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BOUNDARY DETERMINATION OF ELECTROMAGNETIC AND LAMÉ PARAMETERS WITH CORRUPTED DATA

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Abstract. We study boundary determination for an inverse problem associated to the time-harmonic Maxwell equations and another associated to the isotropic elasticity system. We identify the electromagnetic parameters and the Lamé moduli for these two systems from the corresponding boundary measurements. In a first step we reconstruct Lipschitz magnetic permeability, electric permittivity and conductivity on the surface from the ideal boundary measurements. Then, we study inverse problems for Maxwell equations and the isotropic elasticity system assuming that the data contains measurement errors. For both systems, we provide explicit formulas to reconstruct the parameters on the boundary as well as its rate of convergence formula.

1. **Introduction.** In this work we consider the inverse problem of determining the material parameters, specifically electromagnetic parameters or elasticity parameters, at the boundary of a body from knowledge of certain boundary maps of electromagnetic fields or elastic waves. Such boundary determination is usually the preliminary step in solving the inverse problem of determining these parameters inside the body. The prototypical study that inspired that of the electromagnetic and elastic inverse problems is for electrostatics, known as the Calderón problem. In the Calderón problem, one aims at determining the conductivity function σ from the Dirichlet-to-Neumann (DN) map (also known as the voltage-to-current map) associated to the diffusion conductivity equation $\nabla \cdot (\sigma \nabla u) = 0$ for the electrical potential. Extensive studies have been devoted to show the unique determination of the conductivity inside the body, see [26, 21, 2, 9, 11] for example. The work on internal unique determination of electromagnetic and elastic parameters can be found in [22, 23, 10] and [19], respectively. An important direction of generalizing

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the uniqueness result is to obtain uniqueness for parameters with lower regularity, such as Lipschitz conductivities discussed in [9, 11], Lipschitz electromagnetic parameters discussed in [10, 24] and Lipschitz Lamé moduli in [19].

The boundary determination for the Calderón problem was first shown in [18] for smooth conductivities, and later generalized in a series of papers [1, 3, 4, 5]. In particular, the methods in [3, 5] are constructive. A fundamental insight obtained in [27], is that the DN-map Λ_σ is a first order pseudo-differential operator whose full symbol carries all information of the conductivity σ and its derivatives on the boundary. In the case of systems, the available results in this context are due to Joshi-McDowall [15, 20], and Salo-Tzou [25]. In [19], the boundary determination of the Lamé parameters for an isotropic elasticity system has been investigated.

In this paper, we consider the boundary determination of parameters when the boundary of the body is non-smooth, but Lipschitz. Under this assumption, the principal symbol approach in [27, 15] does not directly apply. Instead, we follow the scheme in [3] to show that the boundary values of the electromagnetic parameters in the time-harmonic Maxwell's equations can be uniquely determined by the ideal (noiseless) boundary measurements of electromagnetic fields, formulated as the admittance map for Maxwell's equations.

The second goal of the paper is to provide theoretical analysis for reconstructing the values of the parameters on the boundary assuming *corrupted* boundary measurements. We consider both inverse problems for electromagnetics and elasticity. The corruption of the data is usually a result of discretized approximation by real data with errors. A formulation of such measurements was introduced in [7] for the Dirichlet-to-Neumann map in solving the Calderón problem, where the random white noise was modeled by a random perturbation in the energy bilinear form, that depends on the intensity of the boundary potential and current. Other approaches in handling noises in boundary measurements can be found in [16, 12, 13, 17]. Based on our boundary reconstruction result with ideal data for Maxwell's equations and [19] for elasticity equation, we adopt the approach in [7] to show that the given type of Gaussian noises can be filtered using highly concentrated and oscillatory wave-packets, precisely those used in the ideal reconstruction scheme, when the noise variance is small. The observation in [7] and our work here inspired another paper [8] where the authors present a general framework and theory to solve the inverse problem of recovering the symbol of a pseudo-differential operator from its bilinear form, corrupted by Gaussian white noise that is modeled as a perturbation. More detailed exposition of the idea and the insights of the method are summarized in the following subsections for the two inverse problems individually.

1.1. **Maxwell system.**

We first formulate the inverse problem for Maxwell's equations. Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with a Lipschitz boundary $\partial\Omega$. Consider real-valued functions μ, ε, σ, first in the space $L^\infty(\Omega)$, representing the magnetic permeability, electric permittivity and electric conductivity, respectively. Furthermore, they satisfy

$$\mu(x) \geq \mu_0 > 0, \quad \varepsilon(x) \geq \varepsilon_0 > 0 \text{ and } \sigma(x) \geq 0, \tag{1}$$

almost everywhere (a.e.) $x \in \Omega$, for some positive constants μ_0 and ε_0. Suppose that we have access to the boundary measurements of all electromagnetic waves that are time-harmonic with angular frequency $\omega > 0$. Then, let (E, H) be an electromagnetic field satisfying time-harmonic Maxwell system, either

$$\begin{cases} \operatorname{curl} E - i\omega\mu H = 0 & \text{in } \Omega, \\ \operatorname{curl} H + i\omega\gamma E = 0 & \text{in } \Omega, \\ \nu \times E = f & \text{on } \partial\Omega, \end{cases} \tag{2}$$

or

$$\begin{cases} \operatorname{curl} E - i\omega\mu H = 0 & \text{in } \Omega, \\ \operatorname{curl} H + i\omega\gamma E = 0 & \text{in } \Omega, \\ \nu \times H = g & \text{on } \partial\Omega, \end{cases} \tag{3}$$

where $\gamma := \epsilon + i\sigma/\omega$. It is known that (2) and (3) are well-posed except at a discrete set of frequencies. Note that for real parameters (i.e. $\sigma = 0$), one needs to consider either the vacuum of eigenvalues for the Maxwell operator or replace the following well-defined boundary maps by the Cauchy data set. For the complex parameters (i.e. $\sigma > 0$), there are no real eigenvalues. Throughout this paper, we assume that $\omega > 0$ is not an eigenvalue of (2) and (3). Then the *boundary admittance map* $\Lambda^A_{\mu,\gamma}$ can be defined by

$$\Lambda^A_{\mu,\gamma}(f) = \nu \times H|_{\partial\Omega},$$

where $(E, H) \in H(\operatorname{curl}; \Omega) \times H(\operatorname{curl}; \Omega)$ satisfies the boundary value problem (2). Here $\nu \in (L^\infty(\partial\Omega))^3$ denotes the unit outer normal vector to $\partial\Omega$ and

$$H(\operatorname{curl}; \Omega) = \left\{ u \in (L^2(\Omega))^3 \mid \operatorname{curl} u \in (L^2(\Omega))^3 \right\}.$$

Similarly, one can define the *boundary impedance map* $\Lambda^I_{\mu,\gamma}$ by

$$\Lambda^I_{\mu,\gamma}(g) = \nu \times E|_{\partial\Omega},$$

where $(E, H) \in H(\operatorname{curl}; \Omega) \times H(\operatorname{curl}; \Omega)$ satisfies the boundary value problem (3). In order to reconstruct γ and μ, we need to use the whole boundary information $\Lambda^A_{\mu,\gamma}$ and $\Lambda^I_{\mu,\gamma}$.

The main result for the ideal data case is the unique boundary identifiability of $\mathrm{Lip}(\overline{\Omega})$-parameters μ, γ at frequency ω from boundary measurements

$$\Lambda^A_{\mu,\gamma}, \Lambda^I_{\mu,\gamma} : H^{-1/2}(\mathrm{Div}\,; \partial\Omega) \to H^{-1/2}(\mathrm{Div}\,; \partial\Omega).$$

See (14) in Section 2 for the definition of $H^{-1/2}(\mathrm{Div}\,; \partial\Omega)$.

The following result contains the boundary determination of the electromagnetic parameters without noise.

Theorem 1.1 (Boundary identifiability of electromagnetic parameters)**.** *Let Ω be a bounded domain in $\mathbb{R}^3$, where the boundary $\partial\Omega$ is locally described by the graphs of Lipschitz functions, and $\omega > 0$. Assume that two sets of parameters μ_j and γ_j for $j \in \{1, 2\}$ belong to $\mathrm{Lip}(\overline{\Omega})$, then we have*

(1) **Unique determination.**

$$\Lambda^A_{\mu_1,\gamma_1} = \Lambda^A_{\mu_2,\gamma_2} \text{ implies that } \gamma_1 = \gamma_2 \text{ a.e. on } \partial\Omega$$

and

$$\Lambda^I_{\mu_1,\gamma_1} = \Lambda^I_{\mu_2,\gamma_2} \text{ implies that } \mu_1 = \mu_2 \text{ a.e. on } \partial\Omega.$$

(2) **Pointwise reconstruction.** *For almost every $P \in \partial\Omega$, there exists an explicit sequence of localized boundary data $\{f_N\}_{N=1}^\infty$ supported around P such that*

$$\lim_{N\to\infty} \frac{i}{\omega} \int_{\partial\Omega} \left[\Lambda^{A}_{\mu,\gamma}(f_N|_{\partial\Omega}) \times \nu\right] \cdot \overline{f_N}\ dS = \gamma(P) \tag{4}$$

and

$$\lim_{N\to\infty} \frac{i}{\omega} \int_{\partial\Omega} \left[\overline{\Lambda^{I}_{\mu,\gamma}(f_N|_{\partial\Omega})}\right] \cdot (f_N \times \nu)\ dS = \mu(P). \tag{5}$$

Remark 1. In Theorem 1.1, the conclusion (2) will imply (1) immediately. Therefore, we only prove (2). Note that the boundary data $\{f_N\}_{N=1}^{\infty}$ stands for electric and magnetic fields on $\partial\Omega$ in (4) and (5), respectively.

As mentioned above, since the boundary $\partial\Omega$ is Lipschitz, the principal symbol approach in [15] does not directly apply. Therefore, we adopt the idea from [3]. However, one of the novelties and key ingredients in [3] is the use of Hardy's inequality which seems not to have a clear counterpart in the problem for Maxwell's equations. Instead, we handle the issue by a new technique that involves a duality argument. See the proof of Theorem 2.1.

Our next result provides the analysis for reconstructing the values of the parameters on the boundary assuming corrupted boundary measurements. To be more specific about the modeling of the noise, we consider a complete probability space $(\Pi, \mathcal{H}, \mathbb{P})$, and a countable family $\{X_\alpha : \alpha \in \mathbb{N}^2\}$ of independent complex Gaussian random variables $X_\alpha : \varpi \in \Pi \ \mapsto \ X_\alpha(\varpi) \in \mathbb{C}$ such that

$$\mathbb{E}X_\alpha = 0, \quad \mathbb{E}(X_\alpha \overline{X_\alpha}) = 1, \quad \mathbb{E}(X_\alpha X_\alpha) = 0 \qquad \forall \alpha \in \mathbb{N}^2, \tag{6}$$

with standard expectation of a random variable defined by

$$\mathbb{E}X = \int_{\Pi} X d\mathbb{P}.$$

In [7], the noisy data for the Calderón problem is defined as

$$\mathcal{N}_\sigma(f,g) = \int_{\partial\Omega} \Lambda_\sigma f \overline{g}\ dS + \sum_{\alpha\in\mathbb{N}^2} (f|e_{\alpha_1})(g|e_{\alpha_2}) X_\alpha \qquad f, g \in H^{1/2}(\partial\Omega),$$

where $\alpha = (\alpha_1, \alpha_2)$ and $\{e_n : n \in \mathbb{N}\}$ is an orthonormal basis of $L^2(\partial\Omega)$ and $(\phi|\psi)$ denotes the inner product in $L^2(\partial\Omega, \mathbb{C})$. Here Λ_σ denotes the Dirichlet-to-Neumann map from $H^{1/2}(\partial\Omega)$ to $H^{-1/2}(\partial\Omega)$

$$\Lambda_\sigma \ : \ f \ \mapsto \ \nu \cdot \sigma \nabla u|_{\partial\Omega},$$

where u is the solution to $\nabla \cdot (\sigma \nabla u) = 0$ and $u|_{\partial\Omega} = f$, and ν is the unit outer normal vector on $\partial\Omega$. It is shown in [7] that at almost every point $P \in \partial\Omega$, with a single realization of $\mathcal{N}_\sigma$ at explicit oscillatory boundary inputs f_N (such as the traces of (23)) ($N \in \mathbb{N}$), the boundary value of σ at the point P can be recovered almost surely by

$$\lim_{N\to\infty} \mathcal{N}_\sigma(f_N, f_N) = \sigma(P).$$

Note that the noise introduced in the energy form for the Dirichlet-to-Neumann map above is modeled on $L^2(\partial\Omega)$. In the case of Maxwell's equations, we will see that similar type of noise could be introduced at two different levels: the $H^{-1}(\partial\Omega)$-level which guaranties decay of $\|f_N\|_{(H^{-1}(\partial\Omega))^3}$ in Lipschitz domains, and $L^2(\partial\Omega)$-level where there is not decay of $\|f_N\|_{(L^2(\partial\Omega))^3}$ and we need extra regularity for $\partial\Omega$.

