\mathbb{R}^{n+1})\right\}, \\ \widetilde{\mathbf{H}}^a(\mathcal{O}) &= \text{closure of } C_c^\infty(\mathcal{O}) \text{ in } \mathbb{H}^a(\mathbb{R}^{n+1}).\end{aligned}$$

We also define

$$\|u\|_{\mathbf{H}^a(\mathcal{O})} := \inf\left\{\|v\|_{\mathbf{H}^a(\mathbb{R}^{n+1})} : v|_{\mathcal{O}} = u\right\},$$

and the dual spaces

$$\mathbf{H}^{-a}(\mathcal{O}) = \widetilde{\mathbf{H}}^a(\mathcal{O})^* \quad \text{and} \quad \widetilde{\mathbf{H}}^{-a}(\mathcal{O}) = \left(\mathbf{H}^a(\mathcal{O})\right)^*.$$

On the other hand, we may also consider the parabolic fractional Sobolev space

$$\mathbb{H}^a(\mathbb{R}^{n+1}) := \left\{u \in L^2(\mathbb{R}^{n+1}) : \left(|\xi|^2 + \mathrm{i}\rho\right)^{a/2} \widehat{u}(\rho,\xi) \in L^2(\mathbb{R}^{n+1})\right\},$$

where $\widehat{u}(\xi,\rho) = \int_{\mathbb{R}^{n+1}} e^{-\mathrm{i}(t,x)\cdot(\rho,\xi)} u(x,t)\, dtdx$ denotes the Fourier transform of u with respect to the (t,x)-variable.

In the same time, the graph norm of $\mathbb{H}^a$-functions is given by

$$\|u\|^2_{\mathbb{H}^a(\mathbb{R}^{n+1})} := \int_{\mathbb{R}^{n+1}} \left(1 + \left(|\xi|^4 + |\rho|^2\right)^{1/2}\right)^a |\widehat{u}(\rho,\xi)|^2\, d\rho d\xi. \tag{2.4}$$

One may rewrite

$$\mathbb{H}^a(\mathbb{R}^{n+1}) = \mathbb{H}^{a/2,a}(\mathbb{R}^{n+1}),$$

where the exponents $a/2$ and a denote the fractional derivatives of time and space, respectively. Particularly, when $a = s \in (0,1)$, from [BKS22, Section 3], which is known as the parabolic version of the Kato square root problem introduced in [AEN20], then there holds

$$\mathbb{H}^s(\mathbb{R}^{n+1}) = \mathbf{H}^s(\mathbb{R}^{n+1}), \text{ for } s \in (0,1), \tag{2.5}$$

and we denote

$$\mathbb{H}^s_E := \left\{u \in \mathbb{H}^s(\mathbb{R}^{n+1}) : \mathrm{supp}(u) \subset E\right\},$$

for any closed set $E \subset \mathbb{R}^{n+1}$. We also give a quick review for the fractional Sobolev space $H^s(\mathbb{R}^n)$ for $s \in (0,1)$, which is defined by

$$H^s(\mathbb{R}^n) := \left\{f \in \mathcal{S}'(\mathbb{R}^n) : \|u\|_{H^s(\mathbb{R}^n)} := \left\|\left(1+|\xi|^2\right)^{s/2}\widehat{u}(\xi)\right\|_{L^2(\mathbb{R}^n)} < \infty\right\}. \tag{2.6}$$

Finally, in the rest of the paper, we use $\lesssim$ (resp. $\approx$) to denote that an inequality (resp. equality) holds up to a positive constant whose exact value is irrelevant in our arguments.

2.3. **Well-posedness for the nonlocal parabolic equation.** In [BKS22, LLR20, LLU22], the authors demonstrated that the problem (1.1) is well-posed (for either variable coefficients or constant coefficients cases). More precisely, given any exterior data $f \in \mathbf{H}^s((\Omega_e)_T)$, one can always find a unique solution $u_f \in \mathbf{H}^s(\mathbb{R}^{n+1})$ solving (1.1). Furthermore, we point out the future information will not affect the behavior of solutions, i.e., if $u_f \in \mathbf{H}^s(\mathbb{R}^{n+1})$ is a weak solution to (1.1) in Ω_T, then $u_f(t,x)\chi_{(-T,T)}(t) \in \mathbf{H}^s(\mathbb{R}^{n+1})$ is also a weak solution of (1.1) in Ω_T, where $\chi_{(-T,T)}(t) = \begin{cases} 1 & \text{for } t \in (-T,T) \\ 0 & \text{otherwise}\end{cases}$ denotes the characteristic function. Hence, in the rest of this article, we can always assume that the solution $u_f(t,x)$ of (1.1) is supported for $t \in (-T,T)$ and $x \in \mathbb{R}^n$ without loss of generality. The support assumption also implies that we can assume that $u_f(-T,x) = u_f(T,x) = 0$ for $x \in \mathbb{R}^n$, where u_f is the solution to (1.1). Finally, if $u_f \in \mathbf{H}^s(\mathbb{R}^{n+1})$ is the solution to (1.1), then there holds

$$\|u_f\|_{\mathbf{H}^s(\mathbb{R}^{n+1})} \lesssim \|f\|_{\widetilde{\mathbf{H}}^s((\Omega_e)_T)}, \tag{2.7}$$

which has been derived in [BKS22, Theorem 3.3].

3. Extension problem, duality and auxiliary functions

It is known that the nonlocal parabolic operator $(\partial_t - \nabla\cdot\sigma\nabla)^s$ can be characterized by associated extension problem.

3.1. Extension problems and duality principle.

Let us review a rigorous formulation of the extension problem with respect to the nonlocal parabolic operator $(\partial_t - \nabla\cdot\sigma\nabla)^s$, for $s \in (0,1)$. As in [BKS22, Section 3.1], given an open set $\Sigma \subseteq \mathbb{R}^{n+2}_+ = \mathbb{R}^{n+1}\times(0,\infty)$, consider the energy space

$$\begin{aligned}&\mathcal{L}^{1,2}(\Sigma; y^{1-2s}dtdxdy)\\ &\quad := \left\{\widetilde{u} : \widetilde{u},\, \partial_{x_k}\widetilde{u},\, \partial_y\widetilde{u} \in L^2(\Sigma, y^{1-2s}dtdxdy), \text{ for } k=1,\dots,n\right\},\end{aligned}$$

where the function $\widetilde{u} \in L^2(\Sigma, y^{1-2s}dtdxdy)$ provided that

$$\|\widetilde{u}\|^2_{L^2(\Sigma, y^{1-2s}dtdxdy)} := \int_\Sigma y^{1-2s}\,|\widetilde{u}|^2\, dtdxdy < \infty.$$

Moreover, $\|\cdot\|_{\mathcal{L}^{1,2}(\Sigma;y^{1-2s}dtdxdy)}$ is defined via

$$\|\widetilde{u}\|_{\mathcal{L}^{1,2}(\Sigma;y^{1-2s}dtdxdy)} := \left(\int_\Sigma y^{1-2s}\left(|\widetilde{u}|^2 + |\nabla\widetilde{u}|^2 + |\partial_y\widetilde{u}|^2\right)dtdxdy\right)^{1/2}.$$

Note that $(\partial_t - \nabla\cdot\sigma\nabla)^s$ is already characterized by [BKS22, Theorem 3.1], which is stated below for the sake of completeness.

Proposition 3.1 (Extension problem). *Given $s \in (0,1)$, and $u \in \mathbf{H}^s(\mathbb{R}^{n+1})$. There exists a solution $\widetilde{u}$ to (1.6) which satisfies*

(a) $\lim_{y\to 0}\widetilde{u}(t,x,y) = u(t,x)$ *in* $\mathbf{H}^s(\mathbb{R}^{n+1})$,

(b) $\lim_{y\to 0} y^{1-2s}\partial_y\widetilde{u}(t,x,y) = d_s\,(\partial_t - \nabla\cdot\sigma\nabla)^s u$ *in* $\mathbf{H}^{-s}(\mathbb{R}^{n+1})$*, for some constant d_s depending on $s \in (0,1)$,*

(c) $\|\widetilde{u}\|_{\mathcal{L}^{1,2}(\mathbb{R}^{n+2}_+;y^{1-2s}dtdxdy)} \le C\|u\|_{\mathbf{H}^s(\mathbb{R}^{n+1})}$*, for some constant $C>0$ independent of u and $\widetilde{u}$.*

Since the proof is given by [BKS22, Theorem 3.1], we omit the proof. Here we want to emphasize that the proof is based on the representation formula for the function $\widetilde{u}$. In fact, via [BKS22, Theorem 3.1], the function $\widetilde{u}$ can be written in terms of

$$\widetilde{u}(t,x,y) = \int_0^\infty \int_{\mathbb{R}^n} P^s_y(x,z,\tau)u(t-\tau,z)\,dzd\tau, \tag{3.1}$$

where

$$P^s_y(x,z,\tau) := \frac{1}{2^{2s}\Gamma(s)}\frac{y^{2s}}{\tau^{1+s}}e^{-\frac{y^2}{4\tau}}p(x,z,\tau). \tag{3.2}$$

Here $p(x,z,\tau)$ denote the heat kernel associated to the elliptic operator $\nabla\cdot(\sigma\nabla)$ satisfying (2.1). Note that the constant d_s in (b) can be computed explicitly, which is

$$d_s := \frac{2^{2s-1}\Gamma(s)}{\Gamma(1-s)}. \tag{3.3}$$

Inspired by [CGRU23], we want to derive a duality principle for parabolic equations.

Proposition 3.2 (Duality)**.** *Let $\widetilde{\sigma} \in C^2(\mathbb{R}^n; \mathbb{R}^{(n+1)\times(n+1)})$ be of the form (1.7) for $n \in \mathbb{N}$, $s \in (0,1)$, and $h \in C^0(\mathbb{R}^{n+1})$. Suppose that $u_1 \in C^2(\mathbb{R}^{n+2}_+)$ with $y^{1-2s}\partial_y u_1 \in C^0(\overline{\mathbb{R}^{n+1}_+})$ is a classical solution of*

$$\begin{cases} y^{2s-1}\partial_t u_1 - \nabla_{x,y}\cdot\left(y^{2s-1}\widetilde{\sigma}\nabla_{x,y}u_1\right) = 0 & \text{in } \mathbb{R}^{n+2}_+, \\ \lim_{y\to 0} y^{2s-1}\partial_y u_1 = h & \text{on } \mathbb{R}^{n+1}\times\{0\}. \end{cases} \tag{3.4}$$

Then the function $u_2(t,x,y) := -y^{2s-1}\partial_y u_1(t,x,y)$ is a classical solution of

$$\begin{cases} y^{1-2s}\partial_t u_2 - \nabla_{x,y}\cdot\left(y^{1-2s}\widetilde{\sigma}\nabla_{x,y}u_2\right) = 0 & \text{in } \mathbb{R}^{n+2}_+, \\ u_2 = -h & \text{on } \mathbb{R}^{n+1}\times\{0\}. \end{cases} \tag{3.5}$$

Proof. For $y>0$, one has

$$\begin{aligned} & -y^{1-2s}\partial_t u_2 + \nabla_{x,y}\cdot\left(y^{1-2s}\widetilde{\sigma}\nabla_{x,y}u_2\right) \\ &= y^{1-2s}\partial_t\left(y^{2s-1}\partial_y u_1\right) - \nabla_{x,y}\cdot\left(y^{1-2s}\widetilde{\sigma}\nabla_{x,y}\left(y^{2s-1}\partial_y u_1\right)\right) \\ &= \partial_y\partial_t u_1 - \partial_y\nabla_x\cdot(\sigma\nabla_x u_1) - \partial_y\left\{y^{1-2s}\left[\partial_y\left(y^{2s-1}\partial_y u_1\right)\right]\right\} \\ &= \partial_y\left\{\partial_t u_1 - y^{1-2s}\nabla_{x,y}\cdot\left(y^{2s-1}\widetilde{\sigma}\nabla_{x,y}u_1\right)\right\} \\ &= 0, \end{aligned}$$

where we used (3.4) in the last identity. This shows u_2 solves (3.5) as desired. □

From the relation $u_2(t,x,y) = -y^{2s-1}\partial_y u_1(t,x,y)$, we have that

$$-\partial_y u_1(t,x,y) = y^{1-2s}u_2(t,x,y).$$

By integrating the above equation on (y,∞), we obtain that

$$u_1(t,x,y) = \int_y^\infty \partial_\mu u_1(t,x,\mu)\,d\mu = \int_y^\infty \mu^{1-2s}u_2(t,x,\mu)\,d\mu,$$

where we assume that $\lim_{\mu\to\infty} u_1(t,x,\mu) = 0$. This observation allows us to construct the operator $(\partial_t - \nabla\cdot\sigma\nabla)^{1-s}$ from the operator $(\partial_t - \nabla\cdot\sigma\nabla)^s$.

3.2. **Key functions.** In this section, let us introduce important functions, which play essential roles for our approach. Inspired by Proposition 3.2, let us consider the case $u_2 = u$, where $u \in \mathbf{H}^s(\mathbb{R}^{n+1})$ is the solution to (1.1), and like u_1 to set another function

$$w(t,x,y) := \int_y^\infty \mu^{1-2s}\widetilde{u}(t,x,\mu)\,d\mu, \text{ for } y>0, \tag{3.6}$$

where $\widetilde{u} \in \mathcal{L}^{1,2}(\Sigma; y^{1-2s}dtdxdy)$ is a solution of (1.6) and $u \in \widetilde{\mathbf{H}}^s(W_T)$ is the solution of (1.1). Here $W \subset \mathbb{R}^n$ is a bounded open Lipschitz domain. Meanwhile, by (3.6), it can be seen that the function w is finite for every fixed $y>0$. Furthermore, the function w could be as regular as the leading coefficient $\sigma(x)$ permits. In addition, we will analyze more detailed regularity estimates in Section 4, and one can summarize the limit as $y \to 0$ that

$$v(t,x) = w(t,x,0), \text{ for } (t,x) \in \mathbb{R}^{n+1} \tag{3.7}$$

is well-defined. Here the function v will fulfill

$$v \in L^2(0,T;H^1(\Omega)) \text{ and } \partial_t v \in L^2(0,T;H^{-1}(\Omega)), \tag{3.8}$$

where v is given by (1.9), and rigorous derivations for the property (3.8) will be given in Section 4.

In fact, by using Lemma 4.2 in the next section, we will see that function w given by (3.6) has sufficient decay with respect to both x and y directions, then it will reverse the relation described in Proposition 3.2. More concretely, via direct calculation, we can see that w satisfies the equation

$$\begin{aligned}
&\nabla_{x,y}\cdot\left(y^{2s-1}\widetilde{\sigma}\nabla_{x,y}w(t,x,y)\right)\\
&=y^{2s-1}\int_y^\infty \mu^{1-2s}\nabla\cdot(\sigma\nabla\widetilde{u}(t,x,\mu))\,d\mu-\partial_y\widetilde{u}(t,x,y)\\
&=-y^{2s-1}\int_y^\infty \partial_\mu\left(\mu^{1-2s}\partial_\mu\widetilde{u}(t,x,\mu)\right)dz+y^{2s-1}\int_y^\infty \mu^{1-2s}\partial_t\widetilde{u}(t,x,\mu)\,d\mu\\
&\quad-\partial_y\widetilde{u}(t,x,y)\\
&=y^{2s-1}\partial_t\left(\int_y^\infty \mu^{1-2s}\widetilde{u}(t,x,\mu)\,d\mu\right)\\
&=y^{2s-1}\partial_t w(t,x,y),
\end{aligned} \tag{3.9}$$

where we used the equation (1.6) of $\widetilde{u}$ so that $\widetilde{u}$ has appropriate decay at infinity and the definition (3.6). Combined with the computation (3.7) and (3.9), one can see that the function w given by (3.6) is a solution to

$$\begin{cases} y^{2s-1}\partial_t w-\nabla_{x,y}\cdot\left(y^{2s-1}\widetilde{\sigma}\nabla_{x,y}w\right)=0 & \text{in } \mathbb{R}^{n+2}_+,\\ w(t,x,0)=v(t,x) & \text{in } \mathbb{R}^{n+1}. \end{cases} \tag{3.10}$$

By using the above equation, combined with (3.1), the operator $(\partial_t-\nabla\cdot\sigma\nabla)^{1-s}$ is characterized as the Neumann data of the problem (3.10). In other words, we have

$$\begin{aligned}
(\partial_t-\nabla\cdot\sigma\nabla)^{1-s}v(t,x)&=d_{1-s}\lim_{y\to 0}y^{2s-1}\partial_y w(t,x,y)\\
&=-d_{1-s}\widetilde{u}(t,x,0)\\
&=-d_{1-s}u(t,x)
\end{aligned} \tag{3.11}$$

holds formally, where d_{1-s} is a constant given by (3.3) when the parameter s is replaced by $1-s$. Moreover, via the semigroup property for the nonlocal parabolic operator, one can apply the nonlocal operator $(\partial_t-\nabla\cdot\sigma\nabla)^s$ onto (3.11), then a formal computation yields

$$(\partial_t-\nabla\cdot\sigma\nabla)\,v=-d_{1-s}\,(\partial_t-\nabla\cdot\sigma\nabla)^s\,u \text{ in } \mathbb{R}^{n+1}.$$

To summarize, when u is the solution to (1.1), we have $\widetilde{u}\in\mathcal{L}^{1,2}(\mathbb{R}^{n+2}_+;y^{1-2s}dtdxdy)$ and $(\partial_t-\nabla\cdot\sigma\nabla)\,v=0$ in Ω_T, where $\widetilde{u}$ is defined in (1.6).

4. Regularity estimates of solutions

In this section, we demonstrate that the function $v(t,x)$ defined by (1.9), which has suitable regularity properties. Moreover, with suitable decay estimates at hand, we can make our arguments hold rigorously.

Proposition 4.1. *Let $\Omega,W\subset\mathbb{R}^n$ be bounded open sets with Lipschitz boundaries, for $n\geq 2$ and $T>0$. Let $u\in\mathbf{H}^s(\mathbb{R}^{n+1})$ with compact support in $\overline{(\Omega\cup W)_T}$ and $\widetilde{u}\in\mathcal{L}^{1,2}(\mathbb{R}^{n+2}_+;y^{1-2s}dtdxdy)$ be the corresponding solution of the extension problem (1.6). Let $w:\mathbb{R}^{n+2}_+\to\mathbb{R}$ be the function given by (3.6). Then for $y>0$, the function $w\in L^m(-T,T;L^\infty(\mathbb{R}^n))$ for any $m\in[1,2]$, and the limit $\lim_{y\to 0}w(t,x,y)=w(t,x,0)$ exists with $w(t,x,0)\in L^2(-T,T;H^1(\mathbb{R}^n))$. Moreover, there holds that*

$$\|v_f\|_{L^2(-T,T;H^1(\mathbb{R}^n))}\lesssim\|u_f\|_{\mathbf{H}^s(\mathbb{R}^{n+1})}. \tag{4.1}$$

Before showing Proposition 4.1, we first prove several decay estimates for the solution $\widetilde{u}$ to (1.6) with respect to the y-direction.

Lemma 4.2. *Let $\Omega, W \subset \mathbb{R}^n$ be bounded open sets with Lipschitz boundaries such that $\overline{\Omega} \cap \overline{W} = \emptyset$, for $n \in \mathbb{N}$ and $T > 0$. Given $s \in (0,1)$, let $u \in \mathbf{H}^s(\mathbb{R}^{n+1})$ with $\mathrm{supp}(u) \subset \overline{(\Omega \cup W)_T}$, and $\widetilde{u}$ be the corresponding solution of the extension problem (1.6). Then for any $m, q \in [1,2]$, the function $\widetilde{u}(t,x,y)$ satisfies the following decay estimates:*

(a)

$$\begin{aligned} \|\widetilde{u}(\cdot,x,y)\|_{L^m(\mathbb{R})} &\lesssim y^{-n}\|u\|_{L^m(\mathbb{R};L^1(\mathbb{R}^n))}, \\ \|\nabla \widetilde{u}(\cdot,x,y)\|_{L^m(\mathbb{R})} &\lesssim y^{-n-1}\|u\|_{L^m(\mathbb{R};L^1(\mathbb{R}^n))}, \end{aligned} \tag{4.2}$$

for any $(x,y) \in \mathbb{R}^{n+1}_+ := \mathbb{R}^n \times (0,\infty)$.

(b) For $1 \le p,q,r \le \infty$ with $1 + \frac{1}{r} = \frac{1}{p} + \frac{1}{q}$, $\widetilde{u}$ satisfies

$$\begin{aligned} \|\widetilde{u}(\cdot,\cdot,y)\|_{L^m(\mathbb{R};L^r(\mathbb{R}^n))} &\lesssim y^{\frac{n}{p}-n}\|u\|_{L^m(\mathbb{R};L^q(\mathbb{R}^n))}, \\ \|\nabla_{x,y} \widetilde{u}(\cdot,\cdot,y)\|_{L^m(\mathbb{R};L^r(\mathbb{R}^n))} &\lesssim y^{\frac{n}{p}-n-1}\|u\|_{L^m(\mathbb{R};L^q(\mathbb{R}^n))}. \end{aligned} \tag{4.3}$$

Before proving the lemma, let us offer an auxiliary result, which will be used in the proof of Lemma 4.2.

