OPTIMAL RUNGE APPROXIMATION FOR NONLOCAL WAVE EQUATIONS AND UNIQUE DETERMINATION OF POLYHOMOGENEOUS NONLINEARITIES

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Abstract. The main purpose of this article is to establish the Runge-type approximation in $L^2(0,T;\widetilde{H}^s(\Omega))$ for solutions of linear nonlocal wave equations. To achieve this, we extend the theory of very weak solutions for classical wave equations to our nonlocal framework. This strengthened Runge approximation property allows us to extend the existing uniqueness results for Calderón problems of linear and nonlinear nonlocal wave equations in our earlier works. Furthermore, we prove unique determination results for the Calderón problem of nonlocal wave equations with polyhomogeneous nonlinearities.

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Contents

1. Introduction

In recent years, inverse problems for nonlocal partial differential equations (PDEs) have been extensively studied in the literature. The first work in this field is [GSU20], in which the authors considered the so-called *Calderón problem* for the *fractional Schrödinger equation*

$$((-\Delta)^s + q)u = 0 \text{ in } \Omega, \tag{1.1}$$

where $\Omega \subset \mathbb{R}^n$ is a bounded domain. Here $(-\Delta)^s$ denotes the *fractional Laplacian* for $0 < s < 1$, and $q \in L^\infty(\Omega)$ is a bounded potential. In this problem, one asks whether it is possible to uniquely recover the potential q from the *Dirichlet-to-Neumann (DN) map*

$$\Lambda_q f := (-\Delta)^s u_f\big|_{\Omega_e}, \tag{1.2}$$

where $\Omega_e = \mathbb{R}^n \setminus \overline{\Omega}$ denotes the exterior of Ω, $f\colon \Omega_e \to \mathbb{R}$ is a given Dirichlet data, and $u_f\colon \mathbb{R}^n \to \mathbb{R}$ is the unique solution of (1.1) with $u_f|_{\Omega_e} = f$. In [GSU20], the authors found that the fractional Laplacian satisfies the *unique continuation principle (UCP)* that asserts:

(UCP). *Let $s \in \mathbb{R}_+ \setminus \mathbb{N}$, $t \in \mathbb{R}$ and $V \subset \mathbb{R}^n$ be an open set. If $u \in H^t(\mathbb{R}^n)$ satisfies*

$$u = (-\Delta)^s u = 0 \text{ in } V \text{ then } u \equiv 0 \text{ in } \mathbb{R}^n.$$

In [GSU20] the UCP has been shown for the range $0 < s < 1$ and via iteration by the Laplacian the UCP extends to the range $s \in \mathbb{R}_+ \setminus \mathbb{N}$. Furthermore, they observe that the fractional Laplacian has, as a consequence of a duality argument (Hahn–Banach theorem) and the UCP, the so-called *Runge approximation property*. For any open set $W \subset \Omega_e$, this property can be phrased in two alternative ways:

(i) The *Runge set*

$$\mathcal{R}_W := \left\{ u_f\big|_\Omega ;\, f \in C_c^\infty(W) \right\}$$

is dense in $L^2(\Omega)$, where u_f is the unique solution to (1.2) with exterior value $f \in C_c^\infty(W)$ (cf. [GSU20]).

(ii) The *Runge set*

$$\mathscr{R}_W := \left\{ u_f - f ;\, f \in C_c^\infty(W) \right\}$$

is dense in $\widetilde{H}^s(\Omega)$ (cf. [RS20]).

Together with a suitable integral identity, the result (i) allowed the authors of [GSU20] to uniquely recover bounded potentials, whereas the property (ii) was used in [RS20] to recover certain singular potentials.

The above strategy to establish uniqueness for Calderón-type inverse problems of elliptic or parabolic nonlocal equations has lately been investigated in several research articles, such as [GLX17, CLR20, CLL19, LL22, LL23, LZ23, KLZ24, KLW22, LRZ22, LTZ24, LLU23, CGRU23, LLU23, RZ23, RZ24, CRTZ22, LZ24, FGKU21, Fei21, FKU24]. Some articles of this list consider the detection of linear perturbations as in the above fractional Schrödinger equation (1.1), while others allowed nonlinear perturbations, or even studied the identification of leading order coefficients in the main nonlocal term in the considered PDE.

1.1. The mathematical model and main results.

In this article, we study Calderón type inverse problems for linear and nonlinear *nonlocal wave equations (NWEQs)* formulated generically as

$$\begin{cases} \partial_t^2 u + (-\Delta)^s u + f(x,u) = 0 & \text{in } \Omega_T, \\ u = \varphi & \text{in } (\Omega_e)_T, \\ u(0) = \partial_t u(0) = 0 & \text{in } \Omega, \end{cases} \tag{1.3}$$

where $f\colon \Omega\times\mathbb{R}\to\mathbb{R}$ is a suitable nonlinearity. Here we use the notation

$$A_t := A\times(0,t), \text{ for any } A\subset\mathbb{R}^n \text{ and } t>0.$$

Let us note that nonlocal wave equations such as (1.3) arise in a special case of *peridynamics* — theory of studying dynamics of materials with discontinuities such as fractures (see [Sil16]).

By [LTZ24, Theorem 3.1 and 3.6] the equation (1.3) is well-posed for regular exterior values $\varphi\colon\Omega_e\to\mathbb{R}$, when $f(x,\tau)=q(x)\tau$ with $q\in L^p(\Omega)$, where $1\leq p\leq\infty$ satisfies

$$\begin{cases} n/s\leq p\leq\infty, & \text{if } 2s<n,\\ 2<p\leq\infty, & \text{if } 2s=n,\\ 2\leq p\leq\infty, & \text{if } 2s\geq n,\end{cases} \tag{1.4}$$

or f satisfies the following assumption:

Assumption 1. *We say that a Carathéodory function $f\colon\Omega\times\mathbb{R}\to\mathbb{R}$ is a* weak nonlinearity *if it satisfies the following conditions:*

(i) f has partial derivative $\partial_\tau f$, which is a Carathéodory function, and there exists $a\in L^p(\Omega)$ such that

$$|\partial_\tau f(x,\tau)|\lesssim a(x)+|\tau|^r$$

for all $\tau\in\mathbb{R}$ and a.e. $x\in\Omega$. Here the exponents p and r satisfy the restrictions (1.4) and

$$\begin{cases} 0\leq r<\infty, & \text{if } 2s\geq n,\\ 0\leq r\leq\frac{2s}{n-2s}, & \text{if } 2s<n,\end{cases} \tag{1.5}$$

respectively. Moreover, f fulfills the integrability condition $f(\cdot,0)\in L^2(\Omega)$.

(ii) There is a constant $C_1>0$ such that the function $F\colon\Omega\times\mathbb{R}\to\mathbb{R}$ defined via

$$F(x,\tau)=\int_0^\tau f(x,\rho)\,d\rho$$

satisfies $F(x,\tau)\geq -C_1$ for all $\tau\in\mathbb{R}$ and $x\in\Omega$.

Observe that given a function $0\leq q\in L^\infty(\Omega)$, an example of a nonlinearity f, which satisfies the conditions in Assumption 1, is a fractional power type nonlinearity $f(x,\tau)=q(x)|\tau|^r\tau$ for $r\geq 0$ with r satisfying (1.5). We refer readers to [LTZ24, Section 3] for more details.

Assuming the well-posedness of (1.3) for suitable nonlinearities f, as a generalization of the Calderón problem for the fractional Schrödinger equation, we aim to determine the nonlinearity $f(x,\tau)$ from the DN map Λ_f related to (1.3), which can be formally defined by

$$\Lambda_f\varphi := (-\Delta)^s u_\varphi\big|_{(\Omega_e)_T},$$

where $u_\varphi\colon\mathbb{R}^n_T\to\mathbb{R}$ denotes the unique solution to (1.3) (cf. eq. (1.2)). Next, let us make a few remarks on this inverse problem.

(a) *Linear perturbations:* In [KLW22], the uniqueness of this inverse problem has been established in the case $f(x,u)=q(x)u$ with $q\in L^\infty(\Omega)$. In [LTZ24, Corollary 1.4], it has been shown that uniqueness still holds if $0\leq q\in L^p(\Omega)$ with p satisfying (1.4).

(b) *Semilinear perturbations:* In [LTZ24, Theorem 1.1], we showed that uniqueness holds even when
   - (A) f is a weak nonlinearity (see Assumption 1) with F, r satisfying
     - (A1) $F\geq 0$,
     - (A2) $0<r\leq 1$

(B) and f is $(r+1)-$homogeneous.

All of these results rely on a $L^2(\Omega_T)$ Runge approximation property of linear nonlocal wave equations (cf. (i)):

Proposition 1.1 (Runge approximation, [LTZ24, Proposition 4.1]). *Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain, $W \subset \Omega_e$ an arbitrary open set, $s > 0$ a non-integer and $T > 0$. Suppose that $q \in L^p(\Omega)$ is nonnegative*[1]*, where p is given by (1.4). Consider the* Runge set

$$\mathcal{R}_W := \{u_\varphi|_{\Omega_T} ; \varphi \in C_c^\infty(W_T)\},$$

where $u_\varphi \in C([0,T]; H^s(\mathbb{R}^n)) \cap C^1([0,T]; L^2(\mathbb{R}^n))$ is the unique solution to

$$\begin{cases} \partial_t^2 u + (-\Delta)^s u + qu = 0 & \text{in } \Omega_T, \\ u = \varphi & \text{in } (\Omega_e)_T, \\ u(0) = \partial_t u(0) = 0 & \text{in } \Omega. \end{cases} \tag{1.6}$$

Then $\mathcal{R}_W$ is dense in $L^2(\Omega_T)$.

A main contribution of this article is to settle a question raised in [Zim24, p. 3], namely:

Question 1. *Let us adopt all notation of Proposition 1.1. Is the Runge set*

$$\mathscr{R}_W := \{u_\varphi - \varphi ; \varphi \in C_c^\infty(W_T)\}$$

dense in $L^2(0,T; \widetilde{H}^s(\Omega))$?

By extending the theory of very weak solutions to linear nonlocal wave equations in Section 3, we show in Section 4 the following Runge approximation.

Theorem 1.2 (Runge approximation). *Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain, $W \subset \Omega_e$ an arbitrary open set, $s > 0$ a non-integer, $T > 0$ and $q \in L^p(\Omega)$ with p satisfying the restrictions (1.4). Consider the* Runge set

$$\mathscr{R}_W := \{u_\varphi - \varphi ; \varphi \in C_c^\infty(W_T)\} \subset L^2(0,T; \widetilde{H}^s(\Omega)),$$

where $u_\varphi \in C([0,T]; H^s(\mathbb{R}^n)) \cap C^1([0,T]; L^2(\mathbb{R}^n))$ is the unique solution to (1.6). Then $\mathscr{R}_W$ is dense in $L^2(0,T; \widetilde{H}^s(\Omega))$.

With this Runge approximation and a suitable integral identity, similar to the one demonstrated in [KLW22] or [Zim24], at hand, we can recover L^p-regular linear perturbations, which are not necessarily nonnegative (cf. (a)).

Theorem 1.3. *Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain, $W \subset \Omega_e$ an nonempty open set, $s > 0$ a non-integer, $T > 0$ and $q_j \in L^p(\Omega)$ with p satisfying the restrictions (1.4) for $j = 1, 2$. Let Λ_{q_j} be the DN maps of*

$$\begin{cases} \partial_t^2 u + (-\Delta)^s u + q_j u = 0 & \text{in } \Omega_T, \\ u_j = \varphi & \text{in } (\Omega_e)_T, \\ u_j(0) = \partial_t u_j(0) = 0 & \text{in } \Omega, \end{cases}$$

for $j = 1, 2$, satisfying

$$\Lambda_{q_1}\varphi\big|_{(W_2)_T} = \Lambda_{q_2}\varphi\big|_{(W_2)_T}, \text{ for any } \varphi \in C_c^\infty((W_1)_T). \tag{1.7}$$

Then there holds $q_1 = q_2$ in Ω.

[1]This assumption is included for simplicity and the result remains true, for example, if one assumes instead $q \in L^\infty(\Omega)$.

Next, we present our main results on inverse problems for nonlinear nonlocal wave equations. To this purpose, let us introduce two different types of nonlinearities having a polyhomogeneous structure.

Definition 1.4 (Polyhomogeneous nonlinearities). *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain. Suppose we are given a sequence $\mathfrak{r} := (r_k)_{k=1}^{\infty} \subset \mathbb{R}$ satisfying $0 < r_k < r_{k+1}$ for all $k \in \mathbb{N}$ and let $f\colon \Omega \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function.*

(i) We call f serially $\mathfrak{r}$-polyhomogeneous*, if*

$$f(x,\tau) = \sum_{k\geq 1} f_k(x,\tau),$$

where each expansion coefficient $f_k\colon \Omega\times\mathbb{R}\to\mathbb{R}$ is a Carathéodory function satisfying

$$|f_k(x,\tau)| \leq b_k|\tau|^{r_k+1} \tag{1.8}$$

for some constants $b_k \geq 0$ and is (r_k+1)-homogeneous in the τ-variable.

(ii) We call f asymptotically $\mathfrak{r}$-polyhomogeneous*, denoted as*

$$f(x,\tau) \sim \sum_{k\geq 1} f_k(x,\tau),$$

if each expansion coefficient $f_k\colon \Omega\times\mathbb{R}\to\mathbb{R}$ is a Carathéodory function satisfying (1.8), is (r_k+1)-homogeneous in the τ-variable and for all $N \in \mathbb{N}_{\geq 2}$ there is a constant $C_N > 0$ such that

$$\left| f(x,\tau) - \sum_{k=1}^{N-1} f_k(x,\tau)\right| \leq C_N|\tau|^{r_N+1}, \quad |\tau|\leq 1,\ x\in\Omega. \tag{1.9}$$

With Assumption 1 and Definition 1.4 at hand, we can state our main results for inverse problems of nonlinear nonlocal wave equations.

Theorem 1.5 (Recovery of expansion coefficients). *Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain, $T > 0$ and $s > 0$ a non-integer. Let $W_1, W_2 \subset \Omega_e$ be open sets. Suppose the nonlinearities $f^{(j)}$ satisfy the conditions in Assumption 1 with $F^{(1)}, F^{(2)} \geq 0$, $a^{(1)}, a^{(2)} \in L^\infty(\Omega)$, $r^{(1)} = r^{(2)} = r_\infty$, and are serially or asymptotically $\mathfrak{r}$-polyhomogeneous such that the corresponding (strictly) monotonically increasing sequence $\mathfrak{r} := (r_k)_{k\in\mathbb{N}}$ fulfills $\mathfrak{r} \subset (0, r_\infty]$ (see Definition 1.4). Assume that the DN maps $\Lambda_{f^{(j)}}$ of*

$$\begin{cases} \partial_t^2 u + (-\Delta)^s u + f^{(j)}(x,u) = 0 & \text{in } \Omega_T,\\ u = \varphi & \text{in } (\Omega_e)_T,\\ u(0) = \partial_t u(0) = 0 & \text{in } \Omega, \end{cases}$$

for $j = 1, 2$, coincide;

$$\Lambda_{f^{(1)}}\varphi\big|_{(W_2)_T} = \Lambda_{f^{(2)}}\varphi\big|_{(W_2)_T} \text{ for any } \varphi \in C_c^\infty((W_1)_T). \tag{1.10}$$

(i) If $f^{(j)}$ are serially $\mathfrak{r}$-polyhomogeneous with

$$L_j := \limsup_{k\to\infty} \frac{b_{k+1}^{(j)}}{b_k^{(j)}} < 1, \tag{1.11}$$

then $f^{(1)} = f^{(2)}$.

(ii) If $f^{(j)}$ are asymptotically $\mathfrak{r}$-polyhomogeneous such that (1.9) holds for all $\tau \in \mathbb{R}$ in the cases $2s \leq n$, then $f_k^{(1)}(x,\tau) = f_k^{(2)}(x,\tau)$ for all $x \in \Omega$, $\tau \in \mathbb{R}$ and all $k \in \mathbb{N}$.

1.2. **Comparison to inverse problems for local wave equations.** The inverse problem of recovering coefficient functions in linear wave equations is classical and the first results in this direction were by Belishev and Kurylev using a boundary control method [Bel87, BK92], see also [KKL01]. Nowadays there are also results in reconstructing Riemannian manifolds for linear wave and other equations from partial data boundary measurements, such as in [AKK+04, HLOS18, IKL17, KKLO19, KKL08, KOP18, LO14]. However, in the linear case these results are based on the boundary control method, which requires the (lower order) coefficient functions to be time-independent or real-analytic in time, see [Esk07]. Due to strong unique continuation and Runge approximation properties, in the nonlocal case such restrictions are not needed, and one can consider much lower regularity coefficients, such as in Theorem 1.5 (which also includes the linear case, see also [LTZ24]). Moreover, nonlocal wave equations often enjoy infinite speed of propagation of the solution, and hence one can recover coefficients in larger domains than in the local case, where causality forces restrictions on the possible domains of reconstruction.

Recently, inverse problems for nonlinear (local) wave equations have become mainstream. Let us elaborate on several works in this research field. The nonlinear (self-)interaction of waves will generate new waves, and this effect can be treated as a benefit in solving related inverse problems in the hyperbolic and elliptic settings. The seminal work [KLU18] demonstrated that local measurements can be utilized to recover global topology and differentiable structure uniquely for a semilinear wave equation with a quadratic nonlinearity. Furthermore, general semilinear wave equations on Lorentzian manifolds and related inverse problems were studied for the Einstein-Maxwell equation in [LUW18] and [LUW17], respectively. We also refer readers to [dHUW18, KLOU22, LLL24, LLPMT22, LLPMT21, LLPMT24, WZ19] and references therein for different inverse problems settings.

1.3. **Organization of the paper.** The remainder of this paper is arranged as follows. In Section 2, we introduce several function spaces used in this paper. In Section 3, we show that there exists a unique very weak solution to linear nonlocal wave equations. In Section 4, we prove a stronger version of Runge approximation and use it to solve an inverse problem for linear nonlocal wave equations. Finally, we prove Theorem 1.5 in Section 5.

2. Preliminaries

In this section, we introduce some notation, which will be used throughout the whole article, and use this occasion to recall several basic facts on fractional Sobolev spaces as well as the fractional Laplacian.

Throughout this article, we use the following convention for the Fourier transform

$$\mathcal{F}u(\xi) := \hat{u}(\xi) := \int_{\mathbb{R}^n} u(x)e^{-\mathrm{i}x\cdot\xi}\,dx$$

for functions $u\colon \mathbb{R}^n \to \mathbb{R}$ for example in the Schwartz space $\mathscr{S}(\mathbb{R}^n)$, where i is the imaginary unit. By duality, the Fourier transform can be extended to the space of tempered distributions $\mathscr{S}'(\mathbb{R}^n) = (\mathscr{S}(\mathbb{R}^n))^*$, and we use the same notation for it. We denote the inverse Fourier transform by $\mathcal{F}^{-1}$.

