UNIQUENESS RESULTS FOR INVERSE SOURCE PROBLEMS FOR SEMILINEAR ELLIPTIC EQUATIONS

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Abstract. We study inverse source problems associated to semilinear elliptic equations of the form

$$\Delta u(x) + a(x,u) = F(x)$$

on a bounded domain $\Omega \subset \mathbb{R}^n$, $n \geq 2$. We show that it is possible to use nonlinearity to recover both the source F and the nonlinearity $a(x,u)$ simultaneously and uniquely for a class of nonlinearities. This is in contrast to inverse source problems for linear equations, which always have a natural (gauge) symmetry that obstructs the unique recovery of the source. The class of nonlinearities for which we can uniquely recover the source and nonlinearity, includes a class of polynomials, which we characterize, and exponential nonlinearities.

For general nonlinearities $a(x,u)$, we recover the source $F(x)$ and the Taylor coefficients $\partial_u^k a(x,u)$ up to a gauge symmetry. We recover general polynomial nonlinearities up to the gauge symmetry. Our results also generalize results of [FO20, LLLS20] by removing the assumption that $u \equiv 0$ is a solution. To prove our results, we consider linearizations around possibly large solutions.

Our results can lead to new practical applications, because we show that many practical models do not have the obstruction for unique recovery that has restricted the applicability of inverse source problems for linear models.

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Contents

1. Introduction

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with C^∞-smooth boundary $\partial\Omega$ with $n \geq 2$. In this paper we consider semilinear elliptic equations of the form

$$\begin{cases} \Delta u + a(x,u) = F & \text{in } \Omega, \\ u = f & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $a = a(x,z) : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ is C^∞-smooth in the z-variable. For presentational purposes we also assume that

$$a(x,0) = 0 \text{ in } \Omega. \tag{1.2}$$

This condition is not a restriction of generality as it can be achieved by redefining the source F in (1.1).

Let us assume for now that the boundary value problem (1.1) is well-posed on an open subset $\mathcal{N} \subset C^{2,\alpha}(\partial\Omega)$. In this case, the *Dirichlet-to-Neumann map* (DN map) is defined by the usual assignment

$$\Lambda_{a,F} : \mathcal{N} \to C^{1,\alpha}(\partial\Omega), \qquad f \mapsto \partial_\nu u_f|_{\partial\Omega}\,. \tag{1.3}$$

Here ν denotes the unit outer normal on $\partial\Omega$. In Theorem 2.1 we show that if there exists

$$f_0 \in C^{2,\alpha}(\partial\Omega),$$

such that the equation (1.1) admits a solution $u_0 \in C^{2,\alpha}(\Omega)$ with $u_0|_{\partial\Omega} = f_0$, and

$$0 \text{ is not a Dirichlet eigenvalue of } \Delta + \partial_z a(x,u_0) \text{ in } \Omega,$$

then there is an open neighborhood $\mathcal{N} \subset C^{2,\alpha}(\partial\Omega)$ of f_0 where (1.1) is well-posed in the following sense: For each $f \in \mathcal{N}$ there exists a solution u_f to (1.1) with $u_f|_{\partial\Omega} = f$ and the solution u_f is unique in a fixed neighborhood of $u_0 \in C^{2,\alpha}(\Omega)$. If the sign condition

$$\partial_z a(x,z) \leq 0$$

holds for $x \in \Omega$ and $z \in \mathbb{R}$, the assumptions of Theorem 2.1 will be satisfied and the DN map is well-defined by [GT83]. If F vanishes on Ω, one can take $f_0 \equiv 0$ on $\partial\Omega$. In this case, Theorem 2.1 reduces to similar well-posedness theorems in the literature, such as the one in [LLLS20].

Consider the equation (1.1) for two sets (a_1, F_1) and (a_2, F_2) of coefficients. Let Λ_1 and Λ_2 be the corresponding DN maps defined on $\mathcal{N}_1 \subset C^{2,\alpha}(\partial\Omega)$ and $\mathcal{N}_2 \subset C^{2,\alpha}(\partial\Omega)$, respectively. When we write

$$\Lambda_1(f) = \Lambda_2(f), \text{ for any } f \in \mathcal{N},$$

we have especially assumed that $\mathcal{N} \subset \mathcal{N}_1 \cap \mathcal{N}_2$ and $\mathcal{N} \neq \emptyset$.

- **Inverse source problem:** What can we determine about both a and F from the knowledge of the corresponding DN map $\Lambda_{a,F}$?

For general nonlinearities $a(x,z)$ it is impossible to determine both $a(x,z)$ and $F(x)$ simultaneously from the corresponding DN map. This is due to an inherit *gauge invariance* of the problem, which we will explain later. For inverse source problems of related linear equations, where the aim is to determine a source function from boundary measurements, the gauge invariance of the problem is well-known:

Remark 1.1. *Let us consider the inverse source problem for the linear equation*

$$\begin{cases} \Delta u + qu = F & \text{in } \Omega, \\ u = f & \text{on } \partial\Omega. \end{cases} \tag{1.4}$$

In this inverse problem one asks if the DN map $\Lambda_F : C^\infty(\partial\Omega) \to C^\infty(\partial\Omega)$ associated to the above equation determines F uniquely. We assume here for simplicity that the potential function q is known. In general, the answer to the question is negative due to the following observation. Let u solve (1.4) and let ψ be an arbitrary C^2-function satisfying $\psi|_{\partial\Omega} = \partial_\nu\psi|_{\partial\Omega} = 0$. Let us also define

$$\tilde{u} := u + \psi. \tag{1.5}$$

Consequently, we have $(\tilde{u}|_{\partial\Omega}, \partial_\nu \tilde{u}|_{\partial\Omega}) = (u|_{\partial\Omega}, \partial_\nu u|_{\partial\Omega})$*, and*

$$\begin{aligned}\Delta\tilde{u} + q\tilde{u} =& \Delta(u+\psi) + q(u+\psi) \\ =& F - qu + \Delta\psi + qu + q\psi \\ =& F + \Delta\psi + q\psi.\end{aligned} \tag{1.6}$$

Hence u *and* $\tilde{u}$ *solve the equations* $\Delta u + qu = F$ *and* $\Delta\tilde{u} + q\tilde{u} = F + \Delta\psi + q\psi$ *respectively. Since* u *and* $\tilde{u}$ *also have the same Cauchy data on* $\partial\Omega$*, it follows that the corresponding DN maps are the same:* $\Lambda_F(f) = \Lambda_{F+\Delta\psi+q\psi}(f)$ *on* $\partial\Omega$*. It is thus not possible to determine a source function uniquely from the DN map.*

In this work, we consider different types of nonlinearities, including general ones. For general nonlinearities $a(x,z)$, we prove in Theorem 1.3 that the corresponding DN map determines the quantities

$$\partial_z^k a(x, u_0(x)), \quad x \in \Omega, \quad k \in \mathbb{N}. \tag{1.7}$$

Here u_0 is a solution to (1.1) corresponding to a boundary value f_0. As already evidenced by Remark 1.1, it might not be possible to recover u_0 from the DN map. This means that in general the condition (1.7) does not determine $a(x,z)$, or even its derivatives in the variable z.

Due to the above obstruction to determining $a(x,z)$, and consequently $F(x)$, in general, we mainly focus on nonlinearities $a(x,z)$ of the following special types:

- **General polynomial nonlinearity:**

$$a(x,z) = \sum_{k=1}^{N} a^{(k)}(x) z^k, \quad N \in \mathbb{N}, \tag{1.8}$$

- **Exponential type nonlinearities:**

$$a(x,z) = q(x)e^z \text{ and } a(x,z) = q(x)ze^z, \tag{1.9}$$

- **Sine-Gordon nonlinearity:**

$$a(x,z) = q(x)\sin(z). \tag{1.10}$$

For these nonlinearities, we show that the corresponding inverse source problems are either uniquely solvable or there is a gauge symmetry, which has an explicit form. The fact that there are nonlinearities for which the related inverse source problem is uniquely solvable is in contrast to inverse source problem for linear equations, which always have the gauge symmetry presented in Remark 1.1. That is, nonlinearity can make inverse source problems uniquely solvable.

Quadratic nonlinearity

$$a(x,u) = a^{(1)}(x)u(x) + a^{(2)}(x)u^2(x) \tag{1.11}$$

has a specific form gauge symmetry, which we now derive. For this, let us assume that u solves (1.1), where $a(x,z)$ is as above. Let $\psi \in C^2(\overline{\Omega})$. We denote by $\tilde{a}^{(1)}$, $\tilde{a}^{(2)}$ and $\tilde{F}$ another set of $C^\infty(\overline{\Omega})$-smooth functions corresponding to a quadratic nonlinearity of the form (1.11) and source term for (1.1). These functions can implicitly depend on ψ. Let Λ and $\tilde{\Lambda}$ be the DN maps corresponding to the coefficients without and with tilde signs respectively. If we define

$$\tilde{u} := u + \psi,$$

then we have the chain of equivalences

$$\Delta\tilde{u} + \tilde{a}^{(1)}\tilde{u} + \tilde{a}^{(2)}\tilde{u}^2 = \tilde{F},$$

which is equivalent to

$$\Delta(u+\psi) + \tilde{a}^{(1)}(u+\psi) + \tilde{a}^{(2)}(u+\psi)^2 = \tilde{F},$$

which is also equivalent to

$$\Delta u+\Delta\psi+\tilde{a}^{(1)}u+\tilde{a}^{(1)}\psi+\tilde{a}^{(2)}u^2+2\tilde{a}^{(2)}\psi u+\tilde{a}^{(2)}\psi^2=\tilde{F},$$

which hold in Ω. By using $\Delta u=-a^{(1)}u-a^{(2)}u^2+F$ and equating the powers of u gives the following system

$$\begin{cases} F+\Delta\psi+\tilde{a}^{(1)}\psi+\tilde{a}^{(2)}\psi^2=\tilde{F} & \text{in } \Omega,\\ a^{(1)}=\tilde{a}^{(1)}+2\tilde{a}^{(2)}\psi & \text{in } \Omega,\\ a^{(2)}=\tilde{a}^{(2)} & \text{in } \Omega. \end{cases} \tag{1.12}$$

If the above system is satisfied, then

$$\Delta u+a(x,u)=F \iff \Delta\tilde{u}+a(x,\tilde{u})=\tilde{F}.$$

Consequently, if we additionally require that $\psi|_{\partial\Omega}=\partial_\nu\psi|_{\partial\Omega}=0$, then the DN maps Λ and $\tilde{\Lambda}$ are the same. That is, if we change the coefficients $\left(a^{(1)},a^{(2)},F\right)$ to $\left(\tilde{a}^{(1)},\tilde{a}^{(2)},\tilde{F}\right)$, the DN map is preserved. Thus, it is at best possible to determine coefficients and a source from the DN map up to the gauge conditions (1.12).

• **Earlier works.** Before going into our results in detail, we discuss earlier related works. The standard approach in the study of inverse problems for nonlinear elliptic equations was initiated in [Isa93]. There the author linearized the nonlinear DN map $C^\infty(\partial\Omega)\to C^\infty(\partial\Omega)$. The linearization reduced the inverse problem of a nonlinear equation to an inverse problem of a linear equation, which the author was able to solve by using methods for linear equations. Later, second order linearizations, where data depends on two independent parameters, were used to solve inverse problems for example in [AZ21, CNV19, KN02, Sun96, Sun10, SU97]. We also mention here the work [Isa01] that considers inverse problems for systems of semilinear equations.

For the case $F=0$ in Ω in (1.1), equivalent to $u\equiv 0$ being a solution, inverse problems for semilinear elliptic equations were recently considered in [FO20, LLLS20]. The novelty of these works is that instead viewing nonlinearity as an additional complication in the inverse problem, the works used nonlinearity as a beneficial tool. The method of these two works originates from the seminal work [KLU18], where inverse problems for nonlinear equations were studied in Lorentzian spacetimes. By using the method where nonlinearity is used as a tool, inverse problems for nonlinear equations have been solved in cases where the corresponding inverse problems for linear equations are still open. The method is by now usually called *the higher order linearization method.*

After the works [KLU18, FO20, LLLS20], the literature about inverse problems for nonlinear equations based on the higher order linearization method, has grown substantially. The works [LLLS20, LLLS21, LLST22, KU20b, KU20a, FLL21, HL23] investigated inverse problems for semilinear elliptic equations with general nonlinearities and in the case of partial data. Inverse problems for quasilinear elliptic equations using higher order linearization have been studied in [KKU22, CFK+21, FKU21]. The works [CLLO22, Nur22] studied inverse problems for minimal surface equations on Riemannian surfaces and Euclidean domains. We also mention the works [LL22a, Lin22, LL22b, LL19, LO22, LZ21], where inverse problems for semilinear fractional type equations have been studied.

