MONOTONICITY-BASED INVERSION OF THE FRACTIONAL SCHRÖDINGER EQUATION I. POSITIVE POTENTIALS

BASTIAN HARRACH[†] AND YI-HSUAN LIN[‡]

Abstract. We consider an inverse problem for the fractional Schrödinger equation by using monotonicity formulas. We provide if-and-only-if monotonicity relations between positive bounded potentials and their associated nonlocal Dirichlet-to-Neumann maps. Based on the monotonicity relation, we can prove uniqueness for the nonlocal Calderón problem in a constructive manner. Secondly, we offer a reconstruction method for an unknown obstacles in a given domain. Our method is independent of the dimension and only requires the background solution of the fractional Schrödinger equation.

Key words. Fractional Schrödinger equation, monotonicity method, inverse obstacle problem, shape reconstruction, localized potentials, Calderón's problem, Runge approximation property

AMS subject classifications. 35R30

1. Introduction. In this article we will give a constructive uniqueness result for the Calderón problem for the nonlocal fractional Schrödinger equation and develop a shape reconstruction method to determine unknown obstacles in a given domain. Let Ω be a bounded open set in $\mathbb{R}^n$ with $n \in \mathbb{N}$, and $q \in L^\infty_+(\Omega)$ be a potential, where $L^\infty_+(\Omega)$ consists of all $L^\infty(\Omega)$-functions with positive essential infima. For $s \in (0,1)$, the (nonlocal) Dirichlet problem for the fractional Schrödinger equation is given by

$$\begin{cases} (-\Delta)^s u + qu = 0 & \text{in } \Omega, \\ u = F & \text{in } \Omega_e := \mathbb{R}^n \setminus \overline{\Omega}, \end{cases} \tag{1.1}$$

Note that $(-\Delta)^s$ is a nonlocal operator as $s \in (0,1)$, so that the Dirichlet data is prescribed on the whole complement of Ω and not only on its boundary $\partial\Omega$.

The Dirichlet-to-Neumann (DtN) operator of (1.1)

$$\Lambda(q) : \; H(\Omega_e) \to H(\Omega_e)^*$$

is formally given by

$$\Lambda(q)F := (-\Delta)^s u|_{\Omega_e}, \quad \text{where } u \in H^s(\mathbb{R}^n) \text{ solves (1.1).} \tag{1.2}$$

The precise definition of $(-\Delta)^s$, the DtN map $\Lambda(q)$, and the function spaces are given in Section 2. For further properties for the nonlocal DtN maps, we also refer readers to [20].

The nonlocal fractional Laplacian operator has received considerable attention for its ability to model anomalous stochastic diffusion problems including jumps and longdistance interactions, cf., e.g., [7, 53], and the extensive list of references to applications in the introduction of [12]. Accordingly, inverse problems for the fractional Laplacian

[†]Institute for Mathematics, Goethe-University Frankfurt, Frankfurt am Main, Germany (harrach@math.uni-frankfurt.de)

[‡]Institute for Advanced Study, The Hong Kong University Science and Technology, Hong Kong (yihsuanlin3@gmail.com)

operator appear when an imaging domain is being probed by an anomalous diffusion process. The fractional diffusion model is more complicated than in the standard Brownian motion case modeled by the standard Laplacian ($s = 1$). However, recent works on inverse problems for nonlocal equations (see the references below) indicate that the inverse problem actually becomes easier to solve than in the standard Laplacian case. The present work contributes to this by showing that monotonicity-based reconstruction methods that have been developed for standard diffusion processes can also be applied to the fractional diffusion case and that the methods even become simpler and more powerful.

For a list of recent works on inverse problems for nonlocal equations let us refer to [8, 9, 10, 11, 18, 46], and the review of Salo [59]. Stability questions for the fractional Calderón problem were studied in [55, 57, 58]. Let us point out that the Calderón problem for the fractional Schrödinger equation was first solved by Ghosh, Salo and Uhlmann [20], who showed the global uniqueness result that $\Lambda(q_1) = \Lambda(q_2)$ implies $q_1 = q_2$. Remarkably, it was recently shown that uniqueness in the fractional Calderón problem already holds with a single measurement and with data on arbitrary, possibly disjoint subsets of the exterior, cf. Ghosh, Rüland, Salo, and Uhlmann [19].

Uniqueness proofs for the fractional Calderón problem strongly rely on a *strong uniqueness property*, cf. [20, Theorem 1.2] for the fractional Schrödinger equation and [18, Theorem 1.2] for the nonlocal variable case. The strong uniqueness property states that if $u = (-\Delta)^s u = 0$ (for $0 < s < 1$) in an arbitrary open set in $\mathbb{R}^n$, then $u \equiv 0$ in the entire space $\mathbb{R}^n$ for any $n \in \mathbb{N}$. Note that this property is no longer true for the standard (local) Laplacian case, i.e., for the case $s = 1$. In fact, via this property, one can also derive a nonlocal Runge type approximation property (see [20, Theorem 1.3] or Theorem 3.4), which states that an arbitrary L^2 function can be well approximated by solutions of the fractional Schrödinger equation. Recently, Rüland and Salo [56] studied the fractional Calderón problem under lower regularity assumptions and the stability results for the potentials.

The first effort of this paper is to prove uniqueness for the Calderón problem in a constructive way. We will derive the following *monotonicity formula* in Theorem 4.1: Let $q_0, q_1 \in L^\infty_+(\Omega)$, then

$$q_1 \leq q_0 \quad \text{if and only if} \quad \Lambda(q_1) \leq \Lambda(q_0), \tag{1.3}$$

where $q_1 \leq q_0$ is to be understood pointwise almost everywhere in Ω, and $\Lambda(q_1) \leq \Lambda(q_0)$ is to be understood in the sense of definiteness of quadratic forms (also known as Loewner order), cf. (3.6) in Section 3. Similar monotonicity relations have been widely applied in the study of inverse problems, see [32, 65] for the origins of the monotonicity method combined with the method of localized potentials [65, 22, 31, 23, 2, 32, 25, 3, 33, 28, 48, 66, 14, 15, 34, 64, 68, 4, 21, 30, 26, 29, 24, 61, 69, 16] for a list of recent and related works. Also note that similar arguments involving monotonicity conditions and blow-up arguments have been used in various ways in the study of inverse problems, see, e.g., [1, 36, 37, 42, 43].

The monotonicity relation (1.3) immediately implies a constructive global uniqueness result for the Calderón problem, which is our first main result in this paper, cf. Corollary 4.5: Any $q \in L^\infty_+(\Omega)$ is uniquely determined by $\Lambda(q)$ by the following formula: For $x \in \Omega$ a.e.,

$$q(x) = \sup \{ \psi(x) : \ \psi \text{ positive (density one) simple function}, \Lambda(\psi) \leq \Lambda(q) \}. \tag{1.4}$$

This shows that one can recover an unknown potential with positive infimum by comparing the DtN map with that of simple functions.

The second main result of this paper is on the shape reconstruction (or inclusion detection) problem for the fractional Schrödinger equation. Let $q_0 \in L^\infty_+(\Omega)$ denote a known reference coefficient, and $q_1 \in L^\infty_+(\Omega)$ denote an unknown coefficient function that differs from the reference value q_0 in certain regions. We aim to find these anomalous regions (or scatterers), i.e., the support of $q_1 - q_0$, from the difference of the Dirichlet-to-Neumann-operators $\Lambda(q_1) - \Lambda(q_0)$. We will prove that this can be done *without* solving the fractional Schrödinger equation for potentials other than the reference potentials q_0. More precisely, we will show in Theorem 5.5 that

$$\begin{cases} \operatorname{supp}(q_1 - q_0) & = \bigcap \{ C \subseteq \Omega \text{ closed} : \\ & \qquad \exists \alpha > 0 : \ -\alpha \mathcal{T}_C \leq \Lambda(q_1) - \Lambda(q_0) \leq \alpha \mathcal{T}_C \}, \end{cases} \tag{1.5}$$

where $\mathcal{T}_C := \Lambda'(q_0)\chi_C$, and $\Lambda'(q_0)$ is the Fréchet derivative of the DtN operator $\Lambda(q)$. The test operator $\mathcal{T}_C$ can be easily calculated from knowledge of the solution of the fractional Schrödinger equation with reference potentials q_0. Under the additional definiteness condition that either $q_1 \geq q_0$ or $q_1 \leq q_0$ holds almost everywhere we will also show that the inner support of $q_1 - q_0$ fulfills

$$\operatorname{inn\,supp}(q_1 - q_0) = \bigcup \{ B \subseteq \Omega \text{ open ball} : \ \exists \alpha > 0 : \Lambda(q_1) \leq \Lambda(q_0) - \alpha \mathcal{T}_B \}, \tag{1.6}$$

resp.,

$$\operatorname{inn\,supp}(q_1 - q_0) = \bigcup \{ B \subseteq \Omega \text{ open ball} : \ \exists \alpha > 0 : \Lambda(q_1) \geq \Lambda(q_0) + \alpha \mathcal{T}_B \}, \tag{1.7}$$

cf. Theorem 5.6.

Inverse shape reconstruction problems were intensively studied in the literature, see [38, 49] for the comprehensive introduction and survey. There are several inclusion detection methods, including the enclosure method, the linear sampling method, the probe method and the factorization method, which have been proposed to solve the inclusion detection inverse problem. These methods strongly rely on special solutions of certain differential equations. For example, the special solutions include the complex geometrical optics (CGO) solution, the oscillating decaying (OD) solution and the Wolff solution. In [40, 41, 52, 60, 62, 67]. The authors used the CGO solutions to solve the inverse obstacle problems for different mathematical models for the isotropic problems. However, for the general anisotropic medium, we need to utilize more complicated special solutions such as the OD solutions, see [44, 47, 50, 51]. The Wolff solutions are used to solve the inverse obstacle problem for the p-Laplace equation, see [5, 6]. Our monotonicity-based approach does not require to construct any special solutions to practically determine the inclusions via the formulas (1.5)–(1.7). The proof of these formulas however relies on so-called *localized potentials* [17] , i.e., solutions of the fractional Schrödinger equations with very large energy on a subset of Ω and very low energy elsewhere, see Corollary 3.5.

