AN INVERSE PROBLEM FOR A SEMILINEAR ELLIPTIC EQUATION ON CONFORMALLY TRANSVERSALLY ANISOTROPIC MANIFOLDS

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Abstract. Given a conformally transversally anisotropic manifold (M, g), we consider the semilinear elliptic equation

$$(-\Delta_g + V)u + qu^2 = 0 \quad \text{on } M.$$

We show that an a priori unknown smooth function q can be uniquely determined from the knowledge of the Dirichlet-to-Neumann map associated to the equation. This extends the previously known results of the works [FO20, LLLS21a]. Our proof is based on over-differentiating the equation: We linearize the equation to orders higher than the order two of the nonlinearity qu^2, and introduce non-vanishing boundary traces for the linearizations. We study interactions of two or more products of the so-called Gaussian quasimode solutions to the linearized equation. We develop an asymptotic calculus to solve Laplace equations, which have these interactions as source terms.

Keywords. Inverse problems, boundary determination, semilinear elliptic equation, Riemannian manifold, conformally transversally anisotropic, Gaussian quasimodes, WKB construction.

Contents

1. Introduction

Let (M,g) be a smooth compact Riemannian manifold of dimension $n \geq 3$ with a smooth boundary. We assume that (M,g) is *conformally transversally anisotropic* (CTA), that is to say,

$$M \Subset I \times M_0, \tag{1.1}$$

and the metric g has a smooth extension to $\mathbb{R} \times M_0$ so that

$$g = c(x_1, x')\left(dx_1 \otimes dx_1 + g_0(x')\right), \tag{1.2}$$

where (M_0, g_0) is a compact $(n-1)$-dimensional Riemannian manifold with a smooth boundary ∂M_0 [DSFKSU09]. Let q, V be real-valued smooth functions on M and consider the semi-linear elliptic equation:

$$\begin{cases} (-\Delta_g + V)u + qu^2 = 0 & \text{in } M, \\ u = f & \text{on } \partial M. \end{cases} \tag{1.3}$$

We make the standing assumption that 0 is not a Dirichlet eigenvalue for the operator $-\Delta_g + V$. As shown in [LLLS21a, Proposition 2.1], the Dirichlet problem (1.3) is well-posed for sufficiently small Dirichlet data f. Precisely, given any $\alpha \in (0,1)$, there exists $C, \delta > 0$ such that for all

$$f \in U_\delta = \left\{ h \in C^{2,\alpha}(\partial M) \mid \|f\|_{C^{2,\alpha}(\partial M)} \leq \delta \right\},$$

the Dirichlet problem (1.3) has a unique solution u in the set

$$\left\{ w \in C^{2,\alpha}(M) \mid \|w\|_{C^{2,\alpha}(M)} \leq C\delta \right\}. \tag{1.4}$$

Moreover,

$$\|u\|_{C^{2,\alpha}(M)} \leq C\|f\|_{C^{2,\alpha}(\partial M)}.$$

We define the associated *Dirichlet-to-Neumann* map (DN map in short) for (1.3) by

$$\Lambda_q f = \partial_\nu u|_{\partial M} \quad \text{for} \quad f \in U_\delta,$$

where u is the unique solution to (1.3) that lies in the set (1.4) and ν denotes the unit outward normal vector field on ∂M.

In this paper, we consider the following inverse problem: Given an a priori fixed CTA manifold (M,g) and a smooth zeroth order coefficient V, is it possible to recover an a priori unknown function q given the knowledge of the map Λ_q? We show that this is indeed possible under the following minor technical assumption on the transversal manifold (M_0, g_0):

(H1) *Given any $p \in M_0$, there exists a non-tangential geodesic passing through p that has no self-intersections.*

Precisely, we prove the following uniqueness result.

Theorem 1.1. *Let (M,g) be a conformally transversally anisotropic manifold of the form (1.1)–(1.2) and suppose that (H1) is satisfied. Let $V \in C^\infty(M)$ and assume that zero is not a Dirichlet eigenvalue for $-\Delta_g + V$ on M. Let $q_1, q_2 \in C^\infty(M)$ and assume that for some $\delta > 0$ sufficiently small and for any $f \in U_\delta$*

$$\Lambda_{q_1} f = \Lambda_{q_2} f.$$

Then

$$q_1 = q_2 \quad \text{in } M.$$

We will provide a discussion of the geometric assumption (H1) (and the possibility to remove it entirely) as well as the main novelties of Theorem 1.1 in Section 1.2.

1.1. **Previous literature.** Inverse problems for non-linear partial differential equations is a topic with a vast literature. When the manifold is assumed to be Euclidean, the first result goes back to the work Isakov and Sylvester in [IS94] where the authors considered the equation

$$-\Delta u + F(x,u) = 0,$$

on a Euclidean domain of dimension greater than or equal to three and studied the problem of recovering a class of non-linear functions $F(x,u)$ that satisfy a homogeneity property as well as certain monotonicity and growth conditions on its partial derivatives. The analogous problem in dimension two was first solved by Isakov and Nachman in [IN95]. For further results in Euclidean geometries, we refer the reader to the works [Sun04, Sun10, LLLS21a, LLLS21b, KU20c, HL22] in the context semilinear elliptic equations, to [Sun96, SU97, HS02, MnU20, LLS20, CF20, CFK$^+$21, CF21, CNV19, Sha21] in the context of quasilinear elliptic equations and to [LL22a, Lin21] for fractional semilinear elliptic equations. We also mention the early work [Isa93] and the work [KU20a] on similar results on Euclidean geometries for parabolic equations.

Most of the results discussed above are based on the idea of *higher order linearization* of nonlinear equations. The idea of a first or a second order linearization was initiated by in [Isa93, IS94] and the idea of higher order linearizations was introduced and developed fully by Kurylev, Lassas and Uhlmann [KLU18] in the context of nonlinear hyperbolic equations over Lorentzian geometries. There, the authors showed that in geometric settings, it is possible to solve certain classes of inverse problems for nonlinear hyperbolic equations in a much broader geometric generality compared to analogous inverse problems stated for linear hyperbolic equations. We refer the reader to the works [WZ19, LUW17, LUW18, UZ21b, HUZ21, UZ21a, FLO21, LLPMT21] for more examples of inverse problems for nonlinear hyperbolic equations solved in broad Lorentzian geometries. We also point out the simultaneous recovery results [LLL21, LLLZ21] in inverse problems for semilinear parabolic and hyperbolic equations in the Euclidean space.

Recently, the works [FO20, LLLS21a] introduced a similar higher order linearization approach in the context of semilinear elliptic equations on CTA manifolds. We also refer the reader to the more recent works [KU20b, LLST22] on study of similar inverse problems for nonlinear elliptic equations stated on CTA manifolds. In [FO20, LLLS21a], it was proved that for elliptic semilinear equations of the form

$$-\Delta_g u + F(x,u) = 0 \quad \text{on } M, \tag{1.5}$$

with non-linear functions $F(x,z)$ that depend analytically on z, the problem of recovering the differentials $\partial_z^k F(x,0)$ with $k \geq 3$ is equivalent to the question of injectivity of products of four solutions to the linearized equation

$$(-\Delta_g + V)u = 0 \quad \text{on } M.$$

This density property was subsequently proved in [FO20, LLLS21a] without imposing any geometric assumptions on the transversal manifold (M_0, g_0), through studying products of four Gaussian quasimode solutions to the linear equation. The underlying theme discovered in the latter works is that one can solve inverse problems for nonlinear elliptic equations in CTA manifolds without imposing additional strong assumptions on the transversal manifold (M_0, g_0). This is in sharp contrast to the study of inverse problems for linear elliptic equations on CTA manifolds [DSFKSU09, FKLS16] where additional strong assumptions must be imposed on the transversal manifold such as simplicity or existence of a strictly convex function on (M_0, g_0).

In this paper, we have considered an extension of [FO20, LLLS21a] that allows non-linearities $F(x,u)$ in (1.5) that have a *quadratic* term with respect to the u-variable. As far as we know, the only previous result that is concerned with recovery of quadratic non-linear functions on CTA manifolds is [FO20, Theorem 2] in the context of three and four dimensional CTA manifolds under additional geometric assumptions on the transversal manifold.

1.2. **Outline of the main novelties.** One of the key themes in the recent works that study inverse problems for nonlinear equations of the form

$$-\Delta_g u + qu^m = 0, \quad \text{on } M,$$

on CTA manifolds (M,g) with any integer $m \geq 2$ is the reduction from the problem of recovering the unknown coefficient q to the density problem of showing that the products of $m+1$ harmonic functions on (M,g) forms a dense set in $L^\infty(M)$. This reduction is based on an m-fold linearization argument for the nonlinear equation.

When $m \geq 3$, the latter density problem involves the product of four harmonic functions. Following the arguments of [LLLS21a] and choosing harmonic functions based on Gaussian quasimode constructions near four intersecting geodesics on the transversal factor (M_0, g_0), the density property can be proved. The harmonic functions are called complex geometric optics solutions (CGOs). However, when $m = 2$, one only obtains products of three CGOs corresponding to the Gaussian quasimodes and this is not a sufficiently reach set to conclude our desired density claim.

In this paper, we introduce a method to solve the coefficient determination problem concerning the case $m = 2$, by considering further linearizations of the equation up to fourth order, rather than just considering second order linearizations of the equation. In this sense we *over-differentiate* the nonlinearity, allowing us to *implicitly* obtain products of more harmonic functions. Over-differentiation of nonlinear equations appears in many of the works on inverse problems associated to nonlinear hyperbolic equations, see for example [KLU18, LUW17, LUW18, WZ19, HUZ21]. For example, the seminal work [KLU18] considers a wave equation in the presence of a quadratic nonlinear term and the interaction of linearized solutions is studied through Fourier integral operators, microlocal analysis and conormal singularities. The paper considers fourth order linearization which can be viewed as twice over-differentiating the equation. Heuristically, this is due to the fact that the fourth order of linearization is roughly the first instance where information propagating from a point source type singularity in the interior can be observed at the boundary.

To the best of our knowledge, over-differentiation of elliptic nonlinear equations has not been treated before in the literature since there is no a calculus for studying the interaction of Gaussian quasimodes. By this we mean studying equations of the form

$$-\Delta_g w = f u_1 u_2 \quad \text{on } M, \tag{1.6}$$

where u_1 and u_2 are two CGOs corresponding to Gaussian quasimodes. In our paper, we show that the equation (1.6) can be solved asymptotically with respect to the semi-classical parameter of the Gaussian quasimodes and in doing so we obtain precise closed form expressions for w modulo a small correction term, see Section 4 for details. This will be partly based on a Wentzel–Kramers–Brillouin (WKB in short) type approximation for w as well as a new Carleman estimate on CTA manifolds with boundary terms, see Lemma 4.6. Up to correction terms and normalizations, this means that if the CGOs u_1 and u_2 are of the form $e^{\tau\psi}(a_0 + a_{-1}/\tau + \cdots)$, there is a solution w to (1.6) of the form

$$e^{\tau\Psi}\left(b_{-2}/\tau^2 + b_{-3}/\tau^3 + \cdots\right),$$

where Ψ is the sum of the phase functions of u_1 and u_2 and the coefficients b_{-j} can be determined from the amplitudes of u_1 and u_2.

We mention that our Carleman estimate can have future applications in other problems that require a propagation of smallness argument from the boundary on CTA manifolds. Our calculus for solutions of equations of the type (1.6) can be naturally modified to apply for other equations. For example, it provides an alternative to using singular solutions and the theory of Fourier integral operators for inverse problems for hyperbolic equations. We refer to [HUZ21] for a discussion about the matter. We also mention that while we work in a geometric setting, our over-differentiation method and calculus can have applications in studies of models in $\mathbb{R}^n$, where one is interested in highly localized solutions to nonlinear equations in and outside inverse problems.

Let us also mention that as the correction term in our WKB analysis of w in (1.6) has a non-vanishing trace on ∂M, we need to introduce a variant of the higher order linearization method with a family of Dirichlet data that also depend on additional powers of the involved small parameters (see Section 2.2 and also Section 5). Instead of using boundary values of the form $\sum_i \epsilon_i f_i$, which is standard in the literature, we use for example boundary values of the form

$$\sum_i \epsilon_i f_i + \sum_{i,j} \epsilon_i \epsilon_j f_{ij}.$$

Here f_i and f_{ij} are functions given on the boundary ∂M.

Finally, we remark that the assumption (H1) is only imposed in this paper in order to simplify the presentation of the Gaussian quasimode solutions to the linearized equation (2.4). This allows us to better convey the key ideas discussed above without the additional need to discuss the additional technicality of analyzing self-intersections of geodesics. It is well known that Gaussian quasimodes for equation (2.4) can also be constructed in the presence of self-intersections of geodesics, see for example [FKLS16].

1.3. **Organization of the paper.** The paper is organized as follows. In Section 2, we reduce the setup of our study to a case where the conformal factor c in (1.2) of the CTA manifold is constant 1. There we also review the higher order linearization method, and derive the linearized equations and associated integral identities we use. We review suitable Gaussian quasimodes for the first linearized equations in Section 3. In Section 4, we find solution formulas for the special solutions of the second and third linearized equations. In Section 5 we prove Theorem 1.1 by utilizing these solutions. Finally, we prove a boundary determination result, derive Carleman estimates and compute coefficients related to products of solutions in Appendix.

2. Preliminaries

2.1. **Reduction to the case $c = 1$.** We show that for our purposes we can assume without any loss in generality that $c \equiv 1$. This is standard, see e.g. [DSFKSU09] or [FO20, Section 2.3]. To see this, let us define $\hat{g} = (dx_1)^2 + g$ so that $g = c\hat{g}$. Using the transformation law for changes of the Laplace-Beltrami operator under conformal rescalings of the metric, we write

$$c^{\frac{n+2}{4}}(-\Delta_g u + Vu + qu^2) = -\Delta_{\hat{g}} v + \hat{V} v + \hat{q} v^2, \tag{2.1}$$

where $v = c^{\frac{n-2}{4}} u$, $\hat{V} = cV - (c^{\frac{n-2}{4}} \Delta_g c^{-\frac{n-2}{4}})$ and $\hat{q} = c^{-\frac{n-2}{2}} q$. This shows that there exists a one to one correspondence between solutions to (1.3) with $f \in U_\delta$

and solutions to the following equation

$$\begin{cases} (-\Delta_{\hat{g}} + \hat{V})v + \hat{q}v^2 = 0, & x \in M \\ v = h, & x \in \partial M \end{cases} \tag{2.2}$$

provided that $\|c^{-\frac{n-2}{4}} h\|_{C^{2,\alpha}(\partial M)} \leq \delta$. Hence, the DN map for (1.3) determines the DN map for (2.2). Thus the problem of unique recovery of q from the DN map for (1.3) is equivalent to that of determining $\hat{q}$ from the DN map for (2.2). With this observation in mind, for the remainder of this paper and without loss of generality, we assume that $c \equiv 1$ so that

$$g = dx_1 \otimes dx_1 + g_0.$$

2.2. **Higher order linearization method with boundary values.** In this section, we discuss the higher order linearization method of equation (1.3). Our method is slightly different from the, by now standard, one [LLLS21a, FO20]. The difference is that we include boundary terms, which are not linear in the used small parameters.

Let $\epsilon_i \in \mathbb{R}$ and $f_i, f_{ij}, f_{ijk} \in C^{2,\alpha}(\partial M)$, for some $0 < \alpha < 1$ and for $i, j, k = 1, \ldots, 4$, and $\epsilon = (\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4)$. In the most general case of this paper, we take boundary values f to be of the form

$$f_\epsilon := \sum_{i=1}^{4} \epsilon_i f_i + \sum_{i,j=1}^{4} \epsilon_i \epsilon_j f_{ij} + \sum_{i,j,k=1}^{4} \epsilon_i \epsilon_j \epsilon_k f_{ijk} \quad \text{on} \quad \partial M. \tag{2.3}$$

Observe that the Dirichlet data $f_\epsilon \in U_\delta$ for sufficiently parameters ϵ_i, where U_δ is defined by

$$U_\delta := \left\{ f \in C^{2,\alpha}(\partial M) \,|\, \|f\|_{C^{2,\alpha}(\partial M)} < \delta \right\},$$

for some sufficiently small number $\delta > 0$. By using the implicit function theorem and the Schauder estimate for linear second order elliptic equations, one can show that the solution u_f to the nonlinear equation (1.3) depends smoothly (in the Frechét sense) on the parameters $\epsilon_1, \ldots, \epsilon_4$ (see [FO20, LLLS21a, Section 2] for detailed arguments).

The first linearization of the equation (1.3) at the zero boundary value is

$$\begin{cases} (-\Delta_g + V)v^{(i)} = 0 & \text{in } M, \\ v^{(i)} = f_i & \text{on } \partial M, \end{cases} \tag{2.4}$$

for $i = 1, 2, 3, 4$. Here

$$v^{(i)} := \partial_{\epsilon_i}\big|_{\epsilon=0} u_f,$$

where we have denoted $\epsilon = 0$ for the case $\epsilon_1 = \epsilon_2 = \epsilon_3 = \epsilon_4 = 0$. The second linearization

$$w^{(ij)} := \partial^2_{\epsilon_i \epsilon_j}\Big|_{\epsilon=0} u_f$$

of u_f satisfies the second linearized equation (2.4)

$$\begin{cases} (-\Delta_g + V)w^{(ij)} = -2q v^{(i)} v^{(j)} & \text{in } M, \\ w^{(ij)} = f_{ij} & \text{on } \partial M, \end{cases} \tag{2.5}$$

for different $i, j = 1, 2, 3, 4$, where the functions $v^{(i)} := \partial_{\partial \epsilon_i}\big|_{\epsilon=0} u_f$ are the unique solutions to the first linearized equation

Furthermore, by denoting

$$w^{(ijk)} := \partial^3_{\epsilon_i \epsilon_j \epsilon_k}\Big|_{\epsilon=0} u_f \quad \text{and} \quad w^{(1234)} := \partial^4_{\epsilon_1 \epsilon_2 \epsilon_3 \epsilon_4}\big|_{\epsilon=0} u_f,$$

one can see that they satisfy

$$\begin{cases}(-\Delta_g+V)w^{(ijk)}=-2q\left(v^{(i)}w^{(jk)}+v^{(j)}w^{(ik)}+v^{(k)}w^{(ij)}\right) & \text{in } M,\\ w^{(ijk)}=f_{ijk} & \text{on } \partial M,\end{cases} \tag{2.6}$$

for different $i,j,k=1,2,3,4$, and

$$\begin{cases}(-\Delta_g+V)w^{(1234)}=-2q\left(v^{(1)}w^{(234)}+v^{(2)}w^{(134)}\right.\\ \qquad\qquad +v^{(3)}w^{(124)}+v^{(4)}w^{(123)}\\ \qquad\qquad \left.+w^{(12)}w^{(34)}+w^{(13)}w^{(24)}+w^{(14)}w^{(23)}\right) & \text{in } M,\\ w^{(1234)}=0 & \text{on } \partial M.\end{cases} \tag{2.7}$$

We will construct special solutions for the above linearized equations in Section 3.

2.3. **Integral identities for the inverse problem.** Let us consider two potentials $q_1,q_2\in C^\infty(M)$. Let $v^{(i)}$, $w_\beta^{(ij)}$, $w_\beta^{(ijk)}$ and $w_\beta^{(1234)}$ be the respective solutions of (2.4), (2.5), (2.6) and (2.7), where the index $\beta=1,2$ refers to the potentials $q=q_\beta$, and $i,j,k=1,2,3,4$. We denote by $v^{(5)}\in C^{2,\alpha}(M)$ an additional solution to the linear equation:

$$\begin{cases}(-\Delta_g+V)v^{(5)}=0 & \text{in } M,\\ v^{(5)}=f_5 & \text{on } \partial M.\end{cases}$$

We record the integral identities for the second, third and fourth order linearized equations.