Lemma 4.3. *For any $b, A > 0$, let $f_b(A) := \int_0^\infty \tau^{-(b+1)} e^{-\frac{A}{4\tau}}\,d\tau$, then there holds*

$$\int_0^\infty \tau^{-(b+1)} e^{-\frac{A}{4\tau}}\,d\tau = f_b(1)A^{-b}. \tag{4.4}$$

Proof. By the integration by parts, one has

$$\begin{aligned} f_b(A) &= -\int_0^\infty \frac{d}{d\tau}\left(\frac{1}{b}\tau^{-b}\right) e^{-\frac{A}{4\tau}}\,d\tau \\ &= \frac{1}{b}\int_0^\infty \tau^{-b}\frac{d}{d\tau}\left(e^{-\frac{A}{4\tau}}\right) d\tau \\ &= \frac{A}{4b}\int_0^\infty \tau^{-(b+2)} e^{-\frac{A}{4\tau}}\,d\tau. \end{aligned}$$

Moreover, it is easy to see that $f_b'(A) = -\frac{1}{4}\int_0^\infty \tau^{-(b+2)} e^{-\frac{A}{4\tau}}\,d\tau$. From the preceding computations, one has

$$f_b(A) = -\frac{A}{b} f_b'(A),$$

which implies

$$\frac{f_b'(A)}{f_b(A)} = -\frac{b}{A}.$$

Solving the above ordinary differential equation, one can see that

$$f_b(A) = f_b(1)A^{-b},$$

which proves the assertion. □

With the aid of the preceding lemma, we can show Lemma 4.2.

Proof of Lemma 4.2. For (a), combined with formulas (3.1) of $\widetilde{u}$ and (3.2) in Section 3, one has

$$\widetilde{u}(t,x,y) = c_s y^{2s} \int_{\mathbb{R}^n}\int_0^\infty e^{-\frac{y^2}{4\tau}} p(x,z,\tau) u(t-\tau,z)\,\frac{d\tau}{\tau^{1+s}}\,dz, \tag{4.5}$$

where $p(x,z,\tau)$ denotes the heat kernel satisfying (2.1) and c_s is a constant depending only on s of the form

$$c_s := \frac{1}{2^{2s}\Gamma(s)}. \tag{4.6}$$

Hence, with the heat kernel estimate (2.2) at hand, we can have the pointwise estimate

$$\begin{aligned} \left|\nabla_x^\ell \widetilde{u}(t,x,y)\right| &\lesssim y^{2s}\int_{\mathbb{R}^n}\int_0^\infty \left|\nabla_x^\ell p(x,z,\tau)\right| e^{-\frac{y^2}{4\tau}} \left|u(t-\tau,z)\right| \frac{d\tau}{\tau^{1+s}}dz \\ &\lesssim y^{2s}\int_{\mathbb{R}^n}\int_0^\infty \tau^{-(\frac{n+\ell}{2}+1+s)} e^{-(c\frac{|x-z|^2}{\tau}+\frac{y^2}{4\tau})} \left|u(t-\tau,z)\right| d\tau dz, \end{aligned} \tag{4.7}$$

for $\ell = 0,1$. Note that the right hand side of (4.7) can be viewed as a convolution of space and time variables (one may extend the integrand to be zero in the region $\tau \in (-\infty,0)$).

Recall that the Young's convolution inequality states

$$\|f*g\|_{L^r} = \left\|\int f(\zeta)g(\xi-\zeta)\,d\zeta\right\|_{L^r_\xi} \le \|f\|_{L^p}\|g\|_{L^q}, \text{ with } 1+\frac{1}{r}=\frac{1}{p}+\frac{1}{q}, \tag{4.8}$$

where ζ,ξ could be either space or time variables in the upcoming applications. To make our notation more clear, here we use L^r_ξ to denote the L^r Lebesgue norm with respect to the ξ-variable. Let us take L^m norm in the time variable to (4.7), applying (4.8) for the exponents $p=1$, $q=m\in[1,2]$ with respect to the time-variable, then one can have

$$\begin{aligned} &\left\|\nabla_x^\ell \widetilde{u}(\cdot,x,y)\right\|_{L^m(\mathbb{R})} \\ &\lesssim y^{2s}\int_{\mathbb{R}^n}\left\{\left(\int_0^\infty \tau^{-(\frac{n+\ell}{2}+1+s)} e^{-(c\frac{|x-z|^2}{\tau}+\frac{y^2}{4\tau})}\,d\tau\right)\|u(\cdot,z)\|_{L^m(\mathbb{R})}\right\}dz. \end{aligned} \tag{4.9}$$

Observing that

$$\int_0^\infty \frac{e^{-(c\frac{|x-z|^2}{\tau}+\frac{y^2}{4\tau})}}{\tau^{\frac{n+\ell}{2}+s+1}}\,d\tau \approx \left(|x-z|^2+y^2\right)^{-\frac{n+\ell}{2}-s} \tag{4.10}$$

as shown in (4.4) from Lemma 4.3. Inserting (4.9) and (4.10) into (4.7), we obtain

$$\begin{aligned} &\left\|\nabla_x^\ell \widetilde{u}(\cdot,x,y)\right\|_{L^m(\mathbb{R})} \\ &\lesssim y^{2s}\int_{\mathbb{R}^n}\left\{\left(|x-z|^2+y^2\right)^{-\frac{n+\ell}{2}-s}\|u(\cdot,z)\|_{L^m(\mathbb{R})}\right\}dz, \end{aligned} \tag{4.11}$$

for $\ell=0,1$.

Meanwhile, let us point out that the right hand side of (4.11) can be regarded as the convolution with respect to the space variable. Next, applying (4.8) again to the x-variable, for the exponents $r=\infty$, $p=\infty$ and $q=1$, we have

$$\left\|\nabla_x^\ell \widetilde{u}(\cdot,\cdot,y)\right\|_{L^1(\mathbb{R};L^\infty(\mathbb{R}^n))} \lesssim y^{-n-\ell}\|u\|_{L^1(\mathbb{R}^{n+1})},$$

where we used the fact that

$$\sup_{z\in\mathbb{R}^n}\left(|z|^2+y^2\right)^{-\frac{n+\ell}{2}-s} \le y^{-n-\ell-2s},$$

for $\ell=0,1$. Similar to the computation (4.11), we obtain

$$\left\|\nabla_x^\ell \widetilde{u}(\cdot,\cdot,y)\right\|_{L^m(\mathbb{R};L^r(\mathbb{R}^n))} \lesssim y^{n/p-n-\ell}\|u\|_{L^m(\mathbb{R};L^q(\mathbb{R}^n))},$$

for $m \in [1,2]$, where we used the fact that

$$\left\| \frac{y^{2s}}{(|z|^2+y^2)^{\frac{n+\ell}{2}+s}} \right\|_{L^p(\mathbb{R}^n)} \approx \left(\int_0^\infty \frac{y^{2sp} r^{n-1}}{(r^2+y^2)^{p(\frac{n+\ell}{2}+s)}}\, dr \right)^{1/p} \lesssim y^{n/p-n-\ell}, \tag{4.12}$$

for $\ell = 0,1$. This shows (a).

For (b), we can calculate

$$\begin{aligned} |\partial_y \widetilde{u}(t,x,y)| \lesssim\, & y^{2s-1} \int_{\mathbb{R}^n} \int_0^\infty p(x,z,\tau) e^{-\frac{y^2}{4\tau}} |u(t-\tau,z)| \frac{d\tau}{\tau^{1+s}} dz \\ & + y^{2s+1} \int_{\mathbb{R}^n} \int_0^\infty p(x,z,\tau) e^{-\frac{y^2}{4\tau}} |u(t-\tau,z)| \frac{d\tau}{\tau^{2+s}} dz. \end{aligned} \tag{4.13}$$

Similar as previous cases, we can make use of the Young's convolution inequality again for both terms in the right hand side of (4.13), then direct computations yield that

$$\begin{aligned} & \left\| y^{2s-1} \int_{\mathbb{R}^n} \int_0^\infty p(x,z,\tau) e^{-\frac{y^2}{4\tau}} |u(t-\tau,z)| \frac{d\tau}{\tau^{1+s}} dz \right\|_{L^m(\mathbb{R};L^r(\mathbb{R}^n))} \\ \lesssim & \left\| y^{2s-1} \int_{\mathbb{R}^n} \int_0^\infty \tau^{-(\frac{n}{2}+1+s)} e^{-(c\frac{|x-z|^2}{\tau}+\frac{y^2}{4\tau})} |u(t-\tau,z)|\, d\tau dz \right\|_{L^m(\mathbb{R};L^r(\mathbb{R}^n))} \\ \lesssim & \left\| y^{2s-1} \int_{\mathbb{R}^n} \left(|x-z|^2+y^2 \right)^{-\frac{n}{2}-s} \|u(\cdot,z)\|_{L^m(\mathbb{R})}\, dz \right\|_{L^r(\mathbb{R}^n)} \\ \lesssim & \left\| \frac{y^{2s-1}}{(|z|^2+y^2)^{\frac{n}{2}+s}} \right\|_{L^p(\mathbb{R}^n)} \|u\|_{L^m(\mathbb{R};L^q(\mathbb{R}^n))} \\ \lesssim & \, y^{n/p-n-1} \|u\|_{L^m(\mathbb{R};L^q(\mathbb{R}^n))}, \end{aligned}$$

and similarly,

$$\begin{aligned} & \left\| y^{2s+1} \int_{\mathbb{R}^n} \int_0^\infty p(x,z,\tau) e^{-\frac{y^2}{4\tau}} |u(t-\tau,z)| \frac{d\tau}{\tau^{2+s}} dz \right\|_{L^m(\mathbb{R};L^r(\mathbb{R}^n))} \\ \lesssim & \left\| y^{2s+1} \int_{\mathbb{R}^n} \left(\|x-z\|^2+y^2 \right)^{-\frac{n}{2}-1-s} \|u(\cdot,z)\|_{L^m(\mathbb{R})} \right\|_{L^m(\mathbb{R};L^r(\mathbb{R}^n))} \\ \lesssim & \, y^{n/p-n-1} \|u\|_{L^q(\mathbb{R}^n)}, \end{aligned}$$

where we used the fact (4.12). This proves the assertion. □

We are ready to prove Proposition 4.1.

Proof of Proposition 4.1. Since u is suppoted in $\overline{(\Omega \cup W)_T}$, we assume that there exists a ball $B_R \subset \mathbb{R}^n$ centered at the origin and radius $R>0$ such that $\mathrm{supp}(u) \subset [-T,T] \times B_R$. Let us argue in three steps:

Step 1. Initial regularity.

By (4.7), (4.10) and compact condition of $\overline{\mathrm{supp}(u)}$, we have that

$$\begin{aligned}
\left|\nabla_x^\ell \widetilde{u}(t,x,y)\right| &\lesssim y^{2s}\int_{\mathbb{R}^n}\int_0^\infty \left|\nabla_x^\ell p(x,z,\tau)\right| e^{-\frac{y^2}{4\tau}}\left|u(t-\tau,z)\right| \frac{d\tau}{\tau^{1+s}}dz \\
&\lesssim \underbrace{y^{2s}\int_{\mathbb{R}^n}\int_0^\infty \tau^{-(\frac{n+\ell}{2}+1+s)}e^{-(c\frac{|x-z|^2}{\tau}+\frac{y^2}{4\tau})}\left|u(t-\tau,z)\right| d\tau dz}_{\text{By (2.2)}} \\
&\lesssim y^{2s}\int_{\mathbb{R}^n}\left(\int_0^\infty \tau^{-2(\frac{n+\ell}{2}+1+s)}e^{-2(c\frac{|x-z|^2}{\tau}+\frac{y^2}{4\tau})}\,d\tau\right)^{1/2} \\
&\qquad\cdot\left(\int_0^\infty \left|u(t-\tau,z)\right|^2 d\tau\right)^{1/2} dz \\
&\lesssim \underbrace{y^{2s}\int_{\mathbb{R}^n}\left(|x-z|^2+y^2\right)^{-\frac{n+\ell+1+2s}{2}}\|u(\cdot,z)\|_{L^2(\mathbb{R})}\,dz}_{\text{By (4.4) as } b=n+1+\ell+2s},
\end{aligned}$$

for $\ell=0,1$, where we used the Hölder's inequality and (4.4). In further, we can obtain

$$\left|\nabla_x^\ell \widetilde{u}(t,x,y)\right| \lesssim y^{-n-\ell-1}\|u\|_{L^2(-T,T;L^1(\mathbb{R}^n))},$$

where we used the fact $y^{2s}\left(|x-z|^2+y^2\right)^{-\frac{n+\ell+1+2s}{2}} \leq y^{-n-\ell-1}$ for $\ell=0,1$. Next, we want to check that $w(t,x,y)$ is well-defined for $y>0$. To this end, we can estimate the function w for $y>0$

$$\begin{aligned}
\left|\nabla_x^\ell w(t,x,y)\right| &\leq \int_y^\infty \mu^{1-2s}\left|\nabla_x^\ell \widetilde{u}(t,x,\mu)\right| d\mu \\
&\lesssim \int_y^\infty \mu^{1-2s-n-\ell-1}\|u\|_{L^2(-T,T;L^1(\mathbb{R}^n))}\,d\mu \\
&\lesssim y^{1-2s-n-\ell}\|u\|_{L^2(-T,T;L^1(\mathbb{R}^n))},
\end{aligned} \tag{4.14}$$

for $\ell=0,1$. Since $\overline{\mathrm{supp}(u)}$ is compact and $u\in\mathbf{H}^s(\mathbb{R}^{n+1})$, hence, the right hand side of the estimate (4.14) is finite, for a.e., $(t,x,y)\in\mathbb{R}^{n+2}_+$.

Step 2. L^2-estimate for $v(t,x)=w(t,x,0)$.

First, the Minkowski's integral inequality implies that

$$\begin{aligned}
&\left(\int_{-T}^T\left(\int_y^\infty \mu^{1-2s}\left|\widetilde{u}(t,x,\mu)\right| d\mu\right)^m dt\right)^{1/m} \\
&\leq \int_y^\infty \mu^{1-2s}\left(\int_{-T}^T \left|\widetilde{u}(t,x,\mu)\right|^m dt\right)^{1/m} d\mu \\
&= \int_y^\infty \mu^{1-2s}\left\|\widetilde{u}(\cdot,\cdot,\mu)\right\|_{L^m(-T,T)}\,d\mu,
\end{aligned} \tag{4.15}$$

for $m\in[1,2]$. By using (4.3), we have that

$$\begin{aligned}
&\left\|\nabla_x^\ell w(\cdot,\cdot,y)\right\|_{L^m(-T,T;L^\infty(\mathbb{R}^n))} \\
&\leq \int_y^\infty \mu^{1-2s}\left\|\nabla_x^\ell \widetilde{u}(t,x,\mu)\right\|_{L^m(-T,T;L^\infty(\mathbb{R}^n))}\,d\mu \\
&\lesssim \|u\|_{L^m(-T,T;L^1(\mathbb{R}^n))}\int_y^\infty \mu^{1-2s-n-\ell}\,d\mu \\
&\lesssim y^{2-2s-n-\ell}\|u\|_{L^m(-T,T;L^1(\mathbb{R}^n))},
\end{aligned} \tag{4.16}$$

for $\ell = 0, 1$ and $m \in [1,2]$. Here we use that $u \in \mathbf{H}^s(\mathbb{R}^{n+1})$ is supported in the compact set $\overline{(\Omega \cup W)_T}$ so that $\mathbf{H}^s(\mathbb{R}^{n+1}) \subset L^m(-T,T;L^1(\mathbb{R}^n))$ for $m \in [1,2]$, and $\|u\|_{L^m(\mathbb{R};L^1(\mathbb{R}^n))}$ is finite for $m \in [1,2]$.

Next, we want to prove that $v(t,x) = w(t,x,0) \in L^2(\mathbb{R}^n_T)$. To do so, our goal is to upgrade the right hand side of (4.16) to be independent of the y-variable. Let us write

$$w(t,x,y) = \int_0^\infty \mu^{1-2s}\widetilde{u}(t,x,\mu)\chi_{(y,\infty)}\,d\mu \leq \int_0^\infty \mu^{1-2s}|\widetilde{u}(t,x,\mu)|\,d\mu. \tag{4.17}$$

Define $E(t,x) := \int_0^\infty \mu^{1-2s}|\widetilde{u}(t,x,\mu)|\,d\mu$, then it is not hard to see that $|v(t,x)| \leq E(t,x)$ for all $(t,x) \in \mathbb{R}^n_T$. Next, we want to claim $E(t,x) < \infty$. In order to get $E(t,x) < \infty$ for $(t,x) \in \mathbb{R}^n_T$ almost everywhere (a.e.), we will prove that $\int_{\mathbb{R}^n_T} E(t,x)^2\,dtdx < \infty$. To this end, similar to (4.16), (4.15) and the Fubini's theorem give rises to

$$\begin{aligned}
\left\|\nabla_x^\ell w(\cdot,x,y)\right\|_{L^2(-T,T)} &\lesssim \underbrace{\int_y^\infty \int_{\mathbb{R}^n} \frac{\mu\|u(\cdot,z)\|_{L^2(\mathbb{R})}}{\left(|x-z|^2+\mu^2\right)^{\frac{n+\ell}{2}+s}}\,dzd\mu}_{\text{By (4.11)}} \\
&\leq \int_{\mathbb{R}^n} \|u(\cdot,x+z)\|_{L^2(\mathbb{R})}\left(\int_0^\infty \frac{\mu}{\left(|z|^2+\mu^2\right)^{\frac{n+\ell}{2}+s}}\,d\mu\right)dz \\
&\approx \int_{\mathbb{R}^n} |z|^{2(1-(\frac{n+\ell}{2}+s))}\|u(\cdot,x+z)\|_{L^2(\mathbb{R})}\,dz \\
&= \int_{B_{2R}} \frac{\|u(\cdot,x+z)\|_{L^2(\mathbb{R})}}{|z|^{n+\ell+2s-2}}\,dz,
\end{aligned} \tag{4.18}$$

where B_{2R} denotes the ball in $\mathbb{R}^n$ of radius $2R > 0$ and center at the origin such that $B_R \supset \mathrm{supp}(u)$. Note that the right hand side of (4.18) is independent of $y > 0$.

From (4.18), we will have that

$$\begin{aligned}
&\int_{\mathbb{R}^n_T} E(t,x)^2\,dtdx \\
\lesssim &\int_{B_{2R}}\int_{-T}^T E(t,x)^2\,dtdx + \int_{\mathbb{R}^n\setminus B_{2R}}\int_{-T}^T E(t,x)^2 dtdx \\
\lesssim &\int_{B_{2R}} \left|\int_{\mathbb{R}^n} \frac{\|u(\cdot,z)\|_{L^2(\mathbb{R})}}{|x-z|^{n+2s-2}}\,dz\right|^2 dx + \int_{\mathbb{R}^n\setminus B_{2R}} \left|\int_{\mathbb{R}^n} \frac{\|u(\cdot,z)\|_{L^2(\mathbb{R})}}{|x-z|^{n+2s-2}}\,dz\right|^2 dx \\
\lesssim &\int_{\mathbb{R}^n} \left|\int_{\mathbb{R}^n} \frac{\|u(\cdot,z)\|_{L^2(\mathbb{R})}\chi_{B_{2R}}(x-z)}{|x-z|^{n+2s-2}}\,dz\right|^2 dx + \|u\|^2_{L^2(\mathbb{R}^{n+1})} \\
\approx &\left\|\|u(t,\cdot)\|_{L^2_t(\mathbb{R})} * \left(\chi_{B_R}(\cdot)|\cdot|^{-(n+2s-2)}\right)\right\|^2_{L^2(\mathbb{R}^n)} + \|u\|^2_{L^2(\mathbb{R}^{n+1})} \\
\lesssim &\underbrace{\left\|\chi_{B_{2R}}(\cdot)\,|\cdot|^{-(n+2s-2)}\right\|_{L^1(\mathbb{R}^n)}\|u\|^2_{L^2(\mathbb{R}^{n+1})}}_{\text{By (4.8) for } r=q=2,\ p=1} + \|u\|^2_{L^2(\mathbb{R}^{n+1})} \\
\lesssim &\|u\|^2_{L^2(\mathbb{R}^{n+1})},
\end{aligned} \tag{4.19}$$

where $R > 0$ is a positive constant such that $B_R \supset \Omega'$. Here we used the integrability of the function $\chi_{B_{2R}}(\cdot)\,|\cdot|^{-(n+2s-2)}$ in the last inequality. Let us point out that the right hand side of of the estimate (4.19) is independent of $y \in (0,\infty)$, then we can transfer the estimate (4.17) of $w(t,x,y)$ to $v(t,x)$ in $L^2(\mathbb{R}^{n+1})$ as $y \to 0$ by the Lebesgue dominated convergence theorem.

Step 3. Gradient L^2-estimate.

We want to show that $v(t,x)\in L^2(-T,T;H^1(\Omega'))$. To this end, let us consider the case for $s\in(0,\frac{1}{2})$, and $s\in[\frac{1}{2},1)$.

Step 3a. For $s\in(0,\frac{1}{2})$. By using (4.18) as $\ell=1$, then similar computations as in (4.19) yield that

$$\int_{\Omega'}\int_{-T}^{T}|\nabla v(t,x)|^2\,dtdx \lesssim \left\|\chi_{B_R}(\cdot)\,|\cdot|^{-(n+2s-1)}\right\|_{L^1(\mathbb{R}^n)}\|u\|^2_{L^2(\mathbb{R}^{n+1})}<\infty,$$

since $|\cdot|^{-(n+2s-1)}$ is locally integrable for $s\in(0,\frac{1}{2})$[1]. However, for the case $s\in[\frac{1}{2},1)$, the arguments in the previous step is not enough (since $|\cdot|^{-(n+2s-1)}$ is not locally integrable for $s\in[\frac{1}{2},1)$), so we need more detailed analysis to find desired estimates.