For any $s \in \mathbb{R}$, we let $H^s(\mathbb{R}^n)$ stand for the *fractional Sobolev space*, which consists of all tempered distributions u such that

$$\|u\|_{H^s(\mathbb{R}^n)} := \|\langle D\rangle^s u\|_{L^2(\mathbb{R}^n)} < \infty,$$

where $\langle D\rangle^s$ denotes the Bessel potential of order s with the Fourier symbol $\langle\xi\rangle^s := \left(1+|\xi|^2\right)^{s/2}$. We shall also need the following local versions of the fractional

Sobolev spaces $H^s(\mathbb{R}^n)$:

$$H^s(\Omega) := \{u|_\Omega \ ; \ u \in H^s(\mathbb{R}^n)\},$$
$$\widetilde{H}^s(\Omega) := \text{closure of } C_c^\infty(\Omega) \text{ in } H^s(\mathbb{R}^n),$$

where $\Omega \subset \mathbb{R}^n$ is an open set and $H^s(\Omega)$ naturally endowed with the quotient norm

$$\|u\|_{H^s(\Omega)} := \inf\left\{\|U\|_{H^s(\mathbb{R}^n)} \ ; \ U \in H^s(\mathbb{R}^n) \text{ and } U|_\Omega = u\right\}.$$

Furthermore, we set

$$\widetilde{L}^2(\Omega) := \widetilde{H}^0(\Omega) \quad \text{and} \quad \|\cdot\|_{\widetilde{L}^2(\Omega)} := \|\cdot\|_{\widetilde{H}^0(\Omega)} = \|\cdot\|_{L^2(\mathbb{R}^n)}.$$

Let us also emphasize that if Ω is a Lipschitz domain then one has the following identification

$$(\widetilde{H}^s(\Omega))^* = H^{-s}(\Omega).$$

The *fractional Laplacian* of order $s > 0$ is the homogeneous counterpart of the Bessel potential and hence is the Fourier multiplier

$$(-\Delta)^s u = \mathcal{F}^{-1}\big(|\xi|^{2s}\,\widehat{u}(\xi)\big), \text{ for } u \in \mathscr{S}(\mathbb{R}^n).$$

It is not difficult to see that an equivalent norm on $H^s(\mathbb{R}^n)$ is given by

$$\|u\|^*_{H^s(\mathbb{R}^n)} = \|u\|_{L^2(\mathbb{R}^n)} + \big\|(-\Delta)^{s/2}u\big\|_{L^2(\mathbb{R}^n)}, \tag{2.1}$$

and the fractional Laplacian is a bounded linear operator as a map $(-\Delta)^s\colon H^t(\mathbb{R}^n) \to H^{t-2s}(\mathbb{R}^n)$ for all $s \geq 0$ and $t \in \mathbb{R}$. In fact, one can also write $(-\Delta)^s = (-\Delta)^k(-\Delta)^\alpha$, where $s = k + \alpha$ with $k = \lfloor s \rfloor \in \mathbb{N} \cup \{0\}$ and $\alpha = s - k \in (0,1)$.

Using the Hardy–Littlewood–Sobolev lemma and Hölder's inequality, one can easily see that the following Poincaré inequality holds:

Proposition 2.1 (Poincaré inequality (cf. [RZ23, Lemma 5.4]))**.** *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain. For any $s > 0$, there exists $C > 0$ such that*

$$\|u\|_{L^2(\Omega)} \leq C\big\|(-\Delta)^{s/2}u\big\|_{L^2(\mathbb{R}^n)}, \text{ for any } u \in \widetilde{H}^s(\Omega).$$

Taking into account that (2.1) is an equivalent norm on $\widetilde{H}^s(\Omega)$, the Poincaré inequality (Propositions 2.1) ensures the following simple lemma, which will be used throughout the whole article.

Lemma 2.2. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain and $s \geq 0$. Then an equivalent norm on $\widetilde{H}^s(\Omega)$ is given by*

$$\|u\|_{\widetilde{H}^s(\Omega)} = \big\|(-\Delta)^{s/2}u\big\|_{L^2(\mathbb{R}^n)},$$

which is induced by the inner product

$$\langle u, v\rangle_{\widetilde{H}^s(\Omega)} = \big\langle(-\Delta)^{s/2}u, (-\Delta)^{s/2}v\big\rangle_{L^2(\mathbb{R}^n)}.$$

Finally, let us mention that if X is a Banach space, then we denote by $C^k([a,b]\,;X)$ and $L^p(a,b\,;X)$ ($k \in \mathbb{N}, 1 \leq p \leq \infty$) the space of k-times continuously differentiable functions and the space of measurable functions $u\colon (a,b) \to X$ such that $t \mapsto \|u(t)\|_X \in L^p([a,b])$, respectively. These Banach spaces are endowed with the norms

$$\|u\|_{L^p(a,b\,;X)} := \left(\int_a^b \|u(t)\|_X^p \, dt\right)^{1/p} < \infty,$$
$$\|u\|_{C^k([a,b];X)} := \sup_{0 \leq \ell \leq k} \big\|\partial_t^\ell u\big\|_{L^\infty(a,b;X)}$$

(with the usual modification for $p = \infty$).

3. Existence and uniqueness of very weak solutions to linear nonlocal wave equations

The purpose of this section is to extend the well-established theory of *very weak solutions* to linear wave equations in our nonlocal setting. In Section 3.1, we motivate and present the rigorous definition of these solutions. Afterward, in Section 3.2 we formulate a spectral theoretic lemma, which we need later in Section 3.3 for the construction of very weak solutions to NWEQs without a potential term. In Section 3.4, we then establish via a fixed point argument the well-posedness theory of very weak solutions to linear NWEQs with a nonzero potential. Finally, in Section 3.5 we discuss some properties of very weak solutions. In particular, we show that all weak solutions are very weak solutions, which in turn are distributional solutions.

Throughout the whole section $\Omega \subset \mathbb{R}^n$ denotes a fixed bounded Lipschitz domain and $s \in \mathbb{R}_+ \setminus \mathbb{N}$. As usual $\langle \cdot, \cdot \rangle$ denotes the duality pairing between $\widetilde{H}^s(\Omega)$ and $H^{-s}(\Omega)$. These Hilbert spaces are endowed with the norms $\|\cdot\|_{\widetilde{H}^s(\Omega)}$, introduced in Lemma 2.2, and

$$\|G\|_{H^{-s}(\Omega)} = \sup\left\{ |\langle G, v\rangle|;\, v \in \widetilde{H}^s(\Omega),\, \|v\|_{\widetilde{H}^s(\Omega)} = 1 \right\}.$$

3.1. Definition of very weak solutions.

Next we introduce the notion of very weak solutions to linear NWEQs with possibly a nonzero potential q.

Definition 3.1. *Let $F \in L^2(0,T;H^{-s}(\Omega))$, $u_0 \in \widetilde{L}^2(\Omega)$, $u_1 \in H^{-s}(\Omega)$ and $q \in L^p(\Omega)$, where the exponent p satisfies the restriction (1.4). A function $u\colon \mathbb{R}^n_T \to \mathbb{R}$ is called a* very weak solution *of*

$$\begin{cases} \partial_t^2 u + (-\Delta)^s u + qu = F & \text{in } \Omega_T, \\ u = 0 & \text{in } (\Omega_e)_T, \\ u(0) = u_0, \quad \partial_t u(0) = u_1 & \text{in } \Omega, \end{cases} \tag{3.1}$$

if $u \in C([0,T];\widetilde{L}^2(\Omega)) \cap C^1([0,T];H^{-s}(\Omega))$ satisfies

$$\int_0^T \langle u(t), G(t)\rangle_{L^2(\Omega)}\, dt = \int_0^T \langle F(t), v(t)\rangle_{L^2(\Omega)}\, dt + \langle u_1, v(0)\rangle - \langle u_0, \partial_t v(0)\rangle_{L^2(\Omega)}, \tag{3.2}$$

for all $G \in L^2(0,T;\widetilde{L}^2(\Omega))$, where $v \in C([0,T];\widetilde{H}^s(\Omega)) \cap C^1([0,T];\widetilde{L}^2(\Omega))$ is the unique (weak) solution of the backward equation

$$\begin{cases} \partial_t^2 v + (-\Delta)^s v + qv = G & \text{in } \Omega_T, \\ v = 0 & \text{in } (\Omega_e)_T, \\ v(T) = \partial_t v(T) = 0 & \text{in } \Omega \end{cases} \tag{3.3}$$

(see [LTZ24, Theorem 3.1]*).*

Remark 3.2. *We recall that if $F \in L^2(0,T;\widetilde{L}^2(\Omega))$, $u_0 \in \widetilde{H}^s(\Omega)$ and $u_1 \in \widetilde{L}^2(\Omega)$, then a* weak solution *of (3.1) is a function $u \in C([0,T];\widetilde{H}^s(\Omega)) \cap C^1([0,T];\widetilde{L}^2(\Omega))$ such that*

$$\frac{d}{dt}\langle \partial_t u, w\rangle_{L^2(\Omega)} + \left\langle (-\Delta)^{s/2} u, (-\Delta)^{s/2} w \right\rangle_{L^2(\mathbb{R}^n)} + \langle qu, w\rangle_{L^2(\Omega)} = \langle F, w\rangle_{L^2(\Omega)},$$

for all $w \in \widetilde{H}^s(\Omega)$ in the sense of $\mathscr{D}'((0,T))$. Often, we refer to weak solutions or very weak solutions simply as solutions, because the source term in the relevant PDEs determines which notion of solutions we invoke. Moreover, weak solutions are always very weak solutions as we will see later in Proposition 3.9.

Remark 3.3. *Let us emphasize that the restriction on the exponent p comes from the observation that if $q \in L^p(\Omega)$ and $u \in C([0,T];L^2(\Omega))$, then we have $qu \in L^2(0,T;H^{-s}(\Omega))$ (see (3.33) in the proof of Theorem 3.7).*

Before proceeding, let us give a formal motivation for imposing the identity (3.2). We test the equation $\partial_t^2 u + (-\Delta)^s u + qu = F$ in $H^{-s}(\Omega)$ by the solution $v \in C([0,T];\widetilde{H}^s(\Omega))$ to (3.3) and integrate the resulting identity from $t=0$ to $t=T$. This gives

$$
\begin{aligned}
&\int_0^T \left\langle \partial_t^2 u(t), v(t) \right\rangle \, dt \\
&= -\int_0^T \langle (-\Delta)^s u(t), v(t) \rangle \, dt - \int_0^T \langle qu(t), v(t) \rangle \, dt + \int_0^T \langle F(t), v(t) \rangle \, dt \\
&= -\int_0^T \langle (-\Delta)^s v(t), u(t) \rangle \, dt - \int_0^T \langle qv(t), u(t) \rangle \, dt + \int_0^T \langle F(t), v(t) \rangle \, dt \\
&= \int_0^T \left\langle \partial_t^2 v(t), u(t) \right\rangle \, dt - \int_0^T \langle G(t), u(t) \rangle \, dt + \int_0^T \langle F(t), v(t) \rangle \, dt.
\end{aligned} \tag{3.4}
$$

Using an integration by parts the term on the left-hand side and the first term on the right-hand side can be rewritten as

$$
\begin{aligned}
\int_0^T \left\langle \partial_t^2 u(t), v(t) \right\rangle dt &= -\int_0^T \langle \partial_t u(t), \partial_t v(t) \rangle \, dt + \langle \partial_t u(T), v(T) \rangle - \langle \partial_t u(0), v(0) \rangle \\
&= -\int_0^T \langle \partial_t u(t), \partial_t v(t) \rangle \, dt - \langle u_1, v(0) \rangle \, , \\
\int_0^T \left\langle \partial_t^2 v(t), u(t) \right\rangle dt &= -\int_0^T \langle \partial_t v(t), \partial_t u(t) \rangle \, dt + \langle \partial_t v(T), u(T) \rangle - \langle \partial_t v(0), u(0) \rangle \\
&= -\int_0^T \langle \partial_t u(t), \partial_t v(t) \rangle \, dt - \langle \partial_t v(0), u_0 \rangle \, .
\end{aligned}
$$

Inserting these identities into (3.4), we get (3.2).

Remark 3.4. *Note that the above computations are only formal because the integration by parts identities require better regularity than $\partial_t u \in L^2(0,T;H^{-s}(\Omega))$ for the integral $\int_0^T \langle \partial_t u(t), \partial_t v(t) \rangle \, dt$ to make sense.*

3.2. **A spectral theoretic lemma.** For the construction of very weak solutions to linear nonlocal wave equations, we will need the following elementary spectral theoretic result. Even though the argument is standard, we offer the proof in Appendix A for readers' convenience.

Lemma 3.5. *Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain and $s \in \mathbb{R}_+ \setminus \mathbb{N}$. There exists a sequence of (Dirichlet) eigenvalues of the fractional Laplacian $(-\Delta)^s$ satisfying $0 < \lambda_1 \leq \lambda_2 \leq \dots$ with $\lambda_k \to \infty$ as $k \to \infty$ such that the corresponding eigenfunctions $(\phi_k)_{k\in\mathbb{N}} \subset \widetilde{H}^s(\Omega)$ have the following properties:*

- *(i) $(\phi_k)_{k\in\mathbb{N}}$ is an orthonormal basis of $\widetilde{L}^2(\Omega)$,*
- *(ii) $\left(\lambda_k^{-1/2}\phi_k\right)_{k\in\mathbb{N}}$ is an orthonormal basis of $\widetilde{H}^s(\Omega)$,*
- *(iii) $\left(\lambda_k^{1/2}\phi_k\right)_{k\in\mathbb{N}}$ is an orthonormal basis of $H^{-s}(\Omega)$.*

3.3. **Very weak solutions to linear nonlocal wave equations without potential.** The main purpose of this section is to prove the following well-posedness result.

Theorem 3.6 (Well-posedness of NWEQ with $q=0$)**.** *Let $F\in L^2(0,T;H^{-s}(\Omega))$, $u_0\in \widetilde{L}^2(\Omega)$ and $u_1\in H^{-s}(\Omega)$. Then there exists a unique solution of*

$$\begin{cases}\partial_t^2 u+(-\Delta)^s u=F & \text{in } \Omega_T,\\ u=0 & \text{in } (\Omega_e)_T,\\ u(0)=u_0,\quad \partial_t u(0)=u_1 & \text{in } \Omega.\end{cases} \tag{3.5}$$

Moreover, the following continuity estimate holds

$$\|u(t)\|_{L^2(\Omega)}+\|\partial_t u(t)\|_{H^{-s}(\Omega)}\leq C\big(\|u_0\|_{L^2(\Omega)}+\|u_1\|_{H^{-s}(\Omega)}+\|F\|_{L^2(0,T;H^{-s}(\Omega))}\big), \tag{3.6}$$

for some $C>0$ and for all $0\leq t\leq T$.

Proof. We use the Fourier method to show the existence of a solution to (3.5), that is we make the ansatz

$$u(t)=\sum_{k=1}^\infty c_k(t)\phi_k \tag{3.7}$$

and for later convenience we set

$$u_m(t):=\sum_{k=1}^m c_k(t)\phi_k,$$

for any $m\in\mathbb{N}$. For u to satisfy (3.5) in $H^{-s}(\Omega)$, the coefficient c_k, $k\in\mathbb{N}$, needs to solve the initial value problem

$$\begin{cases}c_k''(t)+\lambda_k c_k(t)=F_k(t),\\ c_k(0)=u_0^k,\ c_k'(0)=u_1^k\end{cases} \tag{3.8}$$

for $0<t<T$, where we set

$$u_0^k=\langle u_0,\phi_k\rangle_{L^2(\Omega)},\ u_1^k=\langle u_1,\phi_k\rangle \text{ and } F_k(t)=\langle F(t),\phi_k\rangle.$$

By Duhamel's principle, for any $k\in\mathbb{N}$, the coefficients c_k are given by

$$\begin{aligned}c_k(t)&=u_0^k\cos\big(\lambda_k^{1/2}t\big)+\lambda_k^{-1/2}u_1^k\sin\big(\lambda_k^{1/2}t\big)\\&\quad+\lambda_k^{-1/2}\int_0^t F_k(\tau)\sin\big(\lambda_k^{1/2}(t-\tau)\big)\,d\tau.\end{aligned} \tag{3.9}$$

Step 1. We first show that for any $t\in[0,T]$, the series in (3.7) converges in $\widetilde{L}^2(\Omega)$. By [Bre11, Corollary 5.10], we only need to ensure that $c_k(t)\in\ell^2$. By Jensen's inequality, we may estimate

$$\begin{aligned}|c_k(t)|^2&\leq 3\bigg(|u_0^k|^2+\lambda_k^{-1}|u_1^k|^2+\lambda_k^{-1}\left|\int_0^t F_k(\tau)\sin\big(\lambda_k^{1/2}(t-\tau)\big)\,d\tau\right|^2\bigg)\\&\leq 3\bigg(|u_0^k|^2+\lambda_k^{-1}|u_1^k|^2+t\lambda_k^{-1}\int_0^t|F_k(\tau)|^2\,d\tau\bigg),\end{aligned} \tag{3.10}$$

for any $k\in\mathbb{N}$. As $u_0\in\widetilde{L}^2(\Omega)$ and $(\phi_k)_{k\in\mathbb{N}}$ is an orthonormal basis in $\widetilde{L}^2(\Omega)$, [Bre11, Corollary 5.10] implies $(u_0^k)_{k\in\mathbb{N}}\in\ell^2$ with

$$\|u_0\|^2_{L^2(\Omega)}=\sum_{k=1}^\infty|u_0^k|^2. \tag{3.11}$$

Similarly, we know by (A.12) that $(\lambda_k^{-1/2}u_1^k)_{k\in\mathbb{N}}\in\ell^2$ with

$$\|u_1\|^2_{H^{-s}(\Omega)}=\sum_{k=1}^\infty\lambda_k^{-1}|u_1^k|^2. \tag{3.12}$$

On the other hand, the formula (A.12) shows that if $G \in L^2(0,T;H^{-s}(\Omega))$, then $(\lambda_k^{-1}G_k)_{k\in\mathbb{N}} \in L^2(0,T;\ell^2)$. Additionally, by Tonelli's theorem, the integral and sum can be exchanged so that

$$\|G\|^2_{L^2(0,T;H^{-s}(\Omega))} = \|\lambda_k^{-1/2}G_k\|^2_{L^2(0,T;\ell^2)} = \sum_{k=1}^{\infty}\lambda_k^{-1}\int_0^T |G_k(t)|^2\,dt. \tag{3.13}$$

