THE ANISOTROPIC CALDERÓN PROBLEM: RIGIDITY NEAR THE EUCLIDEAN METRIC

YI-HSUAN LIN

Abstract. We prove rigidity near the Euclidean metric for the anisotropic Calderón problem on smooth bounded connected domains in dimensions three and higher. Any smooth Riemannian metric sufficiently close to the Euclidean metric in a suitable Hölder norm and having the same Dirichlet-to-Neumann map agrees with it up to a diffeomorphism fixing the boundary pointwise. The result applies to general smooth anisotropic perturbations, without analytic or quasianalytic regularity assumptions, a prescribed conformal class, or a transversal product structure, and requires no convexity of the boundary. Harmonic coordinates and boundary determination reduce the problem to a compactly supported divergence free perturbation of the identity conductivity. Equality of the boundary measurements yields compactly supported correctors, which are controlled by a Carleman estimate on a frequency range determined by the perturbation. The resulting Fourier bound is quadratic in the L^2 norm of the perturbation. A frequency decomposition combines this bound with Sobolev control of the remaining frequencies to yield rigidity.

Contents

1. Introduction

Inverse boundary value problems ask whether an unknown medium can be determined from measurements made at its boundary. Let $\Omega \subset \mathbb{R}^n$ be a smooth bounded domain and let $A = (A^{ij})$

2020 *Mathematics Subject Classification.* 35R30, 35J15, 53C24.

Key words and phrases. Anisotropic Calderón problem, near-Euclidean metric, Dirichlet-to-Neumann map, harmonic coordinates, Carleman estimate, rigidity.

be a real symmetric uniformly elliptic matrix-valued function. For a prescribed boundary value f, let u solve

$$\begin{cases} -\operatorname{div}(A\nabla u) = 0 & \text{in } \Omega, \\ u = f & \text{on } \partial\Omega. \end{cases} \tag{1.1}$$

The associated Dirichlet-to-Neumann (DN) map is $\Lambda_A f = A\nabla u \cdot \nu|_{\partial\Omega}$, where ν is the outward unit normal. The problem was introduced in [Cal80]. For isotropic conductivities $A = \gamma I$, global uniqueness for sufficiently smooth γ in dimensions $n \geq 3$ was proved in [SU87]; see [FSU25] for an introduction and further references.

For anisotropic conductivities, uniqueness in fixed coordinates is impossible. If $F : \overline{\Omega} \to \overline{\Omega}$ is a smooth diffeomorphism satisfying $F|_{\partial\Omega} = \mathrm{Id}$, define $(F_*A)(y) = DF(x)A(x)DF(x)^T/|\det DF(x)|$, where $x = F^{-1}(y)$, then $\Lambda_{F_*A} = \Lambda_A$. In dimensions $n \geq 3$, the conductivity equation is equivalent to a Laplace–Beltrami equation through $A_g = (\det g)^{1/2} g^{-1}$ and $g_A = (\det A)^{1/(n-2)} A^{-1}$. The anisotropic Calderón problem asks whether a boundary fixing change of coordinates is the only obstruction to recovering a Riemannian metric from its DN map. In two dimensions, quasiconformal changes of coordinates reduce anisotropic conductivities to isotropic ones. Global uniqueness up to a boundary fixing quasiconformal change of coordinates holds for bounded measurable uniformly elliptic conductivities on simply connected planar domains [ALP05]; see [FSU25, Section 5.11] for further discussion.

For smooth Riemannian metrics in dimensions $n \geq 3$, the DN map determines the induced boundary metric and the full boundary jet in boundary normal coordinates [LU89, JL05]. Global uniqueness for real analytic metrics was first proved under additional geometric and topological assumptions in [LU89]. These additional assumptions were removed in [LU01]. The Poisson embedding method gives another proof of analytic rigidity and a global construction of the boundary fixing isometry [LLS20]. For smooth metrics, uniqueness in a prescribed conformal class is known on conformally transversally anisotropic manifolds under suitable assumptions on the transversal geometry [FKSU09, FKLS16]. Recent work proves rigidity for quasianalytic metrics and for certain smooth metrics with symmetry or one-sided ordering [CJLT26]. The problem for general smooth metrics in dimensions $n \geq 3$ remains open [FSU25, CJLT26].

Nonuniqueness is known in several related settings. Counterexamples for partial boundary data and metrics with limited regularity are constructed in [DKN20]. More recent examples give smooth nonisometric coefficients with identical full boundary data for equations with prescribed nonconstant potentials or nonzero spectral terms [DEH$^+$26]. These examples concern different data or equations from (1.1) with full boundary measurements.

Singular transformation constructions give another form of nonuniqueness: a singular anisotropic metric can have the same boundary measurements as the Euclidean metric without arising from a smooth boundary fixing change of coordinates [GKLU09]. The theorem below shows that this type of invisibility beyond the diffeomorphism gauge cannot occur for smooth metrics sufficiently close to e.

Related results have been obtained for Lorentzian wave equations. Zeroth-order coefficients can be recovered under curvature assumptions and near the Minkowski geometry [AFO22, AFO25]. For globally hyperbolic Lorentzian metrics that agree with the Minkowski metric outside a compact set, equality of the hyperbolic DN map with the Minkowski one implies rigidity up to a boundary fixing diffeomorphism [ORS26b]. This result follows from a stronger theorem using a formally determined collection of distorted plane wave measurements. For time-independent hyperbolic operators with the same lower-order terms, finitely many plane waves and complementary solutions determine the leading coefficients under geometric assumptions on one of the operators [ORS26a]. Semiglobal uniqueness for Lorentzian metrics is also obtained when one of the two metrics is sufficiently close to the Minkowski metric [ORS26c]. These results use hyperbolic boundary data, while the present paper concerns the elliptic DN map.

A recent near-Euclidean result determines compactly supported scalar potentials on a fixed known three-dimensional metric that is C^4 close to the Euclidean metric and has strictly convex boundary, with a consequence for conformal multiples of the fixed metric [UW26]. The present paper proves nonlinear rigidity at the Euclidean metric for a general smooth anisotropic perturbation of the principal coefficient. The equality $\Lambda_g = \Lambda_e$ forces the metric to be Euclidean up to a boundary fixing diffeomorphism. We assume no convexity of the boundary, prescribed conformal class, transversal product structure, or analytic or quasianalytic regularity. The result holds in every dimension $n \geq 3$, with smallness measured in $C^{k,\alpha}$, where $k = \lfloor n/2 \rfloor + 1$ and $n/2 + 1 - k < \alpha < 1$. Smoothness is used to recover and compare the full boundary jets, while the quantitative argument uses only $\|g - e\|_{C^{k,\alpha}(\overline{\Omega})}$. Throughout the paper, e denotes the Euclidean metric. A closely related rigidity theorem for smooth metrics near the Euclidean metric is proved in [Ste26], with smallness measured in H^s for an integer $s > n/2 + 1$.

Main results. We use the convention $\Delta_g = \operatorname{div}_g \nabla_g$, so that $-\Delta_g$ is nonnegative. For a smooth Riemannian metric g on $\overline{\Omega}$, the DN map is

$$\Lambda_g : C^\infty(\partial\Omega) \to C^\infty(\partial\Omega), \quad f \mapsto \partial_{\nu_g} u\big|_{\partial\Omega},$$

where u is the unique solution of

$$\begin{cases} -\Delta_g u = 0 & \text{in } \Omega, \\ u = f & \text{on } \partial\Omega, \end{cases}$$

and ν_g is the unit outer normal with respect to g. We write $\operatorname{Diff}^\infty(\overline{\Omega})$ for the smooth diffeomorphisms of $\overline{\Omega}$ onto itself, smooth up to the boundary, and Id for the identity map. We write $\operatorname{Sym}^2 \mathbb{R}^n$ for the space of real symmetric $n \times n$ matrices and denote the Euclidean variable by $x = (x^1, \ldots, x^n)$. Repeated coordinate indices are summed from 1 to n.

Set $s_* = \frac{n}{2} + 1$ and $k = \lfloor s_* \rfloor$. The Hölder exponent in the metric statements satisfies $s_* - k < \alpha < 1$, so $k + \alpha > s_*$. When n is even, this condition is $0 < \alpha < 1$; when n is odd, it is $1/2 < \alpha < 1$.

Theorem 1.1 (Rigidity near the Euclidean metric)**.** *Let $\Omega \subset \mathbb{R}^n$ be a smooth bounded connected domain, where $n \geq 3$, and let $s_* - k < \alpha < 1$. There exists $\varepsilon_1 = \varepsilon_1(n, \Omega, \alpha) > 0$ with the following property. Suppose that $g \in C^\infty(\overline{\Omega}; \operatorname{Sym}^2 \mathbb{R}^n)$ is a Riemannian metric satisfying*

$$\|g - e\|_{C^{k,\alpha}(\overline{\Omega})} < \varepsilon_1. \tag{1.2}$$

Assume that

$$\Lambda_g = \Lambda_e. \tag{1.3}$$

For $1 \leq j \leq n$, let y^j solve

$$\begin{cases} -\Delta_g y^j = 0 & \text{in } \Omega, \\ y^j = x^j & \text{on } \partial\Omega \end{cases} \tag{1.4}$$

and set $Y = (y^1, \ldots, y^n)$. Then $Y \in \operatorname{Diff}^\infty(\overline{\Omega})$, $Y|_{\partial\Omega} = \mathrm{Id}$, and

$$g = Y^* e.$$

Moreover, Y is the unique diffeomorphism $F \in \operatorname{Diff}^\infty(\overline{\Omega})$ satisfying $F|_{\partial\Omega} = \mathrm{Id}$ and $g = F^* e$.

The conclusion is the natural one because every boundary fixing diffeomorphism preserves the DN map.

Harmonic coordinate reduction and the conductivity theorem. A related harmonic coordinate argument for the linearized anisotropic Calderón problem appears in [SU91]; see also [Uhl92, Section 9, pp. 190–193]. We use the harmonic map determined by the boundary values of the Cartesian coordinate functions to fix the diffeomorphism freedom in the nonlinear problem. Let y^j solve (1.4), and set $Y = (y^1, \dots, y^n)$. When g is sufficiently close to e, this map is a boundary fixing diffeomorphism. If $\widetilde{g} = Y_* g$ and $\widetilde{A} = (\det \widetilde{g})^{1/2} \widetilde{g}^{-1}$, then the coordinate functions are $\widetilde{g}$-harmonic and

$$\partial_i \widetilde{A}^{ij} = 0 \quad \text{for } 1 \le j \le n. \tag{1.5}$$

Writing $\widetilde{A} = I + H$, (1.5) gives $\operatorname{div} H = 0$. This is an exact condition in the fixed Euclidean target coordinates, not a linearized gauge condition. It follows from the choice of coordinates and is not an additional assumption on the original metric in Theorem 1.1. Moreover, Proposition 5.1 gives $\|H\|_{C^{k,\alpha}(\overline{\Omega})} \le C \|g - e\|_{C^{k,\alpha}(\overline{\Omega})}$, so the transformed conductivity remains close to I in the same Hölder norm.

The key result used after this normalization is the following conductivity theorem. The compact support condition below is an assumption of the conductivity theorem and does not follow directly from the harmonic coordinate normalization. For a matrix field $H = (H^{ij})$, the notation $\operatorname{div} H = 0$ means $\partial_i H^{ij} = 0$ in distributions for every j.

Theorem 1.2 (Rigidity in harmonic gauge)**.** *Let $\Omega \subset \mathbb{R}^n$ be a smooth bounded connected domain, where $n \ge 3$, and let $K \Subset \Omega$ be fixed. There exists $\varepsilon_0 > 0$, depending only on n, Ω, and K, with the following property. Let $A = I + H$ be a real symmetric uniformly elliptic conductivity such that*

$$H \in H^{s_*}(\mathbb{R}^n; \operatorname{Sym}^2 \mathbb{R}^n), \quad \operatorname{supp} H \subset K, \quad \operatorname{div} H = 0.$$

If $\Lambda_A = \Lambda_I$ and $\|H\|_{H^{s_*}(\mathbb{R}^n)} < \varepsilon_0$, then $H = 0$.

For a complex matrix $M = (M_{jk}) \in \mathbb{C}^{n \times n}$, we write $|M| = (\sum_{j,k=1}^n |M_{jk}|^2)^{1/2}$. The L^2 and Sobolev norms of matrix fields are formed using this pointwise matrix norm. Throughout the proof, the matrix L^∞ norm is the operator norm $\|H\|_{L^\infty} := \operatorname{ess\,sup}_{x \in \mathbb{R}^n} \sup_{v \in \mathbb{C}^n,\ |v|=1} |H(x)v|$.

Theorem 1.2 requires smallness in $H^{s_*}(\mathbb{R}^n)$, where $s_* = n/2 + 1$. When $n = 3$, this is the space $H^{5/2}(\mathbb{R}^3)$. The metric theorem uses $C^{k,\alpha}$ smallness with $k = \lfloor s_* \rfloor$ and $\alpha > s_* - k$, which becomes $C^{2,\alpha}$ with $\alpha > 1/2$ in dimension three. The theorem is not applied directly to the perturbation $H = A_{Y_* g} - I$ on the original domain, since the harmonic coordinate normalization gives $\operatorname{div} H = 0$ but does not imply $\operatorname{supp} H \Subset \Omega$. Smooth boundary determination gives an auxiliary boundary fixing gauge in which the metric has the same full boundary jet as e. Comparing this gauge with the canonical harmonic gauge shows that H vanishes to infinite order at the boundary. The perturbation can then be extended by zero to a fixed larger domain. The condition $k + \alpha > s_*$ gives an $H^{s_*}(\mathbb{R}^n)$ bound for this extension in terms of $\|H\|_{C^{k,\alpha}(\overline{\Omega})}$, and Theorem 1.2 applies.

Complex null vectors. We write $\mathsf{i} = \sqrt{-1}$ and extend the Euclidean dot product on $\mathbb{R}^n$ complex bilinearly to $\mathbb{C}^n$, so that $\zeta \cdot \eta = \sum_{j=1}^n \zeta_j \eta_j$ for $\zeta, \eta \in \mathbb{C}^n$ (without an additional complex conjugate). A vector $\zeta \in \mathbb{C}^n$ is called a complex null vector if $\zeta \cdot \zeta = 0$. The notation $|\zeta| = (\sum_{j=1}^n |\zeta_j|^2)^{1/2}$ denotes the usual Hermitian norm on $\mathbb{C}^n$. If $\zeta = \alpha + \mathsf{i}\beta$ with $\alpha, \beta \in \mathbb{R}^n$, then the null condition is equivalent to $\alpha \cdot \beta = 0$ and $|\alpha| = |\beta|$. Since $\Delta e^{\zeta \cdot x} = (\zeta \cdot \zeta) e^{\zeta \cdot x}$, the function $e^{\zeta \cdot x}$ is harmonic whenever ζ is a complex null vector.

Outline of the proof. We first reduce Theorem 1.1 to Theorem 1.2. The canonical harmonic map gives a boundary fixing diffeomorphism Y and a transformed conductivity $I + H$ satisfying $\operatorname{div} H = 0$, together with the estimate $\|H\|_{C^{k,\alpha}(\overline{\Omega})} \le C \|g - e\|_{C^{k,\alpha}(\overline{\Omega})}$. Smooth boundary determination supplies a second boundary fixing diffeomorphism for which the metric has the same full boundary jet as

e. Comparing the two gauges shows that the metric in the canonical harmonic coordinates, and hence H, is flat at the boundary. Extending H by zero to a fixed larger domain produces a smooth compactly supported divergence free tensor. Since $k+\alpha > s_*$, the H^{s_*} norm of this extension is controlled by $\|H\|_{C^{k,\alpha}(\overline{\Omega})}$. A weak gluing argument transfers equality of the DN maps to the outer boundary. Theorem 1.2 then gives $H=0$, so the metric is Euclidean in the harmonic coordinates.

For the proof of Theorem 1.2, set $X = \|H\|_{L^2}$, $\delta = \|H\|_{L^\infty}$, and $M = \|H\|_{H^{s_*}}$. The difficulty is that H changes the principal part of the equation. Conjugation by a harmonic exponential produces terms quadratic in the complex frequency, so the perturbation estimate used here cannot be absorbed at arbitrarily large frequencies for a fixed nonzero H. We instead work on a finite frequency range determined by δ and compare the resulting estimate with the Sobolev tail of the same tensor H.