The structure of this article is given as follows. In Section 2, we provide basic reviews for the fractional Sobolev spaces, the fractional Schrödinger equation and the nonlocal DtN map. In Section 3, we demonstrate the monotonicity formulas and construct the localized potentials for our the fractional Schrödinger equation. In Section 4, we show the converse results of the monotonicity relations, which gives if-and-only-if relations between the DtN maps and positive potentials. In addition, we provide a constructive global uniqueness proof for the Calderón problem by proving (1.4). Finally, in Section 5, we characterize the linearized nonlocal DtN map and derive the inclusion detection formulas (1.5)–(1.7).

2. The Dirichlet-to-Neumann operator for the fractional Schrödinger equation. In this section, we briefly summarize some fundamental definitions and notations on the fractional Schrödinger equation and the associated DtN operator. For $n \in \mathbb{N}$, we denote by

$$\mathscr{F},\ \mathscr{F}^{-1}:\ L^2(\mathbb{R}^n;\mathbb{C}) \to L^2(\mathbb{R}^n;\mathbb{C})$$

the Fourier transform and its inverse on the space of complex-valued L^2-functions, and let $\mathcal{S}(\mathbb{R}^n;\mathbb{C})$ be the Schwartz space of rapidly decreasing complex-valued functions.

For $s \in (0,1)$, the fractional Laplacian is defined by

$$(-\Delta)^s:\ \mathcal{S}(\mathbb{R}^n;\mathbb{C}) \to L^2(\mathbb{R}^n;\mathbb{C}), \quad (-\Delta)^s u := \mathscr{F}^{-1}\left(|\xi|^{2s}\mathscr{F}(u)\right).$$

The fractional Laplacian can be extended to an operator

$$(-\Delta)^s:\ L^2(\mathbb{R}^n;\mathbb{C}) \to \mathcal{S}'(\mathbb{R}^n;\mathbb{C})$$

by setting

$$\langle(-\Delta)^s u, \varphi\rangle_{\mathcal{S}'\times\mathcal{S}} := \langle u, (-\Delta)^s\varphi\rangle_{L^2} \quad \text{for all } u \in L^2(\mathbb{R}^n;\mathbb{C}),\ \varphi \in \mathcal{S}(\mathbb{R}^n;\mathbb{C}),$$

and it can be shown that $(-\Delta)^s u$ will be real-valued for real-valued u (see [12, 45, 63]). Hence, in the following we will always consider the fractional Laplacian as an operator

$$(-\Delta)^s:\ L^2(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^n),$$

and all function spaces in this work are real-valued unless indicated otherwise.

For $0 < s < 1$, the L^2-based fractional Sobolev space is defined by

$$H^s(\mathbb{R}^n) := \{u \in L^2(\mathbb{R}^n):\ (-\Delta)^{s/2}u \in L^2(\mathbb{R}^n)\}$$

and equipped with the scalar product

$$(u,v)_{H^s(\mathbb{R}^n)} := \int_{\mathbb{R}^n} \left((-\Delta)^{s/2}u \cdot (-\Delta)^{s/2}v + uv\right) \mathrm{d}x \quad \text{for all } u, v \in H^s(\mathbb{R}^n).$$

It can be shown that $H^s(\mathbb{R}^n)$ is a Hilbert space (see [12] for instance). Also note that $H^s(\mathbb{R}^n)$ obviously contains the rapidly decreasing Schwartz functions $\mathcal{S}(\mathbb{R}^n)$, and a fortiori all compactly supported C^∞-functions.

For an open set $\Omega \subseteq \mathbb{R}^n$ we define

$$H_0^s(\Omega) := \text{closure of } C_c^\infty(\Omega) \text{ in } H^s(\mathbb{R}^n).$$

Note that this space is sometimes denoted as $\widetilde{H}^s(\Omega)$ in the literature, but in the context of Dirichlet and Neumann boundary value problems it seems more natural to denote this space by $H_0^s(\Omega)$.

We also define the bilinear form

$$\mathscr{B}_q(u,w) := \int_{\mathbb{R}^n} (-\Delta)^{s/2}u \cdot (-\Delta)^{s/2}w \,\mathrm{d}x + \int_\Omega quw \,\mathrm{d}x, \tag{2.1}$$

for any $u, w \in H^s(\mathbb{R}^n)$. We then have the following variational formulation for the fractional Schrödinger equation.

Lemma 2.1. *Let $q \in L^\infty(\Omega)$ and $f \in L^2(\Omega)$. $u \in H^s(\mathbb{R}^n)$ solves (in the sense of distributions)*

$$(-\Delta)^s u + qu = f \quad \text{in } \Omega$$

if and only if $u \in H^s(\mathbb{R}^n)$ satisfies

$$\mathscr{B}_q(u,w) = \int_\Omega fw \,\mathrm{d}x \quad \text{for all } w \in H_0^s(\Omega).$$

Proof. Note that for $u \in H^s(\mathbb{R}^n)$ we can interpret $(-\Delta)^s u$ as a distribution on Ω, and a simple computation shows that

$$\begin{aligned}&\langle (-\Delta)^s u + qu - f, \psi\rangle_{\mathcal{D}'(\Omega)\times\mathcal{D}(\Omega)}\\ =&\int_{\mathbb{R}^n} (-\Delta)^{s/2}u \cdot (-\Delta)^{s/2}\psi \,\mathrm{d}x + \int_\Omega qu\psi \,\mathrm{d}x - \int_\Omega f\psi \,\mathrm{d}x = 0\end{aligned}$$

for all test functions $\psi \in \mathcal{D}(\Omega) = C_c^\infty(\Omega)$, so that the assertion follows by continuous extension. □

We now introduce the Dirichlet trace operator in abstract quotient spaces.

Lemma 2.2. *With $\Omega_e = \mathbb{R}^n \setminus \overline{\Omega}$, we define*

$$\gamma_{\Omega_e} : \; H^s(\mathbb{R}^n) \to H(\Omega_e) := H^s(\mathbb{R}^n)/H_0^s(\Omega), \quad u \mapsto u + H_0^s(\Omega).$$

Then, for all $u, v \in H^s(\mathbb{R}^n)$,

$$\gamma_{\Omega_e} u = \gamma_{\Omega_e} v \quad \text{implies that} \quad u(x) = v(x) \quad \text{for } x \in \Omega_e \text{ a.e.}$$

Proof. If $\gamma_{\Omega_e} u = \gamma_{\Omega_e} v$, then $u - v \in H_0^s(\Omega)$, so that there exists a sequence $(\phi_k)_{k\in\mathbb{N}} \subset C_c^\infty(\Omega)$ with $\phi_k \to u - v$ in $H^s(\mathbb{R}^n)$. In particular, this implies that $\phi_k|_{\Omega_e} = 0$ and $\phi_k \to u - v$ in $L^2(\mathbb{R}^n)$, so that it follows that $u|_{\Omega_e} = v|_{\Omega_e}$ as $L^2(\Omega_e)$-functions. □

For the sake of readability we will write $u|_{\Omega_e}$ instead of $\gamma_{\Omega_e} u$ in the following. Also note that Lemma 2.2 implies that for two functions $F, G \in C_c^\infty(\Omega_e)$, $F = G$ if and only if $F - G \in H_0^s(\Omega)$, so that we can identify $C_c^\infty(\Omega_e)$ with its image in the quotient space $H(\Omega_e)$ and thus consider $C_c^\infty(\Omega_e)$ as a subspace of $H(\Omega_e)$.

We can now state the following result on the solvability of the Dirichlet problem and the definition of Neumann boundary values.

Lemma 2.3. *Let $q \in L_+^\infty(\Omega)$.*

(a) *For every $F \in H(\Omega_e)$ and $f \in L^2(\Omega)$, we have that $u \in H^s(\mathbb{R}^n)$ solves the Dirichlet problem*

$$(-\Delta)^s u + qu = f \quad \text{in } \Omega, \quad u|_{\Omega_e} = F, \tag{2.2}$$

if and only if $u = u^{(0)} + u^{(F)}$, where $u^{(F)} \in H^s(\mathbb{R}^n)$ fulfills $u^{(F)}|_{\Omega_e} = F$ (for $F \in C_c^\infty(\Omega_e)$ we can simply choose $u^{(F)} := F$), and $u^{(0)} \in H_0^s(\Omega)$ solves

$$\mathscr{B}_q(u^{(0)}, w) = -\mathscr{B}_q(u^{(F)}, w) + \int_\Omega f w \, \mathrm{d}x \quad \text{for all } w \in H_0^s(\Omega).$$

The Dirichlet problem (2.2) *is uniquely solvable and the solution $u \in H^s(\mathbb{R}^n)$ depends linearly and continuously on $F \in H(\Omega_e)$ and $f \in L^2(\Omega)$.*

(b) *For a solution $u \in H^s(\mathbb{R}^n)$ of $(-\Delta)^s u + qu = 0$, we define the Neumann exterior data $\mathcal{N}u \in H(\Omega_e)^*$ by*

$$\langle \mathcal{N}u, G \rangle := \mathscr{B}_q(u, v^{(G)}), \quad \text{where } v^{(G)} \in H^s(\mathbb{R}^n) \text{ fulfills } v^{(G)}|_{\Omega_e} = G,$$

where $H(\Omega_e)^*$ is the dual space of $H(\Omega_e)$ and $\langle \cdot, \cdot \rangle := \langle \cdot, \cdot \rangle_{H(\Omega_e)^* \times H(\Omega_e)}$. Then the Dirichlet-to-Neumann operator

$$\Lambda(q) : \; H(\Omega_e) \to H(\Omega_e)^*, \quad F \mapsto \mathcal{N}u,$$

is a symmetric linear bounded operator, where u solves (2.2) *with $f = 0$.*

Proof. Obviously $\mathscr{B}_q$ is a coercive, symmetric, and continuous bilinear form on the Hilbert space $H_0^s(\Omega)$. Thus the assertion follows from a standard application of the Lax-Milgram theorem and the equivalence result in Lemma 2.1. □

