

MONOTONICITY AND LOCAL UNIQUENESS FOR AN ISOTROPIC NONLOCAL ELLIPTIC EQUATION

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ABSTRACT. We extend monotonicity-based inversion methods to an inverse coefficient problem for the isotropic nonlocal elliptic equation

$$(-\nabla \cdot \sigma \nabla)^s u = 0 \quad \text{in } \Omega \subset \mathbb{R}^n,$$

where $0 < s < 1$, $n \geq 3$, and Ω is a bounded open set. We establish a monotonicity relation between the leading coefficient σ and the (partial) exterior Dirichlet-to-Neumann (DN) map. Our main result shows that a monotonicity ordering of the coefficients implies a corresponding ordering of the DN maps. Furthermore, we construct localized potentials for the nonlocal equation, which yield a local uniqueness result for the fractional inverse problem.

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1. INTRODUCTION

Inverse problems for nonlocal operators have attracted considerable attention in recent years, with the fractional Laplacian $(-\Delta)^s$ ($0 < s < 1$) being a key example. This operator emerges naturally in models of anomalous stochastic diffusion, characterized by jumps and long-range interactions, as explored in works such as [BV16, RO16]. In contrast to the classical Laplacian ($s = 1$), which describes standard Brownian motion, the nonlocal nature of the fractional Laplacian introduces significant complexity. Nevertheless, recent progress indicates that inverse problems for nonlocal equations may be more manageable than their local counterparts, thanks to robust properties like unique continuation and Runge approximation.

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The Calderón problem for the fractional Schrödinger equation was first investigated in [GSU20]. Central to this work is the Runge approximation property, which allows any L^2 function to be approximated by solutions to the fractional Schrödinger equation. This property derives from the unique continuation principle (UCP), which asserts that if $u = (-\Delta)^s u = 0$ in any nonempty open subset, then $u \equiv 0$ across $\mathbb{R}^n$. Subsequent research has extended these findings to variable-coefficient nonlocal elliptic operators [GLX17], addressing a challenge that remains unresolved for their local counterparts.

Building on these foundations, a substantial body of literature has developed. Simultaneous determination results are established in [CLL19, CLR20, LL23], while stability estimates are derived in [RS20, KWL22, RS18]. Additionally, monotonicity-based methods for nonlocal inverse problems have been introduced in [HL19, HL20, Lin22]. Further contributions, encompassing both linear and nonlinear settings, are detailed in [LL22, GRSU20, CMRU22, LZ23] and references therein.

Recent advancements have introduced innovative approaches to inverse problems for nonlocal operators. The recovery of leading coefficients has been achieved through novel nonlocal-to-local reductions utilizing the Caffarelli–Silvestre extension, as demonstrated in [CGRU23, Rül25, LLU23, LZ24]. Additionally, local-to-nonlocal reduction is characterized in the transversally anisotropic setting in [LNZ24]. Meanwhile, heat semigroup methods on closed Riemannian manifolds have proven effective for addressing the fractional anisotropic Calderón problem, as explored in [FGKU24, Fei24, FKU24, Lin24, FGK⁺25]. Furthermore, the entanglement principle for nonlocal elliptic operators, investigated in [FKU24, FL24], shows promise for analyzing systems of nonlocal equations. The Calderón problem for the logarithmic Laplacian—a zero-order nonlocal operator—has also been recently addressed in [HLW25]. For a comprehensive overview of inverse problems for nonlocal operators, we refer readers to the recent monograph [LL25].

Mathematical formulation. Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain, for $n \geq 3$, and $0 < s < 1$. Consider the exterior value problem

$$(1.1) \quad \begin{cases} (-\nabla \cdot \sigma \nabla)^s u = 0 & \text{in } \Omega, \\ u = f & \text{in } \Omega_e, \end{cases}$$

where $\sigma = \sigma(x) \in C^2(\mathbb{R}^n)$ satisfying

$$(1.2) \quad 0 < \lambda \leq \sigma(x) \leq \lambda^{-1} \text{ for } x \in \overline{\Omega} \quad \text{and} \quad \sigma = 1 \text{ in } \Omega_e,$$

for some $\lambda \in (0, 1)$, and

$$\Omega_e := \mathbb{R}^n \setminus \overline{\Omega}$$

stands for the exterior domain. Throughout this work, we assume that the condition (1.2) always holds. Let $W \subset \Omega_e$ be a bounded open set with $\overline{\Omega} \cap \overline{W} = \emptyset$, and Λ_σ be the *Dirichlet-to-Neumann* (DN) map of (1.1), which is given by

$$(1.3) \quad \Lambda_\sigma : \tilde{H}^s(W) \rightarrow H^{-s}(W), \quad f \mapsto (-\nabla \cdot \sigma \nabla)^s u^f|_W,$$

where $u^f \in H^s(\mathbb{R}^n)$ is the solution to (1.1). In this work, we are interested in a monotonicity relation between the DN map and leading coefficients σ . Throughout this work, let us assume that σ is a positive bounded scalar function, with $\sigma|_{\Omega_e}$ being known a priori.

In the works [HL19, HL20], the authors demonstrated if-and-only-if monotonicity relations between the DN maps with lower order bounded potentials. To our best knowledge, there is no existing literature to consider such relations between the DN maps with leading coefficients in (1.1). In fact, in Section 3, we are going to prove

that

$$\sigma_1 \geq \sigma_2 \text{ in } \Omega \implies \Lambda_{\sigma_1} \geq \Lambda_{\sigma_2}.$$

Let us make the above sentence meaningful. On the one hand, the relation $\sigma_1 \geq \sigma_2$ in Ω is referred to $\sigma_1(x) \geq \sigma_2(x)$ for almost every (a.e.) $x \in \Omega$. In this work, we have assumed that σ_1, σ_2 satisfy (1.2), so we justify the half-ordering $\sigma_1 \geq \sigma_2$ is referred to $\sigma_1(x) \geq \sigma_2(x)$, for all $x \in \Omega$. On the other hand, we write $\Lambda_{\sigma_1} \geq \Lambda_{\sigma_2}$, if it holds in the quadratic sense that

$$\langle (\Lambda_{\sigma_1} - \Lambda_{\sigma_2}) f, f \rangle \geq 0,$$

for any $f \in C_c^\infty(W)$. Here, $\langle \cdot, \cdot \rangle$ denotes the duality pairing in a suitable sense (see Section 2).

The method to study inverse problems using the combination of monotonicity relations with localized potentials was found in [Geb08]. Due to this remarkable approach and the flexibility of this method, there is some literature in this direction, [AH13, Har09, HS10, Har12, HU13, BHHM17, HU17, BHKS18, GH18, HPS19b, HLL18, SKJ⁺19, HPS19a]. In further, several works build practical reconstruction methods based on monotonicity properties [TR02, HLU15, HU15, HM16, MVVT16, TSV⁺16, Gar17, GS17, SUG⁺17, VMC⁺17, HM18, ZHS18, GS19].

We revisit the fractional Calderón problem for the isotropic nonlocal elliptic equation (1.1). By [CGRU23], as σ is a scalar function fulfilling (1.2), the authors proved the global uniqueness for (1.1). In other words, the isotropic scalar function σ can be determined uniquely by the DN map Λ_σ . In this work, we prove a local uniqueness result of (1.1):

Theorem 1.1 (Local uniqueness). *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with Lipschitz boundary $\partial\Omega$ for $n \geq 3$ and $W \Subset \Omega_e$ be a nonempty open subset. Let $\mathcal{O} \subseteq \bar{\Omega}$ be a connected relatively open subset such that $\mathcal{O} \cap \partial\Omega \neq \emptyset$. Let $\sigma_j \in C^2(\mathbb{R}^n)$ satisfy (1.2), and Λ_{σ_j} be the DN map of*

$$(1.4) \quad \begin{cases} (-\nabla \cdot \sigma_j \nabla)^s u_j = 0 & \text{in } \Omega, \\ u_j = f & \text{in } \Omega_e, \end{cases}$$

for $j = 1, 2$. Suppose

$$(1.5) \quad \text{either } \sigma_1 \leq \sigma_2 \text{ in } \mathcal{O} \quad \text{or} \quad \sigma_1 \geq \sigma_2 \text{ in } \mathcal{O},$$

then

$$\Lambda_{\sigma_1} f|_W = \Lambda_{\sigma_2} f|_W, \quad \text{for any } f \in C_c^\infty(W),$$

implies $\sigma_1 = \sigma_2$ in $\mathcal{O}$.

