

# SIMULTANEOUS RECOVERY OF PIECEWISE ANALYTIC COEFFICIENTS IN A SEMILINEAR ELLIPTIC EQUATION

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**ABSTRACT.** In this short note, we investigate simultaneous recovery inverse problems for semilinear elliptic equations with partial data. The main technique is based on higher order linearization and monotonicity approaches. With these methods at hand, we can determine the diffusion and absorption coefficients together with the shape of a cavity simultaneously by knowing the corresponding localized Dirichlet-Neumann operator.

**Keywords.** Inverse boundary value problems, inverse obstacle problem, semilinear elliptic equations, simultaneous recovery, partial data, higher order linearization, monotonicity method, localized potentials

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## 1. INTRODUCTION

In this note, we investigate some inverse problems for semilinear elliptic equations. Inverse problems for nonlinear partial differential equations (PDEs) have been paid a lot of attention in the past few decades. The main method to study inverse problems for nonlinear PDEs relies on suitable linearization techniques, and the linearization approaches can be traced back to the pioneer work by Isakov [Isa93]. In [Isa93], he demonstrated that the first linearization of the corresponding Dirichlet-to-Neumann (DN) map of the semilinear parabolic equation agrees to the DN map of the associated linearized equation. Hence, related known results on inverse boundary value problems for linear equations can be expected to apply, such that one is able to solve inverse problems for the nonlinear equations.

For the semilinear elliptic equation  $\Delta u + a(x, u) = 0$  in a domain, the inverse problem of determining the coefficient  $a(x, u)$  was treated in [IS94, Sun10], for  $n \geq 3$ , and in [IN95, Sun10, IY13] for  $n = 2$ . In addition, for quasilinear elliptic equations, related inverse boundary value problems have also been studied by [Sun96, SU97, KN02, LW07, MU20, KKV20, CFK<sup>+</sup>21]. Meanwhile, researchers also worked on inverse problems for the degenerate  $p$ -Laplace equation [SZ12, BHKS18], and for the fractional semilinear Schrödinger equation [LL19, LL21, Lin22]. The interior unique determination problem for quasilinear equations on Riemannian manifolds was recently studied in [LLS20, Section 6] via the source-to-solution map. We

also refer the readers to [Sun05, Uhl09] for more introduction and discussions on related inverse problems for nonlinear elliptic equations.

In order to show the results in this work, we utilize a method that has been introduced by [KLU18] for nonlinear hyperbolic equations and developed in [FO20, LLLS21, LLLS20, LLST21]. The method is called the *higher order linearization*, which introduces particular parameters to reduce a semilinear elliptic equation into different linearized elliptic equations. In [FO20, LLLS21], the authors studied inverse boundary value problem with full boundary measurements. In addition, simultaneous recovery inverse problems for semilinear PDEs were also considered in [LLS20, LLST21, LLL21, LLLZ21], and [KU20b, LLLS20, KU20a] studied the Calderón type problem with partial data independently. A similar approach was utilized to study some related inverse problems for fractional semilinear elliptic equations [LL21, Lin22]. It can be noted that uniqueness results for the coefficients of the nonlinear term are easier to obtain than the uniqueness result for the coefficients of the linear term.

In this note, we will not use complex geometrical optics solutions as in several of the previously cited works. Instead, after using the afore-mentioned *higher order linearization* approach, we treat the resulting linearized elliptic equations by a combination of the *monotonicity method* and *localized potentials*. This line of reasoning was initiated by [Geb08] and applied to various inverse problems, such as [Har09, HS10, Har12, AH13, HU13, BHHM17, HU17, BHKS18, GH18, HPS19b, HLL18, SKJ<sup>+</sup>19, HL19, HPS19a]. There are also related works on practical reconstruction methods based on monotonicity properties [TR02, HLU15, HU15, HM16, MVVT16, TSV<sup>+</sup>16, Gar17, GS17, SUG<sup>+</sup>17, VMC<sup>+</sup>17, HM18, ZHS18, GS19, EH21, EH22].

We next introduce the mathematical model in this work. Let  $\Omega \subset \mathbb{R}^n$  be a bounded connected domain with a  $C^\infty$ -smooth boundary  $\partial\Omega$ , for  $n \geq 2$ , and  $D \Subset \Omega$  is an open subset with a  $C^\infty$ -smooth boundary  $\partial D$  such that  $\Omega \setminus \overline{D}$  is connected. Let us consider the following semilinear elliptic equation with diffusion

$$(1.1) \quad \begin{cases} -\nabla \cdot (\sigma(x)\nabla u) + a(x, u) = 0 & \text{in } \Omega \setminus \overline{D}, \\ u = 0 & \text{on } \partial D, \\ u = f & \text{on } \partial\Omega. \end{cases}$$

In particular, when  $D = \emptyset$ , the equation (1.1) becomes

$$(1.2) \quad \begin{cases} -\nabla \cdot (\sigma(x)\nabla u) + a(x, u) = 0 & \text{in } \Omega, \\ u = f & \text{on } \partial\Omega. \end{cases}$$

In this work, we first prove a local well-posedness result for (1.1) under the assumption that the diffusivity  $\sigma = \sigma(x)$ , and the power series terms of the lower order coefficient  $a = a(x, y)$ , are *piecewise real-analytic*, see Assumption 2.1 for the precise definition. Throughout this paper, we also synonymously use the term "analytic" instead of "real-analytic" for the sake of brevity. Note that a local well-posedness of (1.1) was already shown in [FO20, KU20a, LLLS21] under different regularity assumptions. Our case differs from previous results as we allow discontinuities of the coefficients, and we therefore give a detailed proof for the well-posedness result.

To summarize the (local) well-posedness result, let  $\Gamma \subset \partial\Omega$  be a relatively open subset, and  $0 < \alpha < 1$ . We prove that there exists  $\varepsilon > 0$  and  $C > 0$  such that for all Dirichlet boundary data

$$f \in N_\varepsilon := \{f \in C_0^\alpha(\Gamma) : \|f\|_{C^\alpha(\Gamma)} \leq \varepsilon\},$$

there is a unique solution  $u = u_f \in V$  of (1.2) that also satisfies

$$\|u\|_V \leq C\varepsilon,$$

where

$$(1.3) \quad V := \{v \in H^1(\Omega) : -\nabla \cdot (\sigma \nabla v) \in L^\infty(\Omega), v|_{\partial\Omega} \in C_0^\alpha(\Gamma)\},$$

and

$$(1.4) \quad \|v\|_V := \|v\|_{H^1(\Omega)} + \|\nabla \cdot (\sigma \nabla v)\|_{L^\infty(\Omega)} + \|v|_{\partial\Omega}\|_{C^\alpha(\Gamma)}.$$

Moreover, the local well-posedness also holds for (1.1), when the domain  $\Omega$  in (1.3) and (1.4) is replaced by  $\Omega \setminus \bar{D}$ .