Inverse source problems for linear equations that regard determination of both unknown sources and coefficients have attracted recent interest. Applications of them include the photo/thermo-acoustic tomography [LU15], magnetic anomaly detection [DLL19, DLL20] and quantum mechanics [LLM19, LLM21]. In this paper, we are interested in related nonlinear counterparts of the above works considering linear models. Finally, inverse problems of simultaneously recovering for both nonlinearities and initial data have been considered by [LLLZ22] and [LLL21].

In our first result we show that a a general polynomial nonlinearity and a source are determined by the corresponding DN map up to a gauge condition generalizing (1.12). In the following theorem we denote by

$$\binom{m}{k} = \frac{m!}{(m-k)!k!}$$

the usual binomial coefficients. We also include a converse statement to the result.

Theorem 1.1. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with C^∞-smooth boundary $\partial\Omega$, $n \geq 2$. For $j = 1, 2$, let $a_j(x, z)$ be a polynomial of the form*

$$a_j(x, z) = \sum_{k=1}^{N} a_j^{(k)}(x) z^k \quad \text{for } (x, z) \in \overline{\Omega} \times \mathbb{R}, \tag{1.13}$$

for some $N \in \mathbb{N}$, where $a_j^{(k)} \in C^\alpha(\overline{\Omega})$, for $j = 1, 2$ and $k = 1, \ldots, N$. Given $F_j \in C^{0,\alpha}(\overline{\Omega})$ for some $0 < \alpha < 1$. Let Λ_{a_j,F_j} be the DN map of the equation

$$\begin{cases} \Delta u_j + a_j(x, u_j) = F_j & \text{in } \Omega, \\ u_j = f & \text{on } \partial\Omega. \end{cases} \tag{1.14}$$

Suppose that there is an open set $\mathcal{N} \subset C^{2,\alpha}(\partial\Omega)$ such that

$$\Lambda_{a_1,F_1}(f) = \Lambda_{a_2,F_2}(f) \text{ for any } f \in \mathcal{N}. \tag{1.15}$$

Then there exists $\psi \in C^{2,\alpha}(\overline{\Omega})$ with $\psi|_{\partial\Omega} = \partial_\nu \psi|_{\partial\Omega} = 0$ in Ω such that

$$a_1^{(N-k)} = \sum_{m=N-k}^{N} \binom{m}{N-k} a_2^{(m)} \psi^{m-N+k} \quad \text{and} \quad a_1^{(N)} = a_2^{(N)} \quad \text{in } \Omega, \tag{1.16}$$

for $k = 1, 2, \ldots, N$, and

$$F_1 = F_2 - \Delta\psi - \sum_{k=1}^{N} a_2^{(k)} \psi^k. \tag{1.17}$$

Conversely, if (1.16) and (1.17) hold for some $\psi \in C^{2,\alpha}(\overline{\Omega})$ with $\psi|_{\partial\Omega} = \partial_\nu \psi|_{\partial\Omega} = 0$, then $\Lambda_{a_1,F_1}(f) = \Lambda_{a_2,F_2}(f)$ for all $f \in C^{2,\alpha}(\partial\Omega)$ for which either side of the equation is defined.

We remark that in the above theorem it is sufficient that the domain $\mathcal{N}$ of the DN maps is any non-empty open subset of $C^{2,\alpha}(\partial\Omega)$. Especially $\mathcal{N}$ can be arbitrary small in size. The same holds for other results of this paper. We also remark that we could have let N to be finite, but otherwise unknown, in the assumptions of the theorem. That is, N could be initially assumed to be different for the coefficients (a_1, F_1) and (a_2, F_2). The determination result, given by (1.16) and (1.17), is the same also in this case.

We highlight the special cases of quadratic and cubic nonlinearities of the theorem in the following remark.

Remark 1.2. *When $N = 2$ and $N = 3$ the results of Theorem 1.1 are as follows:*

(a) *For $N = 2$, i.e., the quadratic nonlinearity, the condition* (1.15) *implies that*

$$\begin{cases} a_1^{(2)} = a_2^{(2)} =: a^{(2)}, \\ a_1^{(1)} = a_2^{(1)} + 2a^{(2)}\psi, \\ F_1 = F_2 - \Delta\psi - a_1^{(2)}\psi - a^{(2)}\psi^2. \end{cases} \tag{1.18}$$

(b) *For $N = 3$, i.e., the cubic nonlinearity, the condition* (1.15) *implies that*

$$\begin{cases} a_1^{(3)} = a_2^{(3)} =: a^{(3)} & \text{in } \Omega, \\ a_1^{(2)} = a_2^{(2)} + 3a^{(3)}\psi & \text{in } \Omega, \\ a_1^{(1)} = a_2^{(1)} + 2a_2^{(2)}\psi + 3a^{(3)}\psi^2 & \text{in } \Omega, \\ F_1 = F_2 - \Delta\psi - a_2^{(1)}\psi - a_2^{(2)}\psi^2 - a^{(3)}\psi^3 & \text{in } \Omega. \end{cases} \tag{1.19}$$

We will show how one obtains (1.18) *and* (1.19) *in Section 3.*

We note that the condition (1.16) in Theorem 1.1 tells that the highest order coefficient of a polynomial nonlinearity is always uniquely determined. We also mention here that cubic nonlinearities appear for example in Gross–Pitaevskii model for Bose-Einstein condensates [PS16].

With Theorem 1.1 and Remark 1.2 at hand, it is natural to ask if we can obtain strict uniqueness results for both the nonlinear coefficients and sources. This is indeed the case. For example, in view of (1.18), if one knows the linear term a priori, i.e. $a_1^{(1)} = a_2^{(1)}$, and assume $a^{(2)} \neq 0$, then $\psi \equiv 0$. This means that the gauge symmetry of the inverse source problem breaks in the sense that we unique and simultaneous determination of the nonlinearity and the source. Interestingly, the next result shows that inverse source problems for general polynomial nonlinearities are uniquely solvable if the second to highest order coefficient is known.

Theorem 1.2 (Unique recovery general case)**.** *Assume as in Theorem 1.1 and adopt its notation. Suppose additionally that*

$$a_1^{(N-1)} = a_2^{(N-1)} \text{ in } \Omega,$$

and

$$\text{either} \quad a_1^{(N)}(x) \neq 0 \quad \text{or} \quad a_2^{(N)}(x) \neq 0 \quad \text{for all } x \in \Omega.$$

Then all the coefficients are uniquely determined:

$$F_1 \equiv F_2 \quad \text{and} \quad a_1^{(k)} \equiv a_2^{(k)} \text{ in } \Omega, \quad k = 1, 2, \ldots, N.$$

Remark 1.3. *Let us consider Remark 1.2 (a) and assume additionally that*

$$a_1^{(1)} = a_2^{(1)} \text{ in } \Omega$$

and

$$a_1^{(2)}(x) \neq 0 \text{ or } a_2^{(2)}(x) \neq 0 \text{ at any } x \in \Omega.$$

Then also

$$F_1 = F_2 \text{ and } a_1^{(2)} = a_2^{(2)} \text{ in } \Omega.$$

The above remark in particularly states the following. The inverse source problem of recovering F from the DN map of

$$\Delta u + qu + u^2 = F,$$

is uniquely solvable, where q is assumed to be known. This is in contrast to the inverse source problem of $\Delta u + qu = F$, which always has a gauge symmetry by Remark 1.1, even if q is known. Thus, we have provided examples (Theorem 1.2 and Remark 1.3), where nonlinearity can be used to break the gauge symmetry in an inverse source problem.

On the other hand, in case it is a priori known that $F_1 = F_2$, then we also have a uniqueness result:

Corollary 1.4. *Let us adopt the notation and assumptions in Theorem 1.1. If $F_1 = F_2$ in Ω, then we have*

$$a_1^{(k)} = a_2^{(k)} \text{ in } \Omega, \tag{1.20}$$

for $k = 1, 2, \ldots, N$.

The above corollary in particularly says the following. If we consider an inverse problem for the equation

$$\Delta u + qu + u^2 = F, \quad F \text{ known},$$

then we can recover the lower order term q from the DN map.

We also study an inverse source problem for general semilinear elliptic equations and do not assume that the nonlinearity is necessarily a polynomial. In fact, we will prove the next theorem before Theorem 1.1 for convenience.

Theorem 1.3. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with C^∞-smooth boundary $\partial\Omega$, $n \geq 2$. For $j = 1, 2$, let $a_j(\,\cdot\,, z) \in C^\alpha(\overline{\Omega})$ satisfy the condition (1.2) and assume that $a_j(x, z)$ is C^∞-smooth with respect to the z-variable. Given $F_j \in C^{0,\alpha}(\overline{\Omega})$ for some $0 < \alpha < 1$, let Λ_{a_j, F_j} be the DN map of*

$$\begin{cases} \Delta u_j + a_j(x, u_j) = F_j & \text{in } \Omega, \\ u_j = f & \text{on } \partial\Omega. \end{cases} \tag{1.21}$$

Suppose that there is an open set $\mathcal{N} \subset C^{2,\alpha}(\partial\Omega)$ such that

$$\Lambda_{a_1, F_1}(f) = \Lambda_{a_2, F_2}(f) \text{ for any } f \in \mathcal{N}.$$

Then, for any $f_0 \in \mathcal{N}$, we have

$$\partial_z^k a_1(x, u_1^{(0)}(x)) = \partial_z^k a_2(x, u_2^{(0)}(x)), \quad x \in \Omega, \tag{1.22}$$

for any $k \in \mathbb{N}$. Here $u_1^{(0)}$ and $u_2^{(0)}$ are the solutions of (1.21) with boundary condition $u_j^{(0)}\big|_{\partial\Omega} = f_0$.

As a corollary to Theorem 1.3, we do case studies of inverse source problems when the nonlinearity of the model is either of exponential type or $a(x, z) = q(x)\sin(z)$. Exponential type nonlinearities arise for example in mathematical modeling of combustion (see e.g. [Vol14]). The nonlinearity $a(x, z) = q(x)\sin(z)$ corresponds to the sine-Gordon equation. The DN map and inverse problems for the sine-Gordon equation have been considered for example in [BK89, FP12]. The models are chosen so to give examples of cases where the inverse source problem is uniquely solvable, or has an explicit gauge symmetry.

Let q and F belong to $C^{0,\alpha}(\overline{\Omega})$, and consider the semilinear elliptic equations

$$\begin{cases} \Delta u + q(x)e^u = F & \text{in } \Omega, \\ u = f & \text{on } \partial\Omega. \end{cases} \tag{1.23}$$

and

$$\begin{cases} \Delta u + q(x)ue^u = F & \text{in } \Omega, \\ u = f & \text{on } \partial\Omega. \end{cases} \tag{1.24}$$

For the corresponding inverse source problems we assume that both of the above boundary value problems have a solution u_0 for some boundary value f_0 such that 0 is not an eigenvalue of $\Delta + \partial_z a(x, u_0)$. In this case, it follows from Theorem 2.1

that the DN maps $\mathcal{N} \to C^{1,\alpha}(\partial\Omega)$ are defined on an open subset $\mathcal{N} \subset C^{2,\alpha}(\partial M)$ as before by

$$u \mapsto \partial_\nu u_f|_{\partial\Omega}.$$

Here, u_f is the unique solution on a neighborhood of u_0 to either (1.23) or (1.24) depending on which of the two models we are considering. We remark that in the case $q \leq 0$ the above assumptions are satisfied and Theorem 2.1 holds for (1.23) by [GT83, Theorem 15.12]. In this case, also (1.24) has a solution u_0 for a given boundary value $f_0 \in C^{2,\alpha}(\partial\Omega)$ by [GT83, Theorem 15.12]. However, to apply Theorem 2.1, one still needs to assume that 0 is not an eigenvalue of $\Delta + \partial_z a(x, u_0)$. We also remark that if F is assumed to be small enough, the DN maps of (1.23) and (1.24) are well-defined by Proposition 2.1 on a neighborhood of the zero boundary value.

For the nonlinearity $a(x, z) = q(x)e^z$, the inverse source problem is not uniquely solvable due to a gauge symmetry. However, if the nonlinearity is $q(x)ze^z$, and $q(x) \neq 0$ for $x \in \Omega$, the corresponding inverse source problem has a unique solution.

Corollary 1.5. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with C^∞-smooth boundary $\partial\Omega$, $n \geq 2$. Let $q_j \in C^\alpha(\overline{\Omega})$, and suppose additionally that*

Case 1.

$$a_j(x, z) = q_j(x)e^z;$$

Case 2.

$$a_j(x, z) = q_j(x)ze^z,$$

with $q_j \neq 0$ in Ω, for $j = 1, 2$.

Suppose that there is an open $\mathcal{N} \subset C^{2,\alpha}(\partial\Omega)$ such that the corresponding DN maps Λ_{a_j,F_j} of the equation

$$\begin{cases} \Delta u_j + a_j(x, u_j) = F_j & \text{in } \Omega, \\ u_j = f & \text{on } \partial\Omega \end{cases}$$

satisfy

$$\Lambda_{a_1,F_1}(f) = \Lambda_{a_2,F_2}(f) \text{ for any } f \in \mathcal{N}.$$

Then we have:

Case 1. *Gauge symmetry:*

$$q_1 = q_2 e^\psi \quad \text{and} \quad F_1 = F_2 - \Delta\psi \text{ in } \Omega. \tag{1.25}$$

Conversely, if (1.25) holds for some $\psi \in C^{2,\alpha}(\overline{\Omega})$ with $\psi|_{\partial\Omega} = \partial_\nu\psi|_{\partial\Omega} = 0$, then $\Lambda_{a_1,F_1}(f) = \Lambda_{a_2,F_2}(f)$ for all $f \in C^{2,\alpha}(\partial\Omega)$ for which either side of the equation is defined.

