

SIMULTANEOUSLY RECOVERING POTENTIALS AND EMBEDDED OBSTACLES FOR ANISOTROPIC FRACTIONAL SCHRÖDINGER OPERATORS

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ABSTRACT. Let $A \in \text{Sym}(n \times n)$ be an elliptic 2-tensor. Consider the anisotropic fractional Schrödinger operator $\mathcal{L}_A^s + q$, where $\mathcal{L}_A^s := (-\nabla \cdot (A(x)\nabla))^s$, $s \in (0, 1)$ and $q \in L^\infty$. We are concerned with the simultaneous recovery of q and possibly embedded soft or hard obstacles inside q by the exterior Dirichlet-to-Neumann (DtN) map outside a bounded domain Ω associated with $\mathcal{L}_A^s + q$. It is shown that a single measurement can uniquely determine the embedded obstacle, independent of the surrounding potential q . If multiple measurements are allowed, then the surrounding potential q can also be uniquely recovered. These are surprising findings since in the local case, namely $s = 1$, both the obstacle recovery by a single measurement and the simultaneous recovery of the surrounding potential by multiple measurements are long-standing problems and still remain open in the literature. Our argument for the nonlocal inverse problem is mainly based on the strong uniqueness property and Runge approximation property for anisotropic fractional Schrödinger operators.

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1. Introduction.

1.1. Mathematical setup and statement of the main results. Let $\text{Sym}(n \times n)$ signify the space of real-valued $n \times n$ symmetric matrices for $n \geq 2$. Let $A(x) = (a_{ij}(x))_{i,j=1}^n \in \text{Sym}(n \times n)$, $x \in \mathbb{R}^n$. Throughout, it is assumed that A satisfies the following uniform ellipticity condition for some $\gamma \in (0, 1)$,

$$\gamma|\xi|^2 \leq \sum_{i,j=1}^n a_{ij}(x)\xi_i\xi_j \leq \gamma^{-1}|\xi|^2 \quad \text{for all } \xi, x \in \mathbb{R}^n, \quad (1.1)$$

and $A(x)$ is C^∞ -smooth for any $x \in \mathbb{R}^n$. Define \mathcal{L}_A to be the following PDO (partial differential operator),

$$\mathcal{L}_A := -\nabla \cdot (A(x)\nabla),$$

Let $s \in (0, 1)$ be a constant and introduce the following nonlocal PDO,

$$\mathcal{L}_A^s = (-\nabla \cdot (A(x)\nabla))^s,$$

whose rigorous definition shall be given in Section 2.

Let Ω and D be two bounded open sets in \mathbb{R}^n such that $D \Subset \Omega$ and, $\mathbb{R}^n \setminus \overline{\Omega}$ and $\Omega \setminus \overline{D}$ are connected. Let $q \in L^\infty(\Omega \setminus \overline{D})$ be a real-valued function. Physically speaking, q and D , respectively, represent a potential and an embedded impenetrable obstacle inside the potential.

Consider the following nonlocal problem associated with q and D ,

$$\begin{cases} \mathcal{L}_A^s u + qu = 0 & \text{in } \Omega \setminus \overline{D}, \\ \mathcal{B}u = 0 & \text{in } D, \\ u = g & \text{in } \Omega_e := \mathbb{R}^n \setminus \overline{\Omega}, \end{cases} \quad (1.2)$$

where $u \in H^s(\mathbb{R}^n)$ is a weak solution of (1.2) with $g \in H^s(\mathbb{R}^n)$ being an exterior Dirichlet data. In (1.2), $\mathcal{B}u := u$ if D is a *soft* obstacle, and $\mathcal{B}u := \mathcal{L}_A^s u$ if D is a *hard* obstacle. It is known that (1.2) is uniquely solvable if $\{0\}$ is not an eigenvalue of the operator $\mathcal{L}_A^s + q$, in the following sense

$$\begin{cases} \text{if } w \in H^s(\mathbb{R}^n) \text{ solves } (\mathcal{L}_A^s + q)w = 0 \text{ in } \Omega \setminus \overline{D}, \\ w = 0 \text{ in } \Omega_e, \text{ and } \mathcal{B}w = 0 \text{ in } D, \\ \text{then } w \equiv 0. \end{cases} \quad (1.3)$$

Throughout, we assume that $\{0\}$ is not an eigenvalue of $\mathcal{L}_A^s + q$, and hence (1.2) is well-posed. In particular, one has the following well-defined Dirichlet-to-Neumann (DtN) map associated to the nonlocal problem (1.2),

$$\Lambda_{D,q} : \mathbb{H} \rightarrow \mathbb{H}^*,$$

and $\Lambda_{D,q}$ is formally given by

$$\Lambda_{D,q}\psi := \mathcal{L}_A^s u_\psi|_{\Omega_e}, \quad (1.4)$$

where u_ψ is the unique solution to (1.2) with $u_\psi = \psi$ in Ω_e . In the subsequent section, we shall introduce more details of the abstract Banach spaces \mathbb{H} and \mathbb{H}^* . We regard the DtN map $\Lambda_{D,q}$ as the exterior measurement for our inverse problem study. In this article, we are mainly concerned with the recovery of the embedded obstacle $D \Subset \Omega$ and the surrounding potential $q(x) \in L^\infty(\Omega \setminus \bar{D})$ by using the exterior DtN map of $(\mathcal{L}_A^s + q)u = 0$ in $\Omega \setminus \bar{D}$.

For the inverse problem described above, our main results can be stated as follows.

Theorem 1.1. *For $n \geq 2$, let $\Omega \subset \mathbb{R}^n$ be an open bounded set, $D_1, D_2 \Subset \Omega$ be two open subsets of Ω and $\mathcal{O}_1, \mathcal{O}_2 \subset \Omega_e$ be two arbitrary nonempty open sets. Suppose D_j and $q_j \in L^\infty(\Omega \setminus \bar{D}_j)$ satisfy the eigenvalue condition (1.3), $j = 1, 2$. Let Λ_{D_j, q_j} be the DtN maps for the nonlocal equations $(\mathcal{L}_A^s + q_j)u_j = 0$ in $\Omega \setminus \bar{D}_j$ with $u_j = 0$ in D_j for $j = 1, 2$, then the following statements hold.*

1. *For any given $g \in C_c^\infty(\mathcal{O}_1)$ with $g \not\equiv 0$ in \mathcal{O}_1 , if*

$$\Lambda_{D_1, q_1} g|_{\mathcal{O}_2} = \Lambda_{D_2, q_2} g|_{\mathcal{O}_2},$$

then one has $D_1 = D_2$.