Step 3b. For $s\in[\frac{1}{2},1)$. We want to claim Proposition 4.1 hods true in this case. To show this, let $0<R_1<R$, with $\mathrm{supp}(u)\subset(B_{R_1})_T$ and consider a time-independent function $g=g(x)\in C_c^\infty(B_R)$ such that $0\le g\le 1$ and $g=1$ in B_{R_1}. Let $\widetilde{g}=\widetilde{g}(x,y)$ be the extension of g, i.e., $\widetilde{g}$ is the solution to

$$\begin{cases}\nabla_{x,y}\cdot\left(y^{1-2s}\widetilde{\sigma}\nabla_{x,y}\widetilde{g}\right)=0 & \text{for } (x,y)\in\mathbb{R}^{n+1}_+,\\ \widetilde{g}(x,0)=g(x) & \text{for } x\in\mathbb{R}^n,\end{cases}$$

By [ST10], it is known that $\widetilde{g}$ can be expressed by

$$\widetilde{g}(x,y)=c_s y^{2s}\int_{\mathbb{R}^n}\int_0^\infty e^{-\frac{y^2}{4\tau}}p(x,z,\tau)g(z)\,\frac{d\tau}{\tau^{1+s}}dz,$$

where $p(x,z,\tau)$ denotes the heat kernel satisfying (2.1) and c_s is the constant given by (4.6).

Next, as shown in the proof of Lemma 4.2 and the estimate (4.9), with (4.5) at hand, we can get

$$\begin{aligned}&\left|\left(\nabla w(t,x,y)-\nabla\left(\int_y^\infty \mu^{1-2s}\widetilde{g}(x,\mu)\,d\mu\right)u(t,x)\right)\right|\\ =\ & c_s\left|\int_y^\infty \mu\int_{\mathbb{R}^n}\int_0^\infty \frac{e^{-\frac{\mu^2}{4\tau}}\nabla_x p(x,z,\tau)}{\tau^{1+s}}\left[u(t-\tau,z)-g(z)u(t,x)\right]d\tau dz d\mu\right|.\end{aligned}$$

[1]One can see that $\int_{B_R}|x|^{-(n+2s-1)}\,dx$ is bounded for any $s\in(0,\frac{1}{2})$ and $R>0$

To proceed, for $y>0$, by Fubini's theorem, we find

$$
\begin{aligned}
&\int_{-T}^{T}\left|\nabla w(\cdot,x,y)-\nabla\left(\int_y^\infty \mu^{1-2s}\widetilde{g}(x,\mu)\,d\mu\right)u(\cdot,x)\right|^2 dt \\
&\lesssim \int_{-T}^{T}\left|\int_y^\infty \mu\int_{\mathbb{R}^n}\int_0^\infty \frac{e^{-\frac{\mu^2}{4\tau}}\nabla_x p(x,z,\tau)}{\tau^{1+s}}\left|u(t-\tau,z)-g(z)u(t,x)\right|d\tau dz d\mu\right|^2 dt \\
&\approx \int_{-T}^{T}\left|\int_{\mathbb{R}^n}\int_0^\infty \left(\int_y^\infty \frac{\mu}{2\tau}e^{-\frac{\mu^2}{4\tau}}\,d\mu\right)\frac{\nabla_x p(x,z,\tau)}{\tau^s}\left|u(t-\tau,z)-g(z)u(t,x)\right|d\tau dz\right|^2 dt \\
&\leq \underbrace{\int_{-T}^{T}\left|\int_{\mathbb{R}^n}\int_0^\infty \frac{|\nabla_x p(x,z,\tau)|}{\tau^s}\left|u(t-\tau,z)-g(z)u(t,x)\right|d\tau dz\right|^2 dt}_{\text{We use } \int_y^\infty \frac{\mu}{2\tau}e^{-\frac{\mu^2}{4\tau}}\,d\mu=e^{-\frac{y^2}{4\tau}}\leq 1, \text{ for any } \tau,y>0} \\
&\lesssim \underbrace{\int_{-T}^{T}\left|\int_{B_R}\int_0^\infty \frac{|\nabla_x p(x,z,\tau)|}{\tau^s}\left|u(t-\tau,z)-u(t,x)\right|d\tau dz\right|^2 dt\,.}_{\mathrm{supp}(u)\subset (B_{R_1})_T \text{ and } g(z)=1 \text{ for } z\in B_{R_1}}
\end{aligned} \tag{4.20}
$$

By using the triangle inequality $|u(t-\tau,z)-u(t,x)| \leq |u(t-\tau,z)-u(t,z)| + |u(t,z)-u(t,x)|$, we have

$$
\int_{-T}^{T}\left|\nabla w(\cdot,x,y)-\nabla\left(\int_y^\infty \mu^{1-2s}\widetilde{g}(x,\mu)\,d\mu\right)u(\cdot,x)\right|^2 dt \lesssim J_1+J_2,
$$

where

$$
\begin{aligned}
J_1 &:= \int_{-T}^{T}\left|\int_{B_R}\int_0^\infty \frac{|\nabla_x p(x,z,\tau)|}{\tau^s}\left|u(t-\tau,z)-u(t,z)\right|d\tau dz\right|^2 dt, \\
J_2 &:= \int_{-T}^{T}\left|\int_{B_R}\int_0^\infty \frac{|\nabla_x p(x,z,\tau)|}{\tau^s}\left|u(t,z)-u(t,x)\right|d\tau dz\right|^2 dt.
\end{aligned}
$$

Our remaining task is to estimate J_1 and J_2.

Step 3b-1. Estimate for J_1: We consider the quantity $\left(\int_{\Omega'} J_1\,dx\right)^{1/2}$. By Minkowski's integral inequality

$$
\begin{aligned}
&\left(\int_{\Omega'}\int_{-T}^{T}\left|\int_{B_R}\int_0^\infty \frac{|\nabla_x p(x,z,\tau)|}{\tau^s}\left|u(t-\tau,z)-u(t,z)\right|d\tau dz\right|^2 dt dx\right)^{1/2} \\
&\lesssim \int_{B_R}\left(\int_{-T}^{T}\int_{\Omega'}\left|\int_0^\infty \frac{|\nabla_x p(x,z,\tau)|}{\tau^s}\left|u(t-\tau,z)-u(t,z)\right|d\tau\right|^2 dx dt\right)^{1/2} dz.
\end{aligned}
$$

For the integrand in the preceding inequalities, direct computations yield that

$$\begin{aligned}
&\int_{-T}^{T}\int_{\Omega'}\left|\int_0^\infty \frac{|\nabla_x p(x,z,\tau)|}{\tau^s}\,|u(t-\tau,z)-u(t,z)|\,d\tau\right|^2 dxdt \\
=& \int_{-T}^{T}\int_{\Omega'}\left(\int_0^\infty \frac{|\nabla_x p(x,z,\tau)|}{\tau^{\frac{s-1}{2}}}\frac{|u(t-\tau,z)-u(t,z)|}{\tau^{\frac{1+s}{2}}}\,d\tau\right)^2 dxdt \\
\leq& \int_{-T}^{T}\int_{\Omega'}\left(\int_0^\infty \frac{|\nabla_x p(x,z,\tau)|^2}{\tau^{s-1}}\,d\tau\right)\left(\int_0^\infty \frac{|u(t-\tau,z)-u(t,z)|^2}{\tau^{1+s}}\,d\tau\right) dxdt \\
\lesssim& \int_{-T}^{T}\int_{\Omega'}\underbrace{\left(\int_0^\infty e^{-c\frac{|x-z|^2}{\tau}}\tau^{-(n+s)}\,d\tau\right)}_{\text{By (2.2) as } \ell=1}\left(\int_0^\infty \frac{|u(t-\tau,z)-u(t,z)|^2}{\tau^{1+s}}\,d\tau\right) dxdt \\
\lesssim& \int_{\Omega'}\left\{\frac{1}{|x-z|^{2(n+s-1)}}\left(\int_{-T}^{T}\int_0^\infty \frac{|u(t-\tau,z)-u(t,z)|^2}{\tau^{1+s}}\,d\tau dt\right)\right\} dx \\
\lesssim& \left(\int_{\Omega'}\frac{1}{|x-z|^{2(n+s-1)}}dx\right)\|u(\cdot,z)\|^2_{H^{s/2}(\mathbb{R})}.
\end{aligned}$$

Thus, we obtain that

(4.21)

$$\begin{aligned}
&\left(\int_{\Omega'} J_1\,dx\right)^{1/2} \\
\lesssim& \int_{B_R}\left(\int_{-T}^{T}\int_{\Omega'}\left|\int_0^\infty \frac{|\nabla_x p(x,z,\tau)|}{\tau^s}\,|u(t-\tau,z)-u(t,z)|\,d\tau\right|^2 dxdt\right)^{1/2} dz \\
\leq& \int_{B_R}\left\{\left(\int_{\Omega'}\frac{1}{|x-z|^{2(n+s-1)}}dx\right)^{1/2}\|u(\cdot,z)\|_{H^{s/2}(\mathbb{R})}\right\} dz \\
\leq& \underbrace{\left(\int_{B_R}\int_{\Omega'}\frac{1}{|x-z|^{2(n+s-1)}}dxdz\right)^{1/2}\left(\int_{B_R}\|u(\cdot,z)\|^2_{H^{s/2}(\mathbb{R})}\,dz\right)^{1/2}}_{\text{By Hölder's inequality}} \\
\lesssim& \left(\int_{B_R}\|u(\cdot,z)\|^2_{H^{s/2}(\mathbb{R})}dz\right)^{1/2} \\
\leq& \|u\|_{\mathbf{H}^s(\mathbb{R}^{n+1})},
\end{aligned}$$

where we used the fact

$$\int_{B_R}\int_{\Omega'}\frac{1}{|x-z|^{2(n+s-1)}}dxdz<\infty$$

in the above computations.

Step 3b-2. Estimate for J_2: By a straightforward calculation, we can have that

$$\begin{aligned}
&\int_{-T}^{T}\left|\int_{B_R}\int_0^\infty \frac{|\nabla_x p(x,z,\tau)|}{\tau^s}|u(t,z)-u(t,x)|\,d\tau dz\right|^2 dt\\
\lesssim&\underbrace{\int_{-T}^{T}\left|\int_{B_R}\left(\int_0^\infty e^{-c\frac{|x-z|^2}{\tau}}\tau^{-(\frac{n+1}{2}+s)}\,d\tau\right)|u(t,z)-u(t,x)|\,dz\right|^2 dt}_{\text{By (2.2) as } k=1}\\
\lesssim&\underbrace{\int_{-T}^{T}\left|\int_{B_R}\frac{|u(t,z)-u(t,x)|}{|x-z|^{n+2s-1}}\,dz\right|^2 dt}_{\text{By (4.4)}}\\
\leq&\int_{-T}^{T}\left(\int_{B_R}\frac{|u(t,z)-u(t,x)|^2}{|x-z|^{n+2s}}\,dz\right)\underbrace{\left(\int_{B_R}\frac{1}{|x-z|^{n+2s-2}}\,dz\right)}_{\text{This is bounded}}dt\\
\lesssim&\int_{-T}^{T}\int_{B_R}\frac{|u(t,z)-u(t,x)|^2}{|x-z|^{n+2s}}\,dzdt.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\left(\int_{\Omega'}J_2\,dx\right)^{1/2}\\
\lesssim&\left(\int_{\Omega_T'}\left|\int_{B_R}\int_0^\infty\frac{|\nabla_x p(x,z,\tau)|}{\tau^s}|u(t,z)-u(t,x)|\,d\tau dz\right|^2 dtdx\right)^{1/2}\\
\lesssim&\left(\int_{-T}^{T}\int_{\Omega'}\int_{B_R}\frac{|u(t,z)-u(t,x)|^2}{|x-z|^{n+2s}}\,dzdxdt\right)^{1/2}\\
\leq&\left(\int_{-T}^{T}\|u(t,\cdot)\|^2_{H^s(\mathbb{R}^n)}\,dt\right)^{1/2}\\
\lesssim&\|u\|_{\mathbf{H}^s(\mathbb{R}^{n+1})},
\end{aligned}\tag{4.22}$$

where we used $H^s(\mathbb{R}^n)=W^{s,2}(\mathbb{R}^n)$ denotes the fractional Sobolev space of order s, which is characterized in [McL00, Section 3] for instance. Therefore, combined with (4.20), (4.21) and (4.22), we can conclude that

$$\left\|\left(\nabla w(t,x,y)-\nabla\left(\int_y^\infty\mu^{1-2s}\widetilde{g}(x,\mu)\,d\mu\right)u(t,x)\right)\right\|_{L^2(\Omega_T')}\lesssim\|u\|_{\mathbf{H}^s(\mathbb{R}^{n+1})}<\infty,\tag{4.23}$$

for any bounded open set $\Omega'\subset\mathbb{R}^n$ and for any $y>0$ as we want. Note that the upper bound of (4.23) is independent of $y>0$.

The goal is to estimate $\|\nabla w(\cdot,\cdot,0)\|_{L^2(\Omega_T')}$. To this end, we observe that

$$\begin{aligned}
&\left\|\nabla\left(\int_0^\infty\mu^{1-2s}\widetilde{g}(x,\mu)\,d\mu\right)u(t,x)\right\|_{L^2(\Omega_T')}\\
\lesssim&\|u\|_{L^2(\Omega_T')}\left\|\nabla\left(\int_0^\infty\mu^{1-2s}\widetilde{g}(x,\mu)\,d\mu\right)\right\|_{L^\infty(\Omega')}
\end{aligned}$$

by the Hölder's inequality. Note that the function $\int_0^\infty \mu^{1-2s}\widetilde{g}(x,\mu)\,d\mu$ is time-independent, and

$$\left\|\nabla\left(\int_0^\infty \mu^{1-2s}\widetilde{g}(x,\mu)\,d\mu\right)\right\|_{L^\infty(\Omega')} < \infty \tag{4.24}$$

as shown in the proof of [CGRU23, Proposition 6.1]. In particular, we will prove that (4.24) for $n\geq 2$ in the next step. Hence, with these estimates at hand, we can obtain that

$$\begin{aligned}
&\|\nabla w(\cdot,\cdot,0)\|_{L^2(\Omega_T')}\\
&\leq \lim_{y\to 0}\left\|\left(\nabla w(t,x,y)-\nabla\left(\int_y^\infty \mu^{1-2s}\widetilde{g}(x,\mu)\,d\mu\right)u(t,x)\right)\right\|_{L^2(\Omega_T')}\\
&\quad+\left\|\nabla\left(\int_0^\infty \mu^{1-2s}\widetilde{g}(x,\mu)\,d\mu\right)u(t,x)\right\|_{L^2(\Omega_T')}\\
&\lesssim \|u\|_{\mathbf{H}^s(\mathbb{R}^{n+1})}<\infty,
\end{aligned} \tag{4.25}$$

which proves the desired estimate.

Step 4. Auxiliary estimate.

Let us explain that (4.24) holds for $s\in[\frac{1}{2},1)$ and for all $n\geq 2$ for the sake of self-containedness. To this end, our goal is to prove

$$\left|\nabla_x\int_0^\infty\int_{\mathbb{R}^n}\int_0^\infty p(x,z,\tau)\mu e^{-\frac{\mu^2}{4\tau}}\frac{d\tau}{\tau^{1+s}}g(z)\,dzd\mu\right|<\infty,$$

which is equivalent to show

$$\left|\nabla_x\int_{B_R}g(z)\int_0^\infty \tau^{-s}p(x,z,\tau)\,d\tau dz\right|<\infty, \tag{4.26}$$

where we used $\left|\int_0^\infty \frac{\mu}{2\tau}e^{-\frac{\mu^2}{4\tau}}\,d\mu\right|\leq 1$. To this end, we investigate

$$\begin{aligned}
&-\int_{B_R}g(z)\int_\epsilon^\infty \tau^{-s}p(x,z,\tau)\,d\tau dz\\
&\approx -\int_{B_R}g(z)\int_\epsilon^\infty \partial_\tau\left(\tau^{1-s}\right)p(x,z,\tau)\,d\tau dz\\
&=\epsilon^{1-s}\int_{B_R}g(z)p(x,z,\epsilon)\,dz-\underbrace{\int_{B_R}g(z)\lim_{\tau\to\infty}\tau^{1-s}p(x,z,\tau)\,dz}_{(*)}\\
&\quad+\int_{B_R}g(z)\int_\epsilon^\infty \tau^{1-s}\partial_\tau p(x,z,\tau)\,d\tau dz\\
&:=L_1+L_2,
\end{aligned} \tag{4.27}$$

where

$$\begin{aligned}
L_1&:=\epsilon^{1-s}\int_{B_R}g(z)p(x,z,\epsilon)\,dz,\\
L_2&:=\int_{B_R}g(z)\int_\epsilon^\infty \tau^{1-s}\partial_\tau p(x,z,\tau)\,d\tau dz.
\end{aligned}$$

Note that $(*)=0$ in (4.27) can be observed by

$$\left|\lim_{\tau\to\infty}\tau^{1-s}\nabla_x^l p(x,z,\tau)\right|\lesssim \lim_{\tau\to\infty}\tau^{1-s-\frac{n+l}{2}}e^{-c\frac{|x-z|^2}{\tau}}\leq \lim_{\tau\to\infty}\tau^{-s}=0,$$

for any $n\geq 2$ and $l=0,1$.

For L_1, we first note that

$$\int_{B_R} g(z)p(x,z,\epsilon)\,dz \to g(x) \text{ as } \epsilon \to 0,$$

and hence

$$\epsilon^{1-s}\int_{B_R} g(z)p(x,z,\epsilon)\,dz \to 0 \text{ as } \epsilon \to 0, \text{ for } s \in [\frac{1}{2},1).$$

For L_2, by using the equation of the heat kernel $p(x,z,\tau)$, one can see that

$$\begin{aligned}
&\int_{B_R} g(z)\int_\epsilon^\infty \tau^{1-s}\partial_\tau p(x,z,\tau)\,d\tau dz \\
=&\int_{B_R} g(z)\int_\epsilon^\infty \tau^{1-s}\left(\nabla_z\cdot\sigma(z)\nabla_z p(x,z,\tau)\right)d\tau dz \\
=&\int_{B_R} g(z)\left(\nabla_z\cdot\sigma(z)\nabla_z\int_\epsilon^\infty \tau^{1-s}p(x,z,\tau)\right)d\tau dz \\
=&\underbrace{\int_{B_R\setminus B_{R_1}}\left(\nabla_z\cdot\sigma(z)\nabla_z g(z)\right)\int_\epsilon^\infty \tau^{1-s}p(x,z,\tau)\,d\tau dz}_{\text{By integration by part twice and } g(z)=1 \text{ in } B_{R_1}}.
\end{aligned} \tag{4.28}$$

Moreover, applying another heat kernel estimate $|\partial_\tau p(x,z,\tau)| \lesssim \frac{1}{\tau^{\frac{n}{2}+1}}e^{-c\frac{|x-z|^2}{\tau}}$ from [Gri95, equation (0.6)], the integrand in the left hand side of (4.28) has an upper bound that

$$\tau^{1-s}\left|\partial_\tau p(x,z,\tau)\right| \lesssim \frac{1}{\tau^{\frac{n}{2}+1}}e^{-c\frac{|x-z|^2}{\tau}} \leq \frac{1}{\tau^{\frac{n}{2}+1}},$$

which is integrable for $\tau>\epsilon$.

In addition, we also observe that for a.e. $z\in B_R$, then there holds

$$\begin{aligned}
&\left|\nabla_z\cdot\sigma(z)\nabla_z g(z)\right|\left|\int_\epsilon^\infty \tau^{1-s}p(x,z,\tau)\,d\tau\right| \\
\lesssim&\left|\nabla_z\cdot\sigma(z)\nabla_z g(z)\right|\int_\epsilon^\infty \tau^{1-s-\frac{n}{2}}e^{-c\frac{|x-z|^2}{\tau}}\,d\tau \\
\lesssim&\frac{\left|\nabla_z\cdot\sigma(z)\nabla_z g(z)\right|}{|x-z|^{n+2s-4}},
\end{aligned}$$

where the right hand sides of the above bounds are integrable functions of z due to the support and smooth conditions of g. Hence, back to the relation (4.27), the Lebesgue dominated convergence theorem yields that

$$\begin{aligned}
&\int_{B_R} g(z)\int_0^\infty \tau^{-s}p(x,z,\tau)\,d\tau dz \\
\approx&\int_{B_R\setminus B_{R_1}}\nabla_z\cdot\sigma(z)\nabla_z g(z)\int_0^\infty \tau^{1-s}p(x,z,\tau)\,d\tau dz.
\end{aligned}$$

With preceding arguments at hand, we now study (4.26). Let us note that there exists a $\delta>0$ such that $|x-z|>\delta$ for all $z\in B_R\setminus B_{R_1}$. Therefore,

$$\begin{aligned}
\left|\left(\nabla_z\cdot\sigma(z)\nabla_z g(z)\right)\tau^{1-s}\nabla_x p(x,z,\tau)\right| &\lesssim \left|\nabla_z\cdot\sigma(z)\nabla_z g(z)\right|\tau^{-s-\frac{n}{2}}\left|x-z\right|e^{-c\frac{|x-z|^2}{\tau}} \\
&\lesssim \left|\nabla_z\cdot\sigma(z)\nabla_z g(z)\right|\tau^{-s-\frac{n}{2}}\left|x-z\right|e^{-c\frac{\delta^2}{\tau}},
\end{aligned}$$

and

$$\int_{B_R\setminus B_{R_1}}\int_0^\infty \left|\nabla\cdot\sigma\nabla g\right|\tau^{-s-\frac{n}{2}}e^{-c\frac{\delta^2}{\tau}}\,d\tau dz \lesssim \int_{B_R\setminus B_{R_1}}\left|\nabla\cdot\sigma\nabla g\right|\,dz<\infty.$$

As a result, it is not hard to see (4.26) is bounded as we want.