Case 2. *Unique determination:*

$$q_1 = q_2 \quad \text{and} \quad F_1 = F_2 \text{ in } \Omega. \tag{1.26}$$

As the second application of Theorem 1.3, we consider the inverse source problem for the elliptic sine-Gordon equation. Again, let q and F belong to $C^\alpha(\overline{\Omega})$, and assume that the equation

$$\begin{cases} \Delta u + q \sin u = F & \text{in } \Omega, \\ u = f & \text{on } \partial\Omega, \end{cases} \tag{1.27}$$

has a solution for some boundary value $f_0 \in C^{2,\alpha}(\partial\Omega)$ such that 0 is not an eigenvalue of $\Delta + \partial_z a(x, u_0)$. Then the equation is well-posed on a neighborhood $\mathcal{N} \subset C^{2,\alpha}(\partial\Omega)$ of f_0 by Theorem 2.1. The DN map of (1.27) is again defined by

$$\Lambda_{q,F} : \mathcal{N} \to C^{1,\alpha}(\partial\Omega), \qquad u \mapsto \partial_\nu u_f|_{\partial\Omega},$$

where $u_f \in C^{2,\alpha}(\overline{\Omega})$ is the unique solution to (1.27) on a neighborhood of u_{f_0}. If F is assumed to be small enough, the DN map of (1.24) is well defined by Proposition 2.1 on a neighborhood of the zero boundary value.

For the sine-Gordon equation, the inverse source problem is solvable.

Corollary 1.6. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with C^∞-smooth boundary $\partial\Omega$, $n \geq 2$. Let $q \in C^\alpha(\overline{\Omega})$, and suppose additionally that*

$$a_j(x, z) = q_j(x) \sin z, \tag{1.28}$$

for $j = 1, 2$. Suppose that there is an open set $\mathcal{N} \subset C^{2,\alpha}(\partial\Omega)$ such that the corresponding DN maps Λ_{q_j,F_j} of the equation

$$\begin{cases} \Delta u_j + q_j \sin(u_j) = F_j & \text{in } \Omega, \\ u_j = f & \text{on } \partial\Omega. \end{cases} \tag{1.29}$$

satisfy

$$\Lambda_{q_1,F_1}(f) = \Lambda_{q_2,F_2}(f) \text{ for any } f \in \mathcal{N}.$$

Then

$$q_1 = q_2 \quad \text{and} \quad F_1 = F_2 \text{ in } \Omega. \tag{1.30}$$

The paper is organized as follows. In Section 2, we prove a well-posedness result for semilinear elliptic equations with source terms. Moreover, a local well-posedness result is also given in Section 2, and the proof is left in Appendix A. In Section 3, we prove Theorem 1.3 and Remark 1.2. We prove Corollaries 1.5 and 1.6 in Section 4.

2. Preliminaries

In this section, we prove a local well-posedness result for the Dirichlet problem (1.1) on a neighborhood of a given solution. Let $0 < \alpha < 1$ and $\delta > 0$ and denote

$$\mathcal{N}_\delta := \left\{ f \in C^{2,\alpha}(\partial\Omega) : \|f\|_{C^{2,\alpha}(\partial\Omega)} \leq \delta \right\}. \tag{2.1}$$

Note that when the source function F of the equation $\Delta u(x) + a(x, u) = F(x)$ does not vanish, zero function is not a solution to the equation (1.1). This is the main reason why our well-posedness result differs from the usual ones, such as the one in [LLLS21, KU20a].

Theorem 2.1 (Well-posedness)**.** *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with C^∞ boundary $\partial\Omega$ and $n \geq 2$, and $a = a(x, z) : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ is C^∞-smooth in the z-variable with $a(x, 0) = 0$ in $\overline{\Omega}$. Given $\alpha \in (0, 1)$, $F \in C^{2,\alpha}(\overline{\Omega})$ and $f_0 \in C^{2,\alpha}(\partial\Omega)$, suppose that there exists a solution $u_0 \in C^{2,\alpha}(\overline{\Omega})$ to*

$$\begin{cases} \Delta u_0 + a(x, u_0) = F & \text{in } \Omega, \\ u = f_0 & \text{on } \partial\Omega. \end{cases} \tag{2.2}$$

Assume also that

$$0 \text{ is not a Dirichlet eigenvalue of } \Delta + \partial_z a(x, u_0) \text{ in } \Omega. \tag{2.3}$$

Then there are $\delta > 0$ and $C > 0$ such that for any $f \in \mathcal{N}_\delta$ there exists a unique solution $u \in C^{2,\alpha}(\overline{\Omega})$ of

$$\begin{cases} \Delta u + a(x,u) = F & \text{in } \Omega, \\ u = f_0 + f & \text{on } \partial\Omega, \end{cases} \tag{2.4}$$

within the class $\left\{ w \in C^{2,\alpha}(\overline{\Omega}) : \|w - u_0\|_{C^{2,\alpha}(\overline{\Omega})} \leq C \right\}$. Moreover, there are C^∞ Fréchet differentiable maps

$$\begin{aligned} &\mathcal{S} : \mathcal{N}_\delta \to C^{2,\alpha}(\overline{\Omega}), && f \mapsto u, \\ &\Lambda : \mathcal{N}_\delta \to C^{1,\alpha}(\partial\Omega), && f \mapsto \partial_\nu u|_{\partial\Omega}. \end{aligned}$$

Proof. We use the standard method, which uses the implicit function theorem in Banach spaces to prove the theorem. A similar proof can be found from the work [LLLS21] where the source F is assumed to vanish. We refer to that work for additional details of the arguments used. Let

$$\mathcal{B}_1 = C^{2,\alpha}(\partial\Omega), \quad \mathcal{B}_2 = C^{2,\alpha}(\overline{\Omega}), \quad \mathcal{B}_3 = C^{\alpha}(\overline{\Omega}) \times C^{2,\alpha}(\partial\Omega)$$

and assume that u_0 solves (2.2). Consider the map

$$\begin{aligned} \Psi : \mathcal{B}_1 \times \mathcal{B}_2 \to \mathcal{B}_3&, \\ (f,u) \mapsto (\Delta u &+ a(x,u) - F, u|_{\partial\Omega} - (f_0 + f)). \end{aligned}$$

Similar to [LLLS21, Section 2], one can show that the map $u \mapsto a(x,u)$ is a C^∞ map from $C^{2,\alpha}(\overline{\Omega}) \to C^{2,\alpha}(\overline{\Omega})$.

Notice that $\Psi(0, u_0) = (0,0)$, where $u_0 \in C^{2,\alpha}(\overline{\Omega})$ is a solution to (2.2). The first linearization of $\Psi = \Psi(f,u)$ at $(0, u_0)$ in the variable u is

$$D_u \Psi|_{(0,u_0)}(v) = (\Delta v + \partial_z a(x,u_0)v, v|_{\partial\Omega}).$$

This is a homeomorphism $\mathcal{B}_2 \to \mathcal{B}_3$ by the condition (2.3), which is guaranteed by well-posedness and Schauder estimates for linear second order elliptic equations.

Using the implicit function theorem in Banach spaces [RR06, Theorem 10.6 and Remark 10.5] yields that there is $\delta > 0$ and an open ball $\mathcal{N}_\delta \subset C^{2,\alpha}(\partial\Omega)$ and a C^∞ map $\mathcal{S} : \mathcal{N}_\delta \to \mathcal{B}_2$ such that whenever $\|f\|_{C^{2,\alpha}(\partial\Omega)} \leq \delta$ we have

$$\Psi(f, \mathcal{S}(f)) = (0,0).$$

Since $\mathcal{S}$ is smooth and $\mathcal{S}(0) = u_0$, the solution $u = \mathcal{S}(f)$ satisfies

$$\|u\|_{C^{2,\alpha}(\overline{\Omega})} \leq C\|f\|_{C^{2,\alpha}(\partial\Omega)}.$$

Moreover, by the uniqueness statement of the implicit function theorem, by redefining $\delta > 0$ to be smaller if necessary, $u = \mathcal{S}(f)$ is the only solution to $\Psi(f,u) = (0,0)$ whenever $\|f\|_{C^{2,\alpha}(\partial\Omega)} \leq \delta$ and

$$\|u\|_{C^{2,\alpha}(\overline{\Omega})} \leq C.$$

As in [LLLS21], one can check that the solution operator $\mathcal{S} : \mathcal{N}_\delta \to C^{2,\alpha}(\overline{\Omega})$ is a C^∞ map in the Fréchet sense. Since the normal derivative is a linear map $C^{2,\alpha}(\overline{\Omega}) \to C^{1,\alpha}(\partial\Omega)$, then Λ is also a well defined C^∞ map $\mathcal{N}_\delta \to C^{1,\alpha}(\partial\Omega)$. ☐

We remark that if $a(x,z)$ satisfies

$$\partial_z a(x,z) \leq 0$$

for all $x \in \Omega$ and $z \in \mathbb{R}$, then the conditions of Theorem 2.1 are satisfied by [GT83, Theorem 15.12]. Under the assumptions of Theorem 2.1 above, the boundary value problem (1.1) is well-posed in the following sense: There is $f_0 \in C^{2,\alpha}(\partial\Omega)$ and $\delta > 0$ such that for each $f \in f_0 + \mathcal{N}_\delta$ there exists a solution u_f to (1.1) with $u_f|_{\partial\Omega} = f$. The solution u_f is unique on a fixed neighborhood of $u_0 \in C^{2,\alpha}(\Omega)$, where u_0 solves (1.1) with boundary value f_0. In this case the corresponding DN map $f_0 + \mathcal{N}_\delta \to C^{1,\alpha}(\partial\Omega)$ defined by the assignment $f \mapsto \partial_\nu u_f|_{\partial\Omega}$ is well-defined and C^∞ smooth in the Fréchet sense.

Next we give a well-posedness result in the case when the Dirichlet data and the source F are both sufficiently small. We record it to provide an example where the DN map is always defined. Let

$$\mathcal{A}_\varepsilon := \left\{ F \in C^{2,\alpha}(\overline{\Omega}) : \|F\|_{C^\alpha(\overline{\Omega})} \le \varepsilon \right\}.$$

We have following result, whose the proof is placed in the Appendix A.

Proposition 2.1. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with C^∞ boundary $\partial\Omega$ and $n \ge 2$. Assume that $a(x,0) = 0$ and the condition (2.3) hold with $u_0 = 0$. There are $C > 0$, $\varepsilon > 0$ and $\delta > 0$ such that for any $F \in \mathcal{A}_\epsilon$ and $f \in \mathcal{N}_\delta$, then there is a unique solution $u \in C^{2,\alpha}(\overline{\Omega})$ of*

$$\begin{cases} \Delta u + a(x,u) = F & \text{in } \Omega, \\ u = f & \text{on } \partial\Omega, \end{cases} \tag{2.5}$$

within the class $\left\{ w \in C^{2,\alpha}(\overline{\Omega}) : \|w\|_{C^{2,\alpha}(\overline{\Omega})} \le C(\varepsilon + \delta) \right\}$. Moreover, there is a C^∞ Fréchet differentiable map

$$\mathcal{S} : \mathcal{A}_\varepsilon \times \mathcal{N}_\delta \to C^{2,\alpha}(\overline{\Omega}), \quad (F, f) \mapsto u.$$

In particular, for a fixed $F \in \mathcal{A}_\varepsilon$, the map

$$\Lambda_F : \mathcal{N}_\delta \to C^{1,\alpha}(\partial\Omega), \quad f \mapsto \partial_\nu u|_{\partial\Omega}$$

is also C^∞ Fréchet differentiable.

Partly due to the concreteness of presentation, we end this section by an example of unique solvability result in dimension two, where the monotonicity method works well due to the Sobolev embedding $H^1(\Omega) \to L^6(\Omega)$. We note that in the example, we do not need to assume the existence of a solution u_0, the source F does not need to be small, the solutions are globally unique and F can have quite low regularity.

Example 2.2. *In the two-dimensional case, let Ω be a bounded domain with C^∞-smooth boundary $\partial\Omega$. We consider the semilinear equation*

$$\begin{cases} -\Delta u + a^{(3)} u^3 = F & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.6}$$

where $c_0 \le a^{(3)} \in C^\infty(\overline{\Omega})$, for some constant $c_0 > 0$. Given $F \in H^{-1}(\Omega)$, there exists a unique solution $u_F \in H^1(\Omega)$ solving (2.6).