Remark 1.2. *There is an alternative way to show that the nonlocal DN maps Λ_σ of (1.1) determine their local DN maps of*

$$\begin{cases} \nabla \cdot (\sigma \nabla v) = 0 & \text{in } \Omega, \\ v = g \in H^{1/2}(\partial\Omega) & \text{on } \partial\Omega, \end{cases}$$

whenever the condition (1.2) holds. Therefore, combining $\sigma|_{\partial\Omega} = 1$, one can use the well-known result from [SU87] so that the scalar conductivity σ can be determined in the entire domain Ω . In other words, a global uniqueness result for (1.1) can be derived by using this nonlocal-to-local reduction procedure.

Organization of the article. The structure of the paper is as follows. In Section 2, we introduce the function spaces, nonlocal operators, and extension problems that will be used throughout the article. Section 3 is devoted to proving the monotonicity relation between the DN maps and the leading coefficients. A key step in

this analysis is the construction of localized potentials for the extension problem, which is carried out in Section 4. Finally, in Section 5, we combine these ingredients to establish our main results.

2. PRELIMINARIES

2.1. Function spaces. Let us quickly review some function spaces together with the definition of the operator $(-\nabla \cdot \sigma \nabla)^s$, which are introduced in many related articles. Given $0 < s < 1$, the space $H^s(\mathbb{R}^n) = W^{s,2}(\mathbb{R}^n)$ denotes the usual L^2 -based fractional Sobolev space with the given norm

$$\|u\|_{H^s(\mathbb{R}^n)} := \|u\|_{L^2(\mathbb{R}^n)} + [u]_{H^s(\mathbb{R}^n)}$$

where $[\cdot]_{H^s(B)}$

$$[u]_{H^s(B)} := \left(\int_{B \times B} \frac{|u(x) - u(y)|^2}{|x - y|^{n+2s}} dx dy \right)^{1/2},$$

is the seminorm, for any open set $B \subset \mathbb{R}^n$.

Motivated by [GSU20], let $B \subset \mathbb{R}^n$ be a nonempty bounded open set with Lipschitz boundary, $C_c^\infty(B)$ contains all $C^\infty(\mathbb{R}^n)$ -smooth functions supported in B . Given $b \in \mathbb{R}$, let us adopt the following notions

$$\begin{aligned} H^b(B) &:= \{u|_B : u \in H^b(\mathbb{R}^n)\}, \\ \tilde{H}^b(B) &:= \text{closure of } C_c^\infty(B) \text{ in } H^b(\mathbb{R}^n), \\ H_0^b(B) &:= \text{closure of } C_c^\infty(B) \text{ in } H^b(B), \end{aligned}$$

for different fractional Sobolev spaces. $H^b(B)$ is complete in the sense

$$\|u\|_{H^b(B)} := \inf \{ \|w\|_{H^b(\mathbb{R}^n)} : w \in H^b(\mathbb{R}^n) \text{ and } w|_B = u \}.$$

As the exponent $b = s \in (0, 1)$, $H^{-s}(B)$ stands for the dual space of $\tilde{H}^s(B)$, so that $H^{-s}(B)$ can be characterized by

$$H^{-s}(B) = \{u|_B : u \in H^{-s}(\mathbb{R}^n)\} \quad \text{with} \quad \inf_{w \in H^s(\mathbb{R}^n), w|_B = u} \|w\|_{H^s(\mathbb{R}^n)},$$

In addition, we always denote

$$(\tilde{H}^s(B))^* = H^{-s}(B) \quad \text{and} \quad (H^s(B))^* = \tilde{H}^{-s}(B).$$

throughout this paper.

Moreover, we also introduce L^2 -weighted Sobolev spaces for the Caffarelli-Silvestre extension problem. Let $A \subseteq \mathbb{R}_+^{n+1}$ be a nonempty set, $y > 0$, and consider $L^2(A, y^{1-2s})$ as the L^2 -based weighted Sobolev space give by

$$L^2(A, y^{1-2s}) := \{\tilde{u} = \tilde{u}(x, y) : \mathbb{R}_+^{n+1} \rightarrow \mathbb{R} : \|\tilde{u}\|_{L^2(D, y^{1-2s})} < \infty\},$$

where

$$\|\tilde{u}\|_{L^2(A, y^{1-2s})} := \left(\int_A y^{1-2s} |\tilde{u}|^2 dx dy \right)^{1/2}.$$

Define

$$H^1(A, y^{1-2s}) := \{\tilde{u} \in L^2(A, y^{1-2s}) : \nabla_{x,y} \tilde{u} \in L^2(A, y^{1-2s})\},$$

with $\nabla_{x,y} = (\nabla_x, \partial_y) = (\nabla, \partial_y)$ being the total derivative for $(x, y) \in \mathbb{R}^{n+1}$. It is known that the $H^1(A, y^{1-2s})$ has a natural inner product structure that

$$(\tilde{u}, \tilde{v})_{L^2(A, y^{1-2s})} := \int_A y^{1-2s} \tilde{u} \tilde{v} dx dy,$$

for any $A \subseteq \mathbb{R}_+^{n+1}$, so that $\|\tilde{u}\|_{L^2(A, y^{1-2s})}^2 = (\tilde{u}, \tilde{u})_{L^2(A, y^{1-2s})}$. In particular, let us denote another weighted Sobolev space by

$$H_x^1(A, y^{1-2s}) := \{\tilde{u} \in L^2(A, y^{1-2s}) : \nabla \tilde{u} \in L^2(A, y^{1-2s})\},$$

and clearly, $H_x^1(A, y^{1-2s}) \subset H^1(A, y^{1-2s})$.

2.2. Nonlocal operators. We next review the nonlocal elliptic operator $\mathcal{L}^s$ ($0 < s < 1$), where

$$\mathcal{L} := -\nabla \cdot (\sigma \nabla)$$

is a second-order uniformly elliptic operator of divergence form. Note that when we define the nonlocal elliptic operator $\mathcal{L}^s$, we do not need to assume the condition $\sigma = 1$ in Ω_e . It is known that the nonlocal operator $\mathcal{L}^s = (-\nabla \cdot \sigma \nabla)^s$ can be defined by

$$\mathcal{L}^s := \frac{1}{\Gamma(-s)} \int_0^\infty (e^{-t\mathcal{L}} - \text{Id}) \frac{dt}{t^{1+s}},$$

where $e^{-t\mathcal{L}}$ stands for the heat kernel of $\partial_t + \mathcal{L}$ in $\mathbb{R}^n \times (0, \infty)$, and Id denotes the identity map. Moreover, the operator $\mathcal{L}^s$ can be equivalently characterized by the famous Caffarelli-Silvestre [CS07] and Stinga-Torrea extension problems [ST10].

We can also define the bilinear form of the exterior problem (1.1). It is known that the exterior problem (1.1) is well-posed, and the DN map can be defined by the bilinear form

$$(2.1) \quad \langle \mathcal{L}^s u, w \rangle = \frac{1}{2} \int_{\mathbb{R}^n \times \mathbb{R}^n} (u(x) - u(z))(w(x) - w(z)) K_s(x, z) dx dz,$$

where $K_s(x, z)$ is can be derived by the heat kernel

$$(2.2) \quad K_s(x, z) := \frac{1}{\Gamma(-s)} \int_0^\infty p_t(x, z) \frac{dt}{t^{1+s}}.$$

As $p_t(x, z)$ stands for the symmetric heat kernel for $\mathcal{L}$ so that

$$(e^{-t\mathcal{L}} f)(x) = \int_{\mathbb{R}^n} p_t(x, z) f(z) dz, \text{ for } x \in \mathbb{R}^n, t > 0,$$

and

$$\begin{cases} (\partial_t + \mathcal{L})(e^{-t\mathcal{L}} f) = 0 & \text{for } (x, t) \in \mathbb{R}^n \times (0, \infty), \\ (e^{-t\mathcal{L}} f)(x, 0) = f(x) & \text{for } x \in \mathbb{R}^n. \end{cases}$$

The heat kernel $p_t(x, z)$ enjoys pointwise estimates (cf. [Dav90])

$$c_1 e^{-\alpha_1 \frac{|x-z|^2}{t}} t^{-\frac{n}{2}} \leq p_t(x, z) \leq c_2 e^{-\alpha_2 \frac{|x-z|^2}{t}} t^{-\frac{n}{2}}, \text{ for } x, z \in \mathbb{R}^n,$$

for some positive constants c_1, c_2, α_1 and α_2 . Therefore, we can obtain pointwise estimate for $K_s(x, z)$ from the formula (2.2)

$$\frac{C_1}{|x-z|^{n+2s}} \leq K_s(x, z) \leq \frac{C_2}{|x-z|^{n+2s}}, \text{ for } x, z \in \mathbb{R}^n,$$

for some constants $C_1, C_2 > 0$, which ensures the well-posedness of (1.1) (see [GLX17, Section 3] for detailed arguments). This is equivalent to say that given any $f \in H^s(\Omega_e)$, the equation (1.1) admits a unique solution $u \in H^s(\mathbb{R}^n)$. This implies the DN map Λ_σ is well-defined, and there holds

$$\langle \Lambda_\sigma f, g \rangle_{H^{-s}(W) \times \tilde{H}^s(W)} = \frac{1}{2} \int_{\mathbb{R}^n \times \mathbb{R}^n} (u(x) - u(z))(w(x) - w(z)) K_s(x, z) dx dz$$

for any $f, g \in \tilde{H}^s(W)$, which justifies (1.3).

