CORRIGENDUM: RADIATING AND NON-RADIATING SOURCES IN ELASTICITY

EMILIA BLÅSTEN AND YI-HSUAN LIN

Abstract. In this short note, we offer further discussions about the corner scattering of our earlier work [2, Section 3], and we give more details of the proofs of all theorems. In particular we correct a missing smoothness assumption in a lemma for dimension reduction, and we prove that the smoothness is available by elliptic regularity.

Keywords Inverse source problem, elastic waves, Navier's equation, exponential solutions, transmission eigenfunctions

Mathematics Subject Classification (2010): 35P25, 78A46, 74B05 (primary); 51M20 (secondary).

Contents

1. Introduction

In this paper, our aim is to give more detailed proofs of our earlier results in [2], in particular to correct a missing smoothness assumption in Lemma 2.4 and show that it follows from our theorems' assumptions. Let λ, μ be the Lamé constants satisfying the following strong convexity condition

$$\mu > 0 \text{ and } n\lambda + 2\mu > 0 \tag{1.1}$$

in dimensions $n = 2, 3$. Let $\boldsymbol{f} \in \mathbb{C}^n$ be an external force, which is assumed to be compactly supported. More specifically we are interested in forces applied to a subregion, which are denoted by the functions $\boldsymbol{f} = \chi_\Omega \boldsymbol{\varphi}$, where χ_Ω is the characteristic function of a bounded Lipschitz domain Ω in $\mathbb{R}^n$ and $\boldsymbol{\varphi} \in L^\infty(\mathbb{R}^n; \mathbb{C}^n)$. Given an angular frequency $\omega > 0$, let $\boldsymbol{u}(x) = (u_\ell(x))_{\ell=1}^n$ be the displacement vector field. Then the time-harmonic elastic system is

$$\lambda \Delta \boldsymbol{u} + (\lambda + \mu)\nabla\nabla \cdot \boldsymbol{u} + \omega^2 \boldsymbol{u} = \boldsymbol{f} \text{ in } \mathbb{R}^n. \tag{1.2}$$

Via the well-known Helmholtz decomposition in $\mathbb{R}^n \setminus \overline{\Omega}$, one can see that the scattered field can be decomposed as

$$\boldsymbol{u} = \boldsymbol{u}_p + \boldsymbol{u}_s \text{ in } \mathbb{R}^n \setminus \overline{\Omega},$$

with

$$\boldsymbol{u}_p = -\frac{1}{\omega_p^2}\nabla(\nabla \cdot \boldsymbol{u}) \text{ and } \boldsymbol{u}_s = \frac{1}{\omega_s^2}\mathrm{rot}(\mathrm{rot}\boldsymbol{u}),$$

where ω_p and ω_s are the compressional and shear wave numbers, respectively, which are given by

$$\omega_p = \frac{\omega}{\sqrt{\lambda + 2\mu}} \text{ and } \omega_s = \frac{\omega}{\sqrt{\mu}}.$$

Above $\mathrm{rot} = \nabla^\perp$ represents $\frac{\pi}{2}$ clockwise rotation of the gradient when $n = 2$, and $\mathrm{rot} = \nabla \times$ stands for the curl operator when $n = 3$. The vector fields $\boldsymbol{u}_p$ and $\boldsymbol{u}_s$ are called the compressional and shear parts of the scattered vector field $\boldsymbol{u}$, respectively. In addition, recall that $\boldsymbol{f} = 0$ in $\mathbb{R}^n \setminus \overline{\Omega}$. Then $\boldsymbol{u}_p$ and $\boldsymbol{u}_s$ satisfy the Helmholtz equation

$$\begin{aligned} &(\Delta + \omega_p^2)\boldsymbol{u}_p = 0 \text{ and } \mathrm{rot}\boldsymbol{u}_p = 0 \text{ in } \mathbb{R}^n \backslash \overline{\Omega}, \\ &(\Delta + \omega_s^2)\boldsymbol{u}_s = 0 \text{ and } \nabla \cdot \boldsymbol{u}_s = 0 \text{ in } \mathbb{R}^n \backslash \overline{\Omega}. \end{aligned} \tag{1.3}$$

Therefore, for the elastic scattering problem of Equation (1.2), we need to pose the Kupradze radiation condition

$$\lim_{r\to\infty} \left(\frac{\partial \boldsymbol{u}_p}{\partial r} - i\omega_p \boldsymbol{u}_p \right) = 0 \text{ and } \lim_{r\to\infty} \left(\frac{\partial \boldsymbol{u}_s}{\partial r} - i\omega_s \boldsymbol{u}_s \right) = 0, \quad r = |x|, \tag{1.4}$$

uniformly in all directions $\widehat{x} = x/|x|$. Moreover, one can also expand the functions $\boldsymbol{u}_s$ and $\boldsymbol{u}_p$ as

$$\begin{aligned} \boldsymbol{u}_s(x) &= \frac{1}{4\pi} \frac{e^{i\omega_s |x|}}{|x|^{\frac{n-1}{2}}} \boldsymbol{u}_s^\infty(\widehat{x}) + O\left(|x|^{-\frac{n+1}{2}} \right) \text{ as } |x| \to \infty, \\ \boldsymbol{u}_p(x) &= \frac{1}{4\pi} \frac{e^{i\omega_p |x|}}{|x|^{\frac{n-1}{2}}} \boldsymbol{u}_p^\infty(\widehat{x}) + O\left(|x|^{-\frac{n+1}{2}} \right) \text{ as } |x| \to \infty, \end{aligned} \tag{1.5}$$

for $n = 2, 3$, where $\boldsymbol{u}_s^\infty$ and $\boldsymbol{u}_p^\infty$ denote the *transversal* and *longitudinal elastic far fields* radiated by the source $\boldsymbol{f}$. Furthermore, $\boldsymbol{u}_s^\infty$ and $\boldsymbol{u}_p^\infty$ can be explicitly represented by

$$\boldsymbol{u}_s^\infty(\boldsymbol{e}) = \Pi_{\boldsymbol{e}^\perp} \left(\int_{\mathbb{R}^n} e^{-i\omega_s \boldsymbol{e}\cdot y} \boldsymbol{f}(y) dy \right), \ \boldsymbol{u}_p^\infty(\boldsymbol{e}) = \Pi_{\boldsymbol{e}} \left(\int_{\mathbb{R}^n} e^{-i\omega_p \boldsymbol{e}\cdot y} \boldsymbol{f}(y) dy \right),$$

for any unit vector $\boldsymbol{e} \in \mathbb{S}^{n-1}$, where $\Pi_{\boldsymbol{e}}$ is the projection operator with respect to $\boldsymbol{e}$. Notice that the vector fields $\boldsymbol{u}_s^\infty$ and $\boldsymbol{u}_p^\infty$ are the tangential and the normal components of the Fourier transform of $\boldsymbol{f}$ evaluated on $\mathbb{S}^{n-1}$. Note that the elastic far fields (1.5) of the Navier's equation are derived using the Helmholtz decomposition of Equation (1.2) and the far-field patterns for the Helmholtz equations of (1.3), which is allowed by the radiation conditions of Equation (1.4). Let us recall our theorems, which were stated in [2].