The proof is by the monotone operator method, which works well in dimension two. Let us multiply (2.6) by a test function $\varphi \in H_0^1(\Omega)$. Then an integration by parts yields

$$\int_\Omega \nabla u \cdot \nabla \varphi \, dx + \int_\Omega a^{(3)} u^3 \varphi \, dx = \int_\Omega F \varphi \, dx.$$

Let $\mathbf{T} : H_0^1(\Omega) \to H^{-1}(\Omega)$ be the operator given by

$$\langle \mathbf{T}u, \varphi \rangle = \int_\Omega \nabla u \cdot \nabla \varphi \, dx + \int_\Omega a^{(3)} u^3 \varphi \, dx, \text{ for any } \varphi \in H_0^1(\Omega).$$

It is not hard to see that $\mathbf{T}u - F$ is the Frechét derivative of the energy functional

$$\mathbf{E}(u) = \frac{1}{2} \int_\Omega |\nabla u|^2 \, dx + \frac{1}{4} \int_\Omega a^{(3)} u^4 \, dx - \int_\Omega F u \, dx.$$

Since $\Omega \subset \mathbb{R}^2$, the Sobolev space $H^1(\Omega)$ embeds in $L^6(\Omega)$. Using this fact, one can show that the operator $\mathbf{T}$ is bounded, strictly monotone and coercive. Then by applying the classical energy method, the functional $\mathbf{E}$ is coercive and weakly lower *semicontinuous on* $H_0^1(\Omega)$ *(for example, see* [FK14, Theorem 26.11]*). Therefore,* $\mathbf{E}$ *is bounded from below and attains its infimum at some function* $u \in H_0^1(\Omega)$*. Thus* u *is a solution of* (2.6)*. The uniqueness of* u *is a direct result of the strict monotonicity of* $\mathbf{T}$*. We refer to* [BCP22, Theorem 3.1] *for more details about this argument.*

3. Uniqueness for polynomial nonlinearities up to gauge invariances

To better convey the main idea of the proof of Theorem 1.1 regarding general polynomial nonlinearities, let us first consider the simpler cases of quadratic and cubic nonlinearities.

3.1. **Quadratic nonlinearity.** In the introduction we showed that the inverse source problem for

$$\Delta u + a(x,u) = F,$$

$a(x,u)$ is quadratic,

$$a(x,u(x)) = a^{(1)}(x)u(x) + a^{(2)}(x)u^2(x),$$

has a gauge invariance given by the gauge conditions (1.12). We show next that these gauge conditions are the only obstruction to uniqueness in the inverse source problem for quadratic nonlinearities. This is Remark 1.2.

For the quadratic nonlinearity we consider Dirichlet data of the form

$$f := f(x;\epsilon_1,\epsilon_2) := f_0(x) + \epsilon_1 f_1(x) + \epsilon_2 f_2(x) \quad x \in \partial\Omega, \tag{3.1}$$

where where $f_0, f_1, f_2 \in C^{2,\alpha}(\partial\Omega)$, and ϵ_1, ϵ_2 are small real parameters. We now prove Remark 1.2 (a).

Proof of Remark 1.2 (a). By assumption there is $\mathcal{N} \subset C^{2,\alpha}(\partial\Omega)$ such that

$$\Lambda_{a_1,F_1}(f) = \Lambda_{a_2,F_2}(f), \quad f \in \mathcal{N}.$$

Let $f_0 \in \mathcal{N}$, $f_1, f_2 \in C^{2,\alpha}(\partial\Omega)$ and $\epsilon_1, \epsilon_2 > 0$ such that $f_0 + \epsilon_1 f_1 + \epsilon_2 f_2 \in \mathcal{N}$. We apply the higher order linearization method to the equation

$$\begin{cases} \Delta u_j + a_j^{(1)} u_j + a_j^{(2)} u_j^2 = F_j & \text{in } \Omega, \\ u_j = f_0 + \epsilon_1 f_1 + \epsilon_2 f_2 & \text{on } \partial\Omega. \end{cases} \tag{3.2}$$

We denote $\epsilon = (\epsilon_1, \epsilon_2)$, which especially means that $\epsilon = 0$ is equivalent to $\epsilon_1 = \epsilon_2 = 0$. Below the index $j = 1,2$ corresponds to the different sets of coefficients, and an index $\ell = 1,2$ to ϵ_ℓ parameters. Let us denote by $u_j^{(0)}$ the solution to

$$\begin{cases} \Delta u_j^{(0)} + a_j^{(1)} u_j^{(0)} + a_j^{(2)} \left(u_j^{(0)}\right)^2 = F & \text{in } \Omega, \\ u_j^{(0)} = f_0 & \text{on } \partial\Omega. \end{cases} \tag{3.3}$$

With the well-posedness holding on a neighborhood $\mathcal{N}$ of f_0, see Theorem 2.1, we can differentiate (3.2) with respect to ϵ_ℓ, for $\ell = 1,2$. We obtain

$$\begin{cases} \left(\Delta + a_j^{(1)} + 2a_j^{(2)} u_j^{(0)}\right) v_j^{(\ell)} = 0 & \text{in } \Omega, \\ v_j^{(\ell)} = f_\ell & \text{on } \partial\Omega, \end{cases} \tag{3.4}$$

where

$$v_j^{(\ell)} = \partial_{\epsilon_\ell}\big|_{\epsilon=0} u_j,$$

for $j, \ell = 1,2$. It also follows from Theorem 2.1 that we know the DN maps of the equation (3.4) for $j = 1$ and $j = 2$ agree. Thus, by the global uniqueness result for linear inverse boundary value problems (see e.g. [LLLS20, Proposition 2.1] or [SU87] for $n \geq 3$ and [Buk08, BTW19] for $n = 2$), we have

$$Q := a_1^{(1)} + 2a_1^{(2)} u_1^{(0)} = a_2^{(1)} + 2a_2^{(2)} u_2^{(0)} \text{ in } \Omega. \tag{3.5}$$

It then follows by uniqueness of solutions to the Dirichlet problem (3.3) that

$$v^{(\ell)} := v_1^{(\ell)} = v_2^{(\ell)} \text{ in } \Omega,$$

for $\ell = 1, 2$.

We next derive the equation for the second order linearization of (3.2) at $u_j^{(0)}$. For $j = 1, 2$, a straightforward computation shows that

$$\begin{cases} \left(\Delta + a_j^{(1)} + 2a_j^{(2)} u_j^{(0)}\right) w_j + 2a_j^{(2)} v^{(1)} v^{(2)} = 0 & \text{in } \Omega, \\ w_j = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.6}$$

where

$$w_j = \partial^2_{\epsilon_1 \epsilon_2}\big|_{\epsilon=0} u_j.$$

We show next that $a_1^{(2)} = a_2^{(2)}$ in Ω. For that, let us consider $\mathbf{v}^{(\ell)}$ to be the solution of

$$\begin{cases} (\Delta + Q)\, \mathbf{v}^{(\ell)} = 0 & \text{in } \Omega, \\ \mathbf{v}^{(\ell)} = g_\ell & \text{on } \partial\Omega, \end{cases} \tag{3.7}$$

where Q is given in (3.5) and $g_\ell \in H^{1/2}(\partial\Omega)$ will be chosen later for $\ell = 1, 2$. We multiply (3.7) by $\mathbf{v}^{(1)}$. Moreover, by using $\partial_\nu w_1 = \partial_\nu w_2$ on $\partial\Omega$, integration by parts yields

$$\begin{aligned} 0 &= \int_{\partial\Omega} (\partial_\nu w_1 - \partial_\nu w_2)\, \mathbf{v}^{(1)}\, dS \\ &= \int_\Omega \Delta (w_1 - w_2)\, \mathbf{v}^{(1)}\, dx + \int_\Omega \nabla (w_1 - w_2) \cdot \nabla \mathbf{v}^{(1)}\, dx \\ &= \int_\Omega \Delta (w_1 - w_2)\, \mathbf{v}^{(1)}\, dx + \int_{\partial\Omega} (w_1 - w_2) \cdot \partial_\nu \mathbf{v}^{(1)}\, dS \\ &\quad - \int_\Omega (w_1 - w_2)\, \Delta \mathbf{v}^{(1)}\, dx \\ &= \int_\Omega \left(a_1^{(2)} - a_2^{(2)}\right) v^{(1)} v^{(2)} \mathbf{v}^{(1)}\, dx. \end{aligned}$$

Here we used $w_1 - w_2 = 0$ on $\partial\Omega$ and (3.6) and (3.7). By using that products of pairs of complex geometrical optics solutions (CGOs) to (3.4) are dense in $L^1(\Omega)$ for $n \geq 2$, we can choose $v^{(1)}$ and $v^{(2)}$ so that we obtain

$$\left(a_1^{(2)} - a_2^{(2)}\right) \mathbf{v}^{(1)} = 0 \text{ in } \Omega. \tag{3.8}$$

For the construction of the CGOs, see [SU87].

Next we take also $\mathbf{v}^{(1)}$ as a CGO solution and multiply the above identity by yet another CGO solution $\mathbf{v}^{(2)}$ with $\mathbf{v}^{(2)}|_{\partial\Omega} = g_2$, one can integrate the above identity to obtain

$$\int_\Omega \left(a_1^{(2)} - a_2^{(2)}\right) \mathbf{v}^{(1)} \mathbf{v}^{(2)}\, dx = 0.$$

By applying the density of CGOs again shows that

$$a^{(2)} := a_1^{(2)} = a_2^{(2)} \text{ in } \Omega. \tag{3.9}$$

Let us then define $\psi \in C^2(\overline{\Omega})$ as the difference

$$\psi := u_2^{(0)} - u_1^{(0)} \text{ in } \Omega. \tag{3.10}$$

By plugging (3.9) into (3.5), we obtain

$$a_1^{(1)} = a_2^{(1)} + 2a^{(2)}\left(u_2^{(0)} - u_1^{(0)}\right) = a_2^{(1)} + 2a^{(2)}\psi \text{ in } \Omega. \tag{3.11}$$

Moreover, with the relation (3.10) at hand, we calculate

$$\begin{aligned} F_2 =&\Delta u_2^{(0)} + a_1^{(2)} u_2^{(0)} + a_2^{(2)}\left(u_2^{(0)}\right)^2 \\ =&\Delta\left(u_1^{(0)} + \psi\right) + a_1^{(2)}\left(u_1^{(0)} + \psi\right) + a_2^{(2)}\left(u_1^{(0)} + \psi\right)^2 \\ =&\left(F_1 + a_1^{(2)}\psi + a_2^{(2)}\psi^2\right) + \left(a_1^{(2)} - a_1^{(1)} + 2a_2^{(2)}\psi\right)u_1^{(0)} \\ &+ \left(a_2^{(2)} - a_2^{(1)}\right)\left(u_1^{(0)}\right)^2. \end{aligned} \tag{3.12}$$

Here we also utilized (3.3). By using (3.9) and (3.11), we see that $F_2 = F_1 + a_1^{(2)}\psi + a_2^{(2)}\psi^2$ in Ω. Finally, the function ψ of the form (3.10) satisfies $\psi|_{\partial\Omega} = \left(u_2^{(0)} - u_1^{(0)}\right)\Big|_{\partial\Omega} = 0$ and $\partial_\nu\psi|_{\partial\Omega} = \partial_\nu\left(u_2^{(0)} - u_1^{(0)}\right)\big|_{\partial\Omega} = 0$. We have shown

$$\begin{cases} a_1^{(2)} = a_2^{(2)} =: a^{(2)} \\ a_1^{(1)} = a_2^{(1)} + 2a^{(2)}\psi \\ F_1 = F_2 - \Delta\psi - a_1^{(2)}\psi - a^{(2)}\psi^2 \end{cases} \tag{3.13}$$

as desired. □

Remark 3.1. *Note that if the coefficients of quadratic terms vanish, $a_2^{(1)} = a_2^{(2)} = 0$ in Ω, then (3.13) describes the gauge symmetry of inverse source problem for linear equation discussed in Remark 1.1.*

We also remark that in the above proof we could have alternatively used Runge approximation argument to show that $a_1^{(2)} = a_2^{(2)}$ after (3.8). Indeed, if $x_0 \in \Omega$, there is by Runge approximation (see e.g. [LLS20]) a solution $\mathbf{v}^{(1)}$ such that $\mathbf{v}^{(1)}(x_0) \neq 0$. Together with (3.8), and using the above argument for all $x_0 \in \Omega$, shows $a_1^{(2)} = a_2^{(2)}$ in Ω. Runge approximation in similar situations were earlier used in [LLLS21].

As discussed in the introduction, if the linear term of a semilinear equation $\Delta u + a^{(1)}u + a^{(2)}u^2 = F$ is known (i.e., $a^{(1)}$ is known a priori), then the DN map determines the other coefficients of the equation uniquely. This is Remark 1.3.