2.3. The extension problem. Recalling that the extension problem for the non-local operator $\mathcal{L}^s$ with $0 < s < 1$ is

$$(2.3) \quad \begin{cases} \nabla_{x,y} \cdot (y^{1-2s} \tilde{\sigma} \nabla_{x,y} \tilde{u}) = 0 & \text{in } \mathbb{R}_+^{n+1}, \\ \tilde{u}(x, 0) = u(x) & \text{on } \partial \mathbb{R}_+^{n+1} = \mathbb{R}^n, \end{cases}$$

where $\nabla_{x,y} = (\nabla, \partial_y)$ with $\nabla = \nabla_x$. It is known that there holds the following relation holds

$$(2.4) \quad -\lim_{y \rightarrow 0} y^{1-2s} \partial_y \tilde{u} = d_s (-\nabla \cdot \sigma \nabla)^s u \text{ in } \mathbb{R}^n,$$

where $\mathbb{R}_+^{n+1} := \{(x, y) \in \mathbb{R}^{n+1} : x \in \mathbb{R}^n, y > 0\}$,

$$(2.5) \quad d_s = \frac{\Gamma(1-s)}{2^{2s-1}\Gamma(s)} > 0$$

is a constant depending only on $s \in (0, 1)$, and $\tilde{\sigma}$ is an $(n+1) \times (n+1)$ matrix of the form

$$(2.6) \quad \tilde{\sigma}(x) = \begin{pmatrix} \sigma(x) \mathbf{I}_n & 0 \\ 0 & 1 \end{pmatrix}.$$

with the $n \times n$ identity matrix $\mathbf{I}_n$. This type of extension problem was called the Caffarelli-Silvestre or Stinga-Torrea extension problem in the literature.

3. MONOTONICITY RELATION

In this section, we want to show that $\sigma_1 \geq \sigma_2$ a.e. in Ω implies that $\Lambda_{\sigma_1} \geq \Lambda_{\sigma_2}$ in the quadratic sense. More specifically, with the discussions from the previous section, we use the notion

$$\Lambda_{\sigma_1} \geq \Lambda_{\sigma_2} \iff \langle \Lambda_{\sigma_1} f, f \rangle_{H^{-s}(W) \times \tilde{H}^s(W)} \geq \langle \Lambda_{\sigma_2} f, f \rangle_{H^{-s}(W) \times \tilde{H}^s(W)}$$

for any $f \in C_c^\infty(W)$. In what follows, we may use $\langle \cdot, \cdot \rangle \equiv \langle \cdot, \cdot \rangle_{H^{-s}(W) \times \tilde{H}^s(W)}$ to denote the duality pairing, which simplifies the notations, provided that there are no further confusions.

Lemma 3.1 (Monotonicity relations). *Let $\Omega \subset \mathbb{R}^n$ and $W \Subset \Omega_e$ be bounded open sets with Lipschitz boundaries, for $n \in \mathbb{N}$. Let σ_j be a bounded positive coefficient satisfying (1.2), and Λ_{σ_j} be the DN map of (1.4), for $j = 1, 2$. Then there hold that*

$$(3.1) \quad \begin{aligned} & \int_{\Omega \times (0, \infty)} y^{1-2s} (\sigma_1 - \sigma_2) |\nabla \tilde{u}_1^f|^2 dx dy \\ & \leq d_s \langle (\Lambda_{\sigma_1} - \Lambda_{\sigma_2}) f, f \rangle \\ & \leq \int_{\Omega \times (0, \infty)} y^{1-2s} (\sigma_1 - \sigma_2) |\nabla \tilde{u}_2^f|^2 dx dy, \end{aligned}$$

and

$$(3.2) \quad \begin{aligned} & \int_{\Omega \times (0, \infty)} y^{1-2s} \frac{\sigma_2}{\sigma_1} (\sigma_1 - \sigma_2) |\nabla \tilde{u}_2^f|^2 dx dy \\ & \leq d_s \langle (\Lambda_{\sigma_1} - \Lambda_{\sigma_2}) f, f \rangle \\ & \leq \int_{\Omega \times (0, \infty)} y^{1-2s} (\sigma_1 - \sigma_2) |\nabla \tilde{u}_2^f|^2 dx dy, \end{aligned}$$

where $\tilde{u}_j^f$ is the solution to the extension problem (2.3) as $\sigma = \sigma_j$ and $\tilde{u}_j^f(x, 0) = u_j^f(x)$ in $\mathbb{R}^n$, where $u_j^f \in H^s(\mathbb{R}^n)$ is the solution to (1.4) as $j = 2$, with the constant $d_s > 0$ in (3.3).

Remark 3.2. Let us point out that

- (i) As $s = 1$, the monotonicity implication from the conductivity to the DN map is easy to derive using the bilinear form and integration by parts. However, as shown in Section 2, it is known that the bilinear form (2.1) may not be useful to derive this monotonicity relation. Instead, we will use the Caffarelli-Silvestre type extension problem (2.3) to prove the above lemma.
- (ii) Note that the both sides of (3.1) contain solutions $\tilde{u}_1^f$ and $\tilde{u}_2^f$, but in the both sides of (3.2) only consist the solution $\tilde{u}_2^f$. The relation (3.1) is already useful in the determination of the leading coefficients. For further applications in inverse problems (such as inverse obstacle problems), the monotonicity formula (3.2) would be needed. However, we do not pursue this problem in this article.

Proof of Lemma 3.1. Let $\tilde{\sigma}_j$ be of the form (2.6) as $\sigma = \sigma_j$, and $\tilde{u}_j^f$ denote the solution to (2.3) as $\tilde{u} = \tilde{u}_j^f$ with $\tilde{u}_j^f(x, 0) = u_j^f(x)$ in $\mathbb{R}^n$, for $j = 1, 2$. Here $u_j^f \in H^s(\mathbb{R}^n)$ is the solution to (1.4) for $j = 1, 2$. Then there holds that

$$(3.3) \quad -\lim_{y \rightarrow 0} y^{1-2s} \partial_y \tilde{u}_j^f = d_s (-\nabla \cdot \sigma_j \nabla)^s u_j^f \text{ in } \mathbb{R}^n,$$

for $j = 1, 2$, where d_s is a positive constant given by (2.5). On the one hand, there holds

$$(3.4) \quad \langle \Lambda_{\sigma_j} f, f \rangle = \int_W (-\nabla \cdot \sigma_j \nabla)^s u_j^f \cdot f \, dx,$$

for $j = 1, 2$. On the other hand, we have

$$\begin{aligned} 0 &= \int_{\mathbb{R}_+^{n+1}} \nabla_{x,y} \cdot (y^{1-2s} \tilde{\sigma} \nabla_{x,y} \tilde{u}_1^f) \tilde{u}_1^f \, dx dy \\ &= - \int_{\mathbb{R}^n} \left(\lim_{y \rightarrow 0} y^{1-2s} \partial_y \tilde{u}_1^f \right) \tilde{u}_1^f(x, 0) \, dx - \int_{\mathbb{R}_+^{n+1}} y^{1-2s} \tilde{\sigma}_j \nabla_{x,y} \tilde{u}_1^f \cdot \nabla_{x,y} \tilde{u}_1^f \, dx dy \\ &= d_s \underbrace{\int_{\mathbb{R}^n} (-\nabla \cdot \sigma_j \nabla)^s u_1^f \cdot u_1^f \, dx}_{\text{By (2.4)}} - B_{\tilde{\sigma}_1}(\tilde{u}_1^f, \tilde{u}_1^f) \\ &= \underbrace{d_s \int_W (-\nabla \cdot \sigma_j \nabla)^s u_1^f \cdot f \, dx}_{\text{Since } u^f \text{ solves (1.4) and } u_1^f|_{\Omega_c} = f} - B_{\tilde{\sigma}_1}(\tilde{u}_1^f, \tilde{u}_1^f), \end{aligned}$$

which implies that

$$(3.5) \quad B_{\tilde{\sigma}_1}(\tilde{u}_1^f, \tilde{u}_1^f) = d_s \langle \Lambda_{\sigma_1} f, f \rangle,$$

where

$$\begin{aligned} B_{\tilde{\sigma}_j}(\tilde{u}, \tilde{w}) &:= \int_{\mathbb{R}_+^{n+1}} y^{1-2s} \tilde{\sigma}_j \nabla_{x,y} \tilde{u} \cdot \nabla_{x,y} \tilde{w} \, dx dy \\ &= \int_{\mathbb{R}_+^{n+1}} y^{1-2s} \sigma_j \nabla \tilde{u} \cdot \nabla \tilde{w} \, dx dy + \int_{\mathbb{R}_+^{n+1}} y^{1-2s} \partial_y \tilde{u} \partial_y \tilde{w} \, dx dy \end{aligned}$$

is a symmetric bilinear form, for $j = 1, 2$ and any functions $\tilde{u}$ and $\tilde{w}$.