Lemma 2.1 (Integral identities)**.** *Let $\{f_i\}_{i=1}^5,\{f_{ij}\}_{i,j=1}^4,\{f_{i,j,k}\}_{i,j,k=1}^4\subset C^{2,\alpha}(\partial M)$, and for $\epsilon\in\mathbb{R}^4$ in a neighborhood of zero, define f_ϵ via (2.3). The following integral identities hold, for each $i,j,k=1,\dots,4$ and $m=1,\dots,5$.*

(1) The second order integral identity

$$\int_{\partial M}\partial^2_{\epsilon_i\epsilon_j}\Big|_{\epsilon=0}\left(\Lambda_{q_1}f_\epsilon-\Lambda_{q_2}f_\epsilon\right)f_m\,dS=2\int_M(q_1-q_2)v^{(i)}v^{(j)}v^{(m)}\,dV_g. \tag{2.8}$$

(2) The third order integral identity

$$\begin{aligned}&\int_{\partial M}\partial^3_{\epsilon_i\epsilon_j\epsilon_k}\Big|_{\epsilon=0}\left(\Lambda_{q_1}f_\epsilon-\Lambda_{q_2}f_\epsilon\right)f_m\,dS\\ =&2\int_M\left\{q_1\left(v^{(i)}w_1^{(jk)}+v^{(j)}w_1^{(ik)}+v^{(k)}w_1^{(ij)}\right)\right.\\ &\qquad\left.-q_2\left(v^{(i)}w_2^{(jk)}+v^{(j)}w_2^{(ik)}+v^{(k)}w_2^{(ij)}\right)\right\}v^{(m)}\,dV_g.\end{aligned} \tag{2.9}$$

(3) The fourth order integral identity

$$\begin{aligned}&\int_{\partial M}\partial^4_{\epsilon_1\epsilon_2\epsilon_3\epsilon_4}\Big|_{\epsilon=0}\left(\Lambda_{q_1}f_\epsilon-\Lambda_{q_2}f_\epsilon\right)f_5\,dS\\ &=2\int_M\left\{q_1\left(v^{(1)}w_1^{(234)}+v^{(2)}w_1^{(134)}+v^{(3)}w_1^{(124)}+v^{(4)}w_1^{(123)}\right.\right.\\ &\qquad\qquad\left.+w_1^{(12)}w_1^{(34)}+w_1^{(13)}w_1^{(24)}+w_1^{(14)}w_1^{(23)}\right)\\ &\qquad-q_2\left(v^{(1)}w_2^{(234)}+v^{(2)}w_2^{(134)}+v^{(3)}w_2^{(124)}+v^{(4)}w_2^{(123)}\right.\\ &\qquad\qquad\left.\left.+w_2^{(12)}w_2^{(34)}+w_2^{(13)}w_2^{(24)}+w_2^{(14)}w_2^{(23)}\right)\right\}v^{(5)}\,dV_g,\end{aligned} \tag{2.10}$$

Here, the functions $v^{(i)}$, $w_\ell^{(ik)}$ and $w_\ell^{(ijk)}$ are the unique $C^{2,\alpha}(M)$ solutions to (2.4), (2.5) and (2.6) with $q = q_\ell$, respectively, for $\ell = 1,2$, $i,j,k = 1,2,3,4$. Finally, $v^{(5)} \in C^{2,\alpha}(M)$ is the unique solution to (2.4) with Dirichlet boundary data $f = f_5$.

Proof. The proof is based on integration by parts. We only prove (2.8) explicitly. The other two integral identities follow similarly.

(1) Let us consider the second linearized equation (2.5) with $q = q_\beta$ for $\beta = 1,2$. Integrating by parts yields

$$
\begin{aligned}
&\int_{\partial M} \partial^2_{\epsilon_i\epsilon_j}\Big|_{\epsilon=0} \left(\Lambda_{q_1} f_\epsilon - \Lambda_{q_2} f_\epsilon\right) f_m \, dS \\
=&\int_{\partial M} \left(\partial_\nu w_1^{(ij)} - \partial_\nu w_2^{(ij)}\right) f_m \, dS \\
=&\int_M \left(\Delta_g w_1^{(ij)} - \Delta_g w_2^{(ij)}\right) v^{(m)} \, dV_g + \int_M \nabla^g \left(w_1^{(ij)} - w_2^{(ij)}\right) \cdot \nabla^g v^{(m)} \, dV_g \\
=&\int_M V\left(w_1^{(ij)} - w_2^{(ij)}\right) v^{(m)} \, dV_g + 2\int_M (q_1 - q_2) v^{(i)} v^{(j)} v^{(m)} \, dV_g \\
&\quad - \int_M \left(w_1^{(ij)} - w_2^{(ij)}\right) \Delta_g v^{(m)} \, dV_g \\
=&2\int_M (q_1 - q_2) v^{(i)} v^{(j)} v^{(m)} \, dV_g,
\end{aligned}
$$

where we have utilized $w_1^{(ij)} = f_{ij} = w_2^{(ij)}$ on ∂M and $(-\Delta_g + V)v^{(m)} = 0$ in M.

(2) We have

$$
\int_{\partial M} \partial^3_{\epsilon_i\epsilon_j\epsilon_k}\Big|_{\epsilon=0} \left(\Lambda_{q_1} f_\epsilon - \Lambda_{q_2} f_\epsilon\right) f_m \, dS = \int_{\partial M} \left(\partial_\nu w_1^{(ijk)} - \partial_\nu w_2^{(ijk)}\right) f_m \, dS.
$$

The above integration by parts combined with the equations (2.4) and (2.6) results in the claimed identity. Proof of **(3)** is obtained similarly. □

3. Complex geometrical optics and Gaussian beam quasimodes

Let us introduce the *complex geometrical optics* type solutions for the first order linearized equation. These are solutions to the linearized equation (2.4) that concentrate on planes of the form $I \times \gamma$, where I is an interval and γ is an inextendible non-tangential geodesic on M_0. We call them CGOs in short. We also assume in this paper for simplicity that γ does not have self-intersections.

We recall the Gaussian quasimode construction for the equation (2.4) that originated from [FKLS16, Section 3] in the setting of CTA manifolds. We follow the constructions [FO20, Section 4.1, Proposition 5.1] and [LLLS21a, Section 5 and Appendix] that allow a zeroth order term V in (2.4) as well as providing decay estimates in higher order Sobolev spaces. We refer to these works for details of the constructions in this section.

We first consider a unit speed non-tangential geodesic $\gamma : [l_1, l_2] \to M_0$ that connects two points on the boundary ∂M_0. We assume that γ does not have self-intersections for simplicity. We write $(\hat{M}_0, g_0)$ for an artificial smooth extension of (M_0, g_0) into a slightly larger smooth Riemannian manifold and denote by (t, y) the Fermi coordinates in a tubular neighborhood of the geodesic γ, where $t \in [l_1, l_2]$ and $y \in B_{\delta'}(\mathbb{R}^{n-2})$ for some $\delta' > 0$ sufficiently small. We refer the reader to [FKLS16, Section 3] for the details of the construction of Fermi coordinates.

We define the complex parameter

$$
s = \tau + \mathbf{i}\lambda, \quad \tau > 0, \quad \lambda \in \mathbb{R},
$$

where $\mathbf{i} = \sqrt{-1}$, λ is to be viewed as a fixed parameter, and $\tau > 0$ is an asymptotic parameter that tends to infinity. Given any $K > 0$ and $N > 0$, there exists a positive integer N' depending on K, N (see (3.7) for the precise choice) and solutions $v_s \in H^k(M) \subset C^{2,\alpha}(M)$ for $k \in \mathbb{N}$ sufficiently large (especially so that $H^k(M) \subset C^{2,\alpha}(M)$) to the linear equation

$$(-\Delta_g + V)v_s = 0 \quad \text{in } M,$$

of the form

$$v_s(x_1, t, y) = e^{\pm s x_1}\left(\tau^{\frac{n-2}{8}} e^{\mathbf{i}s\psi(t,y)} a_s(x_1, t, y) + r_s(x_1, t, y)\right), \tag{3.1}$$

where χ is a cutoff function supported in a δ'-neighborhood of the origin and each term in the right hand side has certain properties that we will describe next. In what follows next, we describe the construction and properties of the phase term $\psi \in C^\infty(M)$, the amplitude $a_s \in C^\infty(M)$ and the remainder term $r_s \in C^{2,\alpha}(M)$.

The phase function $\psi(t, y)$ satisfies

$$\psi(\gamma(t)) = t, \quad \nabla^g\psi(\gamma(t)) = \dot\gamma(t), \quad \mathrm{Im}(D^2\psi(\gamma(t))) \geq 0, \quad \mathrm{Im}(D^2\psi|_{\dot\gamma(t)^\perp}) > 0. \tag{3.2}$$

More explicitly, in terms of the Fermi coordinates we can write

$$\psi(t, y) = t + \frac{1}{2}\sum_{j,k=1}^{n-2} H_{jk}(t) y_j y_t + O(|y|^3), \tag{3.3}$$

where the complex-valued symmetric matrix $H(t) = (H_{jk}(t))_{j,k=1}^{n-2}$ is given by the expression

$$H(t) = \dot Y(t) Y^{-1}(t), \qquad \text{for any } t \in [l_1, l_2],$$

and Y is a non-degenerate matrix that solves the second order linear differential equation

$$\ddot Y + DY = 0 \qquad \text{for any } t \in [l_1, l_2].$$

Here, the symmetric matrix D is given by $D_{jk} = \frac{1}{2}\partial^2_{jk} g^{11}$ for each $j, k = 1, \ldots, n-2$. The matrix H additionally satisfies

$$\mathrm{Im}(H)(t) > 0 \qquad \text{for any } t \in [l_1, l_2], \tag{3.4}$$

and

$$\det(\mathrm{Im}(H(t))) \cdot |\det Y(t)|^2 = 1.$$

Next we describe the amplitude function in the expansion (3.1). The amplitude $a_s(x_1, t, y)$ is of the form

$$a_s(x_1, t, y) = \left(a_0(t, y) + \frac{a_1^\pm(x_1, t, y)}{s} + \cdots + \frac{a^\pm_{N'-1}(x_1, t, y)}{s^{N'-1}}\right)\chi\left(\frac{|y|}{\delta'}\right), \tag{3.5}$$

where the principal amplitude $a_0(t, y)$ itself is given by the expression

$$a_0(t, y) = a_{0,0}(t) + a_{0,1}(t, y) + \cdots + a_{0,N'-1}(t, y).$$

Here, $a_{0,0}(t)$ is an explicit positive function on γ given by the expression

$$a_{0,0}(t) = (\det Y(t))^{-\frac{1}{2}}, \tag{3.6}$$

and the subsequent terms $a_{0,j}(t, y)$ with $j = 1, 2, \ldots, N' - 1$ are homogeneous polynomials of degree j in the y-coordinates. These terms arise as solutions to certain transport equations along the geodesic γ on M_0.

The remaining amplitude terms $a_k^\pm$, for $k = 1, 2, \ldots, N' - 1$, have analogous expressions of the form

$$a_k^\pm(x_1, t, y) = a^\pm_{k,0}(x_1, t, y) + a^\pm_{k,1}(x_1, t, y) + \cdots + a^\pm_{k,N'-1}(t, y),$$

where $a^{\pm}_{k,j}$ are homogeneous polynomials of degree j in the y-coordinates, for $j = 0, 1, \ldots, N' - 1$. These amplitudes arise as solutions to certain complex transport equations on the plane $y = 0$ on M (see [FO20, Section 4] for more details).

Finally, using [FO20, Proposition 2, Lemma 4] and fixing the order

$$N' = 2 + 2N + 2K, \tag{3.7}$$

for the Gaussian quasimode construction, it follows that given any fixed $K \in \mathbb{N}$, there exists a remainder term r_s in (3.1) in the Sobolev space $H^K(M)$ satisfying the decay estimate

$$\|r_s\|_{H^K(M)} \lesssim \tau^{-N}. \tag{3.8}$$

Remark 3.1. *Let us emphasize that the above estimate has two nice features. Firstly, we obtain precise decay estimates for the remainder term r_s with respect to the large parameter τ. Secondly, note that the regularity of the CGO ansatz v_s is the same as that of r_s and therefore by choosing for example any fixed $K \geq \frac{n}{2} + 4$ and using the Sobolev embedding $H^{\frac{n}{2}+4} \subset C^3(M)$, we may construct remainder terms r_s that are in $C^3(M)$. This would be suitable for us, as we will later use the boundary traces of the CGO solutions above in order to apply Lemma 2.1.*

4. Solutions for the linearized equations

We discussed CGO solutions for the first order linearization of the equation $(-\Delta_g + V)u + qu^2 = 0$ at the zero solution as in the previous section. In this section, we construct solutions for the second and third order linearizations of the equation.

4.1. Solutions for the second order linearization.

In what follows, we assume that geodesics do not have self-intersections. Let $p_0 \in M_0$ and let γ_1 be a nontangential geodesic passing through p_0 in some direction $v \in S_{p_0}M_0$. Here $S_{p_0}M_0$ stands for unit length vectors of $T_{p_0}M_0$. We will use the following definition.

Definition 4.1. *We say that two geodesics intersect* properly *if they intersect and are not reparametrizations of each other.*

Assume that γ_2 is another nontangential geodesic, and that it intersects γ_1 properly at p_0. If $v' \in S_pM_0$ is the velocity vector of γ_2 at p_0, then v' is linearly independent of v due to the uniqueness of geodesics. Due to a compactness argument (see e.g. [LLLS21a]), the geodesics γ_1 and γ_2 can only intersect at a finite number of points.

We consider CGO solutions $v^{(1)} = v^{(1)}_\tau$ and $v^{(2)} = v^{(2)}_\tau$ to the equation (2.4) corresponding to geodesics γ_1 and γ_2, respectively. That is, the CGOs $v^{(k)}$ for $k = 1, 2$ are of the form (3.1):

$$v^{(k)} = e^{\pm s_k x_1}\left(\tau^{\frac{n-2}{8}} e^{\mathbf{i} s_k \psi_k} a^{(k)}_\tau + r^{(k)}_\tau\right), \tag{4.1}$$

where ψ_k, $a^{(k)}_\tau$ and $r^{(k)}_\tau$ have the properties described in the previous section. We have also denoted

$$s_k = c_k\tau + \mathbf{i}\lambda_k, \tag{4.2}$$

where $c_k, \lambda_k \in \mathbb{R}$, and $\tau > 0$ is a (large) parameter.

In the next lemma, we construct solutions for the second linearized equation. After proving the lemma, in Proposition 4.4, we show that if the DN map is known, the boundary value of the solutions can be fixed.

Lemma 4.2. *Let $K, N \in \mathbb{N} \cup \{0\}$. Assume that $v^{(1)}$ and $v^{(2)}$ are CGOs, which correspond to properly intersecting geodesics on M_0 and are of the form (4.1). If the restrictions of the amplitudes $a^{(k)}$ to M_0 are supported in small enough neighborhoods of the geodesics γ_k for $k = 1, 2$, and $N' = N'(K, N)$ is large enough, then the equation*

$$(-\Delta_g + V)w = -2qv^{(1)}v^{(2)} \text{ in } M \tag{4.3}$$

has a smooth solution w up to the boundary ∂M with the following properties: The solution w is of the form

$$w = w_0 + e^{\tau\Psi}R,$$

where

$$w_0 = \tau^{\frac{n-2}{4}} e^{(\pm s_1 \pm s_2)x_1 + \mathbf{i}(s_1\psi_1 + s_1\psi_1)} b_\tau,$$

with

$$\begin{aligned} b_\tau &= \frac{1}{\tau^2}b_{-2} + \frac{1}{\tau^3}b_{-3} + \cdots + \frac{1}{\tau^{2N'}}b_{-2N'}, \\ b_{-2} &= \frac{2q}{(\pm c_1 \pm c_2)^2 - |c_1\nabla^{g_0}\psi_1 + c_2\nabla^{g_0}\psi_2|^2} a_0^{(1)}a_0^{(2)}. \end{aligned} \tag{4.4}$$

The function Ψ is given by

$$\Psi = (\pm c_1 \pm c_2)x_1 + \mathbf{i}c_1\psi_1 + \mathbf{i}c_2\psi_2. \tag{4.5}$$

and $R = R_\tau$ is a remainder term that satisfies

$$\|R_\tau\|_{H^K(M)} \lesssim \tau^{-N}.$$

Proof. We first find an approximate solution for the equation

$$(-\Delta_g + V)\hat{w}_0 = -2qV_\tau^{(1)}V_\tau^{(2)} \text{ in } M, \tag{4.6}$$

where

$$V_\tau^{(k)} = e^{\pm s_k x_1} e^{\mathbf{i}s_k\psi_k} a_\tau^{(k)}.$$

After that, we scale $\hat{w}_0$ and correct it by using either Carleman or elliptic estimates to a solution of (4.3). Here ψ_k and $a^{(k)}$ are constructed with respect to geodesics γ_k that intersect properly on the transversal manifold M_0.

We shorthand our notation and write

$$e^{\pm s_1 x_1} e^{\mathbf{i}s_1\psi_1} e^{\pm s_2 x_1} e^{\mathbf{i}s_2\psi_2} := e^{\tau\Psi}e^{\Lambda},$$

where Ψ is as in (4.5) and

$$\Lambda = \mathbf{i}(\pm\lambda_1 \pm \lambda_2)x_1 - \lambda_1\psi_1 - \lambda_2\psi_2.$$

Using the expressions for the amplitude functions (3.5), the equation (4.6) can be written as

$$(\Delta_g - V)\hat{w}_0 = e^{\tau\Psi}e^{\Lambda} \sum_{k=0}^{2(N'-1)} \frac{E_{-k}}{\tau^k},$$

where the functions $E_{-k} \in C^\infty(M)$, $k = 0, 1, \ldots, 2(N'-1)$, are supported near the intersection points of the geodesics γ_1 and γ_2. We have

$$E_0 = 2qa_0^{(1)}a_0^{(2)}.$$

Let us consider a *WKB ansatz* for $\hat{w}_0$ of the form

$$e^{\tau\Psi}\hat{b}.$$

A direct calculation shows that

$$\begin{aligned} &(\Delta_g - V)\left(e^{\tau\Psi}\hat{b}\right) \\ &\quad = e^{\tau\Psi}\left(\tau^2\langle\nabla^g\Psi, \nabla^g\Psi\rangle\hat{b} + \tau[2\langle\nabla^g\hat{b}, \nabla^g\Psi\rangle + \hat{b}(\Delta_g\Psi)] + (\Delta_g - V)\hat{b}\right), \end{aligned}$$

where $\langle \eta, \zeta \rangle$ denotes the complexified Riemannian inner product. At the center of normal coordinates $\langle \eta, \zeta \rangle = \eta \cdot \zeta = \sum_{i=1}^{n} \eta_i \zeta_i$, for any $\eta, \zeta \in \mathbb{C}^n$. Note that $\langle \cdot, \cdot \rangle$ is not a Hermitian inner product of complex vectors. Especially $\langle \eta, \eta \rangle = 0$ does not imply the complex vector $\eta = 0$. We assume that $\hat{b}_\tau$ is an amplitude function of the form

$$\hat{b}_\tau = \frac{1}{\tau^2}\hat{b}_{-2} + \frac{1}{\tau^3}\hat{b}_{-3} + \cdots + \frac{1}{\tau^N}\hat{b}_{-2N'}. \tag{4.7}$$

At an intersection point of the geodesics, we have by the properties of Gaussian beams (see (3.2)) that

$$\nabla^g \Psi = (\pm c_1 \pm c_2)e_1 + \mathbf{i}c_1 \nabla^{g_0}\psi_1 + \mathbf{i}c_2 \nabla^{g_0}\psi_2 = (\pm c_1 \pm c_2)e_1 + \mathbf{i}(c_1\dot{\gamma}_1 + c_2\dot{\gamma}_2),$$

where $\dot{\gamma}_1$ and $\dot{\gamma}_2$ are the velocity vectors of γ_1 and γ_2 at the intersection point. Here $e_1 = \partial_{x_1} x$, for $x = (x_1, \ldots, x_n)$. Since the geodesics γ_1 and γ_2 intersect properly

$$\begin{aligned}
\langle \nabla^g \Psi, \nabla^g \Psi \rangle &= (\pm c_1 \pm c_2)^2 - |c_1 \nabla^{g_0}\psi_1 + c_2 \nabla^{g_0}\psi_2|^2 \\
&= c_1^2 \pm 2c_1c_2 + c_2^2 - c_1^2 - 2c_1c_2\langle \dot{\gamma}_1, \dot{\gamma}_2 \rangle - c_2^2 \\
&= -2c_1c_2\left(\langle \dot{\gamma}_1, \dot{\gamma}_2 \rangle \mp 1\right) \neq 0
\end{aligned}$$

at the intersection points of the geodesics. By the above and assuming that $a^{(1)}$ and $a^{(2)}$ are supported in small enough neighborhoods of γ_1 and γ_2 we have

$$|\langle \nabla^g \Psi, \nabla^g \Psi \rangle| \geq \text{ constant } > 0 \tag{4.8}$$

on the support of each E_{-k} for all $k = 0, 1, \ldots, 2(N'-1)$.