Proof of Remark 1.3. By assumption and Remark 1.2

$$a^{(1)} = a^{(1)} + 2a^{(2)}\psi$$

and

$$F_1 = F_2 - \Delta\psi - a_1^{(2)}\psi - a^{(2)}\psi^2$$

hold in Ω for some gauge function ψ. Here $a^{(2)} = a_1^{(2)} = a_2^{(2)}$. Since $a^{(2)} \neq 0$ in Ω by assumption, the first identity above shows that $\psi = 0$ in Ω. Substituting $\psi = 0$ to latter identity above shows $F_1 = F_2$ in Ω. □

3.2. **Cubic nonlinearity.** We move on to prove our results about cubic nonlinearities. For $j = 1, 2$, we let

$$a_j(x, z) = a_j^{(1)} z + a_j^{(2)} z^2 + a_j^{(3)} z^3,$$

and let us consider the equation

$$\Delta u_j + a_j^{(1)} u_j + a_j^{(2)} u_j^2 + a_j^{(3)} u_j^3 = F_j \text{ in } \Omega. \tag{3.14}$$

Remark 1.2 (b), which we prove in this section shows that the inverse source problems of the above equation has uniqueness property for both coefficients and source up to a gauge.

Before proving Remark 1.2 (b), let us derive the gauge of the inverse problem. Assume that u_1 solves (3.14) with boundary value $u_1|_{\partial\Omega} = f$. If $\psi \in C^2(\overline{\Omega})$, we denote by $a_2^{(1)}$, $a_2^{(2)}$, $a_3^{(2)}$ and F_2 another set of coefficients and a source, which may depend on ψ. If we denote $u_2 = u_1 + \psi$, then we have the chain of equivalences

$$\Delta u_2 + a_2^{(1)} u_2 + a_2^{(2)} (u_2)^2 + a_2^{(3)} (u_2)^3 = F_2,$$

which is equivalent to

$$\Delta (u_1 + \psi) + a_2^{(1)} (u_1 + \psi) + a_2^{(2)} (u_1 + \psi)^2 + a_2^{(3)} (u_1 + \psi)^3 = F_2,$$

which is also equivalent to

$$\begin{aligned}\Delta u_1 + \Delta\psi + a_2^{(1)} u_1 + a_2^{(1)}\psi + a_2^{(2)} (u_1)^2 + 2a_2^{(2)}\psi u_1 + a_2^{(2)}\psi^2 \\ + a_2^{(3)} \left(u_1^3 + 3u_1^2\psi + 3u_1\psi^2 + \psi^3\right) = F_2,\end{aligned}$$

in Ω. By using $\Delta u_1 = -a_1^{(1)} u_1 - a_1^{(2)} (u_1)^2 - a_1^{(3)} (u_1)^3 + F_1$ in Ω and equating the powers of u gives the following system

$$\begin{cases} F_1 = F_2 - \Delta\psi - a_2^{(1)}\psi - a_2^{(2)}\psi^2 - a_2^{(3)}\psi^3 \\ a_1^{(1)} = a_2^{(1)} + 2a_2^{(2)}\psi + 3a_2^{(3)}\psi^2 \\ a_1^{(2)} = a_2^{(2)} + 3a_2^{(3)}\psi \\ a_1^{(3)} = a_2^{(3)}. \end{cases} \tag{3.15}$$

The above system of equations describes the gauge invariance for the inverse source problem for cubic nonlinearity. If $\psi|_\Omega = \partial_\nu\psi|_{\partial\Omega} = 0$, the above computation shows that corresponding DN maps Λ_{a_1,F_1} and Λ_{a_2,F_2} are the same. It is impossible to uniquely determine the coefficients and sources from the DN map at the same time. There is a gauge symmetry given by (3.15).

We next prove Remark 1.2 (b), which states that the DN map determines the coefficients and source up to the gauge symmetry (3.15).

Proof of Remark 1.2 (b). Let u_j be the solution to

$$\begin{cases} \Delta u_j + a_j^{(1)}(x) u_j + a_j^{(2)}(x) (u_j)^2 + a_j^{(3)} (u_j)^3 = F_j & \text{in } \Omega, \\ u_j = f & \text{on } \partial\Omega, \end{cases} \tag{3.16}$$

for $j = 1, 2$. Let us consider the Dirichlet data

$$f = f(x;\epsilon) = f_0 + \epsilon_1 f_1 + \epsilon_2 f_2 + \epsilon_3 f_3 \quad \text{on} \quad \partial\Omega,$$

where the parameters ϵ_ℓ are real numbers, $f_0 \in \mathcal{N}$ and $f_\ell \in C^{2,\alpha}(\partial\Omega)$, for $\ell = 1, 2, 3$. By assumption $\Lambda_{a_1,F_1}(f) = \Lambda_{a_2,F_2}(f)$, if the parameters ϵ_ℓ are small enough. We denote $\epsilon = (\epsilon_1, \epsilon_2, \epsilon_3)$.

Let us denote by $u_j^{(0)}$ the solution to

$$\begin{cases} \Delta u_j^{(0)} + a_j(x, u_j^{(0)}) = F_j & \text{in } \Omega, \\ u_j^{(0)} = f_0 & \text{on } \partial\Omega. \end{cases}$$

We linearize

$$\begin{cases} \Delta u_j + a_j(x, u_j) = F_j & \text{in } \Omega, \\ u_j = f_0 + \epsilon_1 f_1 + \epsilon_2 f_2 + \epsilon_3 f_3 & \text{on } \partial\Omega. \end{cases}$$

at the solution corresponding to boundary value f_0 for $j=1,2$. The first linearization at f_0 is

$$\begin{cases} \left(\Delta + a_j^{(1)} + 2a_j^{(2)}u_j^{(0)} + 3a_j^{(3)}\left(u_j^{(0)}\right)^2\right) v_j^{(\ell)} = 0 & \text{in } \Omega, \\ v_j^{(\ell)} = f_\ell & \text{on } \partial\Omega, \end{cases} \tag{3.17}$$

where $v_j^{(\ell)} := \partial_{\epsilon_\ell} u_j\big|_{\epsilon=0}$ in Ω. By Theorem 2.1, we know that the DN maps of (3.17) for $j=1$ and $j=2$ agree. By the global uniqueness result for the Calderón problem for linear equations we have

$$Q := a_1^{(1)} + 2a_1^{(2)}u_1^{(0)} + 3a_1^{(3)}\left(u_1^{(0)}\right)^2 = a_2^{(1)} + 2a_2^{(2)}u_2^{(0)} + 3a_2^{(3)}\left(u_2^{(0)}\right)^2 \text{ in } \Omega, \tag{3.18}$$

and by the uniqueness of solutions to the Dirichlet problem (3.17) it follows that

$$v^{(\ell)} := v_1^{(\ell)} = v_2^{(\ell)} \text{ in } \Omega, \tag{3.19}$$

for $\ell = 1,2,3$.

The second linearization reads

$$\begin{cases} (\Delta + Q)\, w_j^{(k\ell)} + 2\left(a_j^{(2)} + 3a_j^{(3)}u_j^{(0)}\right) v^{(k)}v^{(\ell)} = 0 & \text{in } \Omega, \\ w_j^{(k\ell)} = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.20}$$

where $w_j^{(k\ell)} = \partial^2_{\epsilon_k\epsilon_\ell} u_j\big|_{\epsilon=0}$ for $k,\ell \in \{1,2,3\}$ and $j=1,2$. Similar to the proof of Remark 1.2, multiplying (3.20) by the function $\mathbf{v}$ that solves

$$\begin{cases} (\Delta + Q)\,\mathbf{v} = 0 & \text{in } \Omega, \\ \mathbf{v} = \mathbf{g} & \text{on } \partial\Omega, \end{cases} \tag{3.21}$$

where $\mathbf{g} \in H^{1/2}(\partial\Omega)$ is a function to be chosen later. Multiplying (3.20) by the solution $\mathbf{v}$ and integrating by parts show that

$$\int_\Omega \left[\left(a_1^{(2)} + 3a_1^{(3)}u_1^{(0)}\right) - \left(a_2^{(2)} + 3a_2^{(3)}u_2^{(0)}\right)\right] v^{(k)}v^{(\ell)}\mathbf{v}\, dx = 0, \tag{3.22}$$

for $k,\ell = 1,2,3$. Applying an additional density argument as in the proof of Remark 1.2 (a) (or the one described in Remark 3.1), one obtains

$$R := a_1^{(2)} + 3a_1^{(3)}u_1^{(0)} = a_2^{(2)} + 3a_2^{(3)}u_2^{(0)} \text{ in } \Omega. \tag{3.23}$$

The uniqueness of solutions to Dirichlet problem of (3.20) and (3.23) imply

$$w^{(k\ell)} := w_1^{(k\ell)} = w_2^{(k\ell)} \text{ in } \Omega,$$

for any $k,\ell \in \{1,2,3\}$.

Now, a computation shows that the third linearized equation is

$$\begin{cases} (\Delta + Q)\, w_j^{(123)} + 2R\left(w^{(12)}v^{(3)} + w^{(23)}v^{(1)} + w^{(13)}v^{(2)}\right) & \\ \qquad\qquad +6a_j^{(3)}v^{(1)}v^{(2)}v^{(3)} = 0 & \text{in } \Omega, \\ w^{(123)} = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.24}$$

where R is the function given by (3.23). Multiplying (3.24) against the solution $\mathbf{v}$ of (3.21) and integrating by parts produces the identity

$$\int_\Omega \left(a_1^{(3)} - a_2^{(3)}\right) v^{(1)}v^{(2)}v^{(3)}\mathbf{v}\, dx = 0.$$

By choosing $v^{(\ell)}$ ($\ell = 1,2,3$) and $\mathbf{v}$ to be suitable CGO solutions, we conclude via the above integral identity

$$a^{(3)} := a_1^{(3)} = a_2^{(3)} \text{ in } \Omega, \tag{3.25}$$

which proves the first relation in (1.19).

Let us define $\psi \in C^2(\overline{\Omega})$ by

$$\psi = u_2^{(0)} - u_1^{(0)} \text{ in } \Omega. \tag{3.26}$$

Then, the identity (3.23) is equivalent to

$$a_1^{(2)} = a_2^{(2)} + 3a^{(3)}\left(u_2^{(0)} - u_1^{(0)}\right) = a_2^{(2)} + 3a^{(3)}\psi, \tag{3.27}$$

where we utilized (3.25) and (3.26). This shows the second identity in (1.19). By plugging (3.27) into (3.18), direct computations yield

$$\begin{aligned} a_1^{(1)} &= a_2^{(1)} + 2a_2^{(2)}u_2^{(0)} + 3a^{(3)}\left(u_2^{(0)}\right)^2 - 2a_1^{(2)}u_1^{(0)} - 3a^{(3)}\left(u_1^{(0)}\right)^2 \\ &= a_2^{(1)} + 2a_2^{(2)}u_2^{(0)} + 3a^{(3)}\left(u_2^{(0)}\right)^2 - 2a_2^{(2)}u_1^{(0)} - 6a^{(3)}\psi u_1^{(0)} - 3a^{(3)}\left(u_1^{(0)}\right)^2 \\ &= a_2^{(1)} + 2a_2^{(2)}\psi + 3a^{(3)}\psi^2, \end{aligned}$$

which proves the third identity in (1.19). Finally, by inserting (3.26) into the original nonlinear equation (3.16), and equating the powers of $u_2^{(0)}$, yield the last identity in (1.19) as desired. This completes the proof. □

3.3. **Polynomial and general nonlinearity.** In order to prove Theorem 1.1, where the nonlinearity is a general polynomial, it is convenient to prove Theorem 1.3 about general nonlinearities first.

Proof of Theorem 1.3. Let $N \in \mathbb{N}$. By using the higher order linearization method, let us take the Dirichlet data to be of the form

$$f(x) = \sum_{\ell=1}^{N} \epsilon_\ell f_\ell(x), \quad x \text{ in } \partial\Omega,$$

where ϵ_ℓ are parameters such that $|\epsilon_\ell|$ are sufficiently small, and each $f_\ell \in C^{2,\alpha}(\partial\Omega)$, for $\ell = 1, \ldots, N$. We first linearize the equation (1.21) around the solution $u_j^{(0)}$, so that we can have

$$\begin{cases} \left(\Delta + \partial_z a_j(x, u_j^{(0)})\right) v_j^{(\ell)} = 0 & \text{in } \Omega, \\ v_j^{(\ell)} = f_\ell & \text{on } \partial\Omega \end{cases} \tag{3.28}$$

for $j = 1, 2$, and $\ell = 1, \ldots, N$. The uniqueness result for the inverse problem for the linear Shrödinger equation yields again that

$$\partial_z a_1(x, u_1^{(0)}) = \partial_z a_2(x, u_2^{(0)}) \text{ in } \Omega.$$

Moreover, via the uniqueness of solutions, we have $v^{(\ell)} = v_1^{(\ell)} = v_2^{(\ell)}$ in Ω, for $\ell = 1, 2, \ldots, N$.