This estimate can be applied to $F \in L^2(0,T;H^{-s}(\Omega))$. Hence, to sum up, we have

$$\|c_k(t)\|^2_{\ell^2} \leq 3\big(\|u_0\|^2_{L^2(\Omega)} + \|u_1\|^2_{H^{-s}(\Omega)} + T\|F\|^2_{L^2(0,T;H^{-s}(\Omega))}\big).$$

Now, again invoking [Bre11, Corollary 5.10] and Lemma 3.5, we deduce (3.7) converges in $\widetilde{L}^2(\Omega)$ for any $t\in[0,T]$ and

$$\|u(t)\|^2_{L^2(\Omega)} = \|c_k(t)\|^2_{\ell^2} \leq 3\big(\|u_0\|^2_{L^2(\Omega)}+\|u_1\|^2_{H^{-s}(\Omega)}+T\|F\|^2_{L^2(0,T;H^{-s}(\Omega))}\big). \tag{3.14}$$

Step 2. We first show that $u\in C([0,T];\widetilde{L}^2(\Omega))$. To see this, it is enough to show that $u_m \in C([0,T];\widetilde{L}^2(\Omega))$ for $m\in\mathbb{N}$ and $u_m \to u$ in $\widetilde{L}^2(\Omega)$ as $m\to\infty$ uniformly in $0\leq t\leq T$. Note that we have

$$\|u_m(t)-u_m(t')\|_{L^2(\Omega)} \leq \sum_{k=1}^{m}|c_k(t)-c_k(t')|\,,$$

for $t,t'\in[0,T]$ and so $u_m\in C([0,T];\widetilde{L}^2(\Omega))$ as long as $c_k\in C([0,T])$. The first two terms of c_k (see (3.9)) are continuous, hence it only remains to show that

$$d_k(t) := \lambda_k^{-1/2}\int_0^t F_k(\tau)\sin\big(\lambda_k^{1/2}(t-\tau)\big)\,d\tau \in C([0,T]). \tag{3.15}$$

Let us suppose that $t\geq t'$. Then we may calculate

(3.16)

$$\begin{aligned}
&|d_k(t)-d_k(t')|\\
&= \lambda_k^{-1/2}\left|\int_0^t F_k(\tau)\sin\big(\lambda_k^{1/2}(t-\tau)\big)\,d\tau - \int_0^{t'} F_k(\tau)\sin\big(\lambda_k^{1/2}(t'-\tau)\big)\,d\tau\right|\\
&\leq \lambda_k^{-1/2}\int_{t'}^t |F_k(\tau)|\big|\sin\big(\lambda_k^{1/2}(t-\tau)\big)\big|\,d\tau\\
&\quad + \lambda_k^{-1/2}\int_0^{t'} |F_k(\tau)|\big|\sin\big(\lambda_k^{1/2}(t'-\tau)\big)-\sin\big(\lambda_k^{1/2}(t-\tau)\big)\big|\,d\tau\\
&\leq \lambda_k^{-1/2}\int_{t'}^t |F_k(\tau)|\,d\tau\\
&\quad + \lambda_k^{-1/2}\int_0^{t'} |F_k(\tau)|\big|\sin\big(\lambda_k^{1/2}(t'-\tau)\big)-\sin\big(\lambda_k^{1/2}(t-\tau)\big)\big|\,d\tau\\
&\leq |t-t'|^{1/2}\left(\int_{t'}^t \lambda_k^{-1}|F_k(\tau)|^2\,d\tau\right)^{1/2}\\
&\quad + \left(\int_0^{t'}\lambda_k^{-1}|F_k(\tau)|^2\,d\tau\right)^{1/2}\left(\int_0^{t'}\big|\sin\big(\lambda_k^{1/2}(t'-\tau)\big)-\sin\big(\lambda_k^{1/2}(t-\tau)\big)\big|^2\,d\tau\right)^{1/2}\\
&\leq |t-t'|^{1/2}\|F\|_{L^2(0,T;H^{-s}(\Omega))}\\
&\quad + \|F\|_{L^2(0,T;H^{-s}(\Omega))}\left(\int_0^{t'}|\sin(\lambda_k^{1/2}(t'-\tau))-\sin(\lambda_k^{1/2}(t-\tau))|^2\,d\tau\right)^{1/2}.
\end{aligned}$$

In the first inequality, we wrote

$$\sin\big(\lambda_k^{1/2}(t'-\tau)\big) = \big(\sin\big(\lambda_k^{1/2}(t'-\tau)\big)-\sin\big(\lambda_k^{1/2}(t-\tau)\big)\big)+\sin\big(\lambda_k^{1/2}(t'-\tau)\big)$$

and in the fourth inequality used (3.13). Now, let $\epsilon > 0$ and choose first $\rho > 0$ such that $\rho^{1/2}\|F\|_{L^2(0,T;H^{-s}(\Omega))} < \epsilon/2$. Then choose $\delta > 0$ such that

$$\|F\|_{L^2(0,T;H^{-s}(\Omega))}T^{1/2}\delta < \epsilon/2.$$

By uniform continuity of the sine function, we can find $\eta > 0$ such that $|\sin x - \sin y| < \delta$, whenever $|x-y| < \eta$. Now, we set

$$\mu := \min(\rho, \eta/\lambda_k^{1/2}).$$

The above choices show that if $|t-t'| < \mu$ and $t \geq t'$, then

$$\begin{aligned}|d_k(t) - d_k(t')| &\leq \rho^{1/2}\|F\|_{L^2(0,T;H^{-s}(\Omega))} + \|F\|_{L^2(0,T;H^{-s}(\Omega))}\delta(t')^{1/2} \\ &< \epsilon/2 + \|F\|_{L^2(0,T;H^{-s}(\Omega))}\delta T^{1/2} \\ &< \epsilon.\end{aligned}$$

Interchanging the roles of t and t' shows that d_k is uniformly continuous because the constant does not depend on the particular point t or t' (but choice of μ depends on k). Hence, we have shown that $u_m \in C([0,T];\widetilde{L}^2(\Omega))$.

Next, we prove that u_m converges uniformly to u in $\widetilde{L}^2(\Omega)$ on $[0,T]$ as $m \to \infty$. Let $\ell \geq m$, then by (3.10) we deduce that

$$\begin{aligned}\|u_\ell(t) - u_m(t)\|^2_{L^2(\Omega)} &= \left\| \sum_{k=m+1}^{\ell} c_k(t)\phi_k \right\|^2_{L^2(\Omega)} \\ &= \sum_{k=m+1}^{\ell} |c_k(t)|^2 \\ &\leq C \sum_{k=m+1}^{\ell} \left(\left|u_0^k\right|^2 + \lambda_k^{-1}\left|u_1^k\right|^2 + T\lambda_k^{-1}\int_0^T |F_k(\tau)|^2\, d\tau \right),\end{aligned}$$

for some constant $C > 0$. Passing to the limit $\ell \to \infty$ gives

$$\|u(t) - u_m(t)\|^2_{L^2(\Omega)} \leq C \sum_{k=m+1}^{\infty} \left(\left|u_0^k\right|^2 + \lambda_k^{-1}\left|u_1^k\right|^2 + T\lambda_k^{-1}\int_0^T |F_k(\tau)|^2\, d\tau \right)$$

for any $m \in \mathbb{N}$. The right-hand side is independent of $t \in [0,T]$ and by summability of the right-hand side (see (3.11), (3.12) and (3.13)) it needs to go to zero as m tends to infinity. Thus, the convergence $u_m \to u$ in $\widetilde{L}^2(\Omega)$ as $m \to \infty$ is uniform in $t \in [0,T]$.

Step 3. Let us show that $u \in C^1([0,T];H^{-s}(\Omega))$. We first establish that $u_m \in C^1([0,T];H^{-s}(\Omega))$ for any $m \in \mathbb{N}$. Formally, by differentiating c_k, one may compute

(3.17)

$$c_k'(t) = -\lambda_k^{1/2}u_0^k \sin\left(\lambda_k^{1/2}t\right) + u_1^k \cos\left(\lambda_k^{1/2}t\right) + \int_0^t F_k(\tau)\cos\left(\lambda_k^{1/2}(t-\tau)\right) d\tau,$$

for any $k \in \mathbb{N}$. Taking derivatives for the first two terms does not cause any difficulty, but for the integral term, we need to justify it. Using the definition of d_k

from (3.15), for any $t \in [0,T]$ and $h > 0$ such that $t+h \in [0,T]$, one can compute

$$
\begin{aligned}
&\left| \frac{d_k(t+h) - d_k(t)}{h} - \int_0^t F_k(\tau) \cos\left(\lambda_k^{1/2}(t-\tau)\right) d\tau \right| \\
&= \Bigg| \int_t^{t+h} \lambda_k^{-1/2} F_k(\tau) \frac{\sin\left(\lambda_k^{1/2}(t+h-\tau)\right)}{h}\, d\tau \\
&\quad + \int_0^t \lambda_k^{-1/2} F_k(\tau) \\
&\qquad \cdot \left(\frac{\sin\left(\lambda_k^{1/2}(t+h-\tau)\right) - \sin\left(\lambda_k^{1/2}(t-\tau)\right)}{h} - \lambda_k^{1/2} \cos\left(\lambda_k^{1/2}(t-\tau)\right) \right) d\tau \Bigg| \\
&\le \int_t^{t+h} \lambda_k^{-1/2} |F_k(\tau)| \left| \frac{\sin\left(\lambda_k^{1/2}(t+h-\tau)\right)}{h} \right| d\tau \\
&\quad + \int_0^t \lambda_k^{-1/2} |F_k(\tau)| \\
&\qquad \cdot \left| \frac{\sin\left(\lambda_k^{1/2}(t+h-\tau)\right) - \sin\left(\lambda_k^{1/2}(t-\tau)\right)}{h} - \lambda_k^{1/2} \cos\left(\lambda_k^{1/2}(t-\tau)\right) \right| d\tau \\
&=: I_h + II_h.
\end{aligned}
$$

Next, we show that both expressions I_h and II_h vanish as $h \to 0$.

For I_h, by the triangle inequality, we have

$$
\begin{aligned}
I_h &\le \int_t^{t+h} \lambda_k^{-1/2} |F_k(\tau)| \left| \frac{\sin\left(\lambda_k^{1/2}(t+h-\tau)\right) - \sin\left(\lambda_k^{1/2}(t-\tau)\right)}{h} \right| d\tau \\
&\quad + \underbrace{\int_t^{t+h} \lambda_k^{-1/2} |F_k(\tau)| \left| \frac{\sin\left(\lambda_k^{1/2}(t-\tau)\right)}{h} \right| d\tau}_{= \frac{1}{h} \int_t^{t+h} \lambda_k^{-1/2} |F_k(\tau)| |\sin(\lambda_k^{1/2}(t-\tau))| d\tau} \\
&= \int_t^{t+h} \lambda_k^{-1/2} |F_k(\tau)| \left| \frac{\sin\left(\lambda_k^{1/2}(t+h-\tau)\right) - \sin\left(\lambda_k^{1/2}(t-\tau)\right)}{h} \right| d\tau + o(1) \\
&= \int_t^{t+h} |F_k(\tau)| \left| \cos\left(\lambda_k^{1/2}(\eta-\tau)\right) \right| d\tau + o(1) \\
&\le \int_t^{t+h} |F_k(\tau)|\, d\tau + o(1) \\
&= o(1)
\end{aligned}
$$

as $h \to 0$. In the first equality we used that $F_k \in L^2((0,T))$ and Lebesgue's differentiation theorem implies

$$
\frac{1}{h} \int_t^{t+h} \lambda_k^{-1/2} |F_k(\tau)| \left| \sin(\lambda_k^{1/2}(t-\tau)) \right| d\tau \to 0 \text{ as } h \to 0.
$$

In the second equality we applied the mean value theorem, where $\eta \in (t, t+h)$, and in the last equality the absolute continuity of the Lebesgue integral.

On the other hand, the fact that $II_h \to 0$ as $h \to 0$ is a simple application of Lebesgue's dominated convergence theorem. The same argument works out for $h < 0$ and hence the d_k is differentiable with the derivative

$$
d_k'(t) = \int_0^t F_k(\tau) \cos\left(\lambda_k^{1/2}(t-\tau)\right) d\tau. \tag{3.18}
$$

Hence, we have proved the formula (3.17).

It remains to show that $u_m' \in C([0,T];H^{-s}(\Omega))$, but by the same argument as above it is enough to establish $c_k' \in C([0,T])$. The first two terms in c_k' are clearly continuous and hence we only need to show the continuity of d_k' given by the formula (3.18). Let $t \geq t'$. By the same computation as in (3.16) up to replacing sin by cos and forgetting the prefactor $\lambda_k^{-1/2}$, we have

$$|d_k'(t) - d_k'(t')| \leq \int_{t'}^t |F_k(\tau)|\,d\tau + \int_0^{t'} |F_k(\tau)|\,\big|\cos\big(\lambda_k^{1/2}(t'-\tau)\big) - \cos\big(\lambda_k^{1/2}(t-\tau)\big)\big|d\tau.$$

Now, let $\epsilon > 0$. As $F_k \in L^2((0,T))$, by the absolute continuity of the Lebesgue integral, we can find $\delta > 0$ such that the first term is smaller than $\epsilon/2$, whenever $|t-t'| < \delta$. On the other hand, we can find a $\rho > 0$ such that there holds

$$|\cos x - \cos y| < \epsilon/2\|F_k\|_{L^1((0,T))}$$

whenever $|x-y| < \rho$. Let

$$\mu = \min(\delta, \rho/\lambda_k^{1/2}).$$

Hence, if $|t-t'| < \mu$, then we have

$$\big|\lambda_k^{1/2}(t'-\tau) - \lambda_k^{1/2}(t-\tau)\big| = \lambda_k^{1/2}|t-t'| < \rho$$

and hence

$$|d_k'(t) - d_k'(t')| < \epsilon/2 + \frac{\epsilon}{2\|F_k\|_{L^1((0,T))}} \int_0^{t'} |F_k(\tau)|\,d\tau < \epsilon.$$

The very same argument holds when $t \leq t'$ and as all parameters δ, ρ, μ are independent of t and t', we have shown that d_k' are uniformly continuous on $[0,T]$. Hence, we have $u_m \in C^1([0,T];H^{-s}(\Omega))$ for all $m \in \mathbb{N}$.

Now, by Lemma 3.5 and the same arguments as in (3.10), we get

$$\begin{aligned}
\|u_m'(t)\|_{H^{-s}(\Omega)}^2 &= \Big\|\sum_{k=1}^m c_k'(t)\phi_k\Big\|_{H^{-s}(\Omega)}^2 \\
&= \Big\|\sum_{k=1}^m \lambda_k^{-1/2} c_k'(t)(\lambda_k^{1/2}\phi_k)\Big\|_{H^{-s}(\Omega)}^2 \\
&= \sum_{k=1}^m \lambda_k^{-1}|c_k'(t)|^2 \\
&\leq 3\sum_{k=1}^m \lambda_k^{-1}\left(\lambda_k|u_0^k|^2 + |u_1^k|^2 + \left(\int_0^t |F_k(\tau)|\,d\tau\right)^2\right) \\
&\leq 3\sum_{k=1}^m \left(|u_0^k|^2 + \lambda_k^{-1}|u_1^k|^2 + T\lambda_k^{-1}\int_0^T |F_k(\tau)|^2\,d\tau\right).
\end{aligned} \tag{3.19}$$

Using (3.11), (3.12) and (3.13), this implies

$$\|u_m'(t)\|_{H^{-s}(\Omega)}^2 \leq 3\big(\|u_0\|_{L^2(\Omega)}^2 + \|u_1\|_{H^{-s}(\Omega)}^2 + T\|F\|_{L^2(0,T;H^{-s}(\Omega))}^2\big). \tag{3.20}$$

Furthermore, we have

$$\|u_\ell'(t) - u_m'(t)\|_{H^{-s}(\Omega)}^2 = \sum_{k=m+1}^{\ell} \lambda_k^{-1}|c_k'(t)|^2 \leq \sum_{k=m+1}^{\infty} \lambda_k^{-1}|c_k'(t)|^2 \tag{3.21}$$

for all $\ell \geq m$. Observing that the right-hand side goes to zero as $m \to \infty$, we see that $(u_m(t))_{m\in\mathbb{N}} \subset H^{-s}(\Omega)$ is a Cauchy sequence in $H^{-s}(\Omega)$ and therefore converges to some unique limit $w(t) \in H^{-s}(\Omega)$. Passing to the limit $\ell \to \infty$ in (3.21) and using the estimates from (3.19), we get

$$\|w(t) - u_m'(t)\|_{H^{-s}(\Omega)}^2 \leq 3 \sum_{k=m+1}^{\infty} \left(|u_0^k|^2 + \lambda_k^{-1}|u_1^k|^2 + T\lambda_k^{-1} \int_0^T |F_k(\tau)|^2 \, d\tau \right)$$

for any $m \in \mathbb{N}$. The sum on the right-hand side is independent of t and hence the convergence $u_m' \to w$ in $H^{-s}(\Omega)$ as $m \to \infty$ is uniform in $t \in [0, T]$. It is well-known that this implies $u \in C^1([0,T]; H^{-s}(\Omega))$ with $u' = w$. Furthermore, by (3.20) there holds

$$\|u'\|_{L^\infty(0,T;H^{-s}(\Omega))} \leq C \left(\|u_0\|_{L^2(\Omega)} + \|u_1\|_{H^{-s}(\Omega)} + \|F\|_{L^2(0,T;H^{-s}(\Omega))} \right),$$

for some $C > 0$. Notice that this estimate together with (3.14) establishes (3.6).