Remark 2.4. *If $u \in H^{2s}(\mathbb{R}^n)$ solves*

$$(-\Delta)^s u + qu = 0 \quad \text{in } \Omega,$$

then a computation as in the proof of Lemma 2.1 shows that

$$\langle \mathcal{N}u, F \rangle = \int_{\Omega_e} (-\Delta)^s u \cdot v^{(F)} \, \mathrm{d}x$$

where $v^{(F)}|_{\Omega_e} = F$. This motivates to formally write the Neumann boundary values as

$$\mathcal{N}u = (-\Delta)^s u|_{\Omega_e}.$$

Note that $\mathcal{N}u = (-\Delta)^s u|_{\Omega_e} \in L^2(\Omega_e)$ rigorously holds under the additional smoothness condition $u \in H^{2s}(\mathbb{R}^n)$ (not only $u \in H^s(\mathbb{R}^n)$) but no regularity assumptions on the open set $\Omega \subseteq \mathbb{R}^n$ is required. Alternatively, it is also possible to justify the notation $\mathcal{N}u = (-\Delta)^s u|_{\Omega_e}$ without the additional smoothness (i.e., for $u \in H^s(\mathbb{R}^n)$) when regularity assumptions on Ω are imposed, cf. [20].

For a more in-depth discussion on the spaces of Dirichlet and Neumann boundary values for Lipschitz domains we refer readers to [20]. For the results in this work it suffices to use the abstract quotient space definitions given above, which also has the advantage that we can treat arbitrary open sets Ω without any boundary regularity assumptions.

3. Monotonicity relations and localized potentials for the fractional Schrödinger equation. In this section we will derive monotonicity and localized potentials results for the fractional Schrödinger equation

$$(-\Delta)^s u + qu = 0 \quad \text{in } \Omega.$$

3.1. Monotonicity relations. We will first show that increasing the coefficient q increases the Dirichlet-Neumann-operator in the sense of quadratic forms.

LEMMA 3.1. *(Monotonicity relations) Let $n \in \mathbb{N}$, $\Omega \subseteq \mathbb{R}^n$ be an open set, and $s > 0$. For $j = 0, 1$, let $q_j \in L^\infty_+(\Omega)$ and $u_j \in H^s(\mathbb{R}^n)$ be solutions of*

$$\begin{cases} (-\Delta)^s u_j + q_j u_j = 0 & \text{in } \Omega, \\ u_j|_{\Omega_e} = F, \end{cases} \tag{3.1}$$

where $F \in H(\Omega_e)$. Then we have the following monotonicity relations

$$\langle (\Lambda(q_1) - \Lambda(q_0)) F, F \rangle \leq \int_\Omega (q_1 - q_0)|u_0|^2 dx, \tag{3.2}$$

and

$$\langle (\Lambda(q_1) - \Lambda(q_0)) F, F \rangle \geq \int_\Omega (q_1 - q_0)\, |u_1|^2 dx. \tag{3.3}$$

Moreover, we have

$$\langle (\Lambda(q_1) - \Lambda(q_0)) F, F \rangle \geq \int_\Omega \frac{q_0}{q_1} (q_1 - q_0)\, |u_0|^2 dx \tag{3.4}$$

and

$$\langle (\Lambda(q_1) - \Lambda(q_0)) F, F \rangle \leq \int_\Omega \frac{q_1}{q_0} (q_1 - q_0)\, |u_1|^2 dx. \tag{3.5}$$

Proof. The proof is similar to [31, Lemma 2.1]. Note also that the idea goes back to Ikehata, Kang, Seo, and Sheen [35, 39], and that similar results have been obtained and used in many other works on the monotonicity and factorization method.

From the definition of the Dirichlet-Neumann-operator in Lemma 2.3 we have that

$$\begin{aligned} \langle \Lambda(q_0) F, F \rangle &= \mathscr{B}_{q_0}(u_0, u_0), \text{ and} \\ \langle \Lambda(q_1) F, F \rangle &= \mathscr{B}_{q_1}(u_1, u_1) = \mathscr{B}_{q_1}(u_1, u_0) \end{aligned}$$

We thus obtain

$$\begin{aligned} 0 \leq & \mathscr{B}_{q_1}(u_1 - u_0, u_1 - u_0) = \mathscr{B}_{q_1}(u_1, u_1) - 2\mathscr{B}_{q_1}(u_1, u_0) + \mathscr{B}_{q_1}(u_0, u_0) \\ = & - \langle \Lambda(q_1) F, F \rangle + \mathscr{B}_{q_1}(u_0, u_0) \\ = & \langle (\Lambda(q_0) - \Lambda(q_1)) F, F \rangle + \mathscr{B}_{q_1}(u_0, u_0) - \mathscr{B}_{q_0}(u_0, u_0) \\ = & \langle (\Lambda(q_0) - \Lambda(q_1)) F, F \rangle + \int_\Omega (q_1 - q_0)|u_0|^2 dx, \end{aligned}$$

which shows the first assertion (3.2). Interchanging q_0 and q_1 in the above calculation also yields (3.3).

In addition, when $q_j \in L^\infty_+(\Omega)$, one can see

$$\begin{aligned}
&\langle (\Lambda(q_1) - \Lambda(q_0))F, F\rangle = \mathscr{B}_{q_0}(u_0 - u_1, u_0 - u_1) - \int_\Omega (q_0 - q_1)|u_1|^2 dx \\
&= \int_{\mathbb{R}^n} |(-\Delta)^{s/2}(u_1 - u_0)|^2 dx + \int_\Omega \left(q_0(u_1 - u_0)^2 + (q_1 - q_0)|u_1|^2\right) dx \\
&\geq \int_\Omega \left(q_0(u_1 - u_0)^2 + (q_1 - q_0)|u_1|^2\right) dx = \int_\Omega \left(q_1 u_1^2 - 2q_0 u_1 u_0 + q_0 u_0^2\right) dx \\
&= \int_\Omega q_1 \left(u_1 - \frac{q_0}{q_1} u_0\right)^2 dx + \int_\Omega \left(q_0 - \frac{q_0^2}{q_1}\right) |u_0|^2 dx \\
&\geq \int_\Omega \frac{q_0}{q_1} (q_1 - q_0) |u_0|^2 dx,
\end{aligned}$$

which proves (3.4), and interchanging q_0 and q_1 yields (3.5). □

For two functions $q_0, q_1 \in L^\infty_+(\Omega)$, we write $q_0 \geq q_1$ if $q_0(x) \geq q_1(x)$ almost everywhere in Ω. For two operators $\Lambda(q_0), \Lambda(q_1) : \ H(\Omega_e) \to H(\Omega_e)^*$ we write $\Lambda(q_0) \geq \Lambda(q_1)$ if

$$\langle (\Lambda(q_0) - \Lambda(q_1))F, F\rangle \geq 0 \text{ for all } F \in H(\Omega_e). \tag{3.6}$$

With respect to these partial orders, we have the following monotonicity property of the Dirichlet-Neumann-operators.

Corollary 3.2. *For any $q_0, q_1 \in L^\infty_+(\Omega)$,*

$$q_0 \geq q_1 \text{ implies } \Lambda(q_0) \geq \Lambda(q_1). \tag{3.7}$$

Proof. This follows immediately from Lemma 3.1. □

3.2. Localized potentials . In this subsection we will show the existence of localized potentials solutions of the fractional Schrödinger equation that have an arbitrarily high energy on some part of the imaging domain and an arbitrarily low energy on another part. These localized potentials will allow us to control the energy terms in Lemma 3.1 and show a converse of the monotonicity relation (3.7).

For the fractional Schrödinger equation, the existence of localized potentials is a simple consequence from the unique continuation and Runge approximation result shown by Ghosh, Salo and Uhlmann [20], see also [18] for further discussions and [29] for the connection between Runge approximation properties and localized potentials. We use the following unique continuation result from Ghosh, Salo and Uhlmann [20].

Theorem 3.3. *[20, Theorem 1.2] Let $n \in \mathbb{N}$, and $0 < s < 1$. If $u \in H^r(\mathbb{R}^n)$ for some $r \in \mathbb{R}$, and both u and $(-\Delta)^s u$ vanish in the same arbitrary non-empty open set in $\mathbb{R}^n$, then $u \equiv 0$ in $\mathbb{R}^n$.*

Ghosh, Salo and Uhlmann [20, Theorem. 1.3] also showed that this unique continuation result implies the Runge-type approximation property that any $L^2(\Omega)$-function can be approximated by solutions of the fractional Schrödinger equation with exterior Dirichlet data supported on an arbitrarily small open set. Since our formulation slightly differs from [20], and the proof if short and simple, we give the proof for the sake of completeness.

THEOREM 3.4. *Let $n \in \mathbb{N}$, $\Omega \subseteq \mathbb{R}^n$ be an open set, $0 < s < 1$, $q \in L^\infty_+(\Omega)$, and $O \subseteq \Omega_e = \mathbb{R}^n \setminus \overline{\Omega}$ be open. For every $f \in L^2(\Omega)$ there exists a sequence $F_k \in C_c^\infty(O)$, so that the corresponding solutions $u^k \in H^s(\mathbb{R}^n)$ of*

$$(-\Delta)^s u^k + q u^k = 0 \quad \text{in } \Omega, \quad u^k|_{\Omega_e} = F_k,$$

fulfill that $u^k|_\Omega \to f$ in $L^2(\Omega)$.