To proceed, the second linearized equation can be derived as

$$\begin{cases} (\Delta + Q)\, w_j^{(\ell m)} + \partial_z^2 a_j(x, u_j^{(0)}) v^{(\ell)} v^{(m)} = 0 & \text{in } \Omega, \\ w_j^{(\ell m)} = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.29}$$

where $Q := \partial_z a_1(x, u_1^{(0)}) = \partial_z a_2(x, u_2^{(0)})$ in Ω, for $\ell, m = 1, 2, \ldots, N$. Similar as before, consider a solution $\mathbf{v}$ of

$$\begin{cases} (\Delta + Q)\, \mathbf{v} = 0 & \text{in } \Omega, \\ \mathbf{v} = \mathbf{g} & \text{on } \partial\Omega, \end{cases}$$

by multiplying (3.29) by the function $\mathbf{v}$, then an integration by parts formula yields that

$$\int_\Omega \left(\partial_z^2 a_1(x, u_1^{(0)}) - \partial_z^2 a_2(x, u_2^{(0)})\right) v^{(\ell)} v^{(m)} \mathbf{v}\, dx = 0,$$

which shows $\partial_z^2 a_1(x, u_1^{(0)}) = \partial_z^2 a_2(x, u_2^{(0)})$ in Ω by utilizing preceding arguments.

Furthermore, by considering higher order linearized equations and using an induction argument, similar to the ones in the proofs of [LLLS20, Proof of Theorem 1.1] and [KU20a, Proof of Theorem 1.3], it is not hard to show that (1.22) holds for any $k \in \mathbb{N}$, where $u_j^{(0)}$ are the solutions of (3.28), for $j = 1, 2$. As $N \in \mathbb{N}$ was arbitrary, this completes the proof. $\square$

We now prove Theorem 1.1.

Proof of Theorem 1.1. To prove the theorem, we need to show that there is $\psi \in C^{2,\alpha}(\overline{\Omega})$ with $\psi|_{\partial\Omega} = \partial_\nu \psi|_{\partial\Omega} = 0$ such that

$$a_1^{(N-k)} = \sum_{m=N-k}^{N} \binom{m}{N-k} a_2^{(m)} \psi^{m-N+k} \quad \text{in } \Omega, \tag{3.30}$$

for $k = 1, \dots, N$. Since $a_1(x, z)$ and $a_2(x, z)$ are both polynomials of order N, we have by Theorem 1.3

$$a_1^{(N)}(x) = \partial_z^N a_1(x, u_1^{(0)}) = \partial_z^N a_2(x, u_2^{(0)}) = a_2^{(N)}(x)$$

for all $x \in \Omega$. Here $u_j^{(0)}$, $j = 1, 2$, is the solution of (1.14) as $u_j^{(0)}|_{\partial\Omega} = 0$. Thus the claim holds for $k = 0$. We prove the claim by induction. For this, let us assume that (3.30) holds for all $k = 0, \dots, L$. It suffices to show that (1.16) holds for $k = L + 1$.

Using Theorem 1.3 again, we have

$$\partial_z^{N-(L+1)} a_1(x, u_1^{(0)}) = \partial_z^{N-(L+1)} a_2(x, u_2^{(0)}) \quad \text{in } \Omega. \tag{3.31}$$

Since $a_j(x, z)$ is a polynomial in a, this identity is equivalent to

$$\begin{aligned}
&(N-L-1)! a_1^{(N-L-1)} + (N-L)! a_1^{(N-L)} u_1^{(0)} \\
&\quad + \frac{(N-L+1)!}{2!} a_1^{(N-L+1)} \left(u_1^{(0)}\right)^2 + \cdots + \frac{N!}{(L+1)!} a_1^{(N)} \left(u_1^{(0)}\right)^{L+1} \\
&= (N-L-1)! a_2^{(N-L-1)} + (N-L)! a_2^{(N-L)} u_2^{(0)} \\
&\quad + \frac{(N-L+1)!}{2!} a_2^{(N-L+1)} \left(u_2^{(0)}\right)^2 + \cdots + \frac{N!}{(L+1)!} a_2^{(N)} \left(u_2^{(0)}\right)^{L+1}.
\end{aligned}$$

After diving by $(N-L-1)!$ the above reads

$$\begin{aligned}
&\binom{N-L-1}{N-L-1} a_1^{(N-L-1)} + \binom{N-L}{N-L-1} a_1^{(N-L)} u_1^{(0)} \\
&\quad + \binom{N-L+1}{N-L-1} a_1^{(N-L+1)} \left(u_1^{(0)}\right)^2 + \cdots + \binom{N}{N-L-1} a_1^{(N)} \left(u_1^{(0)}\right)^{L+1} \\
&= \binom{N-L-1}{N-L-1} a_2^{(N-L-1)} + \binom{N-L}{N-L-1} a_2^{(N-L)} u_2^{(0)} \\
&\quad + \binom{N-L+1}{N-L-1} a_2^{(N-L+1)} \left(u_2^{(0)}\right)^2 + \cdots + \binom{N}{N-L-1} a_2^{(N)} \left(u_2^{(0)}\right)^{L+1}.
\end{aligned} \tag{3.32}$$

We rewrite (3.32) as

$$
\begin{aligned}
&a_1^{(N-L-1)} + \sum_{k=0}^{L} \binom{N-L+k}{N-L-1} a_1^{(N-L+k)} \left(u_1^{(0)}\right)^{k+1} \\
&= a_2^{(N-L-1)} + \sum_{k=0}^{L} \binom{N-L+k}{N-L-1} a_2^{(N-L+k)} \left(u_2^{(0)}\right)^{k+1}.
\end{aligned} \tag{3.33}
$$

We define

$$
\psi := u_2^{(0)} - u_1^{(0)}. \tag{3.34}
$$

Then $\psi \in C^{2,\alpha}(\Omega)$ and $\psi|_{\partial\Omega} = \partial_\nu \psi|_{\partial\Omega} = 0$. By using the induction assumption, that (3.30) holds for $k = 0, \dots, L$, we write the identity (3.33) as

$$
\begin{aligned}
&a_1^{(N-L-1)} \\
&= a_2^{(N-L-1)} + \sum_{k=0}^{L} \binom{N-L+k}{N-L-1} \left[a_2^{(N-L+k)} \left(u_2^{(0)}\right)^{k+1} - a_1^{(N-L+k)} \left(u_1^{(0)}\right)^{k+1} \right] \\
&= a_2^{(N-L-1)} + \sum_{k=0}^{L} \binom{N-L+k}{N-L-1} \Bigg[a_2^{(N-L+k)} \left(u_1^{(0)} + \psi\right)^{k+1} \\
&\qquad - \sum_{m=N-L+k}^{N} \binom{m}{N-L+k} a_2^{(m)} \psi^{m-N+L-k} \left(u_1^{(0)}\right)^{k+1} \Bigg]
\end{aligned} \tag{3.35}
$$

Here the induction assumption was used in the last equality. By using binomial expansion, the above equality is

$$
\begin{aligned}
&a_1^{(N-L-1)} \\
&= a_2^{(N-L-1)} + \sum_{k=0}^{L} \binom{N-L+k}{N-L-1} \Bigg[a_2^{(N-L+k)} \sum_{\iota=0}^{k+1} \binom{k+1}{\iota} \psi^{\iota} \left(u_1^{(0)}\right)^{k+1-\iota} \\
&\qquad - \sum_{m=N-L+k}^{N} \binom{m}{N-L+k} a_2^{(m)} \psi^{m-N+L-k} \left(u_1^{(0)}\right)^{k+1} \Bigg] \\
&= a_2^{(N-L-1)} + S_1 - S_2.
\end{aligned} \tag{3.36}
$$

Here we have defined

$$
\begin{aligned}
S_1 &:= \sum_{k=0}^{L} \binom{N-L+k}{N-L-1} a_2^{(N-L+k)} \sum_{\iota=0}^{k+1} \binom{k+1}{\iota} \psi^{\iota} \left(u_1^{(0)}\right)^{k+1-\iota}, \\
S_2 &:= \sum_{k=0}^{L} \binom{N-L+k}{N-L-1} \sum_{m=N-L+k}^{N} \binom{m}{N-L+k} a_2^{(m)} \psi^{m-N+L-k} \left(u_1^{(0)}\right)^{k+1}.
\end{aligned} \tag{3.37}
$$

To complete the proof we compare the coefficients of the powers of $u_1^{(0)}$ of S_1 and S_2. We first observe that in the term S_1, the powers of $u_1^{(0)}$ range from 0 to $L+1$. In the tern S_2, the powers of $u_1^{(0)}$ range from 1 to $L+1$. We split the remaining proof into two cases according to powers of $u_1^{(0)}$.

Case 1:

Let us consider the coefficients of the terms $\left(u_1^{(0)}\right)^J$, $J = 1, \dots, L+1$, in S_1 and S_2. We observe that the coefficient of $\left(u_1^{(0)}\right)^J$ in S_1 is

$$\sum_{k=J-1}^{L} \binom{N-L+k}{N-L-1} a_2^{(N-L+k)} \binom{k+1}{k+1-J} \psi^{k+1-J} \tag{3.38}$$

Similarly, the coefficient of $\left(u_1^{(0)}\right)^J$ in S_2 is

$$\begin{aligned} &\binom{N-L+J+1}{N-L-1} \sum_{m=N-L+J-1}^{N} \binom{m}{N-L+J+1} a_2^{(m)} \psi^{m-N+L-J+1} \\ =&\binom{N-L+J-1}{N-L-1} \sum_{k=J-1}^{L} \binom{N-L+k}{N-L+J-1} a_2^{(N-L+k)} \psi^{k+1-J}. \end{aligned} \tag{3.39}$$

On the other hand, a direct computation shows that

$$\binom{N-L+k}{N-L-1}\binom{k+1}{k+1-J} = \binom{N-L+J-1}{N-L-1}\binom{N-L+k}{N-L+J-1},$$

so that (3.38) and (3.39) are the same.

Case 2:

The term S_2 does not contain the zeroth power of $u_1^{(0)}$. We express S_1 as

$$S_1 := S_0 + \widetilde{S},$$

where

$$\begin{aligned} S_0 &:= \sum_{k=0}^{L} \binom{N-L+k}{N-L-1} a_2^{(N-L+k)} \psi^{k+1} \\ \widetilde{S} &:= \sum_{k=0}^{L} \binom{N-L+k}{N-L-1} a_2^{(N-L+k)} \sum_{\iota=0}^{k} \binom{k+1}{\iota} \psi^{\iota} \left(u_1^{(0)}\right)^{k+1-\iota}. \end{aligned} \tag{3.40}$$

By redefining the summation index of S_0, we have

$$S_0 = \sum_{m=N-L}^{N} \binom{m}{N-L-1} a_2^{(m)} \psi^{m-N+L+1}. \tag{3.41}$$

Therefore, by plugging (3.38)–(3.41) into (3.35), we obtain

$$\begin{aligned} a_1^{(N-(L+1))} &= a_2^{(N-(L+1))} + \sum_{m=N-L}^{N} \binom{m}{N-L-1} a_2^{(m)} \psi^{m-N+L+1} \\ &= \sum_{m=N-(L+1)}^{N} \binom{m}{N-L-1} a_2^{(m)} \psi^{m-N+L+1}. \end{aligned}$$

This proves the induction step. It remains to prove (1.17).

Recall that the nonlinearity $a_j(x,z) = \sum_{k=1}^{N} a_j^{(k)} z^k$, for $j = 1, 2$, then we can write $a_1(x, u_1^{(0)})$ in terms of

$$\begin{aligned} a_1(x, u_1^{(0)}) &= \sum_{k=0}^{N-1} a_1^{(N-k)} \left(u_1^{(0)}\right)^{N-k} \\ &= \sum_{k=0}^{N-1} \sum_{m=N-k}^{N} \binom{m}{N-k} a_2^{(m)} \psi^{m-N+k} \left(u_1^{(0)}\right)^{N-k}. \end{aligned} \tag{3.42}$$

On the other hand, one can also express

$$a_2(x, u_2^{(0)}) = \sum_{k=1}^{N} a_2^{(k)} \left(u_2^{(0)}\right)^k = \sum_{k=1}^{N} a_2^{(k)} \sum_{m=0}^{k} \binom{k}{m} \left(u_1^{(0)}\right)^m \psi^{k-m}, \tag{3.43}$$

where we used (3.34) and binomial expansion in the above computation. Similar to the computations of Case 1 in preceding arguments, by comparing the orders of the homogeneous parts $\left(u_1^{(0)}\right)^L$, for $L = 1, 2, \ldots, N$, a direct computation yields that

$$a_2(x, u_2^{(0)}) - a_1(x, u_1^{(0)}) = \sum_{k=1}^{N} a_2^{(k)} \psi^k.$$

Therefore,

$$F_1 - F_2 = \Delta \left(u_1^{(0)} - u_2^{(0)}\right) + a_1(x, u_1^{(0)}) - a_2(x, u_2^{(0)}) = -\Delta\psi - \sum_{k=1}^{N} a_2^{(k)} \psi^k,$$

which shows (1.17). This proves the assertion. □

With Theorem 1.1 at hand, we can prove Theorem 1.2 immediately.