Step 5. Exterior H^1 estimate.

By using the compact support condition of $u \in \mathbf{H}^s(\mathbb{R}^{n+1})$, the Minkowski's integral inequality yields that

$$
\begin{aligned}
\left\|\nabla^\ell v(\cdot,x)\right\|_{L^2(-T,T)} &\leq \left(\int_{-T}^{T}\left(\int_0^\infty y^{1-2s}\left|\nabla^\ell \widetilde{u}(\cdot,x,y)\right| dy\right)^2 dt\right)^{1/2} \\
&\leq \int_0^\infty y^{1-2s}\left\|\nabla^\ell \widetilde{u}(\cdot,x,y)\right\|_{L^2(-T,T)} dy \\
&\lesssim \underbrace{\int_{B_R}\int_0^\infty \frac{y\|u(\cdot,z)\|_{L^2(\mathbb{R})}}{\left(|x-z|^2+y^2\right)^{\frac{n+\ell}{2}+s}}\, dydz}_{\text{By (4.11) and support of } u} \\
&\lesssim \int_{B_R}\frac{\|u(\cdot,z)\|_{L^2(\mathbb{R})}}{|x-z|^{n+\ell+2s-2}}\, dz.
\end{aligned} \tag{4.29}
$$

Therefore, we can obtain

$$
\left\|\nabla^\ell v\right\|^2_{L^2((\mathbb{R}^n\setminus B_{2R})_T)} \lesssim \int_{\mathbb{R}^n\setminus B_{2R}}\left(\int_{B_R}\frac{\|u(\cdot,z)\|_{L^2(\mathbb{R})}}{|x-z|^{n+\ell+2s-2}}\, dz\right)^2 dx \lesssim \|u\|^2_{L^2(\mathbb{R}^{n+1})},
$$

for $\ell = 0,1$, where we used the fact that $x \in \mathbb{R}^n \setminus B_{2R}$ and $z \in B_R$, so that $|x-z| \geq R > 0$.

Step 6. Conclusion.

With the local estimate (4.25) (replacing Ω' by B_{2R}) at hand, we also have

$$
\|\nabla v\|_{L^2(-T,T;H^1(B_{2R}))} \lesssim \|u\|_{\mathbf{H}^s(\mathbb{R}^{n+1})}. \tag{4.30}
$$

We can obtain the desired estimate (4.1) by combining (4.29) and (4.30). This proves the assertion of Proposition 4.1. ☐

5. The key equation

With rigorous analysis in Section 4 at hand, we can obtain the next result, which also makes the computations shown in Section 3.2 rigorously. On the other hand, the equation of v plays a key role to prove Theorem 1.1.

Lemma 5.1. *Given $s \in (0,1)$ and $n \in \mathbb{N}$, let $\widetilde{\sigma} \in C^2(\mathbb{R}^n;\mathbb{R}^{(n+1)\times(n+1)})$ be of the form (1.7). Let $\widetilde{u} \in \mathcal{L}^{1,2}(\mathbb{R}^{n+2}_+; y^{1-2s}dtdxdy)$ be the extension of $u \in \mathbf{H}^s(\mathbb{R}^{n+1})$ (see (1.6)) such that $\overline{\mathrm{supp}(u)} \subset \mathbb{R}^{n+1}$ is compact. Assume that $v \in L^2(-T,T;H^1(\Omega'))$ with $\partial_t v \in L^2(-T,T;H^{-1}(\Omega'))$ for some bounded open Lipschitz set $\Omega' \subset \mathbb{R}^n$, where v is given by (1.9). Then v is a weak solution to*

$$
(\partial_t - \nabla\cdot\sigma\nabla)\, v = (\partial_t - \nabla\sigma\nabla)^s\, u \text{ in } \Omega'_T,
$$

in the weak sense

$$
\int_{\Omega'_T}(v\partial_t\varphi - \sigma(x)\nabla v\cdot\nabla\varphi)\, dtdx = \int_{\mathbb{R}^{n+1}}\varphi \lim_{y\to 0} z^{1-2s}\partial_y\widetilde{u}(t,x,y)\, dtdx, \tag{5.1}
$$

for any test function $\varphi \in C^\infty_c(\Omega'_T)$.

Remark 5.2. *The right hand side in (5.1)*

$$
\int_{\mathbb{R}^{n+1}}\varphi \lim_{y\to 0} y^{1-2s}\partial_y\widetilde{u}(t,x,y)\, dtdx
$$

is understood as $\mathbf{H}^s(\mathbb{R}^{n+1})$-$\mathbf{H}^{-s}(\mathbb{R}^{n+1})$ duality pairing, which is well-defined since the limit $\lim_{y\to 0} y^{1-2s}\partial_y \widetilde{u}(t,x,y)\,dxdt \in \mathbf{H}^{-s}(\mathbb{R}^{n+1})$ due to Proposition 3.1 as $u \in \mathbf{H}^s(\mathbb{R}^{n+1})$.

Proof of Lemma 5.1. Since v is given by (1.9), we have

$$
\begin{aligned}
&\int_{\Omega_T'} (v\partial_t\varphi - \sigma(x)\nabla v\cdot\nabla\varphi)\,dtdx\\
&= \int_{\mathbb{R}^{n+1}} \left(\left(\int_0^\infty y^{1-2s}\widetilde{u}\,dy\right)\partial_t\varphi - \sigma(x)\nabla\left(\int_0^\infty y^{1-2s}\widetilde{u}\,dy\right)\cdot\nabla\varphi\right)dtdx\\
&= \lim_{k\to\infty}\int_{\mathbb{R}^{n+1}}\left(\int_0^\infty y^{1-2s}\widetilde{u}\eta_k(y)\,dy\right)\partial_t\varphi\,dtdx\\
&\quad - \lim_{k\to\infty}\int_{\mathbb{R}^{n+1}}\sigma(x)\left(\nabla\int_0^\infty y^{1-2s}\widetilde{u}\eta_k(y)\,dy\right)\cdot\nabla\varphi\,dxdt,
\end{aligned} \tag{5.2}
$$

for any test function $\varphi \in C_c^\infty([-T,T]\times\Omega')$. Here $\eta_k(y) = \eta(y/k)$, where $\eta : [0,\infty)\to\mathbb{R}$ is a smooth function fulfilling $\eta(y) = \begin{cases} 1 & \text{for } 0\le y<1 \\ 0 & \text{for } y\ge 2\end{cases}$. Notice that the convergences (5.2) follow from the fact that

$$
\begin{aligned}
\left\| \left|\int_0^\infty y^{1-2s}\widetilde{u}\,(1-\eta_k(y))\,dy\right| |\partial_t\varphi| \right\|_{L^1(\Omega_T')} &\lesssim \int_k^\infty y^{1-2s}\|\widetilde{u}\|_{L^1(\mathbb{R};L^\infty(\mathbb{R}^n))}\,dy\\
&\lesssim \underbrace{\int_k^\infty y^{1-2s}y^{-n}\|u\|_{L^1(\mathbb{R};L^1(\mathbb{R}^n))}\,dy}_{\text{By (4.3) as } m=q=1 \text{ and } r=p=\infty}\\
&\le k^{2-2s-n}\|u\|_{L^1(\mathbb{R};L^1(\mathbb{R}^n))}\\
&\lesssim k^{2-2s-n}\|u\|_{L^2(\mathbb{R}^{n+1})},
\end{aligned}
$$

where we used that the support condition $\overline{\mathrm{supp}(u)}$ is compact. Similarly, we can also deduce

$$
\begin{aligned}
&\left\|\sigma(x)\nabla\left(\int_0^\infty y^{1-2s}\widetilde{u}\,(1-\eta_k(y))\,dy\right)\cdot\nabla\varphi\right\|_{L^1(\Omega_T')}\\
&\lesssim \int_k^\infty y^{1-2s}\|\nabla\widetilde{u}\|_{L^1(\mathbb{R};L^\infty(\mathbb{R}^n))}\,dy\\
&\lesssim \int_k^\infty y^{1-2s}y^{-n-1}\|u\|_{L^1(\mathbb{R};L^1(\mathbb{R}^n))}\,dy\\
&\le k^{1-2s-n}\|u\|_{L^1(\mathbb{R};L^1(\mathbb{R}^n))}\\
&\lesssim k^{1-2s-n}\|u\|_{L^2(\mathbb{R}^{n+1})}.
\end{aligned}
$$

Therefore, by the preceding estimates, one can get the convergence

$$
\begin{aligned}
\int_0^\infty y^{1-2s}\widetilde{u}\eta_k(y)\,dy &\to \int_0^\infty y^{1-2s}\widetilde{u}\,dy,\\
\nabla\int_0^\infty y^{1-2s}\widetilde{u}\eta_k(y)\,dy &\to \nabla\int_0^\infty y^{1-2s}\widetilde{u}\,dy,
\end{aligned}
$$

in $L^2(\Omega_T')$ as $k\to\infty$.

Using the regularity of $\widetilde{u}\in\mathcal{L}^{1,2}(\mathbb{R}^{n+2}_+; y^{1-2s}dtdxdy)$ and considering the difference quotient, Lebesgue dominated convergence theorem and Fubini's theorem imply that

$$\begin{aligned}&\int_{\mathbb{R}^{n+1}}\sigma\nabla\left(\int_0^\infty y^{1-2s}\widetilde{u}(t,x,y)\eta_k(y)\,dy\right)\cdot\nabla\varphi\,dtdx\\=&\int_0^\infty\int_{\mathbb{R}^{n+1}}y^{1-2s}\sigma\nabla\widetilde{u}(t,x,y)\cdot\nabla\left(\varphi(t,x)\eta_k(y)\right)dtdxdy.\end{aligned}$$

Since $\widetilde{u}$ is a solution to (1.6), by taking limit as $k\to\infty$ of the preceding equality, an integration by parts yields that

$$\begin{aligned}&\lim_{k\to\infty}\int_0^\infty\int_{\mathbb{R}^{n+1}}y^{1-2s}\sigma\nabla\widetilde{u}(t,x,y)\cdot\nabla\left(\varphi(t,x)\eta_k(y)\right)dtdxdy\\
=&-\lim_{k\to\infty}\int_0^\infty\int_{\mathbb{R}^{n+1}}\nabla\cdot\left(y^{1-2s}\sigma\nabla\widetilde{u}(t,x,y)\right)\left(\varphi(t,x)\eta_k(y)\right)dtdxdy\\
=&\lim_{k\to\infty}\int_0^\infty\int_{\mathbb{R}^{n+1}}\partial_y\left(y^{1-2s}\partial_y\widetilde{u}(t,x,y)\right)\left(\varphi(t,x)\eta_k(y)\right)dtdxdy\\
&-\lim_{k\to\infty}\int_0^\infty\int_{\mathbb{R}^{n+1}}y^{1-2s}\partial_t\widetilde{u}(t,x,y)\left(\varphi(t,x)\eta_k(y)\right)dtdxdy\\
=&-\lim_{k\to\infty}\int_0^\infty\int_{\mathbb{R}^{n+1}}y^{1-2s}\partial_y\widetilde{u}(t,x,y)\partial_y\left(\varphi(t,x)\eta_k(y)\right)dtdxdy\\
&-\int_{\mathbb{R}^{n+1}}\lim_{y\to0}y^{1-2s}\partial_y\widetilde{u}(t,x,y)\varphi(t,x)\,dtdx\\
&+\lim_{k\to\infty}\int_{\mathbb{R}^{n+1}}\left(\int_0^\infty y^{1-2s}\widetilde{u}\eta_k(y)\,dy\right)\partial_t\varphi\,dtdx.\end{aligned}$$

Now, it suffices to show

$$\lim_{k\to\infty}\int_0^\infty\int_{\mathbb{R}^{n+1}}y^{1-2s}\partial_y\widetilde{u}(t,x,y)\partial_y\left(\varphi(t,x)\eta_k(y)\right)dtdxdy=0.$$

To this end, one has that

$$\begin{aligned}&\left|\int_0^\infty\int_{\mathbb{R}^{n+1}}y^{1-2s}\partial_y\widetilde{u}(t,x,y)\varphi(t,x)\partial_y\eta_k(y)\,dtdxdy\right|\\
\leq&\int_k^{2k}\int_{\mathbb{R}^{n+1}}y^{1-2s}\left|\partial_y\widetilde{u}(t,x,y)\right|\left|\varphi(t,x)\right|\left|\partial_y\eta_k(y)\right|dtdxdy\\
\lesssim&\underbrace{\frac{1}{k}\|\varphi\|_{L^\infty(\mathbb{R}^{n+1})}\int_k^{2k}y^{1-2s}\left\|\partial_y\widetilde{u}(\cdot,\cdot,y)\right\|_{L^1(\mathbb{R};L^1(\mathbb{R}^n))}dy}_{\text{By }|\partial_y\eta_k(y)|\lesssim 1/k\text{ for }y>0}\\
\lesssim&\underbrace{\frac{1}{k}\int_k^{2k}y^{1-2s}y^{-1}\|u\|_{L^1(\mathbb{R}^{n+1})}\,dy}_{\text{By (4.3) as }m=1,\ r=p=1\text{ and }q=1}\\
\lesssim&\ k^{-2s}\\
\to&\ 0\text{ as }k\to\infty.\end{aligned}$$

where we utilized the compact support condition for the functions u, φ and η. This proves the assertion. □

With Lemma 5.1 and the regularity results for the function v at hand, we can conclude the next result.

Theorem 5.3 (Key equation). *Let $s, n, \widetilde{\sigma}$ satisfy the assumptions in Lemma 5.1. Suppose that $\Omega, W\subset\mathbb{R}^n$ are nonempty, bounded and open sets with Lipschitz boundaries such that $\overline{\Omega}\cap\overline{W}=\emptyset$. Let $u_f\in\mathbf{H}^s(\mathbb{R}^{n+1})$ be the unique solution to*

(1.1) *with the exterior data* $f \in \widetilde{\mathbf{H}}^s(W_T)$*, and let* $\widetilde{u} \in \mathcal{L}^{1,2}(\mathbb{R}^{n+2}_+; y^{1-2s}dtdxdy)$ *be the extension of* u_f*. Let* v *be given by* (1.9)*, then* $v \in L^2(0,T;H^1(\Omega))$ *with* $\partial_t v \in L^2(0,T;H^{-1}(\Omega))$ *is a weak solution to*

$$(\partial_t - \nabla\cdot\sigma\nabla)\, v = 0 \text{ in } \Omega_T,$$

which is equivalent to

$$\int_{\Omega_T} (v\partial_t\varphi - \sigma\nabla v\cdot\nabla\varphi)\, dtdx = 0,$$

for any $\varphi \in C^\infty_c(\Omega_T)$*.*

6. Density approach

In this section, we want to show that the Cauchy data which are generated from the nonlocal parabolic equation form a dense set in the set of the Cauchy data for the local parabolic equation. The main tool is by using the property of the function v given by (1.9).

Proposition 6.1. *Given* $s\in(0,1)$ *and* $n\geq 2$*, let* $\widetilde{\sigma} \in C^2(\mathbb{R}^n;\mathbb{R}^{(n+1)\times(n+1)})$ *be of the form* (1.7) *with* $\sigma = \mathrm{Id}$ *in* Ω_e*. Suppose that* $\Omega, W \subset \mathbb{R}^n$ *are nonempty, bounded and open sets with Lipschitz boundaries such that* $\overline{\Omega}\cap\overline{W}=\emptyset$*. Let* $\widetilde{u}_f = \widetilde{u}_f(t,x,y) \in \mathcal{L}^{1,2}(\mathbb{R}^{n+2}_+; y^{1-2s}dtdxdy)$ *be the weak solution of*

$$\begin{cases} y^{1-2s}\partial_t\widetilde{u}_f - \nabla_{x,y}\cdot\left(y^{1-2s}\widetilde{\sigma}(x)\nabla_{x,y}\widetilde{u}_f\right) = 0 & \text{in } \mathbb{R}^{n+2}_+, \\ \widetilde{u}_f(t,x,0) = f & \text{on } (\Omega_e)_T, \\ \lim_{y\to 0} y^{1-2s}\partial_y\widetilde{u}_f = 0 & \text{in } \Omega_T, \end{cases}$$

where $f = f(t,x)\in C^\infty_c(W_T)$*. Consider the sets*

$$\mathcal{V} := \left\{ v_f(t,x) := \int_0^\infty y^{1-2s}\widetilde{u}_f(t,x,y)\, dy, \text{ for } f\in C^\infty_c(W_T)\right\},$$
$$\mathcal{V}' := \left\{ v|_{(\partial\Omega)_T} : v\in\mathcal{V}\right\},$$

then $\overline{\mathcal{V}'} = L^2(-T,T;H^{1/2}(\partial\Omega))$*.*

Remark 6.2. *It is worth mentioning that we make use the condition* $\sigma = \mathrm{Id}$ *in* Ω_e *in Proposition 6.1 to connect the nonlocal and the local information. On the other hand, we do not need this condition in the study of the pure nonlocal parabolic operators.*

6.1. **A formal proof for the density result.** With the expression formula (4.5) at hand, for any $t\leq -T$, we can have an initial data condition that

$$\widetilde{u}_f(t,x,y) = c_s y\int_{\mathbb{R}^n}\int_0^\infty e^{-\frac{y^2}{4\tau}} p(x,z,\tau)u_f(t-\tau,z)\,\frac{d\tau}{\tau^{1+s}}dz = 0, \tag{6.1}$$

for $t\leq -T$, where we utilized the fact that u_f is the solution to (1.1) whereas $u(t-\tau,x)\equiv 0$ for $\tau>0$ and $t\leq -T$.

Sketch proof of Proposition 6.1 for $s=1/2$ *and* $\sigma=\mathrm{Id}$. As $s=1/2$ and $\sigma=\mathrm{Id}$, recalling that there holds the condition (6.1), and the rest of the proof is divided into three steps:

Step 1. Interior denseness.

We first prove that the set $\mathcal{V}\subset L^2(-T,T;H^1(\Omega))$ is dense in the set $\mathcal{D}$, where

$$\mathcal{D} := \{v\in L^2(-T,T;H^1(\Omega)) : (\partial_t-\Delta)\, v = 0 \text{ in } \Omega_T \text{ and } v(-T,x)=0 \text{ in } \Omega\}$$

stands for the solution space. By the Hahn-Banach theorem, it is enough to show that if $\psi\in L^2(-T,T;\widetilde{H}^{-1}(\Omega))$ such that $\psi(v_f)=0$ for all $f\in C^\infty_c(W_T)$, then there holds $\psi(v)=0$ for all $v\in\mathcal{D}$. In what follows, for $\psi\in L^2(-T,T;\widetilde{H}^{-1}(\Omega))$. Consider an auxiliary adjoint problem

$$\begin{cases}(\partial_t+\Delta_{x,y})\,w=\psi & \text{in } \mathbb{R}^n_T\times(0,\infty),\\ w=0 & \text{in } (\Omega_e)_T\times\{0\},\\ \partial_y w=0 & \text{in } \Omega_T\times\{0\},\\ w(t,x,y)=0 & \text{for } (x,y)\in\mathbb{R}^{n+1}_+,\ t\geq T.\end{cases} \tag{6.2}$$

The (weak) solvability of the backward heat equation (6.2) can be seen by reversing the time-variable that $t\mapsto -t$ for $t\in[-T,T]$, which will be described in Lemma 6.4 for general cases. Since the function ψ does not have the decay property with respect to the y-variable, By the duality and Fubini's theorem, an integration by parts formula yields that

$$\begin{aligned}0&=\psi(v_f)\\&=\left\langle \psi,\int_0^\infty \widetilde{u}_f(t,x,y)\,dy\right\rangle_{L^2(-T,T;\widetilde{H}^{-1}(\Omega))\times L^2(-T,T;H^1(\Omega))}\\&=\underbrace{\int_0^\infty\int_{\mathbb{R}^n}\int_{-T}^T \widetilde{u}_f\psi\,dtdxdy}_{\text{By the equation (6.2)}}\\&=\int_{\mathbb{R}^n}\int_{-T}^T\widetilde{u}_f\partial_y w(t,x,0)\,dtdx-\int_0^\infty\int_{\mathbb{R}^n}\int_{-T}^T\nabla_{x,y}\widetilde{u}_f\cdot\nabla_{x,y}w\,dtdxdy\\&\quad+\int_0^\infty\int_{\mathbb{R}^n}\int_{-T}^T\widetilde{u}_f\partial_t w\,dtdxdy\\&=\int_{W_T}f\partial_y w(t,x,0)\,dtdx+\underbrace{\int_{\mathbb{R}^n}\int_{-T}^T w(t,x,0)\partial_y\widetilde{u}_f(t,x,0)\,dtdx}_{=0\text{ since }\begin{cases}w(t,x,0)=0 & \text{in } (\Omega_e)_T\\ \partial_y\widetilde{u}_f(t,x,0)=0 & \text{in } \Omega_T\end{cases}}\\&\quad\underbrace{-\int_0^\infty\int_{\mathbb{R}^n}\int_{-T}^T w\,(\partial_t\widetilde{u}_f-\Delta_{x,y}\widetilde{u}_f)\,dtdxdy}_{\text{Integration by parts w.r.t } t\text{ and } w(T,x,y)=\widetilde{u}_f(-T,x,y)=0}\\&=\int_{W_T}f\partial_y w(t,x,0)\,dtdx,\end{aligned} \tag{6.3}$$

where we used that $\widetilde{u}_f$ satisfies (1.6) in the last equality as $s=1/2$. We want to emphasize that for the rigorous argument, one needs to introduce suitable cutoff functions to utilize the equation given by (6.2).