Similarly, we also have

$$(3.6) \quad B_{\tilde{\sigma}_1}(\tilde{u}_1^f, \tilde{u}_2^f) = d_s \langle \Lambda_{\sigma_1} f, f \rangle,$$

Combining (3.4), (3.5) and (3.6), we have

$$\begin{aligned} d_s \langle \Lambda_{\sigma_1} f, f \rangle &= B_{\tilde{\sigma}_1}(\tilde{u}_1^f, \tilde{u}_1^f) = B_{\tilde{\sigma}_1}(\tilde{u}_1^f, \tilde{u}_2^f), \\ d_s \langle \Lambda_{\sigma_2} f, f \rangle &= B_{\tilde{\sigma}_2}(\tilde{u}_2^f, \tilde{u}_2^f). \end{aligned}$$

Direct computations yield that

$$\begin{aligned}
 (3.7) \quad 0 &\leq B_{\tilde{\sigma}_1}(\tilde{u}_1^f - \tilde{u}_2^f, \tilde{u}_1^f - \tilde{u}_2^f) \\
 &= B_{\tilde{\sigma}_1}(\tilde{u}_1^f, \tilde{u}_1^f) - 2B_{\tilde{\sigma}_1}(\tilde{u}_1^f, \tilde{u}_2^f) + B_{\tilde{\sigma}_1}(\tilde{u}_2^f, \tilde{u}_2^f) \\
 &= -d_s \langle \Lambda_{\sigma_1} f, f \rangle + d_s \langle \Lambda_{\sigma_2} f, f \rangle + B_{\tilde{\sigma}_1}(\tilde{u}_2^f, \tilde{u}_2^f) - B_{\tilde{\sigma}_2}(\tilde{u}_2^f, \tilde{u}_2^f),
 \end{aligned}$$

and this implies that

$$\begin{aligned}
 (3.8) \quad &d_s (\langle \Lambda_{\sigma_1} f, f \rangle - \langle \Lambda_{\sigma_2} f, f \rangle) \\
 &\leq \left(\int_{\mathbb{R}_+^{n+1}} y^{1-2s} \sigma_1 |\nabla \tilde{u}_2^f|^2 dx dy + \int_{\mathbb{R}_+^{n+1}} y^{1-2s} |\partial_y \tilde{u}_2^f|^2 dx dy \right) \\
 &\quad - \left(\int_{\mathbb{R}_+^{n+1}} y^{1-2s} \sigma_2 |\nabla \tilde{u}_2^f|^2 dx dy + \int_{\mathbb{R}_+^{n+1}} y^{1-2s} |\partial_y \tilde{u}_2^f|^2 dx dy \right) \\
 &\leq \int_{\mathbb{R}_+^{n+1}} y^{1-2s} (\sigma_1 - \sigma_2) |\nabla \tilde{u}_2^f|^2 dx dy \\
 &= \int_{\Omega \times (0, \infty)} y^{1-2s} (\sigma_1 - \sigma_2) |\nabla \tilde{u}_2^f|^2 dx dy,
 \end{aligned}$$

where we used $\sigma_1 = \sigma_2$ in Ω_e . This proves the right-hand side ordering in (3.1). Interchanging the indices $j = 1, 2$ in (3.8), we can obtain the left-hand side in (3.1).

Finally, for the left-hand side in (3.2), let us interchange σ_1 and σ_2 in (3.7) and (3.8), then we have

$$\begin{aligned}
 &d_s \langle (\Lambda_{\sigma_1} - \Lambda_{\sigma_2}) f, f \rangle \\
 &= \int_{\Omega \times (0, \infty)} y^{1-2s} (\sigma_1 - \sigma_2) |\nabla \tilde{u}_1^f|^2 dx dy \\
 &\quad + \int_{\Omega \times (0, \infty)} y^{1-2s} \sigma_2 |\nabla (\tilde{u}_2^f - \tilde{u}_1^f)|^2 dx dy \\
 &= \int_{\Omega \times (0, \infty)} y^{1-2s} (\sigma_1 |\nabla \tilde{u}_1^f|^2 + \sigma_2 |\nabla \tilde{u}_2^f|^2 - 2\sigma_2 \nabla \tilde{u}_1^f \cdot \nabla \tilde{u}_2^f) dx dy \\
 &= \underbrace{\int_{\Omega \times (0, \infty)} y^{1-2s} \sigma_1 \left| \nabla \tilde{u}_1^f - \frac{\sigma_2}{\sigma_1} \nabla \tilde{u}_2^f \right|^2 dx dy}_{\text{nonnegative}} \\
 &\quad + \int_{\Omega \times (0, \infty)} y^{1-2s} \left(\sigma_2 - \frac{\sigma_2^2}{\sigma_1} \right) |\nabla \tilde{u}_2^f|^2 dx dy \\
 &\geq \int_{\Omega \times (0, \infty)} y^{1-2s} \frac{\sigma_2}{\sigma_1} (\sigma_1 - \sigma_2) |\nabla \tilde{u}_2^f|^2 dx dy,
 \end{aligned}$$

which proves (3.2). $\square$

Corollary 3.3. *Adopting all assumptions in Lemma 3.1, let σ_1, σ_2 be two positive bounded coefficients, then*

$$\sigma_1 \geq \sigma_2 \text{ implies } \Lambda_{\sigma_1} \geq \Lambda_{\sigma_2}.$$

Proof. The proof can be seen using either (3.1) or (3.2). $\square$

Remark 3.4. *From the above discussions, one can see the monotonicity relations depend only on the gradient of certain solutions with respect to the transversal direction (i.e., x -variable), but not $y \in (0, \infty)$.*

4. LOCALIZED POTENTIALS AND RUNGE APPROXIMATION

We first rewrite the extension problem (2.3). If $u \in H^s(\mathbb{R}^n)$ is a solution to (1.1) into

$$(4.1) \quad \begin{cases} \nabla_{x,y} \cdot (y^{1-2s} \tilde{\sigma} \nabla_{x,y} \tilde{u}) = 0 & \text{in } \mathbb{R}_+^{n+1}, \\ \lim_{y \rightarrow 0} y^{1-2s} \partial_y \tilde{u} = 0 & \text{on } \Omega \times \{0\}, \\ \tilde{u}(x, 0) = f(x) & \text{on } \Omega_e \times \{0\}. \end{cases}$$

Thanks to the monotonicity formulas given in the previous section, both sides in (3.1) and (3.2) depend only on the gradient along the x -direction but are independent of $y > 0$. Thus, we can construct localized potentials for the extension problem (4.1) by showing the Runge approximation with respect to the x -variable. More specifically, we have the next result.

Theorem 4.1 (Runge approximation). *Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain for $n \geq 3$, and $B, D \subseteq \bar{\Omega}$ be measurable sets, $B \setminus \bar{D}$ possess positive measure, and $\bar{\Omega} \setminus \bar{D}$ is connected to $\partial\Omega$. Then there exist functions $v = v(x) \in H^1(B \cup D)$, and $0 \neq \beta_1 = \beta_1(y) \in C_c^\infty((0, \infty))$, such that the function $v(x)\beta_1(y)$ can be approximated in the $H_x^1(B \cup D, y^{1-2s})$ -norm by solutions $\tilde{u}$ of (4.1), and satisfies*

$$(4.2) \quad \nabla v|_D \equiv 0 \quad \text{and} \quad \nabla v|_B \neq 0.$$

We will explicitly construct the functions $v(x)$ and $\beta_1(y)$ in the proof of Theorem 4.1. Assuming Theorem 4.1 is true, we can have the following existence of localized potentials.

Corollary 4.2 (Localized potentials). *Let $B, D \subseteq \bar{\Omega}$ be nonempty measurable sets, $B \setminus \bar{D}$ possess positive measure, and $\bar{\Omega} \setminus \bar{D}$ is connected to $\partial\Omega$. Let $W \Subset \Omega_e$ be a nonempty open subset, then there exists a sequence $\{f_k\}_{k=1}^\infty \subset C_c^\infty(W)$ such that*

$$\begin{aligned} \int_{B \times (0, \infty)} y^{1-2s} |\nabla \tilde{u}^{f_k}|^2 dx dy &\rightarrow \infty, \\ \int_{D \times (0, \infty)} y^{1-2s} |\nabla \tilde{u}^{f_k}|^2 dx dy &\rightarrow 0, \end{aligned}$$

as $k \rightarrow \infty$, where $u^{f_k} \in H^1(\mathbb{R}_+^{n+1}, y^{1-2s})$ is the solution to (4.1) with

$$\begin{cases} \lim_{y \rightarrow 0} y^{1-2s} \partial_y \tilde{u}^{f_k} = 0 & \text{in } \Omega, \\ \tilde{u}^{f_k}(x, 0) = f_k(x) & \text{in } \Omega_e, \end{cases}$$

for all $k \in \mathbb{N}$.

