

DETERMINING A NONLINEAR HYPERBOLIC SYSTEM WITH UNKNOWN SOURCES AND NONLINEARITY

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ABSTRACT. This paper is devoted to some inverse boundary problems associated with a time-dependent semilinear hyperbolic equation, where both nonlinearity and sources (including initial displacement and initial velocity) are unknown. It is shown in several generic scenarios that one can uniquely determine the nonlinearity and/or the sources by using passive or active boundary observations. In order to exploit the nonlinearity and the sources simultaneously, we develop a new technique, which combines the observability for linear wave equations and an approximation property with higher order linearization for the semilinear hyperbolic equation.

Keywords: Calderón’s problem, semilinear hyperbolic equation, simultaneous recovery, higher order linearization, complex geometrical optics solutions.

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1. INTRODUCTION

The inverse problems for nonlinear partial differential equations (PDEs) have received considerable attention in the literature. In [Isa93], an inverse boundary problem was proposed for a nonlinear parabolic PDE, and it was shown that the first linearization of the boundary Dirichlet-to-Neumann (DN) map associated with the nonlinear PDE agrees to the DN map of the linearized equation. Hence, results developed for inverse problems of linear PDEs can be applied to solve the inverse problems for many nonlinear PDEs. For the semilinear elliptic equation $\Delta u + a(x, u) = 0$, the inverse problem of determining $a(\cdot, \cdot)$ was investigated in [IS94, Sun10] for dimension $n \geq 3$, and in [IY13, IN95, Sun10] for $n = 2$. Moreover, some inverse problems have been studied for quasilinear elliptic equations in [CFK⁺21, KN02, LW07, MU20, Sun96, SU97], for degenerate elliptic p -Laplacian equations in [BHKS18, SZ12], for fractional semilinear Schrödinger equations in [LL19, LL22, Lin22], and etc. The Calderón type inverse problems for quasilinear PDEs on Riemannian manifolds was recently investigated in [LLS20] by using the Poisson embedding approach. Furthermore, we refer the readers to [Sun05, Uhl09] for more relevant discussions on inverse

problems of nonlinear elliptic equations in the existing developments. Recently, an important method was proposed to study inverse problems for semilinear elliptic equations, which is referred to as the *higher order linearization*, and this method has been applied to tackle some challenging inverse problems ([FLL23, FO20, HL23, KU20a, KU20b, LLS21, LLS20, LLST22]).

The inverse problems for nonlinear hyperbolic equations have also attracted a lot of attention. It turns out that the nonlinear interaction of waves can generate new waves, which are actually beneficial in solving the related inverse problems. In [KLU18], it was shown that the local measurements may uniquely recover global topology and differentiable structure, and the conformal class of the metric g on a globally hyperbolic 4-dimensional Lorentzian manifold, for a wave equation with a quadratic nonlinearity. In [LUW18], inverse problems were investigated for more general semilinear wave equations on Lorentzian manifolds, and in [LUW17], analogous inverse problems were studied for the Einstein-Maxwell equations. We refer to [CLOP22, dHUW18, KLOU22, LLPMT22, LLPMT21, WZ19] and rich references therein for more related studies of inverse problems for hyperbolic PDEs.

The inverse problems mentioned above are mainly concerned with recovering coefficients of the underlying nonlinear PDEs through *active measurements*. In the physical scenario, the PDE coefficients correspond to the unknown medium parameters. The active measurements mean that one actively inputs a certain source into the underlying PDE system to generate the output for the corresponding inverse problem. The input-output pair constitutes a typical measurement data set for many inverse problems including wave probing, nondestructive testing and medical imaging. On the other hand, many inverse problems make use of *passive measurements*, where the measurement data are generated by an unknown source. Inverse problems with passive measurements are usually referred to as the inverse source problems, since the unknown sources are the target objects to be recovered. Typical inverse source problems include those ones from the hazardous radiation detection and the cosmological searching. Recently, the inverse problems by using passive measurements to simultaneously detect the unknown sources and the surrounding mediums, have received considerable studies in the literature, due to their strong backgrounds of practical applications including photo-acoustic and thermo-acoustic tomography [LU15], brain imaging [DLU19], geomagnetic anomaly detection [DLL19, DLL20] and quantum mechanics [LLM19, LLM21]. In fact, in order to achieve the desired simultaneous recovery results, the use of both passive and active measurements was proposed for some of those inverse problems [LLM19, LLM21]. We also refer readers to some related works about inverse problems for nonlinear parabolic and hyperbolic equations, such as [RB22, EEK05, KL21].

Motivated by the studies discussed above, we investigate in this paper inverse boundary problems associated with a time-dependent semilinear hyperbolic equation, where both nonlinearity and sources are unknown. The sources include initial displacement and initial velocity of the nonlinear wave field. It is emphasized that semilinear term considered in our study is more general than those considered in the aforementioned literature on inverse problems for nonlinear hyperbolic equations. In fact, the semilinear terms in our study may contain zeroth- and first-order terms (with respect to the underlying wave function), and both of them may be unknown. It is worth mentioning that this also constitutes one of the novel points of our study compared to most of the existing studies. In the physical situation, the zeroth-order term is in fact a certain source of the hyperbolic system. However, in order to unify and ease the exposition, we mainly refer to the initial data as the sources in our study. We establish in several generic scenarios that one can uniquely determine the nonlinearity or/and the sources by using passive or/and active boundary observations. The major findings can be briefly summarized as follows:

- (1) When the nonlinearity is known, by using the passive measurement, we can establish a quantitative uniqueness result in determining the initial displacement and initial velocity of the wave field from the partial boundary measurement.
- (2) When the nonlinearity is unknown, but belonging to a certain general class, we can also establish the qualitative uniqueness by using passive measurements to determine the initial displacement and initial velocity of the wave field.
- (3) When the initial and boundary data are small enough, and the coefficients are admissible (see Definition 2.6), one can simultaneously recover the initial data as well as the nonlinearity by using the active measurement.

It turns out that the study for simultaneous recovery of both the sources and the nonlinearity becomes radically much more challenging than the case for recovering one of them by assuming the other is known. Finally, we would like to briefly discuss the technical novelties and developments in our study. The high order linearization technique and the nonlinear wave interaction technique mentioned earlier critically rely on the small inputs for inverse problems. The nonlinearity shall successively generate higher order terms (with respect to certain asymptotically small parameters) that can provide more information for the inverse problems. We shall develop techniques following a similar spirit in tackling new inverse problems. On the other hand, it is known that one salient feature for the nonlinear hyperbolic system is the finite-time blowup of solutions. If certain conditions are fulfilled, the blowup may be avoided through boundary inputs in the context of PDE controls [DZZ08]. In this paper, we shall also make use of the controllability properties for semilinear wave equations in studying the associated inverse problems. We believe that the mathematical strategy developed in the current article can be extended to attack other challenging inverse problems in different contexts. Very recently, we also investigate similar problem for nonlinear parabolic systems, and we refer readers to [LLLZ22] for further discussions.

The rest of this paper is organized as follows. In Section 2, we state the main results for the inverse problems. In Section 3, the well-posedness on the initial-boundary value problems of the semilinear hyperbolic equations within different settings of nonlinearities are studied. Section 4 is devoted to the determination of the initial data by using control methods for the hyperbolic equations. In Section 5, an approximation property for the linear wave equations is established. Furthermore, the higher order linearization technique is developed to prove the uniqueness of determining both the nonlinearity and the initial data. Finally, in Appendix A, we present the complex geometrical optics solutions for linear wave equations, which are needed in the proof of the main results.

2. STATEMENT OF MAIN RESULTS

Let $\Omega \subseteq \mathbb{R}^n$ be a nonempty bounded domain with a smooth boundary Γ , for $n \geq 2$. Assume that Γ_0 is a relatively open subset of Γ . Denote by $\nu = (\nu_1, \dots, \nu_n)$ the unit outer normal vector on Γ . For any $T > 0$, set $Q = \Omega \times (0, T)$ and $\Sigma = \Gamma \times (0, T)$. Consider the following initial-boundary value problem of the semilinear wave equation:

$$\begin{cases} u_{tt} - \Delta u + f(x, t, u) = 0 & \text{in } Q, \\ u = h & \text{on } \Sigma, \\ u(x, 0) = \varphi(x), \quad u_t(x, 0) = \psi(x) & \text{in } \Omega, \end{cases} \quad (2.1)$$

where (φ, ψ) is a pair of initial values, h is a boundary value with $\text{supp } h \subseteq \Gamma_0 \times [0, T]$ and $f = f(x, t, s) : Q \times \mathbb{R} \rightarrow \mathbb{R}$ is a given function, so that (2.1) is well-posed. Some local and global well-posedness results for (2.1) will be given in Section 3, respectively.

First, we present the first inverse problem on determining initial values. For any $(\varphi, \psi) \in H_0^1(\Omega) \times L^2(\Omega)$, $h = 0$ and a suitable function f , which guarantees the global well-posedness

of (2.1), introduce the following passive measurement:

$$\Lambda_{\varphi,\psi,f}^0 = \partial_\nu u \Big|_{\Gamma_0 \times (0,T)},$$

where u is the solution to (2.1) associated to $(\varphi, \psi) \in H_0^1(\Omega) \times L^2(\Omega)$ and $h = 0$, and $\partial_\nu u$ denotes the outer normal derivative of u . Physically, (φ, ψ, f) may be regarded as unknown sources defined on Ω and $Q \times \mathbb{R}$, h is a boundary input, and all of them generate a wave field (u, u_t) . If $h = 0$, the wave field is uniquely generated by the sources (φ, ψ, f) . $\Lambda_{\varphi,\psi,f}^0$ encodes the local boundary measurement on Γ_0 of the wave field.

In this paper, we are first concerned with the following inverse problem:

- **Inverse problem 1.** Can we identify unknown functions (φ, ψ, f) by using the passive measurement $\Lambda_{\varphi,\psi,f}^0$?

For this problem, we give some assumptions. Suppose that

$$\Gamma_0 = \left\{ x \in \Gamma \mid (x - x_0) \cdot \nu(x) > 0 \right\} \text{ for some } x_0 \in \mathbb{R}^n \setminus \overline{\Omega}. \quad (2.2)$$

Assume that $T > T^*$, where

$$T^* = 2 \max_{x \in \overline{\Omega}} |x - x_0|. \quad (2.3)$$

Also, introduce the following increasing condition on $f : Q \times \mathbb{R} \rightarrow \mathbb{R}$:

$$\limsup_{s \rightarrow \infty} \frac{\partial_s f(x, t, s)}{\ln|s|} = 0, \quad \text{uniformly for } (x, t) \in \overline{Q}, \quad (2.4)$$

and a set:

$$\mathcal{M}_T = \left\{ f : Q \times \mathbb{R} \rightarrow \mathbb{R} \mid \begin{array}{l} f(x, t, \cdot) \in C^1(\mathbb{R}) \text{ in } Q, \ f(\cdot, \cdot, 0) \in L^2(Q), \\ \text{and (2.4) holds} \end{array} \right\}. \quad (2.5)$$

By Section 3, for any $(\varphi, \psi) \in H_0^1(\Omega) \times L^2(\Omega)$, $h = 0$ and $f \in \mathcal{M}_T$, (2.1) has a unique solution

$$u \in H_0 = C([0, T]; H_0^1(\Omega)) \cap C^1([0, T]; L^2(\Omega)).$$

Moreover, $\partial_\nu u \in L^2(\Sigma)$. Note that any function in $L^\infty(Q; W^{1,\infty}(\mathbb{R}))$ satisfies (2.4).

The uniqueness result of this paper on the **Inverse problem 1** is stated as follows.

Theorem 2.1 (Stability of initial data by passive measurement). *For any $T > T^*$, $f \in \mathcal{M}_T$ and $(\varphi_j, \psi_j) \in H_0^1(\Omega) \times L^2(\Omega)$ ($j = 1, 2$), if $u_j \in H_0$ is the solution to the following semilinear wave equation:*

$$\begin{cases} u_{j,tt} - \Delta u_j + f(x, t, u_j) = 0 & \text{in } Q, \\ u_j = 0 & \text{on } \Sigma, \\ u_j(x, 0) = \varphi_j(x), \ u_{j,t}(x, 0) = \psi_j(x) & \text{in } \Omega, \end{cases} \quad (2.6)$$

then the following quantitative stability estimate holds:

$$\begin{aligned} & \|(\varphi_1 - \varphi_2, \psi_1 - \psi_2)\|_{H_0^1(\Omega) \times L^2(\Omega)} \\ & \leq C(f, u_1, u_2, n, T, \Omega, \Sigma, \Gamma_0) \|\Lambda_{\varphi_1, \psi_1, f}^0 - \Lambda_{\varphi_2, \psi_2, f}^0\|_{L^2(0, T; L^2(\Gamma_0))}, \end{aligned} \quad (2.7)$$

where $C(f, u_1, u_2, n, T, \Omega, \Sigma, \Gamma_0)$ denotes a positive constant depending on $f, u_1, u_2, n, T, \Omega, \Sigma$ and Γ_0 .

Remark 2.1. *Due to the fact that some properties of parabolic equations and hyperbolic equations are fundamentally different, there are many differences between the corresponding inverse problems. One typical feature is that the solution to a hyperbolic equation has a finite propagation speed. Hence, the results on the inverse problems of this paper in determining initial values and nonlinearity of semilinear hyperbolic equations need a sufficient large time T in the space-time domain Q . However, this is not needed in the parabolic case. On the other hand, in determining initial values for semilinear parabolic equations, generally speaking, only conditional stability result may be obtained, i.e., the considered solutions have some bounded restrictions (see our results on semilinear parabolic equations in [LLLZ22]). But the restrictions are removed in the stability results for hyperbolic equations.*

Remark 2.2. *In Theorem 2.1, the quantitative stability estimate (2.7) for the inverse problem of the semilinear hyperbolic equations (2.6) is derived. We notice that in the estimate, the constant C depends on the solutions u_1 and u_2 to the semilinear hyperbolic equations. Such a dependence is always in place from nonlinear PDEs. Here we refer to [EEK05, KM91, KL21] for some earlier and important results on stability estimates for inverse problems associated with parabolic and hyperbolic equations.*

As a corollary of Theorem 2.1, introduce the following set on f :

$$\mathcal{C}_T = \left\{ f : Q \times \mathbb{R} \rightarrow \mathbb{R} \mid \begin{aligned} & f(x, t, s) = f_0(x, t, s)\chi_{[0, T^* + \epsilon]}(t) + g(x, t, s)\chi_{[T^* + \epsilon, T]}(t) \\ & \text{for some } \epsilon > 0 \text{ with } T^* + \epsilon < T \text{ and any given } f_0 \in \mathcal{M}_T, \\ & \text{where } g \in \mathcal{M}_T \end{aligned} \right\}, \quad (2.8)$$

where $\chi_E = \begin{cases} 1 & \text{if } x \in E \\ 0 & \text{otherwise} \end{cases}$ denotes the characteristic function on a set $E \subseteq \mathbb{R}$. The following corollary states that when the nonlinear function $f \in \mathcal{C}_T$, then one can determine the initial data regardless of the nonlinearity $f \in \mathcal{C}_T$.

Corollary 2.3. *For any $T > T^*$, $f_j \in \mathcal{C}_T$ given by (2.8), and $(\varphi_j, \psi_j) \in H_0^1(\Omega) \times L^2(\Omega)$ ($j = 1, 2$). Let $u_j \in H_0$ be the solution to the following semilinear wave equation:*

$$\begin{cases} u_{j,tt} - \Delta u_j + f_j(x, t, u_j) = 0 & \text{in } Q, \\ u_j = 0 & \text{on } \Sigma, \\ u_j(x, 0) = \varphi_j(x), \quad u_{j,t}(x, 0) = \psi_j(x) & \text{in } \Omega, \end{cases} \quad (2.9)$$

and

$$\Lambda_{\varphi_1, \psi_1, f_1}^0 = \Lambda_{\varphi_2, \psi_2, f_2}^0, \quad (2.10)$$

then

$$(\varphi_1, \psi_1) = (\varphi_2, \psi_2) \quad \text{in } \Omega.$$

This means that the passive measurement $\Lambda_{\varphi, \psi, f}^0$ uniquely determines (φ, ψ) , independent of functions f in \mathcal{C}_T .