Starting by defining the corrupted data at the $H^{-1}(\partial\Omega)$-level:

$$\begin{aligned}\mathcal{N}^A_{\mu,\gamma}(f,g) &:= \int_{\partial\Omega} (\Lambda^A_{\mu,\gamma}(f)\times\nu)\cdot\overline{g}\; dS + \sum_{\alpha\in\mathbb{N}^2} (f|\mathbf{e}_{\alpha_1})(g|\mathbf{e}_{\alpha_2})X_\alpha \\ \mathcal{N}^I_{\mu,\gamma}(f,g) &:= \int_{\partial\Omega} \overline{\Lambda^I_{\mu,\gamma}(f)}\cdot(g\times\nu)\; dS + \sum_{\alpha\in\mathbb{N}^2} (f|\mathbf{e}_{\alpha_1})(g|\mathbf{e}_{\alpha_2})X_\alpha\end{aligned} \tag{7}$$

for $f,g\in H^{-1/2}(\mathrm{Div}\,;\partial\Omega)\subset (H^{-1}(\partial\Omega))^3$, where $\{\mathbf{e}_n : n\in\mathbb{N}\}$ is an orthonormal basis of the Hilbert space $(H^{-1}(\partial\Omega))^3$ and $(\phi|\psi)$ here denotes the inner product in $(H^{-1}(\partial\Omega))^3$. Then we have the following reconstruction formula for the Maxwell system with corrupted data.

Theorem 1.2. *Let $\Omega\subset\mathbb{R}^3$ be a bounded Lipschitz domain and μ,ϵ,σ be Lipschitz continuous functions satisfying (1). Let $\mathcal{N}^A_{\mu,\gamma}$ and $\mathcal{N}^I_{\mu,\gamma}$ be the quadratic form given by (7), then for almost every $P\in\partial\Omega$, one has*

(1) **Unique determination.** *There exists explicit boundary data $\{f_N\}_{N=1}^\infty$ in the space $H^{-1/2}(\mathrm{Div}\,;\partial\Omega)$ such that*

$$\lim_{N\to\infty}\mathcal{N}^A_{\mu,\gamma}(f_N,f_N)=\gamma(P),\qquad \lim_{N\to\infty}\mathcal{N}^I_{\mu,\gamma}(f_N,f_N)=\mu(P)$$

almost surely.

(2) **Rates of convergence.** *There exist positive constants C_γ (depending on $\partial\Omega$ and bounds for γ) and C_μ (depending on $\partial\Omega$ and bounds for μ), such that, for every $0<\theta<1$ and $\epsilon>0$, we have*

$$\mathbb{P}\left\{|\mathcal{N}^A_{\mu,\gamma}(f_N,f_N)-\gamma(P)|\le C_\gamma N^{-\theta/2}\right\}\ge 1-\epsilon \text{ for any } N\ge c\epsilon^{-\frac{1}{1-\theta}},$$

where the constant c only depends on $C_{\partial\Omega}$ and θ. A similar estimate holds for μ, that is,

$$\mathbb{P}\left\{|\mathcal{N}^I_{\mu,\gamma}(f_N,f_N)-\mu(P)|\le C_\mu N^{-\theta/2}\right\}\ge 1-\epsilon \text{ for any } N\ge c\epsilon^{-\frac{1}{1-\theta}},$$

where the constant $c>0$ only depends on $C_{\partial\Omega}$ and θ.

Next we consider the problem with error modeled at the $L^2(\partial\Omega)$-level. That is, in the definition (7), we choose $\{\mathbf{e}_n : n\in\mathbb{N}\}$ to be an orthonormal basis of $(L^2(\partial\Omega))^3$ with the inner product $(\phi|\psi)=\int_{\partial\Omega}\phi\cdot\overline{\psi}dS$ and $f,g\in(L^2(\partial\Omega))^3$. To make rigorous sense of this definition, we will assume in this discussion that the boundary of the domain is locally defined by the graph of $C^{1,1}$ functions. In this case, the boundary impedance and admittance maps are well-defined for $f,g\in H^{1/2}(\mathrm{Div}\,,\partial\Omega)$. Unlike the previous case of $(H^{-1}(\partial\Omega))^3$ perturbations, the norm $\|f_N\|_{(L^2(\partial\Omega))^3}$ does not decay as N increases. We actually have $\|f_N\|_{(L^2(\partial\Omega))^3}\le C_{\partial\Omega}$ where $C_{\partial\Omega}$ is a constant depending on the boundary. This is similar to the reconstruction of the normal derivative of the conductivity with corrupted data in [7]; and similarly, our family of solutions can filter out the noise when averaged with respect to the parameter $N^{1/2}$. We then obtain the following result.

Theorem 1.3. *Let $\Omega\subset\mathbb{R}^3$ be a bounded domain whose boundary can be defined by the graphs of $C^{1,1}$-functions, and μ,ε,σ be Lipschitz continuous functions satisfying (1). Let $\mathcal{N}^A_{\mu,\gamma}$ and $\mathcal{N}^I_{\mu,\gamma}$ be the quadratic form given by (7) at the $L^2(\partial\Omega)$-level. Then for every $P\in\partial\Omega$, there exists an explicit family $\{f_t : t\ge 1\}$ in $H^{1/2}(\mathrm{Div}\,,\partial\Omega)$ such that for $N\in\mathbb{N}\backslash\{0\}$ and $T_N := N^{3+3\theta/2}$ with $\theta\in(0,1)$,* *(1)* **Unique determination.**

$$\lim_{N\to\infty}\frac{1}{T_N}\int_{T_N}^{2T_N}\mathcal{N}^A_{\mu,\gamma}(f_{t^2},f_{t^2})\,dt=\gamma(P),\qquad \lim_{N\to\infty}\frac{1}{T_N}\int_{T_N}^{2T_N}\mathcal{N}^I_{\mu,\gamma}(f_{t^2},f_{t^2})\,dt=\mu(P)$$

almost surely.

(2) **Rates of convergence.** *Set*

$$Y_N^A=\frac{1}{T_N}\int_{T_N}^{2T_N}\mathcal{N}^A_{\mu,\gamma}(f_{t^2},f_{t^2})\,dt,\qquad Y_N^I=\frac{1}{T_N}\int_{T_N}^{2T_N}\mathcal{N}^I_{\mu,\gamma}(f_{t^2},f_{t^2})\,dt.$$

There exist positive constants $C_\gamma>0$ (depending on $\partial\Omega$ and bounds for γ) and $C_\mu>0$ (depending on $\partial\Omega$ and bounds for μ), such that, for every $0<\theta<1$ and $\epsilon>0$, we have

$$\mathbb{P}\left\{|Y_N^A-\gamma(P)|\le C_\gamma N^{-\theta/2}\right\}\ge 1-\epsilon \text{ for any } N\ge c_\gamma\epsilon^{-\frac{1}{1-\theta}},$$

and

$$\mathbb{P}\left\{|Y_N^I-\mu(P)|\le C_\mu N^{-\theta/2}\right\}\ge 1-\epsilon \text{ for any } N\ge c_\mu\epsilon^{-\frac{1}{1-\theta}},$$

where the constants c_γ and c_μ depend on θ, $\partial\Omega$, lower bounds for ε_0 and μ_0, and upper bounds for $\|\gamma\|_{\mathrm{Lip}(\overline{\Omega})}$ and $\|\mu\|_{\mathrm{Lip}(\overline{\Omega})}$, respectively.

Remark 2. The reconstruction in Theorem 1.2 can only be ensured for almost every point at the boundary because of the regularity of the domain. However, the reconstruction formula of Theorem 1.3 holds for every point since the domain is assumed to have a $C^{1,1}$ boundary.

If we compare Theorem 1.2 and Theorem 1.3 with the results in [7] for the reconstruction of the conductivity and its normal derivative at the boundary, we can see a couple of similarities. When modeled the noise at the H^{-1}-level, no averaging is required for the reconstruction, as it happened in [7] for the reconstruction of the conductivity. In [7], this was a consequence of the rate of concentration of the supports of the family $\{f_N\}$ around the point to be reconstructed. However, in our Theorem 1.2 this is due to the regularizing effect of the covariance operator associated to the noise in the $H^{-1}(\partial\Omega)$-level. On the other hand, when modeling the noise at the $L^2(\partial\Omega)$-level, we require to perform an average in the parameter $\sqrt{N}$ (since the radius of the support of f_N shrinks as $1/\sqrt{N}$) to overcome the lack of decay of $\|f_N\|_{(L^2(\partial\Omega))^3}$. This was exactly the same situation as in [7] for the reconstruction of the normal derivative of the conductivity at the boundary. In these situations, we have to analyze an oscillatory integral, and isolate appropriately the stationary points. These are the contents of Lemma 3.5. Note that the decaying rate in this lemma suggests that we might still obtain decays in average even if the norms of f_N are increasing as N grows. Consequently, errors modeled in spaces of higher regularities might be potentially filtered.

1.2. **Elasticity system.** For the second system, we consider the boundary determination of the Lamé parameters for the isotropic elasticity equations. Let $\Omega\subset\mathbb{R}^3$ be a bounded domain, $\lambda(x)$ and $\mu(x)$ be the Lamé parameters satisfying the ellipticity condition

$$\mu(x)>0 \text{ and } 3\lambda(x)+2\mu(x)>0 \text{ for all } x\in\overline{\Omega}. \tag{8}$$

The boundary value problem for the isotropic elasticity system is given by

$$\begin{cases} (\nabla\cdot(\mathbf{C}\nabla u))_i = \sum_{j,k,l=1}^3 \frac{\partial}{\partial x_j}\left(C_{ijkl}\frac{\partial}{\partial x_l}u_k\right) = 0 \quad (i=1,2,3) & \text{in } \Omega, \\ u = f & \text{on } \partial\Omega, \end{cases} \tag{9}$$

where $u = (u_1, u_2, u_3)$ is the displacement vector, $\mathbf{C} = (C_{ijkl})_{1\le i,j,k,l\le 3}$ and

$$C_{ijkl} = \lambda\delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) \text{ for } 1 \le i,j,k,l \le 3 \tag{10}$$

is the isotropic elastic four tensor with Kronecker delta δ_{ij}. One can easily see that C_{ijkl} given by (10) satisfies the major and minor symmetries, i.e.,

$$C_{ijkl} = C_{klij} = C_{jikl}, \text{ for } 1 \le i,j,k,l \le 3.$$

The Dirichlet-to-Neumann (DN) map for the isotropic elasticity system is defined by

$$\Lambda_{\mathbf{C}} \; : \; (H^{1/2}(\partial\Omega))^3 \to (H^{-1/2}(\partial\Omega))^3 \text{ with } (\Lambda_{\mathbf{C}}f)_i = \left.\sum_{j,k,l=1}^3 \nu_j C_{ijkl}\frac{\partial u_k}{\partial x_l}\right|_{\partial\Omega} \tag{11}$$

for $i = 1,2,3$, where $u \in (H^1(\Omega))^3$ is the solution to (9) and $\nu = (\nu_1,\nu_2,\nu_3)$ is the unit outer normal on $\partial\Omega$. The inverse problem is whether the elastic tensor $\mathbf{C}$ is uniquely determined by $\Lambda_{\mathbf{C}}$, and to calculate $\mathbf{C}$ of $\Lambda_{\mathbf{C}}$ if $\mathbf{C}$ is determined by $\Lambda_{\mathbf{C}}$. Note that the global uniqueness for the isotropic elasticity system stays open for the three-dimensional case and it was solved in [14] for the two-dimensional case.

The boundary determination of the zeroth order and higher order Lamé moduli was studied by [28] and [19], respectively. In other words, given any $P \in \partial\Omega$ (when $\partial\Omega$ and the Lamé moduli are sufficiently smooth), one can derive reconstruction formulas for the Lamé moduli λ and μ and their derivatives at $P \in \partial\Omega$, from the localized DN map. Now, our goal is to give a similar reconstruction algorithm for the Lamé parameters with corrupted data.

Due to the existence of elliptic regularity theory for this system, the corrupted data for the elastic system is similar to that of the scalar conductivity equation discussed in [7], namely, the random noise is introduced at $(L^2(\partial\Omega))^3$ vector level by introducing the bilinear form with corrupted data

$$\mathcal{N}_{\mathbf{C}}(f,g) := \int_{\partial\Omega} \Lambda_{\mathbf{C}}f\cdot\overline{g}\,dS + \sum_{\alpha\in\mathbb{N}^2}(f|\mathbf{e}_{\alpha_1})(g|\mathbf{e}_{\alpha_2})X_\alpha,$$

for $f,g \in (H^{1/2}(\partial\Omega))^3$, where $\{\mathbf{e}_n : n \in \mathbb{N}\}$ is an orthonormal basis of the Hilbert space $(L^2(\partial\Omega))^3$ and $(\phi|\psi)$ here denotes the inner product in $(L^2(\partial\Omega))^3$. Then our results for the elasticity system is as follows:

Theorem 1.4. *Let $\Omega \subset \mathbb{R}^3$ be a bounded Lipschitz domain. Let $\mathbf{C}$ be a Lipschitz continuous elastic four tensor in $\overline{\Omega}$. Then for almost every $P \in \partial\Omega$, one has*

(1) **Unique determination.** *There exists an explicit boundary data $\{f_N\}_{N=1}^\infty$ in $(H^{1/2}(\partial\Omega))^3$ such that*

$$\lim_{N\to\infty}\mathcal{N}_{\mathbf{C}}(f_N, f_N) = Z(P)$$

almost surely, where $Z(P) = (Z_{ij})_{1\le i,j\le 3}(P)$ with $Z_{ij} = \overline{Z_{ji}}$ for $1 \le i,j \le 3$, and

$$\begin{aligned} Z_{ii} &= \frac{\mu}{\lambda+3\mu}\big(2(\lambda+2\mu) - (\lambda+\mu)\iota_i^2\big), \\ Z_{ij} &= \frac{\mu}{\lambda+3\mu}\big(- (\lambda+\mu)\iota_i\iota_j + \sqrt{-1}(-1)^k 2\mu\,\iota_k\big),\ 1 \le i < j \le 3 \end{aligned} \tag{12}$$

with $(\iota_1, \iota_2, \iota_3) = (\omega_2, -\omega_1, 0)$ *and the index* $k \in \mathbb{N}$ *satisfies the condition* $1 \le k \le 3$, $k \neq i, j$.