Proof. By using Theorem 4.1, there exist functions $v \in H^1(B \cup \bar{D})$, $0 \neq \beta_1 \in C_c^\infty((0, \infty))$, and sequence of solutions $\tilde{v}_{\tilde{f}_k} \in H^1(\mathbb{R}_+^{n+1}, y^{1-2s})$ of

$$\begin{cases} \nabla_{x,y} \cdot (y^{1-2s} \tilde{\sigma} \nabla_{x,y} \tilde{v}_{\tilde{f}_k}) = 0 & \text{in } \mathbb{R}_+^{n+1}, \\ \lim_{y \rightarrow 0} y^{1-2s} \partial_y \tilde{v}_{\tilde{f}_k} = 0 & \text{on } \Omega \times \{0\}, \\ \tilde{v}_{\tilde{f}_k}(x, 0) = \tilde{f}_k(x) & \text{on } \Omega_e \times \{0\}, \end{cases}$$

for all $k \in \mathbb{N}$, such that

$$\int_{B \times (0, \infty)} y^{1-2s} |\nabla \tilde{v}_{\tilde{f}_k}|^2 dx dy \rightarrow \int_{B \times (0, \infty)} y^{1-2s} \beta_1^2 |\nabla v|^2 dx dy > 0,$$

and

$$\int_{D \times (0, \infty)} y^{1-2s} |\nabla \tilde{v}_{\tilde{f}_k}|^2 dx dy \rightarrow \int_{D \times (0, \infty)} y^{1-2s} \beta_1^2 |\nabla v|^2 dx dy = 0.$$

Notice that the value $\int_{D \times (0, \infty)} y^{1-2s} |\nabla \tilde{v} \tilde{f}_k|^2 dx dy > 0$ for all $k \in \mathbb{N}$ thanks to the UCP of (4.1). Let

$$f_k := \frac{\tilde{f}_k}{\sqrt{\int_{D \times (0, \infty)} y^{1-2s} |\nabla \tilde{v} \tilde{f}_k|^2 dx dy}},$$

then $\tilde{u}^{f_k} = \frac{\tilde{v} \tilde{f}_k}{\sqrt{\int_{D \times (0, \infty)} y^{1-2s} |\nabla \tilde{v} \tilde{f}_k|^2 dx dy}}$ solves (4.1) with the boundary data $f = f_k \in C_c^\infty(W)$, for $k \in \mathbb{N}$. This proves the assertion. $\square$

Now, it remains to prove Theorem 4.1.

A formal proof of Theorem 4.1. Let us consider the case $s = 1/2$ and $\sigma = I_n$ (the $n \times n$ identity matrix). For simplicity, let us define $C := B \cup D \subseteq \bar{\Omega}$, such that $\bar{\Omega} \setminus \bar{C}$ is connected to $\partial\Omega$. Without loss of generality, we may assume that ∂C is Lipschitz. Given $\psi \in \tilde{H}^{-1}(C)$, by the duality argument of the Hahn-Banach theorem, we only need to claim

$$(4.3) \quad \int_0^\infty \langle \tilde{u}^f(\cdot, y), \psi(\cdot) \rangle_C dy = 0$$

implies

$$(4.4) \quad \int_0^\infty \langle \tilde{\beta}_1(y) v(\cdot), \psi(\cdot) \rangle_C dy = 0,$$

where $\langle \cdot, \cdot \rangle_C$ denotes the duality pairing between $H^1(C)$ and its dual space $\tilde{H}^{-1}(C)$ ¹. Here the function $v \in H^1(C)$ will satisfy (4.2), and $\tilde{\beta}_1 \in C_c^\infty((0, \infty))$ fulfills $\int_0^\infty \tilde{\beta}_1(y) dy = 1$, $\tilde{\beta}_1 \geq 0$, and $\text{supp}(\tilde{\beta}_1) \subset (1, 2)$. Moreover, let us define $\tilde{\beta}_k(y) = k^{-1} \tilde{\beta}_1(y/k)$ for all $k \in \mathbb{N}$ and $y > 0$, then there holds $\int_0^\infty \tilde{\beta}_k(y) dy = 1$ as well.

To conclude the Hahn-Banach argument, let us consider the adjoint problem

$$(4.5) \quad \begin{cases} \Delta_{x,y} w = \psi & \text{in } \mathbb{R}_+^{n+1}, \\ \lim_{y \rightarrow 0} \partial_y w = 0 & \text{in } \Omega \times \{0\}, \\ w = 0 & \text{in } \Omega_e \times \{0\}. \end{cases}$$

Since $C \subseteq \bar{\Omega}$ is a measurable set with $C \cap \partial\Omega \neq \emptyset$, note that

$$\left| \int_0^\infty \langle \tilde{u}_f, \psi \rangle dy \right| \leq \|\psi\|_{\tilde{H}^{-1}(C)} \int_0^\infty \|\tilde{u}_f(\cdot, y)\|_{H^1(C)} dy < \infty,$$

then the assumption in (4.3) can be rewritten as

$$\begin{aligned} 0 &= \int_0^\infty \langle \tilde{u}^f, \psi \rangle_C dy \\ &= \int_0^\infty \int_{\mathbb{R}^n} (\Delta_{x,y} w) \tilde{u}^f dx dy \\ &= \underbrace{\int_{\mathbb{R}^n} w(x, 0) \left(\lim_{y \rightarrow 0} \partial_y \tilde{u}^f \right) dx}_{:= (I)} + \underbrace{\int_0^\infty \int_{\mathbb{R}^n} w (\Delta_{x,y} \tilde{u}^f) dx dy}_{=0 \text{ since } \tilde{u}^f \text{ solves (4.1)}} \\ &\quad - \int_{\mathbb{R}^n} \left(\lim_{y \rightarrow 0} \partial_y w \right) \tilde{u}^f(x, 0) dx \\ &= - \int_W \left(\lim_{y \rightarrow 0} \partial_y w \right) f dx, \end{aligned}$$

¹Note that the (4.3) is equivalent to $\int_C (\int_0^\infty \tilde{u}^f(x, y) dy) \psi(x) dx$ and (4.4) is equivalent to $\int_C (\int_0^\infty \tilde{\beta}_1(y) v(x) dy) \psi(x) dx$.

where we used both boundary information of $\tilde{u}^f$ and w from the equations (4.1) and (4.5) such that the term $(I) = 0$. Since $f \in C_c^\infty(W)$ is arbitrary, there must hold that $\lim_{y \rightarrow 0} \partial_y w = 0$ in W . Thanks to the (weak) UCP and $\overline{\Omega} \setminus \overline{C}$ is connected to $\partial\Omega$, then it follows that

$$w \equiv 0 \text{ in } (\Omega_e \cup (\Omega \setminus \overline{C})) \times (0, \infty).$$

Particularly, there holds that

$$(4.6) \quad w|_{(\partial\Omega \cup \partial C) \times (0, \infty)} = \partial_\nu w|_{(\partial\Omega \cup \partial C) \times (0, \infty)} = 0,$$

and $\lim_{y \rightarrow 0} \partial_y w = 0$ in $\mathbb{R}^n$.

We can conclude the proof by taking the function $v \in H^1(C)$, which can be extended to an $H^1(\mathbb{R}^n)$ function and satisfies

$$(4.7) \quad \Delta v = 0 \text{ in } C.$$

Similar to [CGRU23, Section 3], we have

$$\begin{aligned} -\langle \psi, v \rangle_C &= -\left\langle \psi, \int_0^\infty \tilde{\beta}_k(y) v \, dy \right\rangle_C \\ &= -\lim_{k \rightarrow \infty} \left\langle \psi, \int_0^\infty \tilde{\beta}_k(y) v \, dy \right\rangle_C \\ (4.8) \quad &= -\lim_{k \rightarrow \infty} \underbrace{\int_{\Omega \times (0, \infty)} \tilde{\beta}_k w \Delta v \, dx dy}_{:= (II)} + \lim_{k \rightarrow \infty} \int_{\Omega \times (0, \infty)} v \partial_y \tilde{\beta}_k \partial_y w \, dx dy \\ &= \lim_{k \rightarrow \infty} k^{-2} \int_{\Omega \times (k, 2k)} v \partial_y \tilde{\beta}_1 \partial_y w \, dx dy \\ &= 0, \end{aligned}$$

where we also used (4.6) to get rid of boundary contributions. The term $(II) = 0$ thanks to the facts $\Delta v = 0$ in C and $w = 0$ in $(\Omega \setminus \overline{C}) \times (0, \infty)$.