Remark 2.4. *Let us remark that:*

- (1) *Theorem 2.1 and Corollary 2.3 show that one may use the passive measurement to determine initial data $(u(0), u_t(0)) = (\psi, \varphi)$, respectively, for a given $f \in \mathcal{M}_T$ or any $f \in \mathcal{C}_T$, where \mathcal{M}_T and \mathcal{C}_T are given in (2.5) and (2.8), respectively.*
- (2) *Note that the requirements on nonlinear functions f in (2.5) and (2.8) are technical. The known observability result for linear wave equations is used to prove the above*

uniqueness results. When f is fixed, the conditions on (2.2), (2.5) and $T > T^*$ assure the coefficient $a \in L^\infty(0, T; L^n(\Omega))$ for the linearized system of (2.1):

$$\tilde{u}_{tt} - \Delta \tilde{u} + a(x, t)\tilde{u}(x, t) = 0,$$

where a is given in (4.3). It satisfies the regularity requirement in the observability result. When f is unknown, it is chosen in the set (2.8). Indeed, this condition divides f into two parts with respect to time. In the first time interval $[0, T^* + \epsilon]$, we may identify initial data by the passive measurement by Theorem 2.1. Hence, there is indeed no requirement on nonlinear function g in the rest time interval $[T^* + \epsilon, T]$ only if it ensures the well-posedness.

Remark 2.5. The results on inverse problems in Theorem 2.1 and Corollary 2.3 may be generalized to the following semilinear hyperbolic equation:

$$\begin{cases} u_{tt} - \nabla \cdot (\sigma \nabla u) + f(x, t, u) = 0 & \text{in } Q, \\ u = h & \text{on } \Sigma, \\ u(x, 0) = \varphi(x), \quad u_t(x, 0) = \psi(x) & \text{in } \Omega, \end{cases}$$

where $\sigma(\cdot) \in C^2(\bar{\Omega}; \mathbb{R}^{n \times n})$ is a positive definite matrix-valued function, which satisfies the following condition:

(H) There exists a positive constant ρ_0 and a positive function $d(\cdot) \in C^2(\bar{\Omega})$ without any critical point in $\bar{\Omega}$, such that for any $(x, \xi^1, \dots, \xi^n) \in \bar{\Omega} \times \mathbb{R}^n$,

$$\sum_{i,j=1}^n \sum_{i',j'=1}^n \left[2\sigma^{ij'}(\sigma^{i'j} d_{x_{i'}})_{x_{j'}} - \sigma_{x_{j'}}^{ij} \sigma^{i'j'} d_{x_{i'}} \right] \xi^i \xi^j \geq \rho_0 \sum_{i,j=1}^n \sigma^{ij}(x) \xi^i \xi^j. \quad (2.11)$$

Also,

$$\Gamma_0 = \left\{ x \in \Gamma \mid \sum_{i,j=1}^n \sigma^{ij}(x) d_{x_i}(x) \nu_j(x) > 0 \right\}. \quad (2.12)$$

The above conditions on σ and Γ_0 are used to guarantee the observability of linear hyperbolic equations.

More importantly, in this paper we will determine coefficient and initial data simultaneously for the semilinear wave equation (2.1). To avoid confusion of notations, we replace f by \tilde{f} and consider the following semilinear wave equation:

$$\begin{cases} u_{tt} - \Delta u + \tilde{f}(x, t, u) = 0 & \text{in } Q, \\ u = h & \text{on } \Sigma, \\ u(x, 0) = \varphi(x), \quad u_t(x, 0) = \psi(x) & \text{in } \Omega. \end{cases} \quad (2.13)$$

For any given pair of initial values (φ, ψ) and a suitable function \tilde{f} , which guarantees the well-posedness of (2.13), define the following the input-output map by

$$\Lambda_{\varphi, \psi, \tilde{f}}(h) = \left(\partial_\nu u \Big|_\Sigma, u(\cdot, T), u_t(\cdot, T) \right), \quad \text{for all } h \in E_\delta, \quad (2.14)$$

where E_δ will be defined later and u is the solution to (2.13) associated to (φ, ψ, h) . If $\Lambda_{\varphi, \psi, \tilde{f}}(h)$ is known for all $h \in E_\delta$, it means that the operator $\Lambda_{\varphi, \psi, \tilde{f}}$ is known and it is called the active measurement. We are concerned with the following inverse problem.

- **Inverse problem 2.** Can we identify unknown initial data and coefficient $(\varphi, \psi, \tilde{f})$ by using the active measurement $\Lambda_{\varphi, \psi, \tilde{f}}$?

To our best knowledge, this simultaneously recovering inverse problem for semilinear wave equations is the first result to be considered in the field.

First, introduce some notations and assumptions. Assume that m is a positive integer and define the energy space E^m as in [LLPMT22, Section 1] and [CB08, Definition 3.5 in page 596]:

$$E^m = \bigcap_{k=0}^m C^k([0, T]; H^{m-k}(\Omega)), \quad (2.15)$$

which is equipped with the norm $\|\cdot\|_{E^m}$ as

$$\|u\|_{E^m} = \sup_{0 \leq t \leq T} \sum_{k=0}^m \left\| \partial_t^k u(\cdot, t) \right\|_{H^{m-k}(\Omega)}, \quad \forall u \in E^m.$$

Inspired by [LLPMT22, Definition 1], we impose the following conditions on \tilde{f} .

Definition 2.6 (Admissible coefficients). *For any $T > T^*$ (in (2.3)), $\tilde{f} = \tilde{f}(x, t, s): Q \times \mathbb{R} \rightarrow \mathbb{R}$ is called an admissible coefficient, if it satisfies:*

(1) *Analyticity on \mathbb{R} :*

$$\begin{cases} \text{the map } s \mapsto \tilde{f}(\cdot, \cdot, s) \text{ is analytic on } \mathbb{R} \text{ with values in } E^{m+1}, \\ \tilde{f}(x, t, 0) = 0, \text{ in } Q. \end{cases} \quad (2.16)$$

This means that \tilde{f} may be written as the Taylor expansion at any $s_0 \in \mathbb{R}$:

$$\tilde{f}(x, t, s) = \sum_{k=0}^{\infty} \tilde{f}^{(k)}(x, t, s_0) \frac{(s - s_0)^k}{k!},$$

where $\frac{\tilde{f}^{(k)}(x, t, s_0)}{k!} = \frac{\partial_s^k \tilde{f}(x, t, s_0)}{k!}$ are Taylor's coefficients at $s_0 \in \mathbb{R}$, for any $k \in \mathbb{N}$.

(2) *Compact support: There exist two positive constants t_1 and t_2 with $T^* < t_1 < t_2 < T$, such that for any $s \in \mathbb{R}$,*

$$\text{supp } \tilde{f}(\cdot, \cdot, s) \subseteq \overline{Q} \times [t_1, t_2]. \quad (2.17)$$

Remark 2.7. *The definition of admissible coefficients is inspired by the need to guarantee the well-posedness of (2.13) in E^{m+1} and the application of complex geometrical optics solutions (see Section 5). Indeed, in order to derive the well-posedness results, the compact support condition on \tilde{f} may be weaken to*

$$\text{supp } \tilde{f}(\cdot, \cdot, s) \subseteq \overline{\Omega} \times (0, T].$$

It suffices to require \tilde{f} to be zero near initial time for the compatibility conditions. The compact support condition (2.17) is technical and it will be used in studying the above inverse problem.

Furthermore, for a positive integer m , define the following function space:

$$\mathcal{N}_m = \left\{ h \in H^m(\Sigma) \mid h \in H_0^{m-k}(0, T; H^k(\Gamma)), \text{ for } k = 0, 1, \dots, m-1 \right\}, \quad (2.18)$$

and for a positive constant δ , set

$$E_\delta = \left\{ h \in \mathcal{N}_{m+1} \mid \|h\|_{H^{m+1}(\Sigma)} < \delta/2 \right\}.$$

By the local well-posedness of the semilinear wave equation (2.13) (see Section 3), for any $m > n + 1$ and an admissible coefficient \tilde{f} , there exists a $\delta > 0$, when $(\varphi, \psi) \in H_0^{m+1}(\Omega) \times H_0^m(\Omega)$ and $h \in \mathcal{N}_{m+1}$ satisfy

$$\|(\varphi, \psi)\|_{H^{m+1}(\Omega) \times H^m(\Omega)} + \|h\|_{H^{m+1}(\Sigma)} < \delta,$$

(2.13) has a unique solution $u \in E^{m+1}$ and $\partial_\nu u \in H^m(\Sigma)$.

Now, we give an answer to **Inverse problem 2** as follows.

Theorem 2.2 (Simultaneous recovery by active measurement). *Assume that $T > T^*$, $m > n + 1$, and \tilde{f}_1 and \tilde{f}_2 are two admissible coefficients. There exists a $\delta > 0$, such that for any $(\varphi_j, \psi_j) \in H_0^{m+1}(\Omega) \times H_0^m(\Omega)$ ($j = 1, 2$) with $\|(\varphi_j, \psi_j)\|_{H^{m+1}(\Omega) \times H^m(\Omega)} < \delta/2$, denote by $u_j \in E^{m+1}$ the solution to the following semilinear wave equation:*

$$\begin{cases} u_{j,tt} - \Delta u_j + \tilde{f}_j(x, t, u_j) = 0 & \text{in } Q, \\ u_j = h & \text{on } \Sigma, \\ u_j(x, 0) = \varphi_j(x), \quad u_{j,t}(x, 0) = \psi_j(x) & \text{in } \Omega, \end{cases} \quad (2.19)$$

for $j = 1, 2$. Let $\Lambda_{\varphi_j, \psi_j, \tilde{f}_j}$ be the input-output map defined via (2.14) for $j = 1, 2$, and if

$$\Lambda_{\varphi_1, \psi_1, \tilde{f}_1}(h) = \Lambda_{\varphi_2, \psi_2, \tilde{f}_2}(h), \quad \text{for all } h \in E_\delta,$$

then

$$\varphi_1 = \varphi_2, \quad \psi_1 = \psi_2 \text{ in } \Omega \quad \text{and} \quad \tilde{f}_1 = \tilde{f}_2 \text{ in } Q \times \mathbb{R}.$$

The proof of Theorem 2.2 is mainly based on the higher order linearization method, which was initiated in [KLU18] for some nonlinear hyperbolic equations. Recently, this method has been extended to many other different problems, such as [KU20a, KU20b, LLLS21, LLLS20] and rich references therein.

Remark 2.8. *Now, we explain main differences between Theorem 2.1 and Theorem 2.2.*

- (1) *In Theorem 2.1, any initial value (φ, ψ) (without smallness conditions) of the initial boundary value problem (2.1) can be determined uniquely, by utilizing the passive measurement, under suitable assumptions on coefficients f .*
- (2) *In Theorem 2.2, by using the active measurement, one can determine small initial data (φ, ψ) and admissible coefficient \tilde{f} simultaneously in the initial boundary value problem (2.13). The smallness conditions in Theorem 2.2 are mainly used to prove the local well-posedness. In the determination of initial data and coefficients, they are not essential.*

Before ending this section, we give a corollary for Theorem 2.2 to show the simultaneously recovering for the following linear wave equation:

$$\begin{cases} u_{tt} - \Delta u + qu = 0 & \text{in } Q, \\ u = h & \text{on } \Sigma, \\ u(x, 0) = \varphi(x), \quad u_t(x, 0) = \psi(x) & \text{in } \Omega, \end{cases} \quad (2.20)$$

where $q \in E^{m+1}$, $\varphi \in H_0^{m+1}(\Omega)$, $\psi \in H_0^m(\Omega)$ and $h \in \mathcal{N}_{m+1}$ for $m > n + 1$. By Lemma 3.2, (2.20) is well-posed with $u \in E^{m+1}$ and $\partial_\nu u \in H^m(\Sigma)$. Due to linearity, we do not need to impose any smallness condition for both initial data and boundary inputs.

Now, for any $\varphi \in H_0^{m+1}(\Omega)$, $\psi \in H_0^m(\Omega)$ and $q \in E^{m+1}$, we define the corresponding input-output map $\Lambda_{\varphi, \psi, q}$ of (2.20) via

$$\Lambda_{\varphi, \psi, q}(h) = \left(\partial_\nu u \Big|_\Sigma, u(\cdot, T), u_t(\cdot, T) \right), \quad \text{for all } h \in \mathcal{N}_{m+1}, \quad (2.21)$$

where $u \in E^{m+1}$ is the solution to (2.20). In order to study this inverse problem by Theorem 2.2, we still assume that t_1 and t_2 are two positive constants with $T^* < t_1 < t_2 < T$ as in Definition 2.6, and $q \in E^{m+1}$ with $\text{supp } q \subseteq \Omega \times [t_1, t_2]$. Then the following result holds, which may be regarded as a corollary of Theorem 2.2 in the case that $\tilde{f}(x, t, u) = q(x, t)u$.

Corollary 2.9 (Simultaneous recovery for linear wave equations). *Assume that $T > T^*$, $m > n + 1$ and $q_j \in E^{m+1}$ with $\text{supp } q_j \subseteq \Omega \times [t_1, t_2]$ for $j = 1, 2$. For any $(\varphi_j, \psi_j) \in H_0^{m+1}(\Omega) \times H_0^m(\Omega)$ ($j = 1, 2$), denote by $u_j \in E^{m+1}$ the solution to the following linear wave equation:*

$$\begin{cases} u_{j,tt} - \Delta u_j + q_j u_j = 0 & \text{in } Q, \\ u_j = h & \text{on } \Sigma, \\ u_j(x, 0) = \varphi_j(x), \quad u_{j,t}(x, 0) = \psi_j(x) & \text{in } \Omega, \end{cases} \quad (2.22)$$

for $j = 1, 2$. Let $\Lambda_{\varphi_j, \psi_j, q_j}$ be the input-output map defined via (2.21) for $j = 1, 2$, and if

$$\Lambda_{\varphi_1, \psi_1, q_1}(h) = \Lambda_{\varphi_2, \psi_2, q_2}(h), \quad \text{for all } h \in E_\delta,$$

then

$$\varphi_1 = \varphi_2, \quad \psi_1 = \psi_2 \text{ in } \Omega \quad \text{and} \quad q_1 = q_2 \text{ in } Q.$$

3. WELL-POSEDNESS OF SEMILINEAR WAVE EQUATIONS

This section is devoted to investigating the well-posedness of the semilinear wave equations (2.1) and (2.13). The global well-posedness for (2.1) under the superlinear increasing condition (2.4) and local well-posedness for (2.13) under admissible coefficients conditions are established, respectively. Throughout this paper, we denote by C a positive constant, which is independent of solutions to involved equations and may be different from line to line. Furthermore, if the constant C depends on some factor, for example, C depends on some real number p , we will write $C = C(p)$.

3.1. Local well-posedness with small data. This subsection is devoted to the well-posedness of the semilinear wave equation (2.13). Similar results have been investigated in some known works for different structures on \tilde{f} , see for instance [NVW20].

To begin with, recall the definition of the energy space

$$E^m = \bigcap_{k=0}^m C^k([0, T]; H^{m-k}(\Omega)).$$

By the Sobolev embedding and [CB08, Definition 3.5], the above space E^m is an algebra, due to

$$\|\phi\psi\|_{E^m} \leq C_m \|\phi\|_{E^m} \|\psi\|_{E^m}, \quad \text{for any } \phi, \psi \in E^m,$$

for any integer $m > n + 1$. Indeed, the algebra property of function spaces plays an essential role in the study of the well-posedness for many nonlinear partial differential equations. For example, in [FO20, KU20a, LLLS21, LLLS20], suitable Hölder continuous spaces were utilized to prove the well-posedness for semilinear elliptic equations.