2. *Furthermore, if*

$$\Lambda_{D_1, q_1} g|_{\mathcal{O}_2} = \Lambda_{D_2, q_2} g|_{\mathcal{O}_2} \text{ for all } g \in C_c^\infty(\mathcal{O}_1),$$

then one has $q_1 = q_2$ in $\Omega \setminus \bar{D}$, where $D := D_j$ for $j = 1, 2$.

Moreover, if we further assume $q(x) \neq 0$ for any $x \in \Omega$, then we have the following unique recovery result for the sound hard case.

Theorem 1.2. *Let Ω, \mathcal{O}_j and D_j, q_j , $j = 1, 2$, be the same as those described in Theorem 1.1. Let Λ_{D_j, q_j} be the DtN maps for the nonlocal equations $(\mathcal{L}_A^s + q_j)u_j = 0$ in $\Omega \setminus \bar{D}_j$ with $\mathcal{L}_A^s u_j = 0$ in D_j for $j = 1, 2$, then the following statements hold.*

1. *We further assume that $q_j(x) \neq 0$ for any $x \in \Omega$ and $j = 1, 2$. For any given $g \in C_c^\infty(\mathcal{O}_1)$ with $g \not\equiv 0$ in \mathcal{O}_1 , if*

$$\Lambda_{D_1, q_1} g|_{\mathcal{O}_2} = \Lambda_{D_2, q_2} g|_{\mathcal{O}_2},$$

then one has $D_1 = D_2$.

2. *Furthermore, if*

$$\Lambda_{D_1, q_1} g|_{\mathcal{O}_2} = \Lambda_{D_2, q_2} g|_{\mathcal{O}_2} \text{ for all } g \in C_c^\infty(\mathcal{O}_1),$$

then one has $q_1 = q_2$ in $\Omega \setminus \bar{D}$, where $D := D_j$ for $j = 1, 2$.

Remark 1.1. In this paper, we define the *nonlocal Neumann derivative* of the solution u to (1.2) as $\mathcal{L}_A^s u$. This notion is used for defining both the hard obstacle D and the exterior measurement data in (1.4). With such a definition, we show the well-posedness of the direct problem (1.2) as well as derive the unique determination results in Theorems 1.1 and 1.2 for the associated inverse problems. Nevertheless, we would like to point out that there are different ways of introducing the nonlocal

Neumann derivative; see for example in [7, Section 3], and the nonlocal Neumann derivative can also be defined by

$$\mathcal{N}_A^s u(x) := \int_{\Omega} (u(x) - u(z)) \mathcal{K}_s(x, z) dz, \quad (1.5)$$

where the integral kernel $\mathcal{K}_s(x, z)$ is given in (2.1) in what follows. There is the following relationship which connects the aforementioned two definitions of the nonlocal Neumann derivatives (cf. [7, Lemma 3.6]),

$$\mathcal{L}_A^s u|_{\Omega_e} = (\mathcal{N}_A^s u - mu + \mathcal{L}_A^s(E_0 u))|_{\Omega_e},$$

where $m(x) := \int_{\Omega} \mathcal{K}_s(x, z) dz$ and $E_0 u = \chi_{\Omega_e} u$. Clearly, one can use $\mathcal{N}_A^s u$ in (1.5) to replace $\mathcal{L}_A^s u$ for a different definition of a hard obstacle in (1.2). This leads to a different forward nonlocal problem, and it would be interesting to consider this forward model as well as the associated inverse problems as those in Theorems 1.1 and 1.2. However, this would significantly change our theoretical framework for the present study and we choose to leave it for a future work.

By the first statement in Theorem 1.1 or 1.2, a single pair of non-trivial Cauchy data $(g, \Lambda_{D, q} g)$ is sufficient to uniquely recover the embedded soft or hard obstacle D , independent of the surrounding potential q . It is also noted that no restrictive regularity assumption is required on the obstacle D . If multiple measurements are used, then both the embedded obstacle and the surrounding potential can be uniquely recovered. We can further show that the recovery of the embedded obstacle can be achieved without knowing whether it is soft or hard. Indeed, by virtue of Theorems 1.1 and 1.2, it suffices for us to establish the following result.

Theorem 1.3. *Let Ω, \mathcal{O}_j and $D_j, q_j, j = 1, 2$, be the same as those described in Theorem 1.1. Let Λ_{D_j, q_j} be the DtN maps for the nonlocal equations $(\mathcal{L}_A^s + q_j) u_j = 0$ in $\Omega \setminus \overline{D_j}$ with either*

$$u_1 = 0 \text{ in } D_1 \text{ and } \mathcal{L}_A^s u_2 = 0 \text{ in } D_2$$

or

$$\mathcal{L}_A^s u_1 = 0 \text{ in } D_1 \text{ and } u_2 = 0 \text{ in } D_2,$$

then the following statements hold.

1. *We further assume that $q_j(x) \neq 0$ for any $x \in \Omega$ and $j = 1, 2$. Then for any given $g \in C_c^\infty(\mathcal{O}_1)$ with $g \not\equiv 0$ in \mathcal{O}_1 , if*

$$\Lambda_{D_1, q_1} g|_{\mathcal{O}_2} = \Lambda_{D_2, q_2} g|_{\mathcal{O}_2},$$

then one has $D_1 = D_2$.

2. *Furthermore, if*

$$\Lambda_{D_1, q_1} g|_{\mathcal{O}_2} = \Lambda_{D_2, q_2} g|_{\mathcal{O}_2} \text{ for all } g \in C_c^\infty(\mathcal{O}_1),$$

then one has $q_1 = q_2$ in $\Omega \setminus \overline{D}$, where $D := D_j$ for $j = 1, 2$.