For a complex null vector ζ, let u_ζ be the A-harmonic solution with boundary value $e^{\zeta\cdot x}$. Equality of the DN maps gives a compactly supported corrector r_ζ such that $u_\zeta = e^{\zeta\cdot x}(1+r_\zeta)$. A Carleman estimate for compactly supported functions gives

$$\|(\nabla+\zeta)r_\zeta\|_{L^2} \le C|\zeta|^2 X, \quad \text{for } 1 \lesssim |\zeta| \lesssim \delta^{-1}.$$

Pairing u_ζ with an explicit harmonic exponential, using $\operatorname{div} H = 0$, and estimating the integral remainder by the Cauchy–Schwarz inequality yields

$$|\widehat{H}(\xi)| \le C(1+|\xi|)X^2, \quad \text{for } |\xi| \lesssim \delta^{-1}.$$

The low-frequency part is bounded by $CR^{s_*}X^2$, while the H^{s_*} norm bounds the remaining tail by $CR^{-s_*}M$. For $X>0$, choose $R = \kappa X^{-1/s_*}$ with a fixed sufficiently small $\kappa > 0$. Gagliardo–Nirenberg interpolation gives $\delta \le CX^{1/s_*}M^{1-1/s_*}$, so $R\delta \le C\kappa M^{1-1/s_*}$ and the chosen radius lies in the available frequency range when M is sufficiently small. Absorbing the low-frequency term gives $X \le CMX$, and smallness of M forces $X=0$.

Organization of the paper. In Section 2, we prove the mixed comparison identity, construct the compactly supported correctors, and derive their conjugated equation. The Carleman estimate and the resulting corrector bounds are established in Section 3. We then use these bounds to obtain the Fourier estimate and prove Theorem 1.2 in Section 4. Section 5 returns to the metric problem. There we construct the canonical harmonic gauge and compare it with the boundary determination. An exterior gluing argument transfers equality of the DN maps to a larger domain, allowing us to apply the conductivity theorem and complete the proof of Theorem 1.1.

2. Exterior normalization and the conjugated equation

The DN map and the mixed identity. Let A be a real symmetric uniformly elliptic conductivity on Ω. All complex-valued dualities and energy pairings below are understood in the complex bilinear sense, with no complex conjugation, unless a conjugate is displayed explicitly. For $f \in H^{1/2}(\partial\Omega)$, let $u_f \in H^1(\Omega)$ be the unique weak solution of (1.1). The DN map is defined weakly by

$$\langle \Lambda_A f, g\rangle = \int_\Omega A\nabla u_f \cdot \nabla v\, dx, \tag{2.1}$$

where $v \in H^1(\Omega)$ is any function whose boundary trace is $v|_{\partial\Omega} = g \in H^{1/2}(\partial\Omega)$. The definition is independent of the extension v. The following is the comparison identity used throughout the proof.

Lemma 2.1 (Mixed comparison identity)**.** *Suppose that $A = I + H$ and $\Lambda_A = \Lambda_I$. Let $u \in H^1(\Omega)$ satisfy $-\operatorname{div}(A\nabla u) = 0$, and let $v \in H^1(\Omega)$ satisfy $-\Delta v = 0$. Then*

$$\int_\Omega H\nabla u\cdot\nabla v\, dx = 0. \tag{2.2}$$

Proof. Let $f = u|_{\partial\Omega}$ and $g = v|_{\partial\Omega}$. By the weak definitions (2.1) of the two DN maps,

$$\int_\Omega A\nabla u \cdot \nabla v \, dx = \langle \Lambda_A f, g\rangle = \langle \Lambda_I f, g\rangle. \tag{2.3}$$

Let u_0 be the harmonic function with trace f. Since v is Euclidean harmonic,

$$\langle \Lambda_I f, g\rangle = \int_\Omega \nabla u_0 \cdot \nabla v \, dx = \int_\Omega \nabla u \cdot \nabla v \, dx. \tag{2.4}$$

The last equality follows because $u - u_0 \in H_0^1(\Omega)$ and v is harmonic. Subtracting (2.4) from (2.3) proves (2.2). □

Remark 2.2. Lemma 2.1 uses only one solution for the unknown conductivity. In the Fourier argument in Section 4, we choose the second factor to be an explicit harmonic exponential. This makes the Fourier variable explicit. There is only one unknown corrector in the integral identity, and no product of two such correctors. The integral remainder can be estimated by pairing H and the differentiated corrector in L^2. Since the corrector estimate is linear in $\|H\|_{L^2}$, the resulting Fourier estimate is quadratic in this norm.

Compactly supported correctors. We now assume that $A = I + H$, $\operatorname{supp} H \subset K \Subset \Omega$, and $\Lambda_A = \Lambda_I$. Let $\zeta \in \mathbb{C}^n$ be a complex null vector. By the definition above, $e^{\zeta\cdot x}$ is harmonic.

Lemma 2.3 (Exterior-normalized complex exponential solutions)**.** *There is a compact set K_1 satisfying $K \Subset \operatorname{int}(K_1) \subset K_1 \Subset \Omega$, depending only on K and Ω, with the following property. For every complex null vector $\zeta \in \mathbb{C}^n$, there is a unique A-harmonic solution $u_\zeta \in H^1(\Omega)$ with boundary trace $e^{\zeta\cdot x}$, and*

$$w_\zeta := u_\zeta - e^{\zeta\cdot x} = 0 \quad \text{in } \Omega \setminus K_1. \tag{2.5}$$

Moreover,

$$u_\zeta = e^{\zeta\cdot x}(1 + r_\zeta), \quad r_\zeta = e^{-\zeta\cdot x} w_\zeta, \quad \operatorname{supp} r_\zeta \subset K_1. \tag{2.6}$$

In particular, $r_\zeta \in H^1(\Omega)$.

Proof. Existence and uniqueness follow from uniform ellipticity. Set $f_\zeta = e^{\zeta\cdot x}|_{\partial\Omega}$. Since u_ζ and $e^{\zeta\cdot x}$ have the same boundary trace, their difference w_ζ belongs to $H_0^1(\Omega)$. The exponential is harmonic because $\zeta\cdot\zeta = 0$, and equality of the DN maps gives $(A\nabla u_\zeta)\cdot\nu = \partial_\nu e^{\zeta\cdot x}$ on $\partial\Omega$ in $H^{-1/2}(\partial\Omega)$.

Let ν be the outward unit normal on $\partial\Omega$. Since $\partial\Omega$ is smooth and compact, we may choose $\rho > 0$, depending only on K and Ω, such that $2\rho < \operatorname{dist}(K, \partial\Omega)$ and the map $\Phi(p,t) = p - t\nu(p)$ is a smooth diffeomorphism from $\partial\Omega \times (-\rho, \rho)$ onto $\mathcal{U} = \{x \in \mathbb{R}^n : \operatorname{dist}(x, \partial\Omega) < \rho\}$. The positive parameter range $0 < t < \rho$ parametrizes $\mathcal{U} \cap \Omega$, while $-\rho < t < 0$ parametrizes $\mathcal{U} \setminus \overline{\Omega}$. Thus, $\mathcal{U}$ is a two-sided collar of the boundary whose interior part is disjoint from K. In particular, $A = I$ and $\Delta w_\zeta = 0$ in $\mathcal{U} \cap \Omega$.

Extend w_ζ by zero outside Ω, and denote the restriction of this extension to $\mathcal{U}$ by $\widetilde{w}_\zeta$. Since $w_\zeta \in H_0^1(\Omega)$, its zero extension belongs to $H^1(\mathbb{R}^n)$. For $\varphi \in C_c^\infty(\mathcal{U})$, extend φ by zero to $\mathbb{R}^n$ and use the same notation. Since $A = I$ wherever $\nabla\varphi$ can be nonzero in Ω, the weak definitions of the DN maps give

$$\int_{\mathcal{U}} \nabla\widetilde{w}_\zeta \cdot \nabla\varphi \, dx = \int_\Omega \big(A\nabla u_\zeta - \nabla e^{\zeta\cdot x}\big) \cdot \nabla\varphi \, dx = \langle (\Lambda_A - \Lambda_I) f_\zeta, \varphi|_{\partial\Omega}\rangle = 0.$$

Thus, $\widetilde{w}_\zeta$ is distributionally harmonic in the entire collar $\mathcal{U}$, including across $\partial\Omega$.

Weyl's lemma shows that $\widetilde{w}_\zeta$ is smooth and harmonic in $\mathcal{U}$. Every connected component of $\mathcal{U}$ contains a nonempty exterior open part, parametrized by $-\rho < t < 0$, on which $\widetilde{w}_\zeta = 0$. Unique continuation for harmonic functions implies $\widetilde{w}_\zeta = 0$ on every component of $\mathcal{U}$. In particular, $w_\zeta = 0$ at every point of Ω whose distance from $\partial\Omega$ is less than ρ.

Define $K_1 = \{x \in \overline{\Omega} : \operatorname{dist}(x, \partial\Omega) \geq \rho/2\}$. This is a compact subset of Ω, depends only on K and Ω, and satisfies $K \Subset \operatorname{int}(K_1)$ by the choice of ρ. Since $\Omega \setminus K_1 \subset \mathcal{U} \cap \Omega$, the preceding vanishing proves (2.5). Finally, multiplication by the smooth nonvanishing function $e^{-\zeta\cdot x}$ preserves $H^1(\Omega)$ and does not change the support. Hence, $r_\zeta = e^{-\zeta\cdot x} w_\zeta \in H^1(\Omega)$, is supported in K_1, and satisfies (2.6). $\square$

The function u_ζ has the usual complex geometrical optics (CGO) form, but no smallness of r_ζ is asserted at this stage. The point of Lemma 2.3 is that the correction has support in a fixed compact set. This support property uses both equality of the DN maps and the fact that $A = I$ near the boundary; it is not a property of the Dirichlet solution for an arbitrary conductivity. Therefore, the Carleman estimate can be applied directly to the actual correction, without multiplying it by a cutoff and introducing additional commutator terms. Its quantitative estimate is proved in Proposition 3.3. After extending A by I and r_ζ by zero outside Ω, the same formula defines a whole space solution, which agrees with $e^{\zeta\cdot x}$ outside K_1.

For a complex null vector $\zeta \in \mathbb{C}^n$, define $D_\zeta r = \nabla r + \zeta r$. Since $w_\zeta = e^{\zeta\cdot x} r_\zeta$, direct differentiation gives

$$\nabla w_\zeta = e^{\zeta\cdot x} D_\zeta r_\zeta. \tag{2.7}$$

The conjugated equation. The divergence free condition gives the following conjugated equation.

Lemma 2.4 (Conjugated divergence-form equation). *Suppose that $A = I + H$ and $\operatorname{div} H = 0$. Then the corrector in Lemma 2.3 satisfies*

$$P_\zeta r_\zeta + Q_{H,\zeta} r_\zeta = \zeta^T H \zeta, \tag{2.8}$$

where $P_\zeta = -\Delta - 2\zeta \cdot \nabla$ *and* $\quad Q_{H,\zeta} r = -\operatorname{div}(H\nabla r) - 2H\zeta \cdot \nabla r - (\zeta^T H \zeta) r.$

Proof. For a smooth scalar function r,

$$e^{-\zeta\cdot x}[-\operatorname{div}(A\nabla)]e^{\zeta\cdot x} r = -\operatorname{div}(A\nabla r) - 2A\zeta \cdot \nabla r - (\zeta^T A \zeta) r, \tag{2.9}$$

where we used $\operatorname{div}(A\zeta) = 0$ from $\operatorname{div} A = 0$. Since $A \in L^\infty(\Omega)$, both sides of (2.9) depend continuously on $r \in H^1(\Omega)$ as distributions. Approximation by smooth functions extends the identity to $H^1(\Omega)$. Apply (2.9) to $1 + r_\zeta$. Since $\zeta \cdot \zeta = 0$ and $A = I + H$, the equation for $u_\zeta = e^{\zeta\cdot x}(1 + r_\zeta)$ becomes $-\Delta r_\zeta - 2\zeta \cdot \nabla r_\zeta - \operatorname{div}(H\nabla r_\zeta) - 2H\zeta \cdot \nabla r_\zeta - (\zeta^T H \zeta) r_\zeta = \zeta^T H \zeta$, which is (2.8). $\square$

3. Carleman estimates for compactly supported functions and corrector bounds

Throughout Sections 3 and 4, we assume all the hypotheses of Theorem 1.2 except for the smallness condition on $\|H\|_{H^{s_*}(\mathbb{R}^n)}$, unless stated otherwise. We use the correctors constructed in Lemma 2.3.

3.1. Semiclassical Sobolev spaces and the free estimate.

For $h > 0$ and $s \in \mathbb{R}$, define

$$\|v\|_{H^s_{\mathrm{scl}}(\mathbb{R}^n)} = \big\|\langle h\xi\rangle^s \widehat{v}(\xi)\big\|_{L^2_\xi}, \quad \langle h\xi\rangle = (1 + h^2|\xi|^2)^{1/2},$$

where the subscript ξ indicates that the L^2 norm is taken in the Fourier variable. In particular, Plancherel's theorem gives

$$\|v\|^2_{H^1_{\mathrm{scl}}} \asymp \|v\|^2_{L^2} + \|h\nabla v\|^2_{L^2}. \tag{3.1}$$

Here and below, $A \asymp B$ means that $cB \leq A \leq CB$ for constants $c, C > 0$ independent of the parameters under consideration. In particular, in (3.1), these constants are independent of h and

v. The L^2 pairing identifies H^{-1}_{scl} with the continuous dual of H^1_{scl}, with norm equivalence constants independent of h. Let

$$\mathcal{Z} = \{a + \mathrm{i}b : \ a, b \in \mathbb{R}^n, \ a \cdot b = 0, \ |a| = |b| = 1\}.$$

This is the set of complex null vectors whose real and imaginary parts both have length one. For $\zeta_0 \in \mathcal{Z}$, set $P_{\zeta_0,h} = -h^2\Delta - 2h\zeta_0 \cdot \nabla$.

Proposition 3.1 (Limiting Carleman estimate for compactly supported functions)**.** *Let $\mathcal{O} \subset \mathbb{R}^n$ be a bounded open set. There exist $h_0 > 0$ and $C > 0$, depending only on n and $\mathcal{O}$, such that*

$$h\|v\|_{H^1_{\mathrm{scl}}} \le C\|P_{\zeta_0,h}v\|_{H^{-1}_{\mathrm{scl}}} \tag{3.2}$$

for every $0 < h < h_0$, every $\zeta_0 \in \mathcal{Z}$, and every $v \in C_c^\infty(\mathcal{O})$. The constants are independent of h, ζ_0, and v. The same estimate (3.2) holds for every $v \in H^1(\mathbb{R}^n)$ satisfying $\operatorname{supp} v \Subset \mathcal{O}$, where $P_{\zeta_0,h}v$ is understood in the distributional sense.

Proof. Choose concentric Euclidean balls $B \Subset \widetilde{B}$, centered at the origin, such that $\overline{\mathcal{O}} \subset B$. Fix $a \in \mathbb{S}^{n-1}$ and set $\psi_a(x) = -a \cdot x$. Since $\nabla\psi_a = -a$ is constant and nonzero, ψ_a is a limiting Carleman weight for the Euclidean metric; see [FKSU09, Section 3] and [KU14, Section 2]. Apply [KU14, Proposition 2.2] on B, using $\widetilde{B}$ as the larger ambient domain and setting the magnetic and electric potentials in that proposition equal to zero. In the notation of the cited result, $L_{0,0} = -\Delta$. Taking the Carleman weight to be ψ_a gives constants $h_a > 0$ and $C_a > 0$ such that

$$h\|v\|_{H^1_{\mathrm{scl}}} \le C_a\big\|e^{-a\cdot x/h}(-h^2\Delta)e^{a\cdot x/h}v\big\|_{H^{-1}_{\mathrm{scl}}} \tag{3.3}$$

for every $v \in C_c^\infty(B)$ and every $0 < h < h_a$. Indeed, the conjugated operator in [KU14, Proposition 2.2] is $e^{\psi_a/h}(h^2L_{0,0})e^{-\psi_a/h}$, which is exactly $e^{-a\cdot x/h}(-h^2\Delta)e^{a\cdot x/h}$.