Proof. For $F \in C_c^\infty(O)$ let $S(F) := u \in H^s(\mathbb{R}^n)$ denote the solution of

$$(-\Delta)^s u + q u = 0 \quad \text{in } \Omega, \quad u|_{\Omega_e} = F, \tag{3.8}$$

i.e. (see Lemma 2.3) $u = u^{(0)} + F$ where $u^{(0)} \in H_0^s(\Omega)$ solves $\mathscr{B}_q(u^{(0)}, w) = -\mathscr{B}_q(F, w)$ for all $w \in H_0^s(\Omega)$.

The assertion follows if we can show that the space of all such solutions

$$\{S(F)|_\Omega : \ F \in C_c^\infty(O)\} \subseteq L^2(\Omega)$$

has trivial $L^2(\Omega)$-orthogonal complement. To that end let

$$f \in \{S(F)|_\Omega : \ F \in C_c^\infty(O)\}^\perp \subseteq L^2(\Omega)$$

and $v \in H^s(\mathbb{R}^n)$ solve

$$(-\Delta)^s v + q v = f \quad \text{in } \Omega, \quad v|_{\Omega_e} = 0,$$

i.e., $v \in H_0^s(\Omega)$ and $\mathscr{B}_q(v, w) = \int_\Omega f w|_\Omega \mathrm{d}x$ for all $w \in H_0^s(\Omega)$. Then for all $F \in C_c^\infty(O)$, the solution $u = S(F)$ of (3.8) fulfills

$$0 = \int_\Omega f u dx = \int_\Omega f(u^{(0)} + F) dx = \int_\Omega f u^{(0)} dx = \mathscr{B}_q(v, u^{(0)}) = -\mathscr{B}_q(v, F).$$

Using Lemma 2.1 with O instead of Ω this yields that

$$(-\Delta)^s v + q v = 0 \quad \text{in } O.$$

Since also $v|_O = 0$ (cf. Lemma 2.2), it follows from Theorem 3.3 that $v \equiv 0$ in $\mathbb{R}^n$ and thus $f = 0$ which proves the assertion. ◻

Theorem 3.4 implies the existence of a localized potentials sequence.

COROLLARY 3.5. *Let $n \in \mathbb{N}$, $\Omega \subseteq \mathbb{R}^n$ be an open set, $0 < s < 1$, $q \in L^\infty_+(\Omega)$, and $O \subseteq \Omega_e = \mathbb{R}^n \setminus \overline{\Omega}$ be an arbitrary open set. For every measurable set $M \subseteq \Omega$ with positive measure, there exists a sequence $F^k \in C_c^\infty(O)$, so that the corresponding solutions $u^k \in H^s(\mathbb{R}^n)$ of*

$$(-\Delta)^s u^k + q u^k = 0 \quad \text{in } \Omega, \quad u^k|_{\Omega_e} = F^k, \text{ for all } k \in \mathbb{N} \tag{3.9}$$

fulfill that

$$\int_M |u^k|^2 \, \mathrm{d}x \to \infty \quad \text{and} \quad \int_{\Omega \setminus M} |u^k|^2 \to 0 \text{ as } k \to \infty.$$

Proof. Using Theorem 3.4 there exists a sequence $\widetilde{F}^k$ so that the corresponding solutions $\widetilde{u}^k|_\Omega$ converge against $\frac{1}{|M|}\chi_M$ in $L^2(\Omega)$, and thus

$$\|\widetilde{u}^k\|^2_{L^2(M)} = \int_M |\widetilde{u}^k|^2 \,\mathrm{d}x \to 1, \quad \text{and} \quad \|\widetilde{u}^k\|^2_{L^2(\Omega\setminus M)} = \int_{\Omega\setminus M} |\widetilde{u}^k|^2 \,\mathrm{d}x \to 0.$$

Without loss of generality, we can assume for all $k \in \mathbb{N}$ that $\widetilde{u}^k \not\equiv 0$. Moreover, by possibly removing a sufficiently small open ball from M (so that the measure of M remains positive), we can assume that $\Omega \setminus M$ contains an open set. Thus, $\|\widetilde{u}^{(k)}\|_{L^2(\Omega\setminus M)} > 0$ follows from Theorem 3.3. Setting

$$F^k := \frac{\widetilde{F}^k}{\|\widetilde{u}^k\|^{1/2}_{L^2(\Omega\setminus M)}}$$

the sequence of corresponding solutions $u^k \in H^s(\mathbb{R}^n)$ of (3.9) has the desired property that

$$\|u^k\|^2_{L^2(M)} = \frac{\|\widetilde{u}^k\|^2_{L^2(M)}}{\|\widetilde{u}^k\|_{L^2(\Omega\setminus M)}} \to \infty, \quad \text{and} \quad \|u^k\|^2_{L^2(\Omega\setminus M)} = \|\widetilde{u}^k\|_{L^2(\Omega\setminus M)} \to 0,$$

as $k \to \infty$. □

The following lemma will also be useful for applying localized potentials in next sections.

Lemma 3.6. *Let $n \in \mathbb{N}$, $\Omega \subseteq \mathbb{R}^n$ be a bounded open set, $0 < s < 1$, and $q_0, q_1 \in L^\infty_+(\Omega)$. Set $D := \operatorname{supp}(q_0 - q_1)$. There exist constants $c, C > 0$ so that, for all $F \in H(\Omega_e)$, the solutions $u_0, u_1 \in H^s(\mathbb{R}^n)$ of*

$$(-\Delta)^s u_j + q_j u_j = 0 \quad \text{in } \Omega, \quad u_j|_{\Omega_e} = F \quad (j = 0, 1)$$

fufill

$$c\|u_1\|_{L^2(D)} \le \|u_0\|_{L^2(D)} \le C\|u_1\|_{L^2(D)}. \tag{3.10}$$

Proof. For the difference $w := u_0 - u_1 \in H^s_0(\Omega)$ we have that

$$0 = \mathscr{B}_{q_0}(u_0, w) = \mathscr{B}_{q_1}(u_1, w) \quad \text{for all } w \in H^s_0(\Omega).$$

With the coercivity constant $\alpha > 0$ of $\mathscr{B}_{q_1}$ we can estimate

$$\begin{aligned}
\alpha\|u_1 - u_0\|^2_{H^s(\Omega)} &\le \mathscr{B}_{q_1}(u_1 - u_0, u_1 - u_0) = -\mathscr{B}_{q_1}(u_0, u_1 - u_0) \\
&= \mathscr{B}_{q_0}(u_0, u_1 - u_0) - \mathscr{B}_{q_1}(u_0, u_1 - u_0) \\
&= \int_\Omega (q_0 - q_1) u_0 (u_1 - u_0) \,\mathrm{d}x \\
&\le \|q_0 - q_1\|_{L^\infty(\Omega)} \|u_0\|_{L^2(D)} \|u_1 - u_0\|_{H^s(\Omega)}.
\end{aligned}$$

Hence,

$$\begin{aligned}\|u_1\|_{L^2(D)} - \|u_0\|_{L^2(D)} &\leq \|u_1 - u_0\|_{L^2(D)} \leq \|u_1 - u_0\|_{H^s(\Omega)} \\ &\leq \frac{1}{\alpha}\|q_0 - q_1\|_{L^\infty(\Omega)}\|u_0\|_{L^2(D)},\end{aligned}$$

which shows that

$$\|u_1\|_{L^2(D)} \leq C\|u_0\|_{L^2(D)} \quad \text{with } C := 1 + \frac{1}{\alpha}\|q_0 - q_1\|_{L^\infty(\Omega)}.$$

The other inequality follows from interchanging q_0 and q_1. ☐

4. Converse monotonicity relations and the nonlocal Calderón problem. This section contains the first main result of this work. We will show that the monotonicity relation between the coefficients and the Dirichlet-to-Neumann operators holds in both directions, and use this to give a monotonicity-based, constructive proof of Ghosh, Salo and Uhlmann's uniqueness result for the Calderón problem for the fractional Schrödinger equation [20].

THEOREM 4.1. *Let $n \in \mathbb{N}$, $\Omega \subseteq \mathbb{R}^n$ be a bounded open set, and $0 < s < 1$. For any two potentials $q_0, q_1 \in L^\infty_+(\Omega)$, we have that*

$$q_0 \leq q_1 \text{ if and only if } \Lambda(q_0) \leq \Lambda(q_1). \tag{4.1}$$

Proof. From Corollary 3.2, we know that $q_0 \leq q_1$ implies $\Lambda(q_0) \leq \Lambda(q_1)$. Hence, it remains to show $\Lambda(q_0) \leq \Lambda(q_1)$ implies that $q_0 \leq q_1$ a.e. in Ω. We will prove this via contradiction and assume that $q_0 \leq q_1$ is not true a.e. in Ω. Then there exists $\delta > 0$ and a measurable set $M \subset \Omega$ with positive measure such that $q_0 - q_1 \geq \delta$ on M. Using the sequence of localized potentials from Corollary 3.5 for the coefficient q_0, and the monotonicity inequality (3.2) from Lemma 3.1, we obtain

$$\begin{aligned}\left\langle (\Lambda(q_1) - \Lambda(q_0))F^k, F^k \right\rangle &\leq \int_\Omega (q_1 - q_0)|u_0^k|^2 \,\mathrm{d}x \\ &\leq \|q_1 - q_0\|_{L^\infty(\Omega\setminus M)}\|u_0^{(k)}\|^2_{L^2(\Omega\setminus M)} - \delta\|u_0^k\|^2_{L^2(M)} \\ &\to -\infty, \text{ as } k \to \infty.\end{aligned}$$

This shows that $\Lambda(q_0) \not\leq \Lambda(q_1)$. ☐

Theorem 4.1 implies global uniqueness for the fractional Calderón problem. But let us stress again, that this has already been proven in [20] and we have used the results from [20] in our proof of the existence of localized potentials.