1.2. Discussions and historical remarks. The study of nonlocal inverse problems has received significant attention in the literature in recent years. The Calderón problem for the fractional Schrödinger equation was first solved by Ghosh, Salo and Uhlmann [8]. Based on the similar idea, [7] and [15] generalized the results to the nonlocal variable case and nonlocal semilinear case, respectively. Note that the global uniqueness results hold for these nonlocal cases for any space dimension $n \geq 2$. The proof of the Calderón problem strongly relies on the *strong uniqueness*

property, and we refer readers to [8, Theorem 1.2] for the fractional Laplace $(-\Delta)^s$ and [7, Theorem 1.2] for the nonlocal variable operator \mathcal{L}_A^s . The strong uniqueness means that: for $s \in (0, 1)$, $u \in H^s(\mathbb{R}^n)$, if $u = \mathcal{L}_A^s u = 0$ in an arbitrary nonempty open set in \mathbb{R}^n , then $u \equiv 0$ in \mathbb{R}^n for any $n \geq 2$. Based on the strong uniqueness property, one can obtain the nonlocal Runge approximation property, which states that any L^2 function can be approximated by a sequence of the solutions to $(\mathcal{L}_A^s + q)u = 0$.

Recently, Rüländ and Salo [30] studied the fractional Calderón problem under lower regularity conditions and established the stability results for the determination of unknown potentials. They [29] proved the optimal logarithmic stability for the corresponding inverse problem associated to the fractional Schrödinger equation. In [9], the authors characterized an if-and-only-if relationship between two positive potentials and their associated DtN maps of the fractional Schrödinger equation. Harrach and Lin [9] also provided a reconstruction algorithm of an unknown inclusion based on the monotonicity method. The nonlocal inverse problems reveal some novel and distinctive features compared to their local counterparts. For the current study of simultaneously recovering unknown potentials with possibly embedded impenetrable obstacles, we also provide some interesting discussions and observations compared with its local counterpart.

When $s = 1$, (1.2) becomes a local problem and in such a case, one should replace the nonlocal condition $\mathcal{B}u = 0$ in D by $\tilde{\mathcal{B}}u = 0$ on ∂D , where $\tilde{\mathcal{B}}u = u$ if D is a soft obstacle and $\tilde{\mathcal{B}}u = \nu^T \cdot A \cdot \nabla u$ if D is a hard obstacle, with ν signifying the exterior unit normal vector to ∂D . The corresponding local DtN map can be readily defined on ∂D , which we still denote by $\Lambda_{D,q}$. The local inverse problem of determining D by $\Lambda_{D,q}$ is usually referred to as the *obstacle problem*. The obstacle problem by a single measurement, namely determining D by a single pair of Cauchy data $(\psi, \Lambda_{D,q}\psi)$ is a well-known and long-standing problem in the inverse scattering theory, which is also known as the Schiffer's problem, particularly for the case $A = I$ and $q = 1$ [4, 12, 22]. There has been extensive study in the literature and significant progress has been achieved in recent years on the Schiffer's problem for the case with general polyhedral obstacles; see [1, 3, 20, 21] and [17, 26, 27] and the references therein, respectively, for related uniqueness and stability studies. Under the restrictive assumption that ∂D is everywhere non-analytic, the Schiffer's problem was solved in [10]. However, for the case with general obstacles, the Schiffer's problem still remains open in the literature. According to Theorems 1.1–1.3, the nonlocal Schiffer's problem has been completely solved in our study. Hence, it would be much interesting to study the connection of the nonlocal and local Schiffer's problems. This might be partly seen by taking the limit $s \rightarrow 1^-$. The simultaneous recovery of an embedded obstacle and an unknown surrounding potential is also a long-standing problem in the literature and closely related to the so-called partial data Calderón problem [5, 11]. The existing unique recovery results were established based on knowing the embedded obstacle to recover the unknown potential [11], or knowing the surrounding potential to recover the embedded obstacle [13, 14, 18, 19, 24], or using multiple spectral data to recover both of them [16].

The rest of the paper is structured as follows. In Section 2, we provide rigorous mathematical formulations of the nonlocal elliptic operator \mathcal{L}_A^s and fractional Sobolev spaces. In Section 3, we study the well-posedness and the associated DtN map for $\mathcal{L}_A^s + q$ with an embedded obstacle. In Section 4, we prove the uniqueness in determining the obstacle D in Ω by using a single exterior measurement. In

Section 5, we prove the global uniqueness in recovering the surrounding potential q . Combining with Section 4 and 5, then we prove Theorem 1.1–1.3.

2. Preliminary knowledge on \mathcal{L}_A^s . In this section, we present some preliminary knowledge on the nonlocal PDO \mathcal{L}_A^s that shall be needed in our inverse problem study. We begin with the definition of fractional Sobolev spaces.

2.1. Fractional Sobolev spaces. For $0 < s < 1$, the fractional Sobolev space is denoted by $H^s(\mathbb{R}^n) := W^{s,2}(\mathbb{R}^n)$, which is the standard L^2 based Sobolev space with the norm

$$\|u\|_{H^s(\mathbb{R}^n)}^2 = \|u\|_{L^2(\mathbb{R}^n)}^2 + \|(-\Delta)^{s/2}u\|_{L^2(\mathbb{R}^n)}^2.$$

The semi-norm $\|(-\Delta)^{s/2}u\|_{L^2(\mathbb{R}^n)}^2$ can also be expressed as

$$\|(-\Delta)^{s/2}u\|_{L^2(\mathbb{R}^n)}^2 = ((-\Delta)^s u, u)_{\mathbb{R}^n},$$

where

$$(-\Delta)^s u = c_{n,s} \text{P.V.} \int_{\mathbb{R}^n} \frac{u(x) - u(y)}{|x - y|^{n+2s}} dy$$

is the standard fractional Laplacian with the constant

$$c_{n,s} = \frac{\Gamma(\frac{n}{2} + s)}{|\Gamma(-s)|} \frac{4^s}{\pi^{n/2}}$$

and P.V. denotes the standard principal value operator (see [6] for detailed description).