Next, we recall a known well-posedness result for the following wave equation:

$$\begin{cases} v_{tt} - \Delta v = g & \text{in } Q, \\ v = h & \text{on } \Sigma, \\ v(x, 0) = \varphi(x), \quad v_t(x, 0) = \psi(x) & \text{in } \Omega. \end{cases} \quad (3.1)$$

The compatibility conditions up to order m^1 mean that

$$\begin{cases} h(x, 0) = \varphi(x), \quad h_t(x, 0) = \psi(x) \quad \text{on } \Gamma, \\ h_{tt}(x, 0) = \Delta\varphi(x) + g(x, 0) \quad \text{on } \Gamma, \\ h_{ttt}(x, 0) = \Delta\psi(x) + g_t(x, 0) \quad \text{on } \Gamma, \\ h_{tttt}(x, 0) = \Delta^2\varphi(x) + \Delta g(x, 0) + g_{tt}(x, 0) \quad \text{on } \Gamma, \end{cases} \quad (3.2)$$

and higher order derivatives of h up to order m with respect to time.

By [KKL01, Theorem 2.45], the following well-posedness result holds for (3.1).

Lemma 3.1. *Let m be a non-negative integer and $T > 0$. For any $\varphi \in H^{m+1}(\Omega)$, $\psi \in H^m(\Omega)$, $h \in H^{m+1}(\Sigma)$ and $g \in L^1(0, T; H^m(\Omega))$ with $\partial_t^m g \in L^1(0, T; L^2(\Omega))$, if the compatibility conditions (3.2) hold, (3.1) admits a unique solution*

$$v \in C([0, T]; H^{m+1}(\Omega)) \cap C^{m+1}([0, T]; L^2(\Omega))$$

and $\partial_\nu v \in H^m(\Sigma)$. Moreover,

$$\begin{aligned} & \|v\|_{C([0, T]; H^{m+1}(\Omega))} + \|v\|_{C^{m+1}([0, T]; L^2(\Omega))} + \|\partial_\nu v\|_{H^m(\Sigma)} \\ & \leq C e^{CT} \left(\|g\|_{L^1(0, T; H^m(\Omega))} + \|\partial_t^m g\|_{L^1(0, T; L^2(\Omega))} \right. \\ & \quad \left. + \|\varphi\|_{H^{m+1}(\Omega)} + \|\psi\|_{H^m(\Omega)} + \|h\|_{H^{m+1}(\Sigma)} \right). \end{aligned}$$

Based on Lemma 3.1, we have the well-posedness result for the following linear wave equation:

$$\begin{cases} v_{tt} - \Delta v + qv = g_1 & \text{in } Q, \\ v = h & \text{on } \Sigma, \\ v(x, 0) = \varphi(x), \quad v_t(x, 0) = \psi(x) & \text{in } \Omega, \end{cases} \quad (3.3)$$

where $q \in E^m$.

Lemma 3.2. *Let $m > n + 1$ and $T > 0$. For any $\varphi \in H_0^{m+1}(\Omega)$, $\psi \in H_0^m(\Omega)$, $h \in \mathcal{N}_{m+1}$ (see (2.18)), $q \in E^m$, and $g_1 \in E^m$ with $\partial_t^k g_1(\cdot, 0) \in H_0^{m-k}(\Omega)$ for $k = 0, 1, \dots, m-2$, (3.3) admits a unique solution*

$$v \in E^{m+1} \quad \text{and} \quad \partial_\nu v \in H^m(\Sigma).$$

Moreover,

$$\begin{aligned} & \|v\|_{E^{m+1}} + \|\partial_\nu v\|_{H^m(\Sigma)} \\ & \leq C e^{CT} \left(\sum_{k=0}^m \|\partial_t^k g_1\|_{L^1(0, T; H^{m-k}(\Omega))} + \|\varphi\|_{H^{m+1}(\Omega)} + \|\psi\|_{H^m(\Omega)} + \|h\|_{H^{m+1}(\Sigma)} \right). \end{aligned} \quad (3.4)$$

Proof. First, we consider the case of $q = 0$. By Lemma 3.1 and the definition of E^{m+1} , it suffices to prove that for any positive integer $k \in (0, m+1)$,

$$\partial_t^k v(\cdot, \cdot) \in C([0, T]; H^{m+1-k}(\Omega)).$$

Set $u = \partial_t^k v$ and it satisfies the following equation:

$$\begin{cases} u_{tt} - \Delta u = \partial_t^k g_1 & \text{in } Q, \\ u = \partial_t^k h & \text{on } \Sigma. \end{cases}$$

¹One needs to check the compatible conditions for $\partial_t^k h(x, 0)$ for $x \in \Gamma$ and for $k = 0, 1, \dots, m$ to get higher order regularity estimates.

Since

$$g_1 \in E^m, \quad h \in H^{m+1}(\Sigma) \quad \text{and} \quad (\varphi, \psi) \in H^{m+1}(\Omega) \times H^m(\Omega),$$

we have that

$$\begin{aligned} u(\cdot, 0) &\in H^{m+1-k}(\Omega), \quad u_t(\cdot, 0) \in H^{m-k}(\Omega), \quad \partial_t^k h \in H^{m+1-k}(\Sigma), \\ \partial_t^k g_1 &\in L^1(0, T; H^{m-k}(\Omega)) \quad \text{and} \quad \partial_t^m g_1 \in L^1(0, T; L^2(\Omega)). \end{aligned}$$

And the compatibility conditions of order $m - k$ hold. By Lemma 3.1,

$$u = \partial_t^k v \in C([0, T]; H^{m+1-k}(\Omega)) \quad \text{for any integer } k \in (0, m + 1).$$

Also, the estimate (3.4) remains true for the case of $q = 0$.

Next, consider the general case of $q \in E^m$. Define the set

$$\mathcal{K}_1 = \left\{ v \in E^m \mid \partial_t^k v(\cdot, 0) \in H_0^{m-k}(\Omega), \text{ for } k = 0, 1, \dots, m-2 \right\}.$$

For any $v \in \mathcal{K}_1$, consider the following wave equation:

$$\begin{cases} w_{tt} - \Delta w = g_1 - qv & \text{in } Q, \\ w = h & \text{on } \Sigma, \\ w(x, 0) = \varphi(x), \quad w_t(x, 0) = \psi(x) & \text{in } \Omega. \end{cases} \quad (3.5)$$

By $g_1, v, q \in E^m$, it follows that $g_1 - qv \in E^m$. Also,

$$\partial_t^k [g_1(\cdot, 0) - (qv)(\cdot, 0)] \in H_0^{m-k}(\Omega), \quad \text{for } k = 0, 1, \dots, m-2.$$

By the well-posedness result in the case of $q = 0$, (3.5) admits a unique solution

$$w \in E^{m+1} \quad \text{and} \quad \partial_\nu w \in H^m(\Sigma).$$

Moreover,

$$\begin{aligned} \|w\|_{E^{m+1}} + \|\partial_\nu w\|_{H^m(\Sigma)} &\leq C e^{CT} \left(\sum_{k=0}^m \|\partial_t^k (g_1 - qv)\|_{L^1(0, T; H^{m-k}(\Omega))} \right. \\ &\quad \left. + \|\varphi\|_{H^{m+1}(\Omega)} + \|\psi\|_{H^m(\Omega)} + \|h\|_{H^{m+1}(\Sigma)} \right). \end{aligned} \quad (3.6)$$

Define the mapping

$$\mathcal{L}_1 : \mathcal{K}_1 \rightarrow \mathcal{K}_1, \quad \mathcal{L}_1(v) = w,$$

where w is the solution to (3.5) associated to $v \in \mathcal{K}_1$. For any $v_1, v_2 \in \mathcal{K}_1$, denote by w_1 and w_2 the associated solutions to (3.5). By (3.6), we obtain that

$$\begin{aligned} &\|w_1 - w_2\|_{E^{m+1}} + \|\partial_\nu (w_1 - w_2)\|_{H^m(\Sigma)} \\ &\leq C e^{CT} \sum_{k=0}^m \|\partial_t^k [q(v_1 - v_2)]\|_{L^1(0, T; H^{m-k}(\Omega))} \\ &\leq CT e^{CT} \sum_{k=0}^m \|\partial_t^k [q(v_1 - v_2)]\|_{C([0, T]; H^{m-k}(\Omega))} \\ &= CT e^{CT} \|q(v_1 - v_2)\|_{E^m} \leq CT e^{CT} \|q\|_{E^m} \|v_1 - v_2\|_{E^m}. \end{aligned}$$

If T is sufficiently small such that $CT e^{CT} \|q\|_{E^m} < 1$, then by the Banach fixed point theorem, \mathcal{L}_1 has a unique fixed point $v \in \mathcal{K}_1$. Since (3.3) is a linear equation, by a rescaling with respect to the time variable, we can get the result for any $T > 0$. \square

The main result of this subsection is stated as follows.

Theorem 3.1 (Local well-posedness). *Assume that $m > n + 1$ and \tilde{f} is an admissible coefficient. Then there exists a $\delta > 0$, such that for any (h, φ, ψ) in the set*

$$U_\delta = \left\{ (h, \varphi, \psi) \in \mathcal{N}_{m+1} \times H_0^{m+1}(\Omega) \times H_0^m(\Omega) \mid \|h\|_{H^{m+1}(\Sigma)} + \|\varphi\|_{H^{m+1}(\Omega)} + \|\psi\|_{H^m(\Omega)} < \delta \right\},$$

(2.13) admits a unique solution $u \in E^{m+1}$ satisfying that $\partial_\nu u \in H^m(\Sigma)$ and

$$\|u\|_{E^{m+1}} + \|u\|_{C(\bar{Q})} + \|\partial_\nu u\|_{H^m(\Sigma)} \leq C \left(\|h\|_{H^{m+1}(\Sigma)} + \|\varphi\|_{H^{m+1}(\Omega)} + \|\psi\|_{H^m(\Omega)} \right), \quad (3.7)$$

where C is a positive constant independent of u, h, φ and ψ . Moreover, the following solution map is C^∞ Fréchet differentiable:

$$S : U_\delta \rightarrow E^{m+1}, \quad (h, \varphi, \psi) \mapsto u.$$

Proof. Similar to arguments in [KU20a, LLLS20], we prove the well-posedness of (2.13) by the implicit function theorem in Banach spaces.

First, set

$$X_1 = \mathcal{N}_{m+1} \times H_0^{m+1}(\Omega) \times H_0^m(\Omega),$$

$$X_2 = \left\{ u \in E^{m+1} \mid u|_\Sigma \in \mathcal{N}_{m+1}, \partial_\nu u \in H^m(\Sigma), u(\cdot, 0) \in H_0^{m+1}(\Omega), u_t(\cdot, 0) \in H_0^m(\Omega), \right.$$

$$\left. u_{tt} - \Delta u \in E^m, \partial_t^k (u_{tt} - \Delta u)(\cdot, 0) \in H_0^{m-k}(\Omega), k = 0, 1, \dots, m-2 \right\},$$

where

$$\|u\|_{X_2} = \|u\|_{E^{m+1}} + \|\partial_\nu u\|_{H^m(\Sigma)} + \|u\|_{H^{m+1}(\Sigma)} + \|u_{tt} - \Delta u\|_{E^m}.$$

Note that X_2 is nonempty and indeed $C_0^\infty(Q) \subseteq X_2$. Meanwhile, let us write

$$X_3 = \left\{ g \in E^m \mid \partial_t^k g(\cdot, 0) \in H_0^{m-k}(\Omega), \forall k = 0, 1, \dots, m-2 \right\} \times X_1.$$

Consider the following map:

$$F : X_1 \times X_2 \rightarrow X_3,$$

$$F(h, \varphi, \psi, u) = \left(u_{tt} - \Delta u + \tilde{f}(x, t, u), u|_\Sigma - h, u(\cdot, 0) - \varphi, u_t(\cdot, 0) - \psi \right), \quad (3.8)$$

where $(h, \varphi, \psi) \in X_1$ and $u \in X_2$. By the condition (2.16) for \tilde{f} , for any positive integer k and positive constant R ,

$$\|\tilde{f}^{(k)}(\cdot, \cdot, 0)\|_{E^{m+1}} \leq \frac{k!}{R^k} \sup_{|s|=R} \|\tilde{f}(\cdot, \cdot, s)\|_{E^{m+1}}. \quad (3.9)$$

Since E^{m+1} is an algebra, it follows that for any $u \in X_2$,

$$\begin{aligned} \|\tilde{f}(\cdot, \cdot, u(\cdot, \cdot))\|_{E^{m+1}} &\leq \sum_{k=0}^{\infty} \frac{1}{k!} \|\tilde{f}^{(k)}(\cdot, \cdot, 0)\|_{E^{m+1}} \|u\|_{E^{m+1}}^k \\ &\leq \sum_{k=0}^{\infty} \frac{1}{R^k} \|u\|_{E^{m+1}}^k \sup_{|s|=R} \|\tilde{f}(\cdot, \cdot, s)\|_{E^{m+1}}. \end{aligned}$$

Choose $R = 2(\|u\|_{E^{m+1}} + 1)$. Then, $\tilde{f}(\cdot, \cdot, u(\cdot, \cdot)) \in E^{m+1}$ and

$$\|\tilde{f}(\cdot, \cdot, u(\cdot, \cdot))\|_{E^{m+1}} \leq C \sup_{|s|=2(\|u\|_{E^{m+1}}+1)} \|\tilde{f}(\cdot, \cdot, s)\|_{E^{m+1}}.$$

By using the definition of X_2 , $u_{tt} - \Delta u \in E^m$ and therefore, $u_{tt} - \Delta u + \tilde{f}(\cdot, \cdot, u(\cdot, \cdot)) \in E^m$. Also, by the admissible coefficient condition on \tilde{f} ,

$$\partial_t^k \tilde{f}(\cdot, \cdot, u(\cdot, \cdot)) \Big|_{t=0} = 0, \quad \text{for any } k = 0, 1, \dots, m-2, \text{ and } u \in X_2.$$

Hence, the map F is well-defined.

Now, we prove that F is locally bounded. Indeed, for any $M > 0$, when $(h, \varphi, \psi, u) \in X_1 \times X_2$ with $\|(h, \varphi, \psi, u)\|_{X_1 \times X_2} \leq M$,

$$\begin{aligned}
 & \|F(h, \varphi, \psi, u)\|_{X_3} \\
 & \leq \|u_{tt} - \Delta u + \tilde{f}(\cdot, \cdot, u(\cdot, \cdot))\|_{E^m} + \|u - h\|_{H^{m+1}(\Sigma)} \\
 & \quad + \|u(\cdot, 0) - \varphi\|_{H^{m+1}(\Omega)} + \|u_t(\cdot, 0) - \psi\|_{H^m(\Omega)} \\
 & \leq \|\tilde{f}(\cdot, \cdot, u(\cdot, \cdot))\|_{E^m} + C\|u\|_{X_2} + \|h\|_{H^{m+1}(\Sigma)} + \|\varphi\|_{H^{m+1}(\Omega)} + \|\psi\|_{H^m(\Omega)} \\
 & \leq C \sup_{|s|=2(1+\|u\|_{E^{m+1}})} \|\tilde{f}(\cdot, \cdot, s)\|_{E^{m+1}} + C\|u\|_{X_2} + \|h\|_{H^{m+1}(\Sigma)} + \|\varphi\|_{H^{m+1}(\Omega)} + \|\psi\|_{H^m(\Omega)} \\
 & \leq C \sup_{|s|=2(1+M)} \|\tilde{f}(\cdot, \cdot, s)\|_{E^{m+1}} + (C+3)M < \infty.
 \end{aligned}$$

Next, we verify the weak holomorphy of F (see [Pos87, page 133]). It is sufficient that for any $(h_0, \varphi_0, \psi_0, u_0), (h, \varphi, \psi, u) \in X_1 \times X_2$, the map

$$\lambda \mapsto F((h_0, \varphi_0, \psi_0, u_0) + \lambda(h, \varphi, \psi, u))$$

is holomorphic in a neighborhood of the origin in with values in X_3 . It suffices to check that the map

$$\lambda \mapsto \tilde{f}(x, t, u_0(x, t) + \lambda u(x, t))$$

is holomorphic in a neighborhood of the origin in \mathbb{C} with values in E^m . This follows from the convergence of the series

$$\sum_{k=0}^{\infty} \frac{\tilde{f}^{(k)}(x, t, 0)}{k!} [u_0(x, t) + \lambda u(x, t)]^k$$

in E^{m+1} , locally uniformly in $\lambda \in \mathbb{C}$. Hence, F is holomorphic in $X_1 \times X_2$.