COROLLARY 4.2. *Let $n \in \mathbb{N}$, $\Omega \subseteq \mathbb{R}^n$ be a bounded open set, and $0 < s < 1$. For any two potentials $q_0, q_1 \in L^\infty_+(\Omega)$,*

$$q_0 = q_1 \text{ if and only if } \Lambda(q_0) = \Lambda(q_1).$$

Proof. This follows immediately from Theorem 4.1. ☐

Moreover, Theorem 4.1 suggests the constructive uniqueness result that q can be reconstructed from $\Lambda(q)$ by taking the supremum over an appropriate class of test functions ψ with $\Lambda(\psi) \leq \Lambda(q)$. For a rigorous formulation of this result we have to be attentive to the somewhat subtle fact that function values on null sets might still affect the supremum when the supremum is taken over uncountably many functions.

Recall that a point $x \in \mathbb{R}^n$ is called of *density one* for E if

$$\lim_{r \to 0} \frac{|B(x,r) \cap E|}{|B(x,r)|} = 1,$$

where $B(x,r)$ stands for the ball of radius r and centered at x. We therefore define the space of *density one simple functions*

$$\Sigma := \left\{ \psi = \sum_{j=1}^m a_j \chi_{M_j} : \ a_j \in \mathbb{R}, \ M_j \subseteq \Omega \text{ is a density one set} \right\},$$

where we call a subset $M \subseteq \Omega$ a *density one set* if it is non-empty, measurable and has Lebesgue density 1 in all $x \in M$. $\Sigma_+ \subseteq \Sigma$ denotes the subset of density one simple functions with positive essential infima on Ω (i.e., where all coefficients a_j are positive).

Lemma 4.3.

(a) Density one sets have positive measure.
(b) Finite intersections of density one sets are density one sets.
(c) If $\psi \in \Sigma$ is nonzero at some point $\hat{x} \in \Omega$, then there exists a density one set M containing $\hat{x}$ so that $\psi(x) = \psi(\hat{x})$ for all $x \in M$.

Proof. Assertion (a) is obvious. To prove (b), let $M_1, M_2 \subseteq \Omega$ be density one sets, and let $x \in M_1 \cap M_2$. Then

$$\lim_{r \to 0} \frac{|B(x,r) \setminus (M_1 \cap M_2)|}{|B(x,r)|} \leq \lim_{r \to 0} \frac{|B(x,r) \setminus M_1|}{|B(x,r)|} + \lim_{r \to 0} \frac{|B(x,r) \setminus M_2|}{|B(x,r)|} = 0,$$

which shows that

$$\lim_{r \to 0} \frac{|B(x,r) \cap M_1 \cap M_2|}{|B(x,r)|} = 1.$$

For the last assertion, let $\psi \in \Sigma$. Then $\psi = \sum_{j=1}^m a_j \chi_{M_j}$ with $a_j \in \mathbb{R}$ and density one sets $M_j \subseteq \Omega$. Then ψ is constant on the intersection of all M_j containing $\hat{x}$, so that (c) follows from (b). □

Lemma 4.3(c) shows that we might interpret the density one simple functions as simple functions where function values that are only attained on a null set are replaced by zero. Note also, that the Lebesgue's density theorem implies that every measurable set agrees almost everywhere with a density one set (see Corollary 3 in Section 1.7 of [13] for instance), and thus every simple function agrees with a density one simple function almost everywhere in Ω.

As before, we write $\psi \leq q$ if $\psi(x) \leq q(x)$ almost everywhere in Ω. We then have the following variant of the simple function approximation theorem.

Lemma 4.4. *For each function $q \in L_+^\infty(\Omega)$ we have that*

$$q(x) = \sup\{\psi(x) : \ \psi \in \Sigma_+, \ \psi \leq q\} \qquad \textit{almost everywhere in } \Omega.$$

Proof. By the standard simple function approximation theorem [54], there exists a sequence $(\psi_k)_{k\in\mathbb{N}}$ of simple functions with $\psi_k \leq q$ and $\|\psi_k - q\|_{L^\infty(\Omega)} \leq 1/k$. $q \in L^\infty_+(\Omega)$ implies that $\psi_k \in L^\infty_+(\Omega)$ for almost all $k \in \mathbb{N}$, and by changing the values of the countably many functions ψ_k on a null set we can assume that $\psi_k \in \Sigma_+$. This shows that

$$q(x) = \lim_{k\to\infty} \psi_k(x) \leq \sup\{\psi(x) : \ \psi \in \Sigma_+, \ \psi \leq q\} \quad \text{almost everywhere in } \Omega.$$

To show equality, it suffices to show that for each $\delta > 0$ the set

$$M := \{x \in \Omega : \ q(x) + \delta < \sup\{\psi(x) : \ \psi \in \Sigma_+, \ \psi \leq q\}\}$$

is a null set. To prove this, assume that M is not a null set for some $\delta > 0$. By removing a null set from M, we can assume that M is a density one set and that $q(x) > 0$ for all $x \in M$. By using the Lusin's theorem (see [54] for instance), all measurable function are approximately continuous at almost every point, M must contain a point $\hat{x}$ in which q is approximately continuous, and thus the set

$$M' := \{x \in \Omega : q(x) \leq q(\hat{x}) + \delta/3\}$$

has density one in $\hat{x}$. (see [13]). Removing a null set, we can assume that M' is a density one set still containing $\hat{x}$.

Moreover, by the definition of M, there must exist a $\psi \in \Sigma_+$ with $\psi \leq q$ and

$$q(\hat{x}) + \frac{2}{3}\delta \leq \psi(\hat{x}).$$

Since $q(\hat{x}) > 0$, there exists a density one set M'' containing $\hat{x}$ where $\psi(x) = \psi(\hat{x})$ for all $x \in M''$.

We thus have that

$$q(x) + \delta/3 \leq q(\hat{x}) + \frac{2}{3}\delta \leq \psi(\hat{x}) = \psi(x) \quad \text{for all } x \in M' \cap M'',$$

with density one sets M' and M'' that both contain $\hat{x}$, so that their intersection possesses positive measure. But this contradicts that $q(x) \geq \psi(x)$ almost everywhere, and thus shows that M is a null set for all $\delta > 0$, and hence

$$q(x) \geq \sup\{\psi(x) : \ \psi \in \Sigma_+, \ \psi \leq q\} \quad \text{almost everywhere in } \Omega.$$

□

COROLLARY 4.5. *Let $n \in \mathbb{N}$, $\Omega \subseteq \mathbb{R}^n$ be an open set, and $0 < s < 1$. A potential $q \in L^\infty_+(\Omega)$ is uniquely determined by $\Lambda(q)$ via the following formula*

$$q(x) = \sup\{\psi(x) : \ \psi \in \Sigma_+, \ \Lambda(\psi) \leq \Lambda(q)\} \quad \textit{almost everywhere in } \Omega.$$

Proof. This follows immediately from Theorem 4.1 and Lemma 4.4. □

5. Shape reconstruction by linearized monontonicity tests . The results in Section 4 show that the coefficient q in the fractional Schrödinger equation

$$(-\Delta)^s u + qu = 0 \quad \text{in } \Omega$$

can be reconstructed from the Dirichlet-to-Neumann operator $\Lambda(q)$ by comparing $\Lambda(q)$ with the DtN map $\Lambda(\psi)$ of (density one) simple functions ψ. A practical implementation of these monotonicity tests would require solving the fractional Schrödinger equation for each utilized simple function ψ.

In this section we will study the shape reconstruction problem of determining regions where a coefficient function $q \in L^\infty_+(\Omega)$ changes from a known reference function $q_0 \in L^\infty_+(\Omega)$ (e.g., q_0 may describe a background coefficient, and q_1 denotes the coefficient function in the presence of anomalies or scatterers). We will show that the support of $q_1 - q_0$ can be reconstructed with *linearized monotonicity tests* [32, 14]. These linearized tests only utilize the solution of the fractional Schrödinger equation with the reference coefficient function $q_0 \in L^\infty_+(\Omega)$. They do not require any other special solutions of the equation.

5.1. Linerization of the Dirichlet-to-Neumann operator. We start by showing Fréchet differentiability of the Dirichlet-to-Neumann operator.

LEMMA 5.1. *Let $n \in \mathbb{N}$, $\Omega \subseteq \mathbb{R}^n$ be a bounded open set, and $0 < s < 1$. The Dirichlet-to-Neumann operator*

$$\Lambda : \mathscr{D}(\Lambda) := L^\infty_+(\Omega) \subset L^\infty(\Omega) \to \mathcal{L}(H(\Omega_e), H(\Omega_e)^*), \quad q \mapsto \Lambda(q),$$

is Fréchet differentiable. At $q \in L^\infty_+(\Omega)$ its derivative is given by

$$\Lambda'(q) : L^\infty(\Omega) \to \mathcal{L}(H(\Omega_e), H(\Omega_e)^*), \quad r \mapsto \Lambda'(q)r,$$
$$\langle (\Lambda'(q)r)F, G\rangle := \int_\Omega r S_q(F) S_q(G)\,dx \quad \textit{for all } r \in L^\infty(\Omega),\ F, G \in H(\Omega_e),$$

where $S_q : H(\Omega_e) \to H^s(\mathbb{R}^n)$, $F \mapsto u$, is the solution operator of the Dirichlet problem

$$(-\Delta)^s u + qu = 0 \text{ in } \Omega \quad \text{and} \quad u|_{\Omega_e} = F.$$

Proof. Let $q \in L^\infty_+(\Omega)$. $\Lambda'(q)$ is a linear bounded operator since S_q is linear and bounded, cf. Lemma 2.3. For sufficiently small $r \in L^\infty(\Omega)$, so that $q + r \in L^\infty_+(\Omega)$, we obtain from the monotonicity relations (3.2) and (3.4) in Lemma 3.1 that for all $F \in H(\Omega_e)$,

$$0 \geq \langle (\Lambda(q+r) - \Lambda(q) - \Lambda'(q)r)\, F, F\rangle \geq \int_\Omega \left(\frac{q}{q+r} r - r \right) |u_q|^2 dx,$$

where $u_q = S_q(F)$.