The Sobolev orders in (3.3) follow from the estimate with a gain of two derivatives in [KU14, Proposition 2.1 and the proof of Proposition 2.2]. Let $\psi_{a,\varepsilon} = \psi_a + \frac{h}{2\varepsilon}\psi_a^2$ be the convexified weight. Taking $s = -1$ in that estimate places the conjugated operator in H^{-1}_{scl} and controls the function in H^1_{scl}, with the factor $h/\sqrt{\varepsilon}$ on the left. The parameter $\varepsilon > 0$ is chosen sufficiently small and then fixed independently of h, so this factor is a fixed multiple of h. To pass back to the original weight, choose $\chi \in C_c^\infty(\widetilde{B})$ such that $\chi = 1$ in a neighborhood of $\overline{B}$. Applying the convexified estimate to $e^{\psi_a^2/(2\varepsilon)}v$ uses the identity $e^{-\psi_{a,\varepsilon}/h}e^{\psi_a^2/(2\varepsilon)}v = e^{-\psi_a/h}v$. Since v is supported in B, multiplication by $e^{\psi_a^2/(2\varepsilon)}$ and its inverse may be replaced by multiplication by the compactly supported smooth functions $\chi e^{\psi_a^2/(2\varepsilon)}$ and $\chi e^{-\psi_a^2/(2\varepsilon)}$. These multiplication operators are bounded on $H^s_{\mathrm{scl}}(\mathbb{R}^n)$ for $-1 \le s \le 1$, with bounds independent of h and $a \in \mathbb{S}^{n-1}$. This explains the H^1_{scl} and H^{-1}_{scl} norms in (3.3). The choice $s = -1$ will be used below because the divergence-form perturbation naturally maps H^1_{scl} into H^{-1}_{scl} when only the L^∞ norm of its coefficient is available.

It remains to show that the constants in (3.3) may be chosen independently of $a \in \mathbb{S}^{n-1}$. Let $e_1 = (1, 0, \dots, 0) \in \mathbb{R}^n$, and let $C = C_{e_1}$ and $h_0 = h_{e_1}$ be the constants supplied by (3.3) for the single direction e_1. For each $a \in \mathbb{S}^{n-1}$, choose an orthogonal map $R_a \in O(n)$ such that $R_a e_1 = a$. Since $R_a^T R_a = I$, one has $R_a^T a = e_1$, and $\psi_a(R_a x) = -a \cdot R_a x = -(R_a^T a) \cdot x = -e_1 \cdot x = \psi_{e_1}(x)$. Define $T_a v = v \circ R_a$. Since B and $\widetilde{B}$ are balls centered at the origin, they are invariant under R_a, and T_a maps $C_c^\infty(B)$ onto itself. Orthogonality gives $\Delta(T_a\phi) = T_a(\Delta\phi)$ for every smooth function ϕ. Moreover, $\widehat{T_a v}(\xi) = \widehat{v}(R_a\xi)$, so the change of variables $\eta = R_a\xi$ and the identity $|R_a\xi| = |\xi|$ give $\|T_a v\|_{H^s_{\mathrm{scl}}} = \|v\|_{H^s_{\mathrm{scl}}}$ for every $s \in \mathbb{R}$.

The conjugated operators are intertwined by T_a. Indeed, $a \cdot R_a x = e_1 \cdot x$, and hence $e^{\pm e_1\cdot x/h}T_a v = T_a(e^{\pm a\cdot x/h}v)$. Combining this identity with $\Delta T_a = T_a\Delta$, we obtain

$$e^{-e_1\cdot x/h}(-h^2\Delta)e^{e_1\cdot x/h}T_a v = e^{-e_1\cdot x/h}(-h^2\Delta)T_a\big(e^{a\cdot x/h}v\big) = T_a\big(e^{-a\cdot x/h}(-h^2\Delta)e^{a\cdot x/h}v\big).$$

For $0 < h < h_0$, apply the special case $a = e_1$ of (3.3) to $T_a v$. The preceding identities give

$$h\|v\|_{H^1_{\mathrm{scl}}} = h\|T_a v\|_{H^1_{\mathrm{scl}}} \le C\big\|e^{-e_1\cdot x/h}(-h^2\Delta)e^{e_1\cdot x/h}T_a v\big\|_{H^{-1}_{\mathrm{scl}}} = C\big\|e^{-a\cdot x/h}(-h^2\Delta)e^{a\cdot x/h}v\big\|_{H^{-1}_{\mathrm{scl}}}.$$

This shows that the same constants $C = C_{e_1}$ and $h_0 = h_{e_1}$ may be used in (3.3) for every $a \in \mathbb{S}^{n-1}$. After decreasing h_0, we may assume that $0 < h_0 \le 1$.

Let $b \in \mathbb{S}^{n-1}$ satisfy $a \cdot b = 0$, and define $M_b v = e^{\mathrm{i}b\cdot x/h}v$. With the Fourier transform convention in Section 4, one has $\widehat{M_b v}(\xi) = \widehat{v}(\xi - b/h)$. After the change of variables $\eta = \xi - b/h$, one obtains

$$\|M_b v\|^2_{H^s_{\mathrm{scl}}} = \int_{\mathbb{R}^n} \langle h\eta + b\rangle^{2s}|\widehat{v}(\eta)|^2\, d\eta.$$

The triangle inequality, applied once with b and once with $-b$, shows that $\langle h\eta+b\rangle \asymp \langle h\eta\rangle$ uniformly in $h > 0$, $\eta \in \mathbb{R}^n$, and $b \in \mathbb{S}^{n-1}$. For $-1 \le s \le 1$, this gives

$$C_s^{-1}\|v\|_{H^s_{\mathrm{scl}}} \le \|M_b v\|_{H^s_{\mathrm{scl}}} \le C_s\|v\|_{H^s_{\mathrm{scl}}}, \tag{3.4}$$

and the same inequalities hold with M_b replaced by M_b^{-1}.

Set $L_{a,h}\cdot = e^{-a\cdot x/h}(-h^2\Delta)(e^{a\cdot x/h}\cdot)$. Direct differentiation gives $L_{a,h} = -h^2\Delta - 2ha\cdot\nabla - |a|^2$, while $M_b^{-1}\nabla(M_b\cdot) = (\nabla + \mathrm{i}b/h)\cdot$. Combining these identities gives

$$M_b^{-1}L_{a,h}M_b = -h^2\Delta - 2h(a+\mathrm{i}b)\cdot\nabla + \big(|b|^2 - |a|^2 - 2\mathrm{i}a\cdot b\big).$$

The zeroth-order term vanishes because $|a| = |b| = 1$ and $a \cdot b = 0$. For $\zeta_0 = a + \mathrm{i}b$, we have

$$M_b^{-1}L_{a,h}M_b = P_{\zeta_0,h} \quad \text{equivalently} \quad L_{a,h}M_b = M_b P_{\zeta_0,h}. \tag{3.5}$$

Let $v \in C_c^\infty(\mathcal{O})$. Since multiplication by M_b does not change the support, one has $M_b v \in C_c^\infty(\mathcal{O}) \subset C_c^\infty(B)$. Applying the estimate (3.3) to $M_b v$ and using the modulation estimates (3.4) and the operator identity (3.5), we obtain

$$h\|v\|_{H^1_{\mathrm{scl}}} \le Ch\|M_b v\|_{H^1_{\mathrm{scl}}} \le C\|L_{a,h}M_b v\|_{H^{-1}_{\mathrm{scl}}} = C\|M_b P_{\zeta_0,h}v\|_{H^{-1}_{\mathrm{scl}}} \le C\|P_{\zeta_0,h}v\|_{H^{-1}_{\mathrm{scl}}}.$$

This proves (3.2) for $v \in C_c^\infty(\mathcal{O})$, with constants independent of $\zeta_0 \in \mathcal{Z}$.

It remains to extend the estimate to compactly supported H^1 functions. Let $v \in H^1(\mathbb{R}^n)$ satisfy $\operatorname{supp} v \Subset \mathcal{O}$, and choose an open set $\mathcal{O}_1$ such that $\operatorname{supp} v \Subset \mathcal{O}_1 \Subset \mathcal{O}$. Standard mollification gives a sequence $v_k \in C_c^\infty(\mathcal{O}_1) \subset C_c^\infty(\mathcal{O})$ such that $v_k \to v$ in $H^1(\mathbb{R}^n)$. The Fourier multiplier of $P_{\zeta_0,h}$ is $p_{\zeta_0,h}(\xi) = h^2|\xi|^2 - 2\mathrm{i}h\zeta_0\cdot\xi$. Since $|\zeta_0| = \sqrt{2}$ on $\mathcal{Z}$, one has $|p_{\zeta_0,h}(\xi)| \le C\langle h\xi\rangle^2$ uniformly for $0 < h \le 1$ and $\zeta_0 \in \mathcal{Z}$. Thus,

$$\|P_{\zeta_0,h}f\|^2_{H^{-1}_{\mathrm{scl}}} = \int_{\mathbb{R}^n}\langle h\xi\rangle^{-2}|p_{\zeta_0,h}(\xi)|^2|\widehat{f}(\xi)|^2\,d\xi \le C\int_{\mathbb{R}^n}\langle h\xi\rangle^2|\widehat{f}(\xi)|^2\,d\xi = C\|f\|^2_{H^1_{\mathrm{scl}}}. \tag{3.6}$$

Thus, the operator $P_{\zeta_0,h} : H^1_{\mathrm{scl}}(\mathbb{R}^n) \to H^{-1}_{\mathrm{scl}}(\mathbb{R}^n)$ is continuous, uniformly for the stated values of h and ζ_0. The convergence $v_k \to v$ in $H^1(\mathbb{R}^n)$ implies convergence in $H^1_{\mathrm{scl}}(\mathbb{R}^n)$ for each fixed $0 < h < h_0$, and (3.6) gives $P_{\zeta_0,h}v_k \to P_{\zeta_0,h}v$ in $H^{-1}_{\mathrm{scl}}(\mathbb{R}^n)$. Passing to the limit in (3.2) proves the final assertion. □

The divergence-form perturbation. For $\zeta_0 \in \mathcal{Z}$, define

$$Q_{H,\zeta_0,h}\cdot = -h^2\operatorname{div}(H\nabla\cdot) - 2hH\zeta_0\cdot\nabla\cdot - (\zeta_0^T H\zeta_0)\cdot\,.$$

Lemma 3.2 (Perturbation estimate)**.** *There is a constant C, depending only on the dimension, such that*

$$\|Q_{H,\zeta_0,h}v\|_{H^{-1}_{\mathrm{scl}}} \le C\|H\|_{L^\infty}\|v\|_{H^1_{\mathrm{scl}}} \tag{3.7}$$

for all $0 < h \leq 1$, $\zeta_0 \in \mathcal{Z}$, *and* $v \in H^1_{\mathrm{scl}}$. *Moreover,*

$$\|\zeta_0^T H \zeta_0\|_{H^{-1}_{\mathrm{scl}}} \leq C\|H\|_{L^2}. \tag{3.8}$$

Proof. Using the distributional definition of the divergence term, for every $\psi \in H^1_{\mathrm{scl}}$ we have

$$|\langle Q_{H,\zeta_0,h}v, \psi\rangle| \leq h^2\|H\|_{L^\infty}\|\nabla v\|_{L^2}\|\nabla\psi\|_{L^2} + 2h\|H\|_{L^\infty}|\zeta_0|\|\nabla v\|_{L^2}\|\psi\|_{L^2} + \|H\|_{L^\infty}|\zeta_0|^2\|v\|_{L^2}\|\psi\|_{L^2}. \tag{3.9}$$

By the definition of the semiclassical norms and Plancherel's theorem, the constants in (3.1) depend only on n. Since $|\zeta_0| = \sqrt{2}$ for $\zeta_0 \in \mathcal{Z}$, the right-hand side of (3.9) is bounded by $C\|H\|_{L^\infty}\|v\|_{H^1_{\mathrm{scl}}}$ $\|\psi\|_{H^1_{\mathrm{scl}}}$, where C depends only on n. Taking the supremum over ψ with $\|\psi\|_{H^1_{\mathrm{scl}}} = 1$ proves (3.7).

For the source term, the Cauchy–Schwarz inequality gives $|\zeta_0^T H(x)\zeta_0| \leq |\zeta_0|^2|H(x)| = 2|H(x)|$. Also, by duality and (3.1), $\|F\|_{H^{-1}_{\mathrm{scl}}} \leq C\|F\|_{L^2}$ for every $F \in L^2(\mathbb{R}^n)$, with C depending only on n. Hence, $\|\zeta_0^T H\zeta_0\|_{H^{-1}_{\mathrm{scl}}} \leq 2C\|H\|_{L^2}$, which proves (3.8). $\square$

3.2. **Corrector estimates in a growing complex frequency range.** Let K_1 be the compact set fixed in Lemma 2.3. Thus, $K \Subset \mathrm{int}(K_1) \subset K_1 \Subset \Omega$, and both w_ζ and r_ζ are supported in K_1 for every complex null vector ζ. We regard r_ζ as its zero extension to $\mathbb{R}^n$. Put

$$X = \|H\|_{L^2(\mathbb{R}^n)}, \quad \delta = \|H\|_{L^\infty}. \tag{3.10}$$

All constants below may depend on n, Ω, and the fixed set K, but are independent of H, X, δ, and the complex frequency ζ, unless stated otherwise. This includes the dependence on K_1, since K_1 was fixed in Lemma 2.3 depending only on K and Ω.

Proposition 3.3 (Corrector estimate)**.** *There are constants $c_0, C, \delta_0 > 0$, depending only on n, Ω, and the fixed set K, with the following properties. If $\delta = 0$, then $H = 0$ almost everywhere, $A = I$, and $r_\zeta = 0$ for every complex null vector ζ. Suppose that $0 < \delta \leq \delta_0$. For every complex null vector $\zeta \in \mathbb{C}^n$ satisfying*

$$|\zeta| \leq c_0\delta^{-1}, \tag{3.11}$$

one has

$$\|D_\zeta r_\zeta\|_{L^2(\mathbb{R}^n)} \leq C|\zeta|(1 + |\zeta|)X. \tag{3.12}$$

More precisely,

$$\|D_\zeta r_\zeta\|_{L^2(\mathbb{R}^n)} \leq C|\zeta|X \quad \textit{if } |\zeta| \leq 1, \tag{3.13}$$

and

$$\|D_\zeta r_\zeta\|_{L^2(\mathbb{R}^n)} \leq C|\zeta|^2X \quad \textit{if } 1 \leq |\zeta| \leq c_0\delta^{-1}. \tag{3.14}$$

Proof. If $\delta = 0$, then $H = 0$ almost everywhere and $A = I$. The functions u_ζ and $e^{\zeta\cdot x}$ solve the same Dirichlet problem with $u_\zeta|_{\partial\Omega} = e^{\zeta\cdot x}|_{\partial\Omega}$. Uniqueness gives $u_\zeta = e^{\zeta\cdot x}$ in $\overline{\Omega}$, so $w_\zeta = 0$ and $r_\zeta = 0$.

Assume from now on that $0 < \delta \leq \delta_0$. Fix a bounded open set $\mathcal{O} \subset \mathbb{R}^n$ such that $K_1 \Subset \mathcal{O}$, and let h_0 be the constant in Proposition 3.1 for this fixed set $\mathcal{O}$. Choose once and for all $R_0 > \max\{1, \frac{\sqrt{2}}{h_0}\}$. We first prove the estimate for $|\zeta| \leq R_0$. Recall that $w_\zeta = u_\zeta - e^{\zeta\cdot x} \in H^1_0(\Omega)$ and

$$-\operatorname{div}(A\nabla w_\zeta) = \operatorname{div}\left(e^{\zeta\cdot x}H\zeta\right) \quad \text{in } \Omega. \tag{3.15}$$

We shall choose $\delta_0 \leq 1/2$. Since H is real symmetric and $\|H\|_{L^\infty} = \delta$, every eigenvalue of $H(x)$ belongs to $[-\delta, \delta]$ for almost every x. Every eigenvalue of $A(x) = I + H(x)$ is at least $1 - \delta \geq 1/2$. In particular, $A(x)z \cdot \overline{z} \geq \frac{1}{2}|z|^2$ for every $z \in \mathbb{C}^n$ and almost every $x \in \Omega$.