With these results at hand, one can define the corresponding (partial) DN operator

$$\Lambda_{\sigma,a}^\Gamma : N_\varepsilon \rightarrow H^{-1/2}(\Gamma), \quad \Lambda_{\sigma,a}^\Gamma(f) := \sigma \partial_\nu u_f|_\Gamma,$$

for some sufficiently small number  $\varepsilon > 0$ , where  $u_f$  is the unique solution of (1.2) and  $\nu$  is the unit outer normal on  $\partial\Omega$ . Likewise, one can also define the DN operator

$$\Lambda_{\sigma,a,D}^\Gamma : N_\varepsilon \rightarrow H^{-1/2}(\Gamma), \quad \Lambda_{\sigma,a,D}^\Gamma(f) := \sigma \partial_\nu u_f|_\Gamma,$$

for some sufficiently small number  $\varepsilon > 0$ , where  $u_f$  is the unique solution of (1.1) and  $\nu$  is the unit outer normal on  $\partial\Omega$ . We then study the following simultaneous recovery inverse problems:

- (1) Can one simultaneously identify  $\sigma$  and  $a$  by knowing the partial measurements  $\Lambda_{\sigma,a}^\Gamma$ ?
- (2) Can one simultaneously identify  $\sigma$ ,  $a$  and  $D$  by knowing the partial measurements  $\Lambda_{\sigma,a,D}^\Gamma$ ?

We will give affirmative answers to both questions in this paper.

The paper is structured as follows. In Section 2, we state our main results of this note, and the proofs are given in Section 3. The main methods depend on suitable linearization and monotonicity methods combining with localized potentials.

## 2. THE MAIN RESULTS

In this section, we will formulate our two main results: The semilinear elliptic equation (1.2) is uniquely solvable (for sufficiently small Dirichlet data), and the associated DN operator uniquely determines the coefficients in equation (1.2).

Our results will be valid under the following assumptions on the domain and the coefficients.

**Assumption 2.1.** *We assume that  $\Omega \subset \mathbb{R}^n$ ,  $n \geq 2$  is a bounded domain with  $C^\infty$ -smooth boundary  $\partial\Omega$ , and that  $\sigma \in L_+^\infty(\Omega)$  is a piecewise analytic function in the sense of [KV85, Section 3]. The function  $a : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$  is assumed to fulfill*

$$a(x, y) = \sum_{k=0}^{\infty} a_k(x) \frac{y^k}{k!},$$

with  $a_k \in L^\infty(\Omega)$ ,  $a_0 = a_1 = 0$ , and  $\sup_{k \geq 2} \|a_k\|_{L^\infty(\Omega)} < \infty$ . Moreover each function  $a_k$  is assumed to be piecewise analytic in the sense of [KV85, Sect. 3].

Note that [KV85, Sect. 3] implies that two piecewise analytic functions are piecewise analytic with respect to the same partition, and this naturally extends to every finite number of piecewise analytic functions. However, Assumption 2.1 contains infinitely many such functions  $\sigma$ ,  $a_k$  ( $k \in \mathbb{N}$ ), and we do not assume that they are piecewise analytic with respect to the same partition.

For our solvability result for the forward problem, we use the following solution space

$$V := \{v \in H^1(\Omega) : -\nabla \cdot (\sigma \nabla v) \in L^\infty(\Omega), v|_{\partial\Omega} \in C^\alpha(\partial\Omega)\}.$$

equipped with the norm

$$\|v\|_V := \|v\|_{H^1(\Omega)} + \|\nabla \cdot (\sigma \nabla v)\|_{L^\infty(\Omega)} + \|v|_{\partial\Omega}\|_{C^\alpha(\partial\Omega)}.$$

Clearly,  $V$  is a Banach space. Moreover, by a result of Li and Vogelius [LV00, Corollary 7.3],

$$(2.1) \quad V \subseteq H^1(\Omega) \cap L^\infty(\Omega) \text{ is continuously embedded.}$$

**Theorem 2.1** (Local well-posedness of the forward problem). *Let  $\Omega \subset \mathbb{R}^n$ ,  $\sigma : \Omega \rightarrow \mathbb{R}$ , and  $a : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$  fulfill assumption 2.1. Then there exists  $\varepsilon > 0$  so that for all*

$$(2.2) \quad f \in N_\varepsilon := \{\phi \in C^\alpha(\partial\Omega) : \|\phi\|_{C^\alpha(\partial\Omega)} < \varepsilon\},$$

there exists a solution  $u \in V \subseteq H^1(\Omega) \cap L^\infty(\Omega)$  to the Dirichlet problem

$$(2.3) \quad \begin{cases} -\nabla \cdot (\sigma \nabla u) + a(x, u) = 0 & \text{in } \Omega, \\ u = f & \text{on } \partial\Omega. \end{cases}$$

Moreover, there exists  $\delta > 0$ , so that, for all  $f \in N_\varepsilon$ , the solution is unique in the set of all

$$H_\delta := \{v \in H^1(\Omega) \cap L^\infty(\Omega) : \|v\|_{H^1(\Omega)} + \|v\|_{L^\infty(\Omega)} \leq \delta\},$$

and that the solution operator

$$\mathcal{S} : N_\varepsilon \rightarrow V \subseteq H^1(\Omega) \cap L^\infty(\Omega), \quad f \mapsto u, \quad \text{where } u \in H_\delta \text{ solves (2.3)}$$

is infinitely differentiable.

Clearly,  $V \subseteq L^\infty(\Omega)$  implies that  $x \mapsto a(x, u)$  is a  $L^\infty(\Omega)$ -function for all  $u \in V$ . Hence, a solution  $u \in V$  of (2.3) has well-defined Neumann boundary values so that we can define the (non-linear) Dirichlet-Neumann-Operator

$$\Lambda_{\sigma,a} : N_\varepsilon \rightarrow H^{-1/2}(\partial\Omega), \quad \Lambda_{\sigma,a}(f) := \sigma \partial_\nu u|_{\partial\Omega},$$

where  $u$  is the (sufficiently small) solution of (2.3).