(2) **Rates of convergence.** *There exists a constant* $C > 0$*, independent of* N*, such that, for every* $0 < \theta < 1$ *and* $\epsilon > 0$*, we have*

$$\mathbb{P}\left\{|\mathcal{N}_{\mathbf{C}}(f_N, f_N) - Z(P)| \le CN^{-\theta/2}\right\} \ge 1-\epsilon \text{ for any } N \ge c\epsilon^{-\frac{1}{1-\theta}}, \tag{13}$$

where the constant $c > 0$ *depends only on* $C_{\partial\Omega}$ *and* θ.

Theorem 1.4 shows that when the domain Ω is Lipschitz and $\mathbf{C}$ is Lipschitz continuous, then one can reconstruct the Lamé moduli at almost every boundary point $P \in \partial\Omega$ in a constructive way.

1.3. **Outline.** The rest of this paper is organized as follows. The reconstruction formulas for Lipschitz parameters μ and γ in Maxwell's equations on a Lipschitz boundary $\partial\Omega$ are given in Section 2. In Section 3, we analyze the reconstruction with corrupted data by random white noise for the Maxwell equations. The analysis for the reconstruction of the Lipschitz Lamé moduli for the isotropic elasticity system with corrupted data is given in Section 4.

2. Boundary determination of electromagnetic parameters.

First, let us define several function spaces and notations.

2.1. **Preliminaries.** Let us begin with some definitions of function spaces, where the impedance map is well-defined. For a bounded Lipschitz domain Ω, we adopt Tartar's definition (see [29] or [6]) of the space

$$\begin{aligned} H^{-1/2}(\mathrm{Div}\,;\partial\Omega) := \Big\{ & u \in (H^{-1/2}(\partial\Omega))^3 \mid \exists\, \eta \in H^{-1/2}(\partial\Omega), \text{ s.t.}, \\ & \int_{\partial\Omega} u\cdot\nabla\phi\, dS = \int_{\partial\Omega} \eta\phi\, dS \text{ for } \ \phi \in H^2(\Omega)\Big\}, \end{aligned} \tag{14}$$

where $(H^{-1/2}(\partial\Omega))^3$ is the dual space of $(H^{1/2}(\partial\Omega))^3$. This implies in a weak sense that $\eta = -\mathrm{Div}\, u$, where Div denotes the surface divergence, and that $\nu\cdot u|_{\partial\Omega} = 0$, based on the identity for u smooth

$$-\int_{\partial\Omega} (\mathrm{Div}\, u)\phi\, dS = \int_{\partial\Omega} u\cdot\nabla\phi\, dS - \int_{\partial\Omega} (u\cdot\nu)(\nabla\phi\cdot\nu)\, dS.$$

We will also define in the same spirit the space for the surface scalar curl

$$\begin{aligned} H^{-1/2}(\mathrm{Curl}\,;\partial\Omega) := \Big\{ & u \in (H^{-1/2}(\partial\Omega))^3 \mid \exists\, \xi \in H^{-1/2}(\partial\Omega), \text{ s.t.}, \\ & \int_{\partial\Omega} (\nu\times u)\cdot\nabla\phi\, dS = \int_{\partial\Omega} \xi\phi\, dS \ \text{ for } \ \phi \in H^2(\Omega) \\ & \text{and } \int_{\partial\Omega} u\cdot\nabla\psi\, dS = 0 \ \text{ for } \ \psi \in H^2(\Omega)\cap H_0^1(\Omega)\Big\}. \end{aligned} \tag{15}$$

Note that the first condition implies in the weak sense that $\xi = -\mathrm{Curl}\, u$, where Curl denotes the surface scalar curl, and the second condition in the definition implies weakly the tangentiality $\nu\cdot u|_{\partial\Omega} = 0$.

Moreover, $H^{-1/2}(\mathrm{Curl}\,;\partial\Omega)$ is the dual of $H^{-1/2}(\mathrm{Div}\,;\partial\Omega)$. It is then shown in [6, 29] that the tangential trace map

$$\begin{aligned}\tau_t : H(\mathrm{curl}\,;\Omega) &\to H^{-1/2}(\mathrm{Div}\,;\partial\Omega)\\ u &\mapsto \nu \times u|_{\partial\Omega}\end{aligned}$$

and the projection map

$$\begin{aligned}\pi_t : H(\mathrm{curl}\,;\Omega) &\to H^{-1/2}(\mathrm{Curl}\,;\partial\Omega)\\ u &\mapsto (\nu \times u|_{\partial\Omega}) \times \nu\end{aligned}$$

are both surjective.

In order to reconstruct the values of the parameters, we begin with the following energy identity, which is obtained by integration by parts

$$\frac{i}{\omega}\int_{\partial\Omega}(\nu\times(\nu\times H))\cdot(\nu\times\overline{E})\,dS = \int_\Omega \gamma|E|^2 - \mu|H|^2\,dx \tag{16}$$

for the solution $(E,H)\in H(\mathrm{curl}\,;\Omega)\times H(\mathrm{curl}\,;\Omega)$ to the Maxwell's equations. Here the boundary integral is the parity of $H^{-1/2}(\mathrm{Div}\,;\partial\Omega)$ and $H^{-1/2}(\mathrm{Curl}\,;\partial\Omega)$.

In the following we use d to denote the dimension number so one can trace the dependence of the convergence rate on d. In all cases considered in this paper including Maxwell system and elasticity system, $d=3$. We denote by $B(x,r)$ the ball centered at x of radius $r>0$ and adopt the coordinate notation $x=(x',x_d)\in\mathbb{R}^{d-1}\times\mathbb{R}$ in d dimensions. Since we will use some results of Brown [3], we will follow his notation.

Given a Lipschitz domain $\Omega\subset\mathbb{R}^d$, for each $P:=(p',p_d)\in\partial\Omega$, we consider a change of variable that flattens the boundary near P

$$(z',z_d) = F(x',x_d) = \big(x'+p', x_d+\phi(x'+p')\big), \tag{17}$$

where $\phi:\mathbb{R}^{d-1}\to\mathbb{R}$ is Lipschitz such that

$$\begin{aligned}B(P,\rho)\cap\partial\Omega &= B(P,\rho)\cap\{z_d=\phi(z')\}\\ B(P,\rho)\cap\Omega &= B(P,\rho)\cap\{z_d>\phi(z')\}\end{aligned}$$

for some $\rho>0$. Let $\widetilde{\Omega}=F^{-1}(\Omega)\subset\mathbb{R}^d$ and $\partial\widetilde{\Omega}$ be its boundary. There exists a $r>0$ such that

$$B(0,2r)\cap\{x_d=0\}\subset F^{-1}\big(B(P,\rho)\cap\partial\Omega\big)\subset\partial\widetilde{\Omega}.$$

Since we are interested in the coefficients at the point P, we focus on reconstructing $\mu(F(0,0))=\mu(p',\phi(p'))$ and $\gamma(F(0,0))=\gamma(p',\phi(p'))$.

Denote

$$M(x) := DF^{-1}(F(x)) = \left(\frac{dx_i}{dz_j}\right)_{i,j}(F(x)). \tag{18}$$

By the change of coordinates (17), we have the right hand side of (16) to be

$$I := \int_\Omega \gamma|E|^2-\mu|H|^2\,dz = \int_{\widetilde{\Omega}}(\widetilde{\gamma}\widetilde{E})\cdot\overline{\widetilde{E}} - (\widetilde{\mu}\widetilde{H})\cdot\overline{\widetilde{H}}\,dx, \tag{19}$$

where

$$\widetilde{\mu}(x) := \mu(F(x))M(x)M(x)^t,\quad \widetilde{\gamma}(x) := \gamma(F(x))M(x)M(x)^t,$$

and

$$\widetilde{E}(x) := (M(x)^t)^{-1}E(F(x)),\qquad \widetilde{H}(x) := (M(x)^t)^{-1}H(F(x)).$$

Furthermore, the electromagnetic field $(\widetilde{E}, \widetilde{H})$ (defined as the pull-back of (E, H) by $F: \widetilde{\Omega} \to \Omega$) satisfies the Maxwell's equations (in the weak sense)

$$\text{curl } \widetilde{E} - i\omega\widetilde{\mu}\widetilde{H} = 0, \quad \text{curl } \widetilde{H} + i\omega\widetilde{\gamma}\widetilde{E} = 0 \quad \text{in } \widetilde{\Omega}. \tag{20}$$

This last point can be justified by checking that curl $\widetilde{E}(x) = M(x)(\text{curl } E)(F(x))$.

We now list a couple of properties of the parameters that are required to apply some results of Brown [3]. First, let us note that $\mu, \gamma \in \text{Lip}(\overline{\Omega})$ satisfy the hypothesis (H1) in [3], that is,

$$|\mu(F(x', x_d)) - \mu(F(x', 0))| + |\gamma(F(x', x_d)) - \gamma(F(x', 0))| \lesssim |x_d| \tag{21}$$

for all $|x'| < 2r$. Regarding the hypothesis H2 in [3], note that

$$\begin{aligned} & s^{1-d} \int_{|y'|<s} |\widetilde{\gamma}(0,0) - \widetilde{\gamma}(y',0)|^2 \, dy' + s^{1-d} \int_{|y'|<s} |\widetilde{\mu}(0,0) - \widetilde{\mu}(y',0)|^2 \, dy' \\ & \lesssim s^2 + s^{1-d} \int_{|y'|<s} |\nabla'\phi(y'+p') - \nabla'\phi(p')|^2 \, dy', \end{aligned}$$

where the limit of the last term on the right-hand side vanishes, when s goes to zero, for almost every p' by the Lebesgue differentiation theorem. Here we denote $\nabla'\phi := (\partial_1\phi, \partial_2\phi)^t$.

Our reconstruction method only will work for points $P \in \partial\Omega$ so that

$$\lim_{s\to 0} s^{1-d} \int_{|y'|<s} |\nabla'\phi(y'+p') - \nabla'\phi(p')|^2 \, dy' = 0 \tag{22}$$

for the corresponding ϕ and p'. As pointed out before, for almost every point in $P \in \partial\Omega$ its corresponding limit in (22) vanishes.

2.2. **Reconstruction of γ.** We first give an explicit reconstruction formula of γ in an admissible point $P \in \partial\Omega$ from the knowledge of the admittance map $\Lambda^A_{\mu,\gamma}$. Recall in [3], a family of functions with special decaying property is constructed as the input of the Dirichlet-to-Neumann map for $\nabla \cdot \sigma\nabla$ to reconstruct σ. More specifically, this family was given by

$$v_N(y) = \eta(N^{1/2}|y'|)\eta(N^{1/2}y_d)e^{N(i\alpha - \vec{e}_d)\cdot y}, \tag{23}$$

where $\vec{e}_d = (0, \cdots, 0, 1) \in \mathbb{R}^d$ and $\eta : \mathbb{R} \to [0,1]$ is a smooth cutoff function which takes value 1 in $B(0, 1/2)$ and 0 outside $B(0,1)$, the vector $\alpha \in \mathbb{R}^d$ can be chosen such that

$$\begin{aligned} & |M(0)^t\alpha| = |M(0)^t\vec{e}_d|, \\ & \alpha \cdot M(0)M(0)^t\vec{e}_d = 0. \end{aligned} \tag{24}$$

An explicit choice of α is given in (37).

We will make an essential use of the gradient fields $\{\nabla v_N\}_N$. More particularly we will choose (E, H) so that their pull-back $(\widetilde{E}, \widetilde{H}) = (\nabla v_N + w_1, w_2)$ with w_1 and w_2 solving

$$\begin{cases} \text{curl } w_1 - i\omega\widetilde{\mu}w_2 = 0 & \text{in } \widetilde{\Omega}, \\ \text{curl } w_2 + i\omega\widetilde{\gamma}w_1 = -i\omega\widetilde{\gamma}\nabla v_N & \text{in } \widetilde{\Omega}, \\ \nu \times w_1 = 0 & \text{on } \partial\widetilde{\Omega}. \end{cases} \tag{25}$$

Note that $\widetilde{\Omega}$ is not necessarily locally described by the graph of Lipschitz functions, so in principle, the theory of well-posedness for (25) should be revisited. In our particular case, the situation is simpler since $\widetilde{\Omega}$ is the pull-back of a domain whose boundary is locally described by the graph of a Lipschitz function. Therefore, it is enough to use the map F to obtain (w_1, w_2) in $\widetilde{\Omega}$ from the corresponding fields in Ω. We will be solving in $\widetilde{\Omega}$ in the rest of the paper, and it will always be justified through the map F.