Testing (3.15) with $\overline{w_\zeta}$ and taking real parts gives

$$\frac{1}{2}\|\nabla w_\zeta\|^2_{L^2(\Omega)} \le \operatorname{Re}\int_\Omega A\nabla w_\zeta\cdot\nabla\overline{w_\zeta}\,dx = -\operatorname{Re}\int_\Omega e^{\zeta\cdot x}H\zeta\cdot\nabla\overline{w_\zeta}\,dx \le |\zeta|\|e^{\zeta\cdot x}H\|_{L^2(K)}\|\nabla w_\zeta\|_{L^2(\Omega)}.$$

If $\|\nabla w_\zeta\|_{L^2(\Omega)} = 0$, then $w_\zeta = 0$ because $w_\zeta \in H^1_0(\Omega)$, and (2.7) gives $D_\zeta r_\zeta = 0$. Otherwise, division by $\|\nabla w_\zeta\|_{L^2(\Omega)}$ gives

$$\|\nabla w_\zeta\|_{L^2(\Omega)} \le C|\zeta|\|e^{\zeta\cdot x}H\|_{L^2(K)}. \tag{3.16}$$

Since the fixed set K_1 is compact, choose $\rho > 0$ such that $K_1 \subset \{x \in \mathbb{R}^n : |x| \le \rho\}$. The inclusion $K \subset K_1$ and the inequality $|e^{\pm\zeta\cdot x}| \le e^{|\operatorname{Re}\zeta||x|}$ give $|e^{\pm\zeta\cdot x}| \le e^{\rho|\operatorname{Re}\zeta|}$ on K_1. In particular, $\|e^{\zeta\cdot x}H\|_{L^2(K)} \le e^{\rho|\operatorname{Re}\zeta|}X$. Lemma 2.3 gives $\operatorname{supp} w_\zeta \subset K_1$, while (2.7) gives $D_\zeta r_\zeta = e^{-\zeta\cdot x}\nabla w_\zeta$. Using (3.16), we obtain

$$\|D_\zeta r_\zeta\|_{L^2(\mathbb{R}^n)} = \|e^{-\zeta\cdot x}\nabla w_\zeta\|_{L^2(K_1)} \le e^{\rho|\operatorname{Re}\zeta|}\|\nabla w_\zeta\|_{L^2(\Omega)} \le C|\zeta|e^{2\rho|\operatorname{Re}\zeta|}X.$$

For $|\zeta| \le R_0$, the inequality $|\operatorname{Re}\zeta| \le |\zeta|$ gives

$$\|D_\zeta r_\zeta\|_{L^2(\mathbb{R}^n)} \le C_{R_0}|\zeta|X, \tag{3.17}$$

where $C_{R_0} = Ce^{2\rho R_0}$. This proves (3.13). If $1 \le |\zeta| \le R_0$, then $|\zeta| \le |\zeta|^2$, and (3.17) also gives

$$\|D_\zeta r_\zeta\|_{L^2(\mathbb{R}^n)} \le C_{R_0}|\zeta|^2X. \tag{3.18}$$

We next consider $|\zeta| \ge R_0$. Write $\zeta = \alpha + \mathrm{i}\beta$ with $\alpha, \beta \in \mathbb{R}^n$. Since $\zeta\cdot\zeta = 0$, one has $\alpha\cdot\beta = 0$ and $|\alpha| = |\beta| =: s$. Set $h = s^{-1}$ and $\zeta_0 = h\zeta = \frac{\alpha}{s} + \mathrm{i}\frac{\beta}{s}$, then $\zeta_0 \in \mathcal{Z}$, $|\zeta_0| = \sqrt{2}$, and

$$h = \frac{\sqrt{2}}{|\zeta|}. \tag{3.19}$$

The choice of R_0 gives $0 < h < h_0$.

Since $r_\zeta = 0$ and $H = 0$ in a neighborhood of $\partial\Omega$, extending r_ζ by zero and A by I introduces no distribution supported on $\partial\Omega$. Multiplying (2.8) by h^2 and using $\zeta = h^{-1}\zeta_0$ gives

$$P_{\zeta_0,h}r_\zeta + Q_{H,\zeta_0,h}r_\zeta = \zeta_0^T H\zeta_0 \tag{3.20}$$

in distributions on $\mathbb{R}^n$. The zero extension of r_ζ belongs to $H^1(\mathbb{R}^n)$ and is supported in the fixed set $K_1 \Subset \mathcal{O}$. Proposition 3.1 applies to this extension. Using (3.20) and Lemma 3.2, we obtain

$$h\|r_\zeta\|_{H^1_{\mathrm{scl}}} \le C\|P_{\zeta_0,h}r_\zeta\|_{H^{-1}_{\mathrm{scl}}} \le C\|\zeta_0^T H\zeta_0\|_{H^{-1}_{\mathrm{scl}}} + C\|Q_{H,\zeta_0,h}r_\zeta\|_{H^{-1}_{\mathrm{scl}}} \le C_1X + C_2\delta\|r_\zeta\|_{H^1_{\mathrm{scl}}}, \tag{3.21}$$

where the constants C_1 and C_2 depend only on n, the fixed set $\mathcal{O}$, and the fixed support sets K and K_1. In particular, they are independent of H, X, δ, ζ, and h. Since $\mathcal{O}$ and K_1 have already been fixed in terms of K and Ω, the constants C_1 and C_2 are fixed before c_0 and δ_0 are chosen.

Choose $c_0 > 0$ so that $C_2c_0 \le 1/\sqrt{2}$. If $|\zeta| \le c_0\delta^{-1}$, then $\delta \le c_0|\zeta|^{-1}$. Formula (3.19) gives

$$C_2\delta \le \frac{C_2c_0}{|\zeta|} \le \frac{1}{\sqrt{2}|\zeta|} = \frac{h}{2}. \tag{3.22}$$

By (3.22), the last term in (3.21) is bounded by $\frac{h}{2}\|r_\zeta\|_{H^1_{\mathrm{scl}}}$. Moving this term to the left and dividing by $h/2$ gives

$$\|r_\zeta\|_{H^1_{\mathrm{scl}}} \le Ch^{-1}X. \tag{3.23}$$

Since $\zeta = h^{-1}\zeta_0$, one has $D_\zeta r_\zeta = h^{-1}(h\nabla + \zeta_0)r_\zeta$. Formula (3.1), the identity $|\zeta_0| = \sqrt{2}$, and (3.23) give

$$\|D_\zeta r_\zeta\|_{L^2(\mathbb{R}^n)} \le h^{-1}\big(\|h\nabla r_\zeta\|_{L^2} + |\zeta_0|\|r_\zeta\|_{L^2}\big) \le Ch^{-1}\|r_\zeta\|_{H^1_{\mathrm{scl}}} \le Ch^{-2}X \le C|\zeta|^2X.$$

This proves (3.14) for $R_0 \le |\zeta| \le c_0\delta^{-1}$. Formula (3.18) gives the same estimate for $1 \le |\zeta| \le R_0$.

Choose $0 < \delta_0 \le \min\{\frac{1}{2}, \frac{c_0}{R_0}\}$. For every $0 < \delta \le \delta_0$, one has $c_0\delta^{-1} \ge R_0$, so the bounded frequency estimate and the Carleman estimate cover the entire interval $1 \le |\zeta| \le c_0\delta^{-1}$. After increasing C to include the fixed constant C_{R_0}, this proves (3.13) and (3.14) with one constant C.

For $|\zeta| \le 1$, the right-hand side in (3.13) is bounded by the right-hand side in (3.12). For $1 \le |\zeta| \le c_0\delta^{-1}$, one has $|\zeta|^2 \le |\zeta|(1+|\zeta|)$. This proves (3.12). □

Remark 3.4. The upper bound in (3.11) is the condition needed to absorb the perturbation term in (3.21). The relation $h = \sqrt{2}/|\zeta|$ turns $C_2\delta \le h/2$ into a bound of the form $|\zeta| \le c_0\delta^{-1}$. In the Fourier argument below, the complex frequency associated with a real frequency ξ has size $|\zeta| = |\xi|/\sqrt{2}$. For $X > 0$, the frequency decomposition uses $R = \kappa X^{-1/s_*}$, where $\kappa > 0$ is fixed and sufficiently small. If $M = \|H\|_{H^{s_*}}$, Lemma 4.6 gives $\delta \le CX^{1/s_*}M^{1-1/s_*}$, and $R\delta \le C\kappa M^{1-1/s_*}$. Smallness of M places this radius in the real-frequency range of Proposition 4.3.

4. Fourier estimates and Sobolev rigidity

We now use the corrector estimate in Proposition 3.3 to control the Fourier transform of H and then combine this estimate with the Sobolev regularity of H. The quantity $\delta = \|H\|_{L^\infty}$ defined in (3.10) continues to use the pointwise matrix operator norm, as specified earlier. In particular, $|Hv| \le \delta|v|$ for every $v \in \mathbb{C}^n$ and almost every $x \in \mathbb{R}^n$.

4.1. **Fourier estimates in a growing frequency range.** We use the Fourier transform convention $\widehat{f}(\xi) = \int_{\mathbb{R}^n} e^{-\mathrm{i}x\cdot\xi} f(x)\,dx$, and apply it componentwise to vector and matrix fields. Since H is compactly supported and belongs to $L^2(\mathbb{R}^n)$, it also belongs to $L^1(\mathbb{R}^n)$, so $\widehat{H}$ is continuous.

For each $1 \le j \le n$, the identity $\partial_i H^{ij} = 0$ gives $\mathrm{i}\sum_{i=1}^n \xi_i \widehat{H}^{ij}(\xi) = 0$. In matrix notation, this says $\xi^T\widehat{H}(\xi) = 0$. The tensor H is symmetric, so $\widehat{H}(\xi)^T = \widehat{H}(\xi)$ and $\widehat{H}(\xi)\xi = 0$ for every $\xi \in \mathbb{R}^n$.

Lemma 4.1 (Null decomposition of a real frequency)**.** *Let $0 \ne \xi \in \mathbb{R}^n$, set $\omega = \xi/|\xi|$, and let $\theta \in \mathbb{S}^{n-1}$ satisfy $\theta\cdot\xi = 0$. Define $\zeta = \frac{|\xi|}{2}(\theta + \mathrm{i}\omega)$ and $\eta = \frac{|\xi|}{2}(-\theta + \mathrm{i}\omega)$. Then ζ and η are complex null vectors, $\zeta + \eta = \mathrm{i}\xi$, and $|\zeta| = |\eta| = |\xi|/\sqrt{2}$. If $M \in \mathbb{C}^{n\times n}$ is symmetric and $M\xi = 0$, then $\zeta^T M\eta = -\frac{|\xi|^2}{4}\theta^T M\theta$.*

Proof. The orthogonality $\theta\cdot\omega = 0$ and the identities $|\theta| = |\omega| = 1$ give $\zeta\cdot\zeta = \eta\cdot\eta = 0$, while the definitions give $\zeta + \eta = \mathrm{i}|\xi|\omega = \mathrm{i}\xi$. The Hermitian norms satisfy $|\zeta|^2 = |\eta|^2 = |\xi|^2/2$. Since $M\xi = 0$, one has $M\omega = 0$. Symmetry also gives $\omega^T M = 0$. Expanding $\zeta^T M\eta$ leaves only the term containing $\theta^T M\theta$, and gives $\zeta^T M\eta = -\frac{|\xi|^2}{4}\theta^T M\theta$. □

Lemma 4.2 (Polarization on the transverse space)**.** *For every $n \ge 2$, there is $C_n > 0$ such that, if $M \in \mathbb{C}^{n\times n}$ is symmetric and $M\xi = 0$ for some $0 \ne \xi \in \mathbb{R}^n$, then*

$$|M| \le C_n \sup\left\{|\theta^T M\theta| : \theta \in \mathbb{R}^n,\ \theta \perp \xi,\ |\theta| = 1\right\}. \tag{4.1}$$

Proof. Set $\omega = \xi/|\xi|$ and choose an orthonormal basis $e_1, \dots, e_{n-1}, e_n$ of $\mathbb{R}^n$ with $e_n = \omega$. Since $M\omega = 0$, the last column of M in this basis vanishes. Symmetry shows that the last row also vanishes. It is enough to estimate the entries $M_{jk} = e_j^T M e_k$ for $1 \le j, k \le n-1$.

Let $Q = \sup\{|\theta^T M\theta| : \theta \in \mathbb{R}^n,\ \theta \perp \xi,\ |\theta| = 1\}$. For each $1 \le j \le n-1$, one has $|M_{jj}| = |e_j^T M e_j| \le Q$. If $j \ne k$, set $\theta_{jk} = (e_j + e_k)/\sqrt{2}$. Symmetry gives $2M_{jk} = 2\theta_{jk}^T M\theta_{jk} - e_j^T M e_j - e_k^T M e_k$, so $|M_{jk}| \le 2Q$. All remaining entries vanish. It follows that $|M|^2 = \sum_{j,k=1}^{n-1} |M_{jk}|^2 \le C_n^2 Q^2$, which proves (4.1). □

Proposition 4.3 (Fourier estimate in a growing frequency range). *Let c_0 and δ_0 be the constants in Proposition 3.3. There are constants $c_1, C > 0$, depending only on n, Ω, and the fixed set K, with the following property. If $0 < \delta \le \delta_0$, then*

$$|\widehat{H}(\xi)| \le C(1+|\xi|)X^2 \quad \text{for every } |\xi| \le c_1\delta^{-1}. \tag{4.2}$$

If $\delta = 0$, then $H = 0$ almost everywhere and $\widehat{H} = 0$.

Proof. Assume that $0 < \delta \le \delta_0$, and choose $c_1 > 0$ so that $c_1 \le \sqrt{2}c_0$. Fix $0 \ne \xi \in \mathbb{R}^n$ with $|\xi| \le c_1\delta^{-1}$, and let $\theta \in \mathbb{S}^{n-1}$ satisfy $\theta\cdot\xi = 0$. Define ζ and η as in Lemma 4.1. Then $|\zeta| = |\xi|/\sqrt{2} \le c_0\delta^{-1}$, so Proposition 3.3 applies to r_ζ.

Using Lemma 2.3, the function $u_\zeta = e^{\zeta\cdot x}(1+r_\zeta)$ is A-harmonic, while $e^{\eta\cdot x}$ is harmonic. Lemma 2.1 gives $0 = \int_\Omega H\nabla u_\zeta\cdot\nabla e^{\eta\cdot x}\,dx$. Since $\nabla u_\zeta = e^{\zeta\cdot x}(\zeta + D_\zeta r_\zeta)$ and $\nabla e^{\eta\cdot x} = e^{\eta\cdot x}\eta$, one has $H\nabla u_\zeta\cdot\nabla e^{\eta\cdot x} = e^{(\zeta+\eta)\cdot x}\big((H\zeta)\cdot\eta + (HD_\zeta r_\zeta)\cdot\eta\big)$. Since $\operatorname{supp} H \subset K$, the mixed identity gives

$$\int_K e^{(\zeta+\eta)\cdot x}(H\zeta)\cdot\eta\,dx = -\int_K e^{(\zeta+\eta)\cdot x}(HD_\zeta r_\zeta)\cdot\eta\,dx. \tag{4.3}$$

Using $H^T = H$, $\zeta+\eta = \mathrm{i}\xi$, the Fourier transform convention, and (4.3), we obtain

$$\begin{aligned}\zeta^T\widehat{H}(-\xi)\eta &= \zeta^T\Big(\int_{\mathbb{R}^n} e^{\mathrm{i}\xi\cdot x}H(x)\,dx\Big)\eta = \int_K e^{\mathrm{i}\xi\cdot x}\zeta^T H(x)\eta\,dx = \int_K e^{(\zeta+\eta)\cdot x}(H(x)\zeta)\cdot\eta\,dx\\ &= -\int_K e^{(\zeta+\eta)\cdot x}(H(x)D_\zeta r_\zeta)\cdot\eta\,dx = -\int_K e^{\mathrm{i}\xi\cdot x}(H(x)D_\zeta r_\zeta)\cdot\eta\,dx.\end{aligned} \tag{4.4}$$

Since $\widehat{H}(-\xi)\xi = 0$ and $\widehat{H}(-\xi)$ is symmetric, Lemma 4.1 gives $\zeta^T\widehat{H}(-\xi)\eta = -\frac{|\xi|^2}{4}\theta^T\widehat{H}(-\xi)\theta$. The Cauchy–Schwarz inequality gives

$$\left|\int_K e^{\mathrm{i}\xi\cdot x}(HD_\zeta r_\zeta)\cdot\eta\,dx\right| \le |\eta|\|H\|_{L^2(K)}\|D_\zeta r_\zeta\|_{L^2(K)} \le |\eta|X\|D_\zeta r_\zeta\|_{L^2(\mathbb{R}^n)}. \tag{4.5}$$

Here the pointwise matrix norm in the L^2 norm is the Frobenius norm fixed above, and $|Hz| \le |H||z|$ for every $z \in \mathbb{C}^n$.