Let us set $\hat{b}_0 = \hat{b}_{-1} = 0$ and define the coefficients $\hat{b}_{-k}$ for $k = 2, \ldots, 2N'$ recursively by the formula

$$\hat{b}_{-k} = \frac{e^{\Lambda}E_{-k+2} - [2\langle \nabla^g \hat{b}_{-k+1}, \nabla^g \Psi \rangle + \hat{b}_{-k+1}\Delta_g \Psi] - (\Delta_g - V)\hat{b}_{-k+2}}{\langle \nabla^g \Psi, \nabla^g \Psi \rangle}. \tag{4.9}$$

specially,

$$\hat{b}_{-2} = e^{\Lambda}\frac{2q}{\langle \nabla^g \Psi, \nabla^g \Psi \rangle}\, a_0^{(1)} a_0^{(2)}.$$

We also see by a recursive inspection that $\hat{b}_k$ is supported on the set where (4.8) holds. Thus $\hat{b}_k$ are well-defined. It follows by re-indexing sums and using $\hat{b}_{-1} = \hat{b}_0 = 0$, such that

$$\begin{aligned}
&(\Delta_g - V)\left(e^{\tau\Psi}\hat{b}_\tau\right) - 2qV_\tau^{(1)}V_\tau^{(2)} \\
=&e^{\tau\Psi}\sum_{k=2}^{2N'}\Big[\tau^{2-k}\langle \nabla^g \Psi, \nabla^g \Psi \rangle \hat{b}_{-k} + \tau^{1-k}[2\langle \nabla^g \hat{b}_{-k}, \nabla^g \Psi \rangle + \hat{b}_{-k}(\Delta_g \Psi)] \\
&\qquad + \tau^{-k}(\Delta_g - V)\hat{b}_{-k} - e^{\Lambda}E_{-k+2}\Big] \\
=& e^{\tau\Psi}\Big(\tau^{-2N'+1}\Big([2\langle \nabla^g \hat{b}_{-2N'}, \nabla^g \Psi \rangle + \hat{b}_{-2N'}(\Delta_g \Psi)] + (\Delta_g - V)\hat{b}_{-2N'+1}\Big) \\
&\qquad + \tau^{-2N'}(\Delta_g - V)\hat{b}_{-2N'}\Big),
\end{aligned} \tag{4.10}$$

where we have used (4.9) to get to the last equality.

Next, we scale and correct $e^{\tau\Psi}\hat{b}_\tau$ so that it solves (4.3). We write

$$w = \tau^{\frac{n-2}{4}}e^{\tau\Psi}\hat{b}_\tau + \hat{R}_\tau.$$

Note that

$$qv^{(1)}v^{(2)} = q\tau^{\frac{n-2}{4}}V_\tau^{(1)}V_\tau^{(2)} + qe^{\pm s_1x_1 \pm s_2x_1}r,$$

where r corresponds to the correction terms $r^{(1)}$ and $r^{(2)}$ and is given by

$$r = r_\tau^{(1)}\tau^{\frac{n-2}{8}}e^{\mathbf{i}s_2\psi_2}a_\tau^{(2)} + r_\tau^{(2)}\tau^{\frac{n-2}{8}}e^{\mathbf{i}s_1\psi_1}a_\tau^{(1)} + r_\tau^{(1)}r_\tau^{(2)}.$$

Hence, $\hat{R}_\tau$ solves

$$(\Delta_g - V)\hat{R}_\tau = -\left[(\Delta_g - V)\tau^{\frac{n-2}{4}}e^{\tau\Psi}\hat{b}_\tau - 2\tau^{\frac{n-2}{4}}qV_\tau^{(1)}V_\tau^{(2)}\right] + 2qe^{\pm s_1x_1\pm s_2x_1}r, \tag{4.11}$$

whenever w solves (4.3). Now, if N' is chosen large enough, i.e.

$$N' \geq 2 + 2N + 4K,$$

then we have $r = \mathcal{O}_{H^K(M)}(\tau^{-N})$ by combining the bounds (3.8) for the correction terms $r_\tau^{(\beta)}$, $\beta = 1, 2$ together with the bounds

$$\tau^{\frac{n-2}{4}}\|e^{is_2\psi_2}\|_{H^K(M)} + \tau^{\frac{n-2}{4}}\|e^{is_2\psi_2}\|_{H^K(M)} \lesssim \tau^K. \tag{4.12}$$

For example, see [FO20, Lemma 4] for the estimate (4.12).

Redefining N' to be larger, if necessary, the equation (4.11) for $\hat{R}_\tau$ together with (4.10) implies

$$(\Delta_g - V)\hat{R}_\tau = e^{\tau(\pm c_1 \pm c_2)x_1}\mathcal{O}_{H^K(M)}(\tau^{-N}).$$

By writing

$$R_\tau = e^{-\tau(\pm c_1\pm c_2)x_1}\hat{R}_\tau \quad \text{and} \quad b_\tau = e^{-\Lambda}\hat{b}_\tau,$$

and in the case that

$$\pm c_1 \pm c_2 \neq 0,$$

the claim in the lemma follows from [FO20, Proposition 2] (also from the Carleman estimates [DSFKSU09, Lemma 4.1, Proposition 4.3]). The claim that R_τ and consequently w are actually smooth functions follows from the previous reference together with the fact that the right hand side of the above equation for $\hat{R}_\tau$ is a smooth function on M. Alternatively, in the case that

$$\pm c_1 \pm c_2 = 0,$$

we may impose zero boundary conditions for R_τ and use standard elliptic estimates to complete the proof of the lemma. □

Remark 4.3. *Let us emphasize that the function w constructed above is globally well-defined on M. Indeed, recalling that $M \subset I \times M_0$ we observe that for each fixed $x^1 \in I$, the principal function w_0 is smoothly defined in x^1 and it is compactly supported and smooth in a small neighborhood of the intersection of the two geodesics in M_0. With regards to the remainder term R_τ, we remark that in the case $\pm c_1 \pm c_2 \neq 0$, R_τ is a smooth function defined on an open manifold U such that $M \Subset U$, which satisfies*

$$\|R_\tau\|_{H^K(U)} \lesssim \tau^{-N},$$

see e.g. [DSFKSU09]. In the case $\pm c_1 \pm c_2 = 0$, R_τ is also smooth due to elliptic regularity and has zero boundary values on ∂M.

Let us next consider the second linearized equation (2.5) for two possibly different potentials q_1 and q_2. We show that if $\Lambda_{q_1} = \Lambda_{q_2}$, then the solutions of Lemma 4.2 corresponding to potentials q_1 and q_2 can be taken to have same boundary values.

Proposition 4.4. *Assume as in Lemma 4.2 and adopt its notation, and assume that $\Lambda_{q_1} = \Lambda_{q_2}$. Then the second linearized equations*

$$(-\Delta_g + V)w^{(\beta)} = -2q_\beta v^{(1)}v^{(2)}, \quad \beta = 1, 2, \tag{4.13}$$

have solutions of the form

$$w^{(\beta)} = w_0^{(\beta)} + e^{\tau\Psi}R^{(\beta)}.$$

Here

$$\begin{aligned} w_0^{(\beta)} &= \tau^{\frac{n-2}{4}} e^{(\pm s_1 \pm s_2)x_1 + \mathbf{i}(s_1\psi_1 + s_2\psi_2)} b^{(\beta)}, \\ b^{(\beta)} &= \tau^{-2} b_{-2}^{(a)} + \cdots + \tau^{-2N'} b_{-2N'}^{(\beta)}, \\ b_{-2}^{(\beta)} &= \frac{2q_\beta}{(\pm c_1 \pm c_2)^2 - |c_1 \nabla^{g_0}\psi_1 + c_2 \nabla^{g_0}\psi_2|^2} a_0^{(1)} a_0^{(2)}. \end{aligned}$$

Moreover $R^{(\beta)} = \mathcal{O}_{L^2(M)}(\tau^{-N})$ *(*$\beta = 1, 2$*) and*

$$w^{(1)}\big|_{\partial M} = w^{(2)}\big|_{\partial M}.$$

In order to prove Proposition 4.4, we need a *boundary determination* result:

Proposition 4.5 (Boundary determination)**.** *For* $m \geq 2$*,* $m \in \mathbb{N}$*, let* (M, g) *be a compact Riemannian manifold with* C^∞ *boundary* ∂M *and consider the boundary value problem*

$$\begin{cases} (-\Delta_g + V)u + qu^m = 0 & \text{in } M, \\ u = f & \text{on } \partial M. \end{cases} \tag{4.14}$$

Assume that the DN map Λ_q *of the equation* (4.14) *is known for small boundary values. Then* Λ_q *determines the formal Taylor series of* q *on the boundary* ∂M*.*

In addition, if $f \in C^{2,\alpha}(\partial M)$ *is so small that* (4.14) *has a unique small solution, the DN map determines the formal Taylor series of the solution* $u = u_f$ *at any point on the boundary.*

We also need the following Carleman estimate with boundary terms.

Lemma 4.6 (Carleman estimate with boundary terms)**.** *Let* (M, g) *be a compact, smooth, transversally anisotropic Riemannian manifold with a smooth boundary. Let* $V \in L^\infty(M)$*. There exists constants* $\tau_0 > 0$ *and* $C > 0$ *depending only on* (M, g) *and* $\|V\|_{L^\infty(M)}$ *such that given any* $|\tau| > \tau_0$*, and any* $v \in C^2(M)$*, there holds*

$$\begin{aligned} C|\tau|\,\|v\|_{L^2(M)} \leq &\|e^{-\tau x_1}(-\Delta_g + V)(e^{\tau x_1} v)\|_{L^2(M)} + |\tau|^{\frac{3}{2}} \|v\|_{W^{2,\infty}(\partial M)} \\ &+ |\tau|^{\frac{3}{2}} \|\partial_\nu v\|_{W^{2,\infty}(\partial M)} + |\tau|^{\frac{3}{2}} \|\partial_\nu^2 v\|_{W^{2,\infty}(\partial M)}, \end{aligned} \tag{4.15}$$

We have placed the proofs of the above two results in the the Appendix A and B, respectively. The proof of Proposition 4.5 uses a standard boundary determination result for linearized second order elliptic equations. The proof of Lemma 4.6 is by integration by parts and using standard elliptic estimates. In this paper, the preceding Carleman estimate with the $L^2(M)$ bound is sufficient in deriving the upper bound for the correction term $R^{(\beta)}$ in Proposition 4.4 for $\beta = 1, 2$; however let us also mention that analogous Carleman estimates with boundary terms can be obtained in higher Sobolev spaces $H^k(M)$, for $k \in \mathbb{N}$.

Proof of Proposition 4.4. Let us first consider the case $\pm c_1 \pm c_2 \neq 0$. By Lemma 4.2 we have a smooth solution of the form

$$w^{(2)} = w_0^{(2)} + e^{\tau\Psi} R^{(2)}$$

for the equation

$$(-\Delta_g + V) w^{(2)} = -2q_2 v^{(1)} v^{(2)}.$$

In general, controlling the boundary value of $R^{(2)}$ is hard. As already mentioned in Remark 4.3, we have that $R^{(2)}$ is a smooth function defined on an open manifold U such that $M \Subset U$, which satisfies

$$\|R^{(2)}\|_{H^K(U)} \leq \frac{C}{\tau^N}$$

if $N' = N'(K,N)$ was chosen large enough.

By redefining K as $K+5/2$ (and thus also redefining also N' larger) and using trace theorem

$$R^{(2)}|_{\partial M} = \mathcal{O}_{H^K(\partial M)}(\tau^{-N}) \tag{4.16}$$

and

$$\partial_\nu R^{(2)}\Big|_{\partial M} = \mathcal{O}_{H^K(\partial M)}(\tau^{-N}), \qquad \partial_\nu^2 R^{(2)}\Big|_{\partial M} = \mathcal{O}_{H^K(\partial M)}(\tau^{-N}). \tag{4.17}$$

Let us then consider the equation (4.13) for q_1 with boundary value $w^{(2)}\big|_{\partial M}$. As 0 is not a Dirichlet eigenvalue of $-\Delta_g + V$ and noting that $w^2|_{\partial M}$ is also smooth, it follows from elliptic regularity (see e.g. [Tay11]), that there is a unique smooth solution $w^{(1)}$ to the equation

$$\begin{cases} (-\Delta_g + V)w^{(1)} = -2q_1 v^{(1)} v^{(2)} & \text{in } M, \\ w^{(1)} = w^{(2)}\big|_{\partial M} & \text{on } \partial M. \end{cases} \tag{4.18}$$

We write

$$w^{(1)} = w_0^{(1)} + e^{\tau\Psi} R^{(1)},$$

where $w_0^{(1)} = \tau^{\frac{n-2}{4}} e^{(\pm s_1 \pm s_2)x_1 + \mathbf{i}(s_1\psi_1 + s_1\psi_1)} b_\tau^{(1)}$ is the WKB ansatz given as in Lemma 4.2 such that

$$(\Delta_g - V)w_0^{(1)} - 2q_1 v^{(1)} v^{(2)} = e^{\tau\Psi} F.$$

Here

$$F = \mathcal{O}_{H^K(M)}(\tau^{-N}), \tag{4.19}$$

which can be derived by making the WKB ansatz $w_0^{(1)}$ precise enough (i.e. N' large enough). Since $w^{(1)}$ solves $(-\Delta_g + V)w^{(1)} = -2q_1 v^{(1)} v^{(2)}$, we have that $R^{(1)}$ solves the conjugated equation

$$e^{-\tau\Psi}(\Delta_g - V)e^{\tau\Psi} R^{(1)} = \mathcal{O}_{H^K(M)}(\tau^{-N}).$$

Unfortunately, we can not directly deduce from standard Carleman estimates that the correction term $\|R^{(1)}\|_{L^2(M)}$ is small.

As matter of fact, in order to obtain that $\|R^{(1)}\|_{L^2(M)}$ is small, we use the assumption $\Lambda_{q_1} = \Lambda_{q_2}$, which implies that the DN maps of the second linearized equations (see equation (2.5)) for q_1 and q_2 are the same, that is to say,

$$\partial^2_{\epsilon_i\epsilon_j} \Lambda_{q_1}(f_\epsilon)\big|_{\epsilon=0} = \partial^2_{\epsilon_i\epsilon_j} \Lambda_{q_1}(f_\epsilon)\big|_{\epsilon=0}.$$

By additionally using the boundary determination result (Proposition 4.5), we have that

$$q_1 = q_2 \quad \text{on } \partial M$$

up to infinite order. The ansatzes $w_0^{(1)}$ and $w_0^{(2)}$ depend on (M,g) and the potentials q_1 and q_2 respectively. The dependence on the potentials is local. That is, the dependence is on pointwise values of the potentials and their derivatives, see (4.4). It follows that

$$w_0^{(1)}\Big|_{\partial M} = w_0^{(2)}\Big|_{\partial M} \tag{4.20}$$

and also

$$\partial_\nu w_0^{(1)}\Big|_{\partial M} = \partial_\nu w_0^{(2)}\Big|_{\partial M} \quad \text{and} \quad \partial_\nu^2 w_0^{(1)}\Big|_{\partial M} = \partial_\nu^2 w_0^{(2)}\Big|_{\partial M}. \tag{4.21}$$

Consequently, by using $w_1|_{\partial M} = w_2|_{\partial M}$, we have that

$$R^{(1)}\Big|_{\partial M} = e^{-\tau\Psi}|_{\partial M}\,(w^{(1)} - w_0^{(1)})\Big|_{\partial M} = e^{-\tau\Psi}|_{\partial M}\,(w^{(2)} - w_0^{(2)})\Big|_{\partial M} = R^{(2)}\Big|_{\partial M}.$$

By (4.16) we thus have that

$$R^{(1)}\big|_{\partial M} = \mathcal{O}_{H^K(\partial M)}(\tau^{-N}).$$

Furthermore, we have $\partial_\nu w_1|_{\partial M} = \partial_\nu w_2|_{\partial M}$ since $\Lambda_{q_1} = \Lambda_{q_2}$. Consequently, by (4.21) we have

$$\begin{aligned}\partial_\nu R^{(1)}\big|_{\partial M} &= \partial_\nu \left(e^{-\tau\Psi}(w^{(1)} - w_0^{(1)})\right)\Big|_{\partial M} = \partial_\nu \left(e^{-\tau\Psi}(w^{(2)} - w_0^{(2)})\right)\Big|_{\partial M} \\ &= \partial_\nu R^{(2)}\big|_{\partial M} = \mathcal{O}_{H^K(\partial M)}(\tau^{-N}).\end{aligned}$$

By the boundary determination result of solutions on the boundary in Proposition 4.5, we have $\partial_\nu^2 w_1\big|_{\partial M} = \partial_\nu^2 w_2\big|_{\partial M}$. Thus, combining (4.17) and (4.21) shows $\partial_\nu^2 R^{(1)}\big|_{\partial M} = \mathcal{O}_{H^K(\partial M)}(\tau^{-N})$. In conclusion, we have that $R^{(1)}$ solves

$$\begin{cases} e^{-\tau\Psi}(\Delta_g - V)e^{\tau\Psi}R^{(1)} = \mathcal{O}_{H^K(M)}(\tau^{-N}) & \text{in } M, \\ \partial_\nu^\ell R^{(1)} = \mathcal{O}_{H^K(\partial M)}(\tau^{-N}) & \text{on } \partial M, \quad \ell = 0,1,2. \end{cases} \tag{4.22}$$

Now, it follows from Lemma 4.6 by taking $K = \frac{n+1}{2}$ and using the Sobolev embedding $H^K(\partial M) \subset L^\infty(\partial M)$, and finally redefining N as $N-2$ that

$$\|R^{(1)}\|_{L^2(M)} = \mathcal{O}(\tau^{-N}).$$

In the remaining case $\pm c_1 \pm c_2 = 0$, the correction terms $R^{(1)}$ and $R^{(2)}$ have zero boundary values by Remark 4.3. Since we also have $w_0^{(1)}|_{\partial M} = w_0^{(2)}|_{\partial M}$ by (4.20), the claim follows also in this case. □

4.2. **Solutions for the third linearization.** In this section, we consider solutions for the third linearizations of $(-\Delta_g + V)u + qu^2 = 0$ at the zero solution. Recalling that the third linearized equation is of the form

$$(-\Delta_g + V)\omega^{(ijk)} = -2q\left(v^{(i)}w^{(jk)} + v^{(j)}w^{(ik)} + v^{(k)}w^{(ij)}\right) \text{ in } M, \tag{4.23}$$

where $v^{(i)}$ and $w^{(jk)}$, are solutions to (2.5) and (2.6), respectively, for different $i,j,k = 1,2,3$. Again, we assume that the solutions $v^{(k)}$ are CGOs of the form (4.1):

$$v^{(k)} = e^{\pm s_k x_1}\left(\tau^{\frac{n-2}{8}} e^{\mathbf{i}s_k\psi_k} a_\tau^{(k)} + r_\tau^{(k)}\right),$$

where ψ_k corresponds to a nontangential geodesics γ_k of (M_0, g_0). Here $s_k = c_k\tau + \mathbf{i}\lambda_k$ also as before. We assume that γ_1, γ_2 and γ_3 intersect at the point p_0. We also assume that the supports of $v^{(k)}$ restricted to M_0 are so small that that the mutual support of $v^{(1)}$, $v^{(2)}$ and $v^{(3)}$ does not intersect the points on the geodesics γ_k where only two of the geodesics intersect. Lastly, we assume that all the pairs of geodesics γ_i and γ_k, $i \neq k$, intersect properly.