Finally, since $v \in H^1(C)$ is a solution to (4.7) with $C = B \cup D$ and $\overline{B} \cap \overline{D} = \emptyset$, one can simply take v to be piecewise defined solution, which is a nonconstant solution in B , and $v = 1$ in D . Therefore, with these special choices at hand, we can imply that (4.2) holds. This demonstrates the ideas of the proof of the density, which is very similar to the case shown in [CGRU23, Section 3]. $\square$

Remark 4.3. From the first identity in (4.8), we can see that

$$\left\langle \psi, \int_0^\infty \tilde{\beta}_k(y) v \, dy \right\rangle_C = 0, \text{ for all } k \in \mathbb{N},$$

which, of course, holds as $k = 1$. The rigorous proof will follow the same idea and approach as in the formal one.

The rigorous proof can be concluded by introducing suitable cutoff functions in both x and y directions, given as in [CGRU23, Section 3].

Proof of Theorem 4.1. The argument is similar to the rigorous proof [CGRU23, Proposition 3.1]. As in the formal proof, let $C = B \cup D$ with $C \cap \partial\Omega \neq \emptyset$, and $\psi \in \tilde{H}^{-1}(C)$. Then we have

$$\left| \int_0^\infty y^{1-2s} \langle \tilde{u}^f(\cdot, y), \psi(\cdot) \rangle_C \, dy \right| \leq \|\psi\|_{\tilde{H}^{-1}(C)} \|\tilde{u}^f\|_{H^1(C, y^{1-2s})} < \infty,$$

and by the duality argument of the Hahn-Banach theorem, it suffices to claim that

$$(4.9) \quad \int_0^\infty y^{1-2s} \langle \tilde{u}^f(\cdot, y), \psi(\cdot) \rangle_C \, dy = 0$$

implies

$$(4.10) \quad \int_0^\infty y^{1-2s} \langle \beta_1(y)v(\cdot), \psi(\cdot) \rangle_C dy = 0,$$

where $\langle \cdot, \cdot \rangle_C$ denotes the duality pairing between $H^1(C)$ and its dual space $\tilde{H}^{-1}(C)$ ². As we shall see, the function $v \in H^1(C)$ could be an arbitrary function satisfying

$$(4.11) \quad \nabla \cdot (\sigma \nabla v) = 0 \text{ in } C.$$

Moreover, β_1 is a suitable cutoff function with respect to the y -direction, which will be given in Appendix A.

To this end, consider the adjoint problem

$$(4.12) \quad \begin{cases} \nabla_{x,y} \cdot (y^{1-2s} \tilde{\sigma} \nabla_{x,y} w) = y^{1-2s} \psi & \text{in } \mathbb{R}_+^{n+1}, \\ \lim_{y \rightarrow 0} y^{1-2s} \partial_y w = 0 & \text{in } \Omega \times \{0\}, \\ w = 0 & \text{in } \Omega_e \times \{0\}. \end{cases}$$

The solvability of the above problem (4.12) can be found in [CGRU23, Section 3], so we do not give further details about it. The rest of the argument is completely the same as the proof of [CGRU23, Proposition 3.1], with suitable cutoff function arguments in both x and y directions; by using (4.9), one can obtain

$$\int_W f \lim_{y \rightarrow 0} y^{1-2s} \partial_y w dx = 0, \quad \text{for any } f \in C_c^\infty(W).$$

This implies that $\lim_{y \rightarrow 0} y^{1-2s} \partial_y w = 0$ in W . Combining with $w = 0$ in $W \times \{0\}$ and $\bar{\Omega} \setminus \bar{C}$ is connected to $\partial\Omega$, the UCP implies that

$$w = 0 \text{ in } (\Omega_e \cup (\Omega \setminus \bar{C})) \times (0, \infty).$$

This particularly implies also implies that $w = \partial_\nu w = 0$ on $(\partial\Omega \cup \partial C) \times (0, \infty)$, and $\lim_{y \rightarrow 0} y^{1-2s} \partial_y w = 0$ in $\mathbb{R}^n$.

Finally, given any function $v \in H^1(C)$ that satisfies (4.11), then the same arguments as in the proof of [CGRU23, Proposition 3.1] and (4.8), the identity (4.10) must hold. In other words, for any solution $v \in H^1(C) = H^1(B \cup D)$ to (4.11), the function $v(x)\beta_1(y)$ can be approximated by solutions $\tilde{u}$ of (2.3) in $H^1(B \cup D, y^{1-2s})$, then this concludes the proof. $\square$

Remark 4.4. *In the proof of [CGRU23, Proposition 3.1], there are more (smooth) cutoff functions involved. However, the proofs of Theorem 4.1 and [CGRU23, Proposition 3.1] are the same, and there are no major changes to the whole argument. We skip the details.*

5. PROOF OF THE LOCAL UNIQUENESS

We are now ready to prove Theorem 1.1.

Proof of Theorem 1.1. We want to claim $\sigma_1 = \sigma_2$ in $\mathcal{O}$ by using the monotonicity relation and localized potentials. Since the condition (1.5) holds, we may assume $\sigma_1 - \sigma_2 \geq 0$ in $\mathcal{O}$, and there exists $B \subseteq \mathcal{O}$ such that

$$(5.1) \quad \sigma_1 - \sigma_2 \geq \delta > 0 \text{ in } B,$$

for some $\delta > 0$. Additionally, we can assume $D := \Omega \setminus \bar{\mathcal{O}}$, such that $B \setminus \bar{D}$ possesses positive measure since $\bar{B} \cap \bar{D} = \emptyset$, and $\bar{\Omega} \setminus \bar{D}$ is connected to $\partial\Omega$.

²Note that the (4.9) is equivalent to $\int_C (\int_0^\infty y^{1-2s} \tilde{u}^f(x, y) dy) \psi(x) dx$ and (4.10) is equivalent to $\int_C (\int_0^\infty y^{1-2s} \beta_1(y) v(x) dy) \psi(x) dx$.

Next, the monotonicity relation (3.2) yields that

$$\begin{aligned}
 0 &= d_s \langle (\Lambda_{\sigma_1} - \Lambda_{\sigma_2}) f_k, f_k \rangle \\
 &\geq \int_{\Omega \times (0, \infty)} y^{1-2s} \frac{\sigma_2}{\sigma_1} (\sigma_1 - \sigma_2) |\nabla \tilde{u}_2^{f_k}|^2 dx dy \\
 (5.2) \quad &\geq -C \int_{D \times (0, \infty)} y^{1-2s} |\nabla \tilde{u}_2^{f_k}|^2 dx dy + c\delta \int_{B \times (0, \infty)} y^{1-2s} |\nabla \tilde{u}_2^{f_k}|^2 dx dy,
 \end{aligned}$$

for some constants $c, C > 0$ independent of $\tilde{u}_2^{f_k}$, where $\{\tilde{u}_2^{f_k}\}_{k \in \mathbb{N}} \subset H^1(\mathbb{R}_+^{n+1}, y^{1-2s})$ are localized potentials constructed by Corollary 4.2. Here $C := \|\frac{\sigma_2}{\sigma_1}(\sigma_1 - \sigma_2)\|_{L^\infty(\Omega)}$ and $c := \min_{\bar{\Omega}} \frac{\sigma_2}{\sigma_1} > 0$. Hence, taking $k \rightarrow \infty$ in the inequality (5.2), we can obtain

$$0 \geq -C \underbrace{\int_{D \times (0, \infty)} y^{1-2s} |\nabla \tilde{u}_2^{f_k}|^2 dx dy}_{\rightarrow 0 \text{ as } k \rightarrow \infty} + c\delta \underbrace{\int_{B \times (0, \infty)} y^{1-2s} |\nabla \tilde{u}_2^{f_k}|^2 dx dy}_{\rightarrow \infty \text{ as } k \rightarrow \infty} \rightarrow +\infty,$$

as $k \rightarrow \infty$, which leads to a contradiction. Therefore, the inequality (5.1) cannot be true, which implies that $\sigma_1 \leq \sigma_2$ in $\mathcal{O}$.