Proof of Theorem 1.2. With the identities (1.16) at hand, as $k = 1$, we have

$$a_1^{(N-1)} = \sum_{m=N-1}^{N} \binom{m}{N-1} a_2^{(m)} \psi^{m-N+1} = a_2^{(N-1)} + N a_2^{(N)} \psi \text{ in } \Omega.$$

Since $a_1^{(N)}(x) = a_2^{(N)}(x) \neq 0$ for all $x \in \Omega$, and $a_1^{(N-1)} = a_2^{(N-1)}$, the preceding equality yields that $\psi = 0$ in Ω. Finally, by applying the (1.16) again as $k = N$, one can prove $F_1 = F_2$ in Ω, which completes the proof. □

We next prove that if the sources F_1 and F_2 are known in Theorem 1.1 and Remark 1.2, then it is possible to determine the coefficients uniquely. We have the following corollary, which we formulate in terms of the general polynomial nonlinearity.

Corollary 3.2. *Let us adopt the notation and assumptions in Theorem 1.1. If $F_1 = F_2$ in Ω, then we have*

$$a_1^{(k)} = a_2^{(k)} \text{ in } \Omega,$$

for $k = 1, 2, \ldots, N$.

Proof. By using (1.17), we have

$$\Delta\psi + \sum_{k=1}^{N} a_2^{(k)} \psi^k = 0 \text{ in } \Omega, \tag{3.44}$$

where $\psi \in C^{2,\alpha}(\overline{\Omega})$ is defined via (3.34), which is a bounded function. Since $a_2^{(k)} \in C^\alpha(\overline{\Omega})$ for $k = 1, 2, \ldots, N$, (3.44) implies that

$$\begin{cases} |\Delta\psi| \leq C|\psi| & \text{in } \Omega, \\ \psi = \partial_\nu \psi = 0 & \text{on } \partial\Omega, \end{cases}$$

for some constant $C > 0$. Applying the unique continuation for differential inequalities (see e.g. [JK85]), one obtains that $\psi = 0$ in Ω. Finally, combining with the relations (1.16), we obtain the uniqueness of coefficients. (To easily see how this final argument goes, see the cubic case and (3.15) first.) □

4. Case studies of Theorem 1.3

In the end of this paper, we study special cases Theorem 1.3, which stated that for general nonlinearities

$$\partial_z^k a_1(x, u_1^{(0)}(x)) = \partial_z^k a_2(x, u_2^{(0)}(x)), \quad x \in \Omega, \quad k \in \mathbb{N}. \tag{4.1}$$

In general, given only the conditions (4.1), it is not clear how explicit relation between the coefficients $(a_1(x,z), F_1(x))$ and $(a_2(x,z), F_2(x))$ in terms of $\psi = u_2^{(0)} - u_1^{(0)}$ one can find. This final section of this paper consider examples where the relation is explicit.

4.1. **Exponential nonlinearity.**

Proof of Theorem Corollary 1.5. We prove cases 1 and 2 separately:
Case 1.

The nonlinearity in this case is $a_j(x,z) = q_j(x)e^z$. Let $u_j^{(0)}$ be the solution to

$$\begin{cases} \Delta u_j^{(0)} + q_j(x) e^{u_j^{(0)}} = F_j & \text{in } \Omega, \\ u_j^{(0)} = f_0 & \text{on } \partial\Omega, \end{cases} \tag{4.2}$$

for $j = 1, 2$. Here $f_0 \in \mathcal{N}$. Using (4.1) with $k = 1$, we have

$$q_1 e^{u_1^{(0)}} = \partial_z a_1(x, u_1^{(0)}) = \partial_z a_2(x, u_2^{(0)}) = q_2 e^{u_2^{(0)}} \text{ in } \Omega. \tag{4.3}$$

On the other hand, by taking $u_2^{(0)} = u_1^{(0)} + \psi$ in Ω, by (4.3) one has $q_1 e^{u_1^{(0)}} = q_2 e^{u_1^{(0)} + \psi}$ which implies $q_1 = q_2 e^{\psi}$ in Ω. Then, by using (4.2), we have

$$F_2 - F_1 = \Delta\big(u_2^{(0)} - u_1^{(0)}\big) + q_2 e^{u_2^{(0)}} - q_1 e^{u_1^{(0)}} = \Delta\psi \text{ in } \Omega,$$

where we have utilized (4.3). This shows (1.25).

For the converse statement, we note that if

$$q_1 = q_2 e^{\psi} \text{ and } F_1 = F_2 - \Delta\psi,$$

and we set $u_2 = u_1 + \psi$, then

$$\begin{aligned} \Delta u_1 + q_1 e^{u_1} = F_1 &\iff \Delta u_2 - \Delta\psi + q_2 e^{\psi} e^{u_2 - \psi} = F_2 - \Delta\psi \\ &\iff \Delta u_2 + q_2 e^{u_2} = F_2. \end{aligned}$$

Since $\psi|_{\partial\Omega} = \partial_\nu|_{\partial\Omega} = 0$, we have the converse statement.
Case 2.

In this case $a_j(x,z) = q_j(x) z e^z$. Let $u_j^{(0)}$ be the solution of

$$\begin{cases} \Delta u_j^{(0)} + q_j u_j^{(0)} e^{u_j^{(0)}} = F_j & \text{in } \Omega, \\ u_j^{(0)} = f_0 & \text{on } \partial\Omega, \end{cases} \tag{4.4}$$

for $j = 1, 2$. The condition (4.1) for $k = 1$ yields

$$Q := q_1 \left(u_1^{(0)} + 1\right) e^{u_1^{(0)}} = q_2 \left(u_2^{(0)} + 1\right) e^{u_2^{(0)}} \text{ in } \Omega, \tag{4.5}$$

and for $k = 2$ it yields

$$q_1 \left(u_1^{(0)} + 2\right) e^{u_1^{(0)}} = q_2 \left(u_2^{(0)} + 2\right) e^{u_2^{(0)}} \text{ in } \Omega. \tag{4.6}$$

Combining (4.5) and (4.6), we obtain

$$q_1 e^{u_1^{(0)}} = q_2 e^{u_2^{(0)}} \quad \text{and} \quad q_1 u_1^{(0)} e^{u_1^{(0)}} = q_2 u_2^{(0)} e^{u_2^{(0)}} \text{ in } \Omega. \tag{4.7}$$

By the first identity of (4.7), we have $q_1 = q_2 e^{u_2^{(0)} - u_1^{(0)}}$ in Ω. The second identity of (4.7) shows that $q_2 u_1^{(0)} e^{u_2^{(0)}} = q_2 u_2^{(0)} e^{u_2^{(0)}}$ in Ω. Since $q_2 \neq 0$ in Ω, we must have

$u_1^{(0)} = u_2^{(0)}$ in Ω, which implies that $F_1 = F_2$ in Ω, where we utilized the equation (4.4). Moreover, by the first identity of (4.7) and $u_1^{(0)} = u_2^{(0)}$ in Ω, we can derive $q_1 = q_2$ in Ω. This proves the assertion. $\square$

If $F_1 = F_2$ in Ω in Corollary 1.5, we have the following uniqueness result regarding the Case 1 in the above corollary.

Corollary 4.1. *Let us assume as in the Case 1 of Corollary 1.5 and adopt its notation. If additionally $F_1 = F_2$, then*

$$q_1 = q_2 \text{ in } \Omega.$$

Proof. Since the source terms in Corollary 1.5 satisfy $F_1 = F_2$ in Ω, it follows from (1.25) that $\Delta\psi = 0$ in Ω with $\psi|_{\partial\Omega} = \partial_\nu \psi|_{\partial\Omega} = 0$. By using the unique continuation principle, we conclude that $\psi \equiv 0$ in Ω. Therefore, combining with (4.3), we must have $q_1 = q_2$ in Ω as desired. $\square$

4.2. **The sine-Gordon equation.** We prove Corollary 1.6.

Proof of Corollary 1.6. We divide the proof into two steps:

Step 1. Gauge invariance.

Let $u_j^{(0)}$ be the solution of

$$\begin{cases} \Delta u_j^{(0)} + q_j \sin(u_j^{(0)}) = F_j & \text{in } \Omega, \\ u_j^{(0)} = f_0 & \text{on } \partial\Omega, \end{cases} \tag{4.8}$$

for $j = 1, 2$ and where $f_0 \in \mathcal{N}$. By Theorem 1.3, we have $\partial_z^k a_1(x, u_1^{(0)}) = \partial_z^k a_2(x, u_2^{(0)})$, for $k = 1, 2$, which implies that

$$q_1 \cos u_1^{(0)} = q_2 \cos u_2^{(0)} \quad \text{and} \quad q_1 \sin u_1^{(0)} = q_2 \sin u_2^{(0)} \text{ in } \Omega. \tag{4.9}$$

By the Euler identity, we have $e^{\mathbf{i}y} = \cos y + \mathbf{i}\sin y$, where $\mathbf{i} = \sqrt{-1}$. Then (4.9) is equivalent to

$$q_1 e^{\mathbf{i}u_1^{(0)}} = q_2 e^{\mathbf{i}u_2^{(0)}} \text{ in } \Omega. \tag{4.10}$$

By defining $\psi = u_2^{(0)} - u_1^{(0)}$, we have that $\psi \in C^{2,\alpha}(\overline{\Omega})$ and $\psi = \partial_\nu \psi = 0$ on $\partial\Omega$. Via the second identity of (4.9) and (4.8), one has

$$\Delta\psi = \Delta\big(u_2^{(0)} - u_1^{(0)}\big) = F_2 - F_1 \text{ in } \Omega,$$

and by (4.10),

$$q_1 e^{\mathbf{i}u_1^{(0)}} = q_2 e^{\mathbf{i}(u_1^{(0)} + \psi)} \text{ in } \overline{\Omega},$$

which implies $q_1 = q_2 e^{\mathbf{i}\psi}$ in $\overline{\Omega}$. Furthermore, since q_1 and q_2 are real-valued functions and ψ is continuous, we must have either $e^{\mathbf{i}\psi} \equiv -1$ or $e^{\mathbf{i}\psi} \equiv 1$ in Ω. Thus

$$q_1 = \pm q_2 \text{ in } \overline{\Omega}. \tag{4.11}$$

It remains to show that

$$e^{\mathbf{i}\psi} = 1 \text{ in } \overline{\Omega}. \tag{4.12}$$

Step 2. Boundary determination.

We show by using boundary determination that $\psi \equiv 1$ in $\overline{\Omega}$. Let ϵ be a small real parameter, $g \in C^{2,\alpha}(\partial\Omega)$ and $f = f_0 + \epsilon g$. By linearizing (1.29) around the solution $u_j^{(0)}$ of (4.8), one has

$$\begin{cases} \left(\Delta + q_j \cos u_j^{(0)}\right) v_j = 0 & \text{in } \Omega, \\ v_j = g & \text{on } \partial\Omega, \end{cases} \tag{4.13}$$

for $j=1,2$. Now, by applying standard boundary determination for the linear Schrödinger equation (4.13), one can determine that

$$q_1\cos(f_0+\epsilon g)=q_1\cos u_1^{(0)}=q_2\cos u_2^{(0)}=q_2\cos(f_0+\epsilon g)\quad \text{ on } \partial\Omega.$$

In particular, for $\epsilon=0$, the above identity shows that

$$q_1\cos(f_0)=q_2\cos(f_0) \text{ on } \partial\Omega.$$

If $\cos(f_0)\equiv 0$ on $\partial\Omega$, we can slightly perturb f_0 so that there is $x_0\in\partial\Omega$ with $\cos(f_0(x_0))\neq 0$ and repeat the above argument again. We deduce that $e^{\mathbf{i}\psi(x_0)}=1$, and since ψ is constant, we conclude that

$$q_1=q_2 \text{ in } \Omega.$$

This proves the claim. □

Appendix A. Proof of Proposition 2.1

Let us prove Proposition 2.1. The proof is almost identical to the proof of Theorem 2.1, but Proposition 2.1 does not exactly follow from Theorem 2.1. A very similar proof can be found from the work [LLLS21, Section 2].

Proof of Proposition 2.1. Let

$$\mathcal{B}_1=C^{2,\alpha}(\partial\Omega),\quad \mathcal{B}_2=C^{\alpha}(\overline{\Omega}),\quad \mathcal{B}_3=C^{2,\alpha}(\overline{\Omega}),\quad \mathcal{B}_4=C^{\alpha}(\overline{\Omega})\times C^{2,\alpha}(\partial\Omega)$$

and consider the map

$$\begin{aligned}\Psi:\ &\mathcal{B}_1\times\mathcal{B}_2\times\mathcal{B}_3\to\mathcal{B}_4,\\ &(f,F,u)\mapsto\left(\Delta u+a(x,u)-F, u|_{\partial\Omega}-f\right).\end{aligned}$$

Similar to [LLLS21, Section 2], one can show that the map $u\mapsto a(x,u)$ is a C^∞ map from $C^{2,\alpha}(\overline{\Omega})\to C^{2,\alpha}(\overline{\Omega})$.