Step 2. Hahn-Banach approach.

Since $f\in C_c^\infty(W_T)$ can be arbitrary, by the equality (6.3), one must have that $\partial_y w=0$ on $W_T\times\{0\}$, where w is a solution to (6.2). By the UCP for second order parabolic equations (for example, see [Sog90])[2], we obtain that

$$w(t,x,y)\equiv 0 \text{ for } (t,x,y)\in(-T,\infty)\times\Omega_e\times(0,\infty), \tag{6.4}$$

which will be used to prove the Hahn-Banach approach.

[2]Since $w=\partial_y w=0$ on $W_T\times\{0\}$, one can extend w by zero to $W_T\times\{y\leq 0\}$ and apply the classical UCP for parabolic equations.

Let $v \in \mathcal{D} \subset L^2(-T,T;H^1(\Omega))$ and $\beta \in C_c^\infty(0,\infty)$ such that

$$\beta \geq 0, \quad \int_0^\infty \beta(y)\,dy = 1 \quad \text{and} \quad \text{supp}(\beta) \subset (1,2).$$

Let us set $\beta_k(y) := 1/k\beta(y/k)$, for $k \in \mathbb{N}$. Recalling that $\psi \in L^2(-T,T;\widetilde{H}^{-1}(\Omega))$ and $v \in \mathcal{D}$, one has

$$\begin{aligned}
&-\psi(v)\\
&= -\psi\left(\int_0^\infty \beta_k(y) v\,dy\right)\\
&= -\lim_{k\to\infty} \psi\left(\int_0^\infty \beta_k(y) v\,dy\right)\\
&= \underbrace{\lim_{k\to\infty}\left\{\int_0^\infty \int_{\Omega_T} \left[\nabla_{x,y}(v\beta_k)\cdot\nabla_{x,y} w - (v\beta_k)\,\partial_t w\right] dtdxdy\right\}}_{\text{By (6.2) and integration by parts}}\\
&= \lim_{k\to\infty}\left[\int_{\Omega_T} \nabla v\cdot\nabla\left(\int_0^\infty \beta_k w\,dy\right) dtdx + \int_0^\infty\int_{\Omega_T} v\partial_y\beta_k\partial_y w\,dtdxdy\right.\\
&\qquad\left. + \int_0^\infty\int_{\Omega_T} (w\beta_k)\,\partial_t v\,dtdxdy\right]\\
&= \underbrace{\lim_{k\to\infty}\left[\int_{(\partial\Omega)_T} (\nabla v\cdot\nu)\left(\int_0^\infty \beta_k w\,dy\right) dtdS_x + \int_0^\infty\int_{\Omega_T} v\partial_y\beta_k\partial_y w\,dtdxdy\right]}_{\text{Since } v\in\mathcal{D}}\\
&= \lim_{k\to\infty}\int_0^\infty\int_{\Omega_T} v\partial_y\beta_k\partial_y w\,dtdxdy,
\end{aligned}$$

where all boundary integrals vanish due to the condition (6.4). Therefore, it suffices to show

$$\lim_{k\to\infty}\int_0^\infty\int_{\Omega_T} v\partial_y\beta_k\partial_y w\,dtdxdy = 0.$$

To this end, by the definition of ζ_k, one has

$$\lim_{k\to\infty}\int_0^\infty\int_{\Omega_T} v\partial_y\beta_k\partial_y w\,dtdxdy = \lim_{k\to\infty}\left(k^{-2}\int_k^{2k}\int_{\Omega_T} v\partial_y\beta\partial_y w\,dtdxdy\right) = 0,$$

which proves the density $\mathcal{V} \subset L^2(-T,T;H^1(\Omega))$ in $\mathcal{D}$ formally. In order to make the preceding derivation rigorously, we need to check that the solution w of (6.2) possesses appropriate bounds and decay.

Step 3. Boundary denseness.

The denseness of $\mathcal{V}'$ can be seen via trace estimates from $L^2(-T,T;H^1(\Omega))$ to $L^2(-T,T;H^{1/2}(\partial\Omega))$. □

Remark 6.3. *We usually consider parabolic problems as initial-boundary value problems. However, for the extension problem (1.6), there is no initial condition proposed with respect to the time-variable. As a matter of fact, suppose that the past time information $u_f(t,x) = 0$ for $t \leq -T$, where u_f is the solution to the equation (1.1), then (6.1) holds for the extension problem (1.6). This implies that the extension problem (1.6) contains an initial data implicitly, which comes from the past time information of (1.1).*

In next subsection, we will make all computations in the subsection rigorously for the case $s \in (0,1)$ and $\sigma \in C^2(\mathbb{R}^n;\mathbb{R}^{n\times n})$ fulfilling the condition (1.2).

6.2. **A rigorous proof for the density result.** Let us begin with the well-posed of a generalized version of (6.2).

Lemma 6.4 (Solvability)**.** *Given $s\in(0,1)$ and $n\geq 2$, let $\widetilde{\sigma}\in C^2(\mathbb{R}^n;\mathbb{R}^{(n+1)\times(n+1)})$ be of the form (1.7). Let $\Omega\subset\mathbb{R}^n$ be a bounded open set with Lipschitz boundary $\partial\Omega$. Given $\psi\in L^2(-T,T;\widetilde{H}^{-1}(\Omega))$, consider the problem*

$$
\begin{cases}
y^{1-2s}\partial_t w-\nabla_{x,y}\cdot\left(y^{1-2s}\widetilde{\sigma}\nabla_{x,y}\right)w=y^{1-2s}\psi & \text{in } \mathbb{R}^n_T\times(0,\infty),\\
w=0 & \text{in } (\Omega_e)_T\times\{0\},\\
\lim_{y\to 0}y^{1-2s}\partial_y w=0 & \text{in } \Omega_T\times\{0\},\\
w(t,x,y)=0 & \text{for } (x,y)\in\mathbb{R}^{n+1}_+,\ t\leq -T.
\end{cases}
\tag{6.5}
$$

Then there exists a solution $w\in\mathcal{L}^{1,2}(\mathbb{R}^{n+2}_+;y^{1-2s}dtdxdy)$ of (6.5) such that

$$
\begin{aligned}
&\int_{\mathbb{R}^{n+2}_+}y^{1-2s}\left(-w\partial_t\varphi+\widetilde{\sigma}\nabla_{x,y}w\cdot\nabla_{x,y}\varphi\right)dtdxdy\\
&=\left\langle\psi,\int_0^\infty y^{1-2s}\varphi(t,x,y)\,dy\right\rangle_{L^2(-T,T;\widetilde{H}^{-1}(\Omega))\times L^2(-T,T;H^1(\Omega))},
\end{aligned}
$$

for any $\varphi\in\mathcal{L}^{1,2}_{c,0}(\mathbb{R}^{n+2}_+;y^{1-2s}dtdxdy)$, where

$$
\begin{aligned}
&\mathcal{L}^{1,2}_{c,0}(\mathbb{R}^{n+2}_+;y^{1-2s}dtdxdy)\\
&:=\Big\{\varphi\in\mathcal{L}^{1,2}(\mathbb{R}^{n+2}_+;y^{1-2s}dtdxdy):\ \varphi(\cdot,x,y)\in H^{s/2}(\mathbb{R})\text{ for }(x,y)\in\mathbb{R}^{n+1}_+,\\
&\qquad\varphi(t,\cdot,\cdot)\text{ has compact support in }\overline{\mathbb{R}^{n+1}_+},\text{ for any }t\in[-T,T],\\
&\qquad\varphi(t,x,y)=0\text{ for }t\geq T\text{ and }(x,y)\in\mathbb{R}^{n+1}_+,\ \varphi|_{(\Omega_e)_T\times\{0\}}=0\Big\}.
\end{aligned}
\tag{6.6}
$$

Remark 6.5. *It is known that $\lim_{y\to0}y^{1-2s}\partial_y w\in\mathbf{H}^{-s}(\mathbb{R}^{n+1})$ by the regularity of w. As a matter of fact, we can obtain that*

$$
\begin{aligned}
&\int_{\mathbb{R}^{n+2}_+}y^{1-2s}\left(-w\partial_t\varphi+\widetilde{\sigma}\nabla_{x,y}w\cdot\nabla_{x,y}\varphi\right)dtdxdy\\
&=\left\langle\psi,\int_0^\infty y^{1-2s}\varphi(t,x,y)\,dy\right\rangle_{L^2(-T,T;\widetilde{H}^{-1}(\Omega))\times L^2(-T,T;H^1(\Omega))}\\
&\quad+\int_{\mathbb{R}^{n+1}}\varphi(t,x,0)\lim_{y\to0}y^{1-2s}\partial_y w\,dtdx,
\end{aligned}
\tag{6.7}
$$

for any $\varphi\in\mathcal{L}^{1,2}_{c}(\mathbb{R}^{n+2}_+;y^{1-2s}dtdxdy)$, where

$$
\begin{aligned}
&\mathcal{L}^{1,2}_{c}(\mathbb{R}^{n+2}_+;y^{1-2s}dtdxdy)\\
&:=\Big\{\varphi\in\mathcal{L}^{1,2}(\mathbb{R}^{n+2}_+;y^{1-2s}dtdxdy):\ \varphi(\cdot,x,y)\in H^{s/2}(\mathbb{R})\text{ for }(x,y)\in\mathbb{R}^{n+1}_+,\\
&\qquad\varphi(t,\cdot,\cdot)\text{ has compact support in }\overline{\mathbb{R}^{n+1}_+},\text{ for any }t\in[-T,T],\\
&\qquad\varphi(t,x,y)=0\text{ for }t\geq T\text{ and }(x,y)\in\mathbb{R}^{n+1}_+\Big\}.
\end{aligned}
$$

We also point out that the last term in (6.7) makes sense due to the fact $\varphi\in\mathcal{L}^{1.2}_c(\mathbb{R}^{n+2}_+;y^{1-2s}dtdxdy)$. As a matter of fact, we will demonstrate that $\varphi(t,x,0)\in\mathbf{H}^s(\mathbb{R}^{n+1})$ for either $\varphi\in\mathcal{L}^{1.2}_{c,0}(\mathbb{R}^{n+2}_+;y^{1-2s}dtdxdy)$ or $\varphi\in\mathcal{L}^{1.2}_{c}(\mathbb{R}^{n+2}_+;y^{1-2s}dtdxdy)$. This will be shown in the proof of Lemma 6.4.

We observe a simple and useful result in the next lemma.

Lemma 6.6. *Let $u = u(t,x)$ be a function satisfy*

$$u \in L^2(\mathbb{R}; H^1(\mathbb{R}^n)) \quad \text{and} \quad \partial_t u \in L^2(\mathbb{R}; H^{-1}(\mathbb{R}^n)). \tag{6.8}$$

Then $u \in \mathbf{H}^s(\mathbb{R}^{n+1}))$ for any $s \in (0,1)$.

Proof. Since we have (6.8), then Fourier transform implies that

$$\begin{aligned} \int_{\mathbb{R}^{n+1}} \left(1+|\xi|^2\right) |\widehat{u}(\rho,\xi)|^2 \, d\rho d\xi < \infty, \\ \int_{\mathbb{R}^{n+1}} \left(1+|\xi|^2\right)^{-1} |\rho|^2 |\widehat{u}(\rho,\xi)|^2 \, d\rho d\xi < \infty, \end{aligned} \tag{6.9}$$

where $\widehat{u}$ denotes the Fourier transform for u with respect to both space and time variables. By utilizing (6.9), one has

$$\int_{\mathbb{R}^{n+1}} |\widehat{u}(\rho,\xi)|^2 \, d\rho d\xi < \infty.$$

Meanwhile, combined with (6.9) and the Hölder's inequality, we get

$$\begin{aligned} &\int_{\mathbb{R}^{n+1}} |\rho| \, |\widehat{u}(\rho,\xi)|^2 \, d\rho d\xi \\ \le& \left(\int_{\mathbb{R}^{n+1}} \left(1+|\xi|^2\right) |\widehat{u}(\rho,\xi)|^2 \, d\rho d\xi \right)^{1/2} \\ &\cdot \left(\int_{\mathbb{R}^{n+1}} \left(1+|\xi|^2\right)^{-1} |\rho|^2 |\widehat{u}(\rho,\xi)|^2 \, d\rho d\xi \right)^{1/2} < \infty. \end{aligned} \tag{6.10}$$

So, $u \in \mathbf{H}^s(\mathbb{R}^{n+1})$ for $s \in (0,1)$. Moreover, (6.9) and (6.10) yield that

$$\int_{\mathbb{R}^{n+1}} \left(1 + \left(|\xi|^4 + |\rho|^2\right)^{1/2}\right)^s |\widehat{u}(\rho,\xi)|^2 \, d\rho d\xi < \infty,$$

for $s \in (0,1)$, which infers $u \in \mathbf{H}^s(\mathbb{R}^{n+1}) = \mathbb{H}^s(\mathbb{R}^{n+1})$ as desired.

□

Now, we are ready to prove Lemma 6.4.

Proof of Lemma 6.4. We split the proof into three steps:

Step 1. Initiation

Notice that $\psi \in L^2(-T,T;\widetilde{H}^{-1}(\Omega))$, so we may assume that $\mathrm{supp}(\psi) \subset \Omega_T \subset \mathbb{R}^n_T$ with $\psi \in L^2(-T,T;H^{-1}(\mathbb{R}^n))$. Note that $\overline{\Omega_T}$ is a compact set in $\mathbb{R}^{n+1}$. Consider the equation

$$\begin{cases} (\partial_t - \nabla\cdot\sigma\nabla)\, u_1 = \psi & \text{in } \mathbb{R}^n_T, \\ u_1(-T,x) = 0 & \text{for } x \in \mathbb{R}^n, \end{cases} \tag{6.11}$$

then we have the preceding equation is solvable by using the variational method, (for instance, see [DL92, Chapter XVIII, Section 3]), that is, there exists a solution $u \in L^2(-T,T;H^1(\mathbb{R}^n))$ to (6.11). Meanwhile, $\nabla\cdot\sigma\nabla u_1 \in L^2(-T,T;H^{-1}(\mathbb{R}^n))$ so that

$$u_1 \in L^2(-T,T;H^1(\mathbb{R}^n)) \quad \text{and} \quad \partial_t u_1 \in L^2(-T,T;H^{-1}(\mathbb{R}^n)). \tag{6.12}$$

To proceed, we want to extend the function $u_1 \in \mathbf{H}^s(\mathbb{R}^{n+1})$ with the same notation. To this end, we extend u_1 by zero to the set outside $\{t \le -T\} \times \mathbb{R}^{n+1}$ by the initial condition in (6.11). For $t \ge T$ and we can define $u_1(t+T,x) = u_1(T-t,x)$ for $t \ge 0$ and $x \in \mathbb{R}^n$. Then the extend $u_1(t,x)$ defined on $\mathbb{R}^{n+1}$ with compact support on $\{-T \le t \le 3T\}$ and satisfy

$$u_1 \in L^2(\mathbb{R};H^1(\mathbb{R}^n)) \quad \text{and} \quad \partial_t u_1 \in L^2(\mathbb{R};H^{-1}(\mathbb{R}^n)).$$

By Lemma 6.6, we can see that the function $u_1 \in \mathbf{H}^s(\mathbb{R}^{n+1})$.

Step 2. Artificial solutions.

Let us set an auxiliary function $\widetilde{u}_1 = \widetilde{u}_1(t,x,y)$, so that $\widetilde{u}_1$ is a constant with respect to the y-direction. In other words, $\widetilde{u}_1(t,x,y) := \widetilde{u}_1(t,x)$, where u_1 is a solution to (6.11). With the aid of (6.1), it is not hard to check that $\widetilde{u}_1$ is a solution to

$$\begin{cases} y^{1-2s}\partial_t \widetilde{u}_1 - \nabla_{x,y}\cdot\left(y^{1-2s}\widetilde{\sigma}\nabla_{x,y}\widetilde{u}_1\right) & \\ \qquad = y^{1-2s}\left(\partial_t u_1 - \nabla\cdot\sigma\nabla u_1\right) = y^{1-2s}\psi & \text{in } \mathbb{R}^n_T\times(0,\infty), \\ \widetilde{u}_1(t,x,y)=0 & \text{for } (x,y)\in\mathbb{R}^{n+1}_+,\ t\leq -T, \end{cases} \tag{6.13}$$

where we utilized the equation (6.11). Let us consider the operator $\mathcal{E}_s$ satisfying

$$\begin{aligned} \mathcal{E}_s : \mathbf{H}^s(\mathbb{R}^{n+1}) &\to \mathcal{L}^{1,2}(\mathbb{R}^{n+2}_+; y^{1-2s}dtdxdy), \\ u_1 &\mapsto \mathcal{E}_s u_1, \end{aligned}$$

which stands for the extension operator. Then $\mathcal{E}_s u_1 = \left(\mathcal{E}_s u_1\right)(t,x,y)$ is a solution to the extension problem

$$\begin{cases} y^{1-2s}\partial_t \mathcal{E}_s u_1 - \nabla_{x,y}\cdot\left(y^{1-2s}\widetilde{\sigma}\nabla_{x,y}\mathcal{E}_s u_1\right) = 0 & \text{in } \mathbb{R}^{n+2}_+, \\ \left(\mathcal{E}_s u_1\right)(t,x,0) = u_1(t,x) & \text{for } (t,x)\in\mathbb{R}^{n+1}. \end{cases} \tag{6.14}$$

Thus, via Proposition 3.1 (b), we have $\lim_{y\to 0} y^{1-2s}\partial_y \mathcal{E}_s u_1 \in \mathbf{H}^{-s}(\mathbb{R}\times\Omega)$. In further, the same computation as shown in (6.1), we can see that $\left(\mathcal{E}_s u_1\right)(t,x,y) = 0$ for $(x,y)\in\mathbb{R}^{n+1}_+$, and $t\leq -T$. Combined with (6.14), one can derive

$$\begin{cases} y^{1-2s}\partial_t \mathcal{E}_s u_1 - \nabla_{x,y}\cdot\left(y^{1-2s}\widetilde{\sigma}\nabla_{x,y}\mathcal{E}_s u_1\right) = 0 & \text{in } \mathbb{R}^{n+2}_+, \\ \left(\mathcal{E}_s u_1\right)(t,x,0) = u_1(t,x) & \text{for } (t,x)\in\mathbb{R}^{n+1}, \\ \left(\mathcal{E}_s u_1\right)(t,x,y) = 0 & \text{for } (x,y)\in\mathbb{R}^{n+1}_+,\ t\leq -T. \end{cases} \tag{6.15}$$

On the other hand, we consider the problem

$$\begin{cases} y^{1-2s}\partial_t u_2 - \nabla_{x,y}\cdot\left(y^{1-2s}\widetilde{\sigma}\nabla_{x,y}u_2\right) = 0 & \text{in } \mathbb{R}^n_T\times(0,\infty), \\ u_2 = 0 & \text{on } (\Omega_e)_T\times\{0\}, \\ \lim_{y\to 0} y^{1-2s}\partial_y u_2 = \lim_{y\to 0} y^{1-2s}\partial_y \mathcal{E}_s u_1 & \text{on } \Omega_T\times\{0\}, \\ u_2(t,x,y) = 0 & \text{for } (x,y)\in\mathbb{R}^{n+1}_+,\ t\leq -T. \end{cases} \tag{6.16}$$

Let us utilize this fact to discuss the solvability of (6.16) by considering that

$$\begin{aligned} &\int_{\mathbb{R}^{n+2}_+} y^{1-2s}\left(-u_2\partial_t\varphi + \widetilde{\sigma}\nabla_{x,y}u_2\cdot\nabla_{x,y}\varphi\right)dtdxdy \\ &= \int_{\mathbb{R}\times\Omega} \varphi(t,x,0)\lim_{y\to 0} y^{1-2s}\partial_y \mathcal{E}_s u_1\, dtdx, \end{aligned}$$

for any $\varphi \in \mathcal{L}^{1,2}_{c,0}(\mathbb{R}^{n+2}_+, y^{1-2s}dtdxdy)$. As we showed before, it is known that $\lim_{y\to 0} y^{1-2s}\partial_y \mathcal{E}_s u_1 \in \mathbf{H}^{-s}(\mathbb{R}^{n+1})$ so that $\lim_{y\to 0} y^{1-2s}\partial_y \mathcal{E}_s u_1\big|_{\mathbb{R}\times\Omega} \in \mathbf{H}^{-s}(\mathbb{R}\times\Omega)$. So, $\varphi(t,x,0)\in\mathbf{H}^s(\mathbb{R}^{n+1})$ is enough to prove the existence of u_2.