Step 4. In this step we show that u is in fact a solution of (3.5). First let us note that by formally applying the Leibniz rule and (3.9), one has

$$\begin{aligned} c_k''(t) &= -\lambda_k u_0^k \cos\left(\lambda_k^{1/2} t\right) - \lambda_k^{1/2} u_1^k \sin\left(\lambda_k^{1/2} t\right) + F_k(t) \\ &\quad - \lambda_k^{1/2} \int_0^t F_k(\tau) \sin\left(\lambda_k^{1/2}(t-\tau)\right) d\tau \\ &= -\lambda_k c_k(t) + F_k(t). \end{aligned} \tag{3.22}$$

Thus, c_k indeed solves (3.8). To see that the first equality sign in formula (3.22) holds, it is enough to show that d_k' is differentiable with derivative

$$d_k''(t) = F_k(t) - \lambda_k^{1/2} \int_0^t F_k(\tau) \sin\left(\lambda_k^{1/2}(t-\tau)\right) d\tau. \tag{3.23}$$

We can repeat the same computation as for the first derivative. This time we have

$$\begin{aligned} &\left| \frac{d_k'(t+h) - d_k'(t)}{h} - F_k(t) + \lambda_k^{1/2} \int_0^t F_k(\tau) \sin\left(\lambda_k^{1/2}(t-\tau)\right) d\tau \right| \\ &\leq \int_t^{t+h} \left| \frac{F_k(\tau) \cos\left(\lambda_k^{1/2}(t+h-\tau)\right) - F_k(t)}{h} \right| d\tau \\ &\quad + \int_0^t |F_k(\tau)| \left| \frac{\cos\left(\lambda_k^{1/2}(t+h-\tau)\right) - \cos\left(\lambda_k^{1/2}(t-\tau)\right)}{h} - \lambda_k^{1/2} \sin\left(\lambda_k^{1/2}(t-\tau)\right) \right| d\tau \\ &=: III_h + IV_h. \end{aligned}$$

The second term IV_h again goes to zero by Lebesgue's dominated convergence theorem. For the first term III_h, we proceed similarly as for I above. This gives

$$\begin{aligned} III_h \leq &\int_t^{t+h} |F_k(\tau)| \left| \frac{\cos\left(\lambda_k^{1/2}(t+h-\tau)\right) - \cos\left(\lambda_k^{1/2}(t-\tau)\right)}{h} \right| d\tau \\ &+ \int_t^{t+h} \left| \frac{F_k(\tau) \cos\left(\lambda_k^{1/2}(t-\tau)\right) - F_k(t)}{h} \right| d\tau. \end{aligned}$$

The second term goes to zero as $h \to 0$ by Lebesgue's differentiation theorem and for the first term, one can use the mean value theorem and the absolute continuity of the integral to find $III_h \to 0$ as $h \to 0$. This proves (3.23) and hence (3.22). From the differential equation for c_k we get $c_k'' \in L^2((0,T))$. Note that as F is not (in general) continuous, we generally do not have $c_k \in C^2([0,T])$. But in fact $c_k' \in H^1((0,T))$ and the fundamental theorem of calculus imply $c_k \in C^{1,1/2}([0,T])$.

Now, we wish to show that u is a solution in the sense of Definition 3.1. To this end, let us assume that $G \in L^2(0,T; \widetilde{L}^2(\Omega))$ and $v \in C([0,T]; \widetilde{H}^s(\Omega)) \cap$

$C^1([0,T];\widetilde{L}^2(\Omega))$ is the unique solution to

$$
\begin{cases}
\partial_t^2 v + (-\Delta)^s v = G & \text{in } \Omega_T, \\
v = 0 & \text{in } (\Omega_e)_T, \\
v(T) = \partial_t v(T) = 0 & \text{in } \Omega.
\end{cases}
\tag{3.24}
$$

Note that from this equation we also have $\partial_t^2 v \in L^2(0,T;H^{-s}(\Omega))$. By Lemma 3.5 we can write

$$
v(t) = \sum_{k=1}^{\infty} \alpha_k(t)\phi_k,
$$

where $\alpha_k = \langle v(t), \phi_k \rangle_{L^2(\Omega)}$. Furthermore, note that by our choice of ϕ_k and the inner product on $H^{-s}(\Omega)$ (see (A.16)), there holds

$$
\alpha_k = \lambda_k^{-1/2} \big\langle v(t), \lambda_k^{-1/2}\phi_k \big\rangle_{\widetilde{H}^s(\Omega)} = \lambda_k^{1/2} \big\langle v(t), \lambda_k^{1/2}\phi_k \big\rangle_{H^{-s}(\Omega)}.
$$

The last equality is shown in (A.17). For later convenience, we let v_m be defined via

$$
v_m(t) = \sum_{k=1}^{m} \alpha_k(t)\phi_k.
$$

Moreover, we know that:

(a) For any $t \in [0,T]$ one has $v_m(t) \to v(t)$ in $\widetilde{H}^s(\Omega)$ as $m \to \infty$.
(b) There holds $\alpha_k \in C^1([0,T]) \cap H^2((0,T))$ for any $k \in \mathbb{N}$.
(c) For any $k \in \mathbb{N}$ the functions α_k solve

$$
\begin{cases}
\alpha_k''(t) + \lambda_k \alpha_k(t) = G_k(t) \\
\alpha_k(T) = \alpha_k'(T) = 0,
\end{cases}
$$

for $0 < t < T$, where $G_k(t) = \langle G(t), \phi_k \rangle_{L^2(\Omega)}$.

In fact, (a) follows from Lemma 3.5 and the regularity of v. The regularity $\alpha_k \in C^1([0,T])$ in (b) follows from $v \in C^1([0,T];\widetilde{L}^2(\Omega))$. The claim that $\alpha_k \in H^2((0,T))$ can be seen as follows. First of all, $v'' \in L^2(0,T;H^{-s}(\Omega))$ implies

$$
-\int_0^T v'(t)\eta'(t)\,dt = \int_0^T v''(t)\eta(t)\,dt \text{ in } H^{-s}(\Omega)
$$

for all $\eta \in C_c^\infty((0,T))$. Acting on this identity by $w \mapsto \lambda_k^{1/2}\langle w, \lambda_k^{1/2}\phi_k \rangle_{H^{-s}(\Omega)} \in (H^{-s}(\Omega))^*$ gives

$$
-\int_0^T \lambda_k^{1/2}\langle v'(t), \lambda_k^{1/2}\phi_k \rangle_{H^{-s}(\Omega)}\eta'(t)\,dt = \int_0^T \lambda_k^{1/2}\langle v''(t), \lambda_k^{1/2}\phi_k \rangle_{H^{-s}(\Omega)}\eta(t)\,dt.
$$

Now, the first factor on the left-hand side is nothing else than α_k', and thus

$$
\alpha_k''(t) = \lambda_k^{1/2}\langle v''(t), \lambda_k^{1/2}\phi_k \rangle_{H^{-s}(\Omega)} \in L^2((0,T)).
$$

This shows that $\alpha_k \in H^2((0,T))$ for $k \in \mathbb{N}$. The endpoint conditions in (c) follow from $v(T) = v'(T) = 0$. From the previous calculation, (A.16) and (3.24), we can compute

$$
\begin{aligned}
\alpha_k''(t) &= \lambda_k^{1/2}\big\langle v''(t), \lambda_k^{1/2}\phi_k\big\rangle_{H^{-s}(\Omega)} \\
&= \lambda_k^{1/2}\big\langle v''(t)-G(t), \lambda_k^{1/2}\phi_k\big\rangle_{H^{-s}(\Omega)} + \lambda_k^{1/2}\big\langle G(t), \lambda_k^{1/2}\phi_k\big\rangle_{H^{-s}(\Omega)} \\
&= \big\langle S(v''(t)-G(t)), S(\lambda_k\phi_k)\big\rangle_{\widetilde{H}^s(\Omega)} + \langle S(G(t)), S(\lambda_k\phi_k)\rangle_{\widetilde{H}^s(\Omega)} \\
&= -\big\langle v(t), \phi_k\big\rangle_{\widetilde{H}^s(\Omega)} + \langle S(G(t)), \phi_k\rangle_{\widetilde{H}^s(\Omega)} \\
&= -\lambda_k\langle v(t), \phi_k\rangle_{L^2(\Omega)} + \langle G(t), \phi_k\rangle_{L^2(\Omega)} \\
&= -\lambda_k\alpha_k(t) + G_k(t),
\end{aligned}
$$

where $S\colon H^{-s}(\Omega) \to \widetilde{H}^s(\Omega)$ is the source-to-solution map for the Dirichlet problem of the fractional Laplacian $(-\Delta)^s$ (see Appendix A for more details). This verifies that α_k satisfies (c) . By $c_k \in C^1([0,T]) \cap H^2((0,T))$, (3.8), (b) and (c) we may calculate

$$
\begin{aligned}
\int_0^T c_k''\alpha_k\,dt &= -\int_0^T c_k'\alpha_k'\,dt + c_k'(T)\alpha_k(T) - c_k'(0)\alpha_k(0) \\
&= -\int_0^T c_k'\alpha_k'\,dt - u_1^k\alpha_k(0)
\end{aligned}
\tag{3.25}
$$

and

$$
\begin{aligned}
\int_0^T c_k\alpha_k''\,dt &= -\int_0^T c_k'\alpha_k'\,dt + c_k(T)\alpha_k'(T) - c_k(0)\alpha_k'(0) \\
&= -\int_0^T c_k'\alpha_k'\,dt - u_0^k\alpha_k'(0).
\end{aligned}
\tag{3.26}
$$

Inserting (3.26) into (3.25) yields

$$
\int_0^T c_k''\alpha_k\,dt = \int_0^T c_k\alpha_k''\,dt + u_0^k\alpha_k'(0) - u_1^k\alpha_k(0).
$$

Hence by (3.8) and (c), we get

$$
\int_0^T (-\lambda_k c_k + F_k)\alpha_k\,dt = \int_0^T c_k(-\lambda_k\alpha_k + G_k)\,dt + u_0^k\alpha_k'(0) - u_1^k\alpha_k(0),
\tag{3.27}
$$

or equivalently

$$
\int_0^T F_k\alpha_k\,dt = \int_0^T c_kG_k\,dt + u_0^k\alpha_k'(0) - u_1^k\alpha_k(0).
\tag{3.28}
$$

Summing this identity from $k=1$ to $k=N$ gives

$$
\begin{aligned}
\int_0^T \big\langle F^{(N)}, v_N\big\rangle dt = \int_0^T &\big\langle u_N, G^{(N)}\big\rangle_{L^2(\Omega)}\,dt \\
&+ \big\langle u_0^{(N)}, v_N'(0)\big\rangle_{L^2(\Omega)} - \big\langle u_1^{(N)}, v_N(0)\big\rangle,
\end{aligned}
\tag{3.29}
$$

where we set

$$
F^{(N)} = \sum_{k=1}^N F_k\phi_k, \quad G^{(N)} = \sum_{k=1}^N G_k\phi_k, \quad u_j^{(N)} = \sum_{k=1}^N u_j^k\phi_k
$$

for $j=0,1$. That (3.28) and (3.29) are equivalent can be seen by Lemm 3.5. Next, note that $G \in L^2(0,T;\widetilde{L}^2(\Omega))$ and Lebesgue's dominated convergence theorem together with Parseval's identity ensure

$$\begin{aligned}\left\|G^{(N)}(t)\right\|_{L^2(\Omega)}^2 &= \sum_{k=1}^{N} \left|\langle G, \phi_k\rangle_{L^2(\Omega)}\right|^2 \\ &\leq \sum_{k=1}^{\infty} \left|\langle G, \phi_k\rangle_{L^2(\Omega)}\right|^2 = \|G(t)\|_{L^2(\Omega)}^2,\end{aligned}$$

which gives

$$G^{(N)} \to G \text{ in } L^2(0,T;\widetilde{L}^2(\Omega))$$

as $N \to \infty$. To see that $F^{(N)} \to F$ in $L^2(0,T;H^{-s}(\Omega))$, let us first observe that

$$\begin{aligned}F_k\phi_k &= \langle F, \phi_k\rangle \phi_k \\ &= \langle SF, \phi_k\rangle_{\widetilde{H}^s(\Omega)}\phi_k \\ &= \langle SF, S(\lambda_k\phi_k)\rangle_{\widetilde{H}^s(\Omega)}\phi_k \\ &= \langle F, \lambda_k\phi_k\rangle_{H^{-s}(\Omega)}\phi_k \\ &= \left\langle F, \lambda_k^{1/2}\phi_k\right\rangle_{H^{-s}(\Omega)}\lambda_k^{1/2}\phi_k.\end{aligned} \tag{3.30}$$

As $\left(\lambda_k^{1/2}\phi_k\right)_{k\in\mathbb{N}}$ is an orthonormal basis in $H^{-s}(\Omega)$, we deduce from (3.30) that there holds

$$F^{(N)}(t) \to F(t) \text{ in } H^{-s}(\Omega)$$

as $N \to \infty$. Thus using Lebesgue's dominated convergence theorem and Parseval's identity for G, we get

$$F^{(N)} \to F \text{ in } L^2(0,T;H^{-s}(\Omega))$$

as $N \to \infty$. So, we can finally pass to the limit in (3.29) to obtain

$$\int_0^T \langle F(t), v(t)\rangle\, dt = \int_0^T \langle u(t), G(t)\rangle_{L^2(\Omega)}\, dt + \langle u_0, v'(0)\rangle_{L^2(\Omega)} - \langle u_1, v(0)\rangle .$$

This establishes that u is a solution to (3.5).

Step 5. In this final step, we show that the constructed solution u is unique. Suppose that $\widetilde{u}$ is another solution, then $U = u - \widetilde{u}$ is a solution to

$$\begin{cases}\partial_t^2 U + (-\Delta)^s U = 0 & \text{in } \Omega_T,\\ U = 0 & \text{in } (\Omega_e)_T,\\ U(0) = \partial_t U(0) = 0 & \text{in } \Omega.\end{cases}$$

By Definition 3.1 this means

$$\int_0^T \langle U(t), G(t)\rangle_{L^2(\Omega)}\, dt = 0$$

for all $G \in L^2(\Omega_T)$, but this clearly implies $U = 0$ and hence $u = \widetilde{u}$. □

3.4. **Very weak solutions to linear nonlocal wave equations with potential.** The purpose of this section is to extend the well-posedness theory of equation (3.5) to linear NWEQs with a nonzero potential.

Theorem 3.7 (Well-posedness nonlocal wave equation with potential)**.** *Let* $F \in L^2(0,T;H^{-s}(\Omega))$*,* $u_0 \in \widetilde{L}^2(\Omega)$ *and* $u_1 \in H^{-s}(\Omega)$*. Furthermore, assume that* $q \in L^p(\Omega)$ *with* p *satisfying the restriction* (1.4)*. Then the problem*

$$\begin{cases}\partial_t^2 u + (-\Delta)^s u + qu = F & \text{in } \Omega_T,\\ u = 0 & \text{in } (\Omega_e)_T,\\ u(0) = u_0, \quad \partial_t u(0) = u_1 & \text{in } \Omega\end{cases} \tag{3.31}$$

has a unique very weak solution $u \in C([0,T];\widetilde{L}^2(\Omega)) \cap C^1([0,T];H^{-s}(\Omega))$.

Proof. Let us first note that $qu \in H^{-s}(\Omega)$ for any $u \in L^2(\Omega)$ as

$$\begin{aligned} \left|\int_\Omega quv\,dx\right| &\leq \|qv\|_{L^2(\Omega)}\|u\|_{L^2(\Omega)} \\ &\leq \|q\|_{L^{n/s}(\Omega)}\|v\|_{L^{\frac{2n}{n-2s}}(\Omega)}\|u\|_{L^2(\Omega)} \\ &\leq C\|q\|_{L^{n/s}(\Omega)}\|v\|_{\widetilde{H}^s(\Omega)}\|u\|_{L^2(\Omega)} \\ &\leq C\|q\|_{L^p(\Omega)}\|v\|_{\widetilde{H}^s(\Omega)}\|u\|_{L^2(\Omega)} \end{aligned} \tag{3.32}$$

for all $v \in \widetilde{H}^s(\Omega)$. The case $p=\infty$ is clear. In the case $\frac{n}{s} \leq p < \infty$ with $2s < n$ we used Hölder's inequality with

$$\frac{1}{2} = \frac{n-2s}{2n} + \frac{s}{n},$$

$L^{r_2}(\Omega) \hookrightarrow L^{r_1}(\Omega)$ for $r_1 \leq r_2$ as $\Omega \subset \mathbb{R}^n$ is bounded, and Sobolev's inequality. In the case $2s > n$ one can use the embedding $H^s(\mathbb{R}^n) \hookrightarrow L^\infty(\mathbb{R}^n)$ and the boundedness of Ω to see that the estimate (3.32) holds. In the case $n = 2s$ one can use the boundedness of the embedding $\widetilde{H}^s(\Omega) \hookrightarrow L^{\bar{p}}(\Omega)$ for all $2 \leq \bar{p} < \infty$, Hölder's inequality and the boundedness of Ω to get the final estimate (3.32). The aforementioned embedding in the critical case follows by [Oza95] and the Poincaré inequality. The above clearly implies that for any $u \in C([0,T];\widetilde{L}^2(\Omega))$, we have $qu \in L^2(0,T;H^{-s}(\Omega))$ with

$$\|qu\|_{L^2(0,T;H^{-s}(\Omega))} \leq C\|q\|_{L^p(\Omega)}\|u\|_{L^2(\Omega_T)}. \tag{3.33}$$

Now, we wish to use a fixed point argument to construct the solution to (3.5).