Moreover, notice that $F(0, 0, 0, 0) = 0$ and $F_u(0, 0, 0, 0) : X_2 \rightarrow X_3$ is defined by

$$F_u(0, 0, 0, 0)w = \left(w_{tt} - \Delta w + \tilde{f}_u(\cdot, \cdot, 0)w, w|_{\Sigma}, w(\cdot, 0), w_t(\cdot, 0) \right), \quad \text{for all } w \in X_2.$$

Indeed, by the definition of X_2 , for any $w \in X_2$, $w_{tt} - \Delta w \in E^m$. Since $\tilde{f}_u(\cdot, \cdot, 0) \in E^{m+1}$, it holds that $w_{tt} - \Delta w + \tilde{f}_u(\cdot, \cdot, 0)w \in E^m$. Also, by the admissible coefficient condition on \tilde{f} ,

$$\partial_t^k [w_{tt} - \Delta w + \tilde{f}_u(\cdot, \cdot, 0)w] (\cdot, 0) \in H_0^{m-k}(\Omega), \quad \forall k = 0, 1, \dots, m-2.$$

Hence, the map $F_u(0, 0, 0, 0)$ is well-defined. Furthermore, by the well-posedness of the linear wave equation (3.3) in Lemma 3.2 with

$$q = \tilde{f}_u(\cdot, \cdot, 0) \quad \text{and} \quad g_1 \in \mathcal{H} = \left\{ g \in E^m \mid \partial_t^k g(\cdot, 0) \in H_0^{m-k}(\Omega), \forall k = 0, 1, \dots, m-2 \right\},$$

$F_u(0, 0, 0, 0)$ is a linear isomorphism from $X_2 \rightarrow X_3$. In fact, for any $g_1 \in \mathcal{H}$, $h \in \mathcal{N}_{m+1}$ and $(\varphi, \psi) \in H_0^{m+1}(\Omega) \times H^m(\Omega)$, the equation (3.3) has a unique solution $w \in E^{m+1}$ and $\partial_\nu w \in H^m(\Sigma)$. Also, by the fact that $g_1 \in \mathcal{H}$, $g_1 = w_{tt} - \Delta w + \tilde{f}_u(\cdot, \cdot, 0)w$,

$$\tilde{f}_u(\cdot, \cdot, 0)w \in E^m \quad \text{and} \quad \partial_t^k [\tilde{f}_u(\cdot, \cdot, 0)w] (\cdot, 0) \in H_0^{m-k}(\Omega), k = 0, 1, \dots, m-2,$$

we have that

$$w_{tt} - \Delta w \in E^m \quad \text{and} \quad \partial_t^k (w_{tt} - \Delta w)(\cdot, 0) \in H_0^{m-k}(\Omega), k = 0, 1, \dots, m-2.$$

By the implicit function theorem in Banach spaces, there exists a $\delta > 0$ and a C^∞ map $S : U_\delta \rightarrow E^{m+1}$, such that for any $(h, \varphi, \psi) \in \mathcal{N}_{m+1} \times H_0^{m+1}(\Omega) \times H_0^m(\Omega)$ satisfying

$$\|h\|_{H^{m+1}(\Sigma)} + \|\varphi\|_{H^{m+1}(\Omega)} + \|\psi\|_{H^m(\Omega)} < \delta,$$

it holds that

$$F(h, \varphi, \psi, S(h, \varphi, \psi)) = (0, 0, 0, 0).$$

Since S is locally Lipschitz continuous and $S(0, 0, 0) = 0$, $u = S(h, \varphi, \psi)$ satisfies that

$$\|u\|_{E^{m+1}} + \|\partial_\nu u\|_{H^m(\Sigma)} \leq C (\|h\|_{H^{m+1}(\Sigma)} + \|\varphi\|_{H^{m+1}(\Omega)} + \|\psi\|_{H^m(\Omega)}).$$

This, combined with the Sobolev embedding theorem, proves the local well-posedness of (2.13) and the estimate (3.7). \square

Note that one may extend the result in Theorem 3.1 to more general hyperbolic equations:

$$u_{tt} - \nabla \cdot (\sigma \nabla u) + \tilde{f}(x, t, u) = 0,$$

where σ is either isotropic or anisotropic. However, in the application of a density result of products of solutions of linear hyperbolic equations, we simply consider the classical wave equation to demonstrate ideas of this approach (see Section 5).

3.2. Global well-posedness of weak solutions. This subsection is devoted to the well-posedness of weak solutions to the following semilinear wave equation:

$$\begin{cases} u_{tt} - \Delta u + f(x, t, u) = 0 & \text{in } Q, \\ u = 0 & \text{on } \Sigma, \\ u(x, 0) = \varphi(x), \quad u_t(x, 0) = \psi(x) & \text{in } \Omega, \end{cases} \quad (3.10)$$

where f satisfies (2.4), and the following conditions:

$$f(x, t, \cdot) \in C^1(\mathbb{R}), \text{ a.e. } (x, t) \in Q \quad \text{and} \quad f(\cdot, \cdot, 0) \in L^2(Q).$$

Under the above assumptions on f , we have the following global well-posedness result for (3.10).

Theorem 3.2 (Global well-posedness). *For any $T > 0$ and $(\varphi, \psi) \in H_0^1(\Omega) \times L^2(\Omega)$, the semilinear wave equation (3.10) admits a unique solution u in the class of*

$$u \in H_0 = C([0, T]; H_0^1(\Omega)) \cap C^1([0, T]; L^2(\Omega)) \quad \text{and} \quad \partial_\nu u \in L^2(\Sigma).$$

Proof. The proof is based on the method of the fixed point theorem. First, assume that $n \geq 3$. Set

$$g(x, t, s) = \begin{cases} \frac{f(x, t, s) - f(x, t, 0)}{s} & \text{for } s \neq 0, \\ \partial_s f(x, t, 0) & \text{for } s = 0, \end{cases} \quad \forall (x, t, s) \in Q \times \mathbb{R}.$$

For any $z \in L^\infty(0, T; L^2(\Omega))$, consider the following linear wave equation:

$$\begin{cases} u_{tt} - \Delta u + a_z(x, t)u + f(x, t, 0) = 0 & \text{in } Q, \\ u = 0 & \text{on } \Sigma, \\ u(x, 0) = \varphi(x), \quad u_t(x, 0) = \psi(x) & \text{in } \Omega, \end{cases} \quad (3.11)$$

where $a_z(x, t) = g(x, t, z(x, t))$. Then we have that $a_z \in L^\infty(0, T; L^p(\Omega))$, for any $p \geq 1$. Indeed, by the condition (2.4), for any $\epsilon \in (0, 1)$, there is a $C_\epsilon > 0$, such that

$$|g(x, t, s)| \leq \epsilon \ln(1 + |s|) + C_\epsilon, \quad \forall (x, t, s) \in Q \times \mathbb{R}.$$

Therefore, for any $z \in L^\infty(0, T; L^2(\Omega))$,

$$\begin{aligned}
 & \sup_{t \in (0, T)} \int_{\Omega} e^{C|a_z(x, t)|} dx \\
 &= \sup_{t \in (0, T)} \int_{\Omega} e^{C|g(x, t, z(x, t))|} dx \leq \sup_{t \in (0, T)} \int_{\Omega} e^{C[\epsilon \ln(1+|z(x, t)|) + C_\epsilon]} dx \\
 &= C(\epsilon) \sup_{t \in (0, T)} \int_{\Omega} [1 + |z(x, t)|]^{C_\epsilon} dx \leq C(\epsilon) \sup_{t \in (0, T)} \int_{\Omega} [1 + |z(x, t)|]^2 dx \\
 &\leq C(\epsilon) \left(1 + \|z\|_{L^\infty(0, T; L^2(\Omega))}^2\right) < \infty,
 \end{aligned} \tag{3.12}$$

where ϵ is a sufficiently small positive constant such that $C\epsilon \leq 2$. Furthermore, similar to arguments in [LZ00], we may obtain

$$e^{C\|a_z\|_{L^\infty(0, T; L^p(\Omega))}} \leq C \left(1 + \sup_{t \in (0, T)} \int_{\Omega} e^{C|a_z(x, t)|} dx\right). \tag{3.13}$$

Indeed,

$$\begin{aligned}
 & e^{C\|a_z\|_{L^\infty(0, T; L^p(\Omega))}} \\
 &= \sum_{j=0}^{\infty} \frac{C^j}{j!} \|a_z\|_{L^\infty(0, T; L^p(\Omega))}^j \\
 &\leq \sum_{j=0}^{p+1} \frac{C^j}{j!} \sup_{t \in (0, T)} \left(\int_{\Omega} |a_z(x, t)|^p dx\right)^{j/p} + \sum_{j=p+1}^{\infty} \frac{C^j}{j!} \sup_{t \in (0, T)} \left(\int_{\Omega} |a_z(x, t)|^p dx\right)^{j/p} \\
 &\leq C(p) \left[1 + \sup_{t \in (0, T)} \left(\int_{\Omega} |a_z(x, t)|^p dx\right)^{\frac{p+1}{p}}\right] + \sum_{j=p+1}^{\infty} \frac{C^j}{j!} \sup_{t \in (0, T)} \left(\int_{\Omega} |a_z(x, t)|^p dx\right)^{j/p} \\
 &\leq C(p) \left[1 + 2 \sum_{j=p+1}^{\infty} \frac{C^j}{j!} \sup_{t \in (0, T)} \left(\int_{\Omega} |a_z(x, t)|^p dx\right)^{j/p}\right] \\
 &\leq C(p) \left[1 + 2 \sum_{j=p+1}^{\infty} \frac{C^j}{j!} \sup_{t \in (0, T)} \left(\int_{\Omega} |a_z(x, t)|^j dx\right) \cdot |\Omega|^{\frac{j}{p}-1}\right] \\
 &\leq C(p) \left[1 + \sum_{j=0}^{\infty} \frac{C_1^j}{j!} \sup_{t \in (0, T)} \int_{\Omega} |a_z(x, t)|^j dx\right] \leq C(p) \left(1 + \sup_{t \in (0, T)} \int_{\Omega} e^{C_1|a_z(x, t)|} dx\right),
 \end{aligned}$$

where $C_1 = C|\Omega|^{1/p}$, $C(p)$ denotes a positive constant, which may be different in different places, and $|\Omega|$ denotes the measure of the set Ω . Combining (3.13) with (3.12), one has that $a_z \in L^\infty(0, T; L^p(\Omega))$.

Next, we prove that the linear wave equation (3.11) admits a unique solution $u \in H_0$ and $\partial_\nu u \in L^2(\Sigma)$. For any $w \in L^1(0, T; L^{2+\gamma_0}(\Omega))$, with γ_0 being a positive constant, which will be determined later, choose $p = 2(2 + \gamma_0)/\gamma_0$. By $a_z \in L^\infty(0, T; L^p(\Omega))$, it holds that $a_z w \in L^1(0, T; L^2(\Omega))$. Consider the following wave equation:

$$\begin{cases} u_{tt} - \Delta u = -a_z(x, t)w - f(x, t, 0) & \text{in } Q, \\ u = 0 & \text{on } \Sigma, \\ u(x, 0) = \varphi(x), \quad u_t(x, 0) = \psi(x) & \text{in } \Omega. \end{cases} \tag{3.14}$$

By Lemma 3.1 with $m = 0$, (3.14) has a unique solution $u \in H_0$ and $\partial_\nu u \in L^2(\Sigma)$. By the Sobolev embedding theorem,

$$u \in C([0, T]; L^{\frac{2n}{n-2}}(\Omega)).$$

When $n = 1$ and $n = 2$, $L^{\frac{2n}{n-2}}(\Omega)$ is replaced by $L^\infty(\Omega)$ and $L^p(\Omega)$ for any $p \geq 1$, respectively. Choose $\gamma_0 = 4/(n-2)$. Then, $2 + \gamma_0 = 2n/(n-2)$ and the following mapping is well-defined:

$$\mathcal{L}_2 : L^\infty(0, T; L^{\frac{2n}{n-2}}(\Omega)) \rightarrow L^\infty(0, T; L^{\frac{2n}{n-2}}(\Omega)), \quad \mathcal{L}_2(w) = u,$$

where $u \in H_0$ is the solution to (3.14). Also, by (2.9),

$$\begin{aligned} & \|u\|_{C([0, T]; H^1(\Omega))} + \|u\|_{C^1([0, T]; L^2(\Omega))} + \|\partial_\nu u\|_{L^2(\Sigma)} \\ & \leq C e^{CT} \left(\|a_z w + f(\cdot, \cdot, 0)\|_{L^1(0, T; L^2(\Omega))} + \|\varphi\|_{H^1(\Omega)} + \|\psi\|_{L^2(\Omega)} + \|h\|_{H^1(\Sigma)} \right). \end{aligned}$$

This implies that

$$\begin{aligned} \|u\|_{L^\infty(0, T; L^{\frac{2n}{n-2}}(\Omega))} & \leq C e^{CT} \left(T \|a_z\|_{L^\infty(0, T; L^p(\Omega))} \|w\|_{L^\infty(0, T; L^{\frac{2n}{n-2}}(\Omega))} + \|f(\cdot, \cdot, 0)\|_{L^1(0, T; L^2(\Omega))} \right. \\ & \quad \left. + \|\varphi\|_{H^1(\Omega)} + \|\psi\|_{L^2(\Omega)} + \|h\|_{H^1(\Sigma)} \right). \end{aligned}$$

Similar to arguments in Lemma 3.2, by the Banach fixed point theorem, (3.11) admits a unique solution $u \in H_0$.