Reviewing that from the trace characterization for fractional Sobolev spaces (for example, see [Tyu14]), it is known that $H^s(\mathbb{R}^n)$ can be viewed as the trace space of $H^1(\mathbb{R}^{n+1}_+, y^{1-2s}dxdy)$ with respect to the space-variable on $\partial\mathbb{R}^{n+1}_+$, for any $n\in\mathbb{N}$. Hence, given any $\varphi\in\mathcal{L}^{1,2}_{c,0}(\mathbb{R}^{n+2}_+; y^{1-2s}dtdxdy)$, one can see that

$$\|\varphi(t,\cdot,0)\|_{H^s(\mathbb{R}^n)} \lesssim \|\varphi(t,\cdot,y)\|_{H^1(\mathbb{R}^{n+1}_+; y^{1-2s}dxdy)}, \tag{6.17}$$

for any fixed $t \in \mathbb{R}$, where $H^s(\mathbb{R}^n)$ denotes the fractional Sobolev space given by (2.6) for $s \in (0,1)$. Taking the $H^{s/2}(\mathbb{R})$ with respect to the time-variable for the inequality (6.17), we can have

$$\|\varphi(t,x,0)\|_{H^{s/2}(\mathbb{R};H^s(\mathbb{R}^n))} \lesssim \|\varphi(t,x,y)\|_{H^{s/2}(\mathbb{R};H^1(\mathbb{R}^{n+1}_+;y^{1-2s}dxdy))}.$$

Now, since the identity (2.5) holds, with the definition (2.4) at hand, then we can get

$$\|\varphi(t,x,0)\|_{\mathbf{H}^s(\mathbb{R}^{n+1})} \approx \|\varphi(t,x,0)\|_{H^{s/2}(\mathbb{R};H^s(\mathbb{R}^n))},$$

which implies that $\varphi(t,x,0) \in \mathbf{H}^s(\mathbb{R}^{n+1})$ if $\varphi \in \mathcal{L}^{1,2}_{c,0}(\mathbb{R}^{n+2}_+, y^{1-2s}dtdxdy)$. Hence, by the preceding discussions, one has the trace relation that

$$\mathcal{L}^{1,2}_{c,0}(\mathbb{R}^{n+2}_+, y^{1-2s}dtdxdy) \hookrightarrow \mathbf{H}^s(\mathbb{R}^{n+1}),$$

so that

$$\mathcal{L}^{1,2}_{c,0}(\mathbb{R}^{n+2}_+, y^{1-2s}dtdxdy) \ni \varphi \mapsto \int_{\mathbb{R}\times\Omega} \varphi \lim_{y\to 0} y^{1-2s}\partial_y \mathcal{E}_s u_1 dtdx$$

is bounded, where the right hand side in the above relation is viewed as the duality pairing between $\mathbf{H}^{-s}(\mathbb{R}\times\Omega)$ and $\mathbf{H}^s(\mathbb{R}\times\Omega)$. Thus, by using the standard variational method, we can also summarize that the problem (6.16) possesses a unique solution $u_2 \in \mathcal{L}^{1,2}(\mathbb{R}^n_T \times (0,\infty), y^{1-2s}dtdxdy)$. It is not hard to extend $u_2(t,x,y)$ for $t > T$ such that $u_2 \in \mathcal{L}^{1,2}(\mathbb{R}^{n+2}_+, y^{1-2s}dtdxdy)$.

Now, let $\varphi \in \mathcal{L}^{1,2}_{c,0}(\mathbb{R}^{n+2}_+; y^{1-2s}dtdxdy)$ be an arbitrary test function, then we have

$$\begin{aligned}
&\int_{\mathbb{R}^{n+2}_+} y^{1-2s}\left(-\mathcal{E}_s u_1 \partial_t \varphi + \widetilde{\sigma}\nabla_{x,y}\mathcal{E}_s u_1 \cdot \nabla_{x,y}\varphi\right) dtdxdy \\
=& \int_{\mathbb{R}^{n+1}} \varphi(t,x,0) \lim_{y\to 0} y^{1-2s}\partial_y \mathcal{E}_s u_1\, dtdx \\
=& \int_{\mathbb{R}\times\Omega} \varphi(t,x,0) \lim_{y\to 0} y^{1-2s}\partial_y \mathcal{E}_s u_1\, dtdx,
\end{aligned}$$

and

$$\begin{aligned}
&\int_{\mathbb{R}^{n+2}_+} y^{1-2s}\left(-u_2 \partial_t \varphi + \widetilde{\sigma}\nabla_{x,y} u_2 \cdot \nabla_{x,y}\varphi\right) dtdxdy \\
=& \int_{\mathbb{R}\times\Omega} \varphi(t,x,0) \lim_{y\to 0} y^{1-2s}\partial_y \mathcal{E}_s u_1\, dtdx.
\end{aligned}$$

Step 3. Construction of solutions.

We want to show

$$w := \widetilde{u}_1 - \mathcal{E}_s u_1 + u_2 \in \mathcal{L}^{1,2}(\mathbb{R}^n_T \times (0,\infty), y^{1-2s}dtdxdy) \tag{6.18}$$

is a solution to the equation (6.5).

In fact, there also holds

$$\begin{aligned}
&\int_{\mathbb{R}^{n+2}_+} y^{1-2s}\left(\partial_t \widetilde{u}_1 \varphi + \widetilde{\sigma}\nabla_{x,y}\widetilde{u}_1 \cdot \nabla_{x,y}\varphi\right) dtdxdy \\
=& \int_{\mathbb{R}^{n+2}_+} y^{1-2s}\left(\partial_t u_1 \varphi + \sigma(x)\nabla u_1 \cdot \nabla\varphi\right) dtdxdy \\
=& \int_{\mathbb{R}^{n+2}_+} \left\{\partial_t u_1 \left(\int_0^\infty y^{1-2s}\varphi\, dy\right) + \sigma(x)\nabla u_1 \cdot \nabla\left(\int_0^\infty y^{1-2s}\varphi\, dy\right)\right\} dtdx \\
=& \left\langle \psi(\cdot,\cdot), \int_0^\infty y^{1-2s}\varphi(\cdot,\cdot,y)\, dy \right\rangle_{L^2(-T,T;H^{-1}(\mathbb{R}^n)),L^2(-T,T;H^1(\mathbb{R}^n))},
\end{aligned} \tag{6.19}$$

where we used the weak formulation of (6.11) as $\tau = T$. Finally, in order to make (6.19) rigorously, we will verify that $\int_0^\infty y^{1-2s}\varphi(\cdot,\cdot,y)\,dy \in L^2(\mathbb{R}; H^1(\mathbb{R}^n))$. By the compact support of φ, for $M > 0$ sufficiently large, there holds

$$\begin{aligned}
&\int_{\mathbb{R}^{n+1}} \left| \nabla \left(\int_0^\infty y^{1-2s}\varphi\,dy \right) \right|^2 dtdx \\
&= \int_{\mathbb{R}^{n+1}} \left| \int_0^M y^{1-2s}\nabla\varphi\,dy \right|^2 dtdx \\
&\le \int_{\mathbb{R}^{n+1}} \left(\int_0^M y^{1-2s}\,dy \right) \left(\int_0^M y^{1-2s}|\nabla\varphi|^2\,dy \right) dtdx \\
&\le \frac{M^{2-2s}}{2-2s} \int_{\mathbb{R}^{n+2}_+} y^{1-2s}|\nabla\varphi|^2\,dtdxdy \\
&< \infty,
\end{aligned}$$

and similar estimates hold for $\int_{\mathbb{R}^{n+1}} \left|\int_0^\infty y^{1-2s}\varphi\,dy\right|^2 dtdx$. This demonstrates that the function $\int_0^\infty y^{1-2s}\varphi\,dy \in L^2(\mathbb{R}; H^1(\mathbb{R}^n))$ can be regarded as a test function in (6.19). Hence, one can conclude that $w := \widetilde{u}_1 - \mathcal{E}_s u_1 + u_2 \in \mathcal{L}^{1,2}(\mathbb{R}^{n+2}_+, y^{1-2s}dtdxdy)$ is a solution to (6.5) as desired. This completes the proof. $\square$

Remark 6.7. *Let us emphasize that*

(a) In fact, Lemma 6.4 also implies the existence of solutions to the following adjoint problem

(6.20)
$$\begin{cases}
y^{1-2s}\partial_t\widetilde{w} + \nabla_{x,y}\cdot\left(y^{1-2s}\widetilde{\sigma}\nabla_{x,y}\right)\widetilde{w} = y^{1-2s}\psi & \text{in } \mathbb{R}^n_T \times (0,\infty), \\
\widetilde{w} = 0 & \text{in } (\Omega_e)_T \times \{0\}, \\
\lim_{y\to 0} y^{1-2s}\partial_y\widetilde{w} = 0 & \text{in } \Omega_T \times \{0\}, \\
\widetilde{w}(t,x,y) = 0 & \text{for } t \ge T \text{ and } (x,y) \in \mathbb{R}^{n+1}_+.
\end{cases}$$

The result can be derived by simply taking the time reversing change of variables $t \mapsto -t$ for $t \in [-T,T]$, i.e., if $w(t,x.y)$ is a solution to (6.5) if and only if $\widetilde{w}(t,x,y) := w(-t,x,y)$ is a solution to the backward equation (6.20).

(b) From the Step 1 in the proof of Lemma 6.4, we know that if u is a parabolic solution satisfying (6.8), then u will satisfy the regularity for the nonlocal parabolic equation, i.e., $u \in \mathbf{H}^s(\mathbb{R}^{n+1})$. This also gives us some hints to relate nonlocal and local parabolic equations.

Proof of Proposition 6.1. Let us prove the density of $\mathcal{V} \subset L^2(-T,T;H^1(\Omega))$ in $\mathcal{D}$, where $\mathcal{D}$ is defined by

$$\begin{aligned}
\mathcal{D} := \{v \in L^2(-T,T;H^1(\Omega)) : (\partial_t - \nabla\cdot\sigma\nabla)\,v = 0 \text{ in } \Omega_T, \\
\text{and } v(-T,x) = 0 \text{ in } \Omega\}.
\end{aligned} \tag{6.21}$$

By the Hahn-Banach theorem, it is enough to show that if $\psi \in L^2(-T,T;\widetilde{H}^{-1}(\Omega))$ with $\psi(v_f) = 0$ for all $f \in C_c^\infty(W_T)$, then there holds $\psi(v) = 0$ for all $v \in \mathcal{D}$. In the rest of the proof, we adopt the notation that $\widetilde{u}_f$ to denote solutions of (1.6) with $\widetilde{u}_f(t,x,0) = u_f(t,x)$ and $v_f = \int_0^\infty y^{1-2s}\widetilde{u}_f\,dy$, where $u_f \in \mathbf{H}^s(\mathbb{R}^{n+1})$ is the solution of (1.3) with the exterior data f.

Step 1. Smooth cutoffs.

Let us consider the auxiliary problem (6.5), which is solvable by Lemma 6.4. We introduce two cutoff functions, one for x-variable and the other for y-variable. On one hand, let $\zeta_k(y) = \zeta(y/k)$ for $k \in \mathbb{N}$, where $\zeta \in C_c^\infty([0,2])$ is a smooth cutoff function satisfying $\zeta \equiv 1$ in a neighborhood of $y = 0$ and $\int_0^\infty y^{1-2s}\zeta(y)\,dy = 1$. In particular, the change of variable yields

$$k^{2s-2}\int_0^\infty y^{1-2s}\zeta_k(y)\,dy = \int_0^\infty y^{1-2s}\zeta(y)\,dy = 1.$$

On the other hand, we may assume that $\overline{\Omega} \cap \overline{W} \subset B_R$, for sufficiently large $R > 0$ as before, where B_R is the ball in $\mathbb{R}^n$ of radius R and center at the origin. Let $\eta_k(x) := \eta(x/k)$ for $k \in \mathbb{N}$, where $\eta \in C_c^\infty(B_{2R})$ is a radial cutoff function such that $\eta = 1$ in B_R. With these smooth cutoff functions at hand, we can see that the

$$\widetilde{u}_{f,k}(t,x,y) := \widetilde{u}_f(t,x,y)\eta_k(x)\zeta_k(y) \in \mathcal{L}_c^{1,2}(\mathbb{R}^{n+2}_+; y^{1-2s}dtdxdy),$$

for any $f \in C_c^\infty(W_T)$.

As in Remark 6.7, we have constructed that the function $\widetilde{w}$ is a solution to the backward equation (6.20). Recall that $\psi \in L^2(-T,T;\widetilde{H}^{-1}(\Omega))$, then

$$\begin{aligned}
0 &= \psi(v_f) \\
&= \lim_{k\to\infty}\left\langle \psi, \int_0^\infty y^{1-2s}\widetilde{u}_{f,k}\,dy\right\rangle_{L^2(-T,T;\widetilde{H}^{-1}(\Omega)),L^2(-T,T;H^1(\Omega))} \\
&= \underbrace{\lim_{k\to\infty}\int_{\mathbb{R}^{n+2}_+}\left[y^{1-2s}\partial_t\widetilde{w} + \nabla_{x,y}\cdot\left(y^{1-2s}\widetilde{\sigma}\nabla_{x,y}\widetilde{w}\right)\right]\widetilde{u}_{f,k}\,dtdxdy}_{\text{By (6.20)}} \\
&= \lim_{k\to\infty}\Big\{\int_{\mathbb{R}^{n+2}_+} y^{1-2s}\widetilde{u}_{f,k}\partial_t\widetilde{w}\,dtdxdy + \int_{\mathbb{R}^{n+1}}\widetilde{u}_{f,k}(t,x,0)\lim_{y\to 0}y^{1-2s}\partial_y\widetilde{w}\,dtdx \\
&\qquad - \lim_{k\to\infty}\int_{\mathbb{R}^{n+2}_+} y^{1-2s}\widetilde{\sigma}\nabla_{x,y}\widetilde{w}\cdot\nabla_{x,y}\widetilde{u}_{f,k}\,dtdxdy\Big\} \\
&= \int_{W_T} f\lim_{y\to 0}y^{1-2s}\partial_y\widetilde{w}\,dtdx + \lim_{k\to\infty}I_k,
\end{aligned} \tag{6.22}$$

where

$$\begin{aligned}
I_k &:= -\int_{\mathbb{R}^{n+2}_+} y^{1-2s}\widetilde{\sigma}\nabla_{x,y}\widetilde{w}\cdot\nabla_{x,y}\widetilde{u}_{f,k}\,dtdxdy + \int_{\mathbb{R}^{n+2}_+} y^{1-2s}\widetilde{u}_{f,k}\partial_t\widetilde{w}\,dtdxdy \\
&= -\int_{\mathbb{R}^{n+2}_+} y^{1-2s}\widetilde{\sigma}\nabla_{x,y}\widetilde{u}_{f,k}\cdot\nabla_{x,y}\widetilde{w}\,dtdxdy + \int_{\mathbb{R}^{n+2}_+} y^{1-2s}\widetilde{u}_{f,k}\partial_t\widetilde{w}\,dtdxdy.
\end{aligned}$$

Next, integration by parts in the time-variable yields that

$$\begin{aligned}
I_k = &-\int_{\mathbb{R}^{n+2}_+} y^{1-2s}\eta_k\zeta_k\widetilde{\sigma}\left(\nabla_{x,y}\widetilde{u}_f\right)\cdot\nabla_{x,y}\widetilde{w}\,dtdxdy \\
&-\int_{\mathbb{R}^{n+2}_+} y^{1-2s}\widetilde{u}_f\widetilde{\sigma}\nabla_{x,y}\left(\eta_k\zeta_k\right)\cdot\nabla_{x,y}\widetilde{w}\,dtdxdy - \underbrace{\int_{\mathbb{R}^{n+2}_+} y^{1-2s}\widetilde{w}\left(\partial_t\widetilde{u}_{f,k}\right)dtdxdy}_{\text{By } \widetilde{w}(T,x,y)=\widetilde{u}_{f,k}(-T,x,y)=0}.
\end{aligned} \tag{6.23}$$

In (6.22), we also point out that the limit holds

$$\int_0^\infty y^{1-2s}\widetilde{u}_{f,k}(t,x,y)\,dy \to \int_0^\infty y^{1-2s}\widetilde{u}_f(t,x,y)\,dy$$

in $L^2(\mathbb{R}; H^1(\Omega))$ as $k \to \infty$. In fact, there holds

$$
\begin{aligned}
&\left\| \int_0^\infty y^{1-2s} \zeta_k(y) \eta_k(x) \widetilde{u}_f(t,x,y)\, dy - \int_0^\infty y^{1-2s} \widetilde{u}_f(t,x,y)\, dy \right\|_{L^2(\mathbb{R};H^1(\Omega))} \\
&\leq \left\| \int_k^\infty y^{1-2s} \left(|\widetilde{u}_f| + |\nabla \widetilde{u}_f| \right) dy \right\|_{L^2(\mathbb{R};L^2(\Omega))} \\
&\lesssim \int_k^\infty y^{1-2s} \left(\|\widetilde{u}_f(\cdot,\cdot,y)\|_{L^2(\mathbb{R};L^\infty(\mathbb{R}^n))} + \|\nabla \widetilde{u}_f(\cdot,\cdot,y)\|_{L^2(\mathbb{R};L^\infty(\mathbb{R}^n))} \right) dy \\
&\lesssim \underbrace{\int_k^\infty y^{1-2s-n} \|u_f(\cdot,\cdot)\|_{L^2(\mathbb{R};L^1(\mathbb{R}^n))}\, dy}_{\text{By (4.2) and for } k \text{ large}} \\
&\lesssim \underbrace{k^{2-2s-n} \|f\|_{\widetilde{\mathbf{H}}^s(W_T)}}_{\text{By using (2.7)}} \\
&\to 0,
\end{aligned}
$$

as $k \to \infty$, for $n \geq 2$, where we used $u_f(t,x) \in \mathbf{H}^s(\mathbb{R}^{n+1})$ has compact support in $\overline{(\Omega \cup W)_T}$. We next analyze the limit of I_k as $k \to \infty$.

Step 2. $\lim\limits_{k \to \infty} I_k = 0$.

We want to claim that $\lim_{k\to\infty} I_k = 0$. The argument is similar to the proof of [CGRU23, Proposition 3.1], we provide detailed derivation for the sake of completeness. Integrating by parts on the right hand in for I_k in (6.23), one has

(6.24)

$$
\begin{aligned}
I_k &= -\int_{\mathbb{R}^{n+2}_+} y^{1-2s} \widetilde{w} \left(\partial_t \widetilde{u}_{f,k} \right) dtdxdy - \int_{\mathbb{R}^{n+2}_+} y^{1-2s} \widetilde{\sigma} \nabla_{x,y} \widetilde{u}_f \cdot \nabla_{x,y} \left(\eta_k \zeta_k \widetilde{w} \right) dtdxdy \\
&\quad + \int_{\mathbb{R}^{n+2}_+} y^{1-2s} \widetilde{w} \widetilde{\sigma} \nabla_{x,y} \widetilde{u}_f \cdot \nabla_{x,y} \left(\eta_k \zeta_k \right) dtdxdy \\
&\quad + \int_{\mathbb{R}^{n+2}_+} \widetilde{w} \nabla_{x,y} \cdot \left(y^{1-2s} \widetilde{u}_f \widetilde{\sigma} \nabla_{x,y} \left(\eta_k \zeta_k \right) \right) dtdxdy \\
&= \underbrace{\int_{\mathbb{R}^{n+1}} \eta_k \widetilde{w}(t,x,0) \lim_{y \to 0} y^{1-2s} \partial_y \widetilde{u}_f\, dtdx}_{(*)} \\
&\quad + \underbrace{\int_{\mathbb{R}^{n+2}_+} \eta_k \zeta_k \widetilde{w} \left[\nabla_{x,y} \cdot \left(y^{1-2s} \widetilde{\sigma} \nabla_{x,y} \widetilde{u}_f \right) - y^{1-2s} \partial_t \widetilde{u}_f \right] dtdxdy}_{=0, \text{ since } \widetilde{u}_f \text{ is a solution to (1.6)}} \\
&\quad + 2 \int_{\mathbb{R}^{n+2}_+} y^{1-2s} \widetilde{w} \widetilde{\sigma} \nabla_{x,y} \left(\eta_k \zeta_k \right) \cdot \nabla_{x,y} \widetilde{u}_f\, dtdxdy \\
&\quad + \int_{\mathbb{R}^{n+2}_+} \widetilde{w} y^{1-2s} \widetilde{u}_f \widetilde{\mathcal{L}} \left(\eta_k \zeta_k \right) dtdxdy \\
&\quad + (1-2s) \int_{\mathbb{R}^{n+2}_+} y^{-2s} \widetilde{w} \widetilde{u}_f \eta_k \partial_y \zeta_k\, dtdxdy \\
&= \int_0^{2k} \int_{B_{2Rk}} \int_{\mathbb{R}} y^{1-2s} \widetilde{w} \\
&\qquad \cdot \left\{ 2 \widetilde{\sigma} \nabla_{x,y} \left(\eta_k \zeta_k \right) \cdot \nabla_{x,y} \widetilde{u}_f + \left(\widetilde{\mathcal{L}} \left(\eta_k \zeta_k \right) + \frac{1-2s}{y} \eta_k \partial_y \zeta_k \right) \widetilde{u}_f \right\} dtdxdy,
\end{aligned}
$$

where we used the notation $\widetilde{\mathcal{L}} := \nabla_{x,y} \cdot \widetilde{\sigma} \nabla_{x,y}$. Here we used the Lemma 6.4 (or Remark 6.7) to guarantee the integral $(*)$ in (6.24) is well-defined

$$
\begin{aligned}
&\left| \int_{\mathbb{R}^{n+1}} \eta_k \widetilde{w}(t,x,0) \left(\lim_{y\to 0} y^{1-2s} \partial_y \widetilde{u}_f \right) dtdx \right| \\
&\lesssim \|\widetilde{w}(\cdot,\cdot,0)\|_{\mathbf{H}^s(\mathbb{R}^{n+1})} \left\| (\partial_t - \nabla\cdot\sigma(x)\nabla)^s \widetilde{u}_f(t,x,0) \right\|_{\mathbf{H}^{-s}(\mathbb{R}^{n+1})} < \infty.
\end{aligned}
$$

In (6.24), the term $(*) = 0$ since $\operatorname{supp}(\widetilde{w}) \bigcap \operatorname{supp}\left(\lim_{y\to 0} y^{1-2s}\partial_y \widetilde{u}_f\right) = \emptyset$.