In order to analyze the solution ansatz for the third linearized equation (4.23), we can simply consider the case $i = 1$, $j = 2$ and $k = 3$. By Lemma 4.2, the equation $(-\Delta_g + V)w^{(23)} = -2qv^{(2)}v^{(3)}$ has a solution of the form

$$w^{(23)} = w_0^{(23)} + e^{\tau\Psi^{(23)}}R^{(23)}.$$

Here $w_0^{(23)}$ is given by the WKB ansatz

$$\begin{aligned} w_0^{(23)} &= \tau^{\frac{n-2}{4}} e^{(\pm s_2 \pm s_3)x_1 + \mathbf{i}(s_2\psi_2 + s_3\psi_3)} b^{(23)}, \\ b^{(23)} &= \tau^{-2} b_{-2}^{(23)} + \cdots + \tau^{-2N} b_{-2N'}^{(23)}, \\ b_{-2}^{(23)} &= \frac{2q}{(\pm c_2 \pm c_3)^2 - |c_2\nabla^{g_0}\psi_2 + c_3\nabla^{g_0}\psi_3|^2} a_0^{(2)} a_0^{(3)}. \end{aligned}$$

We take the solutions $w^{(13)}$ and $w^{(12)}$ to be ones given by similar formulas as $w^{(23)}$. Using these formulas for $w^{(ik)}$ and $v^{(j)}$ we see that (4.23) can be written as

$$(\Delta_g - V)\omega = \tau^{\frac{3(n-2)}{8}} e^{\tau\widetilde{\Psi}}(e^{\widetilde{\Lambda}}H + \rho),$$

where $\omega \equiv \omega^{(123)}$, and

$$\begin{aligned}
\widetilde{\Psi} &= (\pm c_1 \pm c_2 \pm c_3)x_1 + \mathbf{i}c_1\psi_1 + \mathbf{i}c_2\psi_2 + \mathbf{i}c_3\psi_3 \\
\widetilde{\Lambda} &= \mathbf{i}(\pm\lambda_1 \pm \lambda_2 \pm \lambda_3)x_1 - \lambda_1\psi_1 - \lambda_2\psi_2 - \lambda_3\psi_3 \\
H &= \sum_{k=2}^{3N'-1} \frac{H_{-k}}{\tau^k}, \\
\rho &= \mathcal{O}_{H^K(M)}(\tau^{-N}).
\end{aligned} \tag{4.24}$$

The amplitude $H \in C^\infty(M)$ is supported on neighborhoods of the points where all the geodesics γ_1, γ_2 and γ_3 intersect and which do not contain points where only two of the geodesics γ_k intersect. The order $3N' - 1$ of the amplitude H is a consequence of the respective orders $2N'$ and $N' - 1$ of the expansions of $w^{(ij)}$ and $a^{(k)}$. We have also assumed N' to be large enough so that the condition for ρ in (4.24) holds. Meanwhile, the factor $\tau^{\frac{3(n-2)}{8}}$ is a result of the product of the respective normalization factors $\tau^{\frac{n-2}{4}}$ and $\tau^{\frac{n-2}{8}}$ of $w^{(ij)}$ and $v^{(k)}$. The functions H_{-k} depend on q only in terms of the pointwise values q and its derivatives.

By (4.4), the leading order coefficient of H satisfies

$$\begin{aligned}
H_{-2} =& 4q^2 a_0^{(1)} a_0^{(2)} a_0^{(3)} \\
&\times \Bigg(\frac{1}{(\pm c_1 \pm c_2)^2 - |c_1\nabla^{g_0}\psi_1 + c_2\nabla^{g_0}\psi_2|^2} \\
&\quad + \frac{1}{(\pm c_1 \pm c_3)^2 - |c_1\nabla^{g_0}\psi_1 + c_3\nabla^{g_0}\psi_3|^2} \\
&\quad + \frac{1}{(\pm c_2 \pm c_3)^2 - |c_2\nabla^{g_0}\psi_2 + c_3\nabla^{g_0}\psi_3|^2} \Bigg).
\end{aligned} \tag{4.25}$$

If we additionally assume that

$$|\langle \nabla^g \widetilde{\Psi}, \nabla^g \widetilde{\Psi} \rangle| \geq \text{ constant } > 0$$

on the support of H, it makes sense to try an ansatz

$$\tau^{\frac{3(n-2)}{8}} e^{\tau\widetilde{\Psi}} e^{\widetilde{\Lambda}} B$$

for a solution ω of (4.23), where

$$B = \sum_{k=4}^{3N'+1} \frac{B_{-k}}{\tau^k}. \tag{4.26}$$

Here B_{-k}, $k = 4, 3, \ldots, 2(N' + 2)$ are given by the recursive formula

$$B_{-k} = \frac{e^{\widetilde{\Lambda}} H_{-k+2} - [2\langle \nabla^g B_{-k+1}, \nabla^g \Psi\rangle + B_{-k+1}\Delta_g \Psi] - (\Delta_g - V)B_{-k+2}}{\langle \nabla^g \widetilde{\Psi}, \nabla^g \widetilde{\Psi} \rangle} \tag{4.27}$$

and setting $B_{-2} = B_{-3} = 0$. Especially

$$B_{-4} = \frac{H_{-2}}{\langle \nabla^g \widetilde{\Psi}, \nabla^g \widetilde{\Psi} \rangle}, \tag{4.28}$$

where H_{-2} is given in (4.25). The support of B is the mutual support of $v^{(k)}$.

We obtain the following result. We omit the proof as it is a direct adaptation of the proof of Lemma 4.2.

Lemma 4.7. *Let $K, N \in \mathbb{N} \cup \{0\}$. Assume that $v^{(1)}, v^{(2)}, v^{(3)}$ are CGOs of the form (4.1) corresponding to geodesics $\gamma_1, \gamma_2, \gamma_3$ on M_0, respectively, such that the* *pairs of geodesics γ_k and γ_i intersect properly for $i, k = 1, 2, 3$ and $i \neq k$. Assume additionally that $\widetilde{\Psi}$ given by (4.24) satisfies*

$$\langle \nabla^g \widetilde{\Psi}, \nabla^g \widetilde{\Psi} \rangle \neq 0$$

at the points where all the geodesics γ_1, γ_2 and γ_3 intersect. If the restrictions of the amplitudes $a^{(k)}$ of $v^{(k)}$ to M_0 are supported in small enough neighborhoods of the geodesics γ_k, and $N' = N'(K, N)$ is large enough, then the equation

$$(-\Delta_g + V)\omega = -2q \left(v^{(1)} w^{(23)} + v^{(2)} w^{(13)} + v^{(3)} w^{(12)} \right) \text{ in } M, \tag{4.29}$$

where $w^{(ik)}$ is given as in Lemma 4.2 has a smooth solution ω up to the boundary ∂M with the following properties: The solution w is of the form

$$\omega = \omega_0 + e^{\tau \widetilde{\Psi}} \widetilde{R},$$

where the function ω_0 is of the form

$$\omega_0 = \tau^{\frac{3(n-2)}{8}} e^{\tau \widetilde{\Psi}} e^{\widetilde{\Lambda}} B,$$

where $\widetilde{\Lambda}$ and $B = B_\tau$ are given by (4.24) and (4.26) respectively. Especially B_{-4} is given by (4.28). The amplitude B depends on q only in terms of the pointwise values q and its derivatives. The remainder term $\widetilde{R} = \widetilde{R}_\tau$ satisfies

$$\|\widetilde{R}_\tau\|_{H^K(M)} \lesssim \tau^{-N}.$$

As stated, the amplitude B depends on q only in terms of the pointwise values q and its derivatives. Thus, by assuming that we know the DN map of $(-\Delta_g + V)u + qu^2 = 0$, we may determine the value of ω_0 on the boundary by boundary determination result (Proposition 4.5). Consequently, by using the Carleman estimate with boundary terms (Lemma 4.6), we have the following analogous result of Proposition 4.4. Note that

$$\langle \nabla^g \widetilde{\Psi}, \nabla^g \widetilde{\Psi} \rangle = (\pm c_1 \pm c_2 \pm c_3)^2 - |c_1 \nabla^{g_0} \psi_1 + c_2 \nabla^{g_0} \psi_2 + c_3 \nabla^{g_0} \psi_3|^2.$$

Proposition 4.8. *Assume as in Lemma 4.7 and adopt its notation. Assume additionally that $\Lambda_{q_1} = \Lambda_{q_2}$. If the restrictions of the amplitudes $a^{(k)}$ of $v^{(k)}$ to M_0 are supported in small enough neighborhoods of the geodesics γ_k, and $N' = N'(K, N)$ is large enough, then the third linearized equations*

$$(-\Delta_g + V)\omega^{(\beta)} = -2q_\beta \left(v^{(1)} w^{(23)}_\beta + v^{(2)} w^{(13)}_\beta + v^{(3)} w^{(12)}_\beta \right) \text{ in } M, \tag{4.30}$$

where $w^{(ik)}_\beta$, for $\beta = 1, 2$ and different $i, k = 1, 2, 3$, are given as in Proposition 4.4. Moreover, the solution $\omega^{(\beta)}$ is of the form

$$\omega^{(\beta)} = \omega^{(\beta)}_0 + e^{\tau \widetilde{\Psi}} \widetilde{R}^{(\beta)},$$

where

$$\begin{aligned} \omega^{(\beta)}_0 &= \tau^{\frac{3(n-2)}{8}} e^{\tau \widetilde{\Psi}} e^{\widetilde{\Lambda}} B^{(\beta)}, \\ B^{(\beta)} &= \tau^{-4} B^{(\beta)}_{-4} + \cdots + \tau^{-3N'+1} B^{(\beta)}_{-3N'+1}. \end{aligned}$$

Here $\widetilde{\Lambda}$ *and* $\widetilde{\Psi}$ *are given by* (4.24)*. Especially, the quantity* B_{-4} *in* (4.28) *can be written as*

$$
\begin{aligned}
B_{-4}^{(\beta)} =&4q_\beta^3 a_0^{(1)}a_0^{(2)}a_0^{(3)}\frac{1}{(\pm c_1\pm c_2\pm c_3)^2-|c_1\nabla^{g_0}\psi_1+c_2\nabla^{g_0}\psi_2+c_3\nabla^{g_0}\psi_3|^2}\\
&\times\Bigg(\frac{1}{(\pm c_1\pm c_2)^2-|c_1\nabla^{g_0}\psi_1+c_2\nabla^{g_0}\psi_2|^2}\\
&\qquad+\frac{1}{(\pm c_1\pm c_3)^2-|c_1\nabla^{g_0}\psi_1+c_3\nabla^{g_0}\psi_3|^2}\\
&\qquad+\frac{1}{(\pm c_2\pm c_3)^2-|c_2\nabla^{g_0}\psi_2+c_3\nabla^{g_0}\psi_3|^2}\Bigg).
\end{aligned}
\tag{4.31}
$$

and $\widetilde{R}^{(\beta)}=\mathcal{O}_{L^2(M)}(\tau^{-N})$*, for* $\beta=1,2$*. Moreover*

$$\omega^{(1)}\big|_{\partial M}=\omega^{(2)}\big|_{\partial M}.$$

We skip the proof of Proposition 4.8 as it can be obtained from the proof of Proposition 4.4 by replacing w by ω and Ψ by $\widetilde{\Psi}$ etc. The function F in (4.19) in the proof also needs to be replaced by a function of the class $\mathcal{O}_{L^2(M)}(\tau^{-N})$ since $R^{(2)}$ in Proposition 4.4 is $\mathcal{O}_{L^2(M)}(\tau^{-N})$. We remark that by deriving Carleman estimates similar to those in Lemma 4.6 for higher Sobolev spaces, we could in fact have that $\widetilde{R}$ is of the size τ^{-N} also in higher Sobolev spaces $H^K(M)$ by taking N' large enough.

5. Proof of Theorem 1.1

In this section we prove our main result, Theorem 1.1. We will see that it is possible to deduce

$$q_1^2=q_2^2$$

in M from third order linearizations and the DN map of the equation $(-\Delta_g+V)u+qu^2=0$. Our method for the third linearized equation however does not imply $q_1=q_2$ in general. In order to show that

$$q_1=q_2 \text{ in } M,$$

we in fact need to consider fourth order linearized equations. To give a proof of Theorem 1.1, we could consider the fourth order linearization from the beginning. However, we first consider third order linearizations and prove $q_1^2=q_2^2$ to better explain the main ideas of the proof.

5.1. **Proof of $q_1^2=q_2^2$.** Let $p_0\in M_0$, and let γ_1 be a non-tangential geodesic that has no self-intersections. We consider the equation

$$
\begin{cases}
(-\Delta_g+V)u_\beta+q_\beta u_\beta^2=0 & \text{in } M,\\
u_\beta=f & \text{on } \partial M,
\end{cases}
\tag{5.1}
$$

for $\beta=1,2$, where $f=f_\epsilon\in C^{2,\alpha}(\partial M)$ is of the form

$$f_\epsilon:=\sum_{i=1}^4\epsilon_i f_i+\sum_{i,j=1}^4\epsilon_i\epsilon_j f_{ij}\quad \text{on}\quad \partial M. \tag{5.2}$$

Let us recall the linearizations (5.1) from Section 2.2. The first linearization reads

$$
\begin{cases}
(-\Delta_g+V)v_\beta^{(i)}=0 & \text{in } M,\\
v_\beta^{(i)}=f_i & \text{on } \partial M,
\end{cases}
\tag{5.3}
$$

where $v_\beta^{(i)} = \partial_{\epsilon_i}|_{\epsilon=0}\, u_\beta$ for $\beta = 1, 2$, and $i = 1, 2, 3, 4$. By the uniqueness of solutions to (5.3), we obtain

$$v^{(i)} := v_1^{(i)} = v_2^{(i)} \text{ in } M,$$

for $i = 1, 2, 3, 4$. The second linearization of (5.1) satisfies

$$\begin{cases} (-\Delta_g + V)w_\beta^{(ij)} = -2q_\beta v^{(i)} v^{(j)} & \text{in } M, \\ w_\beta^{(ij)} = f_{ij} & \text{on } \partial M, \end{cases} \tag{5.4}$$

where

$$w_\beta^{(ij)} = \partial^2_{\epsilon_i \epsilon_j}\Big|_{\epsilon=0} u_\beta,$$

for $\beta = 1, 2$ and different $i, j \in \{1, 2, 3\}$. Lastly, the third linearization of (5.1) satisfies

$$\begin{cases} (-\Delta_g + V)w_\beta^{(ijk)} = -2q_\beta \left(v^{(i)} w_\beta^{(jk)} + v^{(j)} w_\beta^{(ik)} + v^{(k)} w_\beta^{(ij)} \right) & \text{in } M, \\ w_\beta^{(ikl)} = 0 & \text{on } \partial M, \end{cases} \tag{5.5}$$

where

$$w_\beta^{(ijk)} = \partial^3_{\epsilon_i \epsilon_j \epsilon_k}\Big|_{\epsilon=0} u_\beta.$$

Since $\Lambda_{q_1} = \Lambda_{q_2}$

$$0 = \partial^3_{\epsilon_i \epsilon_j \epsilon_k}\Big|_{\epsilon=0} \left(\Lambda_{q_1} - \Lambda_{q_2}\right)(f_\epsilon) \tag{5.6}$$

Thus, by Lemma 2.1 we have

$$\begin{aligned} 0 = \int_M \Big\{ & q_1 \left(v^{(i)} w_1^{(jk)} + v^{(j)} w_1^{(ik)} + v^{(k)} w_1^{(ij)} \right) \\ & - q_2 \left(v^{(i)} w_2^{(jk)} + v^{(j)} w_2^{(ik)} + v^{(k)} w_2^{(ij)} \right) \Big\} v^{(l)}\, dV_g, \end{aligned} \tag{5.7}$$

where $v^{(i)}$ and $w_\beta^{(jk)}$ are the solutions of (5.3) and (5.4), respectively, for different $i, j, k = 1, 2, 3, 4$ and $\beta = 1, 2$.

We choose $v^{(i)}$ to be CGOs corresponding geodesics on (M_0, g_0), which intersect properly pairwise at p_0. We show that the integrand on the right hand side of (5.7) restricted to a neighborhood of p_0 in M_0 is close to a multiple of the delta function. We let $v^{(1)}$ correspond to the geodesic γ_1 and choose the other 3 geodesics next.

5.2. **Choices of initial vectors for the third linearization.** Let $\delta \in (0, 1)$, and we denote the initial data of γ_1 by $\xi_1 \in S_{p_0}M_0$. We recall that by definition γ_1 is nontangential and has no self-intersections thanks to (H1). By perturbing ξ_1, we find $\xi_2 \in S_{p_0}M_0$ such that the associated geodesic γ_2 is also non-tangential, has no self-intersections (the latter fact follows from the argument in the proof of [DSFKL+18, Lemma 3.1]), and that

$$|\xi_1| = |\xi_2| = 1$$

and

$$\langle \xi_1, \xi_2 \rangle = 1 - \delta.$$

Let us define

$$\xi_3 = -\frac{1}{1+\delta}(\xi_1 + \delta\xi_2) \in S_{p_0}M_0 \quad \text{and} \quad \xi_4 = -\frac{1}{1+\delta}(\delta\xi_1 + \xi_2) \in S_{p_0}M_0.$$

A direct computation shows

$$\sum_{l=1}^4 \xi_l = \xi_1 + \xi_2 - \frac{1}{1+\delta}\xi_1 - \frac{\delta}{1+\delta}\xi_2 - \frac{\delta}{1+\delta}\xi_1 - \frac{1}{1+\delta}\xi_2 = 0. \tag{5.8}$$

We redefine δ smaller, if necessary, so that the geodesics γ_3 and γ_4 corresponding to ξ_3 and ξ_4 are also nontangential and have no self-intersections.

Note that ξ_1 is not proportional to ξ_2 as ξ_1 and ξ_2 are linearly independent. Similarly, for $k = 3, 4$, the vector ξ_k is neither proportional to ξ_1 nor to ξ_2. Lastly, ξ_3 is not proportional to ξ_4. Indeed, if $A \in \mathbb{R}$ is such that $\xi_3 = A\xi_4$, we have that $1 = \delta A$ and $\delta = A$, implying that $\delta = \pm 1$. However, $\delta \in (0, 1)$. This means that all the pairs of the geodesics corresponding to initial data $\xi_k/|\xi_k|$ intersect properly.