Let  $D \Subset \Omega$  be an open set with  $C^\infty$  boundary  $\partial D$  such that  $\Omega \setminus \bar{D}$  is connected. Then the above result also implies local well-posedness of the Dirichlet problem

$$(2.4) \quad \begin{cases} -\nabla \cdot (\sigma \nabla u) + a(x, u) = 0 & \text{in } \Omega \setminus \bar{D}, \\ u = 0 & \text{on } \partial D, \\ u = f & \text{on } \partial\Omega, \end{cases}$$

We denote the corresponding DN operator by

$$\Lambda_{\sigma,a,D} : N_\varepsilon \rightarrow H^{-1/2}(\partial\Omega), \quad \Lambda_{\sigma,a,D}(f) := \sigma \partial_\nu u_f|_{\partial\Omega},$$

where  $u$  is the (sufficiently small) solution (2.4). For an open boundary part  $\Gamma \subseteq \partial\Omega$ , the restriction of  $\Lambda_{\sigma,a}$ , resp.,  $\Lambda_{\sigma,a,D}$ , to  $\Gamma$  is denoted by  $\Lambda_{\sigma,a}^\Gamma$ , resp.,  $\Lambda_{\sigma,a,D}^\Gamma$ .

Note that the well-posedness of (2.4) can be found in [KU20a, Appendix] and [LLS21, Proposition 2.1] in a slightly different settings that required the coefficients to be sufficiently smooth. However, in this paper, we assume piecewise analytic coefficients so that the coefficients may have jumps. The following theorem extends the uniqueness results from [LLS20, Theorem 1.2] and [KU20a, Theorem 1.6] to this setting.

**Theorem 2.2** (Simultaneous recovering of coefficients and obstacle). *Let  $\Omega \subset \mathbb{R}^n$ , and two set of coefficients  $(\sigma, a)$ , and  $(\tilde{\sigma}, \tilde{a})$  each fulfill Assumption 2.1 in connected sets  $\Omega \setminus \bar{D}$  and  $\Omega \setminus \tilde{D}$ , respectively, where  $D, \tilde{D} \Subset \Omega$  are open (possibly empty) sets.*

Let  $\Gamma \subseteq \partial\Omega$  be an open boundary part, and let  $\varepsilon > 0$  be sufficiently small, so that both,  $\Lambda_{\sigma,a,D}^\Gamma$  and  $\Lambda_{\tilde{\sigma},\tilde{a},\tilde{D}}^\Gamma$ , are defined on  $N_\varepsilon$ . Suppose that

$$\Lambda_{\sigma,a,D}^\Gamma(f) = \Lambda_{\tilde{\sigma},\tilde{a},\tilde{D}}^\Gamma(f)$$

for all  $f \in N_\varepsilon$  with  $\text{supp}(f) \subseteq \Gamma$ , then

$$\sigma = \tilde{\sigma}, \quad a = \tilde{a} \quad \text{and} \quad D = \tilde{D}.$$

To our best knowledge, the preceding theorem is a new result. The proof will be given in the next section.

**Remark 2.2.** *Let us emphasize that:*

- (a) *Note that even for the full data case, that is, when  $\Gamma = \partial\Omega$ , Theorem 2.2 is also a new result to our best knowledge. Furthermore, for the case  $\Gamma = \partial\Omega$ , we can reduce the regularity of  $\sigma$ , by using the first linearization and results in [HT<sup>+</sup>13] when  $\sigma$  is Lipschitz continuous for  $n \geq 3$ , and [AP06] when  $\sigma \in L^\infty(\Omega)$  for  $n = 2$ .*
- (b) *Note that Theorem 2.2 also covers the case where one of the sets  $D$  or  $\tilde{D}$  is empty. Hence, the DN operator also uniquely determines whether there is a cavity or not.*

### 3. PROOFS OF MAIN RESULTS

To prove our two main results, we start with the following lemma, which will be used for our results.

**Lemma 3.1.** *The mapping*

$$G : V \rightarrow L^\infty(\Omega), \quad v(x) \mapsto a(x, v(x))$$

*is infinitely differentiable and its  $l$ -th Fréchet derivative fulfills*

$$G^{(l)}(v)(w_1, \dots, w_l) = \sum_{k=0}^{\infty} a_{k+l}(x) \frac{v(x)^k}{k!} w_1 \dots w_l \quad \text{for all } v, w \in V.$$

*Proof.* We also define for  $l \in \mathbb{N}_0$

$$G_l : V \rightarrow L^\infty(\Omega), \quad v(x) \mapsto \sum_{k=0}^{\infty} a_{k+l}(x) \frac{v(x)^k}{k!}$$

Then  $G_0 = G$ , and for all  $v \in V$ ,  $l \in \mathbb{N}_0$ ,  $G_l(v) \in L^\infty(\Omega)$  follows from  $V \subseteq L^\infty(\Omega)$ .

We will prove that  $G_l$  is one-time Fréchet differentiable and that its derivative  $G'_l : V \rightarrow \mathcal{L}(V, L^\infty(\Omega))$  is given by

$$(3.1) \quad G'_l(v)w = M_w G_{l+1}(v) \quad \text{for all } v, w \in V,$$

where, for  $w \in V \subseteq L^\infty(\Omega)$

$$M_w : L^\infty(\Omega) \rightarrow L^\infty(\Omega), \quad u \mapsto wu$$

denotes the continuous linear multiplication operator. Then the assertion follows by trivial induction.