We next estimate I_k. Let $A_k := B_{2Rk} \setminus B_{Rk}$, where $R > 0$ is the radius given from previous steps. With the uniform boundedness of $\widetilde{\sigma}$ and $\nabla\widetilde{\sigma}$ at hand, one can derive

$$
\begin{aligned}
|I_k| &\lesssim \int_0^{2k} \int_{B_{2Rk}} \int_{\mathbb{R}} y^{1-2s} |\widetilde{w}| \left|\nabla_{x,y}(\eta_k \zeta_k)\right| |\nabla_{x,y}\widetilde{u}_f| \, dtdxdy \\
&\quad + \int_0^{2k} \int_{B_{2Rk}} \int_{\mathbb{R}} y^{1-2s} |\widetilde{w}| \left(\widetilde{\mathcal{L}}(\eta_k\zeta_k) + y^{-1}|\eta_k| |\partial_y \zeta_k| \right) |\widetilde{u}_f| \, dtdxdy \\
&\lesssim \int_0^{2k} \int_{B_{2Rk}} \int_{\mathbb{R}} y^{1-2s} |\widetilde{w}| \left(|\nabla \eta_k| + |\partial_y \zeta_k| \right) |\nabla_{x,y}\widetilde{u}_f| \, dtdxdy \\
&\quad + \int_0^{2k} \int_{B_{2Rk}} \int_{\mathbb{R}} y^{1-2s} |\widetilde{w}| \left(|\nabla\cdot\sigma\nabla\eta_k| + \left|\partial_y^2 \zeta_k\right|^2 + y^{-1}|\partial_y\zeta_k| \right) |\widetilde{u}_f| \, dtdxdy \\
&\lesssim I_{1,k}(\widetilde{w}) + I_{2,k}(\widetilde{w}),
\end{aligned}
$$

where

$$
I_{1,k}(\widetilde{w}) := k^{-1} \int_k^{2k} \int_{\mathbb{R}^{n+1}} y^{1-2s} |\widetilde{w}| \left(|\nabla_{x,y}\widetilde{u}_f| + k^{-1}|\widetilde{u}_f| \right) dtdxdy,
$$

and

$$
I_{2,k}(\widetilde{w}) := k^{-1} \int_0^{2k} \int_{A_k} \int_{\mathbb{R}} y^{1-2s} |\widetilde{w}| \left(|\nabla_{x,y}\widetilde{u}_f| + |\widetilde{u}_f| \right) dtdxdy,
$$

where we directly used the fact[3]

$$
|\nabla\cdot\sigma\nabla\eta_k| \lesssim k^{-1}
$$

in $I_{2,k}$, for sufficiently large $k \in \mathbb{N}$. We next estimate $I_{1,k}(\widetilde{w})$ and $I_{2,k}(\widetilde{w})$ separately.

Step 2a. Estimate for $I_{1,k}(\widetilde{w})$. Note that the function $\widetilde{w}$ is constructed from the solution $w = \widetilde{u}_1 - \mathcal{E}_s u_1 + u_2$ given by (6.18) (by reversing time $t \mapsto -t$), and we abuse the notation $\widetilde{w}$ as the form $\widetilde{u}_1 - \mathcal{E}_s u_1 + u_2$ (here we already replace $u_1(t,x)$and $u_2(t,x,y)$ by $u_1(-t,x)$ and $u_2(-t,x,y)$, respectively). Here $\widetilde{u}_1$, $\mathcal{E}_s u_1$ and u_2 are the solutions to (6.13), (6.16) and (6.15), respectively. Let us consider the bound

$$
\int_{\mathbb{R}^{n+1}} |\mathbf{w}(t,x,y)| \left| \nabla_{x,y}^{\ell} \widetilde{u}_f(t,x,y) \right| dtdx
$$

for $\ell = 0, 1$, where the function $\mathbf{w}$ could be any of the functions $\widetilde{u}_1$, $\mathcal{E}_s u_1$ or u_2. By Lemma 4.2 and Lemma 6.4, it is known that both functions $\mathcal{E}_s u_1$ and u_2 have decay in the y-direction, but the function $\widetilde{u}_1$ does not have such decay. Hence, we divide the proof into two parts: $\mathbf{w} = \widetilde{u}_1$ and $\mathbf{w} = \mathcal{E}_s u_1, u_2$.

[3]We want to emphasize that in the elliptic case, one needs to assume σ to be a positive constant matrix in the exterior domain Ω_e in order to obtain sufficiently decay estimates with respect to the parameter k. We refer readers to [CGRU23, page 18] for slightly different arguments.

- For $\mathbf{w}=\widetilde{u}_1$, for $\ell=0,1$, by the Hölder's inequality, we have

$$
\begin{aligned}
&\int_{\mathbb{R}^{n+1}} |\mathbf{w}(t,x,y)| \left|\nabla^\ell_{x,y}\widetilde{u}_f(t,x,y)\right| dtdx \\
&\leq \underbrace{\|\mathbf{w}(\cdot,\cdot,y)\|_{L^2(\mathbb{R}^{n+1})} \left\|\nabla^\ell_{x,y}\widetilde{u}_f(\cdot,\cdot,y)\right\|_{L^2(\mathbb{R}^{n+1})}}_{\text{By (6.12), } \mathbf{w}=u_1\in L^\infty(\mathbb{R};H^1(\mathbb{R}^n))} \\
&\lesssim \underbrace{y^{\frac{n}{2}-n-\ell}\|\mathbf{w}\|_{L^2(\mathbb{R}^{n+1})} \|u_f\|_{L^2(\mathbb{R};L^1(\mathbb{R}^n))}}_{\text{By (4.3)}}.
\end{aligned} \tag{6.25}
$$

By the Hölder's inequality so that $u_f\in \widetilde{\mathbf{H}}^s\left((\Omega\cup W)_T\right)\subset L^1\left((\Omega\cup W)_T\right)$ with

$$
\|u_f\|_{L^2(\mathbb{R};L^1(\mathbb{R}^n))}\lesssim \|f\|_{\widetilde{\mathbf{H}}^s(W_T)}. \tag{6.26}
$$

As a result, we can summarize

$$
\begin{aligned}
I_{1,k}(\widetilde{w}) &\lesssim k^{-1}\|\mathbf{w}\|_{L^\infty(\mathbb{R};L^2(\mathbb{R}^n))}\|f\|_{\widetilde{\mathbf{H}}^s(W_T)}\int_k^{2k} y^{1-\frac{n}{2}-\ell-2s}\,dy \\
&\lesssim k^{1-\frac{n}{2}-\ell-2s}\|\mathbf{w}\|_{L^\infty(\mathbb{R};L^2(\mathbb{R}^n))}\|f\|_{\widetilde{\mathbf{H}}^s(W_T)},
\end{aligned} \tag{6.27}
$$

for $\ell=0,1$, where we use $\mathbf{w}=u_1$ is uniform bounded in $L^\infty(\mathbb{R};L^2(\mathbb{R}^n))$. Via (6.27), one can see that $I_{1,k}\to 0$ as $k\to\infty$ as desired.

- For $\mathbf{w}=\mathcal{E}_s u_1, u_2$, we denote by $\mathbf{w}$ any of the functions $\mathcal{E}_s u_1, u_2$, and for $\ell\in\{0,1\}$. Then the decay estimate (4.3) for the function $\widetilde{u}_f$ implies that

$$
\begin{aligned}
&\int_{\mathbb{R}^{n+1}} |\mathbf{w}(t,x,y)| \left|\nabla^\ell_{x,y}\widetilde{u}_f(t,x,y)\right| dtdx \\
&\leq \|\mathbf{w}(\cdot,\cdot,y)\|_{L^2(\mathbb{R}^{n+1})} \left\|\nabla^\ell_{x,y}\widetilde{u}_f(\cdot,\cdot,y)\right\|_{L^2(\mathbb{R}^{n+1})} \\
&\lesssim \underbrace{y^{-\frac{n}{2}-\ell}\|\mathbf{w}(\cdot,\cdot,y)\|_{L^2(\mathbb{R}^{n+1})} \|u_f\|_{L^2(\mathbb{R};L^1(\mathbb{R}^n))}}_{\text{Applying (4.3) for } r=p=2 \text{ and } q=1},
\end{aligned}
$$

for $\ell=0,1$. Similar to the previous case, we also have $u_f\in\widetilde{\mathbf{H}}^s\left((\Omega\cup W)_T\right)\subset L^2\left((\Omega\cup W)_T\right)$ with (6.26), then the Hölder's inequality with respect to the y-direction yields that

$$
\begin{aligned}
&I_{1,k}(\widetilde{w}) \\
&\lesssim k^{-1}\int_k^{2k} y^{1-2s-\frac{n}{2}-\ell}\|\mathbf{w}(\cdot,\cdot,y)\|_{L^2(\mathbb{R}^{n+1})}\|u_f\|_{L^2(\mathbb{R};L^1(\mathbb{R}^n))}\,dy \\
&\lesssim k^{-1}\|\mathbf{w}\|_{\mathcal{L}^{1,2}(\mathbb{R}^{n+2}_+;y^{1-2s}dtdxdy)}\|f\|_{\widetilde{\mathbf{H}}^s(W_T)}\left(\int_k^{2k} y^{1-n-2\ell-2s}\,dy\right)^{1/2} \\
&\lesssim k^{-\frac{n}{2}-\ell-s}\|\mathbf{w}\|_{\mathcal{L}^{1,2}(\mathbb{R}^{n+2}_+;y^{1-2s}dtdxdy)}\|f\|_{\widetilde{\mathbf{H}}^s(W_T)},
\end{aligned} \tag{6.28}
$$

for $\ell=0,1$, where we have used that $\mathbf{w}$ is a solution of either (6.15) or (6.16) so that $\|\mathbf{w}\|_{\mathcal{L}^{1,2}(\mathbb{R}^{n+2}_+;y^{1-2s}dtdxdy)}<\infty$. Via (6.28), one can also see that $I_{1,k}(\widetilde{w})\to 0$ as $k\to\infty$ as desired.

Therefore, $I_{1,k}(\widetilde{w})\to 0$ as $k\to\infty$.

Step 2b. Estimate for $I_{2,k}(\widetilde{w})$. Recall (4.11), we have for $m\in[1,2]$ that

$$
\begin{aligned}
&\left\|\nabla^\ell_x\widetilde{u}(\cdot,x,y)\right\|_{L^m(\mathbb{R})} \\
&\lesssim y^{2s}\int_{\mathbb{R}^n}\left\{\left(|x-z|^2+y^2\right)^{-\frac{n+\ell}{2}-s}\|u(\cdot,z)\|_{L^m(\mathbb{R})}\right\}dz,
\end{aligned}
$$

for $\ell=0,1$. To proceed, we need to analyze the kernel function $\mathbf{K}_y(x):=\frac{y^{2s}}{(|x|^2+y^2)^{\frac{n}{2}+s}}$ as in Lemma 4.2. Via (4.11), one has

$$\begin{aligned}\|\widetilde{u}_f(t,\cdot,y)\|^r_{L^m(\mathbb{R};L^r(A_k))} &\lesssim y^{2sr}\int_{A_k}\left|\left(\widetilde{u}_f(t,\cdot,0)*\mathbf{K}_y\right)(x)\right|^r\,dx\\ &\lesssim y^{2sr}\int_{A_k}\left(\int_{\Omega\cup W}\frac{\|u_f(\cdot,z)\|_{L^m(\mathbb{R})}}{(|x-z|^2+y^2)^{\frac{n}{2}+s}}\,dz\right)^r dx\\ &\lesssim \frac{y^{2sr}k^n}{(k^2+y^2)^{(\frac{n}{2}+s)r}}\|u_f\|^r_{L^m(\mathbb{R};L^1(\mathbb{R}^n))}.\end{aligned}$$

Similarly, there holds

$$\left\|\nabla^\ell_{x,y}\widetilde{u}_f(\cdot,\cdot,y)\right\|_{L^m(\mathbb{R};L^r(A_k))}\lesssim\frac{y^{2s-\ell}k^{\frac{n}{r}}}{(k^2+y^2)^{\frac{n}{2}+s}}\|u_f\|_{L^m(\mathbb{R};L^1(\mathbb{R}^n))},\tag{6.29}$$

for $\ell=0,1$.

We use analogous strategy as in *Step 2a*, i.e., we derive the estimate by considering two cases:

- For $\mathbf{w}=\widetilde{u}_1$, for $\ell=0,1$, by using the Hölder's inequality as in the previous computation, we have

$$\begin{aligned}&\int_{A_k}\int_{\mathbb{R}}|\mathbf{w}(t,x,y)|\left|\nabla^\ell_{x,y}\widetilde{u}_f(t,x,y)\right|dtdx\\ &\le\|\mathbf{w}\|_{L^2(\mathbb{R}^{n+1})}\left\|\nabla^\ell_{x,y}\widetilde{u}_f(t,x,y)\right\|_{L^2(\mathbb{R};L^2(A_k))}\\ &\lesssim\underbrace{\frac{y^{2s-\ell}k^{\frac{n}{2}}}{(k^2+y^2)^{\frac{n}{2}+s}}\|\mathbf{w}\|_{L^2(\mathbb{R}^{n+1})}\|u_f\|_{L^2(\mathbb{R};L^1(\mathbb{R}^n))}}_{\text{By (6.29) as } m=1,r=2},\end{aligned}$$

  which infers

$$\begin{aligned}&k^{-1}\int_0^{2k}\int_{A_k}\int_{\mathbb{R}}y^{1-2s}|\mathbf{w}(t,x,y)|\left|\nabla^\ell_{x,y}\widetilde{u}_f(t,x,y)\right|dtdxdy\\ &\lesssim\|\mathbf{w}\|_{L^2(\mathbb{R}^{n+1})}\|u_f\|_{L^2(\mathbb{R};L^1(\mathbb{R}^n))}k^{-1+\frac{n}{2}}\int_0^{2k}\frac{y^{1-\ell}}{(k^2+y^2)^{\frac{n}{2}+s}}\,dy\\ &\lesssim\|\mathbf{w}\|_{L^2(\mathbb{R}^{n+1})}\|f\|_{\widetilde{\mathbf{H}}^s(W_T)}\underbrace{k^{-\frac{n}{2}-\ell-2s}\int_0^2\frac{\tau^{1-\ell}}{(1+\tau)^{\frac{n}{2}+s}}\,d\tau}_{\text{Let } y:=k\tau},\end{aligned}\tag{6.30}$$

  where we use $\mathbf{w}=u_1$ is uniform bounded in $L^2(\mathbb{R}^{n+1})$ and $u_f\in\mathbf{H}^s(\mathbb{R}^{n+1})$ is supported in a compact set. Now, since $\int_0^2\frac{\tau^{1-\ell}}{(1+\tau)^{\frac{n}{2}+s}}\,d\tau<\infty$ for $\ell=0,1$, via (6.30), one can see that $I_{2,k}(\widetilde{w})\to 0$ as $k\to\infty$ as we wish.
- For $\mathbf{w}=\mathcal{E}_s u_1,u_2$, we denote by $\mathbf{w}$ any of the functions $\mathcal{E}_s u_1,u_2$, and for $\ell\in\{0,1\}$. Similar to the previous case, the decay estimate (4.3) for the

function $\widetilde{u}_f$ infers

$$\begin{aligned}
&\int_{A_k}\int_{\mathbb{R}} |\mathbf{w}(t,x,y)| \left|\nabla^\ell_{x,y}\widetilde{u}_f(t,x,y)\right| dtdx \\
&\leq \|\mathbf{w}(\cdot,\cdot,y)\|_{L^2(\mathbb{R}^{n+1})} \left\|\nabla^\ell_{x,y}\widetilde{u}_f(t,x,y)\right\|_{L^2(\mathbb{R};L^2(A_k))} \\
&\lesssim \underbrace{\frac{y^{2s-\ell}k^{\frac{n}{2}}}{(k^2+y^2)^{\frac{n}{2}+s}} \|\mathbf{w}(\cdot,\cdot,y)\|_{L^2(\mathbb{R}^{n+1})} \|u_f\|_{L^2(\mathbb{R};L^1(\mathbb{R}^n))}}_{\text{By (4.3) for } \mathbf{w} \text{ and (6.29) for } \widetilde{u}_f \text{ as } m=r=p=2,\ q=1} \\
&\lesssim \underbrace{\frac{y^{2s-\ell}k^{\frac{n}{2}}}{(k^2+y^2)^{\frac{n}{2}+s}} \|\mathbf{w}(\cdot,\cdot,y)\|_{L^2(\mathbb{R}^{n+1})} \|f\|_{\widetilde{\mathbf{H}}^s(W_T)}}_{\text{By using (2.7)}},
\end{aligned}$$

for $\ell=0,1$. In addition, the Hölder's inequality yields that

(6.31)
$$\begin{aligned}
&k^{-1}\int_0^{2k}\int_{A_k}\int_{\mathbb{R}} y^{1-2s} |\mathbf{w}(t,x,y)| \left|\nabla^\ell_{x,y}\widetilde{u}_f(t,x,y)\right| dtdxdy \\
&\lesssim \|\mathbf{w}\|_{\mathcal{L}^{1,2}(\mathbb{R}^{n+2}_+;y^{1-2s}dtdxdy)} \|f\|_{\widetilde{\mathbf{H}}^s(W_T)} k^{-1+\frac{n}{2}} \left(\int_0^{2k} \frac{y^{1+2s-2\ell}}{(k^2+y^2)^{n+2s}}\, dy\right)^{1/2} \\
&\lesssim \|\mathbf{w}\|_{\mathcal{L}^{1,2}(\mathbb{R}^{n+2}_+;y^{1-2s}dtdxdy)} \|f\|_{\widetilde{\mathbf{H}}^s(W_T)} \underbrace{k^{-\frac{1}{2}-\frac{n}{2}-s-\ell}\left(\int_0^{2} \frac{\tau^{1+2s-2\ell}}{(1+\tau^2)^{n+2s}}\, d\tau\right)^{1/2}}_{\text{Let } y:=k\tau},
\end{aligned}$$

where we used $u_f \in \widetilde{\mathbf{H}}^s((\Omega\cup W)_T) \subset L^1((\Omega\cup W)_T)$ and (6.26). Now, since $\int_0^2 \frac{\tau^{1+2s-2\ell}}{(1+\tau^2)^{n+2s}}\, d\tau < \infty$, via (6.31), we have that $I_{2,k}(\widetilde{w}) \to 0$ as $k\to\infty$ as we want.

Therefore, $I_{2,k}(\widetilde{w}) \to 0$ as $k\to\infty$.

In summary, one can conclude that $\lim_{k\to\infty} I_k = 0$. With (6.22) at hand, one has

$$\int_{W_T} f \lim_{y\to 0} y^{1-2s}\partial_y \widetilde{w}\, dtdx = 0.$$

By arbitrary choice of $f\in C^\infty_c(W_T)$, we must have $\lim_{y\to 0} y^{1-2s}\partial_y\widetilde{w} = 0$ in $W_T\times\{0\}$. Moreover, since $\widetilde{w}=0$ in $W_T\times\{0\}$ as well, the unique property for second order parabolic equations (see [Sog90, Corollary 1.2] for instance) yields that $\widetilde{w}\equiv 0$ in $(\Omega_e)_T\times(0,\infty)$. In particular, one has $\widetilde{w}|_{(\partial\Omega)_T\times(0,\infty)} = \widetilde{\sigma}\nabla_{x,y}\widetilde{w}|_{(\partial\Omega)_T\times(0,\infty)} = 0$ in the sense of distribution, and $\lim_{y\to 0} y^{1-2s}\partial_y\widetilde{w} = 0$ in $\mathbb{R}^{n+1}\times\{0\}$.

Let us review auxiliary functions constructed in [CGRU23, Proposition 3.1] that make it convenient for readers. As shown in the proof of [CGRU23, Proposition 3.1], we recall an additional cutoff function to avoid boundary contributions. To this end, let $\mu:[0,1]\to[0,1]$ be a smooth function on $[0,1]$ with $\mu(0)=0$, $\mu(1)=1$. Moreover, one can assume that

$$\int_0^1 \mu(y)\, dy = \frac{1}{2} \quad \text{and} \quad \left|\partial_y^\ell \mu(y)\right| \leq C, \quad y\in(0,1)$$

for $\ell=0,1,2$, where $C>1$ is a constant.

Given $b \in (0,1)$, let $\gamma_b : (-\infty,\infty) \to (0,b)$ be a smooth function defined by

$$\gamma_b = \begin{cases} 0, & \text{if } y < 0, \\ b\mu(y), & \text{if } y \in [0,1], \\ b, & \text{if } y \in [1, \frac{1}{1-b}], \\ b\mu(\frac{2-b}{1-b} - y), & \text{if } y \in [\frac{1}{1-b}, \frac{2-b}{1-b}], \\ 0, & \text{if } y > \frac{2-b}{1-b} = \frac{1}{1-b} + 1. \end{cases}$$

From the construction, it is easy to see that

$$\int_0^\infty \gamma_b(y)\,dy = \frac{b}{1-b} \quad \text{and} \quad \left|\partial_y^\ell \gamma_b\right| \le Cb,$$

for $\ell = 0,1,2$, where $C > 1$ is a constant independent of $b \in (0,1)$. Consider

$$J_{b,k} := \int_0^\infty (y+k)^{1-2s}\gamma_b(y)\,dy = \int_0^{\frac{2-b}{1-b}} (y+k)^{1-2s}\gamma_b(y)\,dy,$$

where $J_{b,k}$ depends continuously on the parameter $b \in (0,1)$.