Next, given any open set U of \mathbb{R}^n and $\eta \in \mathbb{R}$, we consider the following Sobolev spaces,

$$\begin{aligned} H^\eta(U) &:= \{u|_U; u \in H^\eta(\mathbb{R}^n)\}, \\ \tilde{H}^\eta(U) &:= \text{closure of } C_c^\infty(U) \text{ in } H^\eta(\mathbb{R}^n), \\ H_0^\eta(U) &:= \text{closure of } C_c^\infty(U) \text{ in } H^\eta(U), \end{aligned}$$

and

$$H_{\bar{U}}^\eta := \{u \in H^\eta(\mathbb{R}^n); \text{supp}(u) \subset \bar{U}\}.$$

The Sobolev space $H^\eta(U)$ is complete under the graph norm

$$\|u\|_{H^\eta(U)} := \inf \{ \|v\|_{H^\eta(\mathbb{R}^n)}; v \in H^\eta(\mathbb{R}^n) \text{ and } v|_U = u \}.$$

It is known that $\tilde{H}^\eta(U) \subseteq H_0^\eta(U)$, and $H_{\bar{U}}^\eta$ is a closed subspace of $H^\eta(\mathbb{R}^n)$. For more detailed discussions of the fractional Sobolev spaces, we refer to [6, 23].

2.2. Definition of \mathcal{L}_A^s . Let us get into the rigorous mathematical formulation for the problem we study here. Let us begin with the definition of the nonlocal operator \mathcal{L}_A^s , $s \in (0, 1)$ via the spectral characterization of \mathcal{L}_A . Suppose that \mathcal{L}_A is a linear second order self-adjoint elliptic operator, which is densely defined on $L^2(\mathbb{R}^n)$ for $n \geq 2$. There is a unique resolution $E(\lambda)$ of the identity, supported on the spectrum of \mathcal{L}_A which is a subset of $[0, \infty)$, such that

$$\mathcal{L}_A = \int_0^\infty \lambda dE(\lambda)$$

i.e.,

$$\langle \mathcal{L}_A f, g \rangle_{L^2(\mathbb{R}^n)} = \int_0^\infty \lambda dE_{f,g}(\lambda), \quad f \in \text{Dom}(\mathcal{L}_A), g \in L^2(\mathbb{R}^n),$$

where $dE_{f,g}(\lambda)$ is a regular Borel complex measure of bounded variation concentrated on the spectrum of \mathcal{L}_A , such that $d|E_{f,g}|(0, \infty) \leq \|f\|_{L^2(\mathbb{R}^n)} \|g\|_{L^2(\mathbb{R}^n)}$.

If $\phi(\lambda)$ is a real measurable function defined on $[0, \infty)$, then the operator $\phi(\mathcal{L}_A)$ is given formally by

$$\phi(\mathcal{L}_A) = \int_0^\infty \phi(\lambda) dE(\lambda).$$

That is, $\phi(\mathcal{L}_A)$ is an operator with the domain

$$\text{Dom}(\phi(\mathcal{L}_A)) = \left\{ f \in L^2(\mathbb{R}^n) : \int_0^\infty \|\phi(\lambda)\|^2 dE_{f,f}(\lambda) < \infty \right\},$$

defined by

$$\langle \phi(\mathcal{L}_A)f, g \rangle_{L^2(\mathbb{R}^n)} = \left\langle \int_0^\infty \phi(\lambda) dE(\lambda) f, g \right\rangle_{L^2(\mathbb{R}^n)} = \int_0^\infty \phi(\lambda) dE_{f,g}(\lambda).$$

Following that we define the nonlocal elliptic operators \mathcal{L}_A^s , $s \in (0, 1)$ with domain $\text{Dom}(\mathcal{L}_A^s) \subset \text{Dom}(\mathcal{L}_A)$,

$$\mathcal{L}_A^s = \int_0^\infty \lambda^s dE(\lambda) = \frac{1}{\Gamma(-s)} \int_0^\infty (e^{-t\mathcal{L}_A} - \text{Id}) \frac{dt}{t^{1+s}},$$

where $\Gamma(s)$ is the standard Gamma function and $\Gamma(-s) = \frac{\Gamma(1-s)}{-s} < 0$ for $s \in (0, 1)$. Here $e^{-t\mathcal{L}_A}$ ($t \geq 0$) is the heat-diffusion semigroup generated by \mathcal{L}_A with domain $L^2(\mathbb{R}^n)$ and

$$e^{-t\mathcal{L}_A} = \int_0^\infty e^{-t\lambda} dE(\lambda),$$

which enjoys the contraction property in $L^2(\mathbb{R}^n)$ as $\|e^{-t\mathcal{L}_A} f\|_{L^2(\mathbb{R}^n)} \leq \|f\|_{L^2(\mathbb{R}^n)}$. Meanwhile, for any $w \in H^s(\mathbb{R}^n)$, we have

$$\mathcal{L}_A^s w = \frac{1}{\Gamma(-s)} \int_0^\infty (e^{-t\mathcal{L}_A} w(x) - w(x)) \frac{dt}{t^{1+s}}.$$

For more detailed discussions, we refer readers to [25, 28, 31].

In fact, the heat-diffusion semigroup admits a nonnegative symmetric heat kernel $\mathcal{W}_t(x, z)$ for $t > 0$ by integration, that is for any $f \in L^2(\mathbb{R}^n)$

$$e^{-t\mathcal{L}_A} f(x) = \int_\Omega \mathcal{W}_t(x, z) f(\eta) d\eta(z), \quad x \in \mathbb{R}^n,$$

and for any $v, w \in H^s(\mathbb{R}^n)$,

$$(e^{-t\mathcal{L}_A} v, w)_{\mathbb{R}^n} = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \mathcal{W}_t(x, z) v(z) w(x) dz dx = (v, e^{-t\mathcal{L}_A} w)_{\mathbb{R}^n}, \quad t \geq 0.$$

Define

$$\mathcal{H}_s(x, z) = \frac{1}{2|\Gamma(-s)|} \int_0^\infty \mathcal{W}_t(x, z) \frac{dt}{t^{1+s}}, \quad (2.1)$$

which gives the kernel of anisotropic fractional Schrödinger operators and utilizes [2, Theorem 2.4] we have

$$(\mathcal{L}_A^s v, w)_{\mathbb{R}^n} = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} (v(x) - v(z))(w(x) - w(z)) \mathcal{H}_s(x, z) dx dz, \quad (2.2)$$

where we use the fact that $A(x)$ is a bounded matrix-valued function defined in \mathbb{R}^n satisfying (1.1). In addition, the kernel \mathcal{K}_s possesses the following pointwise estimate (see [2, Theorem 2.4] again)

$$\frac{c_1}{|x-z|^{n+2s}} \leq \mathcal{K}_s(x, z) \leq \frac{c_2}{|x-z|^{n+2s}}, \quad x, z \in \mathbb{R}^n, \quad (2.3)$$

for some constants $c_1, c_2 > 0$ depending on A, n and s and $\mathcal{K}_s(x, z) = \mathcal{K}_s(z, x)$ for all $x, z \in \mathbb{R}^n$. We also refer readers to [7] for further discussions of the nonlocal operator \mathcal{L}_A^s .