Clearly, (3.1) defines a continuous linear operator  $G'_l(v) \in \mathcal{L}(V, L^\infty(\Omega))$ . To prove that this is indeed the Fréchet derivative of  $G_l$ , let  $v, w \in V$ ,  $x \in \Omega$ , and define

$$\begin{aligned} \psi_x : \mathbb{R} &\rightarrow \mathbb{R}, \\ \psi_x(t) &:= G_l(v + t(w - v))(x) = \sum_{k=0}^{\infty} a_{k+l}(x) \frac{(v(x) + t(w(x) - v(x)))^k}{k!}. \end{aligned}$$

Then  $\psi$  is infinitely differentiable with

$$\begin{aligned}\psi'_x(t) &= \sum_{k=1}^{\infty} a_{k+l}(x) \frac{(v(x) + t(w(x) - v(x)))^{k-1}}{(k-1)!} (w(x) - v(x)) \\ \psi''_x(t) &= \sum_{k=2}^{\infty} a_{k+l}(x) \frac{(v(x) + t(w(x) - v(x)))^{k-2}}{(k-2)!} (w(x) - v(x))^2.\end{aligned}$$

Using

$$|v(x) + t(w(x) - v(x))| \leq |v(x)| + |w(x)| \quad \text{for all } t \in [0, 1],$$

and that  $a_k$  are uniformly bounded, and that  $V \subseteq L^\infty(\Omega)$  is continuously embedded we have that

$$|\psi''_x(t)| \leq C \|w - v\|_V^2 \exp(\|v\|_V + \|w\|_V) \quad \text{for all } x \in \Omega.$$

Using Taylor's formula

$$|\psi_x(1) - \psi_x(0) - \psi'_x(0)| \leq \frac{1}{2} \max_{\tau \in [0,1]} |\psi''(\tau)|,$$

we thus obtain

$$\|G_l(v) - G_l(w) - G'_l(v)(w - v)\|_{L^\infty(\Omega)} \leq \frac{C}{2} \|w - v\|_V^2 \exp(\|v\|_V + \|w\|_V),$$

so that the assertion is proven.  $\square$

### 3.1. Local well-posedness result for the forward problem.

*Proof of Theorem 2.1.* We will apply the implicit function theorem to the map

$$\begin{aligned}F : C^\alpha(\partial\Omega) \times V &\rightarrow W := L^\infty(\Omega) \times C^\alpha(\partial\Omega), \\ F : (f, v) &\mapsto (-\nabla \cdot (\sigma \nabla v) + a(x, v), v|_{\partial\Omega} - f).\end{aligned}$$

By Lemma 3.1 this mapping is well-defined and infinitely differentiable, and its derivative with respect to  $v \in V$  is the continuous linear operator

$$D_v F(0, 0) : V \rightarrow W$$

with

$$D_v F(0, 0)u = (-\sigma \nabla u, u|_{\partial\Omega}).$$

Given  $w = (w_1, w_2) \in W$  there exists a unique solution  $u \in H^1$  of

$$-\nabla \cdot (\sigma \nabla u) = w_1, \quad \text{and} \quad u|_{\partial\Omega} = w_2.$$

Then  $u \in V$  holds by definition of  $V$ , which shows that  $D_v F(0, 0)$  is surjective. Since  $u$  is the unique solution,  $D_v F(0, 0)$  is also injective. Since also  $F(0, 0) = 0$  is fulfilled, we can apply the implicit function theorem (cf., e.g., [RR06, Sect. 10.1.1]), which yields an infinitely differentiable function

$$S : N_\varepsilon \rightarrow V$$

defined on a neighborhood of the origin  $C^\alpha(\partial\Omega)$ ,

$$N_\varepsilon := \{\phi \in C^\alpha(\partial\Omega) : \|\phi\|_{C^\alpha(\partial\Omega)} < \varepsilon\},$$

so that

$$F(f, S(f)) = 0 \quad \text{for all } f \in N_\varepsilon,$$

and  $S(f)$  is the only such element in a neighborhood of the origin in  $V$ . Since  $F(f, S(f)) = 0$  implies that  $S(f) \in V$  solves (2.3), the existence of a solution in  $V \subseteq H^1(\Omega) \cap L^\infty(\Omega)$  is proven. Moreover, since every solution  $u \in H^1(\Omega) \cap L^\infty(\Omega)$  of (2.3) fulfills

$$u \in V, \quad \text{and} \quad \|u\|_V \leq \|u\|_{H^1(\Omega)} + \sup_{k \geq 2} \|a_k\|_{L^\infty(\Omega)} e^{\|u\|_{L^\infty(\Omega)}} + \|f\|_{C^\alpha(\partial\Omega)},$$

the solution is unique in  $H_\delta$  with sufficiently small  $\delta$ .  $\square$

**3.2. The higher order linearization.** To prove Theorem 2.2 we first derive some auxilliary results on the higher-order derivatives of the solution of (2.3). In the rest of this note, let us fix  $\varepsilon > 0$  to be a sufficiently small number, such that the well-posedness for (1.2) and (1.1) hold, for any  $f \in N_\varepsilon$ .

**Lemma 3.2.** *Let  $f_1, f_2 \in N_\varepsilon$ , and define*

$$F : (-1/2, 1/2) \times (-1/2, 1/2) \rightarrow V, \quad F(t_1, t_2) := \mathcal{S}(t_1 f_1 + t_2 f_2).$$

Then

$$u_1^{(\ell)} := \partial_{t_\ell} F(0, 0) = \partial_{t_\ell} \mathcal{S}(t_1 f_1 + t_2 f_2)|_{t_1=t_2=0} \in V$$

solves

$$\begin{cases} \nabla \cdot (\sigma \nabla u_1^{(\ell)}) = 0 & \text{in } \Omega \\ u_1^{(\ell)} = f_\ell & \text{on } \partial\Omega, \end{cases}$$

for  $\ell = 1, 2$ . Moreover, for all  $m > 1$ ,  $m \in \mathbb{N}$ ,

$$u_m := \partial_{t_1}^2 \partial_{t_2}^{m-2} F(0, 0) = \partial_{t_1}^2 \partial_{t_2}^{m-2} \mathcal{S}(t_1 f_1 + t_2 f_2)|_{t_1=t_2=0} \in V$$

solves

$$\begin{cases} \nabla \cdot (\sigma \nabla u_m) = \partial_{t_1}^2 \partial_{t_2}^{m-2} G(F(t))|_{t_1=t_2=0} & \text{in } \Omega \\ u_m = 0 & \text{on } \partial\Omega. \end{cases}$$

Moreover, for all  $f \in N_\varepsilon$ , the mapping

$$(t_1, t_2) \mapsto \Lambda(t_1 f_1 + t_2 f_2), \quad (-1/2, 1/2) \times (-1/2, 1/2) \rightarrow H^{-1/2}(\partial\Omega)$$

is infinitely differentiable, and

$$\partial_{t_1}^2 \partial_{t_2}^{m-2} \Lambda(t_1 f_1 + t_2 f_2)|_{t_1=t_2=0} = \sigma \partial_\nu u_m|_{\partial\Omega}.$$

