ON UNIQUENESS IN NONLOCAL DIFFUSE OPTICAL TOMOGRAPHY

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Abstract. We investigate the inverse problem of recovering the diffusion and absorption coefficients (σ, q) in the nonlocal diffuse optical tomography equation $(-\operatorname{div}(\sigma\nabla))^s u + qu = 0$ from the (partial) Dirichlet-to-Neumann map. The purpose of this article is twofold:

(i) Firstly, we show that the diffusion coefficient σ and absorption coefficient q can be recovered simultaneously.
(ii) Secondly, we prove that the absorption coefficient q can be determined provided σ is known in a neighborhood of the boundary $\partial\Omega$.

The key ingredients to prove these uniqueness results are the Caffarelli–Silvestre type extension technique and a novel Runge approximation related to solution spaces of two different partial differential equations, which is based on the geometric form of the Hahn–Banach theorem. The results in this work hold for any dimension $n \geq 3$.

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Contents

1. Introduction

In recent years, the study of nonlocal inverse problems attracted interest by many researchers. The first work in this field [GSU20] concerned the unique determination of bounded potentials in the *fractional Schrödinger equation*

$$((-\Delta)^s + q)u = 0 \text{ in } \Omega \tag{1.1}$$

from the related (partial) Dirichlet-to-Neumann (DN) map. Here $\Omega \subset \mathbb{R}^n$ is a bounded domain and $0 < s < 1$. The proof of this uniqueness result relied on two essential ingredients, namely the *unique continuation property* (UCP) of the fractional Laplacian $(-\Delta)^s$ and a *Runge approximation* (see Propositions 2.2 and 2.3), which allows to approximate any function in $L^2(\Omega)$ by solutions to (1.1). By generalizations of these two beautiful and strong results, different articles could solve several inverse problems, which remain open in the local case or there even exist counterexamples to uniqueness. The above inverse problem has been extended into several directions, like determining singular potentials, lower order local perturbations, higher order nonlocal operators or combining the above approach with other techniques as linearization and monotonicity methods (see [BGU21, CLL19, CMR21, CMRU22, GLX17, CLL19, CLR20, FGKU24, HL19, HL20, GRSU20, GU21, HL19, HL20, KRZ23, Lin22, LL22, LL23, LLR20, LLU22, KLW22, RS20, RS18, RZ23, GU21]).

Let us emphasize that the recovery of leading order coefficients for nonlocal operators has also been studied. These can be seen as the nonlocal counterparts to classical Calderón problem [Cal06] or the p-Calderón problem [SZ12]. In the works [CO23, CGRU23, Fei24, FGKU24, GU21, Rül23, LLU22, LLU23, Lin23, RZ22, CRZ22, CRTZ22, RZ24, KLZ22, LRZ22], the authors investigated these types of inverse problems by utilizing either the DN map or the source-to-solution map as measurement operators.

The simultaneous recovery of several coefficients in nonlocal partial differential equations (PDEs) from the related DN map is usually more involved, but there are still a few positive results into this direction. For example, the following results have been obtained:

(i) In [CLR20] the authors showed that the drift term b and the potential q in
$$((-\Delta)^s + b\cdot\nabla + q)\, u = 0 \text{ in } \Omega$$
can be recovered uniquely form the DN map.

(ii) In [LZ23] a unique determination result has been obtained for all coefficients (ρ, q) and kernel K in the nonlocal porous medium equation
$$\rho\partial_t u + L_K\left(u^m\right) + qu = 0 \text{ in } \Omega\times(0,T),$$
where $m > 1$ and L_K is an *elliptic integro-differential operator* of order $2s$, that is L_K is given by
$$L_K u(x) = \text{p.v.}\int_{\mathbb{R}^n} K(x,y)(u(x)-u(y))\,dy \tag{1.2}$$
with
$$K(x,y) = K(y,x) \quad \text{and} \quad \frac{\lambda}{|x-y|^{n+2s}} \leq K(x,y) \leq \frac{\lambda^{-1}}{|x-y|^{n+2s}}$$
for some $\lambda > 0$.