Note also that since $|\xi_1| = |\xi_2| = 1$, we have

$$\begin{aligned}|\xi_3|^2 =& \frac{1}{(1+\delta)^2}\left(|\xi_1|^2 + \delta^2|\xi_2|^2 + 2\delta\langle\xi_1, \xi_2\rangle\right)\\ =& \frac{1}{(1+\delta)^2}\left(|\xi_2|^2 + \delta^2|\xi_1|^2 + 2\delta\langle\xi_2, \xi_1\rangle\right) = |\xi_4|^2.\end{aligned}$$

That is

$$|\xi_3|^2 = |\xi_4|^2 = \frac{1}{(1+\delta)^2}\left(1 + \delta^2 + 2\delta(1-\delta)\right) = \frac{1}{(1+\delta)^2}\left(1 + 2\delta - \delta^2\right) \tag{5.9}$$

Let us then define vectors $\overline{\xi}_k \in TM$, $k = 1, 2, 3, 4$, by

$$\begin{aligned}\overline{\xi}_1 &= |\xi_1|e_1 + \mathbf{i}\xi_1, \quad \overline{\xi}_2 = -|\xi_2|e_1 + \mathbf{i}\xi_2\\ \overline{\xi}_3 &= |\xi_3|e_1 + \mathbf{i}\xi_3, \quad \overline{\xi}_4 = -|\xi_4|e_1 + \mathbf{i}\xi_4.\end{aligned} \tag{5.10}$$

Then

$$\sum_{k=1}^{4}\overline{\xi}_k = 0. \tag{5.11}$$

Note also that

$$\langle\overline{\xi}_k, \overline{\xi}_k\rangle = 0, \quad k = 1, \dots, 4. \tag{5.12}$$

Related to these vectors $\overline{\xi}_k$, we will consider in the proof of Theorem 1.1 CGOs, which can be written of the form

$$v_s^{(k)} = e^{\mathrm{Re}(\overline{\xi}_k)x_1}\left(\tau^{\frac{n-2}{8}}e^{\mathbf{i}|\xi_k|\psi_k}a_s + r_s\right).$$

Here the phase functions ψ_k are constructed with respect to the geodesics γ_k with initial data $\gamma_k(0) = p_0$ and $\dot\gamma_k(0) = \frac{\xi_k}{|\xi_k|}$. We note that

$$\nabla^g\Big(\mathrm{Re}(\overline{\xi}_k)x_1 + \mathbf{i}|\xi_k|\psi_k)\Big)\Big|_{\gamma_k(0)} = \overline{\xi}_k. \tag{5.13}$$

Consequently, the ansatzes $w_0^{(ik)}$ in Lemma 4.2 for the solutions of $(-\Delta_g+V)w^{(ik)} = -2qv^{(i)}v^{(k)}$ have amplitudes with a factor that divides by

$$\left\langle\nabla^g\big(\mathrm{Re}(\overline{\xi}_i + \overline{\xi}_k)x_1 + \mathbf{i}(|\xi_i|\psi_i + |\xi_k|\psi_k)\big), \nabla^g\big(\mathrm{Re}(\overline{\xi}_i + \overline{\xi}_k)x_1 + \mathbf{i}(|\xi_i|\psi_i + |\xi_k|\psi_k)\big)\right\rangle,$$

for different $i, k = 1, 2, 3$. At an intersection point of the geodesics γ_i and γ_k the above equals

$$\langle\overline{\xi}_i + \overline{\xi}_k, \overline{\xi}_i + \overline{\xi}_k\rangle = 2\left\langle\nabla^g\big(\mathrm{Re}(\overline{\xi}_i)x_1 + \mathbf{i}|\xi_i|\psi_i\big), \nabla^g\big(\mathrm{Re}(\overline{\xi}_k)x_1 + \mathbf{i}|\xi_k|\psi_k\big)\right\rangle = 2\left\langle\overline{\xi}_i, \overline{\xi}_k\right\rangle.$$

Here we used (5.13) and (5.12). Motivated by this, we define

$$\mathbf{C}_{ik} := 2\left\langle\overline{\xi}_i, \overline{\xi}_k\right\rangle.$$

The coefficient $\mathbf{C}_{ik}$ can be collectively written as

$$\mathbf{C}_{ik} = 2|\xi_i||\xi_k|\left((-1)^{i+k} - \frac{\langle\xi_i, \xi_k\rangle}{|\xi_i||\xi_k|}\right).$$

We calculate expansions for $\mathbf{C}_{ik}$ for small $\delta > 0$ parameter. A direct computation shows that

$$\langle\xi_1, \xi_3\rangle = \left\langle\xi_1, -\frac{1}{1+\delta}(\xi_1 + \delta\xi_2)\right\rangle = -\frac{1}{1+\delta}(|\xi_1|^2 + \delta\langle\xi_1, \xi_2\rangle) = -\frac{1}{1+\delta}(1 + \delta - \delta^2)$$

and

$$\langle \xi_2, \xi_3\rangle = -\frac{1}{1+\delta}\left(\langle \xi_1, \xi_2\rangle + \delta|\xi_2|^2\right) = -\frac{1}{1+\delta},$$

where we used $\langle \xi_1, \xi_2\rangle = 1-\delta$. We also have

$$\frac{\langle \xi_1, \xi_3\rangle}{|\xi_1||\xi_3|} = \frac{-\frac{1}{1+\delta}(1+\delta-\delta^2)}{\left(\frac{1}{(1+\delta)^2}\left(1+2\delta-\delta^2\right)\right)^{1/2}} = -1+\mathcal{O}(\delta)$$

and

$$\frac{\langle \xi_2, \xi_3\rangle}{|\xi_2||\xi_3|} = \frac{-\frac{1}{1+\delta}}{\left(\frac{1}{(1+\delta)^2}\left(1+2\delta-\delta^2\right)\right)^{1/2}} = -1+\mathcal{O}(\delta).$$

Here we have utilized the Taylor expansions

$$(1+r)^{1/2} = 1+\frac{r}{2}+\mathcal{O}(r^2) \quad \text{and} \quad (1+r)^{-1} = 1-r+\mathcal{O}(r^2),$$

which hold for sufficiently small $|r|$. Combining the above formulas yields

$$\begin{aligned}
\mathbf{C}_{12} &= 2|\xi_1||\xi_2|\left(-1-\frac{\langle \xi_1, \xi_2\rangle}{|\xi_1||\xi_2|}\right) = -4+\mathcal{O}(\delta),\\
\mathbf{C}_{13} &= 2|\xi_1||\xi_3|\left(1-\frac{\langle \xi_1, \xi_3\rangle}{|\xi_1||\xi_3|}\right) = 2\frac{\left(1+2\delta-\delta^2\right)^{1/2}}{1+\delta}\Big(1-\big(-1+\mathcal{O}(\delta)\big)\Big)\\
&= 4+\mathcal{O}(\delta),\\
\mathbf{C}_{23} &= 2|\xi_2||\xi_3|\left(-1-\frac{\langle \xi_2, \xi_3\rangle}{|\xi_2||\xi_3|}\right) = 2\frac{\left(1+2\delta-\delta^2\right)^{1/2}}{1+\delta}\Big(-1-\big(-1+\mathcal{O}(\delta)\big)\Big)\\
&= \mathcal{O}(\delta).
\end{aligned} \tag{5.14}$$

We also remark here that $\mathbf{C}_{ik}\neq 0$, $i\neq k$, for $\delta>0$ since

$$|\mathbf{C}_{ik}| = 2|\xi_i||\xi_k|\left|(-1)^{i+k}-\frac{\langle \xi_i, \xi_i\rangle}{|\xi_i||\xi_k|}\right| \tag{5.15}$$

and

$$\frac{\langle \xi_i, \xi_k\rangle}{|\xi_2||\xi_3|}\in(-1,1),$$

because the pairs of vectors ξ_i and ξ_k are linearly independent. Finally, we note that

$$\left|\frac{1}{\mathbf{C}_{12}}+\frac{1}{\mathbf{C}_{13}}+\frac{1}{\mathbf{C}_{13}}\right| = \left|\frac{1}{-4+\mathcal{O}(\delta)}+\frac{1}{4+\mathcal{O}(\delta)}+\frac{1}{\mathcal{O}(\delta)}\right|\to\infty,$$

when $\delta\to 0$. Thus

$$\frac{1}{\mathbf{C}_{12}}+\frac{1}{\mathbf{C}_{13}}+\frac{1}{\mathbf{C}_{13}}\neq 0, \tag{5.16}$$

for all small enough $\delta>0$.

Remark 5.1. *Let us define a Lorentz metric η for M by the formula*

$$\eta(c_1e_1+V_1, c_2e_1+V_2) := \langle c_1e_1+\mathbf{i}V_1, c_2e_1+\mathbf{i}V_2\rangle_g,$$

where $c_1, c_2\in\mathbb{R}$ and $V_1, V_2\in TM_0$. Note that η is real since V_1 and V_2 are orthogonal to e_1 with respect to the metric g. We required that the vectors $\overline{\xi}_1,\dots,\overline{\xi}_4$ in (5.10) are lightlike vectors with respect to η and sum up to 0. The former requirement is because the corresponding phase functions need to satisfy the complex eikonal equation. The latter requirement is discussed in the next section.

A fact is that three η-lightlike vectors can only sum up to 0, if the parts in TM_0 of two of them are linearly dependent. This would correspond to geodesics that do not *intersect properly. Due to this geometric fact, we overdifferentiate in this paper the nonlinearity qu^2 to obtain integral identities that consider more than three CGOs.*

5.3. **Proof of $q_1^2 = q_2^2$ (continued).** Let us then return to proving $q_1^2 = q_2^2$. Let $\overline{\xi}_k$, $k = 1, 2, 3, 4$, be as in (5.10). We set

$$c_k = |\xi_k| \text{ and}$$

and

$$s_1 = c_1\tau + \mathbf{i}\lambda \quad \text{ and } \quad s_\ell = c_\ell\tau, \quad \text{ for } \quad \ell = 2, 3, 4.$$

Then the corresponding CGOs are of the form

$$\begin{aligned}
v^{(1)} &= e^{(|\xi_1|\tau+\mathbf{i}\lambda)x_1}\left(\tau^{\frac{n-2}{8}}e^{\mathbf{i}(|\xi_1|\tau+\mathbf{i}\lambda)\psi_1}a_1 + r_1\right),\\
v^{(2)} &= e^{-|\xi_2|\tau x_1}\left(\tau^{\frac{n-2}{8}}e^{\mathbf{i}|\xi_2|\tau\psi_2}a_2 + r_2\right),\\
v^{(3)} &= e^{|\xi_3|\tau x_1}\left(\tau^{\frac{n-2}{8}}e^{\mathbf{i}|\xi_3|\tau\psi_3}a_3 + r_3\right),\\
v^{(4)} &= e^{-\xi_4\tau x_1}\left(\tau^{\frac{n-2}{8}}e^{\mathbf{i}|\xi_4|\tau\psi_4}a_4 + r_4\right).
\end{aligned}$$

We may assume that $v^{(k)}$, $k = 1, \ldots, 4$ are supported in small enough neighborhoods of the corresponding geodesics γ_k so that the mutual support of $v^{(k)}$ belongs to neighborhoods of the points where all the geodesics γ_k intersect and where any pair of the geodesics intersect only once. Let us denote the points where all the geodesics γ_k intersect by $p_0, p_1, \ldots, p_Q$.

Let $i \neq j \in \{1, 2, 3, 4\}$, $i \neq j$ and $\beta = 1, 2$. By assumption, the DN maps of the equation (1.3) for the potentials q_1 and q_2 satisfy $\Lambda_{q_1} = \Lambda_{q_2}$. By Proposition 4.4 there are boundary values f_{ij}, which are the same for both q_1 and q_2, such that the solutions of the second linearized equations (5.4) are of the form

$$w_\beta^{(ij)} = w_{0,\beta}^{(ij)} + e^{\tau\Psi^{(ij)}}R_\beta^{(ij)},$$

where the ingredients are as follows:

$$\begin{aligned}
\Psi^{(ij)} &= ((-1)^{1+i}c_i + (-1)^{1+j}c_j)x_1 + \mathbf{i}(c_i\psi_i + c_j\psi_j),\\
w_{0,\beta}^{(ij)} &= \tau^{\frac{n-2}{4}}e^{((-1)^{1+i}s_i+(-1)^{1+j}s_j)x_1+\mathbf{i}(s_i\psi_i+s_j\psi_j)}b_\beta^{(ij)},\\
b_\beta^{(ij)} &= \tau^{-2}b_{-2,\beta}^{(ij)} + \cdots + \tau^{-2N'}b_{-2N',\beta}^{(ij)}\\
b_{-2,\beta}^{(ij)} &= \frac{2q_\beta}{((-1)^{1+i}c_i + (-1)^{1+j}c_j)^2 - |c_i\nabla^{g_0}\psi_i + c_j\nabla^{g_0}\psi_j|^2}a_0^{(i)}a_0^{(j)}.
\end{aligned} \tag{5.17}$$

By (5.13), at points of the form $(x_1, p_0) \in M$ we have

$$b_{-2,\beta}^{(ij)} = \frac{2q_\beta}{\mathbf{C}_{ij}}a_0^{(i)}a_0^{(j)}.$$

Here $a_0^{(i)}$ and $a_0^{(j)}$ are independent of the variable $x_1 \in \mathbb{R}$.

To simplify the following calculations, let us define

$$\Psi_{1234} = \sum_{k=1}^{4}\left((-1)^{1+k}c_kx_1 + \mathbf{i}c_k\psi_k\right) = \mathbf{i}\sum_{k=1}^{4}c_k\psi_k \tag{5.18}$$

and

$$\Lambda_{1234} = \lambda(\mathbf{i}x_1 - \psi_1).$$

Let us observe that

$$\Psi^{(12)} + \Psi^{(34)} = \Psi^{(13)} + \Psi^{(24)} = \Psi^{(14)} + \Psi^{(23)} = \mathbf{i}\sum_{k=1}^{4}c_k\psi_k = \Psi_{1234}. \tag{5.19}$$

Since Ψ_{1234} is purely imaginary at the intersection points p_b, $b = 0, \dots, Q$, the exponentially large linear factors will cancel in terms of the form $v^{(i)} w_\beta^{(jk)} v^{(l)}$ appearing the integral identity for the third linearization (5.5). We also have at the intersection point p_0 of the geodesics that

$$\begin{aligned} \nabla^g(\Psi^{(12)} + \Psi^{(34)})\Big|_{p_0} &= \nabla^g(\Psi^{(13)} + \Psi^{(24)})\Big|_{p_0} = \nabla^g(\Psi^{(14)} + \Psi^{(23)})\Big|_{p_0} \\ &= \mathbf{i} \sum_{k=1}^4 c_k \nabla^g \psi_k \Big|_{p_0} = \mathbf{i}\nabla^g \Psi_{1234}|_{p_0} = \mathbf{i}\sum_{k=1}^4 \xi_k = 0. \end{aligned} \tag{5.20}$$

This implies that p_0 is a critical point of the phase functions of functions of the form $v^{(i)} w_\beta^{(jk)} v^{(l)}$. The critical point is also nondegenerate by (3.2) in Section 3 and thus we will be able to apply stationary phase in the asymptotic parameter τ.

Let us first consider the case p_0 is the only point where all the geodesics $\gamma_1, \dots, \gamma_4$ intersect. With the above preparations and using $\Lambda_{q_1} = \Lambda_{q_2}$ the integral identity (5.7) of the third order linearization reads

$$\begin{aligned} 0 = &\int_M \Big[q_1 \left(v^{(1)} w_1^{(23)} + v^{(2)} w_1^{(13)} + v^{(3)} w_1^{(12)} \right) \\ &\qquad - q_2 \left(v^{(1)} w_2^{(23)} + v^{(2)} w_2^{(13)} + v^{(3)} w_2^{(12)} \right) \Big] v^{(4)} \, dV_g \\ =&\tau^{\frac{n-6}{2}} \int_M e^{\tau \Psi_{1234}} \left[e^{\Lambda_{1234}} a_0^{(1)} a_0^{(2)} a_0^{(3)} a_0^{(4)} (q_1^2 - q_2^2) \left(\mathbf{C}_{12}^{-1} + \mathbf{C}_{13}^{-1} + \mathbf{C}_{23}^{-1} \right) + \overline{R} \right] dV_g, \end{aligned} \tag{5.21}$$

where $\overline{R} = \mathcal{O}_{L^1(M)}(\tau^{-1})$. The factor τ^{-2} arises from the amplitude functions of the solutions $w_\beta^{(ik)}$, see (5.17) and the power $\frac{n-2}{2}$ of τ is the sum of $\frac{3(n-2)}{8}$ and $\frac{n-2}{8}$. By (5.19) the exponentially large factors of the integrand cancel. Recall that the dimension of M_0 is $n-1$.

We multiply the integral identity (5.21) by $\tau^{1/2}$ and τ^2. This achieves the correct normalization $\tau^{\dim(M_0)/2}$ for stationary phase. By (5.20), at the intersection point p_0 of the geodesics γ_k for $x_1 \in I \subset \mathbb{R}$ holds

$$\nabla^g \Psi_{1234}(x_1, p_0) = 0.$$

In normal coordinates $(y^1, \dots, y^{n-1})$ centered at the point p_0 in M_0

$$\operatorname{Re} \Psi_{1234}(y) = \sum_{j,k=1}^{n-1} A_{jk} y^j y^k + \mathcal{O}(|y|^3), \tag{5.22}$$

for some negative definite matrix A by the properties (3.2) of the phase functions. Note also that

$$\tau^{\frac{n-1}{2}} \int_{\mathbb{R}^{n-1}} e^{-\tau |y|^2} \, dy = \mathcal{O}(1) \quad \text{and} \quad \tau^{\frac{n-1}{2}} \int_{\mathbb{R}^{n-1}} |y| e^{-\tau |y|^2} \, dy = \mathcal{O}(\tau^{-\frac{1}{2}}).$$

Thus, stationary phase shows that the limit $\tau \to \infty$ of (5.21) equals

$$\begin{aligned} &c_A \left(a_0^{(1)} a_0^{(2)} a_0^{(3)} a_0^{(4)} \right)\Big|_{p_0} \left(\mathbf{C}_{12}^{-1} + \mathbf{C}_{13}^{-1} + \mathbf{C}_{23}^{-1} \right) \\ &\quad \times \int_{\mathbb{R}} e^{\Lambda_{1234}(x_1, p_0)} (q_1^2(x_1, p_0) - q_2^2(x_1, p_0)) dx_1, \end{aligned}$$

where

$$c_A = \int_{\mathbb{R}^{n-1}} e^{x \cdot Ax} dx \neq 0.$$

We refer to [LLLS21a, Proof of Theorem 5.1, Step 4] for more details on this stationary phase argument. Here we have also used that $a_0^{(k)}$, $k = 1, \dots, 4$, depend only on the transversal variables. We also continued q_1 and q_2 by zero from I to $\mathbb{R}$ in the x_1 variable.

The geodesics γ_k, $k = 1, \dots, 5$ were parametrized so that $\gamma_k(0) = p_0$. Thus $\psi_k(p_0) = 0$ for each $k = 1, \dots, 5$ and we have

$$e^{\Lambda_{1234}(x_1, p_0)} = e^{\mathbf{i}\lambda x_1}.$$

Since $\mathbf{C}_{12}^{-1} + \mathbf{C}_{13}^{-1} + \mathbf{C}_{23}^{-1} \neq 0$ and $a_0^{(k)}|_{\gamma_k} \neq 0$ by (5.16) and (3.6) respectively, combining the above shows that

$$\int_{\mathbb{R}} e^{\lambda x_1} \left(q_1^2(x_1, p_0) - q_2^2(x_1, p_0) \right) dx_1 = 0.$$

Inverting the Fourier transformation in the x_1 variable shows that $q_1^2(x_1, p_0) = q_2^2(x_1, p_0)$. Since p_0 was arbitrary, this completes the proof in the case p_0 was the only point where all the geodesics intersect.

Consider then the remaining case where are several points p_b, $b = 0, \dots, Q$, where all the geodesics $\gamma_1, \gamma_2, \gamma_3, \gamma_4$ intersect. Note also that outside (disjoint) neighborhoods U_b of p_b the function $e^{\tau \Psi_{1234}}$ is exponentially small. We also remark that in a neighborhood of p_b we may write an analogous expression as in (5.22) for some negative definite matrix A. The fact that A is negative definite at p_b is due to property (3.2) for each of the functions ψ_k, $k = 1, 2, 3, 4$ at the point p_b and the fact the geodesics $\gamma_1, \dots, \gamma_4$ intersect transversally at the point p_b.