Theorem 1.1. *Let* $\boldsymbol{f} = \chi_\Omega \boldsymbol{\varphi}$ *for a bounded domain* $\Omega \subset \mathbb{R}^n$, $n \in \{2, 3\}$ *and bounded vector function* $\boldsymbol{\varphi} \in L^\infty(\mathbb{R}^n)$*. Let* $\omega, \mu > 0$, $n\lambda + 2\mu > 0$ *and* $\boldsymbol{u} \in H^2_{loc}(\mathbb{R}^n)$ *satisfy Equation* (1.2) *and the radiation condition of Equation* (1.4)*.*

Assume that Ω *has a corner (2D) or an edge (3D) that can be connected to infinity by a path in* $\mathbb{R}^n \setminus \overline{\Omega}$*, and that* $\boldsymbol{\varphi}$ *is Hölder-continuous near it. If* $\boldsymbol{u}$ *has zero far-field pattern, then* $\boldsymbol{\varphi} = 0$ *on the corner or edge, i.e.* $\boldsymbol{\varphi}$ *is the zero vector. In other words,* f *has no jumps at these locations.*

Theorem 1.2. *Let $n \in \{2,3\}$ and $\Omega, \Omega' \subset \mathbb{R}^n$ be bounded convex polyhedral domains. Let $\boldsymbol{\varphi}, \boldsymbol{\varphi'} \in C^\alpha(\mathbb{R}^n)$, for some $\alpha \in (0,1)$ and have nonzero value on $\partial\Omega, \partial\Omega'$.*

Define $\boldsymbol{f} = \chi_\Omega \boldsymbol{\varphi}$, $\boldsymbol{f'} = \chi_{\Omega'} \boldsymbol{\varphi'}$. Let $\omega, \mu > 0$, $n\lambda + 2\mu > 0$ and $\boldsymbol{u}, \boldsymbol{u'} \in H^2_{loc}(\mathbb{R}^n)$ have elastic sources $\boldsymbol{f}, \boldsymbol{f'}$. In other words they satisfy Equation (1.2) with the radiation condition of Equation (1.4).

If $\boldsymbol{u}$ and $\boldsymbol{u'}$ have the same far-field pattern then $\Omega = \Omega'$ and $\boldsymbol{\varphi} = \boldsymbol{\varphi'}$ at each of their vertices and in three dimensions, edges.

Before stating the last main theorem of [2], let us recall the definition of the interior transmission eigenfunctions.

Definition 1.3 (Interior transmission eigenfunctions)**.** A pair $(\boldsymbol{v}, \boldsymbol{w}) \in L^2(\Omega) \times L^2(\Omega)$ is called *interior transmission eigenfunctions* for the Navier equations with density $V \in L^\infty(\Omega)$ at the *interior transmission eigenvalue* $\omega \in \mathbb{R}_+$ if

$$\begin{cases} \lambda\Delta\boldsymbol{w} + (\lambda+\mu)\nabla\nabla\cdot\boldsymbol{w} + \omega^2\boldsymbol{w} = 0, \\ \lambda\Delta\boldsymbol{v} + (\lambda+\mu)\nabla\nabla\cdot\boldsymbol{v} + \omega^2(1+V)\boldsymbol{v} = 0, \end{cases} \tag{1.6}$$

and $\boldsymbol{v} - \boldsymbol{w} \in H^2(\Omega)$ with $\boldsymbol{v} = \boldsymbol{w}$ and $\boldsymbol{T_\nu v} = \boldsymbol{T_\nu w}$ on $\partial\Omega$. Nothing is imposed on the boundary values of $\boldsymbol{v}, \boldsymbol{w}$ individually.

Above $\boldsymbol{T_\nu}$ is the boundary tration operator.

Definition 1.4. The *boundary traction* operator $\boldsymbol{T_\nu}$ is defined as follows. In the two-dimensional case it is

$$\boldsymbol{T_\nu u} = 2\mu\frac{\partial \boldsymbol{u}}{\partial \boldsymbol{\nu}} + \lambda\boldsymbol{\nu}\nabla\cdot\boldsymbol{u} + \mu\boldsymbol{\nu}^\perp(\partial_2 u_1 - \partial_1 u_2),$$

where $\boldsymbol{\nu} = (\nu_1, \nu_2)$ is a unit outer normal on $\partial\Omega$ and $\boldsymbol{\nu}^\perp := (-\nu_2, \nu_1)$. In the three dimensional case,

$$\boldsymbol{T_\nu u} = 2\mu\frac{\partial \boldsymbol{u}}{\partial \boldsymbol{\nu}} + \lambda\boldsymbol{\nu}\nabla\cdot\boldsymbol{u} + \mu\boldsymbol{\nu} \times (\nabla \times \boldsymbol{u}),$$

where $\boldsymbol{\nu} = (\nu_1, \nu_2, \nu_3)$.

We show the similar conclusion for the interior transmission problem for an elastic material with varying density, with more specifically conditions as follows.

Theorem 1.5. *Let $n \in \{2,3\}$ and $\Omega \subset \mathbb{R}^n$ be a bounded domain. Let $V \in L^\infty(\Omega)$ be the material density, and $\mu > 0$, $n\lambda + 2\mu > 0$ be constant Lamé parameters. Assume that $\omega > 0$ is an interior transmission eigenvalue and $\boldsymbol{v}, \boldsymbol{w} \in L^2(\Omega)$ are the corresponding transmission eigenfunctions defined by Equation (1.6).*

Let x_c be any vertex (2D) or edge point (3D) of $\partial\Omega$ around which V and either one of $\boldsymbol{v}, \boldsymbol{w}$ are C^α smooth in $\overline{\Omega}$, for some $\alpha \in (0, 1/2)$. Then so is the other, and $\boldsymbol{v}(x_c) = \boldsymbol{w}(x_c) = 0$ if $V(x_c) \neq 0$.