Via Theorem 3.6, we can define

$$S\colon C([0,T];\widetilde{L}^2(\Omega)) \to C([0,T];\widetilde{L}^2(\Omega)) \cap C^1([0,T];H^{-s}(\Omega)), \quad v \mapsto u,$$

where u is the solution of

$$\begin{cases} \partial_t^2 u + (-\Delta)^s u = F - qv & \text{in } \Omega_T, \\ u = 0 & \text{in } (\Omega_e)_T, \\ u(0) = u_0, \quad \partial_t u(0) = u_1 & \text{in } \Omega. \end{cases}$$

Assume that $v_1, v_2 \in C([0,T];\widetilde{L}^2(\Omega))$. Since the function $u = S(v^1) - S(v^2)$ solves

$$\begin{cases} \partial_t^2 u + (-\Delta)^s u = -q(v^1 - v^2) & \text{in } \Omega_T, \\ u = 0 & \text{in } (\Omega_e)_T, \\ u(0) = \partial_t u(0) = 0 & \text{in } \Omega, \end{cases}$$

the energy estimate (3.6) (applied for the case $T = t$) yields

$$\|u(t)\|_{L^2(\Omega)} + \|\partial_t u(t)\|_{H^{-s}(\Omega)} \leq C\left\|q(v^1 - v^2)\right\|_{L^2(0,t;H^{-s}(\Omega))},$$

for a.e. $t \in [0,T]$. Hence, by (3.33) we obtain

$$\|u(t)\|_{L^2(\Omega)} \leq C\|q\|_{L^p(\Omega)}\left\|v^1 - v^2\right\|_{L^2(\Omega_t)}, \tag{3.34}$$

for $t \in [0,T]$. Next, introduce for $\theta > 0$ the equivalent norm

$$\|w\|_\theta := \sup_{0 \leq t \leq T} e^{-\theta t}\|w(t)\|_{L^2(\Omega)}$$

on $C([0,T];\widetilde{L}^2(\Omega))$. Then from equation (3.34) we deduce that

$$\begin{aligned}\|u(t)\|_{L^2(\Omega)} &\leq C\|q\|_{L^p(\Omega)}\left\|v^1-v^2\right\|_\theta\left(\int_0^t e^{2\theta\tau}\,d\tau\right)^{1/2}\\ &= \frac{C}{(2\theta)^{1/2}}(e^{2\theta t}-1)^{1/2}\|q\|_{L^p(\Omega)}\left\|v^1-v^2\right\|_\theta\\ &\leq \frac{C}{(2\theta)^{1/2}}e^{\theta t}\|q\|_{L^p(\Omega)}\left\|v^1-v^2\right\|_\theta\end{aligned}$$

for $t\in[0,T]$. Dividing by $e^{\theta t}$ and taking the supremum over $[0,T]$, this implies

$$\|u\|_\theta \leq \frac{C}{(2\theta)^{1/2}}\|q\|_{L^p(\Omega)}\left\|v^1-v^2\right\|_\theta .$$

Remembering that $u=S(v^1)-S(v^2)$ and choosing $\theta>0$ such that

$$C_0 := \frac{C}{(2\theta)^{1/2}}\|q\|_{L^p(\Omega)} < 1,$$

we get

$$\|S(v^1)-S(v^2)\|_\theta \leq C_0\|v^1-v^2\|_\theta$$

and thus S is a contraction on the complete metric space $(C([0,T];\widetilde{L}^2(\Omega)),\|\cdot\|_\theta)$. Therefore, we can apply the Banach fixed point theorem to deduce that S has a unique fixed point $u\in C([0,T];\widetilde{L}^2(\Omega))\cap C^1([0,T];H^{-s}(\Omega))$. Hence, we have shown the existence of a unique very weak solution and we can conclude the proof. □

3.5. **Properties of very weak solutions.** In this section, we establish some relations between various definitions of solutions to linear NWEQs, which can be seen as a consistency test of the introduced notions. In Proposition 3.8 and 3.9, we show that

$$\text{weak solution} \;\Rightarrow\; \text{very weak solution} \;\Rightarrow\; \text{distributional solution}$$

and in Corollary 3.10 that

$$\text{regular very weak solution} = \text{weak solution}.$$

Proposition 3.8 (Distributional solutions)**.** *Let $F\in L^2(0,T;H^{-s}(\Omega))$, $u_0\in\widetilde{L}^2(\Omega)$, $u_1\in H^{-s}(\Omega)$ and $q\in L^p(\Omega)$ with p satisfying the restrictions (1.4). The unique very weak solution of (3.31) is a distributional solution, that is there holds*

(3.35)
$$\int_{\Omega_T} u\left(\partial_t^2\varphi+(-\Delta)^s\varphi+q\varphi\right)dt = \int_0^T\langle F,\varphi\rangle\,dt+\langle u_0,\partial_t\varphi(0)\rangle_{L^2(\Omega)}-\langle u_1,\varphi(0)\rangle,$$

for all $\varphi\in C_c^\infty([0,T)\times\Omega)$.

Proof. Let us note that it is enough to prove the result for $q=0$ as the general case follows by replacing F with $F-qu$.

We use the same notation as in the proof of Theorem 3.6, but this time the α_k are the coefficients in the expansion of φ in the orthonormal basis $(\phi_k)_{k\in\mathbb{N}}$, that is $\alpha_k=\langle\varphi,\phi_k\rangle_{L^2(\Omega)}$. We start with the identity

$$\int_0^T c_k''\alpha_k\,dt = \int_0^T c_k\alpha_k''\,dt + u_0^k\alpha_k'(0)-u_1^k\alpha_k(0)$$

(see (3.27)). Now, using the relation (3.8) we deduce the equality

$$\int_0^T(-\lambda_k c_k+F_k)\,\alpha_k\,dt = \int_0^T c_k\alpha_k''\,dt+u_0^k\alpha_k'(0)-u_1^k\alpha_k(0).$$

This is equivalent to

$$\int_0^T c_k\left(\alpha_k''+\lambda_k\alpha_k\right)dt=\int_0^T F_k\alpha_k\,dt+u_1^k\alpha_k(0)-u_0^k\alpha_k'(0). \tag{3.36}$$

Next note that

$$\begin{aligned}\langle(-\Delta)^s\varphi(t),\phi_k\rangle_{L^2(\Omega)}&=\langle(-\Delta)^s\varphi(t),\phi_k\rangle_{L^2(\mathbb{R}^n)}\\&=\langle(-\Delta)^{s/2}\varphi(t),(-\Delta)^{s/2}\phi_k\rangle_{L^2(\mathbb{R}^n)}\\&=\lambda_k\langle\varphi(t),\phi_k\rangle_{L^2(\Omega)}\\&=\lambda_k\alpha_k(t)\end{aligned}$$

for all $0\leq t\leq T$. Using $\langle\phi_k,\phi_\ell\rangle_{L^2(\Omega)}=\delta_{k\ell}$, we may write

$$\left\langle\sum_{k=1}^m c_k\phi_k,\sum_{\ell=1}^m\langle(-\Delta)^s\varphi,\phi_\ell\rangle_{L^2(\Omega)}\phi_\ell\right\rangle_{L^2(\Omega)}=\sum_{k=1}^m\lambda_k c_k\alpha_k.$$

Since $u,\chi_\Omega(-\Delta)^s\varphi\in L^2(0,T;\widetilde{L}^2(\Omega))$, where χ_Ω is the characteristic function of Ω, the time integral of the left hand side converges to $\langle u,(-\Delta)^s\varphi\rangle_{L^2(\Omega_T)}$. We also have

$$\int_0^T\left\langle\sum_{k=1}^m c_k\phi_k,\sum_{\ell=1}^m\langle\partial_t^2\varphi,\phi_\ell\rangle_{L^2(\Omega)}\phi_\ell\right\rangle_{L^2(\Omega)}dt=\int_0^T\sum_{k=1}^m c_k\alpha_k''\,dt\to\int_{\Omega_T}u\partial_t^2\varphi\,dxdt$$

as $m\to\infty$. Thus, summing the identity (3.36) from $k=1$ to m and passing to the limit $m\to\infty$ yields

$$\int_{\Omega_T}u(\partial_t^2\varphi+(-\Delta)^s\varphi)\,dxdt=\int_0^T\langle F,\varphi\rangle\,dt+\langle u_0,\partial_t\varphi(0)\rangle_{L^2(\Omega)}-\langle u_1,\varphi(0)\rangle.$$

For the convergence of the term involving F we refer to Step 4 in the proof of Theorem 3.6. Hence, we can conclude the proof. □

Proposition 3.9 (Weak solutions are very weak solutions)**.** *Suppose that $F\in L^2(0,T;\widetilde{L}^2(\Omega))$, $u_0\in\widetilde{H}^s(\Omega)$, $u_1\in\widetilde{L}^2(\Omega)$, $q\in L^p(\Omega)$ with p satisfying the restrictions (1.4) and $u\in C([0,T];\widetilde{H}^s(\Omega))\cap C^1([0,T];\widetilde{L}^2(\Omega))$ is a weak solution of (3.31). Then u is a very weak solution of (3.31).*

Proof. Let us prove the result only in the case $q=0$ as the same proof applies in the general case $q\neq0$. For the necessary modifications, we refer the reader to [LTZ24, Proof of Proposition 4.1].

Let $G\in L^2(0,T;\widetilde{L}^2(\Omega))$ and suppose $w\in C([0,T];\widetilde{H}^s(\Omega))\cap C^1([0,T];\widetilde{L}^2(\Omega))$ is the unique solution to

$$\begin{cases}\partial_t^2w+(-\Delta)^sw=G & \text{in }\Omega_T,\\ w=0 & \text{in }(\Omega_e)_T,\\ w(T)=0,\quad\partial_tw(T)=0 & \text{in }\Omega.\end{cases} \tag{3.37}$$

We follow now the proof of [LTZ24, Claim 4.2], that is we consider the parabolic regularized problems

$$\begin{cases}\partial_t^2u+\varepsilon(-\Delta)^s\partial_tu+(-\Delta)^su=F & \text{in }\Omega_T,\\ u=0 & \text{in }(\Omega_e)_T,\\ u(0)=u_0,\quad\partial_tu(0)=u_1 & \text{in }\Omega\end{cases}$$

and

$$\begin{cases}\partial_t^2w-\varepsilon(-\Delta)^s\partial_tw+(-\Delta)^sw=G & \text{in }\Omega_T,\\ w=0 & \text{in }(\Omega_e)_T,\\ w(T)=\partial_tw(T)=0 & \text{in }\Omega\end{cases}$$

for $\varepsilon > 0$. By [Zim24, Theorem 3.1] or [LM12, Chapter 3, Theorem 8.3], these regularized problems have a unique (weak) solution

$$u_\varepsilon \in C([0,T];\widetilde{H}^s(\Omega)) \text{ with } \begin{cases} \partial_t u_\varepsilon \in L^2(0,T;\widetilde{H}^s(\Omega)) \cap C([0,T];\widetilde{L}^2(\Omega)) \\ \partial_t^2 u_\varepsilon \in L^2(0,T;H^{-s}(\Omega)) \end{cases}$$

$$w_\varepsilon \in C([0,T];\widetilde{H}^s(\Omega)) \text{ with } \begin{cases} \partial_t w_\varepsilon \in L^2(0,T;\widetilde{H}^s(\Omega)) \cap C([0,T];\widetilde{L}^2(\Omega)) \\ \partial_t^2 w_\varepsilon \in L^2(0,T;H^{-s}(\Omega)) \end{cases}$$

and as $\varepsilon \to 0$, one has

$$\begin{aligned} u_\varepsilon &\to u \text{ in } C([0,T];\widetilde{H}^s(\Omega)), \\ \partial_t u_\varepsilon &\to \partial_t u \text{ in } C([0,T];\widetilde{L}^2(\Omega)), \\ \partial_t^2 u_\varepsilon &\rightharpoonup \partial_t^2 u \text{ in } L^2(0,T;H^{-s}(\Omega)) \end{aligned} \tag{3.38}$$

(cf. [LM12, Chapter 3, eq. (8.74)]). This ensures that

$$\partial_t^2 u_\varepsilon \overset{*}{\rightharpoonup} \partial_t^2 u \text{ in } L^2(0,T;H^{-s}(\Omega)) \text{ as } \varepsilon \to 0.$$

The convergence results in (3.38) hold for the functions w_ε and w as well. Now, using twice integration by parts, which is allowed by the regularity of the first time derivative of u_ε and w_ε, we obtain

$$\int_0^T \left\langle \partial_t^2 u_\varepsilon, w_\varepsilon \right\rangle dt = \int_0^T \left\langle \partial_t^2 w_\varepsilon, u_\varepsilon \right\rangle dt - \langle u_1, w_\varepsilon(0)\rangle_{L^2(\Omega)} + \langle u_0, \partial_t w_\varepsilon(0)\rangle_{\widetilde{H}^s(\Omega)}$$

for any $\varepsilon > 0$ (cf. [Zim24, eq. (4.1)]). In this computation, we used the final and initial time conditions for w_ε and u_ε, respectively. Now, passing to the limit $\varepsilon \to 0$ gives

$$\int_0^T \left\langle \partial_t^2 u, w \right\rangle dt = \int_0^T \left\langle \partial_t^2 w, u \right\rangle dt - \langle u_1, w(0)\rangle_{L^2(\Omega)} + \langle u_0, \partial_t w(0)\rangle_{L^2(\Omega)}.$$

By (3.31) and (3.37) this is equivalent to

$$\int_0^T \langle F, w\rangle\, dt = \int_0^T \langle G, u\rangle\, dt - \langle u_1, w(0)\rangle_{L^2(\Omega)} + \langle u_0, \partial_t w(0)\rangle_{L^2(\Omega)}.$$

Hence, we can conclude the proof. □

Corollary 3.10 (Regular very weak solutions = weak solutions)**.** *Suppose that $F \in L^2(0,T;\widetilde{L}^2(\Omega))$, $u_0 \in \widetilde{H}^s(\Omega)$, $u_1 \in \widetilde{L}^2(\Omega)$, $q \in L^p(\Omega)$ with p satisfying the restrictions* (1.4) *and u is a very weak solution to* (3.31) *such that $u \in C([0,T];\widetilde{H}^s(\Omega)) \cap C^1([0,T];\widetilde{L}^2(\Omega))$. Then u is a weak solution of* (3.31).

Proof. By the usual well-posedness result the problem (3.31) has a unique weak solution v. Using Proposition 3.9, one sees that v is a very weak solution to the same problem. By uniqueness of very weak solutions it follows that $u = v$, which in turn implies the assertion. □

4. Runge approximation and inverse problem for linear NWEQs

As we mentioned in Section 1, a key ingredient to studying nonlocal inverse problems is based on the Runge approximation. In this section, we establish the proof of Theorem 1.2.

4.1. **Runge approximation.**

Proof of Theorem 1.2. As usual, we show the Runge approximation property by a Hahn–Banach argument. Hence, we need to show that given $F \in L^2(0,T;H^{-s}(\Omega))$ vanishing on $\mathscr{R}_W$, it follows that $F=0$. First observe that if u solves (1.6), then $v = u-\varphi$ is the unique solution to

$$\begin{cases} \partial_t^2 v + (-\Delta)^s v + qv = -(-\Delta)^s\varphi & \text{in } \Omega_T, \\ v=0 & \text{in } (\Omega_e)_T, \\ v(0)=\partial_t v(0)=0 & \text{in } \Omega. \end{cases}$$

Now, by Theorem 3.7 there is a unique solution w of

$$\begin{cases} \partial_t^2 w + (-\Delta)^s w + qw = F & \text{in } \Omega_T, \\ w=0 & \text{in } (\Omega_e)_T, \\ w(T)=\partial_t w(T)=0 & \text{in } \Omega. \end{cases}$$

As $\chi_\Omega(-\Delta)^s\varphi \in L^2(0,T;\widetilde{L}^2(\Omega))$, we can use v as a test function for the equation of w to obtain

$$-\int_0^T \langle w(t), (-\Delta)^s\varphi(t)\rangle_{L^2(\Omega)}\,dt = \int_0^T \langle F(t), v(t)\rangle\,dt$$

(see (3.2)). By assumption, the right-hand side vanishes and hence

$$\int_0^T \langle w(t), (-\Delta)^s\varphi(t)\rangle_{L^2(\Omega)}\,dt = 0.$$

By taking $\varphi(x,t)=\eta(t)\psi(x)$ with $\eta\in C_c^\infty((0,T))$ and $\psi\in C_c^\infty(W)$, this implies

$$(-\Delta)^s w(t) = 0 \quad \text{for } x\in W \text{ and a.e. } t\in[0,T].$$

As $w\in L^2(0,T;\widetilde{L}^2(\Omega))$ we know that $w=0$ in Ω_e and hence the unique continuation principle ensures $w=0$ in $\mathbb{R}^n$. Now according to Proposition 3.8 very weak solutions are distributional solutions and thus we deduce that

$$\int_0^T \langle F,\Phi\rangle\,dt = 0$$

for all $\Phi\in C_c^\infty([0,T)\times\Omega)$ (see (3.35)). This in particular shows that $F=0$ as $C_c^\infty(\Omega_T)$ is dense in $L^2(0,T;\widetilde{H}^s(\Omega))$ and $(L^2(0,T;\widetilde{H}^s(\Omega)))^* = L^2(0,T;H^{-s}(\Omega))$. This proves the assertion. ☐

4.2. **DN maps for NWEQs.**

Let us next recall the rigorous definition of the DN map related to NWEQs.

Definition 4.1 (DN map)**.** *Let $\Omega\subset\mathbb{R}^n$ be a bounded Lipschitz domain, $s>0$ a non-integer, $T>0$ and $q\in L^p(\Omega)$ with p satisfying the restrictions (1.4). Then we define the DN map Λ_q related to*

$$\begin{cases} \partial_t^2 u + (-\Delta)^s u + qu = 0 & \text{in } \Omega_T, \\ u=\varphi & \text{in } (\Omega_e)_T, \\ u(0)=\partial_t u(0)=0 & \text{in } \Omega \end{cases} \tag{4.1}$$

by

$$\langle \Lambda_q\varphi,\psi\rangle := \int_{\mathbb{R}^n_T} (-\Delta)^{s/2}u\,(-\Delta)^{s/2}\psi\,dxdt, \tag{4.2}$$

for all $\varphi,\psi\in C_c^\infty((\Omega_e)_T)$, where $u\in C([0,T];H^s(\mathbb{R}^n))\cap C^1([0,T];L^2(\mathbb{R}^n))$ is the unique solution of (4.1) (see [LTZ24, Theorem 3.1 & Remark 3.2]).

Remark 4.2. *Let us mention for our later study of nonlinear NWEQs that if qu is replaced by a nonlinear function $f(x,u)$ such that $f\colon \Omega\times\mathbb{R}\to\mathbb{R}$ satisfies Assumption 1, then we can still define the DN map Λ_f by (4.2) for all $\varphi,\psi\in C_c^\infty((\Omega_e)_T)$, where this time $u\in C([0,T];H^s(\mathbb{R}^n))\cap C^1([0,T];L^2(\mathbb{R}^n))$ is the unique solution of DN map Λ_f related to (1.3) (see* [LTZ24, Proposition 3.7]*).*

4.3. **Proof of Theorem 1.3.** Before giving the proof of Theorem 1.3, let us introduce the following notation

$$u^\star(x,t)=u(x,T-t)$$

for the time reversal of the function $u\colon\mathbb{R}^n_T\to\mathbb{R}$. We first derive a suitable integral identity (cf. e.g. [KLW22, Lemma 2.4] or [Zim24, Lemm 4.3]) and use our improved Runge approximation (Theorem 1.2) to conclude the desired result.

Lemma 4.3 (Integral identity)**.** *For any $\varphi_1,\varphi_2\in C_c^\infty((W_1)_T)$, there holds that*

$$\langle(\Lambda_{q_1}-\Lambda_{q_2})\varphi_1,\varphi_2^\star\rangle=\int_{\Omega_T}(q_1-q_2)\,(u_1-\varphi_1)\,(u_2-\varphi_2)^\star\,dxdt, \tag{4.3}$$

where u_j is the unique solution of

$$\begin{cases}\left(\partial_t^2+(-\Delta)^s+q_j\right)u=0 & \text{in } \Omega_T\\ u=\varphi_j & \text{in } (\Omega_e)_T,\\ u(0)=\partial_t u(0)=0 & \text{in } \Omega\end{cases}$$

for $j=1,2$.