The corresponding energy (19) for $(\widetilde{E}, \widetilde{H})$ is then given by

$$\begin{aligned} I = &\int_{\widetilde{\Omega}} \gamma(F(y)) \nabla \overline{v_N} \cdot MM^t \nabla v_N \, dy \\ &+ \int_{\widetilde{\Omega}} \gamma(F(y)) \big[2\mathfrak{Re}(\nabla \overline{v_N} \cdot MM^t w_1) + \overline{w_1} \cdot MM^t w_1 \big] dy \\ &+ \int_{\widetilde{\Omega}} \mu(F(y)) \overline{w_2} \cdot MM^t w_2 \, dy. \end{aligned} \tag{26}$$

On the other hand, the tangential boundary condition of the electric field is transformed according to

$$\nu \times E(F(x)) = DF(x) \widetilde{\nu} \times \widetilde{E}(x),$$

where $\widetilde{\nu}(x) = DF(x)^t \nu(F(x))$. For $N^{-1/2} < 2r$ the support of ∇v_N is contained on $\{x_d = 0\} \cap B(0, 2r)$, and the tangential boundary condition there becomes

$$\nu \times E(F(x', 0)) = DF(x', 0) \big[\vec{e}_d \times \nabla v_N(x', 0) \big]. \tag{27}$$

Since H1 and H2 in [3, Lemma 1] are satisfied, the first term of I satisfies

$$\begin{aligned} &\frac{\int_{\widetilde{\Omega}} \gamma(F(y)) \nabla \overline{v_N} \cdot MM^t \nabla v_N \, dy}{N^{\frac{3-d}{2}}} \\ &\to \gamma(p', \phi(p'))(1 + |\nabla' \phi(p')|^2) \int_{\mathbb{R}^{d-1}} \eta(|x'|)^2 \, dx', \end{aligned} \tag{28}$$

as $N \to \infty$.

It turns out that this first term dominates, hence provides the reconstruction of $\gamma(F(0))$ knowing ϕ and η.

Theorem 2.1. *Suppose $\Omega \subset \mathbb{R}^d$ ($d = 3$) is a bounded Lipschitz domain. Let $\mu, \varepsilon, \sigma \in \mathrm{Lip}(\overline{\Omega})$ satisfy (1). Let $P \in \partial\Omega$ be an admissible point with F as in (17). We define*

$$f_N(z) := c_0^{-1/2} (M(y))^t (\nu(y) \times \nabla v_N(y))|_{y = F^{-1}(z)}, \tag{29}$$

where

$$c_0 = (1 + |\nabla' \phi(p')|^2) \int_{\mathbb{R}^{d-1}} \eta(|x'|)^2 \, dx',$$

and M and v_N are given by (18) and (23), respectively. Then

$$I(f_N|_{\partial\Omega}) := \frac{i}{\omega} \int_{\partial\Omega} \big[\Lambda^A_{\mu,\gamma}(f_N|_{\partial\Omega}) \times \nu \big] \cdot \overline{f_N} \, dS \to \gamma(P)$$

as $N \to \infty$.

Proof. To show that the last two terms in (26) are lower order terms, it suffices to show that the $(L^2(\widetilde{\Omega}))^3$-norms of w_1 and w_2 are $o(1)$.

First, we need to consider the dual of the standard regularity estimate for the Maxwell's equations, targeting the L^2-norm of the solution.

Notice that the elliptic condition for the parameters is preserved in the following dual problem: Given $(G_1, G_2) \in (L^2(\widetilde{\Omega}))^6$, except for a discrete set of frequencies, there exists a unique solution $(u_1, u_2) \in H(\mathrm{curl}\,; \widetilde{\Omega}) \times H(\mathrm{curl}\,; \widetilde{\Omega})$ to

$$\begin{cases} \mathrm{curl}\, u_1 + i\omega\overline{\widetilde{\mu}} u_2 = G_1 & \text{in } \widetilde{\Omega}, \\ \mathrm{curl}\, u_2 - i\omega\overline{\widetilde{\gamma}} u_1 = G_2 & \text{in } \widetilde{\Omega}, \\ \nu \times u_1 = 0 & \text{on } \partial\widetilde{\Omega}. \end{cases} \tag{30}$$

Furthermore, we have

$$\|u_1\|_{H(\mathrm{curl}\,;\widetilde{\Omega})} + \|u_2\|_{H(\mathrm{curl}\,;\widetilde{\Omega})} \lesssim \|G_1\|_{(L^2(\widetilde{\Omega}))^3} + \|G_2\|_{(L^2(\widetilde{\Omega}))^3}. \tag{31}$$

Then by integration by parts (duality), we have

$$\left| \int_{\widetilde{\Omega}} w_1 \cdot \overline{G_2} + w_2 \cdot \overline{G_1}\; dy \right| = \left| \int_{\widetilde{\Omega}} (-i\omega\widetilde{\gamma}\nabla v_N) \cdot \overline{u_1}\; dy \right|. \tag{32}$$

It then suffices to show that the right hand side is bounded by $o(1)\|u_1\|_{H(\mathrm{curl}\,;\widetilde{\Omega})}$ since this would imply, using (31),

$$\|w_1\|_{(L^2(\widetilde{\Omega}))^3} + \|w_2\|_{(L^2(\widetilde{\Omega}))^3} \leq o(1).$$

It is worth noticing that in [3], Brown used Hardy's inequality to show a similar estimate

$$\|\nabla \cdot \widetilde{\gamma}\nabla v_N\|_{H^{-1}(\widetilde{\Omega})} = o(1).$$

The main novelty in our approach is to replace the use of Hardy's inequality by a duality argument involving the possibility of writing $\widetilde{\gamma}(0)\nabla e_N$ as the curl of certain vector field L_N.

Start by writing $v_N := \psi_N e_N$ with

$$\psi_N(y) := \eta(N^{1/2}|y'|)\eta(N^{1/2}y_d), \quad e_N(y) = e^{N(i\alpha - \vec{e}_d)\cdot y}.$$

We will estimate the three terms of

$$\widetilde{\gamma}\nabla v_N(y) = \widetilde{\gamma}(y)\nabla\psi_N e_N + \big(\widetilde{\gamma}(y) - \widetilde{\gamma}(0)\big)\psi_N\nabla e_N + \widetilde{\gamma}(0)\psi_N\nabla e_N. \tag{33}$$

For the first two terms, we only need to control their L^2-norms by duality. Then we have

$$\begin{aligned} &\|\widetilde{\gamma}\nabla\psi_N e_N\|^2_{(L^2(\widetilde{\Omega}))^3} \\ &\lesssim \|\nabla\psi_N e_N\|^2_{(L^2(\widetilde{\Omega}))^3} \\ &= N^{\frac{2-d}{2}} \int_{\mathbb{R}^d} e^{-2N^{1/2}y_d}(\eta'(|y'|)^2\eta(y_d)^2 + \eta(|y'|)^2\eta'(y_d)^2)\; dy \\ &\lesssim N^{\frac{2-d}{2}} \int_0^1 e^{-2N^{1/2}y_d} + e^{-2N^{1/2}y_d}\eta'(y_d)^2\; dy_d \\ &\lesssim N^{\frac{2-d}{2}} \left(N^{-1/2} + O(e^{-N^{1/2}})\right) \\ &= O(N^{\frac{1-d}{2}}) = O(N^{-1}). \end{aligned} \tag{34}$$

Similarly, we consider the square of L^2-norm of the second term

$$\begin{aligned} &N^2 \int_{\widetilde{\Omega}} |(\widetilde{\gamma}(y) - \widetilde{\gamma}(0))\,(i\alpha - \vec{e}_d)|^2\, \psi_N^2 e^{-2Ny_d}\; dy \\ &\lesssim N^2 \int_{B(0,N^{-1/2})\times\mathbb{R}^+} |\widetilde{\gamma}(y) - \widetilde{\gamma}(0)|^2\, e^{-2Ny_d}\; dy, \end{aligned} \tag{35}$$

where $B(0, N^{-1/2})$ denotes the ball in $\mathbb{R}^{d-1}$ centered at 0 and radius $N^{-1/2}$. It is convenient to write,

$$\begin{aligned}&\widetilde{\gamma}(y) - \widetilde{\gamma}(0)\\ &= \big(\gamma(F(y)) - \gamma(F(0))\big)M(y)M(y)^t + \gamma(F(0))\big(M(y)M(y)^t - M(0)M(0)^t\big).\end{aligned}$$

Thus, the right-had side of (35) can be bounded by

$$\begin{aligned}&N^2 \int_{B(0,N^{-1/2})\times\mathbb{R}^+} |y'|^2 e^{-2Ny_d}\, dy\\ &\quad + N^2 \int_{B(0,N^{-1/2})\times\mathbb{R}^+} |\nabla'\phi(y'+p') - \nabla'\phi(p')|^2 e^{-2Ny_d}\, dy.\end{aligned} \tag{36}$$

By the (22), we have that the previous sum is $o(1)$. It remains to prove

$$\left| \int_{\widetilde{\Omega}} -i\omega\widetilde{\gamma}(0)\psi_N \nabla e_N \cdot \overline{u_1}\, dx \right| \le o(1)\|u_1\|_{H(\mathrm{curl}\,;\widetilde{\Omega})}.$$

The idea will be to write $\widetilde{\gamma}(0)\nabla e_N$ as the curl of certain vector field L_N. First, we state the explicit expression of the matrices M and MM^t at 0:

$$M(0) = \begin{pmatrix} I_{d-1} & 0 \\ -\nabla'\phi(p')^t & 1 \end{pmatrix}, \quad M(0)M^t(0) = \begin{pmatrix} I_{d-1} & -\nabla'\phi(p') \\ -(\nabla'\phi(p'))^t & 1 + |\nabla'\phi(p')|^2 \end{pmatrix}.$$

Since α is chosen such that $\beta = M(0)^t(i\alpha - \vec{e}_d)$ satisfies $\beta\cdot\beta = 0$, we have that $\widetilde{\gamma}(0)\nabla e_N$ is divergence free, namely,

$$\nabla \cdot (\widetilde{\gamma}(0)\nabla e_N(y)) = 0.$$

Therefore, there must exist a vector field $L_N = L_N(y)$ such that

$$\nabla \times L_N = \widetilde{\gamma}(0)\nabla e_N = N\widetilde{\gamma}(0)(i\alpha - \vec{e}_d)e_N.$$

Next, look for such an L_N. We write an ansatz

$$L_N = \gamma(F(0))(a + ib)e_N$$

and find $a, b \in \mathbb{R}^d$ satisfying the following algebraic equations

$$\begin{aligned}\vec{e}_d \times a + \alpha \times b &= M(0)M(0)^t\vec{e}_d,\\ \alpha \times a - \vec{e}_d \times b &= M(0)M(0)^t\alpha.\end{aligned}$$

It can be verified that in $\mathbb{R}^3$, the choice

$$a = \alpha = \begin{pmatrix} \dfrac{1+|\nabla'\phi|^2}{|\nabla'\phi|}\nabla'\phi \\ \\ |\nabla'\phi| \end{pmatrix}(p'), \qquad b = \begin{pmatrix} -\dfrac{1}{|\nabla'\phi|}\partial_2\phi \\ \dfrac{1}{|\nabla'\phi|}\partial_1\phi \\ 1 \end{pmatrix}(p'), \tag{37}$$

where $\nabla'\phi := (\partial_1\phi, \partial_2\phi)^t$, qualifies and satisfies $\eta\cdot\eta = 0$ and $\eta\cdot\overline{\eta} = 2(1+|\nabla'\phi(p')|^2)$.