(iii) In [Zim23] it is shown that the diffusion and absorption coefficients (γ, q) in the *nonlocal optical tomography equation*
$$L^s_\gamma u + qu = 0 \text{ in } \Omega \tag{1.3}$$
can be uniquely recovered from the DN map. Here, L^s_γ is the integro-differential operator with kernel
$$K(x,y) = C_{n,s}\frac{\gamma^{1/2}(x)\gamma^{1/2}(y)}{|x-y|^{n+2s}}$$
(cf. (1.2)), where $C_{n,s} > 0$ is the usual normalization constant in the definition of the fractional Laplacian $(-\Delta)^s$ and $\gamma\colon \mathbb{R}^n \to \mathbb{R}$ is a uniformly

elliptic function. It is noteworthy that in the endpoint case $s = 1$ the operator L^s_γ becomes the usual conductivity operator $-\operatorname{div}(\gamma\nabla\cdot)$.

Analogous uniqueness statements in the endpoint cases $s = 1$ of (i) and (iii) are generally not true. In this article, we consider a similar PDE as the nonlocal optical tomography equation (1.3), but where the operator is replaced by a fractional power of the conductivity operator $-\operatorname{div}(\sigma\nabla\cdot)$. This model is introduced in the next section.

1.1. **The nonlocal diffuse optical tomography equation.** Here, we describe the PDE considered in this article, present the main results and discuss related models. Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with smooth boundary $\partial\Omega$. Throughout this work, we suppose that the *diffusion coefficient* $\sigma \in C^\infty(\mathbb{R}^n)$ is uniformly elliptic, that is

$$\lambda \leq \sigma(x) \leq \lambda^{-1} \tag{1.4}$$

for some $\lambda \in (0, 1)$, satisfies

$$\sigma(x) = 1 \text{ for } x \in \Omega_e, \tag{1.5}$$

and the *absorption coefficient* or *potential* $q \in L^\infty(\Omega)$ is nonnegative. Given such data, we consider the Dirichlet problem for the *nonlocal diffuse optical tomography equation*

$$\begin{cases} (-\operatorname{div}(\sigma\nabla))^s u + qu = 0 & \text{in } \Omega, \\ u = f & \text{in } \Omega_e, \end{cases} \tag{1.6}$$

where

$$\Omega_e := \mathbb{R}^n \setminus \overline{\Omega}$$

denotes the exterior of Ω and $0 < s < 1$. Here $(-\operatorname{div}(\sigma\nabla))^s$ is an elliptic integro-differential operator, which is rigorously defined in Section 2. By standard methods, one sees that the Dirichlet problem (1.6) is well-posed on the energy space $H^s(\mathbb{R}^n)$. Let $W \subset \Omega_e$ be a nonempty bounded Lipschitz domain, then we can define the (partial) DN map

$$\Lambda^s_{\sigma,q} : \widetilde{H}^s(W) \to H^{-s}(W), \quad f \mapsto (-\operatorname{div}(\sigma\nabla))^s u_f|_W\,, \tag{1.7}$$

where $u_f \in H^s(\mathbb{R}^n)$ is the unique solution to (1.6). Thus, the inverse problem studied in this article can be phrased as:

(IP) Inverse Problem. Can one determine the diffusion and absorption coefficients (σ, q) in Ω from the partial DN map $\Lambda^s_{\sigma,q}$ given by (1.7)?

Before presenting the affirmative results to this nonlocal inverse problem, let us review the situation for the local counterpart. As $s = 1$, it is known that the corresponding inverse problem cannot be solved uniquely, i.e., σ and q cannot be determined uniquely. More concretely, if we consider the *diffuse optical tomography equation*

$$\begin{cases} -\operatorname{div}(\sigma\nabla w) + qw = 0 & \text{in } \Omega, \\ w = g & \text{on } \partial\Omega, \end{cases} \tag{1.8}$$

where $\sigma\colon \overline{\Omega} \to \mathbb{R}$ is the diffusion coefficient and $q\colon \Omega \to \mathbb{R}$ is a bounded absorption coefficient. For instance, when $q \geq 0$ in Ω, the well-posedness of (1.8) guarantees the existence of the (full) DN map, which can be characterized by

$$\Lambda_{\sigma,q} : H^{1/2}(\partial\Omega) \to H^{-1/2}(\partial\Omega), \quad g \mapsto \sigma\nabla w_g \cdot \nu|_{\partial\Omega}\,,$$

where $w_g \in H^1(\Omega)$ is the unique solution to (1.8). Also here the goal is to recover the coefficients (σ, q) from the DN data $\Lambda_{\sigma,q}$, which is unfortunately not possible.