Notice that $\Psi(0,0,0)=(0,0)$, where we have the used condition (1.2). The first linearization of $\Psi=\Psi(f,F,u)$ at $(0,0,0)$ with respect to the variable u is

$$\left.D_u\Psi\right|_{(0,0,0)}(v)=\left(\Delta v+\partial_z a(x,0)v, v|_{\partial\Omega}\right),$$

which is a homeomorphism $\mathcal{B}_3\to\mathcal{B}_4$ by the condition

$$0 \text{ is not a Dirichlet eigenvalue of } \Delta+\partial_z a(x,0) \text{ in } \Omega.$$

This is guaranteed by well-posedness and Schauder estimates for the linear second order elliptic equation .

Now, the implicit function theorem in Banach spaces [RR06, Theorem 10.6 and Remark 10.5] yields that there are $\varepsilon,\delta>0$ and a neighborhood $\mathcal{N}_\delta\times\mathcal{A}_\varepsilon\subset C^{2,\alpha}(\partial\Omega)\times C^{\alpha}(\overline{\Omega})$ and a C^∞ map $\mathcal{S}:\mathcal{N}_\delta\times\mathcal{A}_\varepsilon\to\mathcal{B}_3$ such that

$$\Psi(f,F,\mathcal{S}(f,F))=(0,0),$$

whenever $\|f\|_{C^{2,\alpha}(\partial\Omega)}\leq\delta$ and $\|F\|_{C^{\alpha}(\overline{\Omega})}\leq\varepsilon$. Since $\mathcal{S}$ is smooth and $\mathcal{S}(0,0)=0$, the solution $u=\mathcal{S}(f,F)$ satisfies

$$\|u\|_{C^{2,\alpha}(\overline{\Omega})}\leq C\left(\|f\|_{C^{2,\alpha}(\partial\Omega)}+\|F\|_{C^{\alpha}(\overline{\Omega})}\right).$$

Furthermore, by the uniqueness statement of the implicit function theorem, $u=\mathcal{S}(f,F)$ is the only solution to $\Psi(f,F,u)=(0,0)$ whenever $\|f\|_{C^{2,\alpha}(\partial\Omega)}+\|F\|_{C^{\alpha}(\overline{\Omega})}\leq\delta+\varepsilon$, and

$$\|u\|_{C^{2,\alpha}(\overline{\Omega})}\leq C(\varepsilon+\delta).$$

This can be achieved by redefining $\varepsilon,\delta>0$ to be smaller if necessary. As in [LLLS21], one can check that the solution operator $\mathcal{S}:\mathcal{N}_\delta\times\mathcal{A}_\varepsilon\to C^{2,\alpha}(\overline{\Omega})$ is a C^∞ map in the Fréchet sense. The normal derivative is a linear map $C^{2,\alpha}(\overline{\Omega})\to C^{1,\alpha}(\partial\Omega)$. Thus for a fixed $F\in\mathcal{A}_\varepsilon$, $\Lambda_F: f\mapsto\partial_\nu u_{f,F}$, where $u_{f,F}$ solves $\Delta u_{f,F}+$

$a(x, u_{f,F}) = F$ with $u_{f,F}|_{\partial\Omega} = f$, is also a well defined C^∞ map $\mathcal{N}_\delta \to C^{1,\alpha}(\partial\Omega)$. $\square$

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References

[AZ21] Yernat M Assylbekov and Ting Zhou. Direct and inverse problems for the nonlinear time-harmonic Maxwell equations in Kerr-type media. *J. Spectr. Theory*, 11:1–38, 2021.

[BCP22] Elena Beretta, M Cristina Cerutti, and Dario Pierotti. On a nonlinear model in domains with cavities arising from cardiac electrophysiology. *Inverse Problems*, 38(10):105005, 2022.

[BK89] AB Borisov and VV Kiseliev. Inverse problem for an elliptic sine-Gordon equation with an asymptotic behaviour of the cnoidal-wave type. *Inverse Problems*, 5(6):959, 1989.

[BTW19] Emilia Blåsten, Leo Tzou, and Jenn-Nan Wang. Uniqueness for the inverse boundary value problem with singular potentials in 2D. *Mathematische Zeitschrift*, pages 1–15, 2019.

[Buk08] Alexander Bukhgeim. Recovering a potential from Cauchy data in the two-dimensional case. *Journal of Inverse and Ill-posed Problems, V*, 16(1):19–33, 2008.

[CFK+21] Cătălin I Cârstea, Ali Feizmohammadi, Yavar Kian, Katya Krupchyk, and Gunther Uhlmann. The calderón inverse problem for isotropic quasilinear conductivities. *Advances in Mathematics*, 391:107956, 2021.

[CLLO22] Cătălin I Cârstea, Matti Lassas, Tony Liimatainen, and Lauri Oksanen. An inverse problem for the Riemannian minimal surface equation. *arXiv preprint arXiv:2203.09262*, 2022.

[CNV19] Cătălin I Cârstea, Gen Nakamura, and Manmohan Vashisth. Reconstruction for the coefficients of a quasilinear elliptic partial differential equation. *Applied Mathematics Letters*, 98:121–127, 2019.

[DLL19] Youjun Deng, Jinhong Li, and Hongyu Liu. On identifying magnetized anomalies using geomagnetic monitoring. *Archive for Rational Mechanics and Analysis*, 231(1):153–187, 2019.

[DLL20] Youjun Deng, Jinhong Li, and Hongyu Liu. On identifying magnetized anomalies using geomagnetic monitoring within a magnetohydrodynamic model. *Archive for Rational Mechanics and Analysis*, 235(1):691–721, 2020.

[FK14] Svatopluk Fucik and Alois Kufner. *Nonlinear differential equations*. Elsevier, 2014.

[FKU21] Ali Feizmohammadi, Yavar Kian, and Gunther Uhlmann. An inverse problem for a quasilinear convection–diffusion equation. *Nonlinear Analysis*, in press, 2021.

[FLL21] Ali Feizmohammadi, Tony Liimatainen, and Yi-Hsuan Lin. An inverse problem for a semilinear elliptic equation on conformally transversally anisotropic manifolds. *arXiv preprint arXiv:2112.08305*, 2021.

[FO20] Ali Feizmohammadi and Lauri Oksanen. An inverse problem for a semi-linear elliptic equation in Riemannian geometries. *Journal of Differential Equations*, 269(6):4683–4719, 2020.

[FP12] AS Fokas and Beatrice Pelloni. The Dirichlet-to-Neumann map for the elliptic sine-Gordon equation. *Nonlinearity*, 25(4):1011, 2012.

[GT83] David Gilbarg and Neil S Trudinger. *Elliptic partial differential equations of second order*. Second edition, Springer, 1983.

[HL23] Bastian Harrach and Yi-Hsuan Lin. Simultaneous recovery of piecewise analytic coefficients in a semilinear elliptic equation. *Nonlinear Analysis*, 228:113188, 2023.

[Isa93] Victor Isakov. On uniqueness in inverse problems for semilinear parabolic equations. *Archive for Rational Mechanics and Analysis*, 124(1):1–12, 1993.

[Isa01] Victor Isakov. Uniqueness of recovery of some systems of semilinear partial differential equations. *Inverse Problems*, 17(4):607–618, 2001. Special issue to celebrate Pierre Sabatier's 65th birthday (Montpellier, 2000).

[JK85] David Jerison and Carlos E Kenig. Unique continuation and absence of positive eigenvalues for Schrödinger operators. *Annals of Mathematics*, 121(3):463–488, 1985.

[KKU22] Yavar Kian, Katya Krupchyk, and Gunther Uhlmann. Partial data inverse problems for quasilinear conductivity equations. *Mathematische Annalen*, pages 1–28, 2022.

[KLU18] Yaroslav Kurylev, Matti Lassas, and G. Uhlmann. Inverse problems for Lorentzian manifolds and non-linear hyperbolic equations. *Invent. Math.*, 212(3):781–857, 2018.

[KN02] Hyeonbae Kang and Gen Nakamura. Identification of nonlinearity in a conductivity equation via the Dirichlet-to-Neumann map. *Inverse Problems*, 18:1079–1088, 2002.

[KU20a] Katya Krupchyk and Gunther Uhlmann. Partial data inverse problems for semilinear elliptic equations with gradient nonlinearities. *Mathematical Research Letters*, 27(6):1801–1824, 2020.

[KU20b] Katya Krupchyk and Gunther Uhlmann. A remark on partial data inverse problems for semilinear elliptic equations. *Proc. Amer. Math. Soc.*, 148:681–685, 2020.

[Lin22] Yi-Hsuan Lin. Monotonicity-based inversion of fractional semilinear elliptic equations with power type nonlinearities. *Calculus of Variations and Partial Differential Equations*, 61(5):1–30, 2022.

[LL19] Ru-Yu Lai and Yi-Hsuan Lin. Global uniqueness for the fractional semilinear Schrödinger equation. *Proc. Amer. Math. Soc.*, 147(3):1189–1199, 2019.

[LL22a] Ru-Yu Lai and Yi-Hsuan Lin. Inverse problems for fractional semilinear elliptic equations. *Nonlinear Analysis*, 216:112699, 2022.

[LL22b] Yi-Hsuan Lin and Hongyu Liu. Inverse problems for fractional equations with a minimal number of measurements. *arXiv preprint arXiv:2203.03010*, 2022.

[LLL21] Yi-Hsuan Lin, Hongyu Liu, and Xu Liu. Determining a nonlinear hyperbolic system with unknown sources and nonlinearity. *arXiv preprint arXiv:2107.10219*, 2021.

[LLLS20] Matti Lassas, Tony Liimatainen, Yi-Hsuan Lin, and Mikko Salo. Partial data inverse problems and simultaneous recovery of boundary and coefficients for semilinear elliptic equations. *Revista Matemática Iberoamericana*, 37(4):1553–1580, 2020.

[LLLS21] Matti Lassas, Tony Liimatainen, Yi-Hsuan Lin, and Mikko Salo. Inverse problems for elliptic equations with power type nonlinearities. *Journal de mathématiques pures et appliquées*, 145:44–82, 2021.

[LLLZ22] Yi-Hsuan Lin, Hongyu Liu, Xu Liu, and Shen Zhang. Simultaneous recoveries for semilinear parabolic systems. *Inverse Problems*, 38(11):115006, 2022.

[LLM19] Jingzhi Li, Hongyu Liu, and Shiqi Ma. Determining a random Schröinger equation with unknown source and potential. *SIAM Journal on Mathematical Analysis*, 51(4):3465–3491, 2019.

[LLM21] Jingzhi Li, Hongyu Liu, and Shiqi Ma. Determining a random Schrödinger operator: both potential and source are random. *Communications in Mathematical Physics*, 381(2):527–556, 2021.

[LLS20] Matti Lassas, Tony Liimatainen, and Mikko Salo. The Poisson embedding approach to the Calderón problem. *Mathematische Annalen*, 377(1):19–67, 2020.

[LLST22] Tony Liimatainen, Yi-Hsuan Lin, Mikko Salo, and Teemu Tyni. Inverse problems for elliptic equations with fractional power type nonlinearities. *Journal of Differential Equations*, 306:189–219, 2022.

[LO22] Ru-Yu Lai and Laurel Ohm. Inverse problems for the fractional laplace equation with lower order nonlinear perturbations. *Inverse Problems and Imaging*, 16(2):305–323, 2022.

[LU15] Hongyu Liu and Gunther Uhlmann. Determining both sound speed and internal source in thermo-and photo-acoustic tomography. *Inverse Problems*, 31(10):105005, 2015.

[LZ21] Ru-Yu Lai and Ting Zhou. An inverse problem for non-linear fractional magnetic Schrödinger equation. *arXiv preprint arXiv:2103.08180*, 2021.

[Nur22] Janne Nurminen. An inverse problem for the minimal surface equation. *arXiv preprint arXiv:2203.09272*, 2022.

[PS16] Lev Pitaevskii and Sandro Stringari. *Bose-Einstein condensation and superfluidity*, volume 164. Oxford University Press, 2016.

[RR06] Michael Renardy and Robert C Rogers. *An introduction to partial differential equations*, volume 13. Springer Science & Business Media, 2006.

[SU87] John Sylvester and Gunther Uhlmann. A global uniqueness theorem for an inverse boundary value problem. *Annals of Mathematics*, 125(1):153–169, 1987.

[SU97] Ziqi Sun and Gunther Uhlmann. Inverse problems in quasilinear anisotropic media. *American journal of mathematics*, 119(4):771–797, 1997.

[Sun96] Ziqi Sun. On a quasilinear inverse boundary value problem. *Mathematische Zeitschrift*, 221(1):293–305, 1996.

[Sun10] Ziqi Sun. An inverse boundary-value problem for semilinear elliptic equations. *Electronic Journal of Differential Equations (EJDE)[electronic only]*, 2010:Paper–No, 2010.

[Vol14] Vitaly Volpert. *Elliptic Partial Differential Equations: Volume 2: Reaction-Diffusion Equations*, volume 104. Springer, 2014.

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