The note is organized as follows. In Section 2, we discuss the corner scattering in a plane, and we use the dimensional reduction technique to solve the three-dimensional case. Finally, the proofs of our theorems are in Section 3. The proofs and statements Lemma 2.4 onwards have been updated compared from the corresponding ones in [2].

2. Corner scattering

In the rest of this article, let us write $\boldsymbol{\mathcal{L}} := \lambda\Delta + (\lambda+\mu)\nabla(\nabla\cdot)$ for the second order elliptic operator. In particular, in two dimensions, note that the system of Equation (1.2) can be expressed componentwise as

$$\boldsymbol{\mathcal{L}}\boldsymbol{u} = \begin{pmatrix} \lambda\Delta + (\lambda+\mu)\partial_1^2 & (\lambda+\mu)\partial_1\partial_2 \\ (\lambda+\mu)\partial_1\partial_2 & \lambda\Delta + (\lambda+\mu)\partial_2^2 \end{pmatrix} \boldsymbol{u} = \boldsymbol{f} \text{ in } \mathbb{R}^2. \tag{2.1}$$

From now on we identify $\mathbb{R}^2$ with the complex plane $\mathbb{C}$, and we have the following results, which were shown in [2].

Lemma 2.1. *Let $\Omega \subset \mathbb{C}$ such that $\Omega \cap (\mathbb{R}_- \cup \{0\}) = \emptyset$. Let*

$$\boldsymbol{v}(x) = \begin{pmatrix} \exp(-s\sqrt{z}) \\ i\exp(-s\sqrt{z}) \end{pmatrix} \tag{2.2}$$

where $z = x_1 + ix_2$ and $s \in \mathbb{R}_+$. The complex square root is defined as

$$\sqrt{z} = \sqrt{|z|}\left(\cos\frac{\theta}{2} + i\sin\frac{\theta}{2}\right) \tag{2.3}$$

where $-\pi < \theta \leq \pi$ is the argument of z. Then $\boldsymbol{v}$ satisfies $\boldsymbol{\mathcal{L}}\boldsymbol{v} = 0$ in Ω.

Proposition 2.2. *Let $\boldsymbol{v} : \mathbb{R}^2 \to \mathbb{C}$ be the function given in Lemma 2.1 and define the open sector*

$$\mathcal{K} = \{x \in \mathbb{R}^2 \mid x \neq 0,\, \theta_m < \arg(x_1 + ix_2) < \theta_M\}$$

for angles satisfying $-\pi < \theta_m < \theta_M < \pi$. Then

$$\int_{\mathcal{K}} v_1(x)dx = 6i(e^{-2\theta_M i} - e^{-2\theta_m i})s^{-4}.$$

In addition for $\alpha, h > 0$ and $j \in \{1,2\}$ we have the upper bounds

$$\int_{\mathcal{K}} |v_j(x)|\,|x|^\alpha\,dx \leq \frac{2(\theta_M - \theta_m)\Gamma(2\alpha+4)}{\delta_{\mathcal{K}}^{2\alpha+4}} s^{-2\alpha-4}$$

and

$$\int_{\mathcal{K}\setminus B(0,h)} |v_j(x)|\,dx \leq \frac{6(\theta_M - \theta_m)}{\delta_{\mathcal{K}}^4} s^{-4} e^{-\delta_{\mathcal{K}} s\sqrt{h}/2}.$$

where $\delta_{\mathcal{K}} = \min_{\theta_m < \theta < \theta_M} \cos(\theta/2)$ is a positive constant.

Proposition 2.3. *Let $\Omega \subset \mathbb{R}^2$ be a bounded domain and define the cone*

$$\mathcal{K} = \left\{x \in \mathbb{R}^2 \mid x \neq 0,\, \theta_m < \arg(x_1 + ix_2) < \theta_M\right\} \tag{2.4}$$

with angles $-\pi < \theta_m < \theta_M < \pi$ where $\theta_M \neq \theta_m + \pi$. Assume that $0 \in \partial\Omega$ is the centre of a ball B for which $\Omega \cap B = \mathcal{K} \cap B$.

Given $\alpha \in (0,1)$ and $\boldsymbol{f} \in C^\alpha(\overline{\Omega \cap B})$, let $\boldsymbol{u} \in H^2(\Omega \cap B)$ solve

$$\lambda\Delta\boldsymbol{u} + (\lambda+\mu)\nabla\nabla\cdot\boldsymbol{u} + \omega^2\boldsymbol{u} = \boldsymbol{f} \text{ in } \Omega \cap B, \tag{2.5}$$

for some fixed $\omega > 0$. If $\boldsymbol{u} = 0$ and $\boldsymbol{T_\nu u} = 0$ on $\partial\Omega \cap B$ then $\boldsymbol{f}(0) = 0$.

In what follows, we give more details about the proof of [2, Lemma 3.3]. We denote the range of the various functions explicitly to make it clearer which function is a three-vector and which a two-vector.

Lemma 2.4 (Dimension reduction)**.** *Let D be a locally Lipschitz open set in $\mathbb{R}^2$, $L > 0$ and $\alpha \in (0,1)$ be constants. Given $\boldsymbol{f} \in C^\alpha(\overline{D} \times [-L,L];\mathbb{C}^3)$, let $\boldsymbol{u} \in H^2(D \times (-L,L);\mathbb{C}^3)$ be a solution of*

$$\begin{cases} \boldsymbol{\mathcal{L}}\boldsymbol{u}(x) + \omega^2 \boldsymbol{u}(x) = \boldsymbol{f}(x), & \text{for } x = (x', x_3) \in D \times (-L,L), \\ \boldsymbol{u}(x) = 0,\ \boldsymbol{T_\nu}\boldsymbol{u}(x) = 0 & \text{for } x = (x', x_3) \in \Gamma \times (-L,L), \end{cases} \tag{2.6}$$

where $\Gamma \subset \partial D$ consists of two connected segments, ν is the unit outer normal on $\Gamma \times (-L,L)$, and $\omega, \mu > 0, 3\lambda + 2\mu > 0$. Consider $\phi \in C_c^\infty(-L,L)$ and $\xi \in \mathbb{R}$, and we define the dimension reduction operator $\boldsymbol{R}_\xi$ by

$$\boldsymbol{R}_\xi \boldsymbol{h}(x') := \int_{-L}^{L} e^{-ix_3\xi}\phi(x_3)\boldsymbol{h}(x', x_3)dx_3, \ \text{for } x' \in D.$$