Using that $\Lambda(q)$, $\Lambda(q+r)$, and $\Lambda'(q)r$ are symmetric operators, it follows that

$$\begin{aligned}
&\|\Lambda(q+r) - \Lambda(q) - \Lambda'(q)r\|_{\mathcal{L}(H(\Omega_e),H(\Omega_e)^*)} \\
&= \sup_{\|F\|_{H(\Omega_e)}=1} |\langle (\Lambda(q+r) - \Lambda(q) - \Lambda'(q)r)\, F, F\rangle| \\
&\leq \sup_{\|F\|_{H(\Omega_e)}=1} \int_\Omega \left| \frac{q}{q+r} r - r \right| |u_q|^2 dx \leq \left\| \frac{r^2}{q+r} \right\|_{L^\infty(\Omega)} \sup_{\|F\|_{H(\Omega_e)}=1} \|S_q(F)\|^2_{L^2(\Omega)} \\
&\leq \|r\|_{L^\infty(\Omega)} \left\| \frac{r}{q+r} \right\|_{L^\infty(\Omega)} \|S_q\|_{\mathcal{L}(H(\Omega_e),H^s(\mathbb{R}^n))},
\end{aligned}$$

which shows

$$\lim_{\|r\|_{L^\infty(\Omega)}\to 0} \frac{\|\Lambda(q+r) - \Lambda(q) - \Lambda'(q)r\|_{\mathcal{L}(H(\Omega_e),H(\Omega_e)^*)}}{\|r\|_{L^\infty(\Omega)}} = 0.$$

□

Remark 5.2. *Using the Fréchet derivative, the monotonicity relations* (3.2) *and* (3.4) *in Lemma 3.1 can be written as follows. For all* $q_0, q_1 \in L^\infty_+(\Omega)$

$$\Lambda'(q_0)(q_1 - q_0) \geq \Lambda(q_1) - \Lambda(q_0) \geq \Lambda'(q_0)\left(\frac{q_0}{q_1}(q_1 - q_0)\right).$$

We also have an analogue of the monotonicity result in Theorem 4.1.

Theorem 5.3. *Let* $n \in \mathbb{N}$, $\Omega \subseteq \mathbb{R}^n$ *be a bounded open set, and* $0 < s < 1$. *Then for all* $q \in L^\infty_+(\Omega)$ *and* $r_0, r_1 \in L^\infty(\Omega)$,

$$r_0 \leq r_1 \quad \textit{if and only if} \quad \Lambda'(q)r_0 \leq \Lambda'(q)r_1.$$

Proof. If $r_0 \leq r_1$ then $\Lambda'(q)r_0 \leq \Lambda'(q)r_1$ follows immediately from the characterization of $\Lambda'(q)$ in Lemma 5.1. The converse follows from the same localized potentials argument as in the proof of Theorem 4.1. □

Note that this implies uniqueness of the linearized fractional Calderón problem:

Corollary 5.4. *Let* $n \in \mathbb{N}$, $\Omega \subseteq \mathbb{R}^n$ *be a bounded open set, and* $0 < s < 1$. *For all* $q \in L^\infty_+(\Omega)$, *the Fréchet derivative* $\Lambda'(q)$ *is injective, i.e.*

$$\Lambda'(q)r = 0 \quad \textit{if and only if} \quad r = 0.$$

Proof. This follows immediately from Theorem 5.3. □

5.2. Reconstructing the support of a coefficient change. In this subsection, let $n \in \mathbb{N}$, $\Omega \subseteq \mathbb{R}^n$ be a bounded open set, and $0 < s < 1$. As in the introduction, let $q_0 \in L^\infty_+(\Omega)$ denote a known reference coefficient, and $q_1 \in L^\infty_+(\Omega)$ denote an unknown coefficient function that differs from the reference value q_0 in certain regions. We aim to find these anomalous regions (or scatterers), i.e., the support of $q_1 - q_0$, from the difference of the Dirichlet-to-Neumann-operators $\Lambda(q_1) - \Lambda(q_0)$.

To that end, we introduce, for a measurable subset $M \subseteq \Omega$, the testing operator $\mathcal{T}_M : H(\Omega_e) \to H(\Omega_e)^*$ by setting $T_M := \Lambda'(q_0)\chi_M$. i.e.,

$$\langle \mathcal{T}_M F, G\rangle := \int_M S_{q_0}(F) S_{q_0}(G) dx \quad \text{for all } F, G \in H(\Omega_e), \tag{5.1}$$

where, as in Lemma 5.1, $S_{q_0} : \; H(\Omega_e) \to H^s(\mathbb{R}^n)$, $F \mapsto u_0$, denotes the solution operator of the reference Dirichlet problem

$$(-\Delta)^s u_0 + q_0 u_0 = 0 \text{ in } \Omega \quad \text{and} \quad u_0|_{\Omega_e} = F.$$

The following theorem shows that we can find the support of $q - q_0$ by shrinking closed sets, cf. [32, 16].

Theorem 5.5. *For each closed subset $C \subseteq \Omega$,*

$$\operatorname{supp}(q_1 - q_0) \subseteq C \quad \textit{if and only if} \quad \exists \alpha > 0 : \; -\alpha \mathcal{T}_C \leq \Lambda(q_1) - \Lambda(q_0) \leq \alpha \mathcal{T}_C.$$

Hence,

$$\operatorname{supp}(q_1 - q_0) = \bigcap \{C \subseteq \Omega \textit{ closed}: \; \exists \alpha > 0 : \; -\alpha \mathcal{T}_C \leq \Lambda(q_1) - \Lambda(q_0) \leq \alpha \mathcal{T}_C\}.$$

Proof.

(a) Let $\operatorname{supp}(q_1 - q_0) \subseteq C$. Then every sufficiently large $\alpha > 0$ fulfills

$$q_1 \leq q_0 + \alpha \chi_C.$$

Using Theorem 4.1 and Remark 5.2, we thus obtain

$$\Lambda(q_1) \leq \Lambda(q_0 + \alpha\chi_C) \leq \Lambda(q_0) + \Lambda'(q_0)\alpha\chi_C = \Lambda(q_0) + \alpha \mathcal{T}_C.$$

Moreover, for sufficiently small $\beta > 0$ we also have that

$$q_1 \geq q_0 + (\beta - q_0)\chi_C \quad \text{and} \quad q_0 \geq \beta$$

and thus (using Theorems 4.1, 5.3, and Remark 5.2)

$$\begin{aligned} \Lambda(q_1) - \Lambda(q_0) &\geq \Lambda(q_0 + (\beta - q_0)\chi_C) - \Lambda(q_0) \\ &\geq \Lambda'(q_0) \left(\frac{q_0}{q_0 + (\beta - q_0)\chi_C} (\beta - q_0)\chi_C \right) \\ &\geq -\Lambda'(q_0) \left(\frac{q_0^2}{q_0 + (\beta - q_0)\chi_C} \chi_C \right) \\ &\geq -\frac{1}{\beta} \|q_0\|^2_{L^\infty(\Omega)} \Lambda'(q_0)\chi_C, \end{aligned}$$

which shows that

$$\Lambda(q_1) - \Lambda(q_0) \geq -\alpha \mathcal{T}_C$$

is also fulfilled for sufficiently large $\alpha > 0$.

(b) To show the converse implication, let $\alpha > 0$ fulfill

$$-\alpha\mathcal{T}_C \leq \Lambda(q_1) - \Lambda(q_0) \leq \alpha\mathcal{T}_C.$$

Then we obtain using Remark 5.2

$$\Lambda'(q_0)(-\alpha\chi_C) = -\alpha\mathcal{T}_C \leq \Lambda(q_1) - \Lambda(q_0) \leq \Lambda'(q_0)(q_1 - q_0),$$

so that it follows from Theorem 5.3 that

$$q_1 - q_0 \geq -\alpha\chi_C,$$

and in particular $q_1 - q_0 \geq 0$ almost everywhere on $\Omega \setminus C$.
Likewise we obtain using Remark 5.2

$$\Lambda'(q_0)(\alpha\chi_C) = \alpha\mathcal{T}_C \geq \Lambda(q_1) - \Lambda(q_0) \geq \Lambda'(q_0)\left(\frac{q_0}{q_1}(q_1 - q_0)\right),$$

so that it follows from Teorem 5.3 that

$$\frac{q_0}{q_1}(q_1 - q_0) \leq \alpha\chi_C.$$

Since $q_1, q_0 \in L^\infty_+(\Omega)$ this yields that $q_1 - q_0 \leq 0$ almost everywhere on $\Omega \setminus C$. Hence, $q_1 = q_0$ almost everywhere in the open set $\Omega \setminus C$ and thus $\operatorname{supp}(q_1 - q_0) \subseteq C$.

□

In the definite case that either $q_1 \geq q_0$ or $q_1 \leq q_0$ holds almost everywhere in Ω, we can also use the union of small test balls to characterize the so-called *inner support* of $q_1 - q_0$. The inner support $\operatorname{inn\,supp}(r)$ of a measurable function $r:\ \Omega \to \mathbb{R}$ is defined as the union of all open sets U on which the essential infimum of $|\kappa|$ is positive, cf. [32, Section 2.2].

THEOREM 5.6.

(a) Let $q_1 \leq q_0$. For every open set $B \subseteq \Omega$ and every $\alpha > 0$
(1) $q_1 \leq q_0 - \alpha\chi_B$ implies $\Lambda(q_1) \leq \Lambda(q_0) - \alpha\mathcal{T}_B$.
(2) $\Lambda(q_1) \leq \Lambda(q_0) - \alpha\mathcal{T}_B$ implies $B \subseteq \operatorname{inn\,supp}(q_1 - q_0)$.
Hence,

$$\operatorname{inn\,supp}(q_1 - q_0) = \bigcup\{B \subseteq \Omega \textit{ open ball}:\ \exists \alpha > 0 : \Lambda(q_1) \leq \Lambda(q_0) - \alpha\mathcal{T}_B\}.$$

(b) Let $q_1 \geq q_0$. For every open set $B \subseteq \Omega$ and every $\alpha > 0$
(1) $q_1 \geq q_0 + \alpha\chi_B$ implies $\Lambda(q_1) \geq \Lambda(q_0) + \tilde{\alpha}\mathcal{T}_B$ with $\tilde{\alpha} := \frac{\inf(q_0)\alpha}{\inf(q_0)+\alpha}$.
(2) $\Lambda(q_1) \geq \Lambda(q_0) + \alpha\mathcal{T}_B$ implies $B \subseteq \operatorname{inn\,supp}(q - q_0)$.
Hence,

$$\operatorname{inn\,supp}(q_1 - q_0) = \bigcup\{B \subseteq \Omega \textit{ open ball}:\ \exists \alpha > 0 : \Lambda(q_1) \geq \Lambda(q_0) + \alpha\mathcal{T}_B\}.$$

Proof.