Finally,

$$
\begin{aligned}
&\int_{\widetilde{\Omega}} (\psi_N \widetilde{\gamma}(0) \nabla e_N) \cdot \overline{u_1}\ dy \\
=& \int_{\widetilde{\Omega}} \psi_N (\nabla \times L_N) \cdot \overline{u_1}\ dy \\
=& \int_{\widetilde{\Omega}} L_N \cdot (\psi_N \nabla \times \overline{u_1} + \nabla \psi_N \times \overline{u_1})\ dy \\
\lesssim& \left(\|\psi_N L_N\|_{(L^2(\widetilde{\Omega}))^3} + \|\nabla \psi_N \cdot L_N\|_{L^2(\widetilde{\Omega}))} \right) \|u\|_{H(\mathrm{curl}\,;\widetilde{\Omega})},
\end{aligned} \tag{38}
$$

where we have used that $\nu \times u_1 = 0$ on $\partial\widetilde{\Omega}$. It is then easy to verify, similar to that for (34), $\|\nabla \psi_N \cdot L_N\|_{L^2(\widetilde{\Omega})} = o(1)$. For the other term,

$$
\begin{aligned}
\|\psi_N L_N\|^2_{(L^2(\widetilde{\Omega}))^3} &\lesssim \int_{\widetilde{\Omega}} \eta(N^{1/2}|y'|)^2 \eta(N^{1/2} y_d)^2 e^{-2N y_d}\ dy \\
&= N^{-\frac{d}{2}} \int_{\mathbb{R}^d} \eta(|y'|)^2 \eta(y_d)^2 e^{-2N^{1/2} y_d}\ dy \\
&= O(N^{\frac{-1-d}{2}}) = O(N^{-2}).
\end{aligned}
$$

This completes the proof. ☐

2.3. **Reconstruction of μ.** In order to reconstruct μ, the idea is to let the magnetic energy, namely $\int_\Omega \mu |H|^2\ dz$, dominate. By symmetry of the equations, H should be chosen roughly ∇v_N, for example, by equating them at the boundary. From now on, we utilize the impedance map, that is, the map

$$
\Lambda^I_{\mu,\gamma} : \nu \times H|_{\partial\Omega} \mapsto \nu \times E|_{\partial\Omega},
$$

then similarly to the previous section, we define our indicator functional being

$$
J(f_N|_{\partial\Omega}) := \frac{i}{\omega} \int_{\partial\Omega} \left[\overline{\Lambda^I_{\mu,\gamma}(f_N|_{\partial\Omega})} \right] \cdot (f_N \times \nu)\ dS, \tag{39}
$$

where $f_N = \nu \times \nabla v_N$ as before. This implies

$$
\begin{aligned}
&J(f_N|_{\partial\Omega}) \\
=& \int_\Omega \mu |H|^2 - \gamma |E|^2\ dx \\
=& \int_{\widetilde{\Omega}} \mu(F(y)) \nabla \overline{v_N} \cdot M M^t \nabla v_N\ dy \\
&+ \int_{\widetilde{\Omega}} \mu(F(y)) \left[2\mathfrak{Re}(\nabla \overline{v_N} \cdot M M^t w_2) + \overline{w_2} \cdot M M^t w_2 \right] dy \\
&- \int_{\widetilde{\Omega}} \gamma(F(y)) \overline{w_1} \cdot M M^t w_1\ dy,
\end{aligned}
$$

where $(w_1, w_2) := (\widetilde{E}, \widetilde{H} - \nabla v_N)$ in this section and satisfies

$$
\begin{cases}
\mathrm{curl}\, w_1 - i\omega \widetilde{\mu} w_2 = i\omega \widetilde{\mu} \nabla v_N & \text{in } \widetilde{\Omega}, \\
\mathrm{curl}\, w_2 + i\omega \widetilde{\gamma} w_1 = 0 & \text{in } \widetilde{\Omega}, \\
\nu \times w_2 = 0 & \text{on } \partial\widetilde{\Omega}.
\end{cases} \tag{40}
$$

Following the proof of Theorem 2.1, the equation (32) is replaced by

$$\left|\int_{\widetilde{\Omega}} w_1\cdot\overline{G_2}+w_2\cdot\overline{G_1}\;dy\right| = \left|\int_{\widetilde{\Omega}}(-i\omega\widetilde{\mu}\nabla v_N)\cdot\overline{u_2}\;dy\right| \tag{41}$$

for any $(G_1,G_2)\in(L^2(\widetilde{\Omega}))^6$, where (u_1,u_2) is the unique solution to

$$\begin{cases}\operatorname{curl}u_1+i\omega\widetilde{\mu}u_2=G_1 & \text{in }\widetilde{\Omega},\\ \operatorname{curl}u_2-i\omega\overline{\widetilde{\gamma}}u_1=G_2 & \text{in }\widetilde{\Omega},\\ \nu\times u_2=0 & \text{on }\partial\widetilde{\Omega}.\end{cases}$$

Then it is left to show similarly

$$\left|\int_{\widetilde{\Omega}}(-i\omega\widetilde{\mu}\nabla v_N)\cdot\overline{u_2}\;dy\right| = o(1)\|u_2\|_{H(\operatorname{curl};\widetilde{\Omega})}.$$

The proof is the same as in Theorem 2.1. In particular, the integration by parts in (38) is still valid in this case using the boundary condition $\nu\times u_2|_{\partial\widetilde{\Omega}}=0$.

As a result, we obtain the reconstruction formula for μ.

Theorem 2.2. *Suppose that Ω, μ, ε, σ, $P\in\partial\Omega$ and f_N all satisfy the assumptions in Theorem 2.1. Then we have*

$$\lim_{N\to\infty}J(f_N|_{\partial\Omega})=\mu(P),$$

where $J(f_N|_{\partial\Omega})$ is defined by (39).

Proof of Theorem 1.1. By using all results in Section 2, we can prove Theorem 1.1 immediately. □

3. Boundary determination of electromagnetic parameters with corrupted data.

The main objective of this part is to stably identify boundary values of the unknown electromagnetic coefficients from the boundary measurement corrupted by errors, modeled and handled similarly to that in [7] for the Calderón problem.

First, we give a description of the modeling for the random white noise, first introduced in [7] for the Calderón problem, with modifications adopted to the system of Maxwell's equations with our electromagnetic boundary maps. In particular, the random white noise is introduced to the boundary data on the $H^{-1}(\partial\Omega)$-level as well as on the $L^2(\partial\Omega)$ one.

3.1. Noise modeled on $H^{-1}(\partial\Omega)$.

We start with the fact that $(H^{-1}(\partial\Omega))^3$ is a Hilbert space and let $\{\mathbf{e}_n : n\in\mathbb{N}\}$ be an orthonormal basis of $(H^{-1}(\partial\Omega))^3$. Recall that our bilinear form with corrupted data are defined as

$$\mathcal{N}^A_{\mu,\gamma}(f,g)=\int_{\partial\Omega}(\Lambda^A_{\mu,\gamma}(f)\times\nu)\cdot\overline{g}\;dS+\sum_{\alpha\in\mathbb{N}^2}(f|\mathbf{e}_{\alpha_1})(g|\mathbf{e}_{\alpha_2})X_\alpha \tag{42}$$

$$\mathcal{N}^I_{\mu,\gamma}(f,g)=\int_{\partial\Omega}\overline{\Lambda^I_{\mu,\gamma}(f)}\cdot(g\times\nu)\;dS+\sum_{\alpha\in\mathbb{N}^2}(f|\mathbf{e}_{\alpha_1})(g|\mathbf{e}_{\alpha_2})X_\alpha \tag{43}$$

for $f,g\in H^{-1/2}(\operatorname{Div};\partial\Omega)\subset(H^{-1}(\partial\Omega))^3$, where $\alpha=(\alpha_1,\alpha_2)\in\mathbb{N}^2$ and $(\phi|\psi)$ denotes the inner product in $(H^{-1}(\partial\Omega))^3$.

Then we have the following lemma after replacing $L^2(\partial\Omega)$ by $(H^{-1}(\partial\Omega))^3$ in [7, Lemma 2.3].

Lemma 3.1. *There exists a complete probability space $(\Pi, \mathcal{H}, \mathbb{P})$, and a countable family $\{X_\alpha : \alpha \in \mathbb{N}^2\}$ of independent complex random variables satisfying (6). Moreover, for every $f, g \in (H^{-1}(\partial\Omega))^3$ we have that*

$$\mathbb{E}\left|\sum_{\alpha\in\mathbb{N}^2}(f|\mathbf{e}_{\alpha_1})(g|\mathbf{e}_{\alpha_2})X_\alpha\right|^2 = \|f\|^2_{(H^{-1}(\partial\Omega))^3}\|g\|^2_{(H^{-1}(\partial\Omega))^3}.$$

Since the $(H^{-1}(\partial\Omega))^3$-norm is bounded by the $H^{-1/2}(\mathrm{Div}\,,\partial\Omega)$-norm, immediately, we obtain the boundedness of the operators $\mathcal{N}^A_{\mu,\gamma}$ and $\mathcal{N}^I_{\mu,\gamma}$ from the space $H^{-1/2}(\mathrm{Div}\,;\partial\Omega)\times H^{-1/2}(\mathrm{Div}\,;\partial\Omega)$ to $L^2(\Pi,\mathcal{H},\mathbb{P})$. It gives that $\left|\mathcal{N}^A_{\mu,\gamma}(f,g)\right|$, $\left|\mathcal{N}^I_{\mu,\gamma}(f,g)\right|$ are finite almost surely. Moreover, we have the following decay for the covariance.

Lemma 3.2. *The following estimate holds*

$$\mathbb{E}\left|\sum_{\alpha\in\mathbb{N}^2}(f_N|\mathbf{e}_{\alpha_1})(f_N|\mathbf{e}_{\alpha_2})X_\alpha\right|^2 = \|f_N\|^4_{(H^{-1}(\partial\Omega))^3} \le C_{\partial\Omega}N^{-2}. \tag{44}$$

Proof. The first equality directly comes from Lemma 3.1 and the second inequality is obtained as follows. From (29), one has the equivalent formula

$$f_N(z) = c_0^{-1/2}\nu(z)\times W_N(z), \qquad z\in\partial\Omega,$$

where

$$W_N(z) := (F^{-1})^*(\nabla v_N) = M(y)^t\nabla_y v_N(y)|_{y=F^{-1}(z)}.$$

Here, $\nu(z)$ is the unit outer normal to $\partial\Omega$ while $\nu(y)$ in (29) is the unit outer normal to $\partial\tilde{\Omega}$.

It is easy to verify

$$\nabla_z\times W_N(z) = 0.$$

For $\varphi\in(H^1(\partial\Omega))^3$,

$$\int_\Omega \nabla\times W_N\cdot\varphi_e - W_N\cdot\nabla\times\varphi_e\ dz = \int_{\partial\Omega} f_N\cdot\varphi\ dS,$$

where $\varphi_e\in(H^{3/2}(\Omega))^3$ is the extension such that $\varphi = \nu\times\varphi_e|_{\partial\Omega}\times\nu$. The first term of the left hand side vanishes by above. For the second term of the left hand side, after a change of variable and passing the derivative, we have

$$\begin{aligned}
\int_{\partial\Omega} f_N\cdot\varphi\ dS &= -\int_\Omega W_N\cdot\nabla\times\varphi\ dz \\
&= -\int_\Omega\left(\frac{\partial y}{\partial z}\right)^t\left(\nabla v_N\circ F^{-1}\right)(z)\cdot(\nabla_z\times\varphi(z))\ dz \\
&= -\int_{\tilde{\Omega}}\nabla v_N(y)\cdot(\nabla_y\times\tilde{\varphi}(y))\ \det\left(\frac{\partial z}{\partial y}\right)\ dy \\
&= -\int_{\partial\tilde{\Omega}} v_N\nu\cdot(\nabla_y\times\widetilde{\varphi}(y))\ dS,
\end{aligned}$$

where $\widetilde{\varphi}$ is the push-forward of φ by F given by

$$\widetilde{\varphi} = (M^t)^{-1}(y)\varphi(F(y)).$$

Finally, it is not hard to see that

$$\int_{\partial\Omega} f_N\cdot\varphi\ dS \le C\|v_N\|_{(L^2(\partial\widetilde{\Omega}))^3}\|\varphi\|_{(H^1(\partial\Omega))^3}.$$

Therefore,

$$\|f_N\|_{(H^{-1}(\partial\Omega))^3} \le C\|v_N\|_{(L^2(\partial\widetilde{\Omega}))^3} \le C\|v_N\|_{(H^{1/2}(\widetilde{\Omega}))^3} \le C_{\partial\Omega} N^{-1/2}$$

which gives (44).

□

We state one crucial result from [7] which also works for the vector-valued functions in this paper. This result will lead to the unique determination and the rate of convergence of parameters for both Maxwell and elasticity systems with corrupted data.

Proposition 1 (Lemma 2.5 in [7])**.** *Let (X, Σ, m) be a measure space and $\{f_n\}_{n=1}^{\infty}$ be a vector-valued sequence in $(L^s(X, \Sigma, m))^3$ for $s \in [1, \infty)$. Assume that $f_n \to f$ in $(L^s(X, \Sigma, m))^3$ for some $f \in (L^s(X, \Sigma, m))^3$ and there exists a sequence of positive numbers $\{\lambda_n\}_{n=1}^{\infty} \subset \mathbb{R}_+$ with $\lambda_n \to 0$ as $n \to \infty$ such that*

$$\sum_{n=1}^{\infty} \frac{1}{\lambda_n^s} \int_X |f_n - f|^s dm < \infty.$$

Then one has $f_n \to f$ for almost every $x \in X$.

Suppose furthermore that $m(X) < \infty$. Then, for every $\epsilon > 0$, there exists a $n_0 \in \mathbb{N}$ such that

$$m\{x \in X : \ |f_n(x) - f(x)| \le \lambda_n\} \ge m(X) - \epsilon, \ \text{for } n \ge n_0.$$

Remark 3. The n_0 in the second part of the statement should satisfy

$$\sum_{n=n_0}^{\infty} \frac{1}{\lambda_n^s} \int_X |f_n - f|^s \ dm \le \epsilon.$$

Proof of Theorem 1.2. The part (1) is a consequence of the first part of Proposition 1 to the sequence $\left\{\sum (f_N|\mathbf{e}_{\alpha_1})(\overline{f_N}|\mathbf{e}_{\alpha_2})X_\alpha \ : \ N \in \mathbb{N}\backslash\{0\}\right\}$ with $\lambda_N = N^{-\theta}$.