Then one has $\boldsymbol{R}_\xi \boldsymbol{u} \in H^2(D;\mathbb{C}^3) \cap C^\alpha(\overline{D};\mathbb{C}^3)$. If $\boldsymbol{u} \in W^{2,2/(1-\alpha)}(D \times (-L,L);\mathbb{C}^3)$ then there is a function $\boldsymbol{F}_\xi = \boldsymbol{F}_\xi(x') \in C^\alpha(\overline{D};\mathbb{C}^3)$ such that $\boldsymbol{R}_\xi \boldsymbol{u}$ is a solution of

$$\begin{cases} \widetilde{\boldsymbol{\mathcal{L}}}(\boldsymbol{R}_\xi \boldsymbol{u})(x') + \omega^2 \boldsymbol{R}_\xi \boldsymbol{u}(x') = \boldsymbol{F}_\xi(x') & \text{for } x' \in D, \\ \boldsymbol{R}_\xi \boldsymbol{u}(x') = 0,\ \boldsymbol{T_\nu}\left(\boldsymbol{R}_\xi \boldsymbol{u}'\right)(x') = 0,\ \partial_\nu\left(\boldsymbol{R}_\xi u_3\right)(x') = 0 & \text{for } x' \in \Gamma, \end{cases} \tag{2.7}$$

where

$$\widetilde{\boldsymbol{\mathcal{L}}} := \begin{pmatrix} \lambda\Delta' + (\lambda+\mu)\partial_1^2 & (\lambda+\mu)\partial_1\partial_2 & 0 \\ (\lambda+\mu)\partial_1\partial_2 & \lambda\Delta' + (\lambda+\mu)\partial_2^2 & 0 \\ 0 & 0 & \lambda\Delta' \end{pmatrix} \tag{2.8}$$

with $\Delta' := \partial_1^2 + \partial_2^2$ being the Laplace operator with respect to the x'-variables, and $\boldsymbol{u} = (\boldsymbol{u}', u_3) = (u_1, u_2, u_3)$. Furthermore, we have

$$\boldsymbol{F}_\xi(x') = \boldsymbol{R}_\xi \boldsymbol{f}(x') \ \text{for } x' \in \Gamma. \tag{2.9}$$

Now we abuse the notation to denote that $\boldsymbol{T_\nu u}$ in (2.6) stands for the boundary traction in the three dimension, and $\boldsymbol{T_\nu}\left(\boldsymbol{R}_\xi \boldsymbol{u}'\right)(x')$ in (2.7) denotes the boundary traction of the two dimensional vector $\boldsymbol{u}' = (u_1, u_2)$ evaluated at the point $x' \in \mathbb{R}^2$.

Proof of Lemma 2.4. Denote $\boldsymbol{u} = (u_1, u_2, u_3)$. By using [1, Lemma 3.4] one can conclude that $\boldsymbol{R}_\xi u_\ell \in H^2(D)$ for $\ell = 1,2,3$. It is also in $C^\alpha(\overline{D})$ because $H^2(D)$ embeds into it in two dimensions. Hence, it remains to show that $\boldsymbol{R}_\xi \boldsymbol{u}$ solves Equation (2.7), such that $\boldsymbol{F}_\xi \in C^\alpha(\overline{D};\mathbb{C}^2)$ and (2.9) hold. The beginning of the proof proceeds as in that of [2, Lemma 3.3].

In order to derive the equation for $\boldsymbol{R}_\xi \boldsymbol{u}$, note that in the three-dimensional case, the isotropic elastic operator $\boldsymbol{\mathcal{L}}$ can be rewritten as

$$\boldsymbol{\mathcal{L}} = \begin{pmatrix} \lambda\Delta + (\lambda+\mu)\partial_1^2 & (\lambda+\mu)\partial_1\partial_2 & (\lambda+\mu)\partial_1\partial_3 \\ (\lambda+\mu)\partial_1\partial_2 & \lambda\Delta + (\lambda+\mu)\partial_2^2 & (\lambda+\mu)\partial_2\partial_3 \\ (\lambda+\mu)\partial_1\partial_3 & (\lambda+\mu)\partial_2\partial_3 & \lambda\Delta + (\lambda+\mu)\partial_3^2 \end{pmatrix}, \tag{2.10}$$

then we also have $\widetilde{\boldsymbol{\mathcal{L}}}\boldsymbol{u} + \omega^2 \boldsymbol{u} = \boldsymbol{f} - \boldsymbol{h}(\boldsymbol{u})$, where

$$\boldsymbol{h}(\boldsymbol{u}) = \begin{pmatrix} \lambda\partial_3^2 u_1 + (\lambda+\mu)\partial_3\partial_1 u_3 \\ \lambda\partial_3^2 u_2 + (\lambda+\mu)\partial_3\partial_2 u_3 \\ (2\lambda+\mu)\partial_3^2 u_3 + (\lambda+\mu)\partial_3(\partial_1 u_1 + \partial_2 u_2) \end{pmatrix}. \tag{2.11}$$

The Lebesgue dominated convergence theorem and an integration by parts formula yield that

$$\widetilde{\boldsymbol{\mathcal{L}}}(\boldsymbol{R}_\xi \boldsymbol{u}) + \omega^2 \boldsymbol{R}_\xi \boldsymbol{u} = \boldsymbol{F}_\xi(x') := \boldsymbol{R}_\xi \boldsymbol{f}(x') + I_\xi(x') + II_\xi(x'), \tag{2.12}$$

where

$$\begin{aligned} I_\xi(x') = &- \int_{-L}^{L} e^{-ix_3\xi}\phi''(x_3) \begin{pmatrix} \lambda u_1 \\ \lambda u_2 \\ (2\lambda+\mu)u_3 \end{pmatrix} (x', x_3) dx_3 \\ &+ 2i\xi \int_{-L}^{L} e^{-ix_3\xi}\phi'(x_3) \begin{pmatrix} \lambda u_1 \\ \lambda u_2 \\ (2\lambda+\mu)u_3 \end{pmatrix} (x', x_3) dx_3 \\ &+ \xi^2 \boldsymbol{R}_\xi \begin{pmatrix} \lambda u_1 \\ \lambda u_2 \\ (2\lambda+\mu)u_3 \end{pmatrix} (x'), \end{aligned} \tag{2.13}$$

and

$$\begin{aligned} II_\xi(x') = &- i\xi(\lambda+\mu)\boldsymbol{R}_\xi \begin{pmatrix} \partial_1 u_3 \\ \partial_2 u_3 \\ \partial_1 u_1 + \partial_2 u_2 \end{pmatrix} (x') \\ &+ (\lambda+\mu) \int_{-L}^{L} e^{-ix_3\xi}\phi'(x_3) \begin{pmatrix} \partial_1 u_3 \\ \partial_2 u_3 \\ \partial_1 u_1 + \partial_2 u_2 \end{pmatrix} (x', x_3) dx_3. \end{aligned} \tag{2.14}$$

This gives the first part of (2.7). The following is where more details and some modifications to the proof of [2, Lemma 3.3] are needed.