Similarly, when $\sigma_1 - \sigma_2 \leq 0$ in $\mathcal{O}$, we may assume that $\sigma_1 - \sigma_2 \leq -\delta' < 0$ in $B \subsetneq \mathcal{O}$ for some $\delta' > 0$, this will lead a contradiction too. More concretely, with the same notations as before, one can use the monotonicity relation (3.2) again,

$$\begin{aligned}
 0 &\leq \int_{\Omega \times (0, \infty)} y^{1-2s} (\sigma_1 - \sigma_2) |\nabla \tilde{u}_2^{f_k}|^2 dx dy \\
 &\leq C \int_{D \times (0, \infty)} y^{1-2s} |\nabla \tilde{u}_2^{f_k}|^2 dx dy - \delta' \int_{B \times (0, \infty)} y^{1-2s} |\nabla \tilde{u}_2^{f_k}|^2 dx dy \\
 &\rightarrow -\infty,
 \end{aligned}$$

which leads to a contradiction, too. Thus, $\sigma_1 = \sigma_2$ in B , concluding the proof. $\square$

Remark 5.1. *It would also be interesting to use the monotonicity approach to study the Lipschitz stability for piecewise analytic coefficients and inverse obstacle problems as we mentioned earlier.*

APPENDIX A. THE AUXILIARY FUNCTION

The function introduced in this work, $\beta_1 = \beta_1(y)$, was constructed in the proof of [CGRU23, Proposition 3.1]. For the sake of completeness, let us collect the existing construction of β_k ($k \in \mathbb{N}$) as follows: Given $b \in (0, 1)$, let $\gamma_b : (0, \infty) \rightarrow (0, b)$ be a smooth function with $\text{supp}(\gamma_b) \subset [0, \frac{2-b}{1-b}]$ and $\gamma_b(y) = b$ for $y \in [1, \frac{1}{1-b}]$. One can also assume that

$$\int_0^\infty \gamma_b(y) dy = \frac{b}{1-b} \quad \text{and} \quad |\partial_y^\ell \gamma_b(y)| \leq C,$$

for $\ell = 0, 1, 2$, where $C > 0$ is a constant independent of $b \in (0, 1)$. Let

$$\begin{aligned}
 I_{b,k} &:= \int_0^\infty y^{1-2s} \gamma_b(y-k) dy \\
 &= \int_0^{\frac{2-b}{1-b}} (y+k)^{1-2s} \gamma_b(y) dy \\
 &\in \begin{cases} \frac{bk^{1-2s}}{1-b} \left(1, \left(1 + \frac{2-b}{k(1-b)}\right)^{1-2s}\right), & \text{if } s \in (0, \frac{1}{2}], \\ \frac{bk^{1-2s}}{1-b} \left(\left(1 + \frac{2-b}{k(1-b)}\right)^{1-2s}, 1\right), & \text{if } s \in (\frac{1}{2}, 1) \end{cases}
 \end{aligned}$$

where $I_{b,k}$ depends continuously on $b \in (0, 1)$. From the above analysis, it is evident that the values attained by $I_{b,k}$ can range from arbitrarily large to arbitrarily close to 0. By the continuity of for all $k \in \mathbb{N}$, there exist $b_{k,s} \in (0, 1)$ such that $I_{b_{k,s},k} = 1$. Now, let

$$\beta_k(y) := \gamma_{b_{k,s}}(y - k) \quad \text{and} \quad R_{k,s} := k + \frac{1}{1 - b_{k,s}},$$

then $\beta_k : (0, \infty) \rightarrow [0, 1]$ satisfies

$$\begin{aligned} \text{supp}(\beta_k) &\subset (k, R_{k,s} + 1), \\ \beta_k(y) &= b_{k,s}, \text{ for } y \in (k + 1, R_{k,s}), \\ |\partial_y^\ell \beta_k(y)| &\leq C, \\ \int_0^\infty y^{1-2s} \beta_k(y) dy &= 1, \end{aligned}$$

for $\ell = 0, 1, 2$, and for some constant $C > 0$ independent of $k \in \mathbb{N}$.

STATEMENTS AND DECLARATIONS

Data availability statement. No datasets were generated or analyzed during the current study.

Conflict of Interests. Hereby, we declare there are no conflicts of interest.

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REFERENCES

- [AH13] Lilian Arnold and Bastian Harrach. Unique shape detection in transient eddy current problems. *Inverse Problems*, 29(9):095004, 19, 2013.
- [BHBM17] Andrea Barth, Bastian Harrach, Nuutti Hyvönen, and Lauri Mustonen. Detecting stochastic inclusions in electrical impedance tomography. *Inverse Problems*, 33(11):115012, 18, 2017.
- [BHKS18] Tommi Brander, Bastian Harrach, Manas Kar, and Mikko Salo. Monotonicity and enclosure methods for the p -Laplace equation. *SIAM J. Appl. Math.*, 78(2):742–758, 2018.
- [BV16] Claudia Bucur and Enrico Valdinoci. *Nonlocal diffusion and applications*, volume 20 of *Lecture Notes of the Unione Matematica Italiana*. Springer, [Cham]; Unione Matematica Italiana, Bologna, 2016.
- [CGRU23] Giovanni Covi, Tuhin Ghosh, Angkana Rüland, and Gunther Uhlmann. A reduction of the fractional Calderón problem to the local Calderón problem by means of the Caffarelli-Silvestre extension. *arXiv preprint arXiv:2305.04227*, 2023.
- [CLL19] Xinlin Cao, Yi-Hsuan Lin, and Hongyu Liu. Simultaneously recovering potentials and embedded obstacles for anisotropic fractional Schrödinger operators. *Inverse Probl. Imaging*, 13(1):197–210, 2019.
- [CLR20] Mihajlo Cekic, Yi-Hsuan Lin, and Angkana Rüland. The Calderón problem for the fractional Schrödinger equation with drift. *Cal. Var. Partial Differential Equations*, 59(91), 2020.
- [CMRU22] Giovanni Covi, Keijo Mönkkönen, Jesse Railo, and Gunther Uhlmann. The higher order fractional Calderón problem for linear local operators: Uniqueness. *Adv. Math.*, 399:Paper No. 108246, 2022.
- [CS07] Luis Caffarelli and Luis Silvestre. An extension problem related to the fractional Laplacian. *Comm. Partial Differential Equations*, 32(7-9):1245–1260, 2007.
- [Dav90] Edward Brian Davies. *Heat kernels and spectral theory*, volume 92 of *Cambridge Tracts in Mathematics*. Cambridge University Press, Cambridge, 1990.
- [Fei24] Ali Feizmohammadi. Fractional Calderón problem on a closed Riemannian manifold. *Trans. Amer. Math. Soc.*, 377(4):2991–3013, 2024.