To prove part (2) of Theorem 1.2, again we take $\lambda_N = N^{-\theta/2}$. Applying the second part of Proposition 1 to the sequence $\{\sum (f_N|\mathbf{e}_{\alpha_1})(\overline{f_N}|\mathbf{e}_{\alpha_2})X_\alpha : N \in \mathbb{N}\backslash\{0\}\}$, and using (44), we obtain

$$\mathbb{P}\left\{\left|\sum (f_N|\mathbf{e}_{\alpha_1})(\overline{f_N}|\mathbf{e}_{\alpha_2})X_\alpha\right| \le N^{-\theta/2}\right\} \ge 1 - \epsilon$$

for $N \ge N_0$, where N_0 is as in Remark 3, that is, we need

$$\sum_{N=N_0}^{\infty} \frac{C_{\partial\Omega}^2}{N^{2-\theta}} \le \epsilon.$$

This holds whenever

$$(N_0 - 1)^{1-\theta} > \frac{C_{\partial\Omega}^2}{\epsilon(1-\theta)},$$

which gives $N_0 \ge c\epsilon^{-\frac{1}{1-\theta}}$. Lastly, we see that there exist $C_\gamma > 0$ and $C_\mu > 0$ such that

$$\left\{\left|\sum (f_N|\mathbf{e}_{\alpha_1})(\overline{f_N}|\mathbf{e}_{\alpha_2})X_\alpha\right| \le N^{-\theta/2}\right\} \subset \left\{|\mathcal{N}_{\mu,\gamma}^A(f_N, f_N) - \gamma(P)| \le C_\gamma N^{-\theta/2}\right\}$$

and

$$\left\{\left|\sum (f_N|\mathbf{e}_{\alpha_1})(\overline{f_N}|\mathbf{e}_{\alpha_2})X_\alpha\right| \le N^{-\theta/2}\right\} \subset \left\{|\mathcal{N}_{\mu,\gamma}^I(f_N, f_N) - \mu(P)| \le C_\mu N^{-\theta/2}\right\},$$

respectively. This completes the proof. □

3.2. **Noise modeled on $L^2(\partial\Omega)$.** The noisy admittance and impedance data, $\mathcal{N}^A_{\mu,\gamma}$ and $\mathcal{N}^I_{\mu,\gamma}$ respectively, are defined in the level of $L^2(\partial\Omega)$ exactly in the same way as in (42) and (43) with the exception of some details. The sequence $\{\mathbf{e}_n : n \in \mathbb{N}\}$ is an orthonormal basis of $(L^2(\partial\Omega))^3$, the inner product $(\phi|\psi) = \int_{\partial\Omega} \phi \cdot \overline{\psi} dS$, and finally $f, g \in H^{1/2}(\mathrm{Div}\,, \partial\Omega)$. To make rigorous sense of this definition, we will assume the boundary of the domain to be locally defined by the graph of $C^{1,1}$ functions.

Lemma 3.3. *There exists a complete probability space $(\Pi, \mathcal{H}, \mathbb{P})$, and a countable family $\{X_\alpha : \alpha \in \mathbb{N}^2\}$ of independent complex random variables satisfying (6). Moreover, for every $f, g \in (L^2(\partial\Omega))^3$ we have that*

$$\mathbb{E}\left|\sum_{\alpha\in\mathbb{N}^2} (f|\mathbf{e}_{\alpha_1})(g|\mathbf{e}_{\alpha_2}) X_\alpha\right|^2 = \|f\|^2_{(L^2(\partial\Omega))^3} \|g\|^2_{(L^2(\partial\Omega))^3}.$$

Lemma 3.4. *The following estimate holds*

$$\mathbb{E}\left|\sum_{\alpha\in\mathbb{N}^2} (f_N|\mathbf{e}_{\alpha_1})(f_N|\mathbf{e}_{\alpha_2}) X_\alpha\right|^2 = \|f_N\|^4_{(L^2(\partial\Omega))^3} \leq C_{\partial\Omega}. \tag{45}$$

Proof. To compute the L^2-norm of f_N, we could just take the part of $\partial\Omega$ inside the ball of radius ρ and center P since f_N vanishes outside. This part of $\partial\Omega$ could be flatten and there the following identity would hold if $N^{-1/2} < 2r$

$$f_N = \nu \times E|_{\partial\Omega} = DF(\vec{e}_d \times \nabla v_N)|_{\partial\widetilde{\Omega}}.$$

A straightforward computation shows that

$$\vec{e}_d \times \nabla v_N(x', 0) = e^{iN\alpha\cdot(x',0)}\big[N^{1/2}\vec{e}_d \times \nabla\psi\big(N^{1/2}(x',0)\big) + iN(\vec{e}_d \times \alpha)\psi\big(N^{1/2}(x',0)\big)\big],$$

where $\psi(x) = \eta(|x'|)\eta(x_d)$. On the other hand, note that

$$DF(x', 0) = \begin{bmatrix} I_{d-1} & 0 \\ \nabla'\phi(x' + p') & 1 \end{bmatrix}, \qquad \vec{e}_d \times \alpha = \frac{(1 + |\nabla'\phi(p')|^2)}{|\nabla'\phi(p')|} \begin{bmatrix} -\partial_2\phi(p') \\ \partial_1\phi(p') \\ 0 \end{bmatrix},$$

which implies that $DF(0, 0)(\vec{e}_d \times \alpha) = 0$. Therefore, for $|x'| < 2r$, we have that

$$\begin{aligned} f_N(F(x', 0)) = e^{iN\alpha\cdot(x',0)}\big[&N^{1/2} DF(x', 0)(\vec{e}_d \times \nabla\psi)\big(N^{1/2}(x', 0)\big) \\ &+ iN\big(DF(x', 0) - DF(0, 0)\big)(\vec{e}_d \times \alpha)\psi\big(N^{1/2}(x', 0)\big)\big]. \end{aligned} \tag{46}$$

Thus,

$$\begin{aligned} \|f_N\|_{(L^2(\partial\Omega))^3} \lesssim\ & N^{1/2}\|DF(x', 0)(\vec{e}_d \times \nabla\psi)\big(N^{1/2}(x', 0)\big)\|_{(L^2(\mathbb{R}^2))^3} \\ &+ N\|\big(DF(x', 0) - DF(0, 0)\big)(\vec{e}_d \times \alpha)\psi\big(N^{1/2}(x', 0)\big)\|_{(L^2(\mathbb{R}^2))^3}. \end{aligned}$$

The first term on the right hand side is bounded by a constant independent of N because the rate of shrinking of the support of $(\vec{e}_d \times \nabla\psi)(N^{1/2}x', 0)$. To ensure that the second term is also bounded by a constant independent of N we need an extra cancellation beside the shrinking of the support. This cancellation comes from the inequality $|DF(x', 0) - DF(0, 0)| \lesssim |x'|$, which is a consequence of the fact that $\partial\Omega$ is locally described by $C^{1,1}$ functions. □

Lemma 3.5. *We have that, for $T \geq 1$, there exists a $C > 0$ so that*

$$\mathbb{E}\Big|\frac{1}{T}\int_T^{2T}\sum_{\alpha\in\mathbb{N}^2}(f_{t^2}|\mathbf{e}_{\alpha_1})(\overline{f_{t^2}}|\mathbf{e}_{\alpha_2})X_\alpha\,dt\Big|^2 \leq \frac{C}{T^{2/3}}.$$

The constant C depends on upper bounds for the $C^{1,1}$ norm of the functions describing locally the boundary of $\partial\Omega$.

Proof. One can check that

$$\mathbb{E}\Big|\frac{1}{T}\int_T^{2T}\sum_{\alpha\in\mathbb{N}^2}(f_{t^2}|\mathbf{e}_{\alpha_1})(\overline{f_{t^2}}|\mathbf{e}_{\alpha_2})X_\alpha\,dt\Big|^2 = \frac{1}{T^2}\int_{Q_T}\big|(f_{s^2}|f_{t^2})\big|^2\,d(s,t),$$

where $Q_T = [T, 2T]\times[T, 2T]$. Consider $S \in (0, T/2)$ to be chosen later and split Q_T in the sets

$$\begin{aligned} D(S) &= \{(s,t)\in Q_T : t - S \leq s \leq t + S\},\\ L(S) &= \{(s,t)\in Q_T : T \leq s < t - S\},\\ R(S) &= \{(s,t)\in Q_T : t + S < s \leq 2T\}. \end{aligned}$$

Using Cauchy–Schwarz and the Lemma 3.4, we have that

$$\frac{1}{T^2}\int_{D(S)}\big|(f_{s^2}|f_{t^2})\big|^2\,d(s,t) \lesssim \frac{|D(S)|}{T^2} \simeq \frac{S}{T}, \tag{47}$$

since $|D(S)|$, the Lebesgue measure of $D(S)$ is of the order ST. We are now going to study the other pieces $L(S)$ and $R(S)$. Start by noticing that, using the expression (46), the inner product $(f_{s^2}|f_{t^2})$ can be written as a sum of terms of the form

$$st\int_{\mathbb{R}^2} e^{i(s^2-t^2)\alpha'\cdot x'}a(x';s)b(x';t)\,dx', \tag{48}$$

where $|\partial^\beta a(x';s)| \lesssim (1+s)^{|\beta|}\chi(sx')$ and $|\partial^\beta b(x';t)| \lesssim (1+t)^{|\beta|}\chi(tx')$ with χ a compactly supported function in $\mathbb{R}^2$ and $\beta\in\mathbb{N}^2$ for $|\beta|\leq 1$. Since $D(S)$ contains the stationary points of the oscillatory integral (48), we have that, in $L(S)$ and $R(S)$, its phase is non-stationary. Then, write

$$e^{i(s^2-t^2)\alpha'\cdot x'} = \frac{-i}{|\alpha'|^2(s^2-t^2)}\alpha'\cdot\nabla e^{i(s^2-t^2)\alpha'\cdot x'}$$

in order to count the oscillations. Thus, the absolute value of (48) can be bounded, modulo a multiplicative constant, by

$$\frac{st}{|s^2-t^2|}\int_{\mathbb{R}^2}|\nabla a(x';s)||b(x';t)| + |a(x';s)||\nabla b(x';t)|\,dx'$$

which in term is bounded, again modulo a multiplicative constant, by

$$\frac{1+s+t}{|s^2-t^2|}st\int_{\mathbb{R}^2}\chi(sx')\chi(tx')\,dx' \lesssim \frac{1+s+t}{|s^2-t^2|}. \tag{49}$$

In the last inequality, we have used Cauchy–Schwarz. In $R(S)$, $s^2 - t^2 > 0$ since

$$s^2 - t^2 > (t+S)^2 - t^2 = 2St + S^2 > tS \geq ST.$$

Hence, $|s^2 - t^2| \geq ST$. On the other hand, in $L(S)$, $t^2 - s^2 > 0$ since

$$t^2 - s^2 > t^2 - (t-S)^2 = 2St - S^2 > tS \geq ST.$$

Again, $|s^2 - t^2| \geq ST$. Thus, by the fact that $(f_{s^2}|f_{t^2})$ can be written as a sum of terms of the form (48), and these in turn can be bounded by the right-hand side of (49), we have that

$$\frac{1}{T^2}\int_{L(S)\cup R(S)} \left|(f_{s^2}|f_{t^2})\right|^2 \, d(s,t) \lesssim \frac{|L(S)\cup R(S)|}{T^2}\frac{T^2}{S^2T^2} \lesssim \frac{1}{S^2} \tag{50}$$

since $|L(S)\cup R(S)| \lesssim T^2$. Choosing $S = T^{1/3}$ to make the decays in (47) and (50) of the same order, we have the inequality stated in the lemma. □

Proof of the Theorem 1.3. The proof basically follows the proof of Theorem 1.2 by applying Proposition 1 to the sequence of random variables

$$\left\{ \frac{1}{T_N}\int_{T_N}^{2T_N} \sum_{\alpha\in\mathbb{N}^2} (f_{t^2}|\mathbf{e}_{\alpha_1})(\overline{f_{t^2}}|\mathbf{e}_{\alpha_2})X_\alpha \, dt : N \in \mathbb{N}\backslash\{0\} \right\}$$

and by applying

$$\mathbb{E}\left| \frac{1}{T_N}\int_{T_N}^{2T_N} \sum_{\alpha\in\mathbb{N}^2} (f_{t^2}|\mathbf{e}_{\alpha_1})(\overline{f_{t^2}}|\mathbf{e}_{\alpha_2})X_\alpha \, dt \right|^2 \leq \frac{C}{N^{2+\theta}} \to 0$$

as $N \to \infty$, obtained using Lemma 3.5 and $\lambda_N = N^{-\theta/2}$. Note that the $C_{\partial\Omega}$ (used in control N_0) is replaced by constants in Lemma 3.5, which depend on $\partial\Omega$, lower bounds for ε_0 and μ_0, and upper bounds for $\|\gamma\|_{\mathrm{Lip}(\overline{\Omega})}$ and $\|\mu\|_{\mathrm{Lip}(\overline{\Omega})}$, respectively. □

4. **Boundary determination of Lamé moduli with corrupted data.** In this section, assuming that the data has measurement error as in section 3, we reconstruct the boundary value of Lamé parameters and its rates of convergence formula for the isotropic elasticity system.