Let us show that the boundary condition in (2.7) holds. Since $\boldsymbol{u} = 0$ on $\Gamma \times (-L, L)$, one can easily see $\boldsymbol{R}_\xi \boldsymbol{u} = 0$ on Γ. On the other hand, from $\boldsymbol{u} = 0$ on $\Gamma \times (-L, L)$, we see that $\partial_3 u_1 = \partial_3 u_2 = \partial_3 u_3 = 0$ on $\Gamma \times (-L, L)$ because ∂_3 is along the direction of the boundary. Using this, and noting that the unit outer normal vector is of the form $\boldsymbol{\nu} = (\nu_1, \nu_2, 0)$ on $\Gamma \times (-L, L)$, a direct computation yields that

$$\begin{aligned} 0 &= \boldsymbol{T}_{\boldsymbol{\nu}} \boldsymbol{u} \\ &= \begin{pmatrix} \mu\left(2\partial_1 u_1 \nu_1 + \partial_1 u_2 \nu_2 + \partial_2 u_1 \nu_2\right) + \lambda\left(\partial_1 u_1 \nu_1 + \partial_2 u_2 \nu_1\right) \\ \mu\left(\partial_1 u_2 \nu_1 + 2\partial_2 u_2 \nu_2 + \partial_2 u_1 \nu_1\right) + \lambda\left(\partial_1 u_1 \nu_2 + \partial_2 u_2 \nu_2\right) \\ \mu\left(\partial_1 u_3 + \partial_2 u_3\right) \end{pmatrix} \\ &= \begin{pmatrix} \boldsymbol{T}_{\boldsymbol{\nu}}(\boldsymbol{u}') \\ \mu\left(\partial_1 u_3 + \partial_2 u_3\right) \end{pmatrix} \end{aligned} \tag{2.15}$$

on $\Gamma \times (-L, L)$, where $\boldsymbol{T}_{\boldsymbol{\nu}}(\boldsymbol{u}')$ in the second line of the above equality denotes the traction operator on $\Gamma \subset \mathbb{R}^2$. In addition, the differential operators and components of $\boldsymbol{\nu}$ in (2.15) commute with the dimensional reduction operator $\boldsymbol{R}_\xi$. We apply it to (2.15) and see that (2.7) holds.

By the Minkowsky integral inequality and the Hölder inequality we note that $I_\xi \in H^2(D)$ which embeds into $C^\alpha(\overline{D})$ by the Sobolev embedding. We shall need this argument later and it is a simple generalization of the one in [1, Lemma 3.4], so here it is in more detail: let $\beta \in \mathbb{N}^2$. Then dominated convergence implies that

$$\partial_{x'}^{\beta} \int_{-L}^{L} e^{-ix_3\xi}\psi(x_3)w(x',x_3)dx_3 = \int_{-L}^{L} e^{-ix_3\xi}\psi(x_3)\partial_{x'}^{\beta}w(x',x_3)dx_3 \tag{2.16}$$

for any smooth ψ and w. The Minkowski and Hölder inequalities give then

$$\begin{aligned} &\left\|\partial_{x'}^{\beta} \int_{-L}^{L} e^{-ix_3\xi}\psi(x_3)w(\cdot,x_3)dx_3\right\|_{L^p(D)} \\ \leq& \int_{-L}^{L} \|\psi\|_\infty \left\|\partial_{x'}^{\beta}w(\cdot,x_3)\right\|_{L^p(D)} dx_3 \\ \leq& \|\psi\|_\infty \left(\int_{-L}^{L} dx_3\right)^{1-1/p} \left\|\left\|\partial_{x'}^{\beta}w(\cdot,x_3)\right\|_{L^p(D)}\right\|_{L^p((-L,L),x_3)} \\ =&(2L)^{1-1/p}\|\psi\|_\infty \left\|\partial_{x'}^{\beta}w\right\|_{L^p(D\times(-L,L))} \end{aligned} \tag{2.17}$$

and this can be then extended to any $w \in W^{2,p}(D\times(-L,L))$. In other words, any dimension reduction operator (of the type in the Lemma statement) will map $W^{k,p}(D\times(-L,L)) \to W^{k,p}(D)$ for $k \in \mathbb{N}$ and $1 \leq p < \infty$.

We will show that $\boldsymbol{F}_\xi \in C^\alpha(\overline{D};\mathbb{C}^2)$ next by showing the same for all of its three terms in (2.12). Note that $\boldsymbol{R}_\xi \boldsymbol{f} \in C^\alpha(\overline{D};\mathbb{C}^2)$ by [1, Lemma 3.4] because $\boldsymbol{f}$ is Hölder continuous with the exponent α. We showed that $I_\xi \in C^\alpha(\overline{D};\mathbb{C}^2)$ right before (2.16). We will show that $II_\xi \in C^\alpha(\overline{D};\mathbb{C}^2)$ next. Recall that $\boldsymbol{u} \in W^{2,2/(1-\alpha)}(D\times(-L,L);\mathbb{C}^3)$. This implies that $\partial_{x_k}u_j \in W^{1,2/(1-\alpha)}(D\times(-L,L);\mathbb{C})$ for all $j,k = 1,2,3$. The dimension reduction argument (2.16)–(2.17) implies that $II_\xi \in W^{1,2/(1-\alpha)}(D;\mathbb{C}^2)$ because the components of II_ξ are sums of dimension reduction operators applied to various $\partial_{x_k}u_j$. This space embeds into $C^\alpha(\overline{D};\mathbb{C}^2)$ by Sobolev embedding.