Finally, define a map

$$\mathcal{L}_3 : L^\infty(0, T; L^2(\Omega)) \rightarrow L^\infty(0, T; L^2(\Omega)), \quad \mathcal{L}_3(z) = u,$$

for any $z \in L^\infty(0, T; L^2(\Omega))$, where $u \in H_0$ is the solution to (3.11) associated to $a_z(x, t) = g(x, t, z(x, t))$. In the following, we will prove that the map \mathcal{L}_3 has a unique fixed point in a set $V \subseteq L^\infty(0, T; L^2(\Omega))$. To this aim, for any $t \in [0, T]$, set

$$E(t) = \frac{1}{2} \int_{\Omega} [u_t^2(x, t) + |\nabla u|^2 + u^2(x, t)] dx.$$

Multiplying both sides of the first equation in (3.11) by u_t and integrating in Ω , one has

$$\begin{aligned} E_t(t) & = - \int_{\Omega} [a_z u u_t + f(x, t, 0) u_t] dx + \int_{\Omega} u u_t dx \\ & \leq \|a_z(\cdot, t)\|_{L^n(\Omega)} \|u(\cdot, t)\|_{L^{\frac{2n}{n-2}}(\Omega)} \|u_t(\cdot, t)\|_{L^2(\Omega)} \\ & \quad + \|f(\cdot, t, 0)\|_{L^2(\Omega)} \|u_t(\cdot, t)\|_{L^2(\Omega)} + \|u(\cdot, t)\|_{L^2(\Omega)} \|u_t(\cdot, t)\|_{L^2(\Omega)}. \end{aligned} \tag{3.15}$$

This implies that

$$E_t(t) \leq C \|a_z(\cdot, t)\|_{L^n(\Omega)} E(t) + \|f(\cdot, t, 0)\|_{L^2(\Omega)} E^{\frac{1}{2}}(t) + E(t).$$

Hence,

$$\begin{aligned} \|u\|_{H_0}^2 & \leq C E(t) \\ & \leq C \left[E(0) + \|f(\cdot, \cdot, 0)\|_{L^2(Q)}^2 \right] e^{C(1 + \|a_z\|_{L^1(0, T; L^n(\Omega))})} \\ & \leq C \left[\|\varphi\|_{H^1(\Omega)}^2 + \|\psi\|_{L^2(\Omega)}^2 + \|f(\cdot, \cdot, 0)\|_{L^2(Q)}^2 \right] e^{C(1 + T \|a_z\|_{L^\infty(0, T; L^n(\Omega))})}. \end{aligned}$$

By (3.12) and (3.13), it follows that for a sufficiently small ϵ with $C\epsilon \leq 1$,

$$\begin{aligned} \|u\|_{H_0}^2 &\leq C \left[\|\varphi\|_{H^1(\Omega)}^2 + \|\psi\|_{L^2(\Omega)}^2 + \|f(\cdot, \cdot, 0)\|_{L^2(Q)}^2 \right] \left(1 + \sup_{t \in (0, T)} \int_{\Omega} e^{C|a_z(x, t)|} dx \right) \\ &\leq C \left[\|\varphi\|_{H^1(\Omega)}^2 + \|\psi\|_{L^2(\Omega)}^2 + \|f(\cdot, \cdot, 0)\|_{L^2(Q)}^2 \right] \left(1 + \sup_{t \in (0, T)} \int_{\Omega} |z(x, t)|^{C\epsilon} dx \right) \\ &\leq C \left[\|\varphi\|_{H^1(\Omega)}^2 + \|\psi\|_{L^2(\Omega)}^2 + \|f(\cdot, \cdot, 0)\|_{L^2(Q)}^2 \right] \left(1 + \sup_{t \in (0, T)} \int_{\Omega} |z(x, t)| dx \right) \\ &\leq C \left[\|\varphi\|_{H^1(\Omega)}^2 + \|\psi\|_{L^2(\Omega)}^2 + \|f(\cdot, \cdot, 0)\|_{L^2(Q)}^2 \right] \left(1 + \sup_{t \in (0, T)} \|z(\cdot, t)\|_{L^2(\Omega)} \right). \end{aligned}$$

The above estimate implies that

$$\|u\|_{L^\infty(0, T; L^2(\Omega))}^2 \leq C \left[\|\varphi\|_{H^1(\Omega)}^2 + \|\psi\|_{L^2(\Omega)}^2 + \|f(\cdot, \cdot, 0)\|_{L^2(Q)}^2 \right] (1 + \|z\|_{L^\infty(0, T; L^2(\Omega))}).$$

Therefore, there exists a positive constant C_* , depending on $\|f(\cdot, \cdot, 0)\|_{L^2(Q)}$, $\|\varphi\|_{H^1(\Omega)}$ and $\|\psi\|_{L^2(\Omega)}$, such that

$$\mathcal{L}_3(V) \subseteq V, \quad \text{with } V = \left\{ u \in L^\infty(0, T; L^2(\Omega)) \mid \|u\|_{L^\infty(0, T; L^2(\Omega))} \leq C_* \right\}.$$

Also, \mathcal{L}_3 is compact. By the Schauder fixed point technique, \mathcal{L}_3 has a fixed point u in V , which is the solution to the semilinear wave equation (3.10). Since u is a fixed point of \mathcal{L}_3 , it is a solution to (3.11) associated to some $z \in L^\infty(0, T; L^2(\Omega))$. Hence, $u \in H_0$ and $\partial_\nu u \in L^2(\Sigma)$.

Moreover, suppose that $u_1, u_2 \in C([0, T]; H^1(\Omega)) \cap C^1([0, T]; L^2(\Omega))$ are two solutions to (3.10). Set $u = u_1 - u_2$. Then u satisfies the following wave equation:

$$\begin{cases} u_{tt} - \nabla \cdot (\sigma \nabla u) + a(x, t)u = 0 & \text{in } Q, \\ u = 0 & \text{on } \Sigma, \\ u(x, 0) = u_t(x, 0) = 0 & \text{in } \Omega, \end{cases} \quad (3.16)$$

where $a(x, t) = \int_0^1 f_u(x, t, su_1(x, t) + (1-s)u_2(x, t)) ds$. By the proof of Theorem 2.1 (see (4.3)), the coefficient $a \in L^\infty(0, T; L^n(\Omega))$. Hence, by the classical uniqueness result of linear wave equations, we have directly that $u = 0$, that is, $u_1 = u_2$ in Q . \square

4. UNIQUE DETERMINATION OF INITIAL DATA

In this section, we study the inverse problem on determining initial data for a class of semilinear wave equations by the passive measurement. As preliminaries, we first recall an observability result for the following wave equation:

$$\begin{cases} u_{tt} - \Delta u + a(x, t)u = K(x, t) & \text{in } Q, \\ u = 0 & \text{on } \Sigma, \\ u(x, 0) = \varphi(x), \quad u_t(x, 0) = \psi(x) & \text{in } \Omega, \end{cases} \quad (4.1)$$

where $a \in L^\infty(0, T; L^p(\Omega))$ with $p \geq n$, $K \in L^2(Q)$ and $(\varphi, \psi) \in H_0^1(\Omega) \times L^2(\Omega)$.

Similar to [DZZ08] and [Lü13], one has the following result.

Lemma 4.1. *For any $T > T^*$ (in (2.3)), any solution $u \in C([0, T]; H_0^1(\Omega)) \cap C^1([0, T]; L^2(\Omega))$ to (4.1) satisfies*

$$\begin{aligned} & \|(\varphi, \psi)\|_{H_0^1(\Omega) \times L^2(\Omega)} \\ & \leq C(T, \Omega, n, \Gamma_0) e^{C\|a\|_{L^\infty(0, T; L^p(\Omega))}^{\frac{1}{3/2 - n/p}}} \left(\|\partial_\nu u\|_{L^2(0, T; L^2(\Gamma_0))} + \|K\|_{L^2(Q)} \right), \end{aligned} \quad (4.2)$$

where Γ_0 is given in (2.2).

Next, we give a proof of Theorem 2.1.

Proof of Theorem 2.1. For any $f \in \mathcal{M}_T$ and initial values $(\varphi_1, \psi_1), (\varphi_2, \psi_2) \in H_0^1(\Omega) \times L^2(\Omega)$, denote by u_1 and u_2 , respectively, the corresponding solutions to (2.6) in H_0 . Set $\tilde{u} = u_1 - u_2$. Then $\tilde{u} \in H_0$ satisfies

$$\begin{cases} \tilde{u}_{tt} - \Delta \tilde{u} + f(x, t, u_1) - f(x, t, u_2) = 0 & \text{in } Q, \\ \tilde{u} = 0 & \text{on } \Sigma, \\ \tilde{u}(x, t) = \varphi_1(x) - \varphi_2(x), \quad \tilde{u}_t(x, 0) = \psi_1(x) - \psi_2(x) & \text{in } \Omega. \end{cases}$$

By the mean value theorem,

$$\begin{aligned} & f(x, t, u_1(x, t)) - f(x, t, u_2(x, t)) \\ & = \int_0^1 f_u(x, t, su_1(x, t) + (1-s)u_2(x, t)) ds \cdot [u_1(x, t) - u_2(x, t)]. \end{aligned}$$

It follows that

$$\tilde{u}_{tt} - \Delta \tilde{u} + a(x, t)\tilde{u}(x, t) = 0,$$

where

$$a(x, t) = \int_0^1 f_u(x, t, su_1(x, t) + (1-s)u_2(x, t)) ds \in L^\infty(0, T; L^n(\Omega)). \quad (4.3)$$

Indeed, by (2.4), for $u_1, u_2 \in H_0$ and any $\epsilon \in (0, 1)$, there exists a positive constant C_ϵ , such that

$$|a(x, t)| \leq \epsilon \ln \left(|u_1(x, t)| + |u_2(x, t)| \right) + C_\epsilon.$$

By (3.12) and (3.13) for $p = n$,

$$\begin{aligned} e^{\|a\|_{L^\infty(0, T; L^n(\Omega))}} & \leq C \left(1 + \sup_{t \in (0, T)} \int_\Omega e^{C|a(x, t)|} dx \right) \\ & \leq C + C(\epsilon) \sup_{t \in (0, T)} \int_\Omega e^{C[\epsilon \ln(|u_1(x, t)| + |u_2(x, t)| + 1)]} dx \\ & \leq C(\epsilon) \left[1 + \sup_{t \in (0, T)} \int_\Omega \left(1 + |u_1(x, t)| + |u_2(x, t)| \right)^{C\epsilon} dx \right] \\ & \leq C(\epsilon) \left[1 + \sup_{t \in (0, T)} \int_\Omega \left(1 + |u_1(x, t)| + |u_2(x, t)| \right)^2 dx \right] \\ & \leq C(\epsilon) \left(1 + \|u_1\|_{C([0, T]; L^2(\Omega))}^2 + \|u_2\|_{C([0, T]; L^2(\Omega))}^2 \right), \end{aligned}$$

where $\epsilon > 0$ is a sufficiently small constant, such that $C\epsilon \leq 2$, and $C(\epsilon)$ denotes a positive constant, which depends on ϵ and may be different from line to line.

By Lemma 4.1, for any $p = n$,

$$\|(\varphi_1 - \varphi_2, \psi_1 - \psi_2)\|_{H_0^1(\Omega) \times L^2(\Omega)} \leq C e^{C\|a\|_{L^\infty(0, T; L^n(\Omega))}^2} \|\partial_\nu \tilde{u}\|_{L^2(0, T; L^2(\Gamma_0))},$$

for some positive constant C depending only on T , Ω , n and Γ_0 . This implies that the following quantitative stability result:

$$\begin{aligned} & \|(\varphi_1 - \varphi_2, \psi_1 - \psi_2)\|_{H_0^1(\Omega) \times L^2(\Omega)} \\ & \leq C(f, u_1, u_2, n, T, \Omega, \Gamma_0, \Sigma) \cdot \|\Lambda_{\varphi_1, \psi_1, f}^0 - \Lambda_{\varphi_2, \psi_2, f}^0\|_{L^2(0, T; L^2(\Gamma_0))}, \end{aligned}$$

where $C(f, u_1, u_2, \Omega, n, T, \Gamma_0)$ is a positive constant depending on $n, T, \Omega, \Gamma_0, u_1, u_2$ and f , but independent of (φ_j, ψ_j) , for $j = 1, 2$. \square

On the other hand, there is a counterexample showing that if f is unknown, the passive measurement cannot uniquely determine all unknowns.

Theorem 4.1 (Non-uniqueness). *Suppose that $f_j \in \mathcal{M}_T$ and $(\varphi_j, \psi_j) \in H_0^1(\Omega) \times L^2(\Omega)$ for $j = 1, 2$. Denote by u_j the solution to the following semilinear wave equation:*

$$\begin{cases} u_{j,tt} - \Delta u_j + f_j(x, t, u_j) = 0 & \text{in } Q, \\ u_j = 0 & \text{on } \Sigma, \\ u_j(x, 0) = \varphi_j(x), \quad u_{j,t}(x, 0) = \psi_j(x) & \text{in } \Omega. \end{cases} \quad (4.4)$$

Then there exist two groups of unknown sources $(\varphi_1, \psi_1, f_1), (\varphi_2, \psi_2, f_2) \in H_0^1(\Omega) \times L^2(\Omega) \times \mathcal{M}_T$, such that

$$(\varphi_1, \psi_1, f_1) \neq (\varphi_2, \psi_2, f_2),$$

but

$$\Lambda_{\varphi_1, \psi_1, f_1}^0 = \Lambda_{\varphi_2, \psi_2, f_2}^0.$$

Proof. Assume that two functions $u_1, u_2 \in C^\infty(\overline{Q})$ satisfy that

$$u_1(\cdot, 0) \neq u_2(\cdot, 0) \text{ in a measurable subset of } \Omega \text{ with positive measure,}$$

$$\text{and } u_1(x, t) = u_2(x, t) = 0 \text{ in } \Omega_\epsilon \times [0, T],$$

where $\Omega_\epsilon = \{x \in \Omega \mid \text{dist}(x, \Gamma) < \epsilon\}$. Set

$$F_j(x, t) = -u_{j,tt}(x, t) + \Delta u_j(x, t), \quad \text{for } j = 1, 2 \text{ and } (x, t) \in Q.$$

Then u_j ($j = 1, 2$) are solutions to (2.12) associated to

$$\varphi_j(x) = u_j(x, 0), \quad \psi_j(x) = u_{j,t}(x, 0) \text{ and } f_j(x, t, u_j) = F_j(x, t).$$

Notice that

$$(\varphi_1, \psi_1, f_1) \neq (\varphi_2, \psi_2, f_2),$$

but

$$\partial_\nu u_1 \Big|_{\Gamma_0 \times (0, T)} = \Lambda_{\varphi_1, \psi_1, f_1}^0 = \Lambda_{\varphi_2, \psi_2, f_2}^0 = \partial_\nu u_2 \Big|_{\Gamma_0 \times (0, T)} = 0.$$

\square

Finally, we give a proof of Corollary 2.3.

Proof of Corollary 2.3. For any $f_1, f_2 \in \mathcal{C}_T$, there exists an $f_0 \in \mathcal{M}_T$, such that

$$f_1(x, t, s) = f_2(x, t, s) = f_0(x, t, s) \quad \text{in } \Omega \times [0, T^* + \epsilon] \times \mathbb{R}.$$

Also, by the condition that $\Lambda_{\varphi_1, \psi_1, f_1}^0 = \Lambda_{\varphi_2, \psi_2, f_2}^0$, we have that

$$\Lambda_{\varphi_1, \psi_1, f_0}^0 = \Lambda_{\varphi_2, \psi_2, f_0}^0 \quad \text{on } \Gamma_0 \times [0, T^* + \epsilon].$$

Then, by the results in Theorem 2.1 for $f = f_0$ and $T = T^* + \epsilon$, the conclusion in Corollary 2.3 is true. \square

5. SIMULTANEOUS RECOVERY OF INITIAL DATA AND COEFFICIENTS

In this section, the higher order linearization technique will be used to determine unknown initial value and nonlinear function in the semilinear wave equation (2.13) simultaneously. As preliminaries, based on the observability result in Lemma 4.1, an approximation property for wave equations is given.

First, for any $T > T^*$ (see (2.3)), choose two constants t_1 and t_2 , such that

$$T^* < t_1 < t_2 < T.$$

Then one has the following approximation result.