Proof. Let us start by observing that the function $(u_2-\varphi_2)^\star$ is the unique solution of

$$\begin{cases}\left(\partial_t^2+(-\Delta)^s+q_2\right)u=-(-\Delta)^s\varphi_2^\star & \text{in } \Omega_T\\ u=0 & \text{in } (\Omega_e)_T,\\ u(T)=\partial_t u(T)=0 & \text{in } \Omega\end{cases}$$

and thus by [LTZ24, Claim 4.2] we know that there holds

$$\int_0^T\left\langle\partial_t^2(u_1-\varphi_1),(u_2-\varphi_2)^\star\right\rangle dt=\int_0^T\left\langle\partial_t^2(u_2-\varphi_2)^\star,(u_1-\varphi_1)\right\rangle dt. \tag{4.4}$$

Thus, using the PDEs for $u_1-\varphi_1$ and $(u_2-\varphi_2)^\star$, (4.4) and the symmetry of the fractional Laplacian, we may calculate

$$\begin{aligned}
&\int_{\Omega_T}(q_1-q_2)(u_1-\varphi_1)(u_2-\varphi_2)^\star\,dxdt\\
&=-\int_0^T\left\langle(\partial_t^2+(-\Delta)^s)(u_1-\varphi_1),(u_2-\varphi_2)^\star\right\rangle dt\\
&\quad+\int_0^T\left\langle(\partial_t^2+(-\Delta)^s)(u_2-\varphi_2)^\star,u_1-\varphi_1\right\rangle dt\\
&\quad-\int_0^T\langle(-\Delta)^s\varphi_1,(u_2-\varphi_2)^\star\rangle\,dt+\int_0^T\langle(-\Delta)^s\varphi_2^\star,u_1-\varphi_1\rangle\,dt\\
&=-\int_0^T\langle(-\Delta)^s(u_1-\varphi_1),(u_2-\varphi_2)^\star\rangle\,dt+\int_0^T\langle(-\Delta)^s(u_2-\varphi_2)^\star,u_1-\varphi_1\rangle\,dt\\
&\quad-\int_0^T\langle(-\Delta)^s\varphi_1,(u_2-\varphi_2)^\star\rangle\,dt+\int_0^T\langle(-\Delta)^s\varphi_2^\star,u_1-\varphi_1\rangle\,dt\\
&=-\int_0^T\langle(-\Delta)^s u_2,\varphi_1^\star\rangle\,dt+\int_0^T\langle(-\Delta)^s u_1,\varphi_2^\star\rangle\,dt.
\end{aligned}$$

By definition of the DN map, we deduce that

$$\langle \Lambda_{q_1}\varphi_1, \varphi_2^\star \rangle - \langle \Lambda_{q_2}\varphi_2, \varphi_1^\star \rangle = \int_{\Omega_T} (q_1 - q_2)(u_1 - \varphi_1)(u_2 - \varphi_2)^\star \, dxdt,$$

which completes the proof. □

Proof of Theorem 1.3. We prove Theorem 1.3 by using the Runge argument. Using Theorem 1.2, we can approximate any $\Psi_j \in C_c^\infty(\Omega)$, $j = 1, 2$, in $L^2(0, T; \widetilde{H}^s(\Omega))$ by sequences in the Runge sets $\mathscr{R}_{W_1}$ and $\mathscr{R}_{W_2}$, respectively. Since $q_j \in L^p(\Omega)$ satisfies the estimate (3.32), we may pass in (4.3) to the limit and hence taking the condition (1.7) into account, we arrive at

$$\int_{\Omega_T} (q_1 - q_2)\Psi_1\Psi_2^\star \, dxdt = 0$$

for all $\Psi_1, \Psi_2 \in C_c^\infty(\Omega_T)$. This ensures that $q_1 = q_2$ in Ω as we wanted to show. □

5. Well-posedness and inverse problems for nonlinear NWEQs

In this section, we study the inverse problems for NWEQs with polyhomogeneous nonlinearities. We start in Section 5.1 by showing that for any asymptotically polyhomogeneous nonlinearity the expansion is unique. Then, in Section 5.2, by using similar techniques as in [LTZ24] and our stronger Runge approximation (Theorem 1.2), we demonstrate Theorem 1.5. Let us note that in the case of asymptotically polyhomogeneous nonlinearities, Theorem 1.5 only shows that the expansion coefficients coincide and not the nonlinearities themselves. This could be improved to $f^{(1)} = f^{(2)}$ in the range $2s > n$ by imposing a suitable decay of the constants C_N, $\sum_{k=1}^{N-1} b_k$ as $N \to \infty$, appearing in Definition 1.4, (ii), but we do not investigate this further in this article.

5.1. Uniqueness of asymptotic expansion.

5.1. Uniqueness of asymptotic expansion. Before discussing the proof of Theorem 1.5, let us make the following observation.

Lemma 5.1. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain. Assume that we have given a sequence $(r_k)_{k=1}^\infty \subset \mathbb{R}$ satisfying $0 < r_k < r_{k+1}$ for all $k \in \mathbb{N}$. Let $f\colon \Omega \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function and suppose there holds*

$$f \sim \sum_{k \geq 1} f_k \quad \text{and} \quad f \sim \sum_{k \geq 1} \widetilde{f}_k,$$

for two sequences $(f_k)_{k\in\mathbb{N}}$, $(\widetilde{f}_k)_{k\in\mathbb{N}}$ satisfying the assumptions in Definition 1.4, (ii) for the same sequence $(r_k)_{k\in\mathbb{N}}$ but with possibly different constants C_N and $\widetilde{C}_N$ for $N \in \mathbb{N}_{\geq 2}$. Then we have $f_k = \widetilde{f}_k$ for all $k \in \mathbb{N}$.

Proof. By assumption we may compute

$$\begin{aligned}
\left|f_1(x,1) - \widetilde{f}_1(x,1)\right| &= \tau^{r_1+1}\tau^{-r_1-1}\left|f_1(x,1) - \widetilde{f}_1(x,1)\right| \\
&= \tau^{-r_1-1}\left|f_1(x,\tau) - \widetilde{f}_1(x,\tau)\right| \\
&\leq \tau^{-r_1-1}\left(|f(x,\tau) - \widetilde{f}_1(x,\tau)| + |f(x,\tau) - \widetilde{f}_1(x,\tau)|\right) \\
&\leq \left(C_2 + \widetilde{C}_2\right)\tau^{r_2-r_1}
\end{aligned}$$

for $x \in \Omega$ and $|\tau| \leq 1$. Thus by passing to the limit $\tau \to 0$, we deduce that $f_1(x,1) = \widetilde{f}_1(x,1)$ for all $x \in \Omega$ and by homogeneity $f_1 = \widetilde{f}_1$. Continuing inductively we obtain that $f_k = \widetilde{f}_k$ for all $k \geq 1$. □

5.2. **Recovery of coefficients of polyhomogeneous nonlinearities.** Let us start by recalling the following continuity result on Nemytskii operators.

Lemma 5.2 (Continuity of Nemytskii operators, [AP95, Theorem 2.2] and [Zim24, Lemma 3.6])**.** *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain, $T>0$, and $1 \leq q,p < \infty$. Assume that $f\colon \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function satisfying*

$$|f(x,\tau)| \leq a + b|\tau|^\alpha$$

for some constants $a,b \geq 0$ and $0 < \alpha \leq \min(p,q)$. Then the Nemytskii operator f, defined by

$$f(u)(x,t) := f(x,u(x,t)) \tag{5.1}$$

for all measurable functions $u\colon \Omega_T \to \mathbb{R}$, maps continuously $L^p(\Omega)$ to $L^{p/\alpha}(\Omega)$ and $L^q(0,T;L^p(\Omega))$ into $L^{q/\alpha}(0,T;L^{p/\alpha}(\Omega))$.

From now on we will use the notation (5.1) introduced in the previous lemma.

Proof of Theorem 1.5. Our goal is to show in the first step that the equality of the DN maps for two nonlinearities $f^{(1)}$ and $f^{(2)}$ implies $f^{(1)}(x,v) = f^{(2)}(x,v)$ for any solution to a linear wave equation (see (5.3)). Then in a second step, we use the Runge approximation property of the solutions v in $L^2(0,T;\widetilde{H}^s(\Omega))$ to deduce that all expansion coefficients $f_k^{(1)}$, $f_k^{(2)}$ for $k \in \mathbb{N}$ agree. In the following we will often abbreviate $f(x,u) =: f(u)$.

Let $\varepsilon > 0$. We start by observing that for $j=1,2$ the unique (weak) solution $u_\varepsilon^{(j)}$ of

$$\begin{cases} \partial_t^2 u + (-\Delta)^s u + f^{(j)}(u) = 0 & \text{in } \Omega_T, \\ u = \varepsilon\varphi & \text{in } (\Omega_e)_T, \\ u(0) = \partial_t u(0) = 0 & \text{in } \Omega, \end{cases} \tag{5.2}$$

can be expanded as $u_\varepsilon^{(j)} = \varepsilon v + R_\varepsilon^{(j)}$, where v and $R_\varepsilon^{(j)}$ solve

$$\begin{cases} \partial_t^2 v + (-\Delta)^s v = 0 & \text{in } \Omega_T, \\ v = \varphi & \text{in } (\Omega_e)_T, \\ v(0) = \partial_t v(0) = 0 & \text{in } \Omega \end{cases} \tag{5.3}$$

and

$$\begin{cases} \partial_t^2 R + (-\Delta)^s R = -f^{(j)}\big(u_\varepsilon^{(j)}\big) & \text{in } \Omega_T, \\ R = 0 & \text{in } (\Omega_e)_T, \\ R(0) = \partial_t R(0) = 0 & \text{in } \Omega, \end{cases}$$

respectively. By (1.10), the UCP for the fractional Laplacian and (5.2) guarantee

$$u_\varepsilon := u_\varepsilon^{(1)} = u_\varepsilon^{(2)},\ R_\varepsilon := R_\varepsilon^{(1)} = R_\varepsilon^{(2)} \text{ and } f^{(1)}(u_\varepsilon) = f^{(2)}(u_\varepsilon). \tag{5.4}$$

Furthermore, as shown in [LTZ24, Proof of Theorem 1.1], there holds:

(i) $\big\|f^{(j)}(\cdot,u_\varepsilon)\big\|_{L^2(\Omega_T)} \lesssim \|u_\varepsilon\|_{L^\infty(0,T;H^s(\mathbb{R}^n))}^{r_\infty+1}$,
(ii) $\|R_\varepsilon\|_{L^\infty(0,T;\widetilde{H}^s(\Omega))} + \|\partial_t R_\varepsilon\|_{L^\infty(0,T;L^2(\Omega))} \lesssim \|u_\varepsilon\|_{L^\infty(0,T;H^s(\mathbb{R}^n))}^{r_\infty+1}$
(iii) $\|u_\varepsilon\|_{L^\infty(0,T;H^s(\mathbb{R}^n))} + \|\partial_t u_\varepsilon\|_{L^\infty(0,T;L^2(\mathbb{R}^n))} \lesssim \varepsilon$

In (i) and (ii) we used the fact that $f^{(j)}(x,0) = 0$. For serially polyhomogeneous nonlinearities this is clear and for asymptotically polyhomogeneous nonlinearities $f^{(j)}$ this follows from (1.9) and the homogeneity of the coefficients $f_k^{(j)}$ for $k \in \mathbb{N}$. Therefore, we obtain

$$\|R_\varepsilon\|_{L^\infty(0,T;\widetilde{H}^s(\Omega))} + \|\partial_t R_\varepsilon\|_{L^\infty(0,T;L^2(\Omega))} \lesssim \varepsilon^{r_\infty+1}. \tag{5.5}$$

Now, (5.4) implies

$$f^{(1)}(\varepsilon v+R_\varepsilon)=f^{(2)}(\varepsilon v+R_\varepsilon). \tag{5.6}$$

Our next goal is to inductively recover the terms in the series of expressions

$$\sum_{k\geq 1} f_k^{(j)}(\tau),\quad j=1,2,$$

of $f^{(j)}$ starting from the term $f_1^{(j)}$. We distinguish two cases $2s<n$ and $2s\geq n$ as follows:

Case $2s<n$.

As r_∞ satisfies $0<r_\infty\leq\frac{2s}{n-2s}$, we have

$$1<1+r_\infty\leq\frac{n}{n-2s}<\frac{2n}{n-2s}=:p. \tag{5.7}$$

Using (5.5) and the Sobolev embedding, we see that

$$\varepsilon^{-1}R_\varepsilon\to 0\quad\text{in}\quad L^q(0,T;L^p(\Omega)) \tag{5.8}$$

as $\varepsilon\to 0$ for any $1\leq q\leq\infty$. Recall the assumption that

$$\begin{aligned}\left|f^{(j)}(x,\tau)\right|&\leq A^{(j)}+B^{(j)}|\tau|^{r_\infty+1},\\ \left|f_k^{(j)}(x,\tau)\right|&\leq b_k^{(j)}|\tau|^{r_k+1}\end{aligned}$$

for some constants $A^{(j)},B^{(j)}\geq 0$ (see Assumption 1), which by Lemma 5.2 and (5.7) directly yield that

$$\begin{aligned}f^{(j)}&\colon L^q(0,T;L^p(\Omega))\to L^{\frac{q}{r_\infty+1}}(0,T;L^{\frac{p}{r_\infty+1}}(\Omega)),\\ f_k^{(j)}&\colon L^q(0,T;L^p(\Omega))\to L^{\frac{q}{r_k+1}}(0,T;L^{\frac{p}{r_k+1}}(\Omega)),\quad k\in\mathbb{N},\end{aligned}$$

are continuous as long as q is chosen such that $q\geq r_\infty+1$ and $q\geq r_k+1$, respectively. Here $r_k<r_\ell$, when $k<\ell\leq\infty$, so there holds

$$L^{\frac{q}{r_k+1}}(0,T;L^{\frac{p}{r_k+1}}(\Omega))\subset L^{\frac{q}{r_\ell+1}}(0,T;L^{\frac{p}{r_\ell+1}}(\Omega))$$

and thus

$$f_k^{(j)}\colon L^q(0,T;L^p(\Omega))\to L^{\frac{q}{r_\ell+1}}(0,T;L^{\frac{p}{r_\ell+1}}(\Omega)),\quad k\in\mathbb{N},\ k\leq\ell\leq\infty \tag{5.9}$$

is continuous.

Serially polyhomogeneous nonlinearity for $\mathbf{k=1}$: Multiplying (5.6) by ε^{-r_1-1} and using the homogeneity of $f_k^{(j)}$, we have pointwise the identity

$$\varepsilon^{-r_1-1}f^{(j)}(u_\varepsilon)=\sum_{k=1}^\infty f_k^{(j)}(\varepsilon^{-\frac{r_1+1}{r_k+1}}u_\varepsilon),\quad j=1,2,$$

where

$$\begin{cases}-\frac{r_1+1}{r_k+1}=-1, & \text{if } k=1,\\ -\frac{r_1+1}{r_k+1}>-1, & \text{if } k\geq 2.\end{cases} \tag{5.10}$$

Next, let us denote by $C_S>0$ the optimal Sobolev constant for the embedding $H^s(\mathbb{R}^n)\hookrightarrow L^p(\mathbb{R}^n)$ and by $D>0$ the constant in the estimate (iii). Furthermore, recall that we have

$$\left|f_k^{(j)}(x,\tau)\right|\leq b_k^{(j)}|\tau|^{r_k+1},\quad r_k<r_{k+1}\leq r_\infty \tag{5.11}$$

for $j = 1, 2$. Also note that if $q \geq 1$ is sufficiently large, then

$$1 \leq \frac{p}{r_\infty - r_k} < \infty,\ 1 \leq \frac{q}{r_\infty - r_k} < \infty$$

satisfy

$$\frac{r_\infty + 1}{p} = \frac{r_k + 1}{p} + \frac{r_\infty - r_k}{p} \text{ and } \frac{r_\infty + 1}{q} = \frac{r_k + 1}{q} + \frac{r_\infty - r_k}{q}.$$

Thus, for $0 < \varepsilon \leq 1$, we may compute

(5.12)

$$\begin{aligned}
&\left\| f_k^{(j)}\left(\varepsilon^{-\frac{r_1+1}{r_k+1}} u_\varepsilon\right) \right\|_{L^{\frac{q}{r_\infty+1}} L^{\frac{p}{r_\infty+1}}} \\
&= \varepsilon^{-(r_1+1)} \left\| f_k^{(j)}(u_\varepsilon) \right\|_{L^{\frac{q}{r_\infty+1}} L^{\frac{p}{r_\infty+1}}} \\
&\leq \varepsilon^{-(r_1+1)} |\Omega|^{\frac{r_\infty - r_k}{p}} T^{\frac{r_\infty - r_k}{q}} \| f_k^{(j)}(u_\varepsilon) \|_{L^{\frac{q}{r_k+1}} L^{\frac{p}{r_k+1}}} \quad \text{(by Hölder's inequality)} \\
&\leq \varepsilon^{-(r_1+1)} |\Omega|^{\frac{r_\infty - r_k}{p}} T^{\frac{r_\infty - r_k}{q}} b_k^{(j)} \| u_\varepsilon \|_{L^q L^p}^{r_k+1} \quad \text{(by (5.11))} \\
&\leq \varepsilon^{-(r_1+1)} |\Omega|^{\frac{r_\infty - r_k}{p}} T^{\frac{r_\infty + 1}{q}} b_k^{(j)} \| u_\varepsilon \|_{L^\infty L^p}^{r_k+1} \\
&\leq \varepsilon^{-(r_1+1)} C_s^{r_k+1} |\Omega|^{\frac{r_\infty - r_k}{p}} T^{\frac{r_\infty + 1}{q}} b_k^{(j)} \| u_\varepsilon \|_{L^\infty H^s}^{r_k+1} \quad \text{(by Sobolev's inequality)} \\
&\leq \varepsilon^{r_k - r_1} (C_s D)^{r_k+1} |\Omega|^{\frac{r_\infty - r_k}{p}} T^{\frac{r_\infty + 1}{q}} b_k^{(j)} \quad \text{(by (iii))} \\
&\leq \varepsilon^{r_k - r_1} (\max(1, C_s D))^{r_k+1} (\max(1, |\Omega|))^{\frac{r_\infty - r_k}{p}} T^{\frac{r_\infty + 1}{q}} b_k^{(j)} \\
&\leq (\max(1, C_s D))^{r_\infty+1} (\max(1, |\Omega|))^{\frac{r_\infty}{p}} T^{\frac{r_\infty + 1}{q}} b_k^{(j)} \quad \text{(by (5.11) and } \varepsilon \leq 1) \\
&=: M_k^{(j)},
\end{aligned}$$

where we abbreviated $L^\alpha(0, T; X(U))$ as $L^\alpha X$ for any Banach space $X(U)$ over a spatial domain $U \subset \mathbb{R}^n$ and $1 \leq \alpha \leq \infty$. Since the constants in $M_k^{(j)}$ in front of $b_k^{(j)}$ are independent of k, then (1.11) and the ratio test imply

$$\sum_{k \geq 1} M_k^{(j)} < \infty.$$

Next, by (5.10) and (iii), we see that for $k \geq 2$ and sufficiently large $q \geq 1$ there holds

$$\varepsilon^{-\frac{r_1+1}{r_k+1}} u_\varepsilon \to 0 \text{ in } L^q(0, T; L^p(\Omega))$$

as $\varepsilon \to 0$. Hence, (5.9) and $f_k^{(j)}(0) = 0$ assure that

$$f_k^{(j)}\left(\varepsilon^{-\frac{r_1+1}{r_k+1}} u_\varepsilon\right) \to 0 \text{ in } L^{\frac{q}{r_\infty+1}}(0, T; L^{\frac{p}{r_\infty+1}}(\Omega)) \tag{5.13}$$

as $\varepsilon \to 0$. On the other hand for $k = 1$ we have

$$\varepsilon^{-1} u_\varepsilon \to v \text{ in } L^q(0, T; L^p(\Omega))$$

as $\varepsilon \to 0$ (see (5.8)), which guarantees by (5.9) that

$$f_1^{(j)}(\varepsilon^{-1} u_\varepsilon) \to f(v) \text{ in } L^{\frac{q}{r_\infty+1}}(0, T; L^{\frac{p}{r_\infty+1}}(\Omega)) \tag{5.14}$$

as $\varepsilon \to 0$. Now, using (5.12), (5.13) and (5.14), we can apply the dominated convergence theorem to deduce that

$$\lim_{\varepsilon \to 0} \varepsilon^{-r_1 - 1} f^{(j)}(u_\varepsilon) = \sum_{k \geq 1} \lim_{\varepsilon \to 0} f_k^{(j)}\left(\varepsilon^{-\frac{r_1+1}{r_k+1}} u_\varepsilon\right) = f_1^{(j)}(v)$$

in $L^{\frac{q}{r_\infty+1}}(0, T; L^{\frac{p}{r_\infty+1}}(\Omega))$, which finally implies

$$f_1^{(1)}(v) = f_1^{(2)}(v) \tag{5.15}$$

in $L^{\frac{q}{r_\infty+1}}(0,T;L^{\frac{p}{r_\infty+1}}(\Omega))$.