*Proof.* Note that  $F$ ,  $G$ , and  $\mathcal{S}$  are infinitely differentiable functions by Lemma 3.1, and Theorem 2.1. Moreover, since the trace operator  $u \mapsto u|_{\partial\Omega}$  is a continuous linear function from  $V$  to  $H^{1/2}(\Omega)$ , it follows that

$$u_1^{(\ell)}|_{\partial\Omega} = f_\ell, \text{ for } \ell = 1, 2, \quad \text{and} \quad u_m|_{\partial\Omega} = 0, \text{ for } m \geq 2.$$

Let  $v \in H_0^1(\Omega)$ . Since  $u := \mathcal{S}(t_1 f_1 + t_2 f_2)$  solves (2.3) we have that

$$\begin{aligned} 0 &= \int_{\Omega} \sigma \nabla u \cdot \nabla v \, dx + \int_{\Omega} a(x, u) v \, dx \\ &= \int_{\Omega} \sigma \nabla \mathcal{S}(t_1 f_1 + t_2 f_2) \cdot \nabla v \, dx + \int_{\Omega} G(\mathcal{S}(t_1 f_1 + t_2 f_2)) v \, dx. \end{aligned}$$

Noting that the first summand is a linear continuous functional with respect to  $\mathcal{S}(t_1 f_1 + t_2 f_2) \in V$ , and the second summand is linear and continuous with respect to  $G(\mathcal{S}(t_1 f_1 + t_2 f_2)) \in L^\infty(\Omega)$ , we obtain by differentiation

$$\begin{aligned} 0 &= \int_{\Omega} \sigma \nabla \partial_{t_1}^2 \partial_{t_2}^{m-2} \mathcal{S}(t_1 f_1 + t_2 f_2)|_{t_1=t_2=0} \cdot \nabla v \, dx \\ &\quad + \int_{\Omega} \partial_{t_1}^2 \partial_{t_2}^{m-2} G(\mathcal{S}(t_1 f_1 + t_2 f_2))|_{t_1=t_2=0} v \, dx \\ &= \int_{\Omega} \sigma \nabla u_m \cdot \nabla v \, dx + \int_{\Omega} \partial_{t_1}^2 \partial_{t_2}^{m-2} G(\mathcal{S}(t_1 f_1 + t_2 f_2))|_{t_1=t_2=0} v \, dx. \end{aligned}$$

This proves that, for all  $m > 1$ ,

$$\nabla \cdot (\sigma \nabla u_m) = \partial_{t_1}^2 \partial_{t_2}^{m-2} G(\mathcal{S}(t_1 f_1 + t_2 f_2))|_{t_1=t_2=0}.$$

On the other hand, for  $\ell = 1, 2$ , it follows from  $S(0) = 0$ , and  $G'(0) = 0$ , that

$$\partial_{t_\ell} G(\mathcal{S}(t_1 f_1 + t_2 f_2))|_{t_1=t_2=0} = G'(\mathcal{S}(0)) (\partial_{t_\ell} \mathcal{S}(0)) f_\ell = 0.$$

Hence,  $u_1^{(\ell)} \in V$  solves  $\nabla \cdot (\sigma \nabla u_1^{(\ell)}) = 0$ , for  $\ell = 1, 2$ .

Moreover, for all  $f_1, f_2 \in N_\varepsilon$ , and  $(t_1, t_2) \in (-1/2, 1/2) \times (-1/2, 1/2)$ , the Neumann data  $\Lambda_{\sigma,a}(t_1 f_1 + t_2 f_2) \in H^{-1/2}(\partial\Omega)$  fulfills

$$\langle \Lambda_{\sigma,a}(t_1 f_1 + t_2 f_2), g \rangle = \int_{\Omega} \sigma \nabla F(t_1, t_2) \cdot \nabla v \, dx + \int_{\Omega} G(F(t_1, t_2)) v(x) \, dx$$

for all  $v \in H^1(\Omega)$ . As above, the first summand is a linear continuous functional with respect to  $F(t_1, t_2) \in V$ , and the second summand is linear and continuous with respect to  $G(F(t_1, t_2)) \in L^\infty(\Omega)$ . Hence,  $(t_1, t_2) \mapsto \Lambda(t_1 f_1 + t_2 f_2)$  is infinitely differentiable with respect to  $t_1$  and  $t_2$ , and, for all  $g \in H^{1/2}(\partial\Omega)$ ,

$$\begin{aligned} & \left\langle \partial_{t_1}^2 \partial_{t_2}^{m-2} \Lambda_{\sigma,a}(t_1 f_1 + t_2 f_2) \Big|_{t_1=t_2=0} \cdot g \right\rangle \\ &= \partial_{t_1}^2 \partial_{t_2}^{m-2} \langle \Lambda_{\sigma,a}(t_1 f_1 + t_2 f_2), g \rangle \Big|_{t_1=t_2=0} \\ &= \int_{\Omega} \sigma \nabla u_m \cdot \nabla v \, dx + \int_{\Omega} \partial_{t_1}^2 \partial_{t_2}^{m-2} G(F(t_1, t_2)) \Big|_{t_1=t_2=0} v(x) \, dx \\ &= \langle \sigma \partial_\nu u_m |_{\partial\Omega}, g \rangle, \end{aligned}$$

which proves that  $\partial_{t_1}^2 \partial_{t_2}^{m-2} \Lambda_{\sigma,a}(t_1 f_1 + t_2 f_2) \Big|_{t_1=t_2=0} = \partial_\nu u_m |_{\partial\Omega}$  as desired.  $\square$

**Lemma 3.3.** *Let  $m \in \mathbb{N}$ . There exist numbers  $a_{m,j}^{p_1, p'_1, \dots, p_j, p'_j} \in \mathbb{N}_0$  (depending on  $j = 1, \dots, m$ , and  $p_1, p'_1, \dots, p_j, p'_j \in \mathbb{N} \cup \{0\}$  with  $p_1 + p'_1 + \dots + p_j + p'_j = m$ ) so that*

$$\begin{aligned} & \partial_{t_1}^2 \partial_{t_2}^{m-2} G(F(t_1, t_2)) \\ &= G^{(m)}(F(t_1, t_2)) \underbrace{(\partial_{t_1} F(t_1, t_2), \partial_{t_1} F(t_1, t_2), \partial_{t_2} F(t_1, t_2), \dots, \partial_{t_2} F(t_1, t_2))}_{m\text{-tuples}} \\ &+ \sum_{j=2}^{m-1} \sum_{\substack{p_1, p'_1, \dots, p_j, p'_j \in \mathbb{N} \cup \{0\}, \\ p_1 + \dots + p_j = 2, \quad p'_1 + p'_2 + \dots + p'_j = m-2}} a_{m,j}^{p_1, p'_1, \dots, p_j, p'_j} G^{(j)}(F(t_1, t_2)) \\ &\quad \times \underbrace{\left( \partial_{t_1}^{p_1} \partial_{t_2}^{p'_1} F(t_1, t_2), \dots, \partial_{t_1}^{p_j} \partial_{t_2}^{p'_j} F(t_1, t_2) \right)}_{j\text{-tuples}} \\ &+ G'(F(t_1, t_2)) (\partial_{t_1}^2 \partial_{t_2}^{m-2} F(t_1, t_2)). \end{aligned}$$