Theorem 5.1. *Assume that $q \in E^{m+1}$ with $\text{supp } q \subseteq \bar{\Omega} \times [t_1, t_2]$ for an integer $m > (n+1)/2$. Then for any solution $v \in C([t_1, t_2]; L^2(\Omega)) \cap C^1([t_1, t_2]; H^{-1}(\Omega))$ to*

$$v_{tt} - \Delta v + qv = 0 \quad \text{in } Q,$$

and any $\epsilon > 0$, there exists a solution $V \in C^2(\bar{Q})$ to

$$\begin{cases} V_{tt} - \Delta V + qV = 0 & \text{in } Q, \\ V(x, 0) = V_t(x, 0) = 0 & \text{in } \Omega, \end{cases} \quad (5.1)$$

such that

$$\|V - v\|_{L^2(\Omega \times (t_1, t_2))} < \epsilon.$$

Proof. In order to prove the desired approximation result, it is equivalent to show that

$$X = \left\{ w = V \Big|_{\Omega \times (t_1, t_2)} \mid V \in C^2(\bar{Q}) \text{ is a solution to (5.1)} \right\}$$

is dense in

$$Y = \left\{ v \in C([t_1, t_2]; L^2(\Omega)) \cap C^1([t_1, t_2]; H^{-1}(\Omega)) \mid v_{tt} - \Delta v + qv = 0 \text{ in } \Omega \times (t_1, t_2) \right\}$$

in terms of $L^2(\Omega \times (t_1, t_2))$. By the Hahn-Banach theorem, it suffices to prove the following statement: if $f \in L^2(\Omega \times (t_1, t_2))$ satisfies

$$\int_{t_1}^{t_2} \int_{\Omega} f w \, dx dt = 0, \quad \forall w \in X, \quad (5.2)$$

it follows that

$$\int_{t_1}^{t_2} \int_{\Omega} f v \, dx dt = 0, \quad \forall v \in Y. \quad (5.3)$$

To this aim, let $f \in L^2(\Omega \times (t_1, t_2))$ satisfy (5.2) and set

$$\tilde{f}(x, t) = \begin{cases} f(x, t) & \text{in } \Omega \times (t_1, t_2), \\ 0 & \text{in } \Omega \times ((0, t_1] \cup [t_2, T)). \end{cases}$$

Assume that $\tilde{v} \in H_0$ is the solution to the following backward wave equation:

$$\begin{cases} \tilde{v}_{tt} - \Delta \tilde{v} + q\tilde{v} = \tilde{f} & \text{in } Q, \\ \tilde{v} = 0 & \text{on } \Sigma, \\ \tilde{v}(x, T) = \tilde{v}_t(x, T) = 0 & \text{in } \Omega. \end{cases} \quad (5.4)$$

Then for any solution $V \in C^2(\overline{Q})$ to (5.1) and $w = V|_{\Omega \times (t_1, t_2)}$, one has that

$$\begin{aligned} 0 &= \int_{t_1}^{t_2} \int_{\Omega} f w \, dx dt = \int_Q \tilde{f} V \, dx dt \\ &= \int_Q \left(\tilde{v}_{tt} - \Delta \tilde{v} + q \tilde{v} \right) V \, dx dt = \int_{\Sigma} \partial_{\nu} \tilde{v} \cdot V \, dS dt. \end{aligned} \quad (5.5)$$

Since $V|_{\Sigma}$ can be arbitrary function in $C_0^{\infty}(0, T; C^{\infty}(\Gamma))$, we conclude that the associated solution V to (5.1) satisfies $V \in E^{m+2}$ and therefore, $\partial_{\nu} \tilde{v} = 0$ on Σ . This implies that the solution $\tilde{v} \in H_0$ to (5.4) satisfies

$$\begin{cases} \tilde{v}_{tt} - \Delta \tilde{v} + q \tilde{v} = \tilde{f} & \text{in } Q, \\ \tilde{v} = \partial_{\nu} \tilde{v} = 0 & \text{on } \Sigma, \\ \tilde{v}(x, T) = \tilde{v}_t(x, T) = 0 & \text{in } \Omega. \end{cases}$$

In particular, in the domain $\Omega \times ((0, t_1) \cup (t_2, T))$, $\tilde{v} \in H_0$ satisfies the following wave equation:

$$\begin{cases} \tilde{v}_{tt} - \Delta \tilde{v} + q \tilde{v} = 0 & \text{in } \Omega \times ((0, t_1) \cup (t_2, T)), \\ \tilde{v} = \partial_{\nu} \tilde{v} = 0 & \text{on } \Sigma, \\ \tilde{v}(x, T) = \tilde{v}_t(x, T) = 0 & \text{in } \Omega. \end{cases}$$

By the observability result in Lemma 4.1, we have

$$\tilde{v} \equiv 0 \quad \text{in } \Omega \times (0, t_1).$$

By the uniqueness of solutions to wave equations, we have that

$$\tilde{v} \equiv 0 \quad \text{in } \Omega \times (t_2, T).$$

Hence,

$$\begin{cases} \tilde{v}(\cdot, t_1) = \tilde{v}_t(\cdot, t_1) = \tilde{v}(\cdot, t_2) = \tilde{v}_t(\cdot, t_2) = 0 & \text{in } \Omega, \\ \tilde{v} = \partial_{\nu} \tilde{v} = 0 & \text{on } \Sigma. \end{cases}$$

It follows that

$$\int_{t_1}^{t_2} \int_{\Omega} f v \, dx dt = \int_{t_1}^{t_2} \int_{\Omega} \left(\tilde{v}_{tt} - \Delta \tilde{v} + q \tilde{v} \right) v \, dx dt = 0,$$

for any $v \in C([t_1, t_2]; L^2(\Omega)) \cap C^1([t_1, t_2]; H^{-1}(\Omega))$ with $v_{tt} - \Delta v + qv = 0$ in $\Omega \times (t_2, t_2)$ as desired. This completes the proof of Theorem 5.1. \square

Remark 5.1. *By the proof of Theorem 5.1, the approximation property still holds for the following more general hyperbolic equation:*

$$v_{tt} - \nabla \cdot (\sigma \nabla v) + qv = 0 \quad \text{in } \Omega \times (t_1, t_2),$$

where $\sigma = (\sigma^{ij}(x))_{i,j=1}^n \in C^2(\overline{\Omega}; \mathbb{R}^{n \times n})$ is a symmetric uniformly positive definite matrix-valued function in $\overline{\Omega}$ and (2.11) holds. Also, t_1 and t_2 satisfy $T_* < t_1 < t_2 < T$ for a suitable positive constant T_* .

Finally, we give a proof of Theorems 2.2 and 2.9.

Proof of Theorem 2.2. The whole proof is divided into five parts.

Step 1. Initiation

For $m > (n + 1)/2$, consider the following (lateral) boundary data

$$h(x, t; \epsilon) = \sum_{\ell=1}^M \epsilon_{\ell} g_{\ell} \quad \text{on } \Sigma, \quad (5.6)$$

where $M \in \mathbb{N}$, $g_1, \dots, g_M \in \mathcal{N}_{m+1}$ and $\epsilon = (\epsilon_1, \dots, \epsilon_M)$ is a parameter vector in \mathbb{R}^M with $|\epsilon| = \sum_{\ell=1}^M |\epsilon_{\ell}|$ sufficiently small, such that

$$\left\| \sum_{\ell=1}^M \epsilon_{\ell} g_{\ell} \right\|_{H^{m+1}(\Sigma)} < \frac{\delta}{2}, \quad \text{for the } \delta > 0 \text{ in Theorem 3.1.}$$

For $j = 1, 2$, let $u_j = u_j(x, t; \epsilon) \in E^{m+1}$ be solutions to

$$\begin{cases} u_{j,tt} - \Delta u_j + \tilde{f}_j(x, t, u_j) = 0 & \text{in } Q, \\ u_j = \sum_{\ell=1}^M \epsilon_{\ell} g_{\ell} & \text{on } \Sigma, \\ u_j(x, 0) = \varphi_j(x), \quad u_{j,t}(x, 0) = \psi_j(x) & \text{in } \Omega, \end{cases} \quad (5.7)$$

where $(\varphi_j, \psi_j) \in H_0^{m+1}(\Omega) \times H_0^m(\Omega)$ with $\|(\varphi_j, \psi_j)\|_{H_0^{m+1}(\Omega) \times H_0^m(\Omega)} < \delta/2$ and \tilde{f}_j are admissible coefficients. In particular, when $\epsilon = 0$, $\tilde{u}_j = u_j(\cdot, \cdot; 0)$ are the solutions to

$$\begin{cases} \tilde{u}_{j,tt} - \Delta \tilde{u}_j + \tilde{f}_j(x, t, \tilde{u}_j) = 0 & \text{in } Q, \\ \tilde{u}_j = 0 & \text{on } \Sigma, \\ \tilde{u}_j(x, 0) = \varphi_j, \quad \tilde{u}_{j,t}(x, 0) = \psi_j & \text{in } \Omega. \end{cases} \quad (5.8)$$

We will apply the higher order linearization to the initial-boundary value problem (5.7) around the solution \tilde{u}_j to (5.8) in order to determine informations on \tilde{f}_j for $j = 1, 2$.

Step 2. The first order linearization ($M = 1$)

First, we linearize the equation (5.7) around \tilde{u}_j , where $\tilde{u}_j \in E^{m+1}$ is the solution to (5.8), for $j = 1, 2$. It is easy to show that for $j = 1, 2$ and $\ell = M = 1^2$,

$$v_j^{(\ell)}(x, t) = \lim_{\epsilon_{\ell} \rightarrow 0} \frac{u_j(x, t) - \tilde{u}_j(x, t)}{\epsilon_{\ell}}$$

satisfies the following wave equation:

$$\begin{cases} v_{j,tt}^{(\ell)} - \Delta v_j^{(\ell)} + \tilde{q}_j v_j^{(\ell)} = 0 & \text{in } Q, \\ v_j^{(\ell)} = g_{\ell} & \text{on } \Sigma, \\ v_j^{(\ell)}(x, 0) = v_{j,t}^{(\ell)}(x, 0) = 0 & \text{in } \Omega, \end{cases} \quad (5.9)$$

where

$$\tilde{q}_j(x, t) = \tilde{f}_{j,u}(x, t, \tilde{u}_j(x, t)) \quad \text{in } Q \quad \text{and} \quad \tilde{q}_j \in E^{m+1}.$$

It is worth noting that both \tilde{u}_j and $v_j^{(\ell)}$ in (5.8) and (5.9) are still unknown, respectively, since they involve unknown coefficients and initial data. In this step, we will show that

$$\tilde{f}_{1,u}(x, t, \tilde{u}_1(x, t)) = \tilde{f}_{2,u}(x, t, \tilde{u}_2(x, t)) \quad \text{in } \Omega \times (0, T). \quad (5.10)$$

Recall that we have the same input-output maps

$$\Lambda_{\varphi_1, \psi_1, \tilde{f}_1}^T(h) = \Lambda_{\varphi_2, \psi_2, \tilde{f}_2}^T(h), \quad \text{for any } h \in E_{\delta}.$$

²In fact, the arguments hold for all $\ell = 1, \dots, M$, where we will use in steps 2-5.

Hence, with the same boundary data at hand, one has

$$\begin{aligned} v_1^{(\ell)}(x, 0) &= v_2^{(\ell)}(x, 0), & v_{1,t}^{(\ell)}(x, 0) &= v_{2,t}^{(\ell)}(x, 0), \\ v_1^{(\ell)}(x, T) &= v_2^{(\ell)}(x, T), & v_{1,t}^{(\ell)}(x, T) &= v_{2,t}^{(\ell)}(x, T), \\ v_1^{(\ell)}|_{\Sigma} &= v_2^{(\ell)}|_{\Sigma}, & \partial_\nu v_1^{(\ell)}|_{\Sigma} &= \partial_\nu v_2^{(\ell)}|_{\Sigma}, \end{aligned} \quad (5.11)$$

for $\ell = M = 1$.

Now, subtracting (5.9) with $j = 1, 2$, we have

$$\begin{cases} v_{tt}^{(\ell)} - \Delta v^{(\ell)} + \tilde{q}_1 v^{(\ell)} = (\tilde{q}_2 - \tilde{q}_1) v_2^{(\ell)} & \text{in } Q, \\ v^{(\ell)} = 0 & \text{on } \Sigma, \\ v^{(\ell)}(x, 0) = v_t^{(\ell)}(x, 0) = 0 & \text{in } \Omega, \end{cases} \quad (5.12)$$

where $v^{(\ell)} := v_1^{(\ell)} - v_2^{(\ell)}$. Let $\tilde{v}_1^{(\ell)} \in C^2(\overline{Q})$ be a solution to

$$\tilde{v}_{1,tt}^{(\ell)} - \Delta \tilde{v}_1^{(\ell)} + \tilde{q}_1 \tilde{v}_1^{(\ell)} = 0 \text{ in } Q. \quad (5.13)$$

Multiplying both sides of the first equation in (5.12) by $\tilde{v}_1^{(\ell)}$, by (5.11) and an integration by parts, yields that

$$\int_Q (\tilde{q}_2 - \tilde{q}_1) \tilde{v}_1^{(\ell)} v_2^{(\ell)} dx dt = 0. \quad (5.14)$$

With the admissible conditions in Definition 2.6, (5.14) is equivalent to

$$\int_{t_1}^{t_2} \int_\Omega (\tilde{q}_2 - \tilde{q}_1) \tilde{v}_1^{(\ell)} v_2^{(\ell)} dx dt = 0. \quad (5.15)$$

In fact, by the above arguments, the equality (5.15) still holds for complex-valued solutions $\tilde{v}_1^{(\ell)}$ and $v_2^{(\ell)}$, respectively, to (5.9) with $j = 2$ and (5.13).

On the other hand, let \mathbf{v}_j ($j = 1, 2$) be the complex geometrical optics (CGO) solutions to

$$\mathbf{v}_{j,tt} - \Delta \mathbf{v}_j + \tilde{q}_j \mathbf{v}_j = 0 \text{ in } \Omega \times (t_1, t_2)$$

in the form in Appendix A:

$$\begin{aligned} \mathbf{v}_1(x, t) &= e^{-i\tau[\eta(x)+t]} a_1(x, t) + R_1^{(\tau)}(x, t), \\ \mathbf{v}_2(x, t) &= e^{i\tau[\eta(x)+t]} a_2(x, t) + R_2^{(\tau)}(x, t), \end{aligned} \quad (5.16)$$

where $i = \sqrt{-1}$ denotes the imaginary unit, $\tau \in \mathbb{R}$ with $|\tau| > 1$, $\eta(x) = |x - x_0|$ for an $x_0 \in \overline{\Omega}$ and $a_j(\cdot, \cdot)$ has the form (A.6). Also, $R_j^{(\tau)} \in L^2(Q)$ is the remainder term, which fulfills the conditions (A.1) and (A.2), for $j = 1, 2$.

By the approximation result (Theorem 5.1), there are two sequences of complex-valued functions $\{v_k^1\}_{k \in \mathbb{N}}$ and $\{v_k^2\}_{k \in \mathbb{N}}$, such that for $j = 1, 2$, $v_k^j \in C^2(\overline{Q}; \mathbb{C})$ is the solution to

$$\begin{cases} v_{k,tt}^j - \Delta v_k^j + \tilde{q}_j v_k^j = 0 & \text{in } Q, \\ v_k^j(x, 0) = v_{k,t}^j(x, 0) = 0 & \text{in } \Omega, \end{cases} \quad (5.17)$$

and

$$v_k^j \rightarrow \mathbf{v}_j \quad \text{in } L^2(\Omega \times (t_1, t_2); \mathbb{C}), \text{ as } k \rightarrow \infty. \quad (5.18)$$

Choosing $\tilde{v}_1^{(\ell)} = v_k^1$ and $v_2^{(\ell)} = v_k^2$ in (5.15), and taking limit, as k tends to ∞ , one obtains

$$\int_{t_1}^{t_2} \int_{\Omega} (\tilde{q}_2 - \tilde{q}_1) \mathbf{v}_1 \mathbf{v}_2 \, dx dt = 0. \quad (5.19)$$

It remains to analyze the product $\mathbf{v}_1 \mathbf{v}_2$ of CGO solutions.