3. Nonlocal problems with the embedded obstacles and surrounding potentials . In this section, we give the mathematical formulations for our nonlocal problems.

3.1. Well-Posedness. In the subsequent discussions, we always let $\Omega \subseteq \mathbb{R}^n$ to be a bounded open set and $D \Subset \Omega$ to be an open subset, q to be a potential in $L^\infty(\Omega \setminus \overline{D})$ and $s \in (0, 1)$ to be a constant. Consider the nonlocal Dirichlet problem

$$\begin{cases} (\mathcal{L}_A^s + q)u = f & \text{in } \Omega \setminus \overline{D}, \\ \mathcal{B}u = 0 & \text{in } D, \\ u = g & \text{in } \Omega_e. \end{cases} \quad (3.1)$$

Define the bilinear form $\mathbb{B}_q(\cdot, \cdot)$ by

$$\begin{aligned} \mathbb{B}_q(v, w) : &= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} (v(x) - v(z))(w(x) - w(z))\mathcal{K}_s(x, z) dx dz \\ &+ \int_{\Omega \setminus \overline{D}} q(x)v(x)w(x) dx, \end{aligned} \quad (3.2)$$

for any $v, w \in H^s(\mathbb{R}^n)$. Combining (2.2) and (3.2), we have that

$$\mathbb{B}_q(v, w) = \int_{\mathbb{R}^n} (\mathcal{L}_A^s v)w dx + \int_{\Omega \setminus \overline{D}} qvw dx.$$

Then by using the standard variational formula, the weak solution can be defined by

Definition 3.1. (Weak solution) Let Ω be a bounded open set in \mathbb{R}^n . Given $f \in L^2(\Omega \setminus \overline{D})$ and $g \in H^s(\mathbb{R}^n)$, we call that $u \in H^s(\mathbb{R}^n)$ is a (weak) solution to (3.1) provided that $\tilde{u}_g := u - g \in \tilde{H}^s(\Omega)$ and

$$\mathbb{B}_q(u, \phi) = \int_{\Omega \setminus \overline{D}} f\phi dx \quad \text{for any } \phi \in C_0^\infty(\Omega \setminus \overline{D}), \quad (3.3)$$

with $u - g \in \tilde{H}^s(\Omega)$ or equivalently

$$\mathbb{B}_q(\tilde{u}_g, \phi) = \int_{\Omega \setminus \overline{D}} (f - (\mathcal{L}_A^s + q)g)\phi dx \quad \text{for any } \phi \in C_0^\infty(\Omega \setminus \overline{D}).$$

Next, we have the following well-posedness.

Lemma 3.1. Let $q \in L^\infty(\Omega \setminus \overline{D})$ and $f \in L^2(\Omega \setminus \overline{D})$. $u \in H^s(\mathbb{R}^n)$ solves

$$\mathcal{L}_A^s u + qu = f \quad \text{in } \Omega \setminus \overline{D},$$

(in the sense of distributions) if and only if $u \in H^s(\mathbb{R}^n)$ satisfies

$$\mathbb{B}_q(u, w) = \int_{\Omega \setminus \overline{D}} fwdx \quad \text{for all } w \in \tilde{H}^s(\Omega \setminus \overline{D}).$$

Moreover, when q satisfies the eigenvalue condition (1.3), we have the stability estimate

$$\|u\|_{H^s(\mathbb{R}^n)} \leq C \left(\|f\|_{L^2(\Omega \setminus \overline{D})} + \|g\|_{H^s(\mathbb{R}^n)} \right), \quad (3.4)$$

where $C > 0$ is a constant independent of f and g .

Proof. A straightforward computation shows that

$$\begin{aligned} & \int_{\Omega \setminus \overline{D}} (\mathcal{L}_A^s u + qu - f) w dx \\ &= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} (u(x) - u(z))(w(x) - w(z)) \mathcal{K}_s(x, z) dx dz \\ & \quad + \int_{\Omega \setminus \overline{D}} quw dx - \int_{\Omega \setminus \overline{D}} f w dx \end{aligned}$$

for all $w \in C_c^\infty(\Omega \setminus \overline{D})$. It is easy to see that the bilinear form $\mathbb{B}_q(\cdot, \cdot)$ is bounded, coercive and continuous by using the pointwise estimate (2.3) of the kernel $\mathcal{K}_s(x, z)$, then the stability estimate (3.4) follows from the standard Lax-Milgram theorem (a similar proof has been addressed in [7, Section 3]). This completes the proof. \square

Lemma 3.2. *The solution $u \in H^s(\mathbb{R}^n)$ to (3.1) is independent of the value of $g \in H^s(\mathbb{R}^n)$ in Ω , and it only relies on $g|_{\Omega_e}$.*

Proof. The proof is similar to that of [7, Proposition 3.4] and we skip it. \square

Via Lemma 3.2, we can consider the nonlocal problem (1.2) with Dirichlet data in an abstract quotient space

$$\mathbb{H} := H^s(\mathbb{R}^n) / \widetilde{H}^s(\Omega). \quad (3.5)$$

We also refer readers to [7, 8] for more detailed discussions. Since the solution $u \in H^s(\mathbb{R}^n)$ of (3.1) only depends on the exterior value, in order to simplify notations, we shall consider the Dirichlet data g in the quotient space \mathbb{H} in the subsequent studies.

3.2. The DtN map. We define the associated DtN map for $\mathcal{L}_A^s + q$ via the bilinear form \mathbb{B}_q in (3.3).