Since $|\eta| = |\xi|/\sqrt{2}$, if $|\xi| \le 2$, then (3.12) and $|\zeta| = |\xi|/\sqrt{2}$ give $\|D_\zeta r_\zeta\|_{L^2(\mathbb{R}^n)} \le C|\zeta|(1+|\zeta|)X \le C|\xi|X$. The right-hand side of (4.5) is bounded by $C|\xi|^2X^2$. If $2 \le |\xi| \le c_1\delta^{-1}$, then (3.14) gives $\|D_\zeta r_\zeta\|_{L^2(\mathbb{R}^n)} \le C|\xi|^2X$, and the right-hand side of (4.5) is bounded by $C|\xi|^3X^2$. Using (4.4) and dividing by $|\xi|^2$ gives $|\theta^T\widehat{H}(-\xi)\theta| \le C(1+|\xi|)X^2$. The vector θ was arbitrary among the unit vectors in $\xi^\perp$. Lemma 4.2, applied to $M = \widehat{H}(-\xi)$, gives $|\widehat{H}(-\xi)| \le C(1+|\xi|)X^2$. Replacing ξ by $-\xi$ proves (4.2) for every nonzero frequency in the stated range. Continuity of $\widehat{H}$ gives the estimate at $\xi = 0$. ☐

4.2. Frequency splitting and proof of the conductivity theorem. Low and high Fourier modes.

Lemma 4.4 (Low-frequency estimate). *Under the hypotheses of Proposition 4.3, suppose that $0 < \delta \le \delta_0$. Then*

$$\|\widehat{H}\|_{L^2(|\xi|\le R)} \le CR^{s_*}X^2 \tag{4.6}$$

for every $1 \le R \le c_1\delta^{-1}$.

Proof. Applying Proposition 4.3 and using polar coordinates, we obtain

$$\int_{|\xi|\le R} |\widehat{H}(\xi)|^2\,d\xi \le CX^4\int_0^R (1+r)^2 r^{n-1}\,dr \le CX^4R^{n+2}.$$

The last inequality uses $R \geq 1$. Taking square roots and using $s_* = (n+2)/2$ proves (4.6). □

Lemma 4.5 (Sobolev tail)**.** *Let $m > 0$ and set $M_m = \|H\|_{H^m(\mathbb{R}^n)}$. For every $R \geq 1$,*

$$\|\widehat{H}\|_{L^2(|\xi|>R)} \leq CR^{-m}M_m. \tag{4.7}$$

Proof. On the set $|\xi| > R$, one has $(1+|\xi|^2)^m \geq R^{2m}$. It follows that

$$\int_{|\xi|>R} |\widehat{H}(\xi)|^2 \, d\xi \leq R^{-2m} \int_{\mathbb{R}^n} (1+|\xi|^2)^m |\widehat{H}(\xi)|^2 \, d\xi.$$

Plancherel's theorem gives (4.7). □

4.2.1. *A Fourier proof of the interpolation inequality.*

Lemma 4.6 (Fourier Gagliardo–Nirenberg estimate)**.** *Let $m > n/2$. There is $C = C(n,m) > 0$ such that every scalar function $F \in H^m(\mathbb{R}^n)$ satisfies*

$$\|F\|_{L^\infty} \leq C\|F\|_{L^2}^{1-\frac{n}{2m}} \|F\|_{H^m}^{\frac{n}{2m}}. \tag{4.8}$$

The same estimate holds for every matrix field $F = (F_{jk}) \in H^m(\mathbb{R}^n; \mathbb{C}^{n\times n})$, with the matrix norms fixed above.

Proof. We prove the scalar and matrix-valued estimates at the same time. In the matrix-valued case, every occurrence of $|\widehat{F}(\xi)|$ below denotes $(\sum_{j,k=1}^n |\widehat{F}_{jk}(\xi)|^2)^{1/2}$. Set $X_F = \|F\|_{L^2}$ and $M_F = \|F\|_{H^m}$. If $X_F = 0$, then $F = 0$ almost everywhere. We assume that $X_F > 0$. Since $M_F \geq X_F$, the number $R = (M_F/X_F)^{1/m}$ satisfies $R \geq 1$.

In the scalar case, the Fourier inversion formula gives $\|F\|_{L^\infty} \leq C \int_{\mathbb{R}^n} |\widehat{F}(\xi)| \, d\xi$. For a matrix field, Fourier inversion is applied entrywise. For every complex matrix M, one has $\sup_{|v|=1} |Mv| \leq |M|$. Applying Fourier inversion entrywise and using this inequality gives the same bound for matrix fields.

On the region $|\xi| \leq R$, the Cauchy–Schwarz inequality gives $\int_{|\xi|\leq R} |\widehat{F}(\xi)| \, d\xi \leq CR^{n/2}X_F$. On the region $|\xi| > R$, the weighted Cauchy–Schwarz inequality gives

$$\int_{|\xi|>R} |\widehat{F}(\xi)| \, d\xi \leq M_F \left(\int_{|\xi|>R} (1+|\xi|^2)^{-m} \, d\xi \right)^{1/2} \leq CM_F R^{n/2-m},$$

where the last estimate uses $m > n/2$.

For $R = (M_F/X_F)^{1/m}$, one has $R^{n/2}X_F = X_F^{1-\frac{n}{2m}} M_F^{\frac{n}{2m}}$ and $M_F R^{n/2-m} = X_F^{1-\frac{n}{2m}} M_F^{\frac{n}{2m}}$. Combining the low- and high-frequency estimates proves (4.8). □

4.2.2. *Proof of the conductivity theorem.* The Sobolev order $s_* = n/2+1$ comes from the factor $R^{(n+2)/2}$ in the low-frequency estimate. The quadratic dependence on X allows us to choose the splitting radius in terms of X, rather than δ. At the order s_*, the Sobolev tail then becomes a constant multiple of MX, while Gagliardo–Nirenberg interpolation places the chosen radius in the available frequency range. This identifies the regularity used in the argument; it does not assert that s_* is optimal for the inverse problem.

Proof of Theorem 1.2. Set $X = \|H\|_{L^2(\mathbb{R}^n)}$, $\delta = \|H\|_{L^\infty}$, and $M = \|H\|_{H^{s_*}(\mathbb{R}^n)}$. If $X = 0$ or $\delta = 0$, then $H = 0$ almost everywhere. We assume that $X > 0$ and $\delta > 0$.

Since $s_* > n/2$, applying the Sobolev embedding entrywise gives $\delta \leq C_S M$ for a constant $C_S > 0$. All constants below depend only on n, Ω, and the fixed set K. We choose ε_0 at the end, after these constants have been fixed, and require $C_S\varepsilon_0 \leq \delta_0$. Under the assumption $M < \varepsilon_0$, Proposition 3.3, Proposition 4.3, and Lemma 4.4 are then available.

For every $1 \le R \le c_1\delta^{-1}$, Plancherel's theorem, Lemma 4.4, and Lemma 4.5 with $m = s_*$ give

$$X \le C_0 R^{s_*} X^2 + C_0 R^{-s_*} M \tag{4.9}$$

for a constant $C_0 > 0$. Fix $\kappa > 0$ so small that $C_0\kappa^{s_*} \le 1/2$, and set $R = \kappa X^{-1/s_*}$. We verify that this choice lies in the required range. Since $X \le M$, the condition $M < \varepsilon_0 \le \kappa^{s_*}$ gives $R \ge 1$. Lemma 4.6 with $m = s_*$ gives

$$\delta \le C_2 X^{1-\frac{n}{2s_*}} M^{\frac{n}{2s_*}} = C_2 X^{1/s_*} M^{1-1/s_*},$$

where we used $2s_* = n + 2$. Multiplication by $R = \kappa X^{-1/s_*}$ yields $R\delta \le C_2\kappa M^{1-1/s_*}$. Since $1 - 1/s_* = n/(n+2) > 0$, the inequality $R \le c_1\delta^{-1}$ holds once ε_0 is chosen so that $C_2\kappa\varepsilon_0^{1-1/s_*} \le c_1$.

For this value of R, the first term on the right-hand side of (4.9) is $C_0\kappa^{s_*}X \le X/2$. Moving this term to the left-hand side of (4.9) gives

$$X \le 2C_0 R^{-s_*} M = 2C_0\kappa^{-s_*} M X. \tag{4.10}$$

The constants C_S, δ_0, C_0, κ, C_2, and c_1 are independent of H. Choose $\varepsilon_0 > 0$ so small that $C_S\varepsilon_0 \le \delta_0$, $\varepsilon_0 \le \kappa^{s_*}$, $C_2\kappa\varepsilon_0^{1-1/s_*} \le c_1$, and $2C_0\kappa^{-s_*}\varepsilon_0 < 1$. If $M < \varepsilon_0$, all the estimates above are valid. Since $X > 0$, division of (4.10) by X gives $1 \le 2C_0\kappa^{-s_*}M < 2C_0\kappa^{-s_*}\varepsilon_0 < 1$, which is impossible. We conclude that $X = 0$, and hence $H = 0$. $\square$

5. Metric rigidity in dimensions $n \ge 3$

The metric reduction uses two different boundary fixing diffeomorphisms, and their roles are distinct. The first is the canonical harmonic map Y, obtained by solving a Dirichlet problem for each Cartesian coordinate function. The smallness of $g - e$ makes Y a global diffeomorphism of $\overline{\Omega}$. Its global invertibility allows us to use it as a change of coordinates on the whole domain, while the harmonicity of its components forces the transformed conductivity to be divergence free.

The second diffeomorphism, denoted by F_0, is obtained later from smooth boundary determination. Its only role is to put the metric in a representative whose full boundary jet agrees with the Euclidean metric. The composition $Z = Y \circ F_0$ compares this boundary normalized representative with the canonical harmonic gauge. The conductivity theorem will be applied to the perturbation associated with Y_*g, not to the metric obtained from F_0.

5.1. The canonical global harmonic gauge.

Let g be a smooth Riemannian metric on $\overline{\Omega}$ and set $A_g = (\det g)^{1/2} g^{-1}$. For every smooth function u, $-\operatorname{div}(A_g\nabla u) = (\det g)^{1/2}(-\Delta_g u)$, so the equations $-\Delta_g u = 0$ and $-\operatorname{div}(A_g\nabla u) = 0$ are equivalent.

We use Λ_g for the geometric DN map. If u_f is g-harmonic with boundary value f and V has boundary trace ψ, Green's formula gives

$$\int_{\partial\Omega} (\Lambda_g f)\psi \, dS_g = \int_\Omega A_g \nabla u_f \cdot \nabla V \, dx. \tag{5.1}$$

In particular, when the induced boundary metric is Euclidean, the geometric DN map agrees with the conductivity DN map associated with A_g.

For $1 \le j \le n$, recall that y^j is the unique solution of (1.4), and set $Y = (y^1, \dots, y^n)$. At this stage, Y is only a smooth map from $\overline{\Omega}$ to $\mathbb{R}^n$. The next proposition shows that the near-Euclidean assumption makes this map a global boundary fixing diffeomorphism. Only after this fact has been proved do we use Y as a change of coordinates.

Proposition 5.1 (Canonical global harmonic gauge)**.** *Let $n \ge 3$ and let $s_* - k < \alpha < 1$. There are constants $\varepsilon, C > 0$, depending only on n, Ω, and α, with the following property. Suppose that*

$g \in C^\infty(\overline{\Omega}; \operatorname{Sym}^2 \mathbb{R}^n)$ is a Riemannian metric satisfying $\|g - e\|_{C^{k,\alpha}(\overline{\Omega})} < \varepsilon$. Let $Y = (y^1, \ldots, y^n)$ be the map defined by (1.4). Then $Y \in \operatorname{Diff}^\infty(\overline{\Omega})$, $Y|_{\partial\Omega} = \mathrm{Id}$, and

$$\|Y - \mathrm{Id}\,\|_{C^{k+1,\alpha}(\overline{\Omega})} \le C\|g - e\|_{C^{k,\alpha}(\overline{\Omega})}. \tag{5.2}$$

Set $\widetilde{g} = Y_* g = (Y^{-1})^* g$ and $\widetilde{A} = (\det \widetilde{g})^{1/2}\widetilde{g}^{-1} = Y_* A_g = I + H$. Then $\Lambda_{\widetilde{g}} = \Lambda_g$, $\operatorname{div} H = 0$ in Ω, and

$$\|H\|_{C^{k,\alpha}(\overline{\Omega})} \le C\|g - e\|_{C^{k,\alpha}(\overline{\Omega})}. \tag{5.3}$$

Proof. Let e_j denote the jth standard basis vector of $\mathbb{R}^n$, and set $v^j = y^j - x^j$ and $v = (v^1, \ldots, v^n)$. The boundary conditions for y^j give $v|_{\partial\Omega} = 0$ and $Y = \mathrm{Id} + v$. Since $\operatorname{div}(A_g \nabla y^j) = 0$ and $\nabla x^j = e_j$, the function $v^j \in H^1_0(\Omega)$ satisfies $\operatorname{div}(A_g \nabla v^j) = -\operatorname{div}((A_g - I)e_j)$ in Ω.

In the fixed Euclidean coordinates, let $\mathcal{P} \subset \operatorname{Sym}^2 \mathbb{R}^n$ be the open set of positive definite symmetric matrices, and define $\mathcal{C} : \mathcal{P} \to \operatorname{Sym}^2 \mathbb{R}^n$ by $\mathcal{C}(q) = (\det q)^{1/2} q^{-1}$. Then $A_g(x) = \mathcal{C}(g(x))$ for every $x \in \overline{\Omega}$, while $\mathcal{C}(e) = I$.

After decreasing ε, every matrix $e + \tau(g(x) - e)$, with $x \in \overline{\Omega}$ and $0 \le \tau \le 1$, has all its eigenvalues in the fixed interval $[1/2, 3/2]$. These matrices belong to a fixed compact subset $\mathcal{K} \Subset \mathcal{P}$. For $q \in \mathcal{P}$ and $S \in \operatorname{Sym}^2 \mathbb{R}^n$, direct differentiation gives $D\mathcal{C}(q)[S] = (\det q)^{1/2}\big(\frac{1}{2}\operatorname{tr}(q^{-1}S)q^{-1} - q^{-1}Sq^{-1}\big)$. Since $\mathcal{C}$ is smooth on $\mathcal{P}$, its derivatives of every fixed finite order are uniformly bounded on $\mathcal{K}$. The fundamental theorem of calculus, applied pointwise in x, gives $A_g(x) - I = \int_0^1 D\mathcal{C}\big(e + \tau(g(x) - e)\big)[g(x) - e]\, d\tau$. After differentiating this identity in x up to order k, every resulting term is a product of derivatives of $g - e$ and a derivative of $\mathcal{C}$ evaluated at $e + \tau(g - e)$. Every term contains at least one factor involving a derivative of $g - e$ of order between zero and k. The uniform bounds for the derivatives of $\mathcal{C}$ on $\mathcal{K}$, the product estimates in $C^{k,\alpha}(\overline{\Omega})$, and the corresponding Hölder seminorm estimates give

$$\|A_g - I\|_{C^{k,\alpha}(\overline{\Omega})} \le C\|g - e\|_{C^{k,\alpha}(\overline{\Omega})}. \tag{5.4}$$

The same choice of ε places the eigenvalues of $A_g(x)$ between two fixed positive constants and gives a uniform bound for $\|A_g\|_{C^{k,\alpha}(\overline{\Omega})}$. The constants in the global boundary Schauder and C^0 estimates below may be chosen uniformly for all metrics satisfying the stated smallness condition.