It is easy to see that $I_\xi(x') = 0$ for $x' \in \Gamma$ since $\boldsymbol{u}(x',x_3) = 0$ for $(x',x_3) \in \Gamma\times(-L,L)$. To prove (2.9) it remains to show that that $II_\xi = 0$ on Γ. By denoting $\Gamma := S_1 \cup S_2$, where S_1, S_2 are segments and $\overline{S_1} \cap \overline{S_2} = \{x_0'\}$ is the corner point, we only need to demonstrate that $II_\xi(x') = 0$ on S_1. By choosing suitable boundary normal coordinates, without loss of generality, we may assume that $S_1 \times (-L,L) \subset \operatorname{span}\{e_1,e_2\} \subset \mathbb{R}^3$ with its normal direction $\nu = e_3$. Here $\{e_1,e_2,e_3\}$ forms the standard orthonormal basis in $\mathbb{R}^3$. Recall that $\boldsymbol{u} \in H^2_{loc}(\mathbb{R}^3;\mathbb{C}^3)$, then one has $\frac{\partial u_j}{\partial x_k} \in H^1_{loc}(\mathbb{R}^3)$ for $j,k \in \{1,2,3\}$. Therefore, $\left.\frac{\partial u_j}{\partial x_k}\right|_{\Gamma\times(-L,L)}$ is a well-defined $L^2(\Gamma\times(-L,L))$-function in the trace sense.

Since $\boldsymbol{u} = 0$ on $S_1\times(-L,L)$, we have $\frac{\partial u_j}{\partial x_k} = 0$ for $j = 1,2,3$ and $k = 1,2$. Therefore, by using the boundary traction $\boldsymbol{T_\nu u} = 0$ on $S_1\times(-L,L)$, and that $\mu > 0, \lambda+2\mu > 0$ which follow from the assumptions, one can easily see that $\frac{\partial u_j}{\partial x_k} = 0$ on $\Gamma\times(-L,L)$ for $j,k = 1,2,3$. Similar arguments hold when $x' \in S_2$, which proves that $II_\xi(x') = 0$ on Γ. This demonstrates Equation (2.9). $\square$

Proposition 2.5. *Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with $0 \in \partial\Omega$. Let θ_m, θ_M be the number given by Proposition 2.3 and $\mathcal{K}$ be the cone defined by*

Equation (2.4)*. Suppose that* Ω *has an edge of opening angle* $\theta_M - \theta_m$*, that is, given an origin-centred ball* $B \subset \mathbb{R}^2$ *and there exists* $L > 0$ *such that*

$$(B \times (-L, L)) \cap \Omega = (B \cap \mathcal{K}) \times (-L, L).$$

Given $\boldsymbol{f} \in C^\alpha\big(\overline{(B \times (-L, L)) \cap \Omega}; \mathbb{C}^3\big)$ *for some* $\alpha \in (0, 1)$*, let* $\boldsymbol{u} \in H^2((B \times (-L, L)) \cap \Omega; \mathbb{C}^3)$ *be a solution of* $\boldsymbol{\mathcal{L}u} + \omega^2\boldsymbol{u} = \boldsymbol{f}$ *in* $(B \times (-L, L)) \cap \Omega$ *where* $\omega, \mu > 0, 3\lambda + 2\mu > 0$*. Then*

$$\boldsymbol{u} = \boldsymbol{T_\nu u} = 0 \text{ on } (B \times (-L, L)) \cap \partial\Omega \text{ implies that } \boldsymbol{f}(0) = 0$$

if $\boldsymbol{u} \in W^{2,2/(1-\alpha)}((B \times (-L, L)) \cap \Omega; \mathbb{C}^3)$*.*

Proof. By Lemma 2.4, given any $\xi \in \mathbb{R}$, there are $\boldsymbol{F}_\xi \in C^\alpha(\overline{B \cap \mathcal{K}}; \mathbb{C}^3)$ and a 3-vector $\boldsymbol{U} \in H^2(B \cap \mathcal{K}; \mathbb{C}^3) \cap C^\alpha(\overline{B \cap \mathcal{K}}; \mathbb{C}^3)$ fulfilling $\widetilde{\boldsymbol{\mathcal{L}}}\boldsymbol{U} = \boldsymbol{F}_\xi$ in $B \cap \mathcal{K}$, where $\widetilde{\boldsymbol{\mathcal{L}}}$ is defined by Equation (2.8) and $\boldsymbol{F}_\xi(x') = \boldsymbol{R}_\xi \boldsymbol{f}(x')$ on $B \cap \partial\mathcal{K}$. Splitting $\widetilde{\boldsymbol{\mathcal{L}}}$ into an operator acting on $(\boldsymbol{U}_1, \boldsymbol{U}_2)$ and another acting on $\boldsymbol{U}_3$ this is equivalent to having both

$$\begin{pmatrix} \lambda\Delta' + (\lambda+\mu)\partial_1^2 & (\lambda+\mu)\partial_1\partial_2 \\ (\lambda+\mu)\partial_1\partial_2 & \lambda\Delta' + (\lambda+\mu)\partial_2^2 \end{pmatrix} \begin{pmatrix} \boldsymbol{U}_1 \\ \boldsymbol{U}_2 \end{pmatrix} + \omega^2 \begin{pmatrix} \boldsymbol{U}_1 \\ \boldsymbol{U}_2 \end{pmatrix} = \begin{pmatrix} (\boldsymbol{F}_\xi)_1 \\ (\boldsymbol{F}_\xi)_2 \end{pmatrix} \tag{2.18}$$

and

$$\lambda\Delta'\boldsymbol{U}_3 + \omega^2\boldsymbol{U}_3 = (\boldsymbol{F}_\xi)_3 \tag{2.19}$$

in $B \cap \mathcal{K} \subset \mathbb{R}^2$. Here $\Delta' = \partial_1^2 + \partial_2^2$ is the two-dimensional Laplacian. Note that the operator in (2.18) is the same as in (2.5). Furthermore, Lemma 2.4 shows that

$$\begin{pmatrix} \boldsymbol{U}_1 \\ \boldsymbol{U}_2 \end{pmatrix} = \boldsymbol{T_\nu} \begin{pmatrix} \boldsymbol{U}_1 \\ \boldsymbol{U}_2 \end{pmatrix} = 0, \qquad B \cap \partial\mathcal{K}, \tag{2.20}$$

where $\boldsymbol{T_\nu}$ is the two-dimensional boundary traction, and

$$\boldsymbol{U}_3 = \partial_\nu \boldsymbol{U}_3 = 0, \qquad B \cap \partial\mathcal{K}. \tag{2.21}$$

We are going to deal with the two-dimensional elastic system of Equations (2.18) and (2.20). Note that since $\mu > 0$, we see that $3\lambda + 2\mu > 0$ implies $3\lambda + 3\mu > 0$ and hence also $2\lambda + 2\mu > 0$, so the system represents indeed elasticity. Then Proposition 2.3 implies that $(\boldsymbol{F}_\xi)_1(0) = (\boldsymbol{F}_\xi)_2(0) = 0$. Next, if $\lambda = 0$ in (2.19), we see that $(\boldsymbol{F}_\xi)_3(0) = \omega^2\boldsymbol{U}_3(0) = 0$. If $\lambda \neq 0$, then (2.19), (2.21) and the Helmholtz case from [1, Proposition 3.3] imply* that $(\boldsymbol{F}_\xi)_3(0) = 0$.