*Proof.* This follows by induction using the chain rule for the Fréchet derivative.  $\square$

We also need the following variant of the localized potentials result in [Geb08]:

**Lemma 3.4.** *Let  $D_1, D_2$  be two disjoint non-empty sets, where  $D_1 \subseteq \Omega$  is open,  $D_2 \subseteq \bar{\Omega}$  is closed,  $\Omega \setminus D_2$  is connected, and  $\Gamma \cap \bar{\Omega} \setminus D_2 \neq \emptyset$ . Then there exists a sequence  $(\phi_k)_{k \in \mathbb{N}} \subset C^\alpha(\partial\Omega)$  with  $\text{supp}(\phi_k) \subseteq \Gamma$ , and*

$$\int_{D_2} |v_k|^2 \, dx \rightarrow 0, \quad \text{and} \quad \int_{D_1} |v_k|^2 \, dx \rightarrow \infty,$$

where  $v_k \in H^1(\Omega)$  is the solution of

$$\begin{cases} \nabla \cdot (\sigma \nabla v_k) = 0 & \text{in } \Omega, \\ v_k = \phi_k & \text{on } \partial\Omega, \end{cases}$$

for  $k \in \mathbb{N}$ .



*Proof.* Let  $C_0^\alpha(\Gamma)$  denote the closure of the space of all  $\phi \in C^\alpha(\partial\Omega)$  with  $\phi \subseteq \Gamma$  with respect to the  $C^\alpha(\Gamma)$ -norm. For  $j = 1, 2$ , we define  $A_j \in \mathcal{L}(C_0^\alpha(\Gamma), L^2(D_j))$  by  $A_j : \phi \mapsto v|_{D_j}$ , where  $v \in V$  solves

$$\begin{cases} \nabla \cdot (\sigma \nabla v) = 0 & \text{in } \Omega, \\ v = \phi & \text{on } \partial\Omega. \end{cases}$$

Note that  $V$  is continuously embedded in  $L^2(D_j)$ , so that  $A_j$  are well-defined.

The assertion is proven if we can show that

$$\exists C > 0 : \|A_1\phi\|_{L^2(D_1)} \leq C\|A_2\phi\|_{L^2(D_2)},$$

and, by [Geb08, Lemma 2.5], this is equivalent to proving

$$\mathcal{R}(A'_1) \not\subseteq \mathcal{R}(A'_2).$$

The operators  $A'_j$  are easily checked to map a source term  $\psi \in L^2(D_j)$  to the Neumann boundary values of the solution of  $\nabla \cdot (\sigma \nabla w) = \psi$  with zero Dirichlet data. By a standard unique continuation argument, it then follows that  $0 = R(A'_1) \cap \mathcal{R}(A'_2)$ , which proves the assertion.  $\square$

**3.3. Unique identifiability result for the inverse obstacle problem.** Now we can prove our simultaneously unique identifiability result for the inverse coefficient problem.

*Proof of Theorem 2.2.* We first show the case as  $D = \tilde{D} = \emptyset$ . Let  $\Omega \subset \mathbb{R}^n$ , and two set of coefficients  $(\sigma, a)$ , and  $(\tilde{\sigma}, \tilde{a})$  each fulfill Assumption 2.1. Let  $\epsilon > 0$  be sufficiently small, so that both,  $\Lambda_{\sigma, a}$  and  $\Lambda_{\tilde{\sigma}, \tilde{a}}$ , are defined on  $N_\epsilon$ .

Given  $f \in N_\epsilon$ , we define the operators  $F, G, \mathcal{S}$ , and the functions  $u_m \in V$  ( $m \in \mathbb{N}$ ) as in Lemma 3.2, Lemma 3.1, and Theorem 2.1 using the coefficient pair  $(\sigma, a)$ . The corresponding entities with  $(\sigma, a)$  replaced by  $(\tilde{\sigma}, \tilde{a})$  will be denoted by  $\tilde{F}, \tilde{G}, \tilde{\mathcal{S}}$ , and  $\tilde{u}_m$  ( $m \in \mathbb{N}$ ).

We will show that

(a) If

$$(3.2) \quad \partial_{t_\ell} \Lambda_{\sigma, a}^\Gamma(t_1 f_1 + t_2 f_2)|_\Gamma \Big|_{t_1=t_2=0} = \partial_{t_\ell} \Lambda_{\tilde{\sigma}, \tilde{a}}^\Gamma(t_1 f_1 + t_2 f_2)|_\Gamma \Big|_{t_1=t_2=0},$$

for all  $f_1, f_2 \in N_\epsilon$ , then  $\sigma = \tilde{\sigma}$ , and  $u_1^{(\ell)} = \tilde{u}_1^{(\ell)}$ , for  $\ell = 1, 2$ .

(b) If, for some  $m \geq 2$ ,  $\sigma = \tilde{\sigma}$ ,  $a_j = \tilde{a}_j$ ,  $u_j = \tilde{u}_j$  for all  $j = 1, \dots, m-1$ , and

$$\partial_{t_1}^2 \partial_{t_2}^{m-2} \Lambda_{\sigma, a}^\Gamma(t_1 f_1 + t_2 f_2)|_\Gamma \Big|_{t_1=t_2=0} = \partial_{t_1}^2 \partial_{t_2}^{m-2} \Lambda_{\tilde{\sigma}, \tilde{a}}^\Gamma(t_1 f_1 + t_2 f_2)|_\Gamma \Big|_{t_1=t_2=0},$$

for all  $f_1, f_2 \in N_\epsilon$ , then  $a_m = \tilde{a}_m$ , and  $u_m = \tilde{u}_m$ .