Thus, for different $i, j, k, l = 1, 2, 3, 4$, by normalizing and taking limit $\tau \to \infty$ of (5.21) we obtain

$$\begin{aligned} 0 = \lim_{\tau \to \infty} \sum_{b=0}^{Q} \int_{U_b} \Big[& q_1 \left(v^{(1)} w_1^{(23)} + v^{(2)} w_1^{(13)} + v^{(3)} w_1^{(12)} \right) \\ & - q_2 \left(v^{(1)} w_2^{(23)} + v^{(2)} w_2^{(13)} + v^{(3)} w_2^{(12)} \right) \Big] v^{(4)} \, dV_g \\ = \lim_{\tau \to \infty} \tau^{\frac{n-1}{2}} \sum_{b=0}^{Q} \int_{U_b} \Big[& e^{\Lambda_{1234}} e^{\tau \Psi_{1234}} a_0^{(1)} a_0^{(2)} a_0^{(3)} a_0^{(4)} (q_1^2 - q_2^2) \\ & \times \left(\mathbf{C}_{12}^{-1}(p_b) + \mathbf{C}_{13}^{-1}(p_b) + \mathbf{C}_{23}^{-1}(p_b) \right) \Big] \, dV_g. \end{aligned} \tag{5.23}$$

Here we have denoted

$$\mathbf{C}_{ik}(p_b) = \left\langle \nabla^g \left((-1)^{1+i} c_i x_1 + \mathbf{i} c_i \psi_i \right), \nabla^g \left((-1)^{1+k} c_k x_1 + \mathbf{i} c_k \psi_k \right) \right\rangle \Big|_{p_b} \neq 0$$

Note that $\mathbf{C}_{ik}(p_b) \neq 0$, since γ_i and γ_k, $i \neq k$, intersect properly, cf. (5.15). Therefore, by applying stationary phase to (5.23) it follows that

$$\sum_{b=0}^{Q} \hat{h}_b(\lambda) e^{c_b \lambda} = 0, \quad \lambda \in \mathbb{R}.$$

Here c_b are the distinct geodesic parameter times of γ_1 where $\gamma_1(c_b) = p_b$ and

$$\hat{h}_b(\lambda) := \mathcal{F}_{x_1 \to \lambda} \left(a_0^{(1)} \cdots a_0^{(4)} \left(\mathbf{C}_{12}^{-1} + \mathbf{C}_{13}^{-1} + \mathbf{C}_{23}^{-1} \right) \Big|_{p_b} \left(q_1^2(x_1, p_b) - q_2^2(x_1, p_b) \right) \right),$$

where $\mathcal{F}_{x_1 \to \lambda}$ is the Fourier transform in x_1 variable. By [LLLS21a, Lemma 6.2]

$$h_0 = \cdots = h_Q = 0.$$

Especially $q_1^2(x_1, p_0) = q_2^2(x_1, p_0)$, which concludes the proof of $q_1^2 = q_2^2$ also in the case where there are several points where the geodesics γ_k all intersect.

5.4. **Proof of $q_1 = q_2$ and fourth order linearization.** We proved $q_1^2 = q_2^2$ using third order linearizations of the equation $(-\Delta_g + V)u + qu^2 = 0$. Here we consider fourth order linearizations of the equation and use it to complete the proof of Theorem 1.1. Most of the steps here will be similar to those we used to prove $q_1^2 = q_2^2$. However, the steps are somewhat more complicated.

Let $p_0 \in M_0$, and let γ_1 be a non-tangential geodesic, which has no self-intersections. We recall that the existence of γ_1 is guaranteed, thanks to (H1). Let $\xi_1 \in S_{p_0}M_0$ by the initial data if γ_1. Let us consider the equation

$$\begin{cases} (-\Delta_g + V)u_\beta + q_\beta u_\beta^2 = 0 & \text{in } M, \\ u_\beta = f & \text{on } \partial M, \end{cases} \tag{5.24}$$

this time with boundary values $f \in C^\infty(\partial M)$ of the form (2.3), for $\beta = 1, 2$. The first and second linearized equations are the same as before and read

$$(-\Delta_g + V)v^{(i)} = 0,$$
$$(-\Delta_g + V)w_\beta^{(ij)} = -2q_\beta v^{(i)} v^{(j)}.$$

Here $v^{(i)}$ and $w_\beta^{(ij)}$, $i \neq j \in \{1, \ldots, 4\}$, $\beta = 1, 2$, have boundary values f_i and f_{ij} respectively. The solutions $v^{(i)}$ are the same for both potentials q_1 and q_2. The third order linearizations $w_\beta^{(ijk)}$ now have (possibly) non-zero boundary values and satisfy

$$\begin{cases} (-\Delta_g + V)w_\beta^{(ijk)} = -2q_\beta \left(v^{(i)} w_\beta^{(jk)} + v^{(j)} w_\beta^{(ik)} + v^{(k)} w_\beta^{(ij)} \right) & \text{in } M, \\ w_\beta^{(ijk)} = f_{ijk} & \text{on } \partial M, \end{cases}$$

where $w_\beta^{(ijk)} = \partial^3_{\epsilon_i \epsilon_j \epsilon_k}\big|_{\epsilon=0} u_\beta$, for $\beta = 1, 2$ and different $i, j, k \in \{1, \ldots, 4\}$. The boundary values f_{ijk} are the same for both of the equations (5.5), which correspond to the potentials q_1 and q_2.

The fourth order linearization

$$w_\beta^{(1234)} = \partial^4_{\epsilon_1 \epsilon_2 \epsilon_3 \epsilon_4}\big|_{\epsilon=0} u_\beta$$

is the solution of

$$\begin{cases} (-\Delta_g + V)w_\beta^{(1234)} = -2q \Big(v^{(1)} w_\beta^{(234)} + v^{(2)} w_\beta^{(134)} \\ \qquad\qquad + v^{(3)} w_\beta^{(124)} + v^{(4)} w_\beta^{(123)} \\ \qquad\qquad + w_\beta^{(12)} w_\beta^{(34)} + w_\beta^{(13)} w_\beta^{(24)} + w_\beta^{(14)} w_\beta^{(23)} \Big) & \text{in } M, \\ w_\beta^{(1234)} = 0 & \text{on } \partial M. \end{cases} \tag{5.25}$$

Using $\Lambda_{q_1} = \Lambda_{q_2}$ we have by Lemma 2.1 the integral identity

$$\begin{aligned} 0 = \int_M \Big\{ & q_1 \Big(v^{(1)} w_1^{(234)} + v^{(2)} w_1^{(134)} + v^{(3)} w_1^{(124)} + v^{(4)} w_1^{(123)} \\ & \qquad + w_1^{(12)} w_1^{(34)} + w_1^{(13)} w_1^{(24)} + w_1^{(14)} w_1^{(23)} \Big) \\ & - q_2 \Big(v^{(1)} w_2^{(234)} + v^{(2)} w_2^{(134)} + v^{(3)} w_2^{(124)} + v^{(4)} w_2^{(123)} \\ & \qquad + w_2^{(12)} w_2^{(34)} + w_2^{(13)} w_2^{(24)} + w_2^{(14)} w_2^{(23)} \Big) \Big\} v^{(5)} \, dV_g. \end{aligned} \tag{5.26}$$

We will use five CGOs as the solutions $v^{(k)}$, $k = 1, \dots, 5$. As before, these have the form

$$v^{(k)} = e^{\pm s_k x_1} \left(\tau^{\frac{n-2}{8}} e^{\mathbf{i} s_k \psi_k} a_\tau^{(k)} + r_\tau^{(k)} \right),$$

where $s_k = c_k \tau + \mathbf{i}\lambda_k$. However, the geodesics of (M_0, g_0) corresponding to the phase functions ψ_k will be different from what we used earlier. We choose the geodesics so that each pair of different ones of them intersect properly. This is as before. However, we additionally require the geodesics to be so that

$$(\pm c_i \pm c_j \pm c_k)^2 - |c_i \dot{\gamma}_i + c_j \dot{\gamma}_j + c_k \dot{\gamma}_k|^2 \neq 0,$$

when all the geodesics γ_k intersect. This is the condition $\langle \nabla^g \widetilde{\Psi}, \nabla^g \widetilde{\Psi} \rangle \neq 0$ of Lemma 4.7 and Proposition 4.8.

With suitable choices of other geodesics $\gamma_2, \gamma_3, \gamma_4, \gamma_5$, and coefficients c_k, $k = 1, \dots, 5$, we show that the integrand on the right hand side of (5.26) restricted to a neighborhood of p_0 in M_0 is close to a multiple of the delta function at p_0.

5.5. **Choices of vectors for the fourth order linearization.** The fourth order linearization $w^{(1234)}$ of $(-\Delta_g + V) + qu^2 = 0$ satisfies

$$\begin{aligned}(-\Delta_g + V) w^{(1234)} = -2q \Big(& v^{(1)} w^{(234)} + v^{(2)} w^{(134)} + v^{(3)} w^{(124)} + v^{(4)} w^{(123)} \\ & + w^{(12)} w^{(34)} + w^{(13)} w^{(24)} + w^{(14)} w^{(23)} \Big) \quad \text{in } M.\end{aligned} \tag{5.27}$$

Our aim is to show that the solution $w^{(1234)}$ behaves like $v^{(1)} v^{(2)} v^{(3)} v^{(4)}$ up to a multiplication by an amplitude function for τ sufficiently large.

• **Failed choices of vectors.** Let us first discuss why the earlier vectors $\overline{\xi}_i$, $i = 1, 2, 3, 4$, do not work here. If we use the earlier vectors (5.10) and the corresponding CGOs we will find that for example $w^{(123)}$ in (5.27) solves

$$\begin{aligned}(-\Delta_g + V) w^{(123)} &= -2q \left(v^{(1)} w^{(23)} + v^{(2)} w^{(13)} + v^{(3)} w^{(12)} \right) \\ &= e^{\tau \sum_{j \in \{1,2,3\}} |\xi_j| ((-1)^{j+1} x_1 + \mathbf{i}\psi_j)} \breve{a},\end{aligned} \tag{5.28}$$

where $\breve{a}$ is some amplitude function whose precise form is not important for this discussion. At the intersection points of the geodesics corresponding to ξ_a

$$\nabla^g \left(\sum_{j=1}^{3} |\xi_j| ((-1)^{j+1} x_1 + \mathbf{i}\psi_j) \right) = \sum_{j=1}^{3} \overline{\xi}_j = -\overline{\xi}_4$$

by (5.11). Now, if we try a WKB ansatz of the form $e^{\tau \sum_{a \in \{2,3,4\}} |\xi_a| ((-1)^{a+1} x_1 + \mathbf{i}\psi_a)} \breve{b}$ to solve (5.28), where $\breve{b}$ is an amplitude function, we end up dividing by $\langle \overline{\xi}_4, \overline{\xi}_4 \rangle$, which is 0. Consequently, the ansatz does not work and we need to use vectors that are different than $\overline{\xi}_a$.

• **Successful choices of vectors.** We choose new vectors to define the CGOs $v^{(k)}$, such that we can apply these CGOs to achieve our target. Denote the vectors by

$$\overline{\zeta}_k, \quad k = 1, 2, 3, 4, 5.$$

Let $\delta > 0$ and let ξ_j, $j = 1, 2, 3, 4$, be as in Section 5.2. Especially $\langle \xi_1, \xi_2 \rangle = 1 - \delta$ and $|\xi_1| = |\xi_2| = 1$. Note that the integral identity (5.26) implicitly concerns 5 possibly different $v^{(k)}$. We choose the vectors $\zeta_k \in T_{p_0}M_0$ as follows

$$\zeta_1 = \xi_1, \qquad \zeta_2 = \xi_2,$$
$$\zeta_3 = \left(1 + \sqrt{\frac{2}{2-\delta}}\right)\xi_3, \quad \zeta_4 = \left(1 + \sqrt{\frac{2}{2-\delta}}\right)\xi_4,$$
$$\text{and} \quad \zeta_5 = \sqrt{\frac{2}{2-\delta}}(\xi_1 + \xi_2).$$

Note that $|\zeta_5| = 2$. We define $\overline{\zeta}_k$ by

$$\overline{\zeta}_1 = |\zeta_1|e_1 + \mathbf{i}\zeta_1, \quad \overline{\zeta}_2 = |\zeta_2|e_1 + \mathbf{i}\zeta_2,$$
$$\overline{\zeta}_3 = |\zeta_3|e_1 + \mathbf{i}\zeta_3, \quad \overline{\zeta}_4 = -|\zeta_4|e_1 + \mathbf{i}\zeta_4,$$
$$\text{and} \quad \overline{\zeta}_5 = -|\zeta_5|e_1 + \mathbf{i}\zeta_5.$$

We also define

$$c_k = |\zeta_k|.$$

In particular, we have $c_1 = c_2 = 1$ and $c_5 = 2$. Then

$$\sum_{j=1}^{5} \zeta_j = \xi_1 + \xi_2 - \left(1 + \sqrt{\frac{2}{2-\delta}}\right)\left(\frac{1}{1+\delta}\xi_1 + \frac{\delta}{1+\delta}\xi_2 + \frac{\delta}{1+\delta}\xi_1 + \frac{1}{1+\delta}\xi_2\right)$$
$$+ \sqrt{\frac{2}{2-\delta}}(\xi_1 + \xi_2) = 0. \tag{5.29}$$

and

$$\mathrm{Re}\left(\sum_{j=1}^{5} \overline{\zeta}_j\right) = 0. \tag{5.30}$$

Consequently, the sum of the vectors $\overline{\zeta}_a$ vanishes:

$$\sum_{j=1}^{5} \overline{\zeta}_j = 0. \tag{5.31}$$

The condition (5.31) will imply that the non-stationary phase at p_0 and exponentially growing factors of in the integrand of the integral identity (5.26) will cancel out. We showed in Section 5.2 that the vectors $\xi_1, \ldots, \xi_4$ are pairwise linearly independent. Consequently $\zeta_1, \ldots, \zeta_4$ are pairwise linearly independent. We also see that ζ_5 is not proportional to any of the other vectors $\zeta_1, \ldots, \zeta_4$. It follows that the geodesics of (M_0, g_0) corresponding to $\zeta_1, \ldots, \zeta_5$ intersect properly. Since the vectors $\zeta_2, \ldots, \zeta_5$ can be taken to be up to a scaling arbitrary small perturbations of ξ_1, the corresponding geodesics are nontangential by continuity and they do not have self-interactions. We recall that the latter fact from the end of the proof of [DSFKL+18, Lemma 3.1].

We then consider how solutions to (5.27), which correspond to the CGOs determined by the vectors $\overline{\zeta}_j$, $j = 1, 2, 3, 4$, look like. Let us first note that at the intersection points of the corresponding geodesics

$$(\pm c_i \pm c_j \pm c_k)^2 - |c_i \nabla^{g_0}\psi_i + c_j \nabla^{g_0}\psi_j + c_k \nabla^{g_0}\psi_k|^2 \neq 0 \tag{5.32}$$

for all indices $i, j, k \in \{2, 3, 4\}$ which are all different. Indeed, by (5.31) we note that

$$(\pm c_i \pm c_j \pm c_k)^2 - |c_i \nabla^{g_0}\psi_i + c_j \nabla^{g_0}\psi_j + c_k \nabla^{g_0}\psi_k|^2 = \langle \overline{\zeta}_l + \overline{\zeta}_m, \overline{\zeta}_l + \overline{\zeta}_m \rangle,$$

where $l, m \in \{1,2,3,4,5\}$ are the unique two different indices, which do not belong to the set $\{j,k,l\} \subset \{1,2,3,4,5\}$. Then, we have

$$\langle \overline{\zeta}_l + \overline{\zeta}_m, \overline{\zeta}_l + \overline{\zeta}_m \rangle = 2\langle \overline{\zeta}_l, \overline{\zeta}_m \rangle = 2(\pm c_l c_m - \langle \zeta_l, \zeta_m \rangle) \neq 0,$$

since $|\langle \zeta_l, \zeta_m \rangle| < |\zeta_l||\zeta_m| = c_l c_m$. Here we have the strict inequality since ζ_l and ζ_m are linearly independent.

By (5.32), we may apply Lemma 4.7. Thus, by having the restrictions of the supports of $v^{(k)}$ to M_0 in small enough neighborhoods of the corresponding geodesics, the solution $w^{(123)}$ to third linearization

$$(-\Delta_g + V)w^{(123)} = -2q\left(v^{(1)}w_j^{(23)} + v^{(2)}w^{(13)} + v^{(3)}w_j^{(12)}\right)$$

up to a correction term is given by a WKB ansatz with amplitude of the form (4.26). The leading order coefficient of the ansatz is

$$\begin{aligned} B_{-4}^{(123)} =& 4q^2 a_0^{(1)} a_0^{(2)} a_0^{(3)} \frac{1}{(\pm c_1 \pm c_2 \pm c_3)^2 - |c_1 \nabla^{g_0}\psi_1 + c_2 \nabla^{g_0}\psi_2 + c_3 \nabla^{g_0}\psi_3|^2} \\ &\times \Bigg(\frac{1}{(\pm c_1 \pm c_2)^2 - |c_1 \nabla^{g_0}\psi_1 + c_2\nabla^{g_0}\psi_2|^2} \\ &\quad + \frac{1}{(\pm c_1 \pm c_3)^2 - |c_1 \nabla^{g_0}\psi_1 + c_3\nabla^{g_0}\psi_3|^2} \\ &\quad + \frac{1}{(\pm c_2 \pm c_3)^2 - |c_2 \nabla^{g_0}\psi_2 + c_3\nabla^{g_0}\psi_3|^2} \Bigg). \end{aligned}$$

Note the power 2 of the potential q in $B_{-4}^{(123)}$. Let us define for $i,j,k \in \{1,\dots,4\}$ all different

$$\begin{aligned} &\mathbf{D}_{ij} = \langle \overline{\zeta}_i + \overline{\zeta}_j, \overline{\zeta}_i + \overline{\zeta}_j \rangle, \\ \text{and} \quad &\mathbf{D}_{ijk} = \langle \overline{\zeta}_i + \overline{\zeta}_j + \overline{\zeta}_k, \overline{\zeta}_i + \overline{\zeta}_j + \overline{\zeta}_k \rangle. \end{aligned} \tag{5.33}$$

At an intersection point p_0 of the geodesics, we thus have is

$$B_{-4}^{(123)}(p_0) = 4q^2 a_0^{(1)} a_0^{(2)} a_0^{(3)} \frac{1}{\mathbf{D}_{123}} \left(\frac{1}{\mathbf{D}_{12} + \mathbf{D}_{13} + \mathbf{D}_{23}} \right).$$

We have similar formulas for the leading order coefficients of the ansatzes for $w^{(234)}$, $w^{(134)}$ and $w^{(124)}$. Furthermore, by (5.31) we have

$$\begin{aligned} \mathbf{D}_{123} &= \langle \overline{\zeta}_1 + \overline{\zeta}_2 + \overline{\zeta}_3, \overline{\zeta}_1 + \overline{\zeta}_2 + \overline{\zeta}_3 \rangle = \langle \overline{\zeta}_4 + \overline{\zeta}_5, \overline{\zeta}_4 + \overline{\zeta}_5 \rangle = \mathbf{D}_{45}, \\ \mathbf{D}_{234} &= \langle \overline{\zeta}_1 + \overline{\zeta}_5, \overline{\zeta}_1 + \overline{\zeta}_5 \rangle = \mathbf{D}_{15}, \\ \mathbf{D}_{134} &= \mathbf{D}_{25}, \\ \mathbf{D}_{124} &= \mathbf{D}_{35}. \end{aligned} \tag{5.34}$$

Therefore, by using (5.34), the solution $w^{(1234)}$ to the fourth order linearization will be (up to a correction term) of the form

$$\begin{aligned} &\Bigg[\frac{1}{\mathbf{D}_{15}} \left(\frac{1}{\mathbf{D}_{23} + \mathbf{D}_{24} + \mathbf{D}_{34}} \right) + \frac{1}{\mathbf{D}_{25}} \left(\frac{1}{\mathbf{D}_{13} + \mathbf{D}_{14} + \mathbf{D}_{34}} \right) \\ &\quad + \frac{1}{\mathbf{D}_{35}} \left(\frac{1}{\mathbf{D}_{12} + \mathbf{D}_{14} + \mathbf{D}_{24}} \right) + \frac{1}{\mathbf{D}_{45}} \left(\frac{1}{\mathbf{D}_{12} + \mathbf{D}_{13} + \mathbf{D}_{23}} \right) \\ &\quad + \frac{1}{\mathbf{D}_{12}} \frac{1}{\mathbf{D}_{34}} + \frac{1}{\mathbf{D}_{13}} \frac{1}{\mathbf{D}_{24}} + \frac{1}{\mathbf{D}_{14}} \frac{1}{\mathbf{D}_{23}} \Bigg) \Bigg] \\ &\quad \times e^{\tau \sum_{j \in \{1,2,3,4\}} (\mathrm{Re}(\zeta_j) x_1 + \mathbf{i}|\zeta_j|\psi_j} \widetilde{A}. \end{aligned} \tag{5.35}$$

Here $\widetilde{A}$ is an amplitude function which has (up to a multiplication by a power of τ) the leading order coefficient

$$8q^3 a_0^{(1)} a_0^{(2)} a_0^{(3)} a_0^{(4)}.$$

Note the power 3 of the potential q here.