Expanding the equation for v^j gives $A^{i\ell}_g \partial_i \partial_\ell v^j + (\partial_i A^{i\ell}_g)\partial_\ell v^j = -\partial_i(A^{ij}_g - \delta^{ij})$, where δ^{ij} are the entries of the identity matrix. The leading coefficients belong to $C^{k,\alpha}(\overline{\Omega})$, the first order coefficients belong to $C^{k-1,\alpha}(\overline{\Omega})$, and the right-hand side belongs to $C^{k-1,\alpha}(\overline{\Omega})$. The global boundary Schauder estimate for the zero Dirichlet problem, together with the uniform C^0 estimate, gives

$$\|v^j\|_{C^{k+1,\alpha}(\overline{\Omega})} \le C\|\partial_i(A^{ij}_g - \delta^{ij})\|_{C^{k-1,\alpha}(\overline{\Omega})} \le C\|A_g - I\|_{C^{k,\alpha}(\overline{\Omega})}. \tag{5.5}$$

Summing over $1 \le j \le n$ and using (5.4) proves (5.2). Since g and the boundary data are smooth, smooth elliptic regularity gives $Y \in C^\infty(\overline{\Omega})$.

We next prove that Y is a diffeomorphism of $\overline{\Omega}$. Since $\partial\Omega$ is smooth, the half-space extension construction, applied in boundary charts and combined with a smooth partition of unity, gives a linear extension operator that is bounded on Hölder spaces and preserves smoothness [See64]. Applied componentwise, it gives a bounded extension operator $E : C^{k+1,\alpha}(\overline{\Omega}; \mathbb{R}^n) \to C^{k+1,\alpha}(\mathbb{R}^n; \mathbb{R}^n)$ that maps smooth functions to smooth functions. Fix this operator independently of g. Choose $\chi \in C^\infty_c(\mathbb{R}^n)$ such that $\chi = 1$ in a neighborhood of $\overline{\Omega}$, and set $\widetilde{v} = \chi E v$. Then $\widetilde{v} = v$ on $\overline{\Omega}$, $\widetilde{v}$ has compact support, and $\|\widetilde{v}\|_{C^{k+1,\alpha}(\mathbb{R}^n)} \le C\|v\|_{C^{k+1,\alpha}(\overline{\Omega})}$, where C depends only on n, Ω, α, and the fixed cutoff.

For the derivative matrix, we use $\|D\widetilde{v}\|_{L^\infty(\mathbb{R}^n)} = \sup_{x \in \mathbb{R}^n} \sup_{|\xi|=1} |D\widetilde{v}(x)\xi|$. After decreasing ε, estimates (5.2), (5.4) and (5.5) give $\|D\widetilde{v}\|_{L^\infty(\mathbb{R}^n)} < 1/2$. For $0 \le s \le 1$, define

$$\widetilde{Y}_s(x) = x + s\widetilde{v}(x). \tag{5.6}$$

The line segment formula gives, for every $x, x' \in \mathbb{R}^n$,

$$|\widetilde{Y}_s(x) - \widetilde{Y}_s(x')| \geq \big(1 - s\|D\widetilde{v}\|_{L^\infty(\mathbb{R}^n)}\big)|x - x'| \geq \frac{1}{2}|x - x'|.$$

It follows that $\widetilde{Y}_s$ is injective. Its derivative is $D\widetilde{Y}_s = I + sD\widetilde{v}$. If $(I + sD\widetilde{v}(x))\xi = 0$, then $|\xi| \leq s\|D\widetilde{v}\|_{L^\infty}|\xi| < |\xi|$ unless $\xi = 0$. Thus, $D\widetilde{Y}_s(x)$ is invertible for every x, and $\widetilde{Y}_s$ is a local diffeomorphism.

Since $\widetilde{v}$ has compact support, $\widetilde{Y}_s$ agrees with the identity outside the compact set $K_0 = \operatorname{supp} \widetilde{v}$. For every compact set $L \subset \mathbb{R}^n$, the preimage $\widetilde{Y}_s^{-1}(L)$ is closed and contained in the compact set $K_0 \cup L$, so it is compact. Thus, $\widetilde{Y}_s$ is proper. Its image is open because $\widetilde{Y}_s$ is a local diffeomorphism, and it is closed because $\widetilde{Y}_s$ is proper. The image is nonempty, and $\mathbb{R}^n$ is connected, so $\widetilde{Y}_s(\mathbb{R}^n) = \mathbb{R}^n$. It follows that $\widetilde{Y}_s$ is a global diffeomorphism of $\mathbb{R}^n$. The determinant $\det D\widetilde{Y}_s(x)$ is nonzero and depends continuously on s. Since $\det D\widetilde{Y}_0(x) = 1$, one has $\det D\widetilde{Y}_s(x) > 0$ for every $x \in \mathbb{R}^n$ and every $0 \leq s \leq 1$.

Using (5.6), the boundary condition $\widetilde{v}|_{\partial\Omega} = v|_{\partial\Omega} = 0$ gives $\widetilde{Y}_s(p) = p$ for every $p \in \partial\Omega$. Let $x \in \mathbb{R}^n \setminus \partial\Omega$. The path $s \mapsto \widetilde{Y}_s(x)$ cannot meet $\partial\Omega$. Indeed, if $\widetilde{Y}_s(x) = p \in \partial\Omega$ for some s, then $\widetilde{Y}_s(p) = p$, and injectivity gives $x = p$, which is impossible. Since $\widetilde{Y}_0(x) = x$, this path stays in the connected component of $\mathbb{R}^n \setminus \partial\Omega$ containing x. The domain Ω is connected and is both open and closed in $\mathbb{R}^n \setminus \partial\Omega$, so it is one such component. For every $x \in \Omega$ and $0 \leq s \leq 1$, we have $\widetilde{Y}_s(x) \in \Omega$. This proves $\widetilde{Y}_s(\Omega) \subset \Omega$. Conversely, let $y \in \Omega$ and set $x = \widetilde{Y}_s^{-1}(y)$. The point x cannot lie on $\partial\Omega$, because $\widetilde{Y}_s$ fixes the boundary. The path $r \mapsto \widetilde{Y}_r(x)$ for $0 \leq r \leq s$ avoids $\partial\Omega$ and joins x to y, so x and y belong to the same connected component of $\mathbb{R}^n \setminus \partial\Omega$. Since $y \in \Omega$, we obtain $x \in \Omega$, and this proves $\widetilde{Y}_s(\Omega) = \Omega$. At $s = 1$, the restriction of $\widetilde{Y}_1$ to $\overline{\Omega}$ is $Y = \mathrm{Id} + v$. The global inverse $\widetilde{Y}_1^{-1}$ is smooth on $\mathbb{R}^n$, so its restriction is smooth on $\overline{\Omega}$. We have proved that $Y \in \mathrm{Diff}^\infty(\overline{\Omega})$, that $Y|_{\partial\Omega} = \mathrm{Id}$, and that Y is orientation preserving.

We may now use Y as a global change of coordinates. Write $y = Y(x)$ and set $\widetilde{g} = Y_* g = (Y^{-1})^* g$. Since Y fixes $\partial\Omega$ pointwise, invariance of the geometric DN map gives $\Lambda_{\widetilde{g}} = \Lambda_g$. At $y = Y(x)$, the transformed metric satisfies

$$\widetilde{g}(y) = DY(x)^{-T} g(x) DY(x)^{-1}, \quad \widetilde{g}^{-1}(y) = DY(x) g^{-1}(x) DY(x)^T, \quad (\det \widetilde{g}(y))^{1/2} = \frac{(\det g(x))^{1/2}}{\det DY(x)}.$$

Since $\det DY(x) > 0$, it follows that

$$A_{\widetilde{g}}(y) = (\det \widetilde{g}(y))^{1/2} \widetilde{g}^{-1}(y) = \frac{DY(x) A_g(x) DY(x)^T}{\det DY(x)} = (Y_* A_g)(y).$$

Set $\widetilde{A} = A_{\widetilde{g}} = Y_* A_g$.

We next prove the harmonic gauge condition. For $1 \leq j \leq n$, let $\pi_j(y) = y^j$ be the jth Cartesian coordinate function in the target variables. The identity $\pi_j \circ Y = y^j$ shows that π_j is the transform of the g-harmonic function y^j. Let $\phi \in C_c^\infty(\Omega)$. Since y^j is A_g-harmonic, the function $\phi \circ Y$ may be used as a test function. Using $\nabla_x y^j = DY(x)^T e_j$, $\nabla_x(\phi \circ Y) = DY(x)^T \nabla_y \phi(Y(x))$, and the change of variables $y = Y(x)$, we obtain

$$0 = \int_\Omega A_g(x) \nabla_x y^j(x) \cdot \nabla_x(\phi \circ Y)(x)\, dx = \int_\Omega \widetilde{A}(y) e_j \cdot \nabla_y \phi(y)\, dy = \int_\Omega \widetilde{A}^{ij}(y) \partial_{y_i} \phi(y)\, dy.$$

This identity holds for every $\phi \in C_c^\infty(\Omega)$, so $\partial_{y_i} \widetilde{A}^{ij} = 0$ in distributions for every j. Since $\widetilde{A}$ is smooth, the identities hold classically. Writing $\widetilde{A} = I + H$ gives $\operatorname{div} H = 0$.

It remains to prove (5.3). The estimate for Y^{-1} is needed because the pushforward coefficient is expressed in the y-variable and involves composition with Y^{-1}. Write $Y = \mathrm{Id} + v$. Then $\mathrm{Id} =$

$Y^{-1} + v \circ Y^{-1}$, and hence $Y^{-1} - \mathrm{Id} = -v \circ Y^{-1}$. Differentiating $Y \circ Y^{-1} = \mathrm{Id}$ gives $DY^{-1}(y) = \big[DY(Y^{-1}(y))\big]^{-1}$. Since $\|DY - I\|_{L^\infty(\Omega)} < 1/2$, for every $x \in \Omega$ and $\xi \in \mathbb{R}^n$, one has $|DY(x)\xi| \geq \big(1 - \|DY - I\|_{L^\infty(\Omega)}\big)|\xi| > \frac{1}{2}|\xi|$. It follows that $DY(x)$ is invertible and $|[DY(x)]^{-1}\eta| \leq 2|\eta|$ for every $\eta \in \mathbb{R}^n$, uniformly in x. In particular, DY^{-1} is uniformly bounded. Moreover, $DY^{-1} - I = -\big[(DY)\circ Y^{-1}\big]^{-1}\big[(DY - I)\circ Y^{-1}\big]$, so the first derivative of $Y^{-1} - \mathrm{Id}$ is controlled linearly by $DY - I$. The global lower Lipschitz bound for $\widetilde{Y}_1$ also gives $|\widetilde{Y}_1^{-1}(y) - \widetilde{Y}_1^{-1}(y')| \leq 2|y - y'|$ for all $y, y' \in \mathbb{R}^n$. Thus, its restriction Y^{-1} is uniformly Lipschitz on $\overline{\Omega}$.

Put $a = \|v\|_{C^{k+1,\alpha}(\overline{\Omega})}$ and assume that $a \leq a_0$, where $a_0 \leq 1$ is fixed and sufficiently small. The fixed extension estimate for v and the uniform Lipschitz bound for Y^{-1} give $\|(D^r v)\circ Y^{-1}\|_{C^\alpha(\overline{\Omega})} \leq Ca$ for $1 \leq r \leq k+1$. To start the induction in the Hölder norm, set $B(y) = DY(Y^{-1}(y))$. The matrix identity $B(y)^{-1} - B(z)^{-1} = B(y)^{-1}(B(z) - B(y))B(z)^{-1}$ and the bound $\|B^{-1}\|_{L^\infty} \leq 2$ give $[DY^{-1}]_{C^\alpha} \leq 4[Dv \circ Y^{-1}]_{C^\alpha} \leq Ca$. Together with the preceding estimate for $DY^{-1} - I$, this yields $\|DY^{-1} - I\|_{C^\alpha} \leq Ca$ and $\|DY^{-1}\|_{C^\alpha} \leq C$.

For $2 \leq \ell \leq k+1$, the higher order chain rule applied to $Y \circ Y^{-1} = \mathrm{Id}$ gives $(DY \circ Y^{-1})D^\ell Y^{-1} + \mathcal{R}_\ell = 0$. Each term in $\mathcal{R}_\ell$ is a contraction of $(D^r v)\circ Y^{-1}$ with $D^{k_1}Y^{-1}, \dots, D^{k_r}Y^{-1}$, where $2 \leq r \leq \ell$, $k_i \geq 1$, and $k_1 + \cdots + k_r = \ell$. In particular, every k_i is at most $\ell - 1$. Suppose inductively that the C^α norms of the derivatives of Y^{-1} through order $\ell - 1$ are uniformly bounded. The Hölder product estimate then gives $\|\mathcal{R}_\ell\|_{C^\alpha} \leq C_\ell a$: if the Hölder difference falls on $(D^r v)\circ Y^{-1}$, its seminorm contributes the factor a; if it falls on another factor, the L^∞ norm of $(D^r v)\circ Y^{-1}$ still contributes this factor. Since $(DY \circ Y^{-1})^{-1} = DY^{-1}$, we obtain $D^\ell Y^{-1} = -DY^{-1}\mathcal{R}_\ell$ and $\|D^\ell Y^{-1}\|_{C^\alpha} \leq C_\ell a$. This completes the induction, including the Hölder estimate at order $k+1$.

Finally, $Y^{-1} - \mathrm{Id} = -v \circ Y^{-1}$ gives $\|Y^{-1} - \mathrm{Id}\|_{L^\infty} \leq \|v\|_{L^\infty}$. Combining this with the first derivative estimate and the estimates above gives

$$\|Y^{-1} - \mathrm{Id}\|_{C^{k+1,\alpha}(\overline{\Omega})} \leq C\|Y - \mathrm{Id}\|_{C^{k+1,\alpha}(\overline{\Omega})}. \tag{5.7}$$

The constant is uniform when $\|Y - \mathrm{Id}\|_{C^{k+1,\alpha}(\overline{\Omega})}$ is sufficiently small.

Define $\mathcal{T}(P,Q) = \frac{PQP^T}{\det P}$ for matrices P with positive determinant and symmetric positive definite matrices Q. The map $\mathcal{T}$ is smooth in a neighborhood of (I,I) and satisfies $\mathcal{T}(I,I) = I$. The pushforward formula is $\widetilde{A} = \mathcal{T}(DY, A_g)\circ Y^{-1}$. By (5.7), the maps Y^{-1} remain in a fixed bounded subset of $C^{k+1,\alpha}(\overline{\Omega})$, and composition with Y^{-1} is uniformly bounded on $C^{k,\alpha}(\overline{\Omega})$. Since $C^{k,\alpha}(\overline{\Omega})$ is a Banach algebra and $\mathcal{T}$ is smooth near (I,I), we obtain

$$\begin{aligned}\|H\|_{C^{k,\alpha}(\overline{\Omega})} &= \|\widetilde{A} - I\|_{C^{k,\alpha}(\overline{\Omega})} \leq C\|\mathcal{T}(DY, A_g) - I\|_{C^{k,\alpha}(\overline{\Omega})}\\ &\leq C\big(\|DY - I\|_{C^{k,\alpha}(\overline{\Omega})} + \|A_g - I\|_{C^{k,\alpha}(\overline{\Omega})}\big) \leq C\|g - e\|_{C^{k,\alpha}(\overline{\Omega})},\end{aligned}$$

where the last inequality follows from (5.2) and (5.4). This proves (5.3). □

From now on, Y denotes the fixed global diffeomorphism constructed in Proposition 5.1. It is not an arbitrary coordinate change. Its global invertibility allows the metric and conductivity to be transformed on the whole domain, while the harmonicity of its components gives the divergence free condition for H. The perturbation used in the remainder of the metric argument is always the tensor $H = A_{Y_*g} - I$ associated with this canonical map.