Proposition 3.1. *(DtN map) For $n \geq 2$, let $\Omega \subset \mathbb{R}^n$ be a bounded open set and $D \Subset \Omega$ be an obstacle. Let $0 < s < 1$ and $q \in L^\infty(\Omega \setminus \overline{D})$ satisfy (1.3). Let \mathbb{H} be the abstract space given in (3.5). Define*

$$\langle \Lambda_{D,q} g, h \rangle_{\mathbb{H}^* \times \mathbb{H}} := \mathbb{B}_q(u_g, h), \quad g, h \in \mathbb{H},$$

where $u_g \in H^s(\mathbb{R}^n)$ is the solution to (1.2) with the exterior Dirichlet data g . Then $\Lambda_{D,q} : \mathbb{H} \rightarrow \mathbb{H}^*$ is a bounded linear map. Moreover, we have the following symmetry property for $\Lambda_{D,q}$,

$$\langle \Lambda_{D,q} g, h \rangle_{\mathbb{H}^* \times \mathbb{H}} = \langle \Lambda_{D,q} h, g \rangle_{\mathbb{H}^* \times \mathbb{H}}, \quad g, h \in \mathbb{H}.$$

Proof. Combining with Lemma 3.2, the proof is similar to [7, Proposition 3.5], so we skip it here. \square

Remark 3.1. For any $h \in \mathbb{H}$, there exists $\widehat{h} \in H^s(\mathbb{R}^n)$ such that $\widehat{h} = h$ in Ω_e . A direct calculation shows that

$$\begin{aligned}
(\Lambda_{D,q}g, h)_{\mathbb{H}^* \times \mathbb{H}} &= \mathbb{B}_q(u_g, \widehat{h}) \\
&= \int_{\mathbb{R}^n} \widehat{h}(\mathcal{L}_A^s u_g) dx + \int_{\Omega} q u_g \widehat{h} dx \\
&= \int_{\Omega_e} \widehat{h}(\mathcal{L}_A^s u_g) dx \\
&= \int_{\Omega_e} h(\mathcal{L}_A^s u_g) dx.
\end{aligned} \tag{3.6}$$

Then from (3.6), we have

$$(\Lambda_q g, h)_{\mathbb{H}^* \times \mathbb{H}} = \int_{\Omega_e} h(\mathcal{L}_A^s u_g) dx, \text{ for any } h \in \mathbb{H},$$

which implies that

$$\Lambda_q g = \mathcal{L}_A^s u_g|_{\Omega_e}.$$

The integral identity allows us to solve the nonlocal type inverse problem as a direct consequence of Proposition 3.1. It can be stated as follows.

Lemma 3.3. (*Integral identity*) For $n \geq 2$, let $\Omega \subset \mathbb{R}^n$ be a bounded open set and $D \Subset \Omega$ be an obstacle. Let $s \in (0, 1)$ and $q \in L^\infty(\Omega \setminus \overline{D})$ satisfy (1.3). For any $g_1, g_2 \in \mathbb{H}$, one has

$$\int_{\Omega_e} (\Lambda_{D,q_1} g_1 - \Lambda_{D,q_2} g_1) g_2 dx = \int_{\Omega \setminus \overline{D}} (q_1 - q_2) r_{\Omega \setminus \overline{D}} u_1 r_{\Omega \setminus \overline{D}} u_2 dx$$

where $u_j \in H^s(\mathbb{R}^n)$ solves $(\mathcal{L}_A^s + q_j)u_j = 0$ in $\Omega \setminus \overline{D}$ with $u_j|_{\Omega_e} = g_j$ for $j = 1, 2$, and $r_{\Omega \setminus \overline{D}} u$ refers to the restriction of u on $\Omega \setminus \overline{D}$.

Proof. The proof is similar to [8, Lemma 2.5]. \square

4. Recovery of the obstacle D . In this section, we show that the obstacle D can be uniquely recovered by a single measurement. The following strong uniqueness property shall be needed.

Proposition 4.1. [7, Theorem 1.2] For $n \geq 2$ and $0 < s < 1$. If $u \in H^s(\mathbb{R}^n)$ satisfies $u = \mathcal{L}_A^s u = 0$ in any nonempty open set $U \subset \mathbb{R}^n$, then $u \equiv 0$ in \mathbb{R}^n .

Now we can prove the first statement of Theorem 1.1.

Theorem 4.1. Let Ω be a bounded open set in \mathbb{R}^n , $D_1, D_2 \Subset \Omega$ be two open subsets and $\mathcal{O}_1, \mathcal{O}_2 \subset \Omega_e$ be arbitrary nonempty open sets. Let $q_j \in L^\infty(\Omega \setminus \overline{D})$ satisfy (1.3) and $u_j \in H^s(\mathbb{R}^n)$ be the unique (weak) solution to

$$\begin{cases} \mathcal{L}_A^s u_j + q_j u_j = 0 & \text{in } \Omega \setminus \overline{D}_j, \\ \mathcal{B}u_j = 0 & \text{in } D_j, \end{cases}$$

for $j = 1, 2$. Besides, when $\mathcal{B}u_j = \mathcal{L}_A^s u_j$, we further assume that $q_j(x) \neq 0$ for any $x \in \Omega$, $j = 1, 2$. Suppose that $\Lambda_{D_1, q_1} g = \Lambda_{D_2, q_2} g$ in \mathcal{O}_2 , for any given nonzero $g \in C_c^\infty(\mathcal{O}_1)$ with $u_1 = u_2 = g$ in Ω_e , then $D_1 = D_2$.

Proof. First, we prove that $u_1 = u_2$ in \mathbb{R}^n whenever $\Lambda_{D_1, q_1} g = \Lambda_{D_2, q_2} g$ in \mathcal{O}_2 and $u_1 = u_2 = g$ in Ω_e for the non-identically zero function $g \in C_c^\infty(\mathcal{O}_1)$.