We observe for $0 \le y \le \frac{2-b}{1-b}$ that

$$\begin{cases} k^{1-2s} \le (y+k)^{1-2s} \le (\frac{2-b}{1-b} + k)^{1-2s} = k^{1-2s}\left(1 + \frac{2-b}{k(1-b)}\right)^{1-2s}, & \text{if } s \in (0, \frac{1}{2}], \\ (\frac{2-b}{1-b} + k)^{1-2s} = k^{1-2s}\left(1 + \frac{2-b}{k(1-b)}\right)^{1-2s} \le (y+k)^{1-2s} \le k^{1-2s}, & \text{if } s \in (\frac{1}{2}, 1). \end{cases}$$

Then, we have the estimate for $J_{b,k}$ in the following:

$$\begin{cases} \frac{b}{1-b}k^{1-2s} \le J_{b,k} \le \frac{b}{1-b}k^{1-2s}\left(1 + \frac{2-b}{k(1-b)}\right)^{1-2s}, & \text{if } s \in (0, \frac{1}{2}], \\ \frac{b}{1-b}k^{1-2s}\left(1 + \frac{2-b}{k(1-b)}\right)^{1-2s} \le J_{b,k} \le \frac{b}{1-b}k^{1-2s}, & \text{if } s \in (\frac{1}{2}, 1). \end{cases}$$

One can see that for $b \in (0,1)$, the value $J_{b,k}$ can be both arbitrarily large and arbitrarily close to 0. Hence, by the continuity, for any $k \in \mathbb{N}$, we can find $b_{k,s} \in (0,1)$ such that $J_{b_{k,s},k} = 1$. Consider $\beta_k(y) := \gamma_{b_{k,s}}(y-k)$, and

$$R_{k,s} := k + \frac{1}{1 - b_{k,s}}.$$

For $0 < s < \frac{1}{2}$, we observe that $\left(1 + \frac{2-b}{k(1-b)}\right)^{1-2s} > 1$ and needs

$$bk^{1-2s} \le \frac{b}{1-b}k^{1-2s} \le 1.$$

Thus, if $0 < s < \frac{1}{2}$, then there exists a k_s such that for $k \ge k_s$

$$b_{k,s} \le k^{2s-1}, \quad R_{k,s} = k + \frac{1}{1 - b_{k,s}} \le k + 2. \tag{6.32}$$

Combined with the previous constructions, for $s \in (0,1)$, let us consider the function $\beta_k : (0,\infty) \to [0,1]$ in the following form:

- For $s \neq 1/2$,

$$\begin{cases} \operatorname{supp}(\beta_k) \subseteq (k, R_{k,s} + 1), \\ \beta_k(y) = b_{k,s} \text{ for } y \in (k+1, R_{k,s}), \\ \left|\partial_y^\ell \beta_k(y)\right| \le Cb_{k,s}, \\ \int_0^\infty y^{1-2s}\beta_k(y)\,dy = 1, \end{cases} \tag{6.33}$$

  for any $\ell = 0,1,2$ and $k \in \mathbb{N}$, where the constant $C > 1$ is independent of $k \in \mathbb{N}$;

- For $s = 1/2$, let us consider $\beta \in C_c^\infty(0,\infty)$ such that

$$\beta \geq 0, \quad \int_0^\infty \beta(y)\,dy = 1, \quad \mathrm{supp}(\beta) \subset (1,2), \quad \text{and} \quad \beta_k(y) := 1/k\beta(y/k) \tag{6.34}$$

for $k \in \mathbb{N}$.

We want to use this function β_k to prove the density result by the classical Hahn-Banach argument, and we will handle our arguments in different cases as $s \neq 1/2$ and $s = 1/2$ later.

Let $v \in \mathcal{D} \subset L^2(-T,T;H^1(\Omega))$. Since $v(-T,x) = 0$ in Ω. We let E be an extension with

$$Ev \in L^2(\mathbb{R};H^1(\mathbb{R}^n)) \quad \text{and} \quad \partial_t Ev \in L^2(\mathbb{R};H^{-1}(\mathbb{R}^n)), \tag{6.35}$$

so that the support of $Ev(t,x)$ is contained in $(-T,\widetilde{T}) \times \Omega'$, where $\Omega' \supset \Omega$ is a bounded set in $\mathbb{R}^n$ and $\widetilde{T} > T$. With the condition of $\psi \in L^2(-T,T;\widetilde{H}^{-1}(\Omega))$ at hand, we have

$$\begin{aligned}
\psi(v) &= \left\langle \psi, v\int_0^\infty y^{1-2s}\beta_k\,dy \right\rangle_{L^2(-T,T;H^{-1}(\mathbb{R}^n)),L^2(-T,T;H^1(\mathbb{R}^n))} \\
&= \left\langle \psi, \int_0^\infty y^{1-2s}\beta_k Ev\,dy \right\rangle_{L^2(\mathbb{R};H^{-1}(\mathbb{R}^n)),L^2(\mathbb{R};H^1(\mathbb{R}^n))},
\end{aligned} \tag{6.36}$$

where β_k is given by (6.33). On the other hand, by (6.35), Lemma 6.6 implies that $\beta_k Ev \in \mathcal{L}^{1,2}_{c,0}(\mathbb{R}^{n+2}_+, y^{1-2s}dtdxdy)$ (see the definition (6.6) for $\mathcal{L}^{1,2}_{c,0}(\mathbb{R}^{n+2}_+, y^{1-2s}dtdxdy)$), similar to the computations (6.22), via (6.36), we can deduce

$$\begin{aligned}
&\psi(v) \\
&= \underbrace{\psi\left(\int_0^\infty y^{1-2s}\beta_k(y)v\,dy\right)}_{\text{since } \int_0^\infty y^{1-2s}\beta_k(y)\,dy=1} \\
&= \underbrace{\int_{\mathbb{R}^{n+2}_+}\left[y^{1-2s}\beta_k Ev\partial_t\widetilde{w} + \nabla_{x,y}\cdot\left(y^{1-2s}\widetilde{\sigma}\nabla_{x,y}\widetilde{w}\right)\beta_k Ev\right]dtdxdy}_{\text{since } v(-t,x)=\widetilde{w}(t,x)=0 \text{ for all } t\geq T} \\
&= \int_{\mathbb{R}^{n+2}_+}\left[y^{1-2s}\beta_k Ev\partial_t\widetilde{w} - y^{1-2s}\widetilde{\sigma}\nabla_{x,y}\widetilde{w}\cdot\nabla_{x,y}\left(\beta_k Ev\right)\right]dtdxdy \\
&= \int_{\mathbb{R}^{n+2}_+} y^{1-2s}\beta_k Ev\partial_t\widetilde{w}\,dtdxdy - \int_{\mathbb{R}^{n+2}_+} y^{1-2s}Ev\partial_y\beta_k\partial_y\widetilde{w}\,dtdxdy \\
&\quad - \int_{\mathbb{R}^{n+1}}\sigma\nabla\left(Ev\right)\cdot\nabla\left(\int_0^\infty y^{1-2s}\beta_k\widetilde{w}\,dy\right)dtdx \\
&= \underbrace{\int_{\Omega_T} v\partial_t\left(\int_0^\infty y^{1-2s}\beta_k\widetilde{w}\,dy\right)dtdx - \int_{\Omega_T}\sigma\nabla v\cdot\nabla\left(\int_0^\infty y^{1-2s}\beta_k\widetilde{w}\,dy\right)dtdx}_{(**)} \\
&\quad - \int_0^\infty\int_{\Omega_T} y^{1-2s}v\partial_y\beta_k\partial_y\widetilde{w}\,dtdxdy.
\end{aligned} \tag{6.37}$$

We next want to claim that the term $(**)$ in (6.37) vanishes by making use of the equation of $v \in \mathcal{D}$.

We point out that the function $\int_0^\infty y^{1-2s}\beta_k \widetilde{w}\,dy$ is an admissible test function to this equation, since

$$\begin{aligned}
&\left\| \int_0^\infty y^{1-2s}\beta_k \widetilde{w}\,dy \right\|^2_{L^2(-T,T;H^1(\Omega))} \\
=& \int_{\Omega_T} \left(\int_k^{R_{k,s}+1} y^{1-2s}\beta_k \widetilde{w}\,dy \right)^2 dtdx + \int_{\Omega_T} \left(\int_k^{R_{k,s}+1} y^{1-2s}\beta_k \nabla\widetilde{w}\,dy \right)^2 dtdx \\
\leq& \int_{\Omega_T} \left\{ \left(\int_k^{R_{k,s}+1} y^{1-2s}\beta_k^2\,dy \right) \right. \\
&\qquad \left. \cdot \left[\int_k^{R_{k,s}+1} y^{1-2s}\widetilde{w}^2\,dy + \int_k^{R_{k,s}+1} y^{1-2s} \left|\nabla \widetilde{w}\right|^2\,dy \right] \right\} dtdx \\
\lesssim& \|\widetilde{w}\|^2_{\mathcal{L}^{1,2}(\Omega_T\times(0,R_{k,s}+1),y^{1-2s}dtdxdy)} < \infty,
\end{aligned}$$

where we used the Hölder's inequality. By the UCP $\widetilde{w}=0$ on $(\partial\Omega)_T\times(0,\infty)$, one has $\int_0^\infty y^{1-2s}\beta_k\widetilde{w}\,dy = 0$ on $(\partial\Omega)_T$ so that $\int_0^\infty y^{1-2s}\beta_k\widetilde{w}\,dy \in L^2(-T,T;H_0^1(\Omega))$. Therefore, we can compute

(6.38)
$$\begin{aligned}
&\int_{\Omega_T} v\partial_t \left(\int_0^\infty y^{1-2s}\beta_k\widetilde{w}\,dy \right) dtdx - \int_{\Omega_T} \sigma\nabla v\cdot\nabla \left(\int_0^\infty y^{1-2s}\beta_k\widetilde{w}\,dy \right) dtdx \\
=& \underbrace{\int_{\Omega_T} (-\partial_t v + \nabla\cdot\sigma\nabla v) \left(\int_0^\infty y^{1-2s}\beta_k\widetilde{w}\,dy \right) dtdx}_{\text{Integration by parts and } v(-T,x)=0} \\
&+ \int_{(\partial\Omega)_T} \sigma\nabla v\cdot\nu \left(\int_0^\infty y^{1-2s}\beta_k\widetilde{w}\,dy \right) dtdS = 0,
\end{aligned}$$

since $v\in\mathcal{D}$ (recalling the set $\mathcal{D}$ is defined by (6.21)), and $\widetilde{w}=0$ on $(\partial\Omega)_T\times(0,\infty)$. Insert (6.38) into (6.37), then we get

$$\psi(v) = -\int_0^\infty \int_{\Omega_T} y^{1-2s} v\partial_y\beta_k\partial_y\widetilde{w}\,dtdxdy.$$

The desired result $\psi(v)=0$ can be achieved by passing the limit $k\to\infty$. To this end, note that $\partial_y\widetilde{w} = -\partial_y\mathcal{E}_s u_1 + \partial_y u_2$ (since $\widetilde{u}_1$ is y-independent). Therefore, if $\widetilde{w}$ is any of the functions $\mathcal{E}_s u_1, u_2$, one can estimate as the estimate (6.25) in *Step 2a*. By the Hölder's inequality, one obtains

(6.39)
$$\begin{aligned}
&\int_0^\infty y^{1-2s}\left|\partial_y\beta_k\right| \int_{\Omega_T} \left|v\partial_y\widetilde{w}\right| dtdxdy \\
\leq& \int_0^\infty y^{1-2s}\left|\partial_y\beta_k\right| \|v\|_{L^2(\Omega_T)} \left\|\partial_y\widetilde{w}(\cdot,\cdot,y)\right\|_{L^2(\Omega_T)}\,dy \\
\leq& \|v\|_{L^2(\Omega_T)} \int_0^\infty y^{1-2s}\left|\partial_y\beta_k\right| \left\|\partial_y\widetilde{w}(\cdot,\cdot,y)\right\|_{L^2(\Omega_T)}\,dy.
\end{aligned}$$

To proceed, let us split the case for $s\neq 1/2$ and $s=1/2$:

- For $s \neq 1/2$: Since β_k is given by (6.33), using the numbers $b_{k,s}$, $R_{k,s}$ satisfying (6.33), we can see

(6.40)
$$\begin{aligned}&\int_0^\infty y^{1-2s}\left|\partial_y\beta_k\right|\left\|\partial_y\widetilde{w}(\cdot,\cdot,y)\right\|_{L^2(\Omega_T)}\,dy\\&\lesssim \|\widetilde{w}\|_{\mathcal{L}^{1,2}(\Omega_T\times(0,\infty),y^{1-2s}dtdxdy)}\left(\int_0^\infty y^{1-2s}\left|\partial_y\beta_k\right|^2\,dy\right)^{\frac{1}{2}}\\&\lesssim |b_{k,s}|\left(\int_k^{k+1} y^{1-2s}\,dy+\int_{R_{k,s}}^{R_{k,s}+1} y^{1-2s}\,dy\right)^{\frac{1}{2}}\|\widetilde{w}\|_{\mathcal{L}^{1,2}(\Omega_T\times(0,\infty),y^{1-2s}dtdxdy)}\\&\lesssim |b_{k,s}|\left(k^{1-2s}+R_{k,s}^{1-2s}\right)^{\frac{1}{2}}\|\widetilde{w}\|_{\mathcal{L}^{1,2}(\Omega_T\times(0,\infty),y^{1-2s}dtdxdy)}.\end{aligned}$$

  Taking into account (6.32) for $0<s<\frac{1}{2}$, let us estimate $|b_{k,s}|\left(k^{1-2s}+R_{k,s}^{1-2s}\right)^{\frac{1}{2}}$ in the following:

$$|b_{k,s}|\left(k^{1-2s}+R_{k,s}^{1-2s}\right)^{\frac{1}{2}}\lesssim\begin{cases}k^{\frac{2s-1}{2}}, & \text{if } s\in(0,\frac{1}{2}),\\ k^{\frac{1-2s}{2}}, & \text{if } s\in(\frac{1}{2},1).\end{cases}\tag{6.41}$$

  Let $k\to\infty$, (6.38), (6.39), (6.40) and (6.41) infer that $\psi(v)=0$ holds true for $s\neq 1/2$.
- For $s=1/2$: Since β_k is given by (6.34), by using a similar argument, one has

$$\begin{aligned}&\int_0^\infty \left|\partial_y\beta_k\right|\left\|\partial_y\widetilde{w}(\cdot,\cdot,y)\right\|_{L^2(\Omega_T)}\,dy\\&\lesssim \left\|\partial_y\widetilde{w}(\cdot,\cdot,y)\right\|_{L^2(\Omega_T\times(0,\infty))}\left(\int_k^{2k}\left|\partial_y\beta_k\right|^2\,dy\right)^{1/2}\\&\lesssim k^{-3/2}\left\|\partial_y\widetilde{w}(\cdot,\cdot,y)\right\|_{L^2(\Omega_T\times(0,\infty))}\\&\to 0,\end{aligned}$$

  as $k\to\infty$.

In summary, for both cases $s\neq 1/2$ and $s=1/2$, we conclude that for any $\psi\in L^2(-T,T;\widetilde{H}^{-1}(\Omega))$ with $\psi(v_f)=0$, for all $f\in C_c^\infty(W_T)$, then we also obtain $\psi(v)=0$, for any $v\in\mathcal{D}$. This shows the density result by the Hahn-Banach approach. Hence, $\mathcal{V}\subset L^2(-T,T;H^1(\Omega))$ is dense in $\mathcal{D}$.

Last but not least, we want to show that $\overline{\mathcal{V}'}=L^2(-T,T;H^{1/2}(\partial\Omega))$ holds. To this end, given $g\in L^2(-T,T;H^{1/2}(\partial\Omega))$, and consider the initial-boundary value problem

$$\begin{cases}(\partial_t-\nabla\cdot\sigma\nabla)\,u=0 & \text{in } \Omega_T,\\ u=g & \text{on } (\partial\Omega)_T,\\ u(-T,x)=0 & \text{for } x\in\Omega.\end{cases}$$

By the definition (6.21), one knows that the solution $u\in\mathcal{D}$, then for any $\epsilon>0$, one can always find $v_\epsilon\in\mathcal{V}$, such that $\|u-v_\epsilon\|_{L^2(-T,T;H^1(\Omega))}\leq\epsilon$. By the classical trace estimate, we have

$$\left\|g-v_\epsilon|_{(\partial\Omega)_T}\right\|_{L^2(-T,T;H^{1/2}(\partial\Omega))}\lesssim\|u-v_\epsilon\|_{L^2(-T,T;H^1(\Omega))}\leq\epsilon.$$

This demonstrates that the function $v_\epsilon|_{(\partial\Omega)_T}$ approximates g on $(\partial\Omega)_T$ with respect to the norm $\|\cdot\|_{L^2(-T,T;H^{1/2}(\partial\Omega))}$. Therefore, $\overline{\mathcal{V}'} = L^2(-T,T;H^{1/2}(\partial\Omega))$ holds as desired. This proves the assertion. □

Remark 6.8. *No matter whether σ is isotropic or anisotropic, all the preceding analysis holds.*

7. Proofs of main results

With Proposition 6.1 at hand, we can prove Theorem 1.1.

Proof of Theorem 1.1 and Proposition 1.3. By Proposition 6.1, the operator

$$\begin{aligned} \mathrm{T}_1 : \widetilde{\mathbf{H}}^s(W_T) &\to L^2(-T,T;H^{1/2}(\partial\Omega)), \\ f \mapsto v_f(t,x)|_{(\partial\Omega)_T} &:= \left.\int_0^\infty y^{1-2s}\widetilde{u}_f(t,x,y)\,dy\right|_{(\partial\Omega)_T} \end{aligned}$$

is linear, bounded and has dense range. The DN map $\Lambda_\sigma : L^2(-T,T;H^{1/2}(\partial\Omega)) \to L^2(-T,T;H^{-1/2}(\partial\Omega))$ is continuous, then this implies that

$$\overline{\mathrm{T}\left(\mathcal{C}^s_{\sigma,W_T}\right)}^{L^2(-T,T;H^{1/2}(\partial\Omega))\times L^2(-T,T;H^{-1/2}(\partial\Omega))} = \mathcal{C}_{\sigma,(\partial\Omega)_T},$$

as stated in Proposition 1.3. Additionally, by the UCP, the (partial) nonlocal Cauchy data

$$\left(f|_{W_T}, \left(\partial_t - \nabla\cdot\sigma\nabla\right)^s u_f|_{W_T}\right)$$

determines the (full) local Cauchy data

$$\left(y^{1-2s}u_f\big|_{(\partial\Omega)_T\times(0,\infty)}, y^{1-2s}\sigma\nabla u_f\cdot\nu\big|_{(\partial\Omega)_T\times(0,\infty)}\right).$$

Therefore, the Cauchy data $\left(v_f|_{(\partial\Omega)_T}, \sigma\nabla v_f\cdot\nu|_{(\partial\Omega)_T}\right)$ can be also determined uniquely. By using the density result, this shows that Λ^s_σ determines Λ_σ as desired. This completes the proof. □

Proof of Corollary 1.5. With Theorem 1.1 at hand, the nonlocal (partial) DN map determines the local (full) DN map. Therefore, one can have that the desired uniqueness result by the existing work [CK01] for the local parabolic equation. This completes the proof. □

Proof of Corollary 1.6. By using Theorem 1.1, we only need to consider the local setting. By [GAV12], one can find a diffeomorphism $\Phi : \overline{\Omega} \to \overline{\Omega}$ with $\Phi|_{\partial\Omega} = \mathrm{Id}$, which transforms the parabolic equation (1.10) to (1.11). Moreover, abusing the notation, we denote another diffeomorphism Φ (with the same notation) such that $\Phi : \mathbb{R}^n \to \mathbb{R}^n$ such that $\Phi|_{\mathbb{R}^n\setminus\Omega} = \mathrm{Id}$ (of course such Φ also satisfies $\Phi|_{\partial\Omega} = \mathrm{Id}$). Thus, this Φ satisfies all required assumptions in Corollary 1.6. This proves the assertion. □

Acknowledgments.

- Y.-H. Lin is partially supported by the National Science and Technology Council (NSTC) Taiwan, under the projects 111-2628-M-A49-002 and 112-2628-M-A49-003. Y.-H. Lin is also a Humboldt research fellowship for experienced researcher.
- G. Uhlmann was partially supported by NSF, a Walker Professorship at University of Washington and a Si-Yuan Professorship at Institute for Advanced Study, Hong Kong University of Science and Technology.

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Department of Mathematics, National Cheng- Kung University, Tainan 701, Taiwan.
Email address: cllin2@mail.ncku.edu.tw

Department of Applied Mathematics, National Yang Ming Chiao Tung University, Hsinchu 30050, Taiwan
Email address: yihsuanlin3@gmail.com

Department of Mathematics, University of Washington and Institute for Advanced Study, the Hong Kong University of Science and Technology
Email address: gunther@math.washington.edu