By a direct computation, we have that

$$\int_{t_1}^{t_2} \int_{\Omega} (\tilde{q}_2 - \tilde{q}_1) \mathbf{v}_1 \mathbf{v}_2 \, dx dt = \mathbb{I} + \mathbb{I}_{\tau},$$

where

$$\mathbb{I} = \int_{t_1}^{t_2} \int_{\Omega} (\tilde{q}_2 - \tilde{q}_1) a_1 a_2 \, dx dt$$

and

$$\begin{aligned} \mathbb{I}_{\tau} = \int_{t_1}^{t_2} \int_{\Omega} (\tilde{q}_2 - \tilde{q}_1) \left\{ e^{-i\tau[\eta(x)+t]} a_1(x, t) R_2^{(\tau)}(x, t) \right. \\ \left. + e^{i\tau[\eta(x)+t]} a_2(x, t) R_1^{(\tau)}(x, t) + R_1^{(\tau)}(x, t) R_2^{(\tau)}(x, t) \right\} dx dt. \end{aligned}$$

Since $\tilde{q}_1, \tilde{q}_2, a_1$ and a_2 are bounded in Q , and $\|R_j^{(\tau)}\|_{L^2(\Omega \times (t_1, t_2))} \rightarrow 0$, as $|\tau| \rightarrow \infty$, for $j = 1, 2$, it follows that $\mathbb{I}_{\tau} \rightarrow 0$, as $\tau \rightarrow \infty$. Hence, the integral identity (5.19) implies

$$\int_{t_1}^{t_2} \int_{\Omega} (\tilde{q}_2 - \tilde{q}_1) a_1(x, t) a_2(x, t) \, dx dt = 0. \quad (5.20)$$

It remains to prove that (5.20) implies $\tilde{q}_1 = \tilde{q}_2$ in Q . By applying the similar arguments in [KO19, Section 2], we conclude that $\tilde{q}_1 = \tilde{q}_2$ in Q as desired. Meanwhile, set

$$q(x, t) = \tilde{q}_1(x, t) = \tilde{q}_2(x, t) \text{ in } Q. \quad (5.21)$$

By the uniqueness of solutions to (5.9), one has that

$$v^{(\ell)} := v_1^{(\ell)} = v_2^{(\ell)} \text{ in } Q, \text{ for } \ell = 1. \quad (5.22)$$

Step 3. The second order linearization ($M = 2$)

For $m = 2$, we differentiate (5.7) with respect to different parameters ϵ_1 and ϵ_2 . It is easy to show that the derivatives $w_j^{(2)}$ ($j = 1, 2$) satisfy

$$\begin{cases} w_{j,tt}^{(2)} - \Delta w_j^{(2)} + q(x, t) w_j^{(2)} + \tilde{f}_{j,uu}(x, t, \tilde{u}_j) v^{(1)} v^{(2)} = 0 & \text{in } Q, \\ w_j^{(2)} = 0 & \text{on } \Sigma, \\ w_j^{(2)}(x, 0) = w_{j,t}^{(2)}(x, 0) = 0 & \text{in } \Omega, \end{cases} \quad (5.23)$$

where $q, \tilde{f}_{j,uu}(\cdot, \cdot, \tilde{u}_j) \in E^{m+1}$ and $v^{(1)}, v^{(2)} \in E^{m+1}$ satisfy

$$\begin{cases} v_{tt}^{(\ell)} - \Delta v^{(\ell)} + q(x, t) v^{(\ell)} = 0 & \text{in } Q, \\ v^{(\ell)} = g_{\ell} & \text{on } \Sigma, \\ v^{(\ell)}(x, 0) = v_t^{(\ell)}(x, 0) = 0 & \text{in } \Omega, \end{cases}$$

here $g_1, g_2 \in \mathcal{N}_{m+1}$ are arbitrarily given.

Next, we will prove that

$$\tilde{f}_{1,uu}(x, t, \tilde{u}_1(x, t)) = \tilde{f}_{2,uu}(x, t, \tilde{u}_2(x, t)) \text{ in } Q. \quad (5.24)$$

Similar to the arguments in the first order linearization, with the equal input-output maps at hand, we have that

$$\begin{aligned} w_1^{(2)}(x, 0) &= w_2^{(2)}(x, 0), & w_{1,t}^{(2)}(x, 0) &= w_{2,t}^{(2)}(x, 0), \\ w_1^{(2)}(x, T) &= w_2^{(2)}(x, T), & w_{1,t}^{(2)}(x, T) &= w_{2,t}^{(2)}(x, T), \\ w_1^{(2)}\Big|_{\Sigma} &= w_2^{(2)}\Big|_{\Sigma}, & \partial_\nu w_1^{(2)}\Big|_{\Sigma} &= \partial_\nu w_2^{(2)}\Big|_{\Sigma}. \end{aligned} \quad (5.25)$$

Let $v^{(0)}$ be any solution to

$$\begin{cases} v_{tt}^{(0)} - \Delta v^{(0)} + qv^{(0)} = 0 & \text{in } Q, \\ v^{(0)}(x, 0) = v_t^{(0)}(x, 0) = 0 & \text{in } \Omega. \end{cases} \quad (5.26)$$

By subtracting the equations (5.23) associated to $j = 1, 2$, an integration by parts yields

$$\begin{aligned} & \int_Q \left[\tilde{f}_{1,uu}(x, t, \tilde{u}_1(x, t)) - \tilde{f}_{2,uu}(x, t, \tilde{u}_2(x, t)) \right] v^{(0)} v^{(1)} v^{(2)} dx dt \\ &= \int_{t_1}^{t_2} \int_{\Omega} \left[\tilde{f}_{1,uu}(x, t, \tilde{u}_1(x, t)) - \tilde{f}_{2,uu}(x, t, \tilde{u}_2(x, t)) \right] v^{(0)} v^{(1)} v^{(2)} dx dt = 0, \end{aligned} \quad (5.27)$$

where we have used the admissible conditions for the coefficients.

As in the first step, we consider the CGO solutions \mathbf{v}_1 and \mathbf{v}_2 of the form (5.16). By using the approximation property again, we have that

$$\left[\tilde{f}_{1,uu}(x, t, \tilde{u}_1(x, t)) - \tilde{f}_{2,uu}(x, t, \tilde{u}_2(x, t)) \right] v^{(0)}(x, t) = 0 \quad \text{in } \Omega \times (t_1, t_2).$$

Hence, for any solution $v_{(0)}$ to (5.13),

$$\int_{t_1}^{t_2} \int_{\Omega} \left[\tilde{f}_{1,uu}(x, t, \tilde{u}_1(x, t)) - \tilde{f}_{2,uu}(x, t, \tilde{u}_2(x, t)) \right] v^{(0)} v_{(0)} dx dt = 0.$$

By choosing $v^{(0)}$ and $v_{(0)}$ as the suitable CGO solutions again, we have (5.24) as desired. Furthermore, by the uniqueness of solutions to (5.23), one can immediately obtain

$$w_1^{(2)} = w_2^{(2)} \quad \text{in } Q.$$

Step 4. The higher order linearization ($M > 2$)

By the higher order linearization and the method of induction, we may find M -th order derivative of (5.7) and prove that

$$\partial_u^M \tilde{f}_1(x, t, \tilde{u}_1(x, t)) = \partial_u^M \tilde{f}_2(x, t, \tilde{u}_2(x, t)) \quad \text{in } Q, \quad (5.28)$$

for any $M = 3, 4, \dots$. To this aim, we first assume that

$$\partial_u^k \tilde{f}_1(x, t, \tilde{u}_1(x, t)) = \partial_u^k \tilde{f}_2(x, t, \tilde{u}_2(x, t)) \quad \text{in } Q, \quad \text{for any } k = 1, \dots, M-1.$$

Similar to previous steps, by differentiating (5.7) with respect to $\epsilon_1, \dots, \epsilon_{M-1}$ and ϵ_M , one can obtain

$$\int_{t_1}^{t_2} \int_{\Omega} \left[\partial_u^M \tilde{f}_1(x, t, \tilde{u}_1(x, t)) - \partial_u^M \tilde{f}_2(x, t, \tilde{u}_2(x, t)) \right] v^{(0)} v^{(1)} \dots v^{(M)} dx dt = 0,$$

where $v^{(\ell)}$ ($\ell = 0, 1, \dots, M$) are solutions to (5.26).

Applying the similar approximation properties in Step 3, we have that

$$\int_{t_1}^{t_2} \int_{\Omega} \left[\partial_u^M \tilde{f}_1(x, t, u_1^{(0)}(x, t)) - \partial_u^M \tilde{f}_2(x, t, u_2^{(0)}(x, t)) \right] v^{(0)} \mathbf{v}_1 \mathbf{v}_2 v^{(3)} \dots v^{(M)} dx dt = 0, \quad (5.29)$$

where v_1 and v_2 are CGO solutions of the form (5.16). This implies that

$$\left[\partial_u^M \tilde{f}_1(x, t, u_1^{(0)}(x, t)) - \partial_u^M \tilde{f}_2(x, t, u_2^{(0)}(x, t)) \right] v^{(0)} v^{(3)} \dots v^{(M)} = 0 \quad \text{in } \Omega \times (t_1, t_2).$$

Similar to Step 3, if M is odd, we take successively CGO solutions pairs $v^{(0)}$ and $v^{(3)}, \dots, v^{(M-1)}$ and $v^{(M)}$. Otherwise, we add a CGO solution to this equality, in order to guarantee even solutions to be multiplied together. It follows that

$$\partial_u^M \tilde{f}_1(x, t, \tilde{u}_1(x, t)) = \partial_u^M \tilde{f}_2(x, t, \tilde{u}_2(x, t)) \quad \text{in } Q. \quad (5.30)$$

Step 5. The determination of initial data and coefficients

Recall that \tilde{u}_j ($j = 1, 2$) are the solutions to the following semilinear wave equation:

$$\begin{cases} \tilde{u}_{j,tt} - \Delta \tilde{u}_j + \tilde{f}_j(x, t, \tilde{u}_j) = 0 & \text{in } Q, \\ \tilde{u}_j = 0 & \text{on } \Sigma, \\ \tilde{u}_j(x, 0) = \varphi_j, \quad \tilde{u}_{j,t}(x, 0) = \psi_j & \text{in } \Omega. \end{cases}$$

By the admissible property of \tilde{f}_1 and \tilde{f}_2 ,

$$\begin{aligned} & \tilde{f}_1(x, t, \tilde{u}_1(x, t)) - \tilde{f}_2(x, t, \tilde{u}_2(x, t)) \\ &= \sum_{k=1}^{\infty} \frac{\partial_u^k \tilde{f}_2(x, t, \tilde{u}_2(x, t))}{k!} \left[-\tilde{u}_2(x, t) \right]^k - \sum_{k=1}^{\infty} \frac{\partial_u^k \tilde{f}_1(x, t, \tilde{u}_1(x, t))}{k!} \left[-\tilde{u}_1(x, t) \right]^k \\ &= \sum_{k=1}^{\infty} \frac{\partial_u^k \tilde{f}_1(x, t, \tilde{u}_1(x, t)) (-1)^k}{k!} \left\{ \left[\tilde{u}_2(x, t) \right]^k - \left[\tilde{u}_1(x, t) \right]^k \right\}. \end{aligned} \quad (5.31)$$

Since both \tilde{u}_1 and \tilde{u}_2 are bounded, set $R = \|\tilde{u}_1\|_{L^\infty(Q)} + \|\tilde{u}_2\|_{L^\infty(Q)}$. Then, for any $L > 0$ and $(x, t) \in Q$,

$$\begin{aligned} & \left| \frac{\tilde{f}_1(x, t, \tilde{u}_1(x, t)) - \tilde{f}_2(x, t, \tilde{u}_2(x, t))}{\tilde{u}_1(x, t) - \tilde{u}_2(x, t)} \right| \\ &= \left| \sum_{k=1}^{\infty} \frac{\partial_u^k \tilde{f}_1(x, t, \tilde{u}_1(x, t))}{k!} (-1)^{k+1} \left\{ \left[\tilde{u}_1(x, t) \right]^{k-1} + \left[\tilde{u}_1(x, t) \right]^{k-2} \tilde{u}_2(x, t) + \dots \right. \right. \\ & \quad \left. \left. + \tilde{u}_1(x, t) \left[\tilde{u}_2(x, t) \right]^{k-2} + \left[\tilde{u}_2(x, t) \right]^{k-1} \right\} \right| \\ &\leq \sum_{k=1}^{\infty} \left| \partial_u^k \tilde{f}_1(x, t, \tilde{u}_1(x, t)) \right| \frac{R^{k-1}}{(k-1)!} \\ &\leq \sum_{k=1}^{\infty} \frac{k R^{k-1}}{L^k} \sup_{|s - \tilde{u}_1(x, t)| = L} |\tilde{f}_1(x, t, s)|. \end{aligned}$$

Choose $L = 2(R + 1)$. By the admissibility of \tilde{f}_1 and \tilde{f}_2 ,

$$G(\cdot, \cdot) = \frac{\tilde{f}_1(\cdot, \cdot, \tilde{u}_1(\cdot, \cdot)) - \tilde{f}_2(\cdot, \cdot, \tilde{u}_2(\cdot, \cdot))}{\tilde{u}_1(\cdot, \cdot) - \tilde{u}_2(\cdot, \cdot)} \in L^\infty(Q).$$

Set $w = \tilde{u}_1 - \tilde{u}_2$. It is easy to see that

$$\begin{cases} w_{tt} - \Delta w + Gw = 0 & \text{in } Q, \\ w = 0 & \text{on } \Sigma, \\ w(x, 0) = \varphi_1 - \varphi_2, \quad w_t(x, 0) = \psi_1 - \psi_2 & \text{in } \Omega. \end{cases}$$

By $\Lambda_{\varphi_1, \psi_1, \tilde{f}_1}^T(0) = \Lambda_{\varphi_2, \psi_2, \tilde{f}_2}^T(0)$ and the observability result in Lemma 4.1,

$$\varphi_1 = \varphi_2, \quad \psi_1 = \psi_2 \quad \text{and} \quad \tilde{u}_1 = \tilde{u}_2.$$

By (5.31),

$$\tilde{f}_1(x, t, \tilde{u}_1(x, t)) = \tilde{f}_2(x, t, \tilde{u}_2(x, t)) \quad \text{in } Q.$$

Furthermore, notice that for $j = 1, 2$ and any $(x, t, s) \in Q \times \mathbb{R}$,

$$\tilde{f}_j(x, t, s) = \tilde{f}_j(x, t, \tilde{u}_j(x, t)) + \sum_{k=1}^{\infty} \frac{\partial_u^k \tilde{f}_j(x, t, \tilde{u}_j(x, t))}{k!} (s - \tilde{u}_j(x, t))^k.$$

With (5.30) at hand, this implies that $\tilde{f}_1(x, t, s) = \tilde{f}_2(x, t, s)$ in $Q \times \mathbb{R}$. \square

It remains to prove Corollary 2.9.

Proof of Corollary 2.9. The proof is similar to the one of Theorem 2.2. For $j = 1, 2$, we denote by $u_j \in E^{m+1}$ the solution to

$$\begin{cases} u_{j,tt} - \Delta u_j + q_j u_j = 0 & \text{in } Q, \\ u_j = h & \text{on } \Sigma, \\ u_j(x, 0) = \varphi_j(x), \quad u_{j,t}(x, 0) = \psi_j(x) & \text{in } \Omega, \end{cases}$$

and denote by \tilde{u}_j the solution to

$$\begin{cases} \tilde{u}_{j,tt} - \Delta \tilde{u}_j + q_j \tilde{u}_j = 0 & \text{in } Q, \\ \tilde{u}_j = 0 & \text{on } \Sigma, \\ \tilde{u}_j(x, 0) = \varphi_j(x), \quad \tilde{u}_{j,t}(x, 0) = \psi_j(x) & \text{in } \Omega. \end{cases}$$

Let $v_j = u_j - \tilde{u}_j$ and $v_j \in E^{m+1}$ is the solution to

$$\begin{cases} v_{j,tt} - \Delta v_j + q_j v_j = 0 & \text{in } Q, \\ v_j = h & \text{on } \Sigma, \\ v_j(x, 0) = 0, \quad v_{j,t}(x, 0) = 0 & \text{in } \Omega, \end{cases}$$

for $j = 1, 2$. By applying the same technique as in the proof of Theorem 2.2, we first are able to determine $q_1 = q_2$ in Q . Then, by the observability result in Lemma 4.1, we can derive that $\varphi_1 = \varphi_2$ and $\psi_1 = \psi_2$ in Ω as desired. This proves the assertion. \square

Remark 5.2. *It is worth mentioning that the boundedness of CGO solutions plays an essential role in the proof of Theorem 2.2. With the higher order linearization technique at hand, one can expect the integral identity (5.29) holds, with products of M solutions of the (linear) wave equation.*

- (1) *In the elliptic case, one may choose $v^{(1)}$ and $v^{(2)}$ as suitable CGO solutions, such that $\{v^{(1)}v^{(2)}\}$ forms a dense subset in $L^1(Q)$. Meanwhile, by applying the maximum principle for the second order linear elliptic equations, it is not hard to construct bounded positive solutions $v^{(3)}, \dots, v^{(M)}$, such that one can easily derive the global uniqueness result.*
- (2) *In the hyperbolic case, we do not have the maximum principle and therefore, we do not know the sign and boundedness of certain solutions $v^{(3)}, \dots, v^{(M)}$ in the integral identity (5.29). Hence, we seek for the CGO solutions. Indeed, when the CGO solutions are of the form (5.16), they are bounded in Q . We refer the readers to Appendix A for more details about CGO solutions used in our work.*

• **Conclusion.**

In this work, we use different measurements to study related inverse problems.