(a) If $q_1 \leq q_0 - \alpha\chi_B$, then we obtain using Theorem 5.3, and Remark 5.2 that

$$\Lambda(q_1) - \Lambda(q_0) \leq \Lambda'(q_0)(q_1 - q_0) \leq -\alpha\Lambda'(q_0)\chi_B = -\alpha\mathcal{T}_B.$$

On the other hand, if $\Lambda(q) \leq \Lambda(q_0) - \alpha \mathcal{T}_B$ then we obtain from Remark 5.2 that

$$-\alpha \Lambda'(q_0)\chi_B = -\alpha \mathcal{T}_B \geq \Lambda(q_1) - \Lambda(q_0) \geq \Lambda'(q_0)\left(\frac{q_0}{q_1}(q_1 - q_0)\right)$$

so that it follows from Theorem 5.3 that

$$-\alpha \chi_B \geq \frac{q_0}{q_1}(q_1 - q_0).$$

Hence, $q_0 - q_1 \geq \frac{\inf(q_1)}{\sup(q_0)}\alpha$ almost everywhere on B and thus $B \subseteq \operatorname{inn\,supp}(q_1 - q_0)$.

(b) If $q_1 \geq q_0 + \alpha\chi_B$, then we obtain using Theorems 4.1, 5.3, and Remark 5.2 that

$$\begin{aligned}\Lambda(q_1) - \Lambda(q_0) &\geq \Lambda(q_0 + \alpha\chi_B) - \Lambda(q_0)\\ &\geq \Lambda'(q_0)\left(\frac{q_0}{q_0 + \alpha\chi_B}\alpha\chi_B\right) = \Lambda'(q_0)\left(\left(1 - \frac{\alpha}{q_0 + \alpha}\right)\alpha\chi_B\right)\\ &\geq \Lambda'(q_0)\left(\left(1 - \frac{\alpha}{\inf(q_0) + \alpha}\right)\alpha\chi_B\right) = \frac{\inf(q_0)\alpha}{\inf(q_0) + \alpha}\mathcal{T}_B.\end{aligned}$$

On the other hand, if $\Lambda(q_1) \geq \Lambda(q_0) + \alpha\mathcal{T}_B$ then we obtain from Remark 5.2 that

$$\alpha\Lambda'(q_0)\chi_B = \alpha\mathcal{T}_B \leq \Lambda(q_1) - \Lambda(q_0) \leq \Lambda'(q_0)(q_1 - q_0),$$

so that it follows from Theorem 5.3 that

$$\alpha\chi_B \leq q_1 - q_0,$$

and thus $B \subseteq \operatorname{inn\,supp}(q_1 - q_0)$.

□

6. Discussion and Outlook. We have shown an if-and-only-if monotonicity relation between a positive potential $q \in L^\infty_+(\Omega)$ in the fractional Schrödinger equation, and the associated Dirichlet-to-Neumann operator $\Lambda(q)$ (cf. Theorem 4.1)

$$q_0 \leq q_1 \text{ if and only if } \Lambda(q_0) \leq \Lambda(q_1).$$

From this we obatined a constructive uniqueness result for the Calderón problem for the fractional Schrödinger equation. The potential is uniquely determined by the simple reconstruction formula (cf. Corollary 4.5)

$$q(x) = \sup\left\{\psi(x):\ \psi \text{ positive (density one) simple function}, \Lambda(\psi) \leq \Lambda(q)\right\}.$$

Let us give some remarks on a possible practical implementation of our results. First of all, let us stress that the localized potentials used in this work can be created with Dirichlet data supported in arbitrarily small open subsets $\emptyset \neq O \subseteq \Omega_e$, cf. Corollary 3.5. Hence, all results in this work remain valid if the full data DtN is replaced by the partial data DtN

$$\Lambda(q):\ H^s_0(O) \to H^{-s}(O),$$

where $H_0^s(O)$ is the closure of $C_c^\infty(O)$ in $H^s(\mathbb{R}^n)$, and $H^{-s}(O) := H_0^s(O)'$.

For a numerical implementation, one could choose a family of characteristic functions $\chi_1, \dots, \chi_M$ for disjoint density one sets (e.g., a pixel partition) $P_1, \dots, P_M \subseteq \Omega$, $M \in \mathbb{N}$, and determine

$$\alpha_m := \sup\{\alpha \in \mathbb{R} : \ \Lambda(\alpha_m \chi_m) \leq \Lambda(q)\}.$$

Then, $\psi = \sum_{m=1}^M \alpha_m \chi_m$ is the largest piecewise-constant function (on the given partition) with $\psi \leq q$. Analogously, one could obtain a piecewise-constant function approximation q from above. A numerical implementation of this approach would be computationally rather expensive as it requires solving the fractional Schrödinger equation for a large number of sets P_m and contrast levels α (though these solutions could be precomputed in advance).

A computationally more efficient approach can be used for detecting regions where the potential q differs from a known reference function q_0. The support of this change can be determined by shrinking closed sets according to the formula (cf. Theorem 5.5)

$$\operatorname{supp}(q - q_0) = \bigcap \{C \subseteq \Omega \text{ closed} : \ \exists \alpha > 0 : \ -\alpha \mathcal{T}_C \leq \Lambda(q) - \Lambda(q_0) \leq \alpha \mathcal{T}_C\},$$

where the operator $\mathcal{T}_C$ can be calculated from integrating the solution of the fractional Schrödinger equation for the reference potential q_0 over the set C, and no other PDE solutions are required for this approach. Moreover, the inner support of the potential change can be calculated by comparing $\Lambda(q) - \Lambda(q_0)$ with T_B for open balls B, cf. Theorem 5.6.

Algorithms based on linearized monotonicity tests have been successfully applied to the standard Laplacian case ($s = 1$), cf. the works cited in the introduction. Among these works, let us mention the recent papers [15, 29] that show reconstructions on simulated and real-life measurement data, and discuss practical implementation issues and the regularization of measurement errors.

For the standard Laplacian case, monotonicity-based reconstruction methods have recently been extended to the Schrödinger (or Helmholtz) equation with general (not-necessarily positive) potential function $q \in L^\infty(\Omega)$, cf. [29, 21], and monotonicity arguments were also used to prove stability results, cf. [27, 24, 61]. It can be expected that these results can also be extended to the fractional diffusion case, and we aim to study these questions in future research.

Acknowledgment. Yi-Hsuan Lin is partially supported by MOST of Taiwan under the project 160-2917-I-564-048.

REFERENCES

[1] G. Alessandrini. Singular solutions of elliptic equations and the determination of conductivity by boundary measurements. *J. Differential Equations*, 84(2):252–272, 1990.

[2] L. Arnold and B. Harrach. Unique shape detection in transient eddy current problems. *Inverse Problems*, 29(9):095004, 2013.

[3] A. Barth, B. Harrach, N. Hyvönen, and L. Mustonen. Detecting stochastic inclusions in electrical impedance tomography. *Inverse Problems*, 33(11):115012, 2017.

[4] T. Brander, B. Harrach, M. Kar, and M. Salo. Monotonicity and enclosure methods for the p-Laplace equation. *SIAM J. Appl. Math.*, 78(2):742–758, 2018.

[5] T. Brander, M. Kar, and M. Salo. Enclosure method for the p-Laplace equation. *Inverse Problems*, 31(4):045001, 2015.

[6] T. Brander, B. von Harrach, M. Kar, and M. Salo. Monotonicity and enclosure methods for the p-Laplace equation. *arXiv preprint arXiv:1703.02814*, 2017.

[7] C. Bucur and E. Valdinoci. *Nonlocal diffusion and applications*, volume 20. Springer, 2016.

[8] X. Cao, Y.-H. Lin, and H. Liu. Simultaneously recovering potentials and embedded obstacles for anisotropic fractional Schrödinger operators. *Inverse Problems and Imaging*, 13(1):197–210, 2019.

[9] X. Cao and H. Liu. Determining a fractional Helmholtz system with unknown source and medium parameter. *arXiv preprint arXiv:1803.09538*, 2018.

[10] M. Cekić, Y.-H. Lin, and A. Rüland. The Calderón problem for the fractional Schrödinger equation with drift. *arXiv preprint arXiv:1810.04211*, 2018.

[11] G. Covi. Inverse problems for a fractional conductivity equation. *arXiv preprint arXiv:1810.06319*, 2018.

[12] E. Di Nezza, G. Palatucci, and E. Valdinoci. Hitchhiker's guide to the fractional Sobolev spaces. *Bulletin des Sciences Mathématiques*, 136(5):521–573, 2012.

[13] L. C. Evans and R. F. Gariepy. *Measure theory and fine properties of functions*. CRC press, 2015.

[14] H. Garde. Comparison of linear and non-linear monotonicity-based shape reconstruction using exact matrix characterizations. *Inverse Problems in Science and Engineering*, pages 1–18, 2017.

[15] H. Garde and S. Staboulis. Convergence and regularization for monotonicity-based shape reconstruction in electrical impedance tomography. *Numerische Mathematik*, 135(4):1221–1251, 2017.