From this point of view our **(IP)** can be seen as a nonlocal version of the diffuse optical tomography problem.

This type problem arises in steady state diffusion optical tomography, where light propagation is characterized by a diffusion approximation and the excitation frequency is set to zero. For a complete description of optical tomography including the derivation of (1.8), we refer the reader to the articles [Arr99] and [AL98].

Moreover, one may observe that the classical *Liouville transformation* $w \mapsto \sqrt{\sigma}w$, maps every solution w of the optical tomography equation

$$-\operatorname{div}(\sigma\nabla w) + qw = 0 \text{ in } \Omega$$

to a solution $\widetilde{w} = \sqrt{\sigma}w$ of the Schrödinger equation

$$-\Delta\widetilde{w} + V\widetilde{w} = 0 \text{ in } \Omega \quad \text{with} \quad V = \frac{\Delta\sqrt{\sigma}}{\sqrt{\sigma}} + \frac{q}{\sigma}.$$

The well-known counterexamples of Arridge and Lionheart [AL98] are based on this observation. More precisely, if (σ_1, q_1) with σ_1 uniformly elliptic, $q_1 \geq 0$ is given, then (σ_2, q_2) with $\sigma_2 = \sigma_0 + \sigma_1$, $q_2 = q_0 + q_1$ has the same DN data, whenever (σ_0, q_0) satisfies

(i) $\sigma_0 \geq 0$ and $\sigma_0 = 0$ in a neighborhood of $\partial\Omega$,
(ii) and the perturbation q_0 is given by

$$q_0 = \sigma_2\left(\frac{\Delta\sqrt{\sigma_1}}{\sqrt{\sigma_1}} - \frac{\Delta\sqrt{\sigma_2}}{\sqrt{\sigma_2}} + \frac{q_1}{\sigma_1}\right) - q_1 \tag{1.9}$$

(see also [Zim23]). So, even in the case $\sigma = 1$ in a neighborhood of $\partial\Omega$, one may loose uniqueness, and therefore, it seems impossible to determine σ and q simultaneously.

In contrast, in this work, we establish the following simultaneous determination result, which is completely different to the local case.

Theorem 1.1 (Global uniqueness)**.** *Let $\Omega, W \subset \mathbb{R}^n$ be bounded domains with smooth boundaries such that $\overline{\Omega} \cap \overline{W} = \emptyset$. Suppose that for $j = 1, 2$, the diffusion coefficient $\sigma_j \in C^\infty(\mathbb{R}^n)$ satisfies (1.4)–(1.5) and the absorption coefficient $q_j \in L^\infty(\Omega)$ is nonnegative. If the DN maps $\Lambda^s_{\sigma_j, q_j}$ associated to the problem*

$$\begin{cases} \left(-\operatorname{div}(\sigma_j\nabla)\right)^s u + q_j u = 0 & \text{in } \Omega, \\ u = f & \text{in } \Omega_e \end{cases} \tag{1.10}$$

satisfy

$$\Lambda^s_{\sigma_1, q_1} f = \Lambda^s_{\sigma_2, q_2} f \text{ in } W \tag{1.11}$$

for all $f \in C^\infty_c(W)$, then there holds

$$\sigma_1 = \sigma_2 \text{ and } q_1 = q_2 \text{ in } \Omega.$$

Let us remark that the unique recovery of $\sigma_1 = \sigma_2$ in Ω with $q = 0$ has been studied in [CGRU23]. In this work, the authors investigated a new reduction method via the Caffarelli-Silvestre type extension, which leads to the local DN map can be determined by the nonlocal DN map. This idea is also of help in our nonlocal diffuse optical tomography problem. Conversely, when σ is given, the unique determination of the potential q in (1.6) has been studied in [GLX17], which remains open for $n \geq 3$ for their local counterpart. Moreover, a similar unique determination result has been found in [Zim23] for the nonlocal optical tomography problem (see above (iii)), but in contrast to the problem treated in this article there are several crucial differences:

(a) In analogy with the local case, the Liouville transformation $u \mapsto \gamma^{1/2}u$ maps the unique solution u of the nonlocal optical tomography equation (1.3) to the unique solution v of the fractional Schrödinger equation

$$((-\Delta)^s + Q_\gamma)v = 0 \text{ in } \Omega \text{ with } Q_\gamma = -\frac{(-\Delta)^s(\sqrt{\gamma}-1)}{\sqrt{\gamma}} + \frac{q}{\gamma}.$$

(b) One does not need the assumption that the potential q is compactly contained in Ω.

(c) In [Zim23], it is shown that the assumptions on the potential and diffusion coefficients are sharp as otherwise one may construct counterexamples.

Let us note that in the problem in [Zim23] as well as in the nonlocal diffuse optical tomography problem (1.6), considered in this article, one can first recover the diffusion coefficient σ or γ, respectively, and then the potential q.

Furthermore, we found an interesting localization phenomena for the inverse problem of the nonlocal diffuse optical tomography equation, namely that one can determine q in a certain neighborhood of the boundary, whenever σ is known a priori in the same region without using full information of σ.

Theorem 1.2 (Local uniqueness)**.** *Let $\Omega, W \subset \mathbb{R}^n$ be bounded domains with Lipschitz boundaries such that $\overline{\Omega} \cap \overline{W} = \emptyset$. Suppose that for $j = 1, 2$, the diffusion coefficient $\sigma_j \in C^\infty(\mathbb{R}^n)$ satisfies (1.4)–(1.5) and the absorption coefficient $q_j \in C^0(\Omega)$ is nonnegative, for $j = 1, 2$. If one has $\sigma_1 = \sigma_2$ in a nonempty open, connected neighborhood $\mathcal{N} \subset \overline{\Omega}$ of $\partial\Omega$, then (1.11) implies that $q_1 = q_2$ in $\mathcal{N} \cap \Omega$.*

If $\Lambda^s_{\sigma_1,q_1} = \Lambda^s_{\sigma_2,q_2}$ would imply $\Lambda_{\sigma_1,q_1} = \Lambda_{\sigma_2,q_2}$, then the counterexamples (1.9) suggest that Theorem 1.2 holds. It is noteworthy that our proof of Theorem 1.2 is purely nonlocal without using the reduction method to the local equation. On the other hand, such a reduction could not be used to establish Theorem 1.1 by the ill-posedness of the local diffuse optical tomography problem.

1.2. **Ideas of the proof.** To prove the global uniqueness result (Theorem 1.1), we consider the Caffarelli-Silvestre (CS) type extension for the nonlocal operator $(-\operatorname{div}(\sigma\nabla))^s$ with $s \in (0, 1)$, that is

$$\begin{cases} \operatorname{div}_{x,y}\left(y^{1-2s}\Sigma(x)\nabla_{x,y}\mathcal{P}^s_\sigma u\right) = 0 & \text{in } \mathbb{R}^{n+1}_+, \\ \mathcal{P}^s_\sigma u(x,0) = u(x) & \text{on } \mathbb{R}^n, \end{cases} \tag{1.12}$$

and it is well-known that there holds

$$-\lim_{y\to 0} y^{1-2s}\partial_y \mathcal{P}^s_\sigma u_f = d_s \left(-\operatorname{div}(\sigma\nabla)\right)^s u \text{ in } \mathbb{R}^n, \tag{1.13}$$

where $\mathbb{R}^{n+1}_+ := \left\{(x,y) \in \mathbb{R}^{n+1} : x \in \mathbb{R}^n \text{ and } y > 0\right\}$, d_s is a constant depending only on $s \in (0, 1)$, and Σ is an $(n + 1) \times (n + 1)$ matrix of the form

$$\Sigma(x) = \begin{pmatrix} \sigma(x)\mathbf{1}_n & 0 \\ 0 & 1 \end{pmatrix}. \tag{1.14}$$

Here $\mathbf{1}_n$ denotes the $n \times n$ identity matrix. This type extension problem was first studied in [ST10] for the variable (matrix-valued) coefficients case.