5.2. **Boundary determination and comparison of the two gauges.** Boundary determination provides a second boundary fixing diffeomorphism used only to normalize the full boundary jet. We record the required statement. For a smooth tensor T, the notation $j^\infty_{\partial\Omega}T = 0$ means that all derivatives of its coordinate coefficients vanish on $\partial\Omega$ in every boundary chart, or equivalently that T vanishes to infinite order at the boundary.

Proposition 5.2 (Boundary determination)**.** *Let $g \in C^\infty(\overline{\Omega}; \operatorname{Sym}^2 \mathbb{R}^n)$ be a Riemannian metric and suppose that (1.3) holds. Then g and e induce the same metric on $\partial\Omega$, meaning that $\xi^T g(x)\eta = \xi \cdot \eta$ for every $x \in \partial\Omega$ and all $\xi, \eta \in \mathbb{R}^n$ satisfying $\xi \cdot \nu(x) = \eta \cdot \nu(x) = 0$. Moreover, there exists $F_0 \in \operatorname{Diff}^\infty(\overline{\Omega})$ such that $F_0|_{\partial\Omega} = \operatorname{Id}$ and*

$$j^\infty_{\partial\Omega}(F_0^* g - e) = 0. \tag{5.8}$$

Proof. The principal symbol calculation for the DN map [LU89, Proposition 1.3] shows that (1.3) determines the induced boundary metric. In particular, g and e induce the same metric on $\partial\Omega$.

For $0 \le s \le 1$, set $g_s = (1-s)e + sg$. The matrix $g_s(x)$ is positive definite for every $x \in \overline{\Omega}$, so $(g_s)_{0 \le s \le 1}$ is a smooth family of Riemannian metrics. Let ν be the outward unit normal on $\partial\Omega$ with respect to e, and let ν_s be the outward unit normal with respect to g_s. In the fixed Euclidean coordinates, $\nu_s(p) = g_s(p)^{-1}\nu(p)/\sqrt{\nu(p)^T g_s(p)^{-1}\nu(p)}$. Define $\Phi_s(p,t) = \exp_p^{g_s}(-t\nu_s(p))$. Thus, $\Phi_s(p,t)$ is the point reached at time t by the g_s unit speed geodesic starting from p in the inward normal direction. The vector field ν_s and the geodesic equation depend smoothly on (s,p). Compactness of $[0,1] \times \partial\Omega$ allows us to choose $\varepsilon > 0$ such that every Φ_s is a smooth embedding of $\partial\Omega \times [0,\varepsilon)$ onto a neighborhood of $\partial\Omega$ in $\overline{\Omega}$, and the map $(s,p,t) \mapsto \Phi_s(p,t)$ is smooth. Write $\Phi_e = \Phi_0$ and $\Phi_g = \Phi_1$.

In these normal parametrizations, the endpoint metrics have the forms $\Phi_e^* e = dt^2 + h_e(t)$ and $\Phi_g^* g = dt^2 + h_g(t)$, where $h_e(t)$ and $h_g(t)$ are smooth families of metrics on $\partial\Omega$. Equality of the DN maps implies equality of their full symbols. For scalar functions, the boundary normal recursion in [JL05, Theorem 1.1(ii)] gives $\partial_t^m h_g(0) = \partial_t^m h_e(0)$ for every $m \ge 0$. These are equalities of smooth symmetric tensors on $\partial\Omega$.

Set $U_e = \Phi_e(\partial\Omega \times [0,\varepsilon))$ and define $F_{\mathrm{col}} : U_e \to \overline{\Omega}$ by $F_{\mathrm{col}}(\Phi_e(p,t)) = \Phi_g(p,t)$. Equivalently, $F_{\mathrm{col}} = \Phi_g \circ \Phi_e^{-1}$. Since $\Phi_e(p,0) = \Phi_g(p,0) = p$, the map F_{col} fixes $\partial\Omega$ pointwise. Moreover, $\Phi_e^*(F_{\mathrm{col}}^* g - e) = h_g(t) - h_e(t)$. Every normal derivative of the right-hand side vanishes at $t = 0$. Since each such normal derivative is the zero tensor field on $\partial\Omega$, all of its tangential derivatives vanish as well. It follows that

$$j^\infty_{\partial\Omega}(F_{\mathrm{col}}^* g - e) = 0. \tag{5.9}$$

Choose $\chi \in C^\infty([0,\varepsilon))$ equal to 1 for $0 \le t \le \varepsilon/2$ and to 0 for $3\varepsilon/4 \le t < \varepsilon$. Set $U_s = \Phi_s(\partial\Omega \times [0,\varepsilon))$, define $V_s(\Phi_s(p,t)) = \chi(t)\partial_s\Phi_s(p,t)$ on U_s, and extend V_s by zero to $\overline{\Omega} \setminus U_s$. Since χ vanishes near the inner edge of the collar, this defines a smooth time dependent vector field on $\overline{\Omega}$. Moreover, $\Phi_s(p,0) = p$ is independent of s, so $V_s(p) = \partial_s\Phi_s(p,0) = 0$ for every $p \in \partial\Omega$.

For each $x \in \overline{\Omega}$, define $\widetilde{f}_s(x)$ by the initial value problem

$$\begin{cases} \partial_s \widetilde{f}_s(x) = V_s(\widetilde{f}_s(x)), & 0 \le s \le 1, \\ \widetilde{f}_0(x) = x. \end{cases} \tag{5.10}$$

Boundary points give constant solutions, so uniqueness implies $\widetilde{f}_s(p) = p$ for every $p \in \partial\Omega$. The same uniqueness, applied backward in time, prevents an interior trajectory from reaching the boundary in finite time. Smoothness of V_s and compactness of $\overline{\Omega}$ give solutions for all $0 \le s \le 1$, with smooth dependence on (s,x). Moreover, for each fixed s, solving $\partial_r\gamma(r) = V_r(\gamma(r))$ backward from $\gamma(s) = y$ gives the smooth inverse $\widetilde{f}_s^{-1}(y) = \gamma(0)$. Thus, $\widetilde{f}_s \in \operatorname{Diff}^\infty(\overline{\Omega})$ and $\widetilde{f}_s|_{\partial\Omega} = \operatorname{Id}$.

Fix $p \in \partial\Omega$ and $0 \le t \le \varepsilon/2$. Since $\chi(t) = 1$, the curve $\gamma(s) = \Phi_s(p,t)$ satisfies $\partial_s\gamma(s) = V_s(\gamma(s))$ and $\gamma(0) = \Phi_e(p,t)$. Uniqueness in (5.10) gives $\widetilde{f}_s(\Phi_e(p,t)) = \Phi_s(p,t)$. In particular, $F_0 = \widetilde{f}_1$ satisfies $F_0(\Phi_e(p,t)) = \Phi_g(p,t)$ for $0 \le t \le \varepsilon/2$, so it agrees with F_{col} near $\partial\Omega$. Formula (5.9) now proves (5.8). $\square$

The conductivity associated with F_0^*g need not be divergence free. We use this auxiliary gauge only to prove boundary flatness in the canonical harmonic coordinates, without requiring any quantitative estimate for F_0.

Lemma 5.3 (Boundary flatness in the canonical harmonic gauge)**.** *Assume the hypotheses of Proposition 5.1 and suppose that $\Lambda_g = \Lambda_e$. Let Y be the canonical harmonic map constructed there, set $\widetilde{g} = Y_* g$, and write $A_{\widetilde{g}} = I + H$. Then*

$$j^\infty_{\partial\Omega}(\widetilde{g} - e) = 0, \quad j^\infty_{\partial\Omega} H = 0. \tag{5.11}$$

Proof. Let F_0 be given by Proposition 5.2, and set $g_0 = F_0^* g$ and $Z = Y \circ F_0$. Then $\Lambda_{g_0} = \Lambda_e$ and

$$j^\infty_{\partial\Omega}(g_0 - e) = 0. \tag{5.12}$$

The change of coordinates gives $\Delta_{g_0} Z^j = 0$ in Ω and $Z^j = x^j$ on $\partial\Omega$. Since $g_0 = e$ on the boundary, their unit normals agree there. Thus, equality of the DN maps gives $\partial_{\nu_{g_0}} Z^j = \Lambda_{g_0}(x^j|_{\partial\Omega}) = \Lambda_e(x^j|_{\partial\Omega}) = \partial_{\nu_{g_0}} x^j$. This implies that $w^j = Z^j - x^j$ has zero Dirichlet and Neumann data and satisfies

$$\Delta_{g_0} w^j = f^j, \quad f^j = -\Delta_{g_0} x^j, \quad j^\infty_{\partial\Omega} f^j = 0, \tag{5.13}$$

where the flatness of f^j follows from (5.12) and $\Delta_e x^j = 0$.

In a boundary chart (x', t) with $t = 0$ on $\partial\Omega$, the zero Dirichlet data imply that all tangential derivatives of w^j vanish there. The zero Neumann data then give $\partial_t w^j = 0$, since the unit normal has a nonzero component in the transverse direction ∂_t. Let $a^{tt} > 0$ be the coefficient of ∂_t^2 in Δ_{g_0}. Suppose that all derivatives of w^j involving at most $m+1$ derivatives in t vanish on the boundary. Applying ∂_t^m to (5.13) and restricting to $t = 0$ gives $a^{tt}\partial_t^{m+2} w^j = 0$: all remaining terms involve at most $m+1$ derivatives in t of w^j or a derivative of the flat source f^j. Tangential differentiation and induction yield

$$j^\infty_{\partial\Omega}(Z - \mathrm{Id}) = 0, \tag{5.14}$$

with the jet understood componentwise.

The identity $Z^{-1} - \mathrm{Id} = -(Z - \mathrm{Id}) \circ Z^{-1}$ also gives $j^\infty_{\partial\Omega}(Z^{-1} - \mathrm{Id}) = 0$. Since $\widetilde{g} = Z_* g_0$, we have $\widetilde{g}(y) = DZ^{-1}(y)^T g_0(Z^{-1}(y)) DZ^{-1}(y)$. The chain and product rules, together with (5.12) and (5.14), give $j^\infty_{\partial\Omega}(\widetilde{g} - e) = 0$. The smooth dependence of $A_{\widetilde{g}} = (\det \widetilde{g})^{1/2} \widetilde{g}^{-1}$ on $\widetilde{g}$ then gives $j^\infty_{\partial\Omega} H = 0$. □

Hence, the perturbation H in the canonical harmonic gauge is divergence free, satisfies (5.3), and vanishes to infinite order at the boundary. The auxiliary diffeomorphism F_0 is not used below.

5.3. **Exterior extension and transfer of the DN map.** From now on, H always denotes the perturbation determined by the canonical harmonic gauge $\widetilde{g} = Y_* g$, so that $A_{\widetilde{g}} = I + H$. Fix a smooth bounded connected domain Ω_{e} satisfying $\overline{\Omega} \Subset \Omega_{\mathrm{e}}$, and define

$$H_{\mathrm{e}}(x) = \begin{cases} H(x) & \text{if } x \in \Omega, \\ 0 & \text{if } x \in \Omega_{\mathrm{e}} \setminus \Omega, \end{cases} \qquad A_{\mathrm{e}} = I + H_{\mathrm{e}}. \tag{5.15}$$

The two definitions agree on $\partial\Omega$ because $H|_{\partial\Omega} = 0$.

Lemma 5.4 (Exterior extension and transfer of the DN map)**.** *Assume the hypotheses of Lemma 5.3. Then $H_{\mathrm{e}} \in C_c^\infty(\Omega_{\mathrm{e}}; \mathrm{Sym}^2\, \mathbb{R}^n)$, $\operatorname{supp} H_{\mathrm{e}} \subset \overline{\Omega} \Subset \Omega_{\mathrm{e}}$, $\operatorname{div} H_{\mathrm{e}} = 0$, and A_{e} is a real symmetric uniformly elliptic conductivity. Moreover,*

$$\Lambda^{\partial\Omega_{\mathrm{e}}}_{A_{\mathrm{e}}} = \Lambda^{\partial\Omega_{\mathrm{e}}}_{I}. \tag{5.16}$$

After extending H_{e} by zero outside Ω_{e}, one also has

$$\|H_{\mathrm{e}}\|_{H^{s_*}(\mathbb{R}^n)} \le C \|g - e\|_{C^{k,\alpha}(\overline{\Omega})}, \tag{5.17}$$

where $C > 0$ depends only on n, Ω, and α.

Proof. Formula (5.11) shows that all derivatives of H vanish on $\partial\Omega$, and the piecewise definition (5.15) is smooth across the interface. Since $H_{\mathrm{e}} = 0$ on $\Omega_{\mathrm{e}} \setminus \Omega$, its support is contained in $\overline{\Omega} \Subset \Omega_{\mathrm{e}}$. The tensor H_{e} belongs to $C_c^\infty(\Omega_{\mathrm{e}}; \mathrm{Sym}^2 \mathbb{R}^n)$. The matrix A_{e} equals $A_{\widetilde{g}}$ in Ω and I outside Ω, so it is real symmetric and uniformly elliptic.

We verify the divergence condition across the interface. Let $\phi \in C_c^\infty(\Omega_{\mathrm{e}}; \mathbb{R}^n)$. Since $\operatorname{div} H = 0$ in Ω and $H = 0$ on $\partial\Omega$, integration by parts gives

$$\int_{\Omega_{\mathrm{e}}} H_{\mathrm{e}}^{ij} \partial_i \phi_j \, dx = \int_\Omega H^{ij} \partial_i \phi_j \, dx = \int_{\partial\Omega} H^{ij} \nu_i \phi_j \, dS - \int_\Omega (\partial_i H^{ij}) \phi_j \, dx = 0.$$

This proves $\operatorname{div} H_{\mathrm{e}} = 0$ in distributions. Since H_{e} is smooth, the identity also holds classically.

We next identify the conductivity DN map on the inner boundary. Since Y fixes $\partial\Omega$ pointwise, the invariance of the geometric DN map gives $\Lambda_{\widetilde{g}} = \Lambda_g = \Lambda_e$. Formula (5.11) gives $\widetilde{g} = e$ on $\partial\Omega$, so the induced boundary measures for $\widetilde{g}$ and e agree. Applying the weak identity (5.1) to $\widetilde{g}$ and to e identifies the geometric DN maps with the conductivity DN maps associated with $I + H$ and I. We obtain

$$\Lambda_{I+H}^{\partial\Omega} = \Lambda_I^{\partial\Omega}. \tag{5.18}$$

Fix $f \in H^{1/2}(\partial\Omega_{\mathrm{e}})$, and let $u \in H^1(\Omega_{\mathrm{e}})$ be the A_{e}-harmonic function with boundary trace f on $\partial\Omega_{\mathrm{e}}$. Set $q = u|_{\partial\Omega}$. The restriction of u to Ω is $(I + H)$-harmonic with boundary trace q. Let $v \in H^1(\Omega)$ be the Euclidean harmonic function with the same trace q, and define

$$U = \begin{cases} v & \text{in } \Omega, \\ u & \text{in } \Omega_{\mathrm{e}} \setminus \overline{\Omega}. \end{cases}$$

The two pieces have the same trace on $\partial\Omega$, so $U \in H^1(\Omega_{\mathrm{e}})$.

Let $\varphi \in C_c^\infty(\Omega_{\mathrm{e}})$. Since u satisfies the global weak equation in Ω_{e} and $A_{\mathrm{e}} = I$ in $\Omega_{\mathrm{e}} \setminus \overline{\Omega}$, we have

$$\int_{\Omega_{\mathrm{e}} \setminus \overline{\Omega}} \nabla u \cdot \nabla \varphi \, dx = -\int_\Omega (I + H) \nabla u \cdot \nabla \varphi \, dx = -\langle \Lambda_{I+H} q, \varphi|_{\partial\Omega} \rangle.$$

Since v is Euclidean harmonic in Ω, we have $\int_\Omega \nabla v \cdot \nabla \varphi \, dx = \langle \Lambda_I q, \varphi|_{\partial\Omega} \rangle$. Adding these identities and using (5.18) gives $\int_{\Omega_{\mathrm{e}}} \nabla U \cdot \nabla \varphi \, dx = \langle \Lambda_I q, \varphi|_{\partial\Omega} \rangle - \langle \Lambda_{I+H} q, \varphi|_{\partial\Omega} \rangle = 0$. The function U is Euclidean harmonic in Ω_{e} and has boundary trace f on $\partial\Omega_{\mathrm{e}}$. Uniqueness for the Dirichlet problem shows that U is the harmonic function with boundary value f.