Hereafter, we will consider the problem in $\mathbb{R}^3$. Let $\Omega \subset \mathbb{R}^3$ be a bounded domain, $\lambda(x)$ and $\mu(x)$ be the Lamé parameters satisfying the strong convexity condition in (8). The regularity assumptions of the boundary $\partial\Omega$ and the Lamé parameters (λ, μ) will be described later.

We use the same notations as in Section 2. Given $P = (p', p_3) \in \partial\Omega$ and $x = (x', x_3)$, let $\phi : \mathbb{R}^2 \to \mathbb{R}$ be the Lipschitz function and $(z', z_3) = F(x', x_3) = (x' + p', x_3 + \phi(x' + p'))$ be the boundary flatten map near $P \in \partial\Omega$. The matrix M is defined in (18) with $\det M(x) = 1$. Let $\widetilde{\Omega} = F^{-1}(\Omega)$.

Let u be the solution to the elasticity system (9) associated to the tensor $\mathbf{C}$. By a change of coordinates, the function $\widetilde{u}(x) := u(F(x))$ solves a new elasticity system

$$\nabla\cdot(\widetilde{\mathbf{C}}\nabla\widetilde{u}) = 0 \text{ in } \widetilde{\Omega}, \tag{51}$$

where we have utilized that

$$0 = \int_\Omega \mathbf{C}\nabla u : \nabla\phi \, dz = \int_{\widetilde{\Omega}} \widetilde{\mathbf{C}}\nabla\widetilde{u} : \nabla\widetilde{\phi} \, dx, \text{ for any smooth test function } \phi.$$

Here $\tilde{\mathbf{C}}$ is the elastic tensor expressed as

$$\widetilde{\mathbf{C}}(x) = M(x) \otimes \mathbf{C}(F(x)) \otimes M(x)^t, \tag{52}$$

where $\otimes$ denotes the multiplication between a fourth-order rank tensor and a matrix. In particular, the function $\widetilde{\mathbf{C}} = (\widetilde{C}_{iqkp})_{1\leq i,q,k,p\leq 3}$ can be explicitly written as

$$\widetilde{C}_{iqkp} = \sum_{l,j=1}^{3} C_{ijkl} \frac{\partial x_p}{\partial z_l} \frac{\partial x_q}{\partial z_j}\bigg|_{z=F(x)}. \tag{53}$$

Moreover, $\widetilde{\mathbf{C}}$ satisfies the strong convexity condition (8), but with a different positive lower bound. Note that the new elastic tensor $\widetilde{\mathbf{C}}$ will lose the minor symmetric property, that is,

$$C_{ijkl} = C_{ijlk} = C_{jikl}, \text{ for } 1 \leq i, j, k, l \leq 3,$$

but we can still reconstruct its coefficients at the boundary. Use a change of variable again, then we have

$$\int_{\partial\Omega} \Lambda_{\mathbf{C}} f \cdot \overline{f}\, dS = \int_{\Omega} \mathbf{C}\nabla u : \nabla \overline{u}\, dz = \int_{\widetilde{\Omega}} \widetilde{\mathbf{C}}\nabla \widetilde{u} : \nabla \overline{\widetilde{u}}\, dx, \tag{54}$$

where $\Lambda_{\mathbf{C}}$ is the Dirichlet-to-Neumann map defined by (11) and : denotes the Frobenius product between two matrices.

4.1. **Approximate solution and elliptic estimate.** We first give a reconstruction formula for the Lamé parameters λ and μ on the surface.

Recall that $\eta : \mathbb{R} \to [0, 1]$ be a smooth cutoff function given in Section 2. Let $\omega \in \mathbb{R}^3$, depending on x', be chosen such that

$$\begin{aligned} &|M(x', 0)^t\omega| = |M(x', 0)^t\vec{e}_3|,\\ &\omega \cdot M(x', 0)M(x', 0)^t\vec{e}_3 = 0, \end{aligned} \tag{55}$$

where $\vec{e}_3 = (0, 0, 1)$.

Given any vector $\mathbf{a} = (a_1, a_2, a_3) \in \mathbb{C}^3$, for any integer $N \geq 1$, we define a family of approximation solutions of $\widetilde{u}$ by

$$\widetilde{G}_N(y) = \eta(N^{1/2}|y'|)\eta(N^{1/2}y_3)e^{N(\sqrt{-1}\omega-\vec{e}_3)\cdot(y-(x',0))}\mathbf{a}$$

in a similar spirit for the Maxwell system in section 2.2, see also [28, Section 2.3.2.1]. Because of the need to use i as a summation index, we let $\sqrt{-1}$ denote the imaginary unit. From now on, without loss of generality, we assume that $x' = 0$. Then ω satisfies (55) and $\omega = (\omega_1, \omega_2, 0)$. Similar to the notations introduced in section 2, we denote

$$\psi_N(y) = \eta(N^{1/2}|y'|)\eta(N^{1/2}y_3), \quad e_N(y) = e^{N(\sqrt{-1}\omega-\vec{e}_3)\cdot y},$$

then we can express $\widetilde{G}_N$ as

$$\widetilde{G}_N(y) = \psi_N(y)e_N(y)\mathbf{a}. \tag{56}$$

In what follows, we first apply the gradient of the approximate solution $\{\nabla\widetilde{G}_N\}_{N=1}^{\infty}$ in the integral (57) in Lemma 4.1 and then find out that its first term dominates the whole behavior. This observation will play an essential role in providing the reconstruction formula for $\widetilde{\mathbf{C}}(0)$ in section 4.2 assuming the boundary measurements are corrupted.

Lemma 4.1. *Let λ, μ be the Lipschitz continuous Lamé moduli satisfying the strong convexity condition (8). The four tensor $\widetilde{\mathbf{C}}$ is defined by (52) and $\nabla'\phi(p')$ exists. Then we have*

$$
\begin{aligned}
&\int_{\widetilde{\Omega}} \widetilde{\mathbf{C}} \nabla \widetilde{G}_N : \nabla \overline{\widetilde{G}_N}\; dy \qquad (57)\\
=& \sum_{i,j,k,l=1}^{3} C_{ijkl}(F(0)) a_k \overline{a_i} \left(\sum_{p,q=1}^{2} \frac{\partial y_p}{\partial z_l}(0) \frac{\partial y_q}{\partial z_j}(0) \omega_p \omega_q + \frac{\partial y_3}{\partial z_l}(0) \frac{\partial y_3}{\partial z_j}(0) \right)\\
&\times \int_{\mathbb{R}^2} \eta(|y'|)^2 dy' + O\left(e^{-\frac{1}{2}N^{1/2}}\right)\\
&+ O\left(N^{-1/2} + \left(N \int_{|y'|\le N^{-1/2}} |\nabla'\phi(y'+p') - \nabla'\phi(p')|^2 \; dy' \right)^{1/2} \right),
\end{aligned}
$$

where $\widetilde{G}_N$ is the approximation solution defined by (56) and recall that $\nabla'\phi := (\partial_1\phi, \partial_2\phi)^t$.

For the flat case (i.e., $0 \in \partial\Omega$ with $\Omega = \{z_3 > 0\}$ near 0), the previous lemma was proved in [28, Section 2]. The first term in the right hand side of (57) is the dominant term of the boundary determination, while the remaining parts are lower order terms. For the completeness of the paper, we provide a detailed proof below.

Proof of Lemma 4.1. Following the idea in the proof of [3, Lemma 1], we first note that

$$
\int_{\widetilde{\Omega}} \widetilde{\mathbf{C}} \nabla \widetilde{G}_N : \nabla \overline{\widetilde{G}_N}\; dy = \sum_{i,q,k,p=1}^{3} \int_{\widetilde{\Omega}} \widetilde{C}_{iqkp}(y) \frac{\partial (\widetilde{G}_N)_k}{\partial y_p} \frac{\partial \overline{(\widetilde{G}_N)_i}}{\partial y_q}\; dy, \qquad (58)
$$

where $\widetilde{G}_N = ((\widetilde{G}_N)_1, (\widetilde{G}_N)_2, (\widetilde{G}_N)_3)$. For $k = 1, 2, 3$, by a direct computation, the partial derivatives of $(\widetilde{G}_N)_k$ are

$$
\frac{\partial (\widetilde{G}_N)_k}{\partial y_p} = \left(N^{1/2}\eta'(N^{1/2}|y'|)\eta(N^{1/2}y_3)\frac{y_p}{|y'|} + \sqrt{-1}N\omega_p\psi_N(y) \right) e_N(y) a_k, \qquad (59)
$$

for $p = 1, 2$ and

$$
\frac{\partial (\widetilde{G}_N)_k}{\partial y_3} = \left(N^{1/2}\eta(N^{1/2}|y'|)\eta'(N^{1/2}y_3) - N\psi_N(y) \right) e_N(y) a_k. \qquad (60)
$$

Next, substituting (53), (59) and (60) into the identity (58), then one obtain

$$
\begin{aligned}
&\int_{\widetilde{\Omega}} \widetilde{\mathbf{C}} \nabla \widetilde{G}_N : \nabla \overline{\widetilde{G}_N}\; dy\\
=& \sum_{i,j,k,l,q,p=1}^{3} \int_{\widetilde{\Omega}} C_{ijkl}(F(y)) \frac{\partial y_p}{\partial z_l} \frac{\partial y_q}{\partial z_j} \frac{\partial (\widetilde{G}_N)_k}{\partial y_p} \frac{\partial \overline{(\widetilde{G}_N)_i}}{\partial y_q}\; dy\\
=:& \; I + II + III + IV,
\end{aligned}
$$

where

$$\begin{aligned}
I = N^2 \sum_{i,j,k,l=1}^{3} \int_{\widetilde{\Omega}} C_{ijkl}(F(0)) \left(\sum_{p,q=1}^{2} \frac{\partial y_p}{\partial z_l}(0) \frac{\partial y_q}{\partial z_j}(0) \omega_p \omega_q + \frac{\partial y_3}{\partial z_l}(0) \frac{\partial y_3}{\partial z_j}(0) \right) \\
\times \eta(N^{1/2}|y'|)^2 \eta(N^{1/2} y_3)^2 e^{-2Ny_3} a_k \overline{a_i} \, dy, \\
II = N^2 \sum_{i,k=1}^{3} \left[\sum_{p,q=1}^{2} \int_{\widetilde{\Omega}} \left(\widetilde{C}_{iqkp}(y) - \widetilde{C}_{iqkp}(0) \right) \omega_p \omega_q + \int_{\widetilde{\Omega}} \left(\widetilde{C}_{i3k3}(y) - \widetilde{C}_{i3k3}(0) \right) \right] \\
\times \eta(N^{1/2}|y'|)^2 \eta(N^{1/2} y_3)^2 a_k \overline{a_i} \, dy, \\
III = N^{3/2} \sum_{i,j,k,l=1}^{3} \int_{\widetilde{\Omega}} C_{ijkl}(F(y)) \Big(-2 \sum_{p=1}^{2} \frac{\partial y_p}{\partial z_l} \frac{\partial y_3}{\partial z_j} \psi_N(y) \eta'(N^{1/2}|y'|) \\
\times \eta(N^{1/2} y_3) \frac{y_p}{|y'|} - 2 \frac{\partial y_3}{\partial z_l} \frac{\partial y_3}{\partial z_j} \psi_N(y) \eta(N^{1/2}|y'|) \eta'(M y_3) \Big) e^{-2Ny_3} a_k \overline{a_i} \, dy,
\end{aligned}$$

and

$$\begin{aligned}
IV = N \sum_{i,j,k,l=1}^{3} \int_{\widetilde{\Omega}} C_{ijkl}(F(y)) \Big(\sum_{p,q=1}^{2} \frac{\partial y_p}{\partial z_l} \frac{\partial y_q}{\partial z_j} \eta'(N^{1/2}|y'|)^2 \eta(N^{1/2} y_3)^2 \frac{y_p y_q}{|y'|^2} \\
+ \sum_{p=1}^{2} \frac{\partial y_p}{\partial z_l} \frac{\partial y_3}{\partial z_j} 2\eta'(N^{1/2}|y'|) \eta(N^{1/2}|y'|) \eta(N^{1/2} y_3) \eta'(N^{1/2} y_3) \frac{y_p}{|y'|} \\
+ \frac{\partial y_3}{\partial z_l} \frac{\partial y_3}{\partial z_j} \eta'(N^{1/2} y_3)^2 \eta(N^{1/2}|y'|)^2 \Big) e^{-2Ny_3} a_k \overline{a_i} \, dy.
\end{aligned}$$