Clearly, this proves Theorem 2.2 by induction, since we assumed that  $a_0 = 0 = \tilde{a}_0$  and  $a_1 = 0 = \tilde{a}_1$ .

To show (a), note that (3.2) implies that the local DN operator for the linear elliptic equation  $\nabla \cdot (\sigma \nabla v) = 0$  is the same for the two coefficients  $\sigma$  and  $\tilde{\sigma}$ . This implies  $\sigma = \tilde{\sigma}$  by the classical Kohn-Vogelius result [KV85], and the uniqueness of solutions yield that  $u_1^{(\ell)} = \tilde{u}_1^{(\ell)}$  in  $\Omega$ , for  $\ell = 1, 2$ .

To prove (b), note that

$$G'(F(0, 0))(\partial_{t_1}^2 \partial_{t_2}^{m-2} F(0, 0)) = 0 = \tilde{G}'(\tilde{F}(0, 0))(\partial_{t_1}^2 \partial_{t_2}^{m-2} \tilde{F}(0, 0))$$

since  $F(0, 0) = 0 = \tilde{F}(0, 0)$  and  $G'(0) = 0 = \tilde{G}'(0)$ . Moreover,

$$\partial_{t_1}^{p_1} \partial_{t_2}^{p_2} F(0, 0) = u_p = \tilde{u}_p = \partial_{t_1}^{p_1} \partial_{t_2}^{p_2} \tilde{F}(0, 0), \quad \text{for all } p_1 + p_2 = 1, \dots, m-1,$$

and, for all  $w_1, \dots, w_j$ ,

$$\begin{aligned} G^{(j)}(F(0, 0))(w_1, \dots, w_l) &= a_j(x) w_1 \dots w_l \\ &= \tilde{a}_j(x) w_1 \dots w_l = \tilde{G}^{(j)}(\tilde{F}(0))(w_1, \dots, w_l). \end{aligned}$$

Using Lemma 3.3 we thus obtain

$$\begin{aligned} & \partial_{t_1}^2 \partial_{t_2}^{m-2} G(F(t_1, t_2)) \Big|_{t_1=t_2=0} - \partial_{t_1}^2 \partial_{t_2}^{m-2} \tilde{G}(\tilde{F}(t_1, t_2)) \Big|_{t_1=t_2=0} \\ &= G^{(m)}(0) (\partial_{t_1} F(t_1, t_2), \partial_{t_1} F(t_1, t_2), \partial_{t_2} F(t_1, t_2), \dots, \partial_{t_2} F(t_1, t_2)) \Big|_{t_1=t_2=0} \\ & \quad - \tilde{G}^{(m)}(0) (\partial_{t_1} \tilde{F}(t_1, t_2), \partial_{t_1} \tilde{F}(t_1, t_2), \partial_{t_2} \tilde{F}(t_1, t_2), \dots, \partial_{t_2} \tilde{F}(t_1, t_2)) \Big|_{t_1=t_2=0} \\ &= a_m u_1^m - \tilde{a}_m \tilde{u}_1^m = (a_m - \tilde{a}_m) u_1^m. \end{aligned}$$

Hence,  $u_m - \tilde{u}_m \in V$  solves

$$\begin{aligned} (3.3) \quad & 0 = -\nabla \cdot (\sigma \nabla u_m) + \partial_{t_1}^2 \partial_{t_2}^{m-2} G(F(t)) \Big|_{t_1=t_2=0} \\ & \quad + \nabla \cdot (\sigma \nabla \tilde{u}_m) - \partial_{t_1}^2 \partial_{t_2}^{m-2} G(\tilde{F}(t)) \Big|_{t_1=t_2=0} \\ & = -\nabla \cdot (\sigma \nabla (u_m - \tilde{u}_m)) + (a_m - \tilde{a}_m) \left( u_1^{(1)} \right)^2 \left( u_1^{(2)} \right)^{m-2}. \end{aligned}$$

For all  $g \in C^\alpha(\Gamma)$  with  $\text{supp}(g) \subset \Gamma$ , we thus obtain

$$\begin{aligned} (3.4) \quad & 0 = \left\langle \partial_{t_1}^2 \partial_{t_2}^{m-2} \Lambda_{\sigma, a}(t_1 f_1 + t_2 f_2) \Big|_{\Gamma} \Big|_{t_1=t_2=0}, g \right\rangle \\ & \quad - \left\langle \partial_{t_1}^2 \partial_{t_2}^{m-2} \Lambda_{\tilde{\sigma}, \tilde{a}}(t_1 f_1 + t_2 f_2) \Big|_{\Gamma} \Big|_{t_1=t_2=0}, g \right\rangle \\ & = \langle \sigma \partial_\nu (u_m - \tilde{u}_m) |_\Gamma, g \rangle - \langle \sigma \partial_\nu \tilde{u}_m |_\Gamma, g \rangle \\ & = \int_\Omega \sigma \nabla (u_m - \tilde{u}_m) \cdot \nabla v_1 \, dx + \int_\Omega (a_m - \tilde{a}_m) \left( u_1^{(1)} \right)^2 \left( u_1^{(2)} \right)^{m-2} v_1 \, dx \\ & = \int_\Omega (a_m - \tilde{a}_m) \left( u_1^{(1)} \right)^2 \left( u_1^{(2)} \right)^{m-2} v_1 \, dx, \end{aligned}$$

where  $v_1 \in V$  solves

$$\begin{cases} \nabla \cdot (\sigma \nabla v_1) = 0 & \text{in } \Omega, \\ v_1 = g & \text{on } \partial\Omega. \end{cases}$$

We will now show that this implies  $a_m = \tilde{a}_m$ . Clearly this also implies  $u_m = \tilde{u}_m$  by using (3.3) and  $u_m|_{\partial\Omega} = f = \tilde{u}_m|_{\partial\Omega}$ .

Assume that this is not the case. Since  $a_m$  and  $\tilde{a}_m$  are piecewise analytic, we can choose two disjoint non-empty sets  $D_1, D_2$ , where  $D_1 \subseteq \Omega$  is open,  $D_2 \subseteq \bar{\Omega}$  is closed,  $\Omega \setminus D_2$  is connected, and  $\Gamma \cap \bar{\Omega} \setminus D_2 \neq \emptyset$  such that either

- (i)  $a_m|_{\Omega \setminus D_2} \geq \tilde{a}_m|_{\Omega \setminus D_2}$  and  $(a_m - \tilde{a}_m)|_{D_1} \in L_+^\infty(D_1)$ , or
- (ii)  $a_m|_{\Omega \setminus D_2} \leq \tilde{a}_m|_{\Omega \setminus D_2}$  and  $(\tilde{a}_m - a_m)|_{D_1} \in L_+^\infty(D_1)$ ,

cf. [HU13, Appendix A] for a proof for  $\Gamma = \partial\Omega$  that also holds for arbitrarily small open boundary pieces  $\Gamma \subset \partial\Omega$ . Without loss of generality we will assume that (i) holds true in the following.