Next, we consider the A_2-weighted y integral of the CS type extension

$$U(x) := \int_0^\infty y^{1-2s}\mathcal{P}^s_\sigma u(x,y)\,dy. \tag{1.15}$$

Integrating (1.12) with respect to the y-direction yields that

$$
\begin{aligned}
0 &= \int_0^\infty \operatorname{div}_{x,y}\left(y^{1-2s}\Sigma\nabla_{x,y}\mathcal{P}^s_\sigma u\right)\, dy \\
&= \operatorname{div}\left(\sigma\nabla\left(\int_0^\infty y^{1-2s}\mathcal{P}^s_\sigma u\, dy\right)\right) + \int_0^\infty \partial_y\left(y^{1-2s}\partial_y\mathcal{P}^s_\sigma u\right) dy \\
&= \operatorname{div}(\sigma\nabla U) - \lim_{y\to 0} y^{1-2s}\partial_y\mathcal{P}^s_\sigma u,
\end{aligned} \tag{1.16}
$$

which implies

$$
-\operatorname{div}(\sigma\nabla U) = -\lim_{y\to 0} y^{1-2s}\partial_y\mathcal{P}^s_\sigma u = \underbrace{d_s\left(-\operatorname{div}(\sigma\nabla)\right)^s u}_{\text{By using (1.13)}},
$$

where we have assumed a suitable decay of $\mathcal{P}^s_\sigma u$ such that $\lim_{y\to\infty} y^{1-2s}\partial_y\mathcal{P}^s_\sigma u = 0$ at moment (This fact will be justified in Section 3). In particular, when u_f is the solution to (1.6), the preceding derivations yield that

$$
\operatorname{div}\left(\sigma\nabla U_f\right) = d_s q(x) u_f \text{ in } \Omega, \tag{1.17}
$$

where U_f is the function defined in (1.15) with $u = u_f$.

Note that the left hand side of (1.17) is a local differential operator, and the right hand side comes from the nonlocal information of (1.6). Via (1.17), one can see that the function U_f may not solve the classical conductivity equation $\operatorname{div}(\sigma\nabla v) = 0$, since the source term qu_f in (1.17) may not be zero in general. Surprisingly, we are able to prove that that any function in the set

$$
\left\{v \in H^1(\Omega) : \operatorname{div}(\sigma\nabla v) = 0 \text{ in } \Omega\right\},
$$

can be approximated by a sequence of functions in

$$
\{U_f : \text{ for any } f \in C^\infty_c(W)\} \subset H^1(\Omega)
$$

with respect to the $H^1(\Omega)$-norm. This result rests on the *geometric form of the Hahn–Banach theorem* (see Proposition 4.4 for detailed arguments). This novel reduction allows us to recover the leading coefficient σ firstly, without using any knowledge of the potential q. When σ is known, we can directly apply the global uniqueness result [GLX17] to determine the potential q.

On the other hand, by this approximation property, the corresponding (local) DN map of (1.17) could be formally given by

$$
\begin{aligned}
&\Lambda_{\sigma,q} : H^{1/2}(\partial\Omega) \to H^{-1/2}(\partial\Omega), \\
&\underbrace{\int_0^\infty y^{1-2s}\mathcal{P}^s_\sigma u_f(x,y)\, dy\Big|_{\partial\Omega}}_{=U_f|_{\partial\Omega}} \mapsto \underbrace{\sigma\nabla\left(\int_0^\infty y^{1-2s}\mathcal{P}^s_\sigma u_f(x,y)\, dy\right)\cdot\nu\Big|_{\partial\Omega}}_{=\sigma\nabla U_f\cdot\nu|_{\partial\Omega}}.
\end{aligned}
$$

Thus, in this work, we aim to determine the local DN map $\Lambda_{\sigma,q}$ by the nonlocal one $\Lambda^s_{\sigma,q}$, even though the equation of U_f involves more complicated elliptic equations, but the approximation property will help us to get rid of the zero order potential term.