Finally, recall that

$$0 = \boldsymbol{F}_\xi(0) = \boldsymbol{R}_\xi \boldsymbol{f}(0) = \int_{-L}^{L} e^{-ix_3\xi}\phi(x_3)\boldsymbol{f}(0, x_3)\, dx_3,$$

for any smooth cut-off functions $\phi(x_3) \in C_c^\infty((-L, L))$ and for any $\xi \in \mathbb{R}$. The Fourier inversion formula implies that $\boldsymbol{f}(0) = 0$.

□

*They have $\omega = 0$, but it is not an issue. We can set $f = (\boldsymbol{F}_\xi)_3/\lambda - \omega^2\boldsymbol{U}_3/\lambda$, $u = \boldsymbol{U}_3$ and $u' = f' = 0$ in that proposition.

3. Proof of Theorems

In the end of this note, we prove our theorems which stated in Section 1.

Proof of Theorem 1.1. Rellich's lemma for the Helmholtz equation (see e.g. [3, Lemma 2.11]) and the unique continuation principle imply that $\boldsymbol{u}_p = \boldsymbol{u}_s = 0$ in the connected component of $\mathbb{R}^n \setminus \overline{\Omega}$ that reaches infinity. Hence $\boldsymbol{u} = 0$ and $\boldsymbol{T_\nu u} = 0$ on the boundary of the corner or edge. The 2D case follows from Proposition 2.3. For the 3D case, it would follow from Proposition 2.5 if $\boldsymbol{u}$ would be smoother. However we see that $\boldsymbol{u} \in W^{2,p}_{loc}(\mathbb{R}^3;\mathbb{C}^3)$, for any $p \in (1,\infty)$ by [4, Theorem 7.3] and this is enough. □

Proof of Theorem 1.2. By Rellich's lemma for the Helmholtz equation again and the unique continuation principle, one must have $\boldsymbol{u}_p = \boldsymbol{u}'_p$, $\boldsymbol{u}_s = \boldsymbol{u}'_s$ in $\mathbb{R}^n \setminus \overline{\Omega \cup \Omega'}$. Without loss of generality, we may assume $\Omega \not\subset \Omega'$. Then by convexity there is a corner (2D) or edge (3D) point $x_c \in \partial\Omega \setminus \overline{\Omega'}$. Since $\boldsymbol{u} = \boldsymbol{u}'$ outside $\overline{\Omega \cup \Omega'}$ we have $\boldsymbol{u} = \boldsymbol{u}'$ and $\boldsymbol{T_\nu u} = \boldsymbol{T_\nu u}'$ on $\partial\Omega$ near x_c. Set $\boldsymbol{w} = \boldsymbol{u} - \boldsymbol{u}'$. We have

$$\lambda\Delta\boldsymbol{w} + (\lambda+\mu)\nabla\nabla\cdot\boldsymbol{w} + \omega^2\boldsymbol{w} = \boldsymbol{f}$$

in $\mathbb{R}^n$ near x_c where $\boldsymbol{f}' = 0$, with $\boldsymbol{w} \in H^2$. The interior elliptic regularity of [4, Theorem 7.3] implies that $\boldsymbol{w} \in W^{2,2/(1-\alpha)}$ in a smaller neighbourhood of x_c. Proposition 2.3 and Proposition 2.5 — the latter requiring the additional integrability from the previous sentence — imply that $\boldsymbol{\varphi}(x_c) = 0$. But this is a contradiction since $\boldsymbol{\varphi} \neq 0$ on $\partial\Omega$. Hence $\Omega \subset \Omega'$. The same proof with Ω, Ω' switched shows that $\Omega' \subset \Omega$. Therefore, $\Omega = \Omega'$.

Next, let x_c be a vertex (2D) or an edge point (3D) of $\partial\Omega = \partial\Omega'$. If $\boldsymbol{w} = \boldsymbol{u} - \boldsymbol{u}'$ then this time

$$\lambda\Delta\boldsymbol{w} + (\lambda+\mu)\nabla\nabla\cdot\boldsymbol{w} + \omega^2\boldsymbol{w} = \boldsymbol{f} - \boldsymbol{f}'$$

in $\mathbb{R}^n$ near x_c with $\boldsymbol{w} \in H^2$. As above, [4, Theorem 7.3] implies that $\boldsymbol{w} \in W^{2,2/(1-\alpha)}$ in a smaller neighbourhood of x_c. Rellich's lemma for the Helmholtz equation and the unique continuation principle for the Navier equations imply that $\boldsymbol{w} = 0$ and $\boldsymbol{T_\nu w} = 0$ on $\partial\Omega$ near x_c in this case too. Proposition 2.3 and Proposition 2.5 imply $\boldsymbol{f} = \boldsymbol{f}'$ at x_c. □

Finally, we can prove the third main theorem in this paper.

Proof of Theorem 1.5. Move coordinates so that $x_c = 0$ for this proof. In two and three dimensions H^2 embeds into C^α if $0 < \alpha < 1/2$. So $\boldsymbol{u} = \boldsymbol{v} - \boldsymbol{w}$ is Hölder-continuous in the neighbourhood of the corner or edge[†] and thus both $\boldsymbol{v}$ and $\boldsymbol{w}$ are too, since one of them is in C^α near the corner or edge by assumption.

Set $\boldsymbol{f} = -\omega^2 V\boldsymbol{v}$ and $\boldsymbol{u} = \boldsymbol{v} - \boldsymbol{w}$. These functions satisfy

$$\mathcal{L}\boldsymbol{u} + \omega^2\boldsymbol{u} := \lambda\Delta\boldsymbol{u} + (\lambda+\mu)\nabla\nabla\cdot\boldsymbol{u} + \omega^2\boldsymbol{u} = \boldsymbol{f} \tag{3.1}$$

with $\boldsymbol{u} \in H^2(\Omega;\mathbb{C}^n)$, $\boldsymbol{u} = \boldsymbol{T_\nu u} = 0$ on $\partial\Omega$, and $\boldsymbol{f} \in L^2(\overline{\Omega};\mathbb{C}^n)$.