- (1) By using the passive measurement, we are able to determine initial data, whenever zeroth order coefficients are known a priori. On the other hand, the unique determination for initial data cannot hold when nonlinearities are unknown. However, if the unknown nonlinearities belong to certain classes, one can still determine the initial data via the passive measurement.
- (2) By imposing the admissible conditions on coefficients, one can recover initial data and coefficients simultaneously via a hyperbolic type approximation property and the completeness products of solutions to wave equations.
- (3) The nonlinearity helps us to study the simultaneous recovery inverse problem. In our approach, when we used the first linearization, the unknown initial data disappears in the first linearized wave equation (5.9).
- (4) To our best knowledge, for the linear counterpart, Corollary 2.9 would be the first result for the simultaneous recovery for both initial data and zero order coefficients. Furthermore, via the proofs of Theorem 2.2 and Corollary 2.9, one can see that the smallness conditions for the semilinear wave equation is needed only for the local well-posedness, but not in the study of inverse problems.

APPENDIX A. COMPLEX GEOMETRICAL OPTICS (CGO) SOLUTIONS

In this section, let us review the known complex geometrical optics (CGO) solutions for wave equations with potential. Even though there might be some other references already proved the existence of CGO solutions, we follow the ideas of Kian-Oksanen [KO19] and give exponential type solutions to a wave equation for the sake of self-containedness of this work.

Let $\Omega \subseteq \mathbb{R}^n$ be a bounded domain with smooth boundary Γ , for $n \geq 2$ and t_1, t_2 be two positive numbers with $t_1 < t_2$ ³. For any $q \in E^{m+1}$, consider the wave equation:

$$v_{tt} - \Delta v + qv = 0 \quad \text{in } \Omega \times (t_1, t_2)$$

and its CGO solutions of the form

$$v(x, t) = a(x, t)e^{i\tau[\eta(x)+t]} + R^{(\tau)}(x, t) \quad \text{in } \Omega \times (t_1, t_2),$$

where τ is a real number with $|\tau| > 1$, $\eta(x) = |x - x_0|$ for an $x_0 \in \mathbb{R}^n \setminus \bar{\Omega}$, and $R^{(\tau)}$ satisfies

$$\begin{cases} R_{tt}^{(\tau)} - \Delta R^{(\tau)} + qR^{(\tau)} = -e^{i\tau(|x-x_0|+t)}(a_{tt} - \Delta a + qa) & \text{in } \Omega \times (t_1, t_2), \\ R^{(\tau)} = 0 & \text{on } \Gamma \times (t_1, t_2), \\ R^{(\tau)}(x, t_1) = R_t^{(\tau)}(x, t_1) = 0 & \text{in } \Omega, \end{cases} \quad (\text{A.1})$$

and

$$\lim_{|\tau| \rightarrow \infty} \|R^{(\tau)}\|_{L^2(\Omega \times (t_1, t_2))} = 0. \quad (\text{A.2})$$

Furthermore, $a(\cdot, \cdot)$ satisfies

$$2a_t - 2\nabla\eta \cdot \nabla a - \Delta\eta a = 0 \quad \text{in } \Omega \times (t_1, t_2). \quad (\text{A.3})$$

Without loss of generality, we may assume that $x_0 = 0$ in the following arguments. By (A.3), $a(\cdot, \cdot)$ satisfies the following equation:

$$a_t - \frac{x}{|x|} \cdot \nabla a - \frac{n-1}{2|x|} a = 0. \quad (\text{A.4})$$

³In the applications, the numbers t_1, t_2 are given by Definition 2.6.

As in [KO19], we write $x \in \mathbb{R}^n$ in terms of the polar coordinate $(r, \theta) \in [0, \infty) \times \mathbb{S}^{n-1}$ and the metric takes the form $g(r, \theta) = dr^2 + g_0(r, \theta)$. Then, (A.4) becomes

$$a_t - a_r - \frac{b_r}{4b} \cdot a = 0, \quad (\text{A.5})$$

where

$$b(r, \theta) = \det g_0(r, \theta).$$

For any $h = h(\theta) \in C^\infty(\mathbb{S}^{n-1})$, $\chi(\cdot) \in C^\infty(\mathbb{R})$ and $\mu > 0$, set

$$a(r, \theta, t) = e^{-\frac{\mu(r+t)}{2}} \chi(r+t) h(\theta) b(r, \theta)^{-\frac{1}{4}}. \quad (\text{A.6})$$

Then $a(\cdot, \cdot)$ in (A.6) is the desired solution to the transport equation (A.5). Similarly, $v(x, t) = a(x, t) e^{-i\tau[\eta(x)+t]} + R^{(\tau)}(x, t)$ is also the CGO solution to the wave equation.

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REFERENCES

- [BHKS18] Tommi Brander, Bastian Harrach, Manas Kar, and Mikko Salo. Monotonicity and enclosure methods for the p -Laplace equation. *SIAM J. Appl. Math.*, 78(2):742–758, 2018.
- [CB08] Yvonne Choquet-Bruhat. *General relativity and the Einstein equations*. OUP Oxford, 2008.
- [CFK⁺21] Cătălin I Cârstea, Ali Feizmohammadi, Yavar Kian, Katya Krupchyk, and Gunther Uhlmann. The Calderón inverse problem for isotropic quasilinear conductivities. *Advanced in Mathematics, to appear*, 2021.
- [CLOP22] Xi Chen, Matti Lassas, Lauri Oksanen, and Gabriel Paternain. Detection of Hermitian connections in wave equations with cubic non-linearity. *J. Eur. Math. Soc. (JEMS)*, 24(7):2191–2232, 2022.
- [dHUUW18] Maarten de Hoop, Gunther Uhlmann, and Yiran Wang. Nonlinear interaction of waves in elastodynamics and an inverse problem. *Mathematische Annalen*, pages 1–31, 2018.
- [DLL19] Youjun Deng, Jinhong Li, and Hongyu Liu. On identifying magnetized anomalies using ge-magnetic monitoring. *Archive for Rational Mechanics and Analysis*, 231(1):153–187, 2019.
- [DLL20] Youjun Deng, Jinhong Li, and Hongyu Liu. On identifying magnetized anomalies using ge-magnetic monitoring within a magnetohydrodynamic model. *Archive for Rational Mechanics and Analysis*, 235(1):691–721, 2020.
- [DLU19] Youjun Deng, Hongyu Liu, and Gunther Uhlmann. On an inverse boundary problem arising in brain imaging. *Journal of Differential Equations*, 267(4):2471–2502, 2019.
- [DZZ08] Thomas Duyckaerts, Xu Zhang, and Enrique Zuazua. On the optimality of the observability inequalities for parabolic and hyperbolic systems with potentials. *Annales de l’Institut Henri Poincaré C, Analyse non linéaire*, 25(1):1–41, 2008.
- [EEK05] Herbert Egger, Heinz W. Engl, and Michael V. Klibanov. Global uniqueness and Hölder stability for recovering a nonlinear source term in a parabolic equation. *Inverse Problems*, 21(1):271–290, 2005.
- [FLL23] Ali Feizmohammadi, Tony Liimatainen, and Yi-Hsuan Lin. An inverse problem for a semilinear elliptic equation on conformally transversally anisotropic manifolds. *Ann. PDE*, 9(2):Paper No. 12, 54, 2023.
- [FO20] Ali Feizmohammadi and Lauri Oksanen. An inverse problem for a semi-linear elliptic equation in Riemannian geometries. *Journal of Differential Equations*, 269(6):4683–4719, 2020.

- [HL23] Bastian Harrach and Yi-Hsuan Lin. Simultaneous recovery of piecewise analytic coefficients in a semilinear elliptic equation. *Nonlinear Analysis*, 228:113188, 2023.
- [IN95] Victor Isakov and A Nachman. Global uniqueness for a two-dimensional elliptic inverse problem. *Trans. of AMS*, 347:3375–3391, 1995.
- [IS94] Victor Isakov and John Sylvester. Global uniqueness for a semilinear elliptic inverse problem. *Communications on Pure and Applied Mathematics*, 47(10):1403–1410, 1994.
- [Isa93] Victor Isakov. On uniqueness in inverse problems for semilinear parabolic equations. *Archive for Rational Mechanics and Analysis*, 124(1):1–12, 1993.
- [IY13] Oleg Imanuvilov and Masahiro Yamamoto. Unique determination of potentials and semilinear terms of semilinear elliptic equations from partial Cauchy data. *J. Inverse Ill-Posed Probl.*, 21(1):85–108, 2013.
- [KKL01] Alexander Kachalov, Yaroslav Kurylev, and Matti Lassas. *Inverse boundary spectral problems*. CRC Press, 2001.
- [KL21] Michael V. Klibanov and Jingzhi Li. *Inverse problems and Carleman estimates—global uniqueness, global convergence and experimental data*, volume 63 of *Inverse and Ill-posed Problems Series*. De Gruyter, Berlin, [2021] ©2021.
- [KLOU22] Yaroslav Kurylev, Matti Lassas, Lauri Oksanen, and Gunther Uhlmann. Inverse problem for Einstein-scalar field equations. *Duke Math. J.*, 171(16):3215–3282, 2022.
- [KLU18] Yaroslav Kurylev, Matti Lassas, and Gunther Uhlmann. Inverse problems for Lorentzian manifolds and non-linear hyperbolic equations. *Inventiones mathematicae*, 212(3):781–857, 2018.
- [KM91] Michael V. Klibanov and Joseph Malinsky. Newton-Kantorovich method for three-dimensional potential inverse scattering problem and stability of the hyperbolic Cauchy problem with time-dependent data. *Inverse Problems*, 7(4):577–596, 1991.
- [KN02] Hyeonbae Kang and Gen Nakamura. Identification of nonlinearity in a conductivity equation via the Dirichlet-to-Neumann map. *Inverse Problems*, 18(4):1079, 2002.
- [KO19] Yavar Kian and Lauri Oksanen. Recovery of time-dependent coefficient on Riemannian manifold for hyperbolic equations. *International Mathematics Research Notices*, 2019(16):5087–5126, 2019.
- [KU20a] Katya Krupchyk and Gunther Uhlmann. Partial data inverse problems for semilinear elliptic equations with gradient nonlinearities. *Mathematical Research Letters*, 27(6):1801–1824, 2020.
- [KU20b] Katya Krupchyk and Gunther Uhlmann. A remark on partial data inverse problems for semilinear elliptic equations. *Proceedings of the American Mathematical Society*, 148(2):681–685, 2020.
- [Lin22] Yi-Hsuan Lin. Monotonicity-based inversion of fractional semilinear elliptic equations with power type nonlinearities. *Calculus of Variations and Partial Differential Equations*, 61(5):188, 2022.
- [LL19] Ru-Yu Lai and Yi-Hsuan Lin. Global uniqueness for the fractional semilinear schrödinger equation. *Proceedings of the American Mathematical Society*, 147(3):1189–1199, 2019.
- [LL22] Ru-Yu Lai and Yi-Hsuan Lin. Inverse problems for fractional semilinear elliptic equations. *Nonlinear Analysis*, 216:112699, 2022.
- [LLLS20] Matti Lassas, Tony Liimatainen, Yi-Hsuan Lin, and Mikko Salo. Partial data inverse problems and simultaneous recovery of boundary and coefficients for semilinear elliptic equations. *Revista Matemática Iberoamericana*, 37(4):1553–1580, 2020.
- [LLLS21] Matti Lassas, Tony Liimatainen, Yi-Hsuan Lin, and Mikko Salo. Inverse problems for elliptic equations with power type nonlinearities. *Journal de mathématiques pures et appliquées*, 145:44–82, 2021.
- [LLLZ22] Yi-Hsuan Lin, Hongyu Liu, Xu Liu, and Shen Zhang. Simultaneous recoveries for semilinear parabolic systems. *Inverse Problems*, 38(11):115006, 2022.
- [LLM19] Jingzhi Li, Hongyu Liu, and Shiqi Ma. Determining a random Schrödinger equation with unknown source and potential. *SIAM Journal on Mathematical Analysis*, 51(4):3465–3491, 2019.
- [LLM21] Jingzhi Li, Hongyu Liu, and Shiqi Ma. Determining a random Schrödinger operator: both potential and source are random. *Communications in Mathematical Physics*, 381(2):527–556, 2021.
- [LLPMT21] Matti Lassas, Tony Liimatainen, Leyter Potenciano-Machado, and Teemu Tyni. Stability estimates for inverse problems for semi-linear wave equations on Lorentzian manifolds. *arXiv preprint arXiv:2106.12257*, 2021.
- [LLPMT22] Matti Lassas, Tony Liimatainen, Leyter Potenciano-Machado, and Teemu Tyni. Uniqueness, reconstruction and stability for an inverse problem of a semi-linear wave equation. *J. Differential Equations*, 337:395–435, 2022.

- [LLS20] Matti Lassas, Tony Liimatainen, and Mikko Salo. The Poisson embedding approach to the Calderón problem. *Mathematische Annalen*, 377(1):19–67, 2020.
- [LLST22] Tony Liimatainen, Yi-Hsuan Lin, Mikko Salo, and Teemu Tyni. Inverse problems for elliptic equations with fractional power type nonlinearities. *Journal of Differential Equations*, 306:189–219, 2022.
- [Lü13] Qi Lü. Observability estimate and state observation problems for stochastic hyperbolic equations. *Inverse Problems*, 29(9):095011, 2013.
- [LU15] Hongyu Liu and Gunther Uhlmann. Determining both sound speed and internal source in thermo-and photo-acoustic tomography. *Inverse Problems*, 31(10):105005, 2015.
- [LUW17] Matti Lassas, Gunther Uhlmann, and Yiran Wang. Determination of vacuum space-times from the Einstein-Maxwell equations. *arXiv preprint arXiv:1703.10704*, 2017.
- [LUW18] Matti Lassas, Gunther Uhlmann, and Yiran Wang. Inverse problems for semilinear wave equations on Lorentzian manifolds. *Communications in Mathematical Physics*, 360:555–609, 2018.
- [LW07] Xiaosheng Li and Jenn-Nan Wang. Determination of viscosity in the stationary Navier-Stokes equations. *J. Differential Equations*, 242(1):24–39, 2007.
- [LZ00] Liangyu Li and Xu Zhang. Exact controllability for semilinear wave equations. *Journal of mathematical analysis and applications*, 250(2):589–597, 2000.
- [MU20] Claudio Munoz and Gunther Uhlmann. The Calderón problem for quasilinear elliptic equations. *Annales de l’Institut Henri Poincaré C, Analyse non linéaire*, 37(5):1143–1166, 2020.
- [NVW20] Gen Nakamura, Manmohan Vashisth, and Michiyuki Watanabe. Inverse initial boundary value problem for a non-linear hyperbolic partial differential equation. *Inverse Problems*, 37(1):015012, 2020.
- [Pos87] Jurgen Poschel. *Inverse spectral theory*. Academic Press, 1987.
- [RB22] V. G. Romanov and T. V. Bugueva. An inverse problem for a nonlinear wave equation. *Sib. Zh. Ind. Mat.*, 25(2):83–100, 2022.
- [SU97] Ziqi Sun and Gunther Uhlmann. Inverse problems in quasilinear anisotropic media. *American Journal of Mathematics*, 119(4):771–797, 1997.
- [Sun96] Ziqi Sun. On a quasilinear inverse boundary value problem. *Math. Z.*, 221(2):293–305, 1996.
- [Sun05] Ziqi Sun. Conjectures in inverse boundary value problems for quasilinear elliptic equations. *Cubo*, 7(3):65–73, 2005.
- [Sun10] Ziqi Sun. An inverse boundary-value problem for semilinear elliptic equations. *Electronic Journal of Differential Equations (EJDE)[electronic only]*, 37:1–5, 2010.
- [SZ12] Mikko Salo and Xiao Zhong. An inverse problem for the p-Laplacian: boundary determination. *SIAM journal on mathematical analysis*, 44(4):2474–2495, 2012.
- [Uhl09] Gunther Uhlmann. Electrical impedance tomography and Calderón’s problem. *Inverse Problems*, 25:123011, 2009.
- [WZ19] Yiran Wang and Ting Zhou. Inverse problems for quadratic derivative nonlinear wave equations. *Communications in Partial Differential Equations*, 44(11):1140–1158, 2019.

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