1.3. **Organization of the paper.** Our article is organized as follows. In Section 2, we recall fractional Sobolev spaces and basic properties of the involved nonlocal operators. In Section 3, we introduce weighted Sobolev spaces and associated extension problem for our nonlocal operators. In Section 4, we derive the relation between the nonlocal and the local problems. Moreover, we also provide the key approximation result in the same section. Using this Runge type approximation, we prove Theorem 1.1 in Section 5. Finally, in the same section, we also establish the proof of Theorem 1.2.

2. The nonlocal problem

In this section, we review some known properties of the nonlocal operators considered in this article.

2.1. **Fractional Sobolev spaces.** Let us start by recalling the definition of the fractional Sobolev spaces and the fractional Laplacian.

We denote by $\mathscr{S}(\mathbb{R}^n)$ and $\mathscr{S}'(\mathbb{R}^n)$ the space of Schwartz functions and tempered distributions, respectively. We define the Fourier transform $\mathcal{F}\colon \mathscr{S}(\mathbb{R}^n) \to \mathscr{S}(\mathbb{R}^n)$ by

$$\mathcal{F}f(\xi) := \int_{\mathbb{R}^n} f(x) e^{-\mathrm{i}x\cdot\xi}\, dx,$$

which is occasionally also denoted by $\widehat{f}$, and $\mathrm{i} = \sqrt{-1}$. By duality it can be extended to the space of tempered distributions and will again be denoted by $\mathcal{F}u = \widehat{u}$, where $u \in \mathscr{S}'(\mathbb{R}^n)$, and we denote the inverse Fourier transform by $\mathcal{F}^{-1}$.

Given $s \in \mathbb{R}$, the L^2-based fractional Sobolev space $H^s(\mathbb{R}^n)$ is the set of all tempered distributions $u \in \mathscr{S}'(\mathbb{R}^n)$ such that

$$\|u\|_{H^s(\mathbb{R}^n)} := \|\langle D\rangle^s u\|_{L^2(\mathbb{R}^n)} < \infty,$$

where $\langle D\rangle^s$ is the Bessel potential operator of order s having Fourier symbol $(1+|\xi|^2)^{s/2}$.

Next recall that the fractional Laplacian of order $s \geq 0$ is given as a Fourier multiplier via

$$(-\Delta)^s u = \mathcal{F}^{-1}\left(|\xi|^{2s}\,\widehat{u}(\xi)\right),$$

for $u \in \mathscr{S}'(\mathbb{R}^n)$ whenever the right-hand side is well-defined or as a singular integral by

$$(-\Delta)^s u(x) = C_{n,s}\text{p.v.}\int_{\mathbb{R}^n} \frac{u(x)-u(y)}{|x-y|^{n+2s}}\, dy,$$

where $C_{n,s} > 0$ is a given constant depending only on n, s and p.v. denotes the Cauchy principal value.

It is known that for $s \geq 0$ an equivalent norm on $H^s(\mathbb{R}^n)$ is given by

$$\|u\|^*_{H^s(\mathbb{R}^n)} = \left(\|u\|^2_{L^2(\mathbb{R}^n)} + \|(-\Delta)^{s/2}u\|^2_{L^2(\mathbb{R}^n)}\right)^{1/2}.$$

Next we introduce some local variants of the above fractional Sobolev spaces. If $\Omega \subset \mathbb{R}^n$ is an open set and $s \in \mathbb{R}$, then we set

$$\begin{aligned} H^s(\Omega) &:= \{\, u|_\Omega \mid u \in H^s(\mathbb{R}^n)\}, \\ \widetilde{H}^s(\Omega) &:= \text{closure of } C_c^\infty(\Omega) \text{ in } H^s(\mathbb{R}^n). \end{aligned}$$