[†]In this theorem Ω is not necessarily smooth enough for Sobolev embedding to hold globally.

In the two-dimensional case Proposition 2.3 implies that $\boldsymbol{f}(0) = 0$. Let us consider the case $n = 3$ next. Let $B \subset \mathbb{R}^2$ and $L > 0$ be as in Lemma 2.4 but small enough that

(1) a slightly larger ball B_m and length L_m also satisfy those assumptions, and
(2) $V, \boldsymbol{v}$ are C^α in $\overline{(B_m \times (-L_m, L_m)) \cap \Omega}$.

Denote $U_m = B_m \times (-L_m, L_m)$ and $U = B \times (-L, L)$. Thus we have $\boldsymbol{u} \in H^2(U_m \cap \Omega; \mathbb{C}^3)$, $\boldsymbol{f} \in C^\alpha(\overline{U_m \cap \Omega}; \mathbb{C}^3)$ and (3.1) there too.

We will extend $\boldsymbol{u}$ to the whole U_m next and show that the extension is in $H^2(U_m; \mathbb{C}^3)$. For $h \in L^1(U_m \cap \Omega)$ let $E_0 h$ be the extension of h by zero to $U_m \setminus \overline{\Omega}$. Let us show that ∂_j and E_0 commute for $h \in H^1(U_m \cap \Omega)$ with $h = 0$ on $U_m \cap \partial\Omega$. Let $\phi \in C_0^\infty(U_m)$. Weak derivatives and integration by parts yield that

$$
\begin{aligned}
\langle \partial_j E_0 h, \phi \rangle &= -\langle E_0 h, \partial_j \phi \rangle \\
&= -\int_{U_m \cap \Omega} h \partial_j \phi \, dx \\
&= \int_{U_m \cap \Omega} \partial_j h \phi \, dx - \int_{\partial(U_m \cap \Omega)} h \nu_j \phi \, d\sigma \\
&= \langle E_0 \partial_j h, \phi \rangle
\end{aligned}
\tag{3.2}
$$

because $h = 0$ on $U_m \cap \partial\Omega$ and $\phi = 0$ on ∂U_m. Hence $\partial_j E_0 h = E_0 \partial_j h$.

The final paragraph of the proof of Lemma 2.4 applies here too, and it implies that $u_i = \partial_j u_i = 0$ on $U_m \cap \partial\Omega$ for all i, j. Because $u_i \in H^2(U_m \cap \Omega; \mathbb{C})$ and the boundary conditions, we see that both u_i and $\partial_j u_i$ satisfy the conditions required of h above. Hence $\partial_j E_0 u_i = E_0 \partial_j u_i \in L^2(U_m; \mathbb{C})$ and $\partial_k \partial_j E_0 u_i = E_0 \partial_k \partial_j u_i \in L^2(U_m; \mathbb{C})$ for all j, k. Thus $E_0 u_i \in H^2(U_m; \mathbb{C})$ for all i.

It follows that $E_0 \boldsymbol{u} \in H^2(U_m; \mathbb{C}^3)$ and

$$
\boldsymbol{\mathcal{L}} E_0 \boldsymbol{u} = E_0 \boldsymbol{f} - \omega^2 E_0 \boldsymbol{u}, \qquad \text{in } U_m. \tag{3.3}
$$

Note that $E_0 \boldsymbol{u} \in H^2(U_m; \mathbb{C}^3) \hookrightarrow L^\infty(U_m; \mathbb{C}^3)$ by the Sobolev embedding. Also $\boldsymbol{f} \in L^\infty(U_m \cap \Omega; \mathbb{C}^3)$ so $E_0 \boldsymbol{f} \in L^\infty(U_m; \mathbb{C}^3)$. Thus the right-hand side above is in $L^\infty(U_m; \mathbb{C}^3)$. Interior elliptic regularity [4, Theorem 7.3] implies that in $U \Subset U_m$ we have $\boldsymbol{u} \in W^{2,p}(U)$ for any $2 < p < \infty$, so in particular also for $p = 2/(1-\alpha)$.

We have thus that $\boldsymbol{f} \in C^\alpha(\overline{U \cap \Omega}; \mathbb{C}^3)$, $\boldsymbol{u} \in W^{2,2/(1-\alpha)}(U \cap \Omega; \mathbb{C}^3)$ and (3.1) with the zero boundary Dirichlet and traction conditions. Proposition 2.5 implies that $\boldsymbol{f}(0) = 0$, i.e. $\boldsymbol{f}$ vanishes at the given point on the edge. If $V(0) \neq 0$ then $\boldsymbol{v}(0) = 0$ and since $\boldsymbol{v} = \boldsymbol{w}$ on $\partial\Omega$, so is $\boldsymbol{w}(0) = 0$. ☐

Acknowledgments. The authors would like to thank Prof. Huaian Diao and Prof. Hongyu Liu for fruitful discussions and helpful suggestions. The work of the first author was supported by the Academy of Finland (decision 312124) and in part by the Estonian grant PRG 832. The work of the second author is currently partially supported by the Ministry of Science and Technology Taiwan, under the Columbus Program: MOST-109-2636-M-009-006.

References

[1] E. Blåsten. Nonradiating sources and transmission eigenfunctions vanish at corners and edges. *SIAM Journal on Mathematical Analysis*, 50(6):6255–6270, 2018.

[2] E. Blåsten and Y.-H. Lin. Radiating and non-radiating sources in elasticity. *Inverse Problems*, 35(1):015005, 2018.

[3] D. Colton and R. Kress. *Inverse acoustic and electromagnetic scattering theory*, volume 93 of *Applied Mathematical Sciences*. Springer-Verlag, Berlin, second edition, 1998.

[4] M. Giaquinta. *Introduction to regularity theory for nonlinear elliptic systems.* Lectures in Mathematics ETH Zürich. Birkhäuser Verlag, Basel, 1993.

Department of Mathematics and Systems Analysis, Aalto University, FI-00076 Aalto, Finland.

Email address: emilia.blasten@iki.fi

Department of Applied Mathematics, National Yang Ming Chiao Tung University & National Chiao Tung University, 30050, Hsinchu, Taiwan.

Email address: yihsuanlin3@gmail.com