We will show that I is the dominant term and II, III, IV are remainder terms in the following arguments. We first estimate I. By using the integration by parts with respect to the y_3 variable and applying change of variables, we obtain

$$\begin{aligned}
I = \sum_{i,j,k,l=1}^{3} C_{ijkl}(F(0)) a_k \overline{a_i} \left(\sum_{p,q=1}^{2} \frac{\partial y_p}{\partial z_l}(0) \frac{\partial y_q}{\partial z_j}(0) \omega_p \omega_q + \frac{\partial y_3}{\partial z_l}(0) \frac{\partial y_3}{\partial z_j}(0) \right) \\
\times \int_{\mathbb{R}^2} \eta(y')^2 \, dy' + O\left(e^{-\frac{1}{2} N^{1/2}} \right).
\end{aligned}$$

Secondly, by using change of variables again and following a similar argument as in the proof of [3, Lemma 1], one can derive that

$$III = O\left(N^{-1/2} \right) \text{ and } IV = O\left(N^{-1/2} \right).$$

Finally, for the second term II, the triangle inequality yields that

$$|II| \leq II_1 + II_2, \tag{61}$$

where

$$
\begin{aligned}
II_1 &\lesssim N^2\|\nabla F^{-1}\|_\infty^2 \sum_{i,j,k,l=1}^{3} \int_{\widetilde{\Omega}} |C_{ijkl}(F(y)) - C_{ijkl}(F(y',0))| \\
&\quad \times \eta(N^{1/2}|y'|)^2\eta(N^{1/2}y_3)^2 e^{-2Ny_3}\, dy, \\
II_2 &\lesssim N \sum_{i,k=1}^{3} \int_{\mathbb{R}^2} \Big(\sum_{p,q=1}^{2} |\widetilde{C}_{iqkp}(y',0) - \widetilde{C}_{iqkp}(0)| \\
&\quad + |\widetilde{C}_{i3k3}(y',0) - \widetilde{C}_{i3k3}(0)|\Big)\eta(N^{1/2}|y'|)^2\, dy',
\end{aligned}
$$

for some constant $C > 0$ independent of N. Here we have utilized that $|\omega_p| \leq 1$ for $p = 1, 2$ (recalling that $\omega = (\omega_1, \omega_2, 0)$ is a unit vector) and a_j's are complex numbers for $j = 1, 2, 3$.

To establish (61), we will estimate II_1 and II_2 separately. For II_1, we choose a constant $\lambda > 0$ and split the region of integral into two parts, namely, $\{y_3 > \lambda\}$ and $\{y_3 < \lambda\}$. Thus, one obtains, by following a similar argument as in (36), that

$$
|II_1| \lesssim o(1) \tag{62}
$$

when $N \to \infty$.

On the other hand, for II_2, by Cauchy-Schwartz inequality, one can derive

$$
\begin{aligned}
|II_2| &\lesssim \Big(N \int_{|y'|\leq N^{-1/2}} |\widetilde{C}_{iqkp}(y',0) - \widetilde{C}_{iqkp}(0)|^2 + |\widetilde{C}_{i3k3}(y',0) - \widetilde{C}_{i3k3}(0)|^2\, dy'\Big)^{1/2} \\
&\leq \Big(N \int_{|y'|\leq N^{-1/2}} |\nabla'\phi(y'+p') - \nabla'\phi(p')|^2\, dy'\Big)^{1/2}.
\end{aligned}
$$

Using (22), it leads to

$$
|II_2| \lesssim o(1). \tag{63}
$$

We substitute (62) and (63) into (61). We combine the estimates for I to IV, then we complete the proof.

□

We denote

$$
\kappa := \int_{\mathbb{R}^2} \eta(|y'|)^2\, dy',
$$

by a direct computation and let $N \to \infty$, then the main term I satisfies

$$
\begin{aligned}
I &\to \kappa \sum_{i,j,k,l=1}^{3} C_{ijkl}(F(0)) a_k \overline{a_i} \left(\sum_{p,q=1}^{2} \frac{\partial y_p}{\partial z_l}(0)\frac{\partial y_q}{\partial z_j}(0)\omega_p\omega_q + \frac{\partial y_3}{\partial z_l}(0)\frac{\partial y_3}{\partial z_j}(0) \right) \\
&= \kappa \sum_{i,j=1}^{3} Z_{ij}(P) a_i \overline{a_j},
\end{aligned} \tag{64}
$$

where $Z(P) = (Z_{ij})_{1\leq i,j\leq 3}(P)$ is the 2-tensor defined by (12). For more detailed analysis about the boundary reconstruction for the isotropic elasticity system without noise, we refer readers to [28, Section 2].

Similar to [7, Lemma 2.2], we have an analogues result for the elasticity system.

Lemma 4.2. *Let* $\mathbf{C}$ *be a Lipschitz continuous isotropic elastic tensor given by* (10), *which satisfies* (8). *Let* $\widetilde{\mathbf{C}}$ *be the elastic four tensor defined by* (52) *and* $\nabla'\phi(p')$ *exists. Let* $\widetilde{r}_N$ *be the solution of*

$$\begin{cases}\nabla\cdot(\widetilde{\mathbf{C}}\nabla\widetilde{r}_N)=-\nabla\cdot(\widetilde{\mathbf{C}}\nabla\widetilde{G}_N) & \text{in } \widetilde{\Omega},\\ \widetilde{r}_N=0 & \text{on } \partial\widetilde{\Omega},\end{cases}$$

where $\widetilde{G}_N\in(H^1(\widetilde{\Omega}))^3$ *is the approximate solution defined by* (56). *If* $\nabla'\phi(p')$ *exists, then one has*

$$\begin{aligned}&\|\nabla\widetilde{r}_N\|_{(L^2(\widetilde{\Omega}))^3}\\ \lesssim\ & N^{-1/2}+N^{1/2}\left(\int_{|y'|\le N^{-1/2}}|\nabla'\phi(y'+p')-\nabla'\phi(p')|^2\,dy'\right)^{1/2},\end{aligned}\tag{65}$$

for some constant $C>0$ *independent of* $\widetilde{G}_N$ *and* $\widetilde{r}_N$.

Proof. The estimate (65) holds by using the standard elliptic regularity estimate of $\widetilde{r}_N$, Hardy's inequality for $\widetilde{u}_N$ and Lemma 4.1. The detailed proof is the same as the one of [3, Lemma 2], thus we refer the interested readers to [3]. □

4.2. **Proof of Theorem 1.4.** Let us consider the function u_N with $\widetilde{u}_N=F^*u_N$ and define $\widetilde{u}_N:=\kappa^{-1/2}(\widetilde{G}_N+\widetilde{r}_N)$, then $\widetilde{u}_N\in(H^1(\widetilde{\Omega}))^3$ is the solution of

$$\nabla\cdot(\widetilde{\mathbf{C}}\nabla\widetilde{u}_N)=0 \text{ in } \widetilde{\Omega}\quad\text{ with }\quad \widetilde{u}_N=\widetilde{G}_N \text{ on } \partial\widetilde{\Omega}.\tag{66}$$

Denote $f_N=u_N|_{\partial\Omega}$. From formula (54), one has

$$\kappa^{-1}\int_{\partial\widetilde{\Omega}}\Lambda_{\widetilde{\mathbf{C}}}\widetilde{G}_N\cdot\overline{\widetilde{G}_N}\,dS=\int_{\widetilde{\Omega}}\widetilde{\mathbf{C}}\nabla\widetilde{u}_N:\nabla\overline{\widetilde{u}_N}\,dx.\tag{67}$$

Recall that $(\Pi,\mathcal{H},\mathbb{P})$ is a complete probability space, and $\{X_\alpha:\alpha\in\mathbb{N}^2\}$ is a countable family of independent complex Gaussian random variables $X_\alpha:\varpi\in\Pi\mapsto X_\alpha(\varpi)\in\mathbb{C}$ as in Section 3 such that (6) holds with standard expectation of a random variable defined by $\mathbb{E}X=\int_\Pi X\,d\mathbb{P}$. Let $\{\mathbf{e}_n:n\in\mathbb{N}\}$ be an orthonormal basis of $(L^2(\partial\Omega))^3$, then we define the noisy data for the isotropic elasticity system via the bilinear form

$$\mathcal{N}_{\mathbf{C}}(f,g):=\int_{\partial\Omega}\Lambda_{\mathbf{C}}f\cdot\overline{g}\,dS+\sum_{\alpha\in\mathbb{N}^2}(f|\mathbf{e}_{\alpha_1})(g|\mathbf{e}_{\alpha_2})X_\alpha,\tag{68}$$

for $f,g\in(H^{1/2}(\partial\Omega))^3$, where $\alpha=(\alpha_1,\alpha_2)$ and $(W|w)=\int_{\partial\Omega}W\cdot\overline{w}dS\in\mathbb{C}$, for any $W,w\in(L^2(\Omega))^3$.

Next, by change of variables, (67), Lemma 4.1, Lemma 4.2 and the Cauchy-Schwarz inequality, the equation (68) yields that

$$\begin{aligned}\mathcal{N}_{\mathbf{C}}(f_N,f_N)&=\kappa^{-1}\int_{\partial\widetilde{\Omega}}\Lambda_{\widetilde{\mathbf{C}}}\widetilde{G}_N\cdot\overline{\widetilde{G}_N}\,dS+\sum_{\alpha\in\mathbb{N}^2}(f_N|\mathbf{e}_{\alpha_1})(\overline{f_N}|\mathbf{e}_{\alpha_2})X_\alpha\\ &=\sum_{i,j=1}^3 Z_{ij}(P)a_i\overline{a_j}+\sum_{\alpha\in\mathbb{N}^2}(f_N|\mathbf{e}_{\alpha_1})(\overline{f_N}|\mathbf{e}_{\alpha_2})X_\alpha\\ &\quad+O\left(N^{-1/2}+\mathcal{E}(M)\right),\end{aligned}\tag{69}$$

where Z_{ij} is given by (12) and $\mathcal{E}(M)$ is an error term given by

$$\mathcal{E}(M) := N^{1/2} \left(\int_{|y'| \leq N^{-1/2}} |\nabla' \phi(y' + p') - \nabla' \phi(p')|^2 \, dy' \right)^{1/2}.$$

We then state the following proposition by replacing $L^2(\partial\Omega)$ by $(L^2(\partial\Omega))^3$ in Lemma 2.3 of [7].

Proposition 2. *Let* $\mathbf{C}$ *be the isotropic elastic tensor and* Ω *be a bounded Lipschitz domain in* $\mathbb{R}^3$*. Then there is a complete probability space* $(\Pi, \mathcal{H}, \mathbb{P})$*, and a countable family* $\left\{ X_\alpha : \alpha \in \mathbb{N}^2 \right\}$ *of independent complex random variables satisfying* (6)*. In addition, for any* $f, g \in (L^2(\partial\Omega))^3$*, we have*

$$\mathbb{E} \left| \sum_{\alpha \in \mathbb{N}^2} (f|\mathbf{e}_{\alpha_1})(g|\mathbf{e}_{\alpha_2}) X_\alpha \right|^2 = \|f\|^2_{(L^2(\partial\Omega))^3} \|g\|^2_{(L^2(\partial\Omega))^3}. \tag{70}$$

Furthermore, the corrupted data

$$\mathcal{N}_{\mathbf{C}} : (H^{1/2}(\partial\Omega))^3 \times (H^{1/2}(\partial\Omega))^3 \to L^2(\Pi, \mathcal{H}, \mathbb{P})$$

and the following estimate holds

$$\mathbb{E} \left| \mathcal{N}_{\mathbf{C}}(f, g) \right|^2 \leq C \left(1 + \|\lambda\|_{L^\infty(\Omega)} + \|\mu\|_{L^\infty(\Omega)} \right) \|f\|^2_{(H^{1/2}(\partial\Omega))^3} \|g\|^2_{(H^{1/2}(\partial\Omega))^3}, \tag{71}$$

for any $f, g \in (H^{1/2}(\partial\Omega))^3$*, and for some constant* $C > 0$ *depending on* $\partial\Omega$*. In particular,* (71) *implies that* $|\mathcal{N}_{\mathbf{C}}(f, g)| < \infty$ *almost surely.*

From (70), one can obtain

$$\mathbb{E} \left| \sum_{\alpha \in \mathbb{N}^2} (f_N|\mathbf{e}_{\alpha_1})(f_N|\mathbf{e}_{\alpha_2}) X_\alpha \right|^2 = \|f_N\|^4_{(L^2(\partial\Omega))^3} \leq C_{\partial\Omega} N^{-2}, \tag{72}$$

for some constant $C_{\partial\Omega} > 0$ depending only on the Lipschitz function ϕ, where the last inequality comes from the definition of the oscillating boundary data.

Under some suitable assumptions on the boundary $\partial\widetilde{\Omega}$, the last term in (69) converges to zero as $N \to \infty$. Thus, the Lamé parameters λ and μ at $P \in \partial\Omega$ can be reconstructed from $\mathcal{N}_{\mathbf{C}}(f_N, f_N)$ and (72) by taking $N \to \infty$.

Proof of Theorem 1.4. Following the argument of [7] or of Section 3, one can obtain the boundary determination as well as the rate of convergence for Lamé moduli, which finishes the proof of Theorem 1.4. □

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