Similar to Section 5.2, where we showed that the factor (5.16) of the third order linearization is non-zero, we may show that the coefficient in the brackets of (5.35), call it $\mathbf{E}_\delta$ is not zero. We have:

Lemma 5.2. *The quantity*

$$\begin{aligned}\mathbf{E}_\delta = {} & \frac{1}{\mathbf{D}_{15}}\left(\frac{1}{\mathbf{D}_{23}+\mathbf{D}_{24}+\mathbf{D}_{34}}\right)+\frac{1}{\mathbf{D}_{25}}\left(\frac{1}{\mathbf{D}_{13}+\mathbf{D}_{14}+\mathbf{D}_{34}}\right)\\ & +\frac{1}{\mathbf{D}_{35}}\left(\frac{1}{\mathbf{D}_{12}+\mathbf{D}_{14}+\mathbf{D}_{24}}\right)+\frac{1}{\mathbf{D}_{45}}\left(\frac{1}{\mathbf{D}_{12}+\mathbf{D}_{13}+\mathbf{D}_{23}}\right)\\ & +\frac{1}{\mathbf{D}_{12}}\frac{1}{\mathbf{D}_{34}}+\frac{1}{\mathbf{D}_{13}}\frac{1}{\mathbf{D}_{24}}+\frac{1}{\mathbf{D}_{14}}\frac{1}{\mathbf{D}_{23}}=\mathcal{O}(\delta^{-3})\neq 0,\end{aligned} \tag{5.36}$$

for all sufficiently small $\delta>0$.

The proof of the lemma is elementary, but involves rather long calculations. We have placed the proof in Appendix C.

5.6. **Proof of $q_1=q_2$ (continued).** Let us then return to proving $q_1=q_2$. Let $\overline{\zeta}_k$, $k=1,\dots,5$, be as in Section 5.5 above. We have

$$c_k=|\zeta_k|$$

and we set

$$s_1=c_1\tau+\mathbf{i}\lambda,\ \text{and}\ s_k=c_k,\quad k=2,3,4,5.$$

The CGOs corresponding to vectors $\overline{\zeta}_5$ are of the form

$$\begin{aligned} v^{(1)} &= e^{(|\zeta_1|\tau+\mathbf{i}\lambda)x_1}\left(\tau^{\frac{n-2}{8}}e^{\mathbf{i}(|\zeta_1|\tau+\mathbf{i}\lambda)\psi_1}a_1+r_1\right),\\ v^{(2)} &= e^{|\zeta_2|\tau x_1}\left(\tau^{\frac{n-2}{8}}e^{\mathbf{i}|\zeta_2|\tau\psi_2}a_2+r_2\right),\\ v^{(3)} &= e^{|\zeta_3|\tau x_1}\left(\tau^{\frac{n-2}{8}}e^{\mathbf{i}|\zeta_3|\tau\psi_3}a_3+r_3\right),\\ v^{(4)} &= e^{-|\zeta_4|\tau x_1}\left(\tau^{\frac{n-2}{8}}e^{\mathbf{i}|\zeta_4|\tau\psi_4}a_4+r_4\right),\\ v^{(5)} &= e^{-|\zeta_5|\tau x_1}\left(\tau^{\frac{n-2}{8}}e^{\mathbf{i}|\zeta_5|\tau\psi_5}a_5+r_5\right).\end{aligned} \tag{5.37}$$

Since $\Lambda_{q_1}(f_\epsilon)=\Lambda_{q_2}(f_\epsilon)$, by Propositions 4.4 and 4.8 there are boundary values f_{ij} and f_{ijk}, $i,j,k=1,2,3,4$, such that the solutions of the second linearized equations (5.4) and third linearized equations

$$\begin{cases}(-\Delta_g+V)\omega_\beta^{(ijk)}=-2q_\beta\left(v^{(i)}w_\beta^{(jk)}+v^{(j)}w_\beta^{(ik)}+v^{(k)}w_\beta^{(ij)}\right) & \text{in } M,\\ \omega_\beta^{(ijk)}=f_{ijk} & \text{on } \partial M,\end{cases} \tag{5.38}$$

for $\beta=1,2$, and i,j,k all different, which are of the form

$$\begin{aligned} w_\beta^{(ij)} &= w_{0,\beta}^{(ij)}+e^{\tau\Psi^{(ij)}}R_\beta^{(ij)},\\ \text{and}\quad \omega_\beta^{(ijk)} &= \omega_{0,\beta}^{(ijk)}+e^{\tau\widetilde{\Psi}^{(ijk)}}\widetilde{R}_\beta^{(ijk)}.\end{aligned}$$

For given $K,N\in\mathbb{N}\cup\{0\}$, the correction terms $R_\beta^{(ij)}$ and $\widetilde{R}_\beta^{(ijkl)}$ can be assumed to be $\mathcal{O}_{L^2(M)}(\tau^{-N})$ by taking the amplitude expansions of the CGOs $v^{(k)}$, $k=1,2,3,4$ to be precise enough (i.e. N' large enough). We refer to Propositions 4.4 and 4.8 for the specifics of $w_\beta^{(ij)}$ and $w_\beta^{(ijk)}$.

The phase functions $\Psi^{(ij)}$ and $\widetilde{\Psi}^{(ijk)}$ satisfy at the point p_0 where all the geodesics $\gamma_1, \ldots, \gamma_5$ intersect

$$\begin{aligned} \Psi^{(ij)}(p_0) &= \overline{\zeta}_i + \overline{\zeta}_j, \\ \widetilde{\Psi}^{(ijk)}(p_0) &= \overline{\zeta}_i + \overline{\zeta}_j + \overline{\zeta}_k. \end{aligned} \tag{5.39}$$

The leading order coefficients of the amplitudes of $w_\beta^{(ik)}$ and $\omega_\beta^{(ikl)}$ are

$$\begin{aligned} b_{-2,\beta}^{(ik)}(p_0) &= \frac{2q_\beta}{\mathbf{D}_{ik}} a_0^{(i)} a_0^{(k)}, \\ B_{-4,\beta}^{(ikl)}(p_0) &= 4q_\beta^2 a_0^{(i)} a_0^{(k)} a_0^{(l)} \frac{1}{\mathbf{D}_{ikl}} \left(\frac{1}{\mathbf{D}_{ik} + \mathbf{D}_{il} + \mathbf{D}_{kl}} \right). \end{aligned} \tag{5.40}$$

Let us denote Ψ_{12345} as the sum of all the phase functions of $v^{(1)}, \ldots, v^{(5)}$ in (5.37), where τ is a parameter. More precisely, Ψ_{12345} is given as

$$\Psi_{12345} = \Big(|\zeta_1|+|\zeta_2|+|\zeta_3|-|\zeta_4|-|\zeta_5|\Big)x_1+\mathbf{i}\Big(|\zeta_1|\psi_1+|\zeta_2|\psi_2+|\zeta_3|\psi_3+|\zeta_4|\psi_4+|\zeta_5|\psi_5\Big).$$

Let us also set

$$\Lambda_{1234} = \lambda(\mathbf{i}x_1 - \psi_1).$$

At the point p_0 where all the geodesics $\gamma_1, \ldots, \gamma_5$ intersect

$$\nabla^g \Psi_{12345}(x_1, p_0) = 0 \tag{5.41}$$

for $x \in I \subset \mathbb{R}$ by (5.29), and

$$\mathrm{Re}(\Psi_{12345})(x_1, p_0) = 0 \tag{5.42}$$

by (5.30). The condition (5.42) implies that Ψ_{12345} is not exponentially growing in τ. Moreover, by (5.41) we have that p_0 is a critical point of Ψ_{12345}. By the properties of ψ_k, the point p_0 is also nondegenerate, see (3.2).

We multiply the right hand side of the integral identity of the fourth order linearization (5.26) by $\tau^4\tau^{1/2}$ and take the limit $\tau \to \infty$. In the case p_0 is the only point where all the geodesics $\gamma_1, \ldots, \gamma_5$ intersect, by stationary phase the limit tends to

$$0 = c_{\tilde{A}} \mathbf{E}_\delta \left(a_0^{(1)} a_0^{(2)} a_0^{(3)} a_0^{(4)} a_0^{(5)} \right)\Big|_{p_0} \int_{\mathbb{R}} e^{i\lambda x_1} (q_1^3(x_1, p_0) - q_2^3(x_1, p_0))\, dx_1,$$

where $\mathbf{E}_\delta$ is the coefficient of $w^{(1234)}$ in Lemma 5.2 and $c_{\tilde{A}} \neq 0$ is given by a similar formula as c_A in Section 5.3. Here we also used that $a_0^{(1)}, \ldots, a_0^{(5)}$ are independent of x_1. By Lemma 5.2, the coefficient $\mathbf{E}_\delta \neq 0$ for all small enough $\delta > 0$. Inverting, the Fourier transformation in the variable x_1 shows that $q_1^3(x_1, p_0) = q_2^3(x_1, p_0)$. Thus

$$q_1(x_1, p_0) = q_2(x_1, p_0)$$

for $x_1 \in \mathbb{R}$. If there were several points where $\gamma_1, \ldots, \gamma_5$ intersect, we argue similarly as in Section 5.3 by using [LLLS21a, Lemma 6.2]. Since p_0 was arbitrary, this completes the proof.

Appendix A. Boundary determination

We prove that the DN map of the semilinear elliptic equation

$$(-\Delta_g + V)u + qu^m = 0 \text{ in } M, \quad u = f \text{ on } \partial M$$

on a compact smooth Riemannian manifold with boundary determines the formal Taylor series (the jet) of the coefficient q (in the boundary normal coordinates) on the boundary. Here, $m \geq 2$ is an integer, and V and q are smooth functions on M. We assume also that zero is not a Dirichlet eigenvalue for the operator $-\Delta_g + V$ on M.

We expect this result to be well-known to experts on the field, but could not find a reference on it, so we offer detailed presentation and its proof.

Proposition A.1 (Boundary determination). *For $m \geq 2$, $m \in \mathbb{N}$, let (M,g) be a compact Riemannian manifold with C^∞ boundary ∂M and consider the boundary value problem*

$$\begin{cases} (-\Delta_g + V)u + qu^m = 0 & \text{in } M, \\ u = f & \text{on } \partial M, \end{cases} \tag{A.1}$$

where $V, q \in C^\infty(M)$. Assume that the DN map Λ_q of the equation (A.1) is known for small boundary values. Then Λ_q determines the formal Taylor series of q on the boundary ∂M.

In addition, if $f \in C^\infty(\partial M)$ is so small that (A.1) has a unique small solution, the DN map determines the formal Taylor series of the solution $u = u_f$ at any point on the boundary.

Proof. Determination of Taylor expansion of q:

We first investigate solutions of our semilinear elliptic equation could be C^∞-smooth due to the following observations. Let $f \in C^\infty(\partial M)$. We consider boundary values $f_0, f \in C^\infty(\partial M)$ and $f_t = f_0 + tf$ and assume that $\|f_0\|_{C^{2,\alpha}(\partial M)}$ and $|t|$ are sufficiently small so that the DN maps at f_0 and f_t are both well-defined. We denote by u_0 and u_t, the unique solutions of (A.1) with boundary data f_0 and f_t on ∂M, respectively. In addition, since $V \in C^\infty(M)$ and $f, f_0 \in C^\infty(\partial M)$, by elliptic regularity u_t and u_0 are $C^\infty(M)$ functions.

By linearizing the equation (A.1) at $t = 0$, we obtain

$$\begin{cases} -\Delta_g v + \left(V + mqu_0^{m-1}\right) v = 0 & \text{in } M, \\ v = f & \text{on } \partial M, \end{cases} \tag{A.2}$$

where $v = \lim_{t \to 0} \dfrac{u_t - u_0}{t}$ and u_0 solves

$$\begin{cases} (-\Delta_g + V)u_0 + qu_0^m = 0 & \text{in } M, \\ u_0 = f_0 & \text{on } \partial M. \end{cases} \tag{A.3}$$

Moreover, v is the solution of

$$\begin{cases} -\Delta_g v + \widetilde{q} v = 0 & \text{in } M, \\ v = f & \text{on } \partial M. \end{cases}$$

where

$$\widetilde{q} := V + mqu_0^{m-1} \text{ in } M. \tag{A.4}$$

Note that $\widetilde{q} \in C^\infty(M)$, since $u_0 \in C^\infty(M)$ by elliptic regularity.

Since we know the DN map of the boundary value problem (A.1), we know the DN map of the linearized problem (A.2). This is proven in [LLLS21a, Proposition 2.1], where it is shown that the DN map is C^∞ in the Fréchet sense. (See also the similar result [LL22b, Theorem 2.1], which deals with local well-posedness and linearizations of (A.1) at f_0 not identically 0.) It follows by [DSFKSU09, Theorem 8.4.] that we know the formal Taylor series of $\widetilde{q}$ on ∂M. In particular, by choosing

$$u_0 = f_0 = \varepsilon_0 > 0 \text{ on } \partial M,$$

for some sufficiently small constant $\varepsilon_0 > 0$, and noting that

$$q = \frac{\widetilde{q} - V}{m\varepsilon_0^{m-1}} \text{ on } \partial M,$$

it follows that we know q on the boundary ∂M.

Next we determine first order derivatives of q on the boundary. Given a point $x_0 \in \partial M$, let $x = (x_1, \dots, x_n) \in \partial M$ be boundary normal coordinates near $x = x_0$ in M. Differentiating (A.4) yields

$$\begin{aligned}\partial_{x_n} \widetilde{q} =& m \partial_{x_n} (u_0^{m-1}) q + m(\partial_{x_n} q) u_0^{m-1} + \partial_{x_n} V \\ =& m(m-1) u_0^{m-2} (\partial_{x_n} u_0) q + m(\partial_{x_n} q) u_0^{m-1} + \partial_{x_n} V.\end{aligned} \tag{A.5}$$

Since we have already determined the Taylor series of $\tilde{q}$ on the boundary and

$$\partial_{x_n} u_0 = \Lambda_q(f_0),$$

we may determine $\partial_{x_n} q$ by solving it from (A.5). Since we also know the derivatives of q in tangential directions x_k, where $k = 1, \dots, n-1$, we have determined all first order derivatives of q on the boundary.

To determine higher order derivatives of q on the boundary, we follow an argument similar to [LLS16, Lemma 3.4]. On a neighborhood of x_0 in M we may write

$$Qu_0 := (-\Delta_g + V) u_0 + q u_0^m = -\partial_{x_n}^2 u_0 + P u_0,$$

where P is a non-linear partial differential operator containing derivatives in x' up to order 2 and in x_n up to order 1. The coefficients of P depend on pointwise values of q. By expressing

$$\partial_{x_n}^2 = P - Q$$

we obtain

$$\partial_{x_n}^2 u_0 = P u_0 - Q u_0 = P u_0. \tag{A.6}$$

Since we already know the quantities

$$u_0,\ \partial_{x'} u_0,\ \partial_{x'}^2 u_0,\ \partial_{x_n} u_0,\ \partial_{x'} \partial_{x_n} u_0,\ q,\ \partial_{x'} q \text{ and } \partial_{x_n} q, \tag{A.7}$$

it follows from (A.6) that the second derivative $\partial_{x_n}^2 u_0$ can be also determined. By using this and differentiating (A.5), we may determine second order derivatives of q on the boundary. The higher order derivatives of q on the boundary can be determined by differentiating (A.6) and using (A.5) in succession, and by using induction.

Determination of Taylor expansions of solutions: Let then $f \in C^\infty(\partial M)$ be small enough so that (A.1) has a unique small solution $u = u_f$. Since we have determined the formal Taylor series of q on the boundary, the formal Taylor series of u on the boundary is determined by differentiating (A.6) with u in place of u_0. $\square$

Appendix B. Proof of the Carleman estimate with boundary terms

In this section, we proceed to prove Lemma 4.6. Let (M, g) be a compact, smooth, transversally anisotropic Riemannian manifold with a smooth boundary and let $V \in L^\infty(M)$. There exists constants $\tau_0 > 0$ and $C > 0$ depending only on (M, g) and $\|V\|_{L^\infty(M)}$ such that given any $|\tau| > \tau_0$ and any $v \in C^2(M)$, the following Carleman estimate holds

$$\begin{aligned}\|e^{-\tau x_1}(-\Delta_g + V)(e^{\tau x_1} v)\|_{L^2(M)} + |\tau|^{\frac{3}{2}} \|v\|_{W^{2,\infty}(\partial M)} + |\tau|^{\frac{3}{2}} \|\partial_\nu v\|_{W^{1,\infty}(\partial M)} \\ + |\tau|^{\frac{3}{2}} \|\partial_\nu^2 v\|_{L^\infty(\partial M)} \geq C|\tau| \, \|v\|_{L^2(M)},\end{aligned} \tag{B.1}$$

Proof of Lemma 4.6. We may assume without loss of generality that v is real-valued and also that $\tau > 0$. The proof for the case $\tau < 0$ follows analogously. Throughout this proof, we use the notation C to stand for a generic positive constant that is independent of the parameter τ. We also write $\hat{v}$ to stand for a C^2-extension of the function v into a slightly larger manifold $\hat{M} \Subset \mathbb{R}\times M_0$ with smooth boundary, such that $v \in C^2_c(\hat{M})$ and that there holds

$$\|\hat{v}\|_{W^{2,\infty}(\hat{M}\setminus M)} \le C(\|\partial_\nu^2 v\|_{W^{2,\infty}(\partial M)} + \|\partial_\nu v\|_{W^{2,\infty}(\partial M)} + \|v\|_{W^{2,\infty}(\partial M)}), \tag{B.2}$$

for some constant $C > 0$, only depending on $(\hat{M}, g)$. In order to prove the latter estimate, let us consider the normal coordinate system $(y_1, \dots, y_n) = (y_1, y')$ near ∂M in $\mathbb{R}\times M_0$ where we are assuming that ∂M is given by $\{y_1 = 0\}$, and the metric g near ∂M is given in these coordinates via the expression

$$g = (dy_1)^2 + g'(y_1, y'),$$

where g' can be viewed as a family of smooth Riemannian metrics on ∂M, smoothly depending on y' for all $|y'| < \delta$ sufficiently small. We make the convention that $y_1 > 0$ on $\hat{M}\setminus M$. Let us now define $\hat{v}$ on $\hat{M}$ via

$$\hat{v} = v \quad \text{on } M, \tag{B.3}$$

and

$$\hat{v}(y) = \left(v(0, y') + y_1\, \partial_\nu v(0, y') + \frac{y_1^2}{2}\, \partial_\nu^2 v(0, y')\right) \eta(y_1), \qquad y \in (0, \delta)\times \partial M, \tag{B.4}$$

where η is a smooth non-negative function such that $\eta(t) = 1$ for all $|t| \le \frac{\delta}{2}$ and $\eta = 0$ for all $|y_1| \ge \delta$. It is straightforward to see that $\hat{v} \in C^2_c(\hat{M})$. The claimed estimate (B.2) now follows from the definition (B.4).