By definition, $U = u$ throughout $\Omega_{\mathrm{e}} \setminus \overline{\Omega}$. This region contains a full neighborhood of $\partial\Omega_{\mathrm{e}}$, and $A_{\mathrm{e}} = I$ there. The outer conormal derivative of u agrees with the normal derivative of U on $\partial\Omega_{\mathrm{e}}$. Since this holds for every boundary value f, formula (5.16) follows.

Finally, we prove the quantitative estimate for the zero extension. Set $\sigma = s_* - k$, so $\sigma = 0$ when n is even and $\sigma = 1/2$ when n is odd. By (5.11), for every multi-index β with $|\beta| \le k$, the distributional derivative $D^\beta H_{\mathrm{e}}$ is the zero extension of $D^\beta H$. Since Ω is bounded, the integer-order Sobolev norm satisfies $\|H_{\mathrm{e}}\|_{H^k(\mathbb{R}^n)} \le C \|H\|_{C^k(\overline{\Omega})}$. This proves the required estimate when $\sigma = 0$.

Suppose that $\sigma = 1/2$, and let $|\beta| = k$. Write $F = D^\beta H_{\mathrm{e}}$ and $f = D^\beta H$. The field f belongs to $C^\alpha(\overline{\Omega})$ and vanishes on $\partial\Omega$. Its zero extension satisfies

$$\|F\|_{C^\alpha(\mathbb{R}^n)} \le C \|H\|_{C^{k,\alpha}(\overline{\Omega})}. \tag{5.19}$$

To verify the seminorm bound across the boundary, let $x \in \Omega$ and $y \notin \Omega$, and choose $p \in \partial\Omega$ with $|x - p| = \operatorname{dist}(x, \partial\Omega)$. Since $f(p) = 0$ and $|x - p| \le |x - y|$, we have $|F(x) - F(y)| = |f(x) - f(p)| \le [f]_{C^\alpha(\overline{\Omega})} |x - y|^\alpha$. If both points lie in Ω, the original Hölder estimate applies; if both lie outside Ω, the difference is zero. This proves (5.19).

For $0<\sigma<1$, Plancherel's theorem and the change of variables in the Fourier integral give

$$\int_{\mathbb{R}^n} \frac{\|F(\cdot+z)-F\|^2_{L^2(\mathbb{R}^n)}}{|z|^{n+2\sigma}}\,dz = c_{n,\sigma}\int_{\mathbb{R}^n}|\xi|^{2\sigma}|\widehat{F}(\xi)|^2\,d\xi, \tag{5.20}$$

where $c_{n,\sigma}>0$. Indeed, the Fourier multiplier of $F(\cdot+z)-F(\cdot)$ is $e^{\mathrm{i}z\cdot\xi}-1$, and integration of its squared modulus against $|z|^{-n-2\sigma}\,dz$ gives a positive constant times $|\xi|^{2\sigma}$.

The support of $F(\cdot+z)-F(\cdot)$ is contained in $(\overline{\Omega}-z)\cup\overline{\Omega}$, whose measure is at most $2|\Omega|$. For $|z|\le 1$, formula (5.19) gives $\|F(\cdot+z)-F(\cdot)\|^2_{L^2}\le C\|H\|^2_{C^{k,\alpha}(\overline{\Omega})}|z|^{2\alpha}$. For $|z|>1$, translation invariance of the L^2 norm gives $\|F(\cdot+z)-F(\cdot)\|^2_{L^2}\le 4\|F\|^2_{L^2}\le C\|H\|^2_{C^{k,\alpha}(\overline{\Omega})}$. Hence,

$$\int_{\mathbb{R}^n}\frac{\|F(\cdot+z)-F\|^2_{L^2}}{|z|^{n+2\sigma}}\,dz\le C\|H\|^2_{C^{k,\alpha}(\overline{\Omega})}\left(\int_0^1 r^{2\alpha-2\sigma-1}\,dr+\int_1^\infty r^{-2\sigma-1}\,dr\right)\le C\|H\|^2_{C^{k,\alpha}(\overline{\Omega})}, \tag{5.21}$$

where the last inequality uses $\alpha>\sigma$.

Applying (5.20) and (5.21) to every derivative of order k, and using the equivalence of $|\xi|^{2k}$ with $\sum_{|\beta|=k}|\xi^\beta|^2$, gives $\|H_{\mathrm{e}}\|_{H^{k+\sigma}(\mathbb{R}^n)}\le C\|H\|_{C^{k,\alpha}(\overline{\Omega})}$. Together with the case $\sigma=0$ and Proposition 5.1, this proves

$$\|H_{\mathrm{e}}\|_{H^{s_*}(\mathbb{R}^n)}\le C\|H\|_{C^{k,\alpha}(\overline{\Omega})}\le C\|g-e\|_{C^{k,\alpha}(\overline{\Omega})},$$

where the constants depend only on n, Ω, and α. This proves (5.17). $\square$

5.4. **Proof of the rigidity result.**

Proof of Theorem 1.1. Let $\varepsilon>0$ be the smallness threshold in Proposition 5.1. Choose a smooth bounded connected domain Ω_{e} with $\overline{\Omega}\Subset\Omega_{\mathrm{e}}$ and a compact set K_{e} satisfying $\overline{\Omega}\subset K_{\mathrm{e}}\Subset\Omega_{\mathrm{e}}$. These choices are fixed throughout the proof and depend only on Ω. Let $\varepsilon_0>0$ be the threshold in Theorem 1.2, with the domain Ω and the compact set K in that theorem replaced by Ω_{e} and K_{e}, respectively, and let C_{e} be the constant in (5.17). Choose $\varepsilon_1>0$ so that $\varepsilon_1\le\varepsilon$ and $C_{\mathrm{e}}\varepsilon_1<\varepsilon_0$.

Suppose that the assumptions of Theorem 1.1 hold. Proposition 5.1 gives the canonical boundary fixing diffeomorphism Y and the conductivity $A_{\widetilde{g}}=I+H$ associated with $\widetilde{g}=Y_*g$. Lemma 5.3 shows that H vanishes to infinite order at $\partial\Omega$. Lemma 5.4 gives an extension H_{e} satisfying $\operatorname{supp}H_{\mathrm{e}}\subset\overline{\Omega}\subset K_{\mathrm{e}}$, $\operatorname{div}H_{\mathrm{e}}=0$, and $\Lambda^{\partial\Omega_{\mathrm{e}}}_{I+H_{\mathrm{e}}}=\Lambda^{\partial\Omega_{\mathrm{e}}}_{I}$. Using (1.2), it also gives

$$\|H_{\mathrm{e}}\|_{H^{s_*}(\mathbb{R}^n)}\le C_{\mathrm{e}}\|g-e\|_{C^{k,\alpha}(\overline{\Omega})}<C_{\mathrm{e}}\varepsilon_1<\varepsilon_0.$$

Theorem 1.2 gives $H_{\mathrm{e}}=0$. In particular, $A_{\widetilde{g}}=I$ in Ω. Since $A_{\widetilde{g}}=(\det\widetilde{g})^{1/2}\widetilde{g}^{-1}$, taking determinants gives $\det A_{\widetilde{g}}=(\det\widetilde{g})^{(n-2)/2}$. Thus, for $n\ge 3$, $\widetilde{g}=(\det A_{\widetilde{g}})^{1/(n-2)}A_{\widetilde{g}}^{-1}$. Since $A_{\widetilde{g}}=I$, we obtain $\widetilde{g}=e$. Finally, $\widetilde{g}=Y_*g=(Y^{-1})^*g$ implies $g=Y^*e$.

It remains to prove the uniqueness assertion. Let $F\in\mathrm{Diff}^\infty(\overline{\Omega})$ satisfy $F|_{\partial\Omega}=\mathrm{Id}$ and $g=F^*e$. For each $1\le j\le n$, the function $F^j=x^j\circ F$ satisfies $-\Delta_gF^j=(-\Delta_ex^j)\circ F=0$ and $F^j=x^j$ on $\partial\Omega$. The function y^j defining the canonical harmonic map Y solves the same Dirichlet problem. Dirichlet uniqueness gives $F^j=y^j$ for every $1\le j\le n$, so $F=Y$. $\square$

Corollary 5.5 (Rigidity in harmonic coordinates)**.** *Under the assumptions of Theorem 1.1, suppose in addition that*

$$\partial_i\big((\det g)^{1/2}g^{ij}\big)=0\quad \textit{for } 1\le j\le n. \tag{5.22}$$

Then $g=e$.

Proof. Each Cartesian coordinate function x^j is g-harmonic under (5.22) and has the same boundary value as y^j. Dirichlet uniqueness gives $y^j = x^j$ for every $1 \leq j \leq n$, so $Y = \mathrm{Id}$. The proof of Theorem 1.1 gives $Y_*g = e$, and hence $g = e$. □

Remark 5.6 (The canonical rigidity diffeomorphism). For a fixed metric g, the boundary fixing diffeomorphism in Theorem 1.1 is uniquely determined and is equal to the canonical harmonic map Y. This does not remove the usual diffeomorphism invariance of the anisotropic Calderón problem. Indeed, if $F \in \mathrm{Diff}^\infty(\overline{\Omega})$ fixes the boundary and $g_F = F^*e$, then $\Lambda_{g_F} = \Lambda_e$, while the canonical harmonic map associated with g_F is precisely F. The boundary data determine the metric only up to a boundary fixing diffeomorphism, whereas the harmonic construction identifies the unique rigidity map after a particular representative g has been fixed.

Statements and declarations

Data availability statement. No datasets were generated or analyzed in this study.

Conflict of interest. The author declares no conflict of interest.

Acknowledgments. The author is deeply grateful to Mikko Salo for suggesting a direct comparison with the Calderón exponential solution, which provided an important starting point for the argument, and for generously reading the manuscript and offering valuable comments.

After completing this work, the author became aware of the preprint by P. Stefanov [Ste26], which establishes a closely related rigidity result for the anisotropic Calderón problem near the Euclidean metric. The present work was carried out independently.

The author is partially supported by the National Science and Technology Council (NSTC), Taiwan, under the project 113-2115-M-A49-017-MY3. The author also acknowledges financial support from the Alexander von Humboldt Foundation through the Henriette Herz Scouting Programme, hosted by Universität Duisburg-Essen, Germany. The author acknowledges the use of AI tools. All mathematical arguments and proofs in the final manuscript were checked and written by the author.

References

[AFO22] Spyros Alexakis, Ali Feizmohammadi, and Lauri Oksanen. Lorentzian Calderón problem under curvature bounds. *Invent. Math.*, 229(1):87–138, 2022.

[AFO25] Spyros Alexakis, Ali Feizmohammadi, and Lauri Oksanen. Lorentzian Calderón problem near the Minkowski geometry. *J. Eur. Math. Soc. (JEMS)*, 27(9):3771–3792, 2025.

[ALP05] Kari Astala, Matti Lassas, and Lassi Päivärinta. Calderón's inverse problem for anisotropic conductivity in the plane. *Comm. Partial Differential Equations*, 30(1–3):207–224, 2005.

[Cal80] Alberto P. Calderón. On an inverse boundary value problem. In *Seminar on Numerical Analysis and Its Applications to Continuum Physics*, number 12 in Coleção Atas, pages 65–73. Sociedade Brasileira de Matemática, Rio de Janeiro, 1980.

[CJLT26] Lu Chen, Yan Jiang, Hongyu Liu, and Longyue Tao. Quasianalyticity and geometric rigidity in anisotropic Calderón's problem. *Preprint*, 2026. arXiv:2608.15813.

[DEH+26] Thierry Daudé, Alberto Enciso, Bernard Helffer, Niky Kamran, and François Nicoleau. A sharp regularity threshold for uniqueness in Riemannian Calderón-type problems, 2026. arXiv:2605.21705.

[DKN20] Thierry Daudé, Niky Kamran, and François Nicoleau. On nonuniqueness for the anisotropic Calderón problem with partial data. *Forum Math. Sigma*, 8:e7, 2020.

[FKLS16] David Dos Santos Ferreira, Yaroslav Kurylev, Matti Lassas, and Mikko Salo. The Calderón problem in transversally anisotropic geometries. *J. Eur. Math. Soc. (JEMS)*, 18(11):2579–2626, 2016.

[FKSU09] David Dos Santos Ferreira, Carlos E. Kenig, Mikko Salo, and Gunther Uhlmann. Limiting Carleman weights and anisotropic inverse problems. *Invent. Math.*, 178(1):119–171, 2009.

[FSU25] Joel S. Feldman, Mikko Salo, and Gunther Uhlmann. *The Calderón Problem: An Introduction*, volume 253 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2025.

[GKLU09] Allan Greenleaf, Yaroslav Kurylev, Matti Lassas, and Gunther Uhlmann. Invisibility and inverse problems. *Bull. Amer. Math. Soc. (N.S.)*, 46(1):55–97, 2009.

[JL05] Mark S. Joshi and William R. B. Lionheart. An inverse boundary value problem for harmonic differential forms. *Asymptot. Anal.*, 41(2):93–106, 2005.

[KU14] Katya Krupchyk and Gunther Uhlmann. Uniqueness in an inverse boundary problem for a magnetic Schrödinger operator with a bounded magnetic potential. *Comm. Math. Phys.*, 327(3):993–1009, 2014.

[LLS20] Matti Lassas, Tony Liimatainen, and Mikko Salo. The Poisson embedding approach to the Calderón problem. *Math. Ann.*, 377(1–2):19–67, 2020.

[LU89] John M. Lee and Gunther Uhlmann. Determining anisotropic real-analytic conductivities by boundary measurements. *Comm. Pure Appl. Math.*, 42(8):1097–1112, 1989.

[LU01] Matti Lassas and Gunther Uhlmann. On determining a Riemannian manifold from the Dirichlet-to-Neumann map. *Ann. Sci. Ècole Norm. Sup. (4)*, 34(5):771–787, 2001.

[ORS26a] Lauri Oksanen, Rakesh, and Mikko Salo. Fixed angle inverse scattering with non-constant velocity. *Preprint*, 2026. arXiv:2608.13670.

[ORS26b] Lauri Oksanen, Rakesh, and Mikko Salo. Rigidity in the Lorentzian Calderón problem with formally determined data. *Comm. Amer. Math. Soc.*, 6:749–792, 2026.

[ORS26c] Lauri Oksanen, Rakesh, and Mikko Salo. Semiglobal uniqueness for the Lorentzian Calderón problem. *Preprint*, 2026. arXiv:2608.13116.

[See64] R. Thomas Seeley. Extension of C^∞ functions defined in a half space. *Proc. Amer. Math. Soc.*, 15:625–626, 1964.

[Ste26] Plamen Stefanov. Local rigidity of the Euclidean metric for the anisotropic Calderón problem. Preprint, arXiv:2609.13400, 2026.

[SU87] John Sylvester and Gunther Uhlmann. A global uniqueness theorem for an inverse boundary value problem. *Ann. of Math. (2)*, 125(1):153–169, 1987.

[SU91] John Sylvester and Gunther Uhlmann. Inverse problems in anisotropic media. In *Inverse Scattering and Applications*, volume 122 of *Contemporary Mathematics*, pages 105–117. American Mathematical Society, Providence, RI, 1991.

[Uhl92] Gunther Uhlmann. Inverse boundary value problems and applications. In *Méthodes semi-classiques Volume 1 – École d'Été (Nantes, juin 1991)*, number 207 in Astérisque, pages 153–211. Société mathématique de France, 1992.

[UW26] Gunther Uhlmann and Yiran Wang. The Calderón problem for near-Euclidean metrics. *Preprint*, 2026. arXiv:2606.26540.

Department of Applied Mathematics, National Yang Ming Chiao Tung University, Hsinchu, Taiwan; Fakultät für Mathematik, University of Duisburg-Essen, Essen, Germany

Email address: yihsuanlin3@gmail.com