Let $w := u_1 - u_2 \in \tilde{H}^s(\Omega)$, then w solves

$$\begin{cases} \mathcal{L}_A^s w + q_1 w = (q_2 - q_1) u_2 & \text{in } \Omega \setminus (\overline{D_1} \cup \overline{D_2}), \\ w = 0 & \text{in } \Omega_e. \end{cases}$$

Note that the set $\Omega \setminus (\cup_{i=1}^2 \overline{D_i})$ is a non-empty open set. From the condition $\Lambda_{D_1, q_1} g = \Lambda_{D_2, q_2} g$ in \mathcal{O}_2 and $\Lambda_{D_j, q_j} g = \mathcal{L}_A^s u_j|_{\Omega_e}$, one can see that

$$\mathcal{L}_A^s w = \mathcal{L}_A^s(u_1 - u_2) = 0 \text{ in } \mathcal{O}_2 \subset \Omega_e.$$

In particular, we have $w \in H^s(\mathbb{R}^n)$ such that $w = \mathcal{L}_A^s w = 0$ in \mathcal{O}_2 . By applying the strong uniqueness property (Proposition 4.1), we obtain $w \equiv 0$ in \mathbb{R}^n , which shows $u_1 = u_2$ in \mathbb{R}^n .

Second, we claim that $D_1 = D_2$ in \mathbb{R}^n by using contradiction arguments. Suppose that $D_1 \neq D_2$. Without loss of generality, we can assume that there exists a nonempty open subset $M \Subset D_2 \setminus \overline{D_1}$. Then we have the following two cases.

Case 1.

$$\begin{cases} \text{Either } u_1 = 0 \text{ in } D_1 \text{ or } \mathcal{L}_A^s u_1 = 0 \text{ in } D_1, \\ u_2 = 0 \text{ in } D_2. \end{cases}$$

By using the condition $u_1 = u_2$ in \mathbb{R}^n , we know that $u_1 = u_2 = 0$ in $M \Subset D_2$. Applying the nonlocal elliptic equation for u_1 in M , it is readily seen that

$$\mathcal{L}_A^s u_1 = u_1 = 0 \text{ in } M.$$

Utilizing the strong uniqueness property again, we obtain that $u_1 \equiv 0$ in \mathbb{R}^n .

Case 2.

$$\begin{cases} \text{Either } u_1 = 0 \text{ in } D_1 \text{ or } \mathcal{L}_A^s u_1 = 0 \text{ in } D_1, \\ \mathcal{L}_A^s u_2 = 0 \text{ in } D_2. \end{cases}$$

Recall that $u_1 = u_2$ in \mathbb{R}^n , then $\mathcal{L}_A^s u_1 = \mathcal{L}_A^s u_2$ in \mathbb{R}^n by using a direct calculation. Hence, $\mathcal{L}_A^s u_1 = \mathcal{L}_A^s u_2 = 0$ in $M \Subset D_2 \setminus \overline{D_1}$. By using the equation of u_1 and $q_1(x) \neq 0$ for $x \in \Omega$, we have

$$u_1 = \mathcal{L}_A^s u_1 = 0 \text{ in } M.$$

Therefore, we have $u_1 \equiv 0$ in \mathbb{R}^n by the strong uniqueness property.

However, in either Case 1 or Case 2, the conclusion $u_1 \equiv 0$ in \mathbb{R}^n contradicts to the fact that $u_1 = g$ in Ω_e with a non-identically zero exterior data g . This proves the first part of Theorem 1.1 to Theorem 1.3 by using a single exterior measurement. \square

Remark 4.1. Indeed, we do not need to use any information about the solution w in $\Omega \setminus (\overline{D_1} \cup \overline{D_2})$. We only utilize the strong uniqueness of w in the exterior domain Ω_e , which is a powerful tool in dealing with the nonlocal type inverse problems.

5. Recovery of the surrounding potential q . In this section, we prove the uniqueness in determining the surrounding potential q in $\Omega \setminus \overline{D}$.

5.1. Runge approximation property. We shall make essential use of the following Runge approximation property for solutions to the nonlocal elliptic equation. If $q \in L^\infty(\Omega \setminus \overline{D})$ satisfies the eigenvalue condition (1.3), we denote the solution operator Φ_q by:

$$\Phi_q : \mathbb{H} \rightarrow H^s(\mathbb{R}^n), g \rightarrow u$$

where $\mathbb{H} := H^s(\mathbb{R}^n)/\tilde{H}^s(\Omega)$ is the abstract space of exterior values, and $u \in H^s(\mathbb{R}^n)$ is the unique solution to $(\mathcal{L}_A^s + q)u = 0$ in $\Omega \setminus \overline{D}$ with $\mathcal{B}u = 0$ in D and $u - g \in \tilde{H}^s(\Omega)$.

Lemma 5.1. *Let $\Omega \subseteq \mathbb{R}^n$ be a bounded open set and $D \Subset \Omega$ be an open subset. Assume that $s \in (0, 1)$ and $q \in L^\infty(\Omega \setminus \overline{D})$ satisfies (1.3). Let \mathcal{O} be any open subset of Ω_e . Consider the set*

$$\mathbb{A} := \{u|_{\Omega \setminus \overline{D}} : u = \Phi_q g, g \in C_c^\infty(\mathcal{O})\} \cap \{\mathcal{B}u = 0 \text{ in } D\}.$$

Then \mathbb{A} is dense in $L^2(\Omega \setminus \overline{D})$.

Proof. The proof follows a similar argument to that of [7, Lemma 5.7]. For the completeness of this paper, we present a detailed proof in what follows.

By the Hahn-Banach theorem, it is sufficient to show that for any $v \in L^2(\Omega \setminus \overline{D})$ with $\int_{\Omega \setminus \overline{D}} v w \, dx = 0$ for all $w \in \mathbb{A}$, then it must satisfy $v \equiv 0$ in $\Omega \setminus \overline{D}$. If v is a such function, which means v satisfies

$$\int_{\Omega \setminus \overline{D}} v \cdot r_{\Omega \setminus \overline{D}} \Phi_q g \, dx = 0, \text{ for any } g \in C_c^\infty(\mathcal{O}). \quad (5.1)$$

We claim that

$$\int_{\Omega \setminus \overline{D}} v \cdot r_{\Omega \setminus \overline{D}} \Phi_q g \, dx = -\mathbb{B}_q(\phi, g), \text{ for any } g \in C_c^\infty(\mathcal{O}), \quad (5.2)$$

where $\phi \in H^s(\mathbb{R}^n)$ is the solution given by Lemma 3.1 to

$$(\mathcal{L}_A^s + q)\phi = v \text{ in } \Omega \setminus \overline{D}, \quad \phi \in \tilde{H}^s(\Omega \setminus \overline{D}).$$