- [FGK⁺25] Ali Feizmohammadi, Tuhin Ghosh, Katya Krupchyk, Angkana Rüland, Johannes Sjöstrand, and Gunther Uhlmann. Fractional anisotropic Calderón problem with external data. *arXiv preprint arXiv:2502.00710*, 2025.
- [FGKU24] Ali Feizmohammadi, Tuhin Ghosh, Katya Krupchyk, and Gunther Uhlmann. Fractional anisotropic Calderón problem on closed Riemannian manifolds. *J. Diff. Geom., to appear*, 2024.
- [FKU24] Ali Feizmohammadi, Katya Krupchyk, and Gunther Uhlmann. Calderón problem for fractional Schrödinger operators on closed Riemannian manifolds. *arXiv preprint arXiv:2407.16866*, 2024.
- [FL24] Ali Feizmohammadi and Yi-Hsuan Lin. Entanglement principle for the fractional Laplacian with applications to inverse problems. *arXiv preprint arXiv:2412.13118*, 2024.
- [Gar17] Henrik Garde. Comparison of linear and non-linear monotonicity-based shape reconstruction using exact matrix characterizations. *Inverse Problems in Science and Engineering*, pages 1–18, 2017.
- [Geb08] Bastian Gebauer. Localized potentials in electrical impedance tomography. *Inverse Probl. Imaging*, 2(2):251–269, 2008.
- [GH18] Roland Griesmaier and Bastian Harrach. Monotonicity in inverse medium scattering on unbounded domains. *SIAM J. Appl. Math.*, 78(5):2533–2557, 2018.
- [GLX17] Tuhin Ghosh, Yi-Hsuan Lin, and Jingni Xiao. The Calderón problem for variable coefficients nonlocal elliptic operators. *Comm. Partial Differential Equations*, 42(12):1923–1961, 2017.
- [GRSU20] Tuhin Ghosh, Angkana Rüland, Mikko Salo, and Gunther Uhlmann. Uniqueness and reconstruction for the fractional Calderón problem with a single measurement. *J. Funct. Anal.*, 279(1):108505, 42, 2020.
- [GS17] Henrik Garde and Stratos Staboulis. Convergence and regularization for monotonicity-based shape reconstruction in electrical impedance tomography. *Numerische Mathematik*, 135(4):1221–1251, 2017.
- [GS19] Henrik Garde and Stratos Staboulis. The regularized monotonicity method: detecting irregular indefinite inclusions. *Inverse Probl. Imaging*, 13(1):93–116, 2019.
- [GSU20] Tuhin Ghosh, Mikko Salo, and Gunther Uhlmann. The Calderón problem for the fractional Schrödinger equation. *Anal. PDE*, 13(2):455–475, 2020.
- [Har09] Bastian Harrach. On uniqueness in diffuse optical tomography. *Inverse Problems*, 25(5):055010, 14, 2009.
- [Har12] Bastian Harrach. Simultaneous determination of the diffusion and absorption coefficient from boundary data. *Inverse Probl. Imaging*, 6(4):663–679, 2012.
- [HL19] Bastian Harrach and Yi-Hsuan Lin. Monotonicity-based inversion of the fractional Schrödinger equation I. Positive potentials. *SIAM J. Math. Anal.*, 51(4):3092–3111, 2019.
- [HL20] Bastian Harrach and Yi-Hsuan Lin. Monotonicity-based inversion of the fractional Schrödinger equation II. General potentials and stability. *SIAM J. Math. Anal.*, 52(1):402–436, 2020.
- [HLL18] Bastian Harrach, Yi-Hsuan Lin, and Hongyu Liu. On localizing and concentrating electromagnetic fields. *SIAM J. Appl. Math.*, 78(5):2558–2574, 2018.
- [HLU15] Bastian Harrach, Eunjung Lee, and Marcel Ullrich. Combining frequency-difference and ultrasound modulated electrical impedance tomography. *Inverse Problems*, 31(9):095003, 25, 2015.
- [HLW25] Bastian Harrach, Yi-Hsuan Lin, and Tobias Weth. The Calderón problem for the logarithmic Schrödinger equation. *J. Differential Equations*, 444:113665, 2025.
- [HM16] Bastian Harrach and Mach Nguyet Minh. Enhancing residual-based techniques with shape reconstruction features in electrical impedance tomography. *Inverse Problems*, 32(12):125002, 21, 2016.
- [HM18] Bastian Harrach and Mach Nguyet Minh. Monotonicity-based regularization for phantom experiment data in electrical impedance tomography. In *New Trends in Parameter Identification for Mathematical Models*, pages 107–120. Springer, 2018.
- [HPS19a] Bastian Harrach, Valter Pohjola, and Mikko Salo. Dimension bounds in monotonicity methods for the Helmholtz equation. *SIAM J. Math. Anal.*, 51(4):2995–3019, 2019.
- [HPS19b] Bastian Harrach, Valter Pohjola, and Mikko Salo. Monotonicity and local uniqueness for the Helmholtz equation. *Anal. PDE*, 12(7):1741–1771, 2019.
- [HS10] Bastian Harrach and Jin Keun Seo. Exact shape-reconstruction by one-step linearization in electrical impedance tomography. *SIAM J. Math. Anal.*, 42(4):1505–1518, 2010.
- [HU13] Bastian Harrach and Marcel Ullrich. Monotonicity-based shape reconstruction in electrical impedance tomography. *SIAM J. Math. Anal.*, 45(6):3382–3403, 2013.

- [HU15] Bastian Harrach and Marcel Ullrich. Resolution guarantees in electrical impedance tomography. *IEEE Trans. Med. Imaging*, 34:1513–1521, 2015.
- [HU17] Bastian Harrach and Marcel Ullrich. Local uniqueness for an inverse boundary value problem with partial data. *Proc. Amer. Math. Soc.*, 145(3):1087–1095, 2017.
- [KLW22] Pu-Zhao Kow, Yi-Hsuan Lin, and Jenn-Nan Wang. The Calderón problem for the fractional wave equation: uniqueness and optimal stability. *SIAM J. Math. Anal.*, 54(3):3379–3419, 2022.
- [Lin22] Yi-Hsuan Lin. Monotonicity-based inversion of fractional semilinear elliptic equations with power type nonlinearities. *Calc. Var. Partial Differential Equations*, 61(5):Paper No. 188, 30, 2022.
- [Lin24] Yi-Hsuan Lin. The fractional anisotropic Calderón problem for a nonlocal parabolic equation on closed Riemannian manifolds. *arXiv preprint arXiv:2410.17750*, 2024.
- [LL22] Ru-Yu Lai and Yi-Hsuan Lin. Inverse problems for fractional semilinear elliptic equations. *Nonlinear Anal.*, 216:Paper No. 112699, 21, 2022.
- [LL23] Yi-Hsuan Lin and Hongyu Liu. Inverse problems for fractional equations with a minimal number of measurements. *Commun. Anal. Comput.*, 1(1):72–93, 2023.
- [LL25] Yi-Hsuan Lin and Hongyu Liu. *Inverse Problems for Integro-differential Operators*, volume 222 of *Applied Mathematical Sciences*. Springer, Cham, 2025.
- [LLU23] Ching-Lung Lin, Yi-Hsuan Lin, and Gunther Uhlmann. The Calderón problem for nonlocal parabolic operators: A new reduction from the nonlocal to the local. *arXiv preprint arXiv:2308.09654*, 2023.
- [LNZ24] Yi-Hsuan Lin, Gen Nakamura, and Philipp Zimmermann. The Calderón problem for the schrödinger equation in transversally anisotropic geometries with partial data. *arXiv preprint arXiv:2408.08298*, 2024.
- [LZ23] Yi-Hsuan Lin and Philipp Zimmermann. Unique determination of coefficients and kernel in nonlocal porous medium equations with absorption term. *arXiv preprint arXiv:2305.16282*, 2023.
- [LZ24] Yi-Hsuan Lin and Philipp Zimmermann. Approximation and uniqueness results for the nonlocal diffuse optical tomography problem. *arXiv preprint arXiv:2406.06226*, 2024.
- [MVVT16] Antonio Maffucci, Antonio Vento, Salvatore Ventre, and Antonello Tamburrino. A novel technique for evaluating the effective permittivity of inhomogeneous interconnects based on the monotonicity property. *IEEE Transactions on Components, Packaging and Manufacturing Technology*, 6(9):1417–1427, 2016.
- [RO16] Xavier Ros-Oton. Nonlocal elliptic equations in bounded domains: a survey. *Publ. Mat.*, 60(1):3–26, 2016.
- [RS18] Angkana Rüland and Mikko Salo. Exponential instability in the fractional Calderón problem. *Inverse Problems*, 34(4):045003, 21, 2018.
- [RS20] Angkana Rüland and Mikko Salo. The fractional Calderón problem: low regularity and stability. *Nonlinear Anal.*, 193:111529, 56, 2020.
- [Rül25] Angkana Rüland. Revisiting the Anisotropic Fractional Calderón Problem. *Int. Math. Res. Not. IMRN*, 2025(5):rna036, 2025.
- [SKJ⁺19] Jin Keun Seo, Kang Cheol Kim, Ariungerel Jargal, Kyounghun Lee, and Bastian Harrach. A learning-based method for solving ill-posed nonlinear inverse problems: a simulation study of lung EIT. *SIAM J. Imaging Sci.*, 12(3):1275–1295, 2019.
- [ST10] Pablo Raúl Stinga and José Luis Torrea. Extension problem and Harnack’s inequality for some fractional operators. *Comm. Partial Differential Equations*, 35(11):2092–2122, 2010.
- [SU87] John Sylvester and Gunther Uhlmann. A global uniqueness theorem for an inverse boundary value problem. *Ann. of Math. (2)*, 125(1):153–169, 1987.
- [SUG⁺17] Zhiyi Su, Lalita Udpa, Gaspare Giovinco, Salvatore Ventre, and Antonello Tamburrino. Monotonicity principle in pulsed eddy current testing and its application to defect sizing. In *Applied Computational Electromagnetics Society Symposium-Italy (ACES), 2017 International*, pages 1–2. IEEE, 2017.
- [TR02] Antonello Tamburrino and Guglielmo Rubinacci. A new non-iterative inversion method for electrical resistance tomography. *Inverse Problems*, 18(6):1809–1829, 2002. Special section on electromagnetic and ultrasonic nondestructive evaluation.
- [TSV⁺16] Antonello Tamburrino, Zhiyi Sua, Salvatore Ventre, Lalita Udpa, and Satish S Udpa. Monotonicity based imang method in time domain eddy current testing. *Electromagnetic Nondestructive Evaluation (XIX)*, 41:1, 2016.

- [VMC⁺17] Salvatore Ventre, Antonio Maffucci, François Caire, Nechtan Le Lostec, Antea Perrotta, Guglielmo Rubinacci, Bernard Sartre, Antonio Vento, and Antonello Tamburino. Design of a real-time eddy current tomography system. *IEEE Transactions on Magnetics*, 53(3):1–8, 2017.
- [ZHS18] Liangdong Zhou, Bastian Harrach, and Jin Keun Seo. Monotonicity-based electrical impedance tomography for lung imaging. *Inverse Problems*, 34(4):045005, 25, 2018.

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