2.2. **Nonlocal elliptic operators.** Here, we recall the definition of the nonlocal operator $(-\operatorname{div}(\sigma\nabla))^s$, $0 < s < 1$, and discuss some of its properties. In this article we restrict our attention to the isotropic case, where $\sigma\colon \mathbb{R}^n \to \mathbb{R}$ is a smooth uniformly elliptic function such that $\sigma|_{\Omega_e} = 1$. In this setting it is known that there exists a symmetric kernel $K^s_\sigma(x,y)$ comparable to the fractional Laplacian $(-\Delta)^s$, that is

$$\frac{c}{|x-y|^{n+2s}} \leq K^s_\sigma(x,y) \leq \frac{C}{|x-y|^{n+2s}}, \tag{2.1}$$

such that $(-\operatorname{div}(\sigma\nabla))^s$ defined via

(2.2)

$$\langle(-\operatorname{div}(\sigma\nabla))^s u, v\rangle_{H^{-s}(\mathbb{R}^n)\times H^s(\mathbb{R}^n)} = \int_{\mathbb{R}^{2n}} (u(x)-u(y))(v(x)-v(y))K^s_\sigma(x,y)\, dxdy,$$

for all $u,v\in H^s(\mathbb{R}^n)$ induces a bounded linear operator from $H^s(\mathbb{R}^n)$ to $H^{-s}(\mathbb{R}^n)$. Moreover, using the symmetry of K_σ^s it is easy to see that the operator $(-\operatorname{div}(\sigma\nabla))^s$ is symmetric, that is one has

$$\langle(-\operatorname{div}(\sigma\nabla))^s u,v\rangle_{H^{-s}(\mathbb{R}^n)\times H^s(\mathbb{R}^n)}=\langle(-\operatorname{div}(\sigma\nabla))^s v,u\rangle_{H^{-s}(\mathbb{R}^n)\times H^s(\mathbb{R}^n)}$$

for all $u,v\in H^s(\mathbb{R}^n)$.

By standard arguments, one easily shows that under the above assumptions on σ and $0\le q\in L^\infty(\Omega)$, the Dirichlet problem (1.6) is well-posed for any $f\in H^s(\mathbb{R}^n)$, that is there exists a unique function $u\in H^s(\mathbb{R}^n)$ satisfying $u-f\in\widetilde{H}^s(\Omega)$ and

$$B^s_{\sigma,q}(u,\varphi):=\langle(-\operatorname{div}(\sigma\nabla)^s u,\varphi\rangle_{H^{-s}(\mathbb{R}^n)\times H^s(\mathbb{R}^n)}+\int_\Omega qu\varphi\,dx=0 \tag{2.3}$$

for all $\varphi\in\widetilde{H}^s(\Omega)$. More concretely, one can use the kernel estimates (2.1) and the fractional Poincaré inequality to deduce the boundedness and coercivity of the bilinear form (2.3) on $\widetilde{H}^s(\Omega)$, then the Lax-Milgram theorem yields the well-posedness result as desired. Furthermore, if $f_1,f_2\in H^s(\mathbb{R}^n)$ satisfy $f_1-f_2\in\widetilde{H}^s(\Omega)$, then the corresponding unique solutions to (1.6) coincide. Hence, the exterior value to solution map is well-defined on the abstract trace space $X_s=H^s(\mathbb{R}^n)/\widetilde{H}^s(\Omega)$ and for exterior value f the unique solution will be denoted by u_f in the rest of this article. Note that the well-posedness of (1.6) also holds for general potential without sign assumption. Instead, one could consider suitable eigenvalue condition[1] to prove the well-posedness. However, in this work, we simply introduce the condition $q\ge 0$ in Ω for the self-consistency.