In other words, $\mathbb{B}_q(\phi, w) = \int_{\Omega \setminus \overline{D}} v \cdot r_{\Omega \setminus \overline{D}} w \, dx$ for any $w \in \tilde{H}^s(\Omega \setminus \overline{D})$. To prove (5.2), let $g \in C_c^\infty(\mathcal{O})$, and we denote $u_g := \Phi_q g \in \tilde{H}^s(\mathbb{R}^n)$ such that $u_g - g \in \tilde{H}^s(\Omega)$. Then we have

$$\int_{\Omega \setminus \overline{D}} v \cdot r_{\Omega \setminus \overline{D}} \Phi_q g \, dx = \int_{\Omega \setminus \overline{D}} v \cdot r_{\Omega \setminus \overline{D}} (u_g - g) \, dx = \mathbb{B}_q(\phi, u_g - g) = -\mathbb{B}_q(\phi, g),$$

in which we have used the fact that u_g is a solution and $\phi \in \tilde{H}^s(\Omega \setminus \overline{D})$.

Combining (5.1) and (5.2), we can obtain that

$$\mathbb{B}_q(\phi, g) = 0, \text{ for any } g \in C_c^\infty(\mathcal{O}).$$

Using the fact that $r_{\Omega \setminus \overline{D}} g = 0$, since $g \in C_c^\infty(\mathcal{O})$ we can derive that

$$\int_{\mathbb{R}^n} \mathcal{L}_A^s \phi \cdot g \, dx = 0 \text{ for any } g \in C_c^\infty(\mathcal{O}),$$

and thus we obtain that $\phi \in H^s(\mathbb{R}^n)$ satisfies

$$\mathcal{L}_A^s \phi|_{\mathcal{O}} = \phi|_{\mathcal{O}} = 0.$$

By the strong uniqueness property again, we know that $\phi \equiv 0$ in \mathbb{R}^n and also $v \equiv 0$ in $\Omega \setminus \overline{D}$. This finishes the proof. \square

Remark 5.1. It is easy to see that the soft or hard condition $\mathcal{B}u = 0$ in D does not affect the conclusion of the previous lemma.

5.2. Proof of the uniqueness in determining q . From the equal DtN maps, by the first statements of Theorems 1.1–1.3, we know that the embedded obstacle D is uniquely recovered. Next, we prove the global uniqueness in determining the potential $q \in L^\infty(\Omega \setminus \overline{D})$.

Theorem 5.1. *For $n \geq 2$, let Ω be a bounded open set in \mathbb{R}^n , $D \Subset \Omega$ be an open subset and $\mathcal{O}_1, \mathcal{O}_2 \subset \Omega_e$ be two arbitrary nonempty open sets. Let $q_j \in L^\infty(\Omega \setminus \overline{D})$ satisfy (1.3) and $u_j \in H^s(\mathbb{R}^n)$ be the unique (weak) solution to*

$$\begin{cases} \mathcal{L}_A^s u_j + q_j u_j = 0 & \text{in } \Omega \setminus \overline{D}, \\ \mathcal{B}u_j = 0 & \text{in } D. \end{cases}$$

Assume that Λ_{D, q_j} are the DtN maps with respect to $(\mathcal{L}_A^s + q_j)u_j = 0$ for $j = 1, 2$. If

$$\Lambda_{D, q_1} g|_{\mathcal{O}_2} = \Lambda_{D, q_2} g|_{\mathcal{O}_2},$$

for all $g \in C_c^\infty(\mathcal{O}_1)$ with $u_1 = u_2 = g$ in Ω_e , then one can conclude that

$$q_1 = q_2 \text{ in } \Omega \setminus \overline{D}.$$

Proof. Since $\Lambda_{D, q_1} g|_{\mathcal{O}_2} = \Lambda_{D, q_2} g|_{\mathcal{O}_2}$ for all $g \in C_c^\infty(\mathcal{O}_1)$, where $\mathcal{O}_1, \mathcal{O}_2$ are the open sets of Ω_e , substituting this condition into the integral identity in Lemma 3.3, we have

$$\int_{\Omega \setminus \overline{D}} (q_1 - q_2) u_1 u_2 \, dx = 0, \quad (5.3)$$

where $u_j \in H^s(\mathbb{R}^n)$ are the solutions to $(\mathcal{L}_A^s + q_j)u_j = 0$ in $\Omega \setminus \overline{D}$ with the associated exterior values $g_j \in C_c^\infty(\mathcal{O}_1)$, for $j = 1, 2$ respectively.

Given $\varphi \in L^2(\Omega \setminus \overline{D})$, by Proposition 5.1, suppose that there exists two sequences $(u_j^{(k)})_{k \in \mathbb{N}}$ for $j = 1, 2$ of functions in $H^s(\mathbb{R}^n)$, which satisfy

$$\begin{aligned} (\mathcal{L}_A^s + q_1)u_1^{(k)} &= (\mathcal{L}_A^s + q_2)u_2^{(k)} = 0 \text{ in } \Omega \setminus \overline{D}, \\ \mathcal{B}u_1^{(k)} &= 0 \text{ and } \mathcal{B}u_2^{(k)} = 0 \text{ in } D, \\ u_j^{(k)} &= g_j^{(k)} \text{ in } \Omega_e, \text{ for some exterior data } g_j^{(k)} \in C_c^\infty(\mathcal{O}_1), \\ r_{\Omega \setminus \overline{D}} u_1^{(k)} &= \varphi + r_1^{(k)}, r_{\Omega \setminus \overline{D}} u_2^{(k)} = 1 + r_2^{(k)}, \text{ for any } k \in \mathbb{N}, \end{aligned}$$

where $r_1^{(k)}, r_2^{(k)} \rightarrow 0$ in $L^2(\Omega \setminus \overline{D})$ as $k \rightarrow \infty$. Substituting these solutions into the integral identity (5.3) and taking the limit as $k \rightarrow \infty$, we can deduce that

$$\int_{\Omega \setminus \overline{D}} (q_1 - q_2) \varphi \, dx = 0$$

Since $\varphi \in L^2(\Omega \setminus \overline{D})$ is arbitrary, we readily see that $q_1 = q_2$ in $\Omega \setminus \overline{D}$. This also completes the second parts of Theorems 1.1–1.3. \square

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