Asymptotically polyhomogeneous nonlinearity for $\mathbf{k=1}$: In this case, using Definition 1.4 (ii), we get for $N=2$ that

$$\left|f^{(j)}(u_\varepsilon)-f_1^{(j)}(u_\varepsilon)\right|\leq C_2\,|u_\varepsilon|^{r_2+1}.$$

Multiplying the above inequality by ε^{-r_1-1}, recalling $r_2>r_1$, and using (iii) we get in the $L^{\frac{q}{r_2+1}}L^{\frac{p}{r_2+1}}$-norm

$$\left\|\varepsilon^{-r_1-1}f^{(j)}(u_\varepsilon)-f_1^{(j)}(\varepsilon^{-1}u_\varepsilon)\right\|_{L^{\frac{q}{r_2+1}}L^{\frac{p}{r_2+1}}}\lesssim\varepsilon^{r_2-r_1}\to 0 \tag{5.16}$$

as $\varepsilon\to 0$. Therefore, first using the continuity of Nemytskii operators (see (5.9)) and then (5.16) we obtain

$$f_1^{(j)}(v)=\lim_{\varepsilon\to 0}f_1^{(j)}(\varepsilon^{-1}u_\varepsilon)=\lim_{\varepsilon\to 0}\varepsilon^{-r_1-1}f^{(j)}(u_\varepsilon)$$

in $L^{\frac{q}{r_2+1}}(0,T;L^{\frac{p}{r_2+1}}(\Omega))$. This implies

$$f_1^{(1)}(v)=f_1^{(2)}(v) \tag{5.17}$$

in $L^{\frac{q}{r_2+1}}(0,T;L^{\frac{p}{r_2+1}}(\Omega))$.

Next, let $\Psi\in C_c^\infty(\Omega_T)$. By Theorem 1.2, there exist a sequence $(\psi_k)_{k\in\mathbb{N}}\subset C_c^\infty((W_1)_T)$ such that the unique solutions $(v_k)_{k\in\mathbb{N}}$ of

$$\begin{cases}\partial_t^2 v_k+(-\Delta)^s v_k=0 & \text{in } \Omega_T,\\ v_k=\psi_k & \text{in } (\Omega_e)_T,\\ v_k(0)=\partial_t v_k(0)=0 & \text{in } \Omega\end{cases}$$

satisfy $v_k-\psi_k\to\Psi$ in $L^2(0,T;\widetilde{H}^s(\Omega))$ as $k\to\infty$. Up to extracting a subsequence, we have by Sobolev's embedding theorem that there holds

$$v_k(t)\to\Psi(t)\quad\text{in}\quad L^p(\Omega)$$

for a.e. $t\in[0,T]$. Hence, by Lemma 5.2, Hölder's inequality, and (5.15) or (5.17), we get

$$f_1^{(1)}(\Psi(t))=\lim_{k\to\infty}f_1^{(1)}(v_k(t))=\lim_{k\to\infty}f_1^{(2)}(v_k(t))=f_1^{(2)}(\Psi(t))$$

in $L^{\frac{p}{r_\infty+1}}(\Omega)$ (or $L^{\frac{p}{r_2+1}}(\Omega)$) for a.e. $t\in[0,T]$. As $f_1^{(j)}$ are Carathéodory functions this needs to hold for all $t\in[0,T]$ and hence

$$f_1^{(1)}(x,\Psi(x,t))=f_1^{(2)}(x,\Psi(x,t))$$

for all $(x,t)\in\Omega_T$. Now, let us fix $t_0\in(0,T)$ and $x_0\in\Omega$. Then we choose $\Psi(x,t)=\eta(t)\Phi(x)$ with $\eta\in C_c^\infty((0,T))$ and $\Phi\in C_c^\infty(\Omega)$, where η,Φ satisfy $\eta(t)=1$ in a neighborhood of t_0 and $\Phi(x)=1$ in a neighborhood of x_0. Therefore, evaluating the previous relation at $t=t_0$ we obtain

$$f_1^{(1)}(x,\Phi(x))=f_1^{(2)}(x,\Phi(x))$$

for a.e. $x\in\Omega$. This gives $f_1^{(1)}(x_0,1)=f_1^{(2)}(x_0,1)$. Now, the homogeneity assumptions on $f_k^{(j)}$ ensures that $f_1^{(1)}(x,\rho)=f_1^{(2)}(x,\rho)$ for all $x\in\Omega$ and $\rho\in\mathbb{R}$.

Using a similar approach, one can inductively recover the higher-order terms. In fact, one can argue as follows.

Serially polyhomogeneous nonlinearity for $\mathbf{k\geq 2}$: First, we define

$$f^{(j),2}:=\sum_{k\geq 2}f_k^{(j)}$$

and then by $f_1^{(1)} = f_1^{(2)}$ as well as (5.4), we deduce that

$$f^{(1),2}(u_\varepsilon) = f^{(2),2}(u_\varepsilon).$$

Repeating the same proof above, but this time multiplying with ε^{-r_2-1}, we deduce $f_2^{(1)} = f_2^{(2)}$. Thus, iteratively, we get $f_k^{(1)} = f_k^{(2)}$ for any $k \in \mathbb{N}$.

Asymptotically polyhomogeneous nonlinearity for $\mathbf{k \geq 2}$: Using $f_1^{(1)} = f_1^{(2)}$ and (1.9) for $N = 3$, we get

$$\begin{aligned}
&\|(\varepsilon^{-r_2-1} f^{(1)}(u_\varepsilon) - f_2^{(1)}(\varepsilon^{-1}u_\varepsilon)) - (\varepsilon^{-r_2-1} f^{(2)}(u_\varepsilon) - f_2^{(2)}(\varepsilon^{-1}u_\varepsilon))\|_{L^{\frac{q}{r_3+1}} L^{\frac{p}{r_3+1}}} \\
&\leq \sum_{j=1,2} \|\varepsilon^{-r_2-1}(f^{(j)}(u_\varepsilon) - f_1^{(j)}(u_\varepsilon) - f_2^{(j)}(u_\varepsilon))\|_{L^{\frac{q}{r_3+1}} L^{\frac{p}{r_3+1}}} \\
&\lesssim \varepsilon^{r_3-r_2} \to 0
\end{aligned}$$

as $\varepsilon \to 0$. Taking into account (5.9), this guarantees

$$\begin{aligned}
f_2^{(1)}(v) - f_2^{(2)}(v) &= \lim_{\varepsilon\to 0}(f_2^{(1)}(\varepsilon^{-1}u_\varepsilon) - f_2^{(2)}(\varepsilon^{-1}u_\varepsilon)) \\
&= \lim_{\varepsilon\to 0} \varepsilon^{-r_2-1}(f^{(1)}(u_\varepsilon) - f^{(2)}(u_\varepsilon)) = 0
\end{aligned}$$

in $L^{\frac{q}{r_3+1}} L^{\frac{p}{r_3+1}}$. Now, one can repeat the above argument to find that $f_2^{(1)} = f_2^{(2)}$. Therefore, we iteratively get $f_k^{(1)} = f_k^{(2)}$ for all $k \in \mathbb{N}$.

Case $2s \geq n$.

The proof is almost the same as the Sobolev embedding guarantees that we have $H^s(\mathbb{R}^n) \hookrightarrow L^p(\mathbb{R}^n)$ for any $2 \leq p < \infty$ (see [Oza95] for the critical case $2s = n$). Moreover, let us note that in the supercritical case $2s > n$, we only need (1.9) for $|\tau| \leq 1$ by the Sobolev embedding $H^s(\mathbb{R}^n) \hookrightarrow L^\infty(\mathbb{R}^n)$ and the estimate (iii).

Hence, we have shown that the expansion coefficients of the nonlinearities $f^{(j)}$, $j = 1, 2$, coincide in both cases and we can conclude the proof. □

Appendix A. Proof of Lemma 3.5

In this appendix, we provide the proof of the spectral theoretic lemma, Lemma 3.5. We again denote by $\langle \cdot, \cdot \rangle$ the duality pairing between $\widetilde{H}^s(\Omega)$ and $H^{-s}(\Omega)$, where the spaces $\widetilde{H}^s(\Omega)$, $H^{-s}(\Omega)$ are endowed with the norms $\|\cdot\|_{\widetilde{H}^s(\Omega)}$ and $\|\cdot\|_{H^{-s}(\Omega)}$ as before.

Proof of Lemma 3.5. We start by constructing the sequence of eigenvalues $(\lambda_k)_{k\in\mathbb{N}}$.

Let $L\colon \widetilde{L}^2(\Omega) \to \widetilde{L}^2(\Omega)$ be the compact self-adjoint operator given by $L = K \circ S$, where

$$S\colon \widetilde{L}^2(\Omega) \to \widetilde{H}^s(\Omega), \quad F \mapsto u$$

is the source-to-solution map of the problem

$$\begin{cases} (-\Delta)^s u = F & \text{in } \Omega, \\ u = 0 & \text{in } \Omega_e \end{cases}$$

and $K\colon \widetilde{H}^s(\Omega) \to \widetilde{L}^2(\Omega)$ denotes the usual inclusion, which is compact by the Rellich–Kondrachov theorem. Note that the solution map S is well-defined and continuous by the Lax–Milgram theorem. Hence, it is clear that L is compact.

The operator L is also self-adjoint, because the related bilinear form to $(-\Delta)^s$ is symmetric. Furthermore, if $F \in \widetilde{L}^2(\Omega)$ and $u = LF$, then we have

$$\langle LF, F\rangle_{L^2(\Omega)} = \langle u, F\rangle_{L^2(\Omega)} = \left\langle (-\Delta)^{s/2}u, (-\Delta)^{s/2}u\right\rangle_{L^2(\mathbb{R}^n)} = \|u\|^2_{\widetilde{H}^s(\Omega)} \geq 0.$$

If $F \neq 0$, then we have $\langle LF, F\rangle_{L^2(\Omega)} > 0$ as otherwise u would vanish and hence $F = 0$. Therefore, L is positive definite with $\ker L = \{0\}$. By the spectral theory for compact self-adjoint operators we deduce that $\sigma(L) = \sigma_p(L) \subset \mathbb{R}_+$ is at most countable with accumulation point $\mu = 0$. Here, $\sigma_p(L)$ denotes the point spectrum of L, that is the set of eigenvalues. Moreover, for any $\mu \in \sigma_p(L)$ its related eigenspace $\ker(L-\mu)$ is finite dimensional. Next, observe that $\mu > 0$ is an eigenvalue of L if and only if $\lambda = 1/\mu$ is an eigenvalue for $(-\Delta)^s$ and $F \in \widetilde{L}^2(\Omega)$ is an eigenfunction of L with eigenvalue $\mu > 0$ if and only if $u := SF \in \widetilde{H}^s(\Omega)$ is an eigenfunction of $(-\Delta)^s$ with eigenvalue $\lambda = 1/\mu$. Therefore, we may conclude that $\sigma_p((-\Delta)^s)$ is an unbounded countable sequence and the corresponding eigenspaces are finite-dimensional.

Step 1. The first eigenvalue.

Let us define

$$\lambda_1 = \inf\left\{\|u\|^2_{\widetilde{H}^s(\Omega)}\,;\, u \in \widetilde{H}^s(\Omega),\ \|u\|_{L^2(\Omega)} = 1\right\}.$$

We assert that $\lambda_1 > 0$ is the smallest eigenvalue associated to $(-\Delta)^s$. To see this, let $(u_k)_{k\in\mathbb{N}} \subset \widetilde{H}^s(\Omega)$ be a minimizing sequence, that is

$$\|u_k\|_{L^2(\Omega)} = 1 \text{ and } \lim_{k\to\infty} \|u_k\|^2_{\widetilde{H}^s(\Omega)} = \lambda_1.$$

In particular, this implies that $(u_k)_{k\in\mathbb{N}} \subset \widetilde{H}^s(\Omega)$ is uniformly bounded and hence up to extracting a subsequence there exists $\phi_1 \in \widetilde{H}^s(\Omega)$ such that $u_k \rightharpoonup \phi_1$ in $\widetilde{H}^s(\Omega)$ as $k \to \infty$. Up to extraction of a further subsequence, we can assume by the Rellich–Kondrachov theorem that $u_k \to \phi_1$ in $\widetilde{L}^2(\Omega)$ as $k \to \infty$ (we still denote the subsequence by $(u_k)_{k\in\mathbb{N}}$). The latter condition guarantees $\|\phi_1\|_{L^2(\Omega)} = 1$. Additionally, the lower semicontinuity of weak convergence ensures that $\|\phi_1\|^2_{\widetilde{H}^s(\Omega)} = \lambda_1$. Thus, $\phi_1 \in \widetilde{H}^s(\Omega)$ is a minimizer of the convex functional $u \mapsto \|u\|^2_{\widetilde{H}^s(\Omega)}$, whose Euler–Lagrange equation is

$$\left\langle (-\Delta)^{s/2}\phi_1, (-\Delta)^{s/2}v\right\rangle_{L^2(\mathbb{R}^n)} = \lambda_1 \langle \phi_1, v\rangle_{L^2(\mathbb{R}^n)}$$

for all $v \in \widetilde{H}^s(\Omega)$ (see [KRZ23, Theorem 2.1]). This is nothing else than that $\lambda_1 > 0$ is an eigenvalue and $\phi_1 \in \widetilde{H}^s(\Omega)$ is a related eigenfunction. That is, ϕ_1 solves

$$\begin{cases} (-\Delta)^s u = \lambda_1 u & \text{in } \Omega, \\ u = 0 & \text{in } \Omega_e. \end{cases}$$

Next, we show that $\lambda_1 > 0$ is the smallest eigenvalue. For this purpose, assume that $\lambda > 0$ is any eigenvalue with normalized eigenfunction $\psi \in \widetilde{H}^s(\Omega)$. Then we have

$$\|\psi\|_{L^2(\Omega)} = 1 \text{ and } \|\psi\|^2_{\widetilde{H}^s(\Omega)} = \lambda.$$

By the definition of λ_1, we get $\lambda \geq \lambda_1$.

Step 2. The k-th eigenvalue.

Let $k\geq 2$. Then we define

$$\begin{aligned}\lambda_k = \inf\Big\{ \|u\|^2_{\widetilde{H}^s(\Omega)}\,;\, u\in \widetilde{H}^s(\Omega),\, \|u\|_{L^2(\Omega)}=1, \\ \langle u,\phi_\ell\rangle_{L^2(\Omega)} = 0 \text{ for } 1\leq \ell\leq k-1\Big\},\end{aligned} \tag{A.1}$$

where $\phi_1,\dots,\phi_{k-1}$ are the normalized eigenfunctions corresponding to the eigenvalues $\lambda_1,\dots,\lambda_{k-1}$ and for all $1\leq\ell\leq k-1$ we have

$$\langle \phi_\ell,\phi_m\rangle_{L^2(\Omega)} = 0 \text{ for } 1\leq m\leq \ell-1.$$

Let us assume that that statement holds for $k-1$ and we aim to prove that it holds for k. As above we take a minimizing sequence $(u_\ell)_{\ell\in\mathbb{N}}$ of (A.1) and up to subtracting a subsequence, we can assume that

$$u_\ell \rightharpoonup \phi_k \quad \text{in} \quad \widetilde{H}^s(\Omega) \quad \text{and} \quad u_\ell\to\phi_k \quad \text{in} \quad \widetilde{L}^2(\Omega)$$

as $\ell\to\infty$ for some $\phi_k\in\widetilde{H}^s(\Omega)$. Furthermore, one can easily see that there holds

$$\|\phi_k\|_{L^2(\Omega)}=1,\ \langle\phi_k,\phi_\ell\rangle_{L^2(\Omega)}=0 \text{ for } 1\leq\ell\leq k-1 \text{ and } \lambda_k=\|\phi_k\|^2_{\widetilde{H}^s(\Omega)}. \tag{A.2}$$

Thus, ϕ_k is a minimizer with $\|\phi_k\|^2_{\widetilde{H}^s(\Omega)}=\lambda_k$. Next, let us define

$$w_t := \phi_k + tv - \sum_{\ell=1}^{k-1}\langle \phi_k+tv,\phi_\ell\rangle_{L^2(\Omega)}\phi_\ell = \phi_k + t\Big(v-\sum_{\ell=1}^{k-1}\langle v,\phi_\ell\rangle_{L^2(\Omega)}\phi_\ell\Big)$$

for $t\in\mathbb{R}$ and $v\in\widetilde{H}^s(\Omega)\setminus\{0\}$. Thus, we may estimate

$$\begin{aligned}\|w_t\|_{L^2(\Omega)} &\geq \|\phi_k\|_{L^2(\Omega)} - |t|\Big\|v-\sum_{\ell=1}^{k-1}\langle v,\phi_\ell\rangle_{L^2(\Omega)}\phi_\ell\Big\|_{L^2(\Omega)} \\ &\geq 1-|t|\Big(\|v\|_{L^2(\Omega)}+\sum_{\ell=1}^{k-1}|\langle v,\phi_\ell\rangle_{L^2(\Omega)}|\|\phi_\ell\|_{L^2(\Omega)}\Big) \\ &\geq 1-k|t|\|v\|_{L^2(\Omega)}>0\end{aligned}$$

as long as $|t|<\frac{1}{k}\|v\|_{L^2(\Omega)}$, where we used that $\|\phi_\ell\|_{L^2(\Omega)}=1$ for $1\leq\ell\leq k$. Therefore, we can define

$$\widetilde{w}_t = \frac{w_t}{\|w_t\|_{L^2(\Omega)}}$$

for $|t|<\frac{1}{k}\|v\|_{L^2(\Omega)}$. Using (A.2) and setting $\widetilde{v}=v-\sum_{\ell=1}^{k-1}\langle v,\phi_\ell\rangle_{L^2(\Omega)}\phi_\ell$, we deduce that there holds

$$\frac{d}{dt}\Big|_{t=0}\|w_t\|^2_{L^2(\Omega)} = 2\langle\phi_k,\widetilde{v}\rangle_{L^2(\Omega)} = 2\langle\phi_k,v\rangle_{L^2(\Omega)},$$

and

$$\begin{aligned}0 &= \frac{d}{dt}\Big|_{t=0}\|\widetilde{w}_t\|^2_{\widetilde{H}^s(\Omega)} \\ &= \frac{d}{dt}\Big|_{t=0}\frac{\|w_t\|^2_{\widetilde{H}^s(\Omega)}}{\|w_t\|^2_{L^2(\Omega)}} \\ &= \frac{d}{dt}\Big|_{t=0}\|w_t\|^2_{\widetilde{H}^s(\Omega)} - \lambda_k\frac{d}{dt}\Big|_{t=0}\|w_t\|^2_{L^2(\Omega)} \\ &= 2\big(\langle(-\Delta)^{s/2}\phi_k,(-\Delta)^{s/2}\widetilde{v}\rangle_{L^2(\mathbb{R}^n)} - \lambda_k\langle\phi_k,v\rangle_{L^2(\Omega)}\big) \\ &= 2\big(\langle(-\Delta)^{s/2}\phi_k,(-\Delta)^{s/2}v\rangle_{L^2(\mathbb{R}^n)} - \lambda_k\langle\phi_k,v\rangle_{L^2(\Omega)}\big).\end{aligned}$$

In the last equality, we used that ϕ_ℓ for $1 \leq \ell \leq k-1$ are eigenfunctions of the fractional Laplacian and in $\widetilde{L}^2(\Omega)$ orthogonal to ϕ_k by (A.2). The above computation shows that λ_k is an eigenvalue and ϕ_k a corresponding eigenfunction.