We define

$$P_\tau v = e^{-\tau x_1}\Delta_g(e^{\tau x_1} v), \tag{B.5}$$

and note that

$$P_\tau v = \partial_{x_1}^2 v + \Delta_{g_0} v + 2\tau \partial_{x_1} v + \tau^2 v.$$

We claim that

$$\left|\int_M P_\tau v\, \partial_{x_1} v\, dV_g\right| + C\tau^2 \|v\|^2_{W^{2,\infty}(\partial M)} + C\tau^2\|\partial_\nu v\|^2_{W^{2,\infty}(\partial M)} + C\tau^2\|\partial_\nu^2 v\|^2_{W^{2,\infty}(\partial M)} \ge 2\tau \|\partial_t v\|^2_{L^2(M)}. \tag{B.6}$$

To show (B.6) we begin by writing

$$\int_M P_\tau v\, \partial_{x_1} v\, dV_g = 2\tau \int_M |\partial_{x_1} v|^2\, dV_g + \underbrace{\int_M \partial_{x_1}^2 v\, \partial_{x_1} v\, dV_g}_{\mathrm{I}} + \underbrace{\int_M \Delta_{g_0} v\, \partial_{x_1} v\, dV_g}_{\mathrm{II}} + \underbrace{\int_M \tau^2 v\, \partial_{x_1} v\, dV_g}_{\mathrm{III}}.$$

Note that $M \Subset \mathbb{R}\times M_0$ and $dV_g = dx_1\, dV_{g_0}$. We can use integration by parts to bound each of the terms I–III as follows. For I, we first note that

$$\int_{\hat{M}} \partial_{x_1}^2 \hat{v}\, \partial_{x_1}\hat{v}\, dV_g = \frac{1}{2}\int_{\hat{M}} \partial_{x_1}(|\partial_{x_1}\hat{v}|^2)\, dV_g = 0.$$

Together with the estimate (B.2), we obtain

$$|\mathrm{I}| = \left|\int_{\hat{M}\setminus M} \partial_{x_1}^2 \hat{v}\, \partial_{x_1}\hat{v}\, dV_g\right| \le C\left(\|\partial_\nu^2 v\|^2_{W^{2,\infty}(\partial M)} + \|\partial_\nu v\|^2_{W^{2,\infty}(\partial M)} + \|v\|^2_{W^{2,\infty}(\partial M)}\right).$$

For II, since $[\partial_{x_1}, \Delta_g] = 0$ on $(\hat{M}, g)$, we may apply integration by parts again to deduce that

$$\int_{\hat{M}} \Delta_{g_0}\hat{v}\, \partial_{x_1}\hat{v}\, dV_g = 0.$$

Thus, using (B.2), we can show analogously to term I that

$$|\mathrm{II}| \leq C \left(\|\partial_\nu^2 v\|^2_{W^{2,\infty}(\partial M)} + \|\partial_\nu v\|^2_{W^{2,\infty}(\partial M)} + \|v\|^2_{W^{2,\infty}(\partial M)} \right).$$

Finally for the term III we first note that

$$\tau^2 \int_M \hat{v}\, \partial_{x_1} \hat{v}\, dV_g = \frac{\tau^2}{2} \int_M \partial_{x_1}(\hat{v}^2)\, dV_g = 0.$$

Thus, using (B.2), we have

$$|\mathrm{III}| \leq C\tau^2 \left(\|\partial_\nu^2 v\|^2_{W^{2,\infty}(\partial M)} + \|\partial_\nu v\|^2_{W^{2,\infty}(\partial M)} + \|v\|^2_{W^{2,\infty}(\partial M)} \right).$$

Combining the previous three bounds yields the claimed estimate (B.6). Using (B.6) and applying the Cauchy-Schwarz inequality

$$\left| \int_M P_\tau v\, \partial_{x_1} v\, dV_g \right| \leq \frac{1}{4\tau} \|P_\tau v\|^2_{L^2(M)} + \tau \|\partial_{x_1} v\|^2_{L^2(M)},$$

we deduce that

$$\begin{aligned} \|P_\tau v\|^2_{L^2(M)} + C\tau^3 \|v\|^2_{W^{2,\infty}(\partial M)} + C\tau^3 \|\partial_\nu v\|^2_{W^{2,\infty}(\partial M)} \\ + C\tau^3 \|\partial_\nu^2 v\|^2_{W^{2,\infty}(\partial M)} \geq \tau^2 \|\partial_{x_1} v\|^2_{L^2(M)}, \end{aligned} \tag{B.7}$$

We recall that by the standard Poincaré inequality on $\hat{M}$, there exists $C > 0$ such that

$$\|\partial_{x_1} w\|_{L^2(\hat{M})} \geq C \|w\|_{L^2(\hat{M})} \qquad \forall w \in C^1_0(\hat{M}).$$

Also, analogously to the proof of the estimate (B.2), given any $r \in C^1(M)$, there is a C^1-extension of r into $\hat{M}$ such that $\hat{r} \in C^1_c(\hat{M})$ and there holds

$$\|\hat{r}\|_{W^{1,\infty}(\hat{M}\setminus M)} \leq C(\|\partial_\nu r\|_{W^{1,\infty}(\partial M)} + \|r\|_{W^{1,\infty}(\partial M)}), \tag{B.8}$$

for some constant $C > 0$ only depending on $(\hat{M}, g)$. Combining the latter two bounds, we deduce that given any $v \in C^1(M)$ there holds

$$\|\partial_{x_1} v\|_{L^2(M)} \geq C_1 \|v\|_{L^2(M)} - C_2 \|v\|_{W^{1,\infty}(\partial M)} - C_3 \|\partial_\nu v\|_{W^{1,\infty}(\partial M)}, \tag{B.9}$$

for all $v \in C^1(M)$, where the positive constants C_1, C_2 and C_3 only depend on (M, g).

Via the bounds (B.7)–(B.9), we deduce that

$$\begin{aligned} \|(P_\tau - V)v\|^2_{L^2(M)} + C\tau^3 \|v\|^2_{W^{2,\infty}(\partial M)} + C\tau^3 \|\partial_\nu v\|^2_{W^{2,\infty}(\partial M)} \\ + C\tau^3 \|\partial_\nu^2 v\|^2_{W^{2,\infty}(\partial M)} \geq \tau^2 \|v\|^2_{L^2(M)}. \end{aligned} \tag{B.10}$$

This proves the assertion. □

Appendix C. Computations of $\mathbf{D}_{ik}$

In the end of this paper, we compute the values $\mathbf{D}_{ik}$, for different sub-indices $i, k \in \{1, 2, 3, 4, 5\}$. Recalling that

$$\begin{aligned} &\zeta_1 = \xi_1, \qquad \zeta_2 = \xi_2, \\ &\zeta_3 = \left(1 + \sqrt{\frac{2}{2-\delta}}\right) \xi_3, \quad \zeta_4 = \left(1 + \sqrt{\frac{2}{2-\delta}}\right) \xi_4, \\ &\zeta_5 = \sqrt{\frac{2}{2-\delta}} (\xi_1 + \xi_2), \end{aligned}$$

and

$$\begin{aligned}&\overline{\zeta}_1=|\zeta_1|e_1+\mathbf{i}\zeta_1, \quad \overline{\zeta}_2=|\zeta_2|e_1+\mathbf{i}\zeta_2,\\&\overline{\zeta}_3=|\zeta_3|e_1+\mathbf{i}\zeta_3, \quad \overline{\zeta}_4=-|\zeta_4|e_1+\mathbf{i}\zeta_4,\\&\overline{\zeta}_5=-|\zeta_5|e_1+\mathbf{i}\zeta_5,\end{aligned}$$

where

$$\begin{aligned}&|\xi_1|=|\xi_2|=1, \quad \langle\xi_1,\xi_2\rangle=1-\delta,\\&\xi_3=-\frac{1}{1+\delta}(\xi_1+\delta\xi_2), \quad \xi_4=-\frac{1}{1+\delta}(\delta\xi_1+\xi_2).\end{aligned}$$

Via straightforward computations, we have

$$\begin{aligned}&\langle\xi_1,\xi_2\rangle=1-\delta, \quad \langle\xi_1,\xi_3\rangle=-\frac{1+\delta-\delta^2}{1+\delta}, \quad \langle\xi_1,\xi_4\rangle=-\frac{1}{1+\delta},\\&\langle\xi_2,\xi_3\rangle=-\frac{1}{1+\delta}, \quad \langle\xi_2,\xi_4\rangle=-\frac{1+\delta-\delta^2}{1+\delta}, \text{ and } \langle\xi_3,\xi_4\rangle=\frac{1+\delta+\delta^2-\delta^3}{(1+\delta)^2}.\end{aligned}$$

By

$$\mathbf{D}_{ij}=\langle\overline{\zeta}_i+\overline{\zeta}_j,\overline{\zeta}_i+\overline{\zeta}_j\rangle,$$

for different $i,k\in\{1,2,3,4,5\}$, direct computations yield that

$$\begin{aligned}\mathbf{D}_{12}=&\left(|\zeta_1|e_1+\mathbf{i}\zeta_1+|\zeta_2|e_1+\mathbf{i}\zeta_2\right)\cdot\left(|\zeta_1|e_1+\mathbf{i}\zeta_1+|\zeta_2|e_1+\mathbf{i}\zeta_2\right)\\=&2\left(|\zeta_1||\zeta_2|-\langle\zeta_1,\zeta_2\rangle\right)\\=&2\left(|\xi_1||\xi_2|-\langle\xi_1,\xi_2\rangle\right)=2\delta,\end{aligned}\tag{C.1}$$

$$\begin{aligned}\mathbf{D}_{13}=&\left(|\zeta_1|e_1+\mathbf{i}\zeta_1+|\zeta_3|e_1+\mathbf{i}\zeta_3\right)\cdot\left(|\zeta_1|e_1+\mathbf{i}\zeta_1+|\zeta_3|e_1+\mathbf{i}\zeta_3\right)\\=&2\left(|\zeta_1||\zeta_3|-\langle\zeta_1,\zeta_3\rangle\right)\\=&2\left(1+\sqrt{\frac{2}{2-\delta}}\right)\left(|\xi_1||\xi_3|-\langle\xi_1,\xi_3\rangle\right)\\=&2\left(1+\sqrt{\frac{2}{2-\delta}}\right)\left(2+2\delta+\mathcal{O}(\delta^2)\right),\end{aligned}\tag{C.2}$$

$$\begin{aligned}\mathbf{D}_{14}=&\left(|\zeta_1|e_1+\mathbf{i}\zeta_1-|\zeta_4|e_1+\mathbf{i}\zeta_4\right)\cdot\left(|\zeta_1|e_1+\mathbf{i}\zeta_1-|\zeta_4|e_1+\mathbf{i}\zeta_4\right)\\=&-2\left(1+\sqrt{\frac{2}{2-\delta}}\right)\left(|\xi_1||\xi_4|+\langle\xi_1,\xi_4\rangle\right)\\=&-2\left(1+\sqrt{\frac{2}{2-\delta}}\right)\frac{\delta+\mathcal{O}(\delta^2)}{1+\delta},\end{aligned}\tag{C.3}$$

$$\begin{aligned}\mathbf{D}_{15}=&\left(|\zeta_1|e_1+\mathbf{i}\zeta_1-|\zeta_5|e_1+\mathbf{i}\zeta_5\right)\cdot\left(|\zeta_1|e_1+\mathbf{i}\zeta_1-|\zeta_5|e_1+\mathbf{i}\zeta_5\right)\\=&-2\left(|\zeta_1||\zeta_5|+\langle\zeta_1,\zeta_5\rangle\right)\\=&-2\sqrt{\frac{2}{2-\delta}}\left(|\xi_1||\xi_1+\xi_2|+\langle\xi_1,\xi_1+\xi_2\rangle\right)\\=&-8+\frac{\delta}{2}+\mathcal{O}(\delta^2),\end{aligned}\tag{C.4}$$

$$
\begin{aligned}
\mathbf{D}_{23} =& (|\zeta_2|e_1 + \mathbf{i}\zeta_2 + |\zeta_3|e_1 + \mathbf{i}\zeta_3) \cdot (|\zeta_2|e_1 + \mathbf{i}\zeta_2 + |\zeta_3|e_1 + \mathbf{i}\zeta_3) \\
=& 2\left(1 + \sqrt{\frac{2}{2-\delta}}\right)(|\xi_2||\xi_3| - \langle \xi_2, \xi_3 \rangle) \\
=& 2\left(1 + \sqrt{\frac{2}{2-\delta}}\right)\left(\frac{2+\delta+\mathcal{O}(\delta^2)}{1+\delta}\right).
\end{aligned}
\tag{C.5}
$$

In order to compute $\mathbf{D}_{24}$ more carefully, let us recall the Taylor expansion of $\sqrt{1+\delta} = 1 + \frac{\delta}{2} - \frac{\delta^2}{8} + \mathcal{O}(\delta^3)$, then we have

$$
\begin{aligned}
\mathbf{D}_{24} =& (|\zeta_2|e_1 + \mathbf{i}\zeta_2 - |\zeta_4|e_1 + \mathbf{i}\zeta_4) \cdot (|\zeta_2|e_1 + \mathbf{i}\zeta_2 - |\zeta_4|e_1 + \mathbf{i}\zeta_4) \\
=& -2\,(|\zeta_2||\zeta_4| + \langle \zeta_2, \zeta_4 \rangle) \\
=& -2\left(1 + \sqrt{\frac{2}{2-\delta}}\right)(|\xi_2||\xi_4| + \langle \xi_2, \xi_4 \rangle) \\
=& -2\left(1 + \sqrt{\frac{2}{2-\delta}}\right)\left(\sqrt{1+2\delta-\delta^2} - (1+\delta-\delta^2)\right) \\
=& -2\left(1 + \sqrt{\frac{2}{2-\delta}}\right)\left(1+\delta-\delta^2+\mathcal{O}(\delta^3) - (1+\delta-\delta^2)\right) \\
=& -2\left(1 + \sqrt{\frac{2}{2-\delta}}\right)\frac{\mathcal{O}(\delta^3)}{1+\delta},
\end{aligned}
\tag{C.6}
$$

$$
\begin{aligned}
\mathbf{D}_{25} =& (|\zeta_2|e_1 + \mathbf{i}\zeta_2 - |\zeta_5|e_1 + \mathbf{i}\zeta_5) \cdot (|\zeta_2|e_1 + \mathbf{i}\zeta_2 - |\zeta_5|e_1 + \mathbf{i}\zeta_5) \\
=& -2\,(|\zeta_2||\zeta_5| + \langle \zeta_2, \zeta_5 \rangle) \\
=& -2\sqrt{\frac{2}{2-\delta}}\left(\sqrt{4-2\delta} + 2 - \delta\right) \\
=& -8 + \delta + \mathcal{O}(\delta^2),
\end{aligned}
\tag{C.7}
$$

$$
\begin{aligned}
\mathbf{D}_{34} =& (|\zeta_3|e_1 + \mathbf{i}\zeta_3 - |\zeta_4|e_1 + \mathbf{i}\zeta_4) \cdot (|\zeta_3|e_1 + \mathbf{i}\zeta_3 - |\zeta_4|e_1 + \mathbf{i}\zeta_4) \\
=& -2\,(|\zeta_3||\zeta_4| + \langle \zeta_3, \zeta_4 \rangle) \\
=& -2\left(1 + \sqrt{\frac{2}{2-\delta}}\right)^2 (|\xi_3||\xi_4| + \langle \xi_3, \xi_4 \rangle) \\
=& -2\left(1 + \sqrt{\frac{2}{2-\delta}}\right)^2 \frac{1}{(1+\delta)^2}\left(2+3\delta-\delta^3\right),
\end{aligned}
\tag{C.8}
$$

$$
\begin{aligned}
\mathbf{D}_{35} =& (|\zeta_3|e_1 + \mathbf{i}\zeta_3 - |\zeta_5|e_1 + \mathbf{i}\zeta_5) \cdot (|\zeta_3|e_1 + \mathbf{i}\zeta_3 - |\zeta_5|e_1 + \mathbf{i}\zeta_5) \\
=& -2\left(|\zeta_3||\zeta_5| + \langle \zeta_3, \zeta_5\rangle\right) \\
=& -\frac{2}{1+\delta}\left(1+\sqrt{\frac{2}{2-\delta}}\right)\sqrt{\frac{2}{2-\delta}}\left(\sqrt{4-2\delta}(1+\delta+\mathcal{O}(\delta^2)) - 2 - \delta + \delta^2\right) \\
=& -\frac{2}{1+\delta}\left(1+\sqrt{\frac{2}{2-\delta}}\right)\sqrt{\frac{2}{2-\delta}} \\
&\quad \times \left((2-\frac{\delta}{2}+\mathcal{O}(\delta^2))(1+\delta+\mathcal{O}(\delta^2)) - 2 - \delta + \delta^2\right) \\
=& -\frac{1}{1+\delta}\left(1+\sqrt{\frac{2}{2-\delta}}\right)\sqrt{\frac{2}{2-\delta}}\left(\delta+\mathcal{O}(\delta^2)\right)
\end{aligned} \tag{C.9}
$$

and similarly,

$$
\begin{aligned}
\mathbf{D}_{45} =& (-|\zeta_4|e_1 + \mathbf{i}\zeta_4 - |\zeta_5|e_1 + \mathbf{i}\zeta_5) \cdot (-|\zeta_4|e_1 + \mathbf{i}\zeta_4 - |\zeta_5|e_1 + \mathbf{i}\zeta_5) \\
=& 2\left(|\zeta_4||\zeta_5| - \langle \zeta_4, \zeta_5\rangle\right) \\
=& 2\left(1+\sqrt{\frac{2}{2-\delta}}\right)\sqrt{\frac{2}{2-\delta}}\left(|\xi_4||\xi_1+\xi_2| - \langle \xi_4, \xi_1+\xi_2\rangle\right) \\
=& \frac{2}{1+\delta}\left(1+\sqrt{\frac{2}{2-\delta}}\right)\sqrt{\frac{2}{2-\delta}}\left(\sqrt{4-2\delta}(1+\delta+\mathcal{O}(\delta^2)) + 2 + \delta - \delta^2\right) \\
=& \frac{2}{1+\delta}\left(1+\sqrt{\frac{2}{2-\delta}}\right)\sqrt{\frac{2}{2-\delta}}\left(4+\frac{5}{2}\delta+\mathcal{O}(\delta^2)\right).
\end{aligned} \tag{C.10}
$$

Proof of Lemma 5.2. With (C.1)–(C.10) at hand, let us split the analysis into two cases.

(1) By using (C.5), (C.6) and (C.8), we have that $\frac{1}{\mathbf{D}_{23}+\mathbf{D}_{24}+\mathbf{D}_{34}}$ is a bounded as $\delta \to 0$. Similarly, (C.2), (C.3) and (C.8) imply that $\frac{1}{\mathbf{D}_{13}+\mathbf{D}_{14}+\mathbf{D}_{34}}$ is also bounded as $\delta \to 0$. Similarly $\frac{1}{\mathbf{D}_{12}+\mathbf{D}_{13}+\mathbf{D}_{23}}$ is bounded as $\delta \to 0$. On the other hand, by (C.1), (C.3) and (C.6), we observe that $\frac{1}{\mathbf{D}_{12}+\mathbf{D}_{14}+\mathbf{D}_{24}} = \mathcal{O}(\delta^{-1})$. Meanwhile, $\mathbf{D}_{15}^{-1}$, $\mathbf{D}_{25}^{-1}$ and $\mathbf{D}_{45}^{-1}$ are bounded as $\delta \to 0$, but $\mathbf{D}_{35}^{-1} = \mathcal{O}(\delta^{-1})$.

(2) Similarly, $\frac{1}{\mathbf{D}_{12}}\frac{1}{\mathbf{D}_{34}} = \mathcal{O}(\delta^{-1})$, $\frac{1}{\mathbf{D}_{13}}\frac{1}{\mathbf{D}_{24}} = \mathcal{O}(\delta^{-3})$ and $\frac{1}{\mathbf{D}_{14}}\frac{1}{\mathbf{D}_{23}} = \mathcal{O}(\delta^{-1})$.

Therefore, combining the above, we conclude that

$$
\begin{aligned}
\mathbf{E}_\delta =& \left| \frac{1}{\mathbf{D}_{15}}\left(\frac{1}{\mathbf{D}_{23}+\mathbf{D}_{24}+\mathbf{D}_{34}}\right) + \frac{1}{\mathbf{D}_{25}}\left(\frac{1}{\mathbf{D}_{13}+\mathbf{D}_{14}+\mathbf{D}_{34}}\right)\right. \\
&+ \frac{1}{\mathbf{D}_{35}}\left(\frac{1}{\mathbf{D}_{12}+\mathbf{D}_{14}+\mathbf{D}_{24}}\right) + \frac{1}{\mathbf{D}_{45}}\left(\frac{1}{\mathbf{D}_{12}+\mathbf{D}_{13}+\mathbf{D}_{23}}\right) \\
&\left. + \frac{1}{\mathbf{D}_{12}}\frac{1}{\mathbf{D}_{34}} + \frac{1}{\mathbf{D}_{13}}\frac{1}{\mathbf{D}_{24}} + \frac{1}{\mathbf{D}_{14}}\frac{1}{\mathbf{D}_{23}} \right| \\
\geq& \frac{C_0}{\delta^3} - \frac{C_1}{\delta^2} - C_2 > 0,
\end{aligned}
$$

for all sufficiently small $\delta > 0$, where C_0, C_1 and C_2 are some positive constants independent of δ. Hence, the coefficient $\mathbf{E}_\delta = \mathcal{O}(\delta^{-3}) \neq 0$ for all sufficiently small $\delta > 0$. $\square$

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