[16] H. Garde and S. Staboulis. The regularized monotonicity method: Detecting irregular indefinite inclusions. *Inverse Probl. Imaging*, 13(1):93–116, 2019.

[17] B. Gebauer. Localized potentials in electrical impedance tomography. *Inverse Probl. Imaging*, 2(2):251–269, 2008.

[18] T. Ghosh, Y.-H. Lin, and J. Xiao. The Calderón problem for variable coefficients nonlocal elliptic operators. *Communications in Partial Differential Equations*, 42(12):1923–1961, 2017.

[19] T. Ghosh, A. Rüland, M. Salo, and G. Uhlmann. Uniqueness and reconstruction for the fractional Calderón problem with a single measurement. *arXiv preprint arXiv:1801.04449*, 2018.

[20] T. Ghosh, M. Salo, and G. Uhlmann. The Calderón problem for the fractional Schrödinger equation. *arXiv preprint arXiv:1609.09248*, 2016.

[21] R. Griesmaier and B. Harrach. Monotonicity in inverse medium scattering on unbounded domains. *SIAM J. Appl. Math*, 78(5):2533–2557, 2018.

[22] B. Harrach. On uniqueness in diffuse optical tomography. *Inverse Problems*, 25:055010 (14pp), 2009.

[23] B. Harrach. Simultaneous determination of the diffusion and absorption coefficient from boundary data. *Inverse Probl. Imaging*, 6(4):663–679, 2012.

[24] B. Harrach. Uniqueness and Lipschitz stability in electrical impedance tomography with finitely many electrodes. *Inverse Problems*, 2018.

[25] B. Harrach, E. Lee, and M. Ullrich. Combining frequency-difference and ultrasound modulated electrical impedance tomography. *Inverse Problems*, 31(9):095003, 2015.

[26] B. Harrach, Y.-H. Lin, and H. Liu. On localizing and concentrating electromagnetic fields. *SIAM J. Appl. Math*, 78(5):2558–2574, 2018.

[27] B. Harrach and H. Meftahi. Global uniqueness and lipschitz-stability for the inverse robin transmission problem. *arXiv preprint arXiv:1808.01806*, 2018.

[28] B. Harrach and M. N. Minh. Enhancing residual-based techniques with shape reconstruction features in electrical impedance tomography. *Inverse Problems*, 32(12):125002, 2016.

[29] B. Harrach and M. N. Minh. Monotonicity-based regularization for phantom experiment data in electrical impedance tomography. In *New Trends in Parameter Identification for Mathematical Models*, pages 107–120. Springer, 2018.

[30] B. Harrach, V. Pohjola, and M. Salo. Monotonicity and local uniqueness for the Helmholtz equation. *Anal. PDE*, to appear.

[31] B. Harrach and J. K. Seo. Exact shape-reconstruction by one-step linearization in electrical impedance tomography. *SIAM Journal on Mathematical Analysis*, 42(4):1505–1518, 2010.

[32] B. Harrach and M. Ullrich. Monotonicity-based shape reconstruction in electrical impedance tomography. *SIAM Journal on Mathematical Analysis*, 45(6):3382–3403, 2013.

[33] B. Harrach and M. Ullrich. Resolution guarantees in electrical impedance tomography. *IEEE Trans. Med. Imaging*, 34:1513–1521, 2015.

[34] B. Harrach and M. Ullrich. Local uniqueness for an inverse boundary value problem with partial

data. *Proceedings of the American Mathematical Society*, 145(3):1087–1095, 2017.
[35] M. Ikehata. Size estimation of inclusion. *Journal of Inverse and Ill-Posed Problems*, 6(2):127–140, 1998.
[36] M. Ikehata. Identification of the shape of the inclusion having essentially bounded conductivity. *Journal of Inverse and Ill-Posed Problems*, 7(6):533–540, 1999.
[37] V. Isakov. On uniqueness of recovery of a discontinuous conductivity coefficient. *Comm. Pure Appl. Math.*, 41(7):865–877, 1988.
[38] V. Isakov. *Inverse problems for partial differential equations*, volume 127. Springer, 2006.
[39] H. Kang, J. K. Seo, and D. Sheen. The inverse conductivity problem with one measurement: stability and estimation of size. *SIAM Journal on Mathematical Analysis*, 28(6):1389–1405, 1997.
[40] M. Kar and M. Sini. Reconstruction of interfaces using CGO solutions for the Maxwell equations. *J. Inverse Ill-Posed Probl.*, 22(2):169–208, 2014.
[41] C. Kenig, M. Salo, and G. Uhlmann. Inverse problems for the anisotropic Maxwell equations. *Duke Math. J.*, 157(2):369–419, 2011.
[42] R. V. Kohn and M. Vogelius. Determining conductivity by boundary measurements. *Communications on Pure and Applied Mathematics*, 37(3):289–298, 1984.
[43] R. V. Kohn and M. Vogelius. Determining conductivity by boundary measurements II. Interior results. *Communications on Pure and Applied Mathematics*, 38(5):643–667, 1985.
[44] R. Kuan, Y.-H. Lin, and M. Sini. The enclosure method for the anisotropic Maxwell system. *SIAM Journal on Mathematical Analysis*, 47(5):3488–3527, 2015.
[45] M. Kwaśnicki. Ten equivalent definitions of the fractional Laplace operator. *Fractional Calculus and Applied Analysis*, 20(1):7–51, 2017.
[46] R.-Y. Lai and Y.-H. Lin. Global uniqueness for the fractional semilinear Schrödinger equation. *Proceedings of the American Mathematical Society*, 2018.
[47] Y.-H. Lin. Reconstruction of penetrable obstacles in the anisotropic acoustic scattering. *Inverse problems and imging*, 10(3):765–780, 2016.
[48] A. Maffucci, A. Vento, S. Ventre, and A. Tamburrino. A novel technique for evaluating the effective permittivity of inhomogeneous interconnects based on the monotonicity property. *IEEE Transactions on Components, Packaging and Manufacturing Technology*, 6(9):1417–1427, 2016.
[49] G. Nakamura and R. Potthast. *Inverse Modeling - An introduction to the theory and methods of inverse problems and data assimilation*. IOP Publishing, Bristol UK, 2015.
[50] G. Nakamura, G. Uhlmann, and J.-N. Wang. Oscillating-decaying solutions, Runge approximation property for the anisotropic elasticity system and their applications to inverse problems. *J. Math. Pures Appl. (9)*, 84(1):21–54, 2005.
[51] G. Nakamura, G. Uhlmann, and J.-N. Wang. Oscillating-decaying solutions for elliptic systems. In *Inverse problems, multi-scale analysis and effective medium theory*, volume 408 of *Contemp. Math.*, pages 219–230. Amer. Math. Soc., Providence, RI, 2006.
[52] G. Nakamura and K. Yoshida. Identification of a non-convex obstacle for acoustical scattering. *Journal of Inverse and Ill-posed Problems*, 15(6):611–624, 2007.
[53] X. Ros-Oton. Nonlocal elliptic equations in bounded domains: a survey. *arXiv preprint arXiv:1504.04099*, 2015.
[54] H. L. Royden and P. Fitzpatrick. *Real analysis*, volume 32. Macmillan New York, 1988.
[55] A. Rüland and M. Salo. The fractional Calderón problem: low regularity and stability. *arXiv preprint arXiv:1708.06294*, 2017.
[56] A. Rüland and M. Salo. The fractional Calderón problem problem: lower regularity and stability. *arXiv preprint arXiv:1708.06294*, 2017.
[57] A. Rüland and M. Salo. Exponential instability in the fractional Calderón problem. *Inverse Problems*, 34(4):045003, 2018.
[58] A. Rüland and E. Sincich. Lipschitz stability for the finite dimensional fractional Calderón problem with finite cauchy data. *arXiv preprint arXiv:1805.00866*, 2018.
[59] M. Salo. The fractional Calderón problem. *arXiv preprint arXiv:1711.06103*, 2017.
[60] M. Salo and J.-N. Wang. Complex spherical waves and inverse problems in unbounded domains. *Inverse Problems*, 22(6):2299–2309, 2006.
[61] J. K. Seo, K. C. Kim, A. Jargal, K. Lee, and B. Harrach. A learning-based method for solving ill-posed nonlinear inverse problems: a simulation study of lung eit. *arXiv preprint arXiv:1810.10112*, 2018.
[62] M. Sini and K. Yoshida. On the reconstruction of interfaces using complex geometrical optics solutions for the acoustic case. *Inverse Problems*, 28(5): 055013, 2012.
[63] E. M. Stein. *Singular integrals and differentiability properties of functions (PMS-30)*, volume 30. Princeton university press, 2016.

[64] Z. Su, L. Udpa, G. Giovinco, S. Ventre, and A. Tamburrino. Monotonicity principle in pulsed eddy current testing and its application to defect sizing. In *Applied Computational Electromagnetics Society Symposium-Italy (ACES), 2017 International*, pages 1–2. IEEE, 2017.

[65] A. Tamburrino and G. Rubinacci. A new non-iterative inversion method for electrical resistance tomography. *Inverse Problems*, 18(6):1809, 2002.

[66] A. Tamburrino, Z. Sua, S. Ventre, L. Udpa, and S. S. Udpa. Monotonicity based imang method in time domain eddy current testing. *Electromagnetic Nondestructive Evaluation (XIX)*, 41:1, 2016.

[67] G. Uhlmann and J.-N. Wang. Reconstructing discontinuities using complex geometrical optics solutions. *SIAM J. Appl. Math.*, 68(4):1026–1044, 2008.

[68] S. Ventre, A. Maffucci, F. Caire, N. Le Lostec, A. Perrotta, G. Rubinacci, B. Sartre, A. Vento, and A. Tamburrino. Design of a real-time eddy current tomography system. *IEEE Transactions on Magnetics*, 53(3):1–8, 2017.

[69] L. Zhou, B. Harrach, and J. K. Seo. Monotonicity-based electrical impedance tomography for lung imaging. *Inverse Problems*, 34(4):045005, 2018.