Next, we assert that $\lambda_k \to \infty$ as $k \to \infty$. Suppose for the sake of contradiction that λ_k is uniformly bounded, so that also $(\phi_k)_{k\in\mathbb{N}} \subset \widetilde{H}^s(\Omega)$ is uniformly bounded. Thus, up to extracting a subsequence, $(\phi_k)_{k\in\mathbb{N}}$ converges in $\widetilde{L}^2(\Omega)$ and in particular is a Cauchy sequence. But then by the above construction, we have $\|\phi_k - \phi_m\|^2_{L^2(\Omega)} = \|\phi_k\|^2_{L^2(\Omega)} + \|\phi_m\|^2_{L^2(\Omega)} + 2\langle\phi_k, \phi_m\rangle_{L^2(\Omega)} = 2$, for $k \neq m$ and thus $(\phi_k)_{k\in\mathbb{N}}$ cannot be Cauchy, a contradiction. Therefore, we necessarily have $\lambda_k \to \infty$ as $k \to \infty$.

Step 3. Proof of (i).

We already know that $(\phi_k)_{k\in\mathbb{N}}$ is an orthonormal system in $\widetilde{L}^2(\Omega)$. So, we only need to establish that the linear span of $(\phi_k)_{k\in\mathbb{N}}$ is dense in $\widetilde{L}^2(\Omega)$. As $\widetilde{H}^s(\Omega)$ is dense in $\widetilde{L}^2(\Omega)$ and $(\phi_k)_{k\in\mathbb{N}} \subset \widetilde{H}^s(\Omega)$, it is enough to show that every function in $\widetilde{H}^s(\Omega)$ can be approximated by elements in the linear span of $(\phi_k)_{k\in\mathbb{N}}$. So, let $v \in \widetilde{H}^s(\Omega)$ be any fixed function and define

$$v_k = v - \sum_{\ell=1}^{k-1} \langle v, \phi_\ell\rangle_{L^2(\Omega)} \phi_\ell = v - \sum_{\ell=1}^{k-1} \lambda_\ell^{-1} \langle v, \phi_\ell\rangle_{\widetilde{H}^s(\Omega)} \phi_\ell \tag{A.3}$$

for any $k \geq 2$. By orthonormality of $(\phi_k)_{k\in\mathbb{N}}$ in $\widetilde{L}^2(\Omega)$, we get $\langle v_k, \phi_\ell\rangle_{L^2(\Omega)} = 0$ for any $1 \leq \ell \leq k-1$. By formula (A.1) this yields

$$\|v_k\|^2_{\widetilde{H}^s(\Omega)} \geq \lambda_k \|v_k\|^2_{L^2(\Omega)}. \tag{A.4}$$

Now, using the orthonormality of $(\phi_\ell)_{1\leq\ell\leq k-1}$, we may compute

$$\begin{aligned}
\|v\|^2_{\widetilde{H}^s(\Omega)} &= \left\| v_k + \sum_{\ell=1}^{k-1} \lambda_\ell^{-1} \langle v, \phi_\ell\rangle_{\widetilde{H}^s(\Omega)} \phi_\ell \right\|^2_{\widetilde{H}^s(\Omega)} \\
&= \|v_k\|^2_{\widetilde{H}^s(\Omega)} + 2\sum_{\ell=1}^{k-1} \lambda_\ell^{-1} \langle v, \phi_\ell\rangle_{\widetilde{H}^s(\Omega)} \langle v_k, \phi_\ell\rangle_{\widetilde{H}^s(\Omega)} \\
&\quad + \sum_{\ell=1}^{k-1}\sum_{\ell'=1}^{k-1} \lambda_\ell^{-1}\lambda_{\ell'}^{-1} \langle v, \phi_\ell\rangle_{\widetilde{H}^s(\Omega)} \langle v, \phi_{\ell'}\rangle_{\widetilde{H}^s(\Omega)} \langle \phi_\ell, \phi_{\ell'}\rangle_{\widetilde{H}^s(\Omega)} \\
&= \|v_k\|^2_{\widetilde{H}^s(\Omega)} + \sum_{\ell=1}^{k-1} \lambda_\ell^{-1} \left| \langle v, \phi_\ell\rangle_{\widetilde{H}^s(\Omega)} \right|^2 \\
&\geq \|v_k\|^2_{\widetilde{H}^s(\Omega)}.
\end{aligned} \tag{A.5}$$

Thus, by (A.4) we obtain

$$\|v_k\|^2_{L^2(\Omega)} \leq \lambda_k^{-1} \|v_k\|^2_{\widetilde{H}^s(\Omega)} \leq \lambda_k^{-1} \|v\|^2_{\widetilde{H}^s(\Omega)}.$$

Passing to the limit, this implies $v_k \to 0$ in $\widetilde{L}^2(\Omega)$. Hence, we have

$$v = \sum_{\ell=1}^{\infty} \langle v, \phi_\ell\rangle_{L^2(\Omega)} \phi_\ell = \sum_{\ell=1}^{\infty} \langle v, \lambda_\ell^{-1/2}\phi_\ell\rangle_{\widetilde{H}^s(\Omega)} (\lambda_\ell^{-1/2}\phi_\ell)$$

in $\widetilde{L}^2(\Omega)$.

Step 4. Proof of (ii).

First note that $\big(\lambda_k^{-1/2}\phi_k\big)_{k\in\mathbb{N}} \subset \widetilde{H}^s(\Omega)$ is orthonormal. This is a direct consequence of the above construction. It remains to show the density of the linear span of $\big(\lambda_k^{-1/2}\phi_k\big)_{k\in\mathbb{N}}$ in $\widetilde{H}^s(\Omega)$. Let us fix $v \in \widetilde{H}^s(\Omega)$ and suppose that the sequence $(v_k)_{k\geq 2}$ is defined as in (A.3). From the estimate (A.5) we know that $(v_k)_{k\geq 2}$ is uniformly bounded in $\widetilde{H}^s(\Omega)$ and thus up to extracting a subsequence, we get $v_k \rightharpoonup w$ in $\widetilde{H}^s(\Omega)$ for some $w \in \widetilde{H}^s(\Omega)$. The compact embedding $\widetilde{H}^s(\Omega) \hookrightarrow \widetilde{L}^2(\Omega)$ now gives $w = 0$ as we already know from the previous step that $v_k \to 0$ in $\widetilde{L}^2(\Omega)$ as $k \to \infty$. As for any subsequence, there is a further subsequence with this property, we know that the whole sequence weakly converges in $\widetilde{H}^s(\Omega)$ to this limit $w = 0$. By Mazur's lemma, there exists a sequence of convex linear combinations

$$w_\ell = \sum_{k=2}^{\ell} a_k^{(\ell)} v_k, \quad 0 \leq a_k^{(\ell)} \leq 1, \quad \sum_{k=2}^{\ell} a_k^{(\ell)} = 1$$

such that $w_\ell \to 0$ in $\widetilde{H}^s(\Omega)$. Note that by (A.3) we have

$$w_\ell = v - \sum_{k=2}^{\ell}\sum_{m=1}^{k-1} a_k^{(\ell)} \big\langle v, \lambda_m^{-1/2}\phi_m\big\rangle_{\widetilde{H}^s(\Omega)} \big(\lambda_m^{-1/2}\phi_m\big)$$

and thus

$$W_\ell = \sum_{k=2}^{\ell}\sum_{m=1}^{k-1} a_k^{(\ell)} \big\langle v, \lambda_m^{-1/2}\phi_m\big\rangle_{\widetilde{H}^s(\Omega)} \big(\lambda_m^{-1/2}\phi_m\big) \to v$$

in $\widetilde{H}^s(\Omega)$ as $\ell \to \infty$. As the functions W_ℓ clearly belong to the span of $\big(\lambda_k^{-1/2}\phi_k\big)_{k\in\mathbb{N}}$, we may conclude the proof.

Step 5. Proof of (iii).

Note that for any $G \in H^{-s}(\Omega)$ and $v \in \widetilde{H}^s(\Omega)$ we have by (ii) the identity

$$\langle G, v\rangle = \sum_{k=1}^{\infty} \big\langle v, \lambda_k^{-1/2}\phi_k\big\rangle_{\widetilde{H}^s(\Omega)} \big\langle G, \lambda_k^{-1/2}\phi_k\big\rangle.$$

Using the Cauchy–Schwartz inequality, we get

$$\begin{aligned}
|\langle G, v\rangle| &\leq \sum_{k=1}^{\infty} \big|\big\langle v, \lambda_k^{-1/2}\phi_k\big\rangle_{\widetilde{H}^s(\Omega)}\big| \lambda_k^{-1/2} |G_k| \\
&\leq \left(\sum_{k=1}^{\infty} \big|\big\langle v, \lambda_k^{-1/2}\phi_k\big\rangle_{\widetilde{H}^s(\Omega)}\big|^2\right)^{1/2} \left(\sum_{k=1}^{\infty} \lambda_k^{-1}|G_k|^2\right)^{1/2} \\
&= \|v\|_{\widetilde{H}^s(\Omega)} \left(\sum_{k=1}^{\infty} \lambda_k^{-1}|G_k|^2\right)^{1/2},
\end{aligned}$$

where we have again put $G_k = \langle G, \phi_k\rangle$ and used [Bre11, Corollary 5.10]. Hence,

$$\|G\|_{H^{-s}(\Omega)} \leq \left(\sum_{k=1}^{\infty} \lambda_k^{-1}|G_k|^2\right)^{1/2}. \tag{A.6}$$

Next, let $v \in \widetilde{H}^s(\Omega)$ be the unique solution to

$$\begin{cases} (-\Delta)^s v = G & \text{in } \Omega, \\ v = 0 & \text{in } \Omega_e, \end{cases} \tag{A.7}$$

which exists by the Lax–Milgram theorem, and satisfies

$$\|v\|_{\widetilde{H}^s(\Omega)} \leq \|G\|_{H^{-s}(\Omega)}. \tag{A.8}$$

By Plancherel's theorem, the left-hand side can be written as

$$\|v\|^2_{\widetilde{H}^s(\Omega)} = \sum_{k=1}^{\infty} \left|\left\langle v, \lambda_k^{-1/2}\phi_k\right\rangle_{\widetilde{H}^s(\Omega)}\right|^2. \tag{A.9}$$

As v solves (A.7), we get

$$\begin{aligned} \left\langle v, \lambda_k^{-1/2}\phi_k\right\rangle_{\widetilde{H}^s(\Omega)} &= \lambda_k^{-1/2}\left\langle (-\Delta)^{s/2}v, (-\Delta)^{s/2}\phi_k\right\rangle_{L^2(\mathbb{R}^n)} \\ &= \lambda_k^{-1/2}\left\langle G, \phi_k\right\rangle \\ &= \lambda_k^{-1/2} G_k. \end{aligned} \tag{A.10}$$

Taking into account (A.8) and (A.9), we get

$$\begin{aligned} \left(\sum_{k=1}^{\infty} \lambda_k^{-1}\left|G_k\right|^2\right)^{1/2} &= \left(\sum_{k=1}^{\infty} \left|\left\langle v, \lambda_k^{-1/2}\phi_k\right\rangle_{\widetilde{H}^s(\Omega)}\right|^2\right)^{1/2} \\ &= \|v\|_{\widetilde{H}^s(\Omega)} \\ &\leq \|G\|_{H^{-s}(\Omega)}. \end{aligned} \tag{A.11}$$

Thus, we may conclude that for any $G \in H^{-s}(\Omega)$, we have

$$\|G\|_{H^{-s}(\Omega)} = \left(\sum_{k=1}^{\infty} \lambda_k^{-1}\left|G_k\right|^2\right)^{1/2}. \tag{A.12}$$

Next, we assert that for any $G \in H^{-s}(\Omega)$ we have

$$G = \sum_{k=1}^{\infty} G_k\phi_k \quad \text{in} \quad H^{-s}(\Omega), \tag{A.13}$$

where $G_k = \langle G, \phi_k\rangle$, for $k \in \mathbb{N}$. Again let $v \in \widetilde{H}^s(\Omega)$ be the unique solution of (A.7). Then from (A.6) and (A.11), we know that

$$\|v\|_{\widetilde{H}^s(\Omega)} = \|G\|_{H^{-s}(\Omega)}. \tag{A.14}$$

Therefore the source-to-solution map $S\colon H^{-s}(\Omega) \to \widetilde{H}^s(\Omega)$ related to (A.7) is an isometric isomorphism. In fact, surjectivity follows by using $G = (-\Delta)^s v \in H^{-s}(\mathbb{R}^n) \hookrightarrow H^{-s}(\Omega)$ for given $v \in \widetilde{H}^s(\Omega)$ as a source. By (ii), we already know that

$$v = \sum_{k=1}^{\infty} \left\langle v, \lambda_k^{-1/2}\phi_k\right\rangle_{\widetilde{H}^s(\Omega)} \lambda_k^{-1/2}\phi_k$$

in $\widetilde{H}^s(\Omega)$ for any $v \in \widetilde{H}^s(\Omega)$. As $SG = v$ and S^{-1} is a bounded linear map by the Banach isomorphism theorem, we deduce that

$$G = S^{-1}v = \sum_{k=1}^{\infty} \lambda_k^{-1/2}\left\langle v, \lambda_k^{-1/2}\phi_k\right\rangle_{\widetilde{H}^s(\Omega)} S^{-1}\phi_k \quad \text{in} \quad H^{-s}(\Omega).$$

As $S^{-1}\phi_k = \lambda_k\phi_k$, we get by (A.10) the identity

$$\begin{aligned} G &= \sum_{k=1}^{\infty} \lambda_k^{1/2}\left\langle v, \lambda_k^{-1/2}\phi_k\right\rangle_{\widetilde{H}^s(\Omega)}\phi_k \\ &= \sum_{k=1}^{\infty} G_k\phi_k \quad \text{in} \quad H^{-s}(\Omega). \end{aligned} \tag{A.15}$$

This verifies the assertion (A.13). Observe that the bilinear form

$$\langle G, H\rangle_{H^{-s}(\Omega)} := \langle SG, SH\rangle_{\widetilde{H}^s(\Omega)} \tag{A.16}$$

for $G, H \in H^{-s}(\Omega)$ defines an inner product on $H^{-s}(\Omega)$ and the induced norm coincides with the dual norm $\|\cdot\|_{H^{-s}(\Omega)}$ (see (A.14)). Note that

$$\begin{aligned}\langle \phi_k, \phi_\ell\rangle_{H^{-s}(\Omega)} &= \langle S\phi_k, S\phi_\ell\rangle_{\widetilde{H}^s(\Omega)} = \lambda_k^{-1}\lambda_\ell^{-1}\langle \phi_k, \phi_\ell\rangle_{\widetilde{H}^s(\Omega)} \\ &= \lambda_\ell^{-1}\langle \phi_k, \phi_\ell\rangle_{L^2(\Omega)} = \lambda_k^{-1}\delta_{k,\ell}\end{aligned}$$

for any $k, \ell \in \mathbb{N}$. Hence, $\left(\lambda_k^{1/2}\phi_k\right)_{k\in\mathbb{N}}$ is orthonormal in $H^{-s}(\Omega)$. By the definition of the isomorphism S, $S\phi_k = \lambda_k^{-1}\phi_k$ and (A.16), we get

$$\begin{aligned} G_k &= \langle G, \phi_k\rangle \\ &= \langle SG, \phi_k\rangle_{\widetilde{H}^s(\Omega)} \\ &= \langle SG, S(\lambda_k\phi_k)\rangle_{\widetilde{H}^s(\Omega)} \\ &= \lambda_k \langle G, \phi_k\rangle_{H^{-s}(\Omega)}. \end{aligned} \tag{A.17}$$

Finally, by (A.15) this implies

$$G = \sum_{k=1}^{\infty} \left\langle G, \lambda_k^{1/2}\phi_k\right\rangle_{H^{-s}(\Omega)} \lambda_k^{1/2}\phi_k \quad \text{in} \quad H^{-s}(\Omega),$$

which in turn implies that $\left(\lambda_k^{1/2}\phi_k\right)_{k\in\mathbb{N}}$ is an orthonormal basis in $H^{-s}(\Omega)$. □

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