For  $m \in \mathbb{N}$ , let us choose a non-negative, but not identically zero, function  $\psi \in N_\varepsilon \subset C^\alpha(\partial\Omega)$  with  $\text{supp}(\psi) \subseteq \Gamma$ . By the strong maximum principle, the corresponding solution  $w \in V$  of  $\nabla \cdot (\sigma \nabla w) = 0$  in  $\Omega$ , with  $w|_{\partial\Omega} = \psi$ , will be positive inside  $\Omega$ . Then we can use the localized potentials result in Lemma 3.4 to obtain a sequence  $(\phi_k)_{k \in \mathbb{N}} \subset C^\alpha(\partial\Omega)$  with  $\text{supp}(\phi_k) \subseteq \Gamma$ , and

$$\int_{D_2} w_{1,k}^2 \, dx \rightarrow 0, \quad \text{and} \quad \int_{D_1} w_{1,k}^2 \, dx \rightarrow \infty,$$

where  $w_{1,k} \in V$  solves  $\nabla \cdot (\sigma \nabla w_{1,k}) = 0$  in  $\Omega$  with  $w_{1,k}|_{\partial\Omega} = \phi_k$ .

Using the Dirichlet data

$$f_{1,k} := \frac{\varepsilon}{2\|\phi_k\|_{C^\alpha(\partial\Omega)}} \phi_k \in N_\varepsilon, \quad f_2 = \psi \quad \text{and} \quad g_k := \left( \frac{2\|\phi_k\|_{C^\alpha(\partial\Omega)}}{\varepsilon} \right)^2 \psi,$$

such that solutions of the first linearized equation satisfy  $u_{1,k}|_{\partial\Omega} = f_{1,k}$ ,  $u_2|_{\partial\Omega} = f_2$  and  $v_{1,k}|_{\partial\Omega} = g_k$ , then we can obtain from (3.4) that

$$\begin{aligned} 0 &= \int_{\Omega} (a_m - \tilde{a}_m) u_{1,k}^2 u_2^{m-2} v_{1,k} \, dx \\ &\geq \int_{D_2} (a_m - \tilde{a}_m) u_{1,k}^2 u_2^{m-2} v_{1,k} \, dx + \int_{D_1} (a_m - \tilde{a}_m) u_{1,k}^2 u_2^{m-2} v_{1,k} \, dx \rightarrow \infty, \end{aligned}$$

as  $k \rightarrow \infty$ . Here we have used the nonnegative of  $\psi$  such that  $u_2$  and  $v_{1,k}$  are positive for any  $k \in \mathbb{N}$ . This contradiction shows that (b) holds.

On the other hand, if one of  $D$  or  $\tilde{D}$  is a nonempty set, similar to the arguments of the previous case, one can determine that  $\sigma = \tilde{\sigma}$  in  $\Omega \setminus (\overline{D \cup \tilde{D}})$  by applying the boundary determination to piecewise analytic functions. Let us denote that  $u_1$  and  $\tilde{u}_1$  to be the solution of  $\nabla \cdot (\sigma \nabla u_1) = 0$  and  $\nabla \cdot (\tilde{\sigma} \nabla \tilde{u}_1) = 0$  in  $\Omega$ , respectively. By using the strategy introduced in [LLLS20], let  $G$  be the connected component of  $\Omega \setminus (\overline{D \cup \tilde{D}})$  whose boundary contains  $\partial\Omega$ . Let  $U := u_1 - \tilde{u}_1$ , then  $U$  is a solution of

$$\begin{cases} \nabla \cdot (\sigma \nabla U) = 0 & \text{in } G, \\ U = \sigma \partial_\nu U = 0 & \text{on } \Gamma, \end{cases}$$

where we have utilized  $\Lambda_{\sigma,a,D}^\Gamma(f) = \Lambda_{\tilde{\sigma},\tilde{a},\tilde{D}}^\Gamma(f)$ , for any  $f \in N_\varepsilon$ . By the unique continuation for second order elliptic equations, one has that  $U \equiv 0$  in  $G$ . Therefore,

$$(3.5) \quad u_1 = \tilde{u}_1 \text{ in } G.$$

We next prove  $D = \tilde{D}$  via a contradiction argument. Suppose not, i.e.,  $D \neq \tilde{D} \Subset \Omega$ , and assume that  $\tilde{D} \neq \emptyset$ . By using [LLLS20, Lemma A.1] or [KU20a, Section 4], without loss of generality, we may assume that there exists a point  $x_1$  such that

$$(3.6) \quad x_1 \in \partial G \cap (\Omega \setminus \overline{D}) \cap \partial \tilde{D}.$$

$\tilde{u}_1(x_1) = 0$  since  $x_1 \in \partial \tilde{D}$ . By (3.5), we have  $u_1(x_1) = 0$ . Note that  $x_1$  is an interior point of the open set  $\Omega \setminus \overline{D}$ . Consider the boundary values  $u_1|_\Gamma \geq 0$  such that  $u_1|_\Gamma \not\equiv 0$ . Now, since  $u_1(x_1) = 0$ , by the maximum principle, we have that  $u_1 \equiv 0$  in the connected open set  $\Omega \setminus \overline{D}$ , which contradicts with the nonzero boundary condition on  $\Gamma$ . Therefore, the conclusion  $D = \tilde{D}$  must hold. Furthermore, we have by (3.5) that

$$(3.7) \quad u_1 = \tilde{u}_1 \text{ in } \Omega \setminus \overline{D},$$

as desired. Finally, by repeating the same arguments as in the case (1), we can show that  $a = \tilde{a}$  in  $(\Omega \setminus \overline{D}) \times \mathbb{R}$  as we wish. This proves the assertion.  $\square$

From the proof of Theorem 2.1, one can see that if one of  $D$  or  $\tilde{D}$  is an empty set, then  $D = \tilde{D} = \emptyset$  immediately.

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