2.3. **DN map.** With the well-posedness of (1.6) at hand, we can define the DN map rigorously by using the bilinear form (2.3) induced by $(-\operatorname{div}(\sigma\nabla))^s+q$ (see (2.2)). For any $0<s<1$ and (σ,q) as in Section 2.2, we define the DN map $\Lambda^s_{\sigma,q}\colon X_s\to X_s^*$ via

$$\left\langle\Lambda^s_{\sigma,q}f,g\right\rangle=B_{\sigma,q}\left(u_f,v_g\right), \tag{2.4}$$

where $u_f\in H^s(\mathbb{R}^n)$ is the unique solution to (1.6) and $v_g\in H^s(\mathbb{R}^n)$ is any representative of g. This immediately implies the following simple lemma:

Lemma 2.1. *Let $\Omega,W\subset\mathbb{R}^n$ be bounded open sets such that $W\subset\Omega_e$ and $0<s<1$. Assume that $\sigma_1,\sigma_2\in C^\infty(\mathbb{R}^n)$ satisfy the conditions of Section 2.2 and $q_1,q_2\in L^\infty(\Omega)$ are given nonnegative potentials. Then there holds*

$$\left\langle\Lambda^s_{\sigma_1,q_1}f_1,f_2\right\rangle-\left\langle\Lambda^s_{\sigma_2,q_2}f_1,f_2\right\rangle=\left(B_{\sigma_1,q_1}-B_{\sigma_2,q_2}\right)\left(u^{(1)}_{f_1},u^{(2)}_{f_2}\right) \tag{2.5}$$

and

$$\left\langle\Lambda^s_{\sigma_1,q_1}f_1,f_2\right\rangle-\left\langle\Lambda^s_{\sigma_2,q_2}f_1,f_2\right\rangle=B_{\sigma_1,q_1}\left(u^{(1)}_{f_1},u^{(2)}_{f_2}\right)-B_{\sigma_2,q_2}\left(u^{(2)}_{f_1},u^{(2)}_{f_2}\right) \tag{2.6}$$

for all $f_1,f_2\in C_c^\infty(W)$, where $u^{(j)}_{f_k}\in H^s(\mathbb{R}^n)$ is the unique solution to (1.6) with $\sigma=\sigma_j$, $q=q_j$ and exterior value f_k for $1\le j,k\le 2$.

Proof. First note that using (2.4) and the symmetry of the bilinear form B_{σ_j,q_j} we have

$$\left\langle\Lambda^s_{\sigma_2,q_2}f_1,f_2\right\rangle=B_{\sigma_2,q_2}\left(u^{(2)}_{f_1},u^{(2)}_{f_2}\right)=B_{\sigma_2,q_2}\left(u^{(2)}_{f_2},u^{(2)}_{f_1}\right)=\left\langle\Lambda^s_{\sigma_2,q_2}f_2,f_1\right\rangle$$

[1]For instance, the eigenvalue condition can be written as 0 is not a Dirichlet eigenvalue of $(-\operatorname{div}(\sigma\nabla))^s+q$ in Ω.

for all $f_1, f_2 \in X_s$. Again using (2.4) and the symmetry of B_{σ_2,q_2}, we get

$$\begin{aligned}\left\langle \Lambda^s_{\sigma_1,q_1} f_1, f_2\right\rangle - \left\langle \Lambda^s_{\sigma_2,q_2} f_1, f_2\right\rangle &= \left\langle \Lambda^s_{\sigma_1,q_1} f_1, f_2\right\rangle - \left\langle \Lambda^s_{\sigma_2,q_2} f_2, f_1\right\rangle \\ &= B_{\sigma_1,q_1}\left(u^{(1)}_{f_1}, u^{(2)}_{f_2}\right) - B_{\sigma_2,q_2}\left(u^{(2)}_{f_2}, u^{(1)}_{f_1}\right) \\ &= \left(B_{\sigma_1,q_1} - B_{\sigma_2,q_2}\right)\left(u^{(1)}_{f_1}, u^{(2)}_{f_2}\right).\end{aligned}$$

This completes the proof of the identity (2.5). Next, note that (2.4) implies

$$\begin{aligned}\left\langle \Lambda^s_{\sigma_1,q_1} f_1, f_2\right\rangle &= B_{\sigma_1,q_1}\left(u^{(1)}_{f_1}, u^{(2)}_{f_2}\right), \\ \left\langle \Lambda^s_{\sigma_2,q_2} f_1, f_2\right\rangle &= B_{\sigma_2,q_2}\left(u^{(2)}_{f_1}, u^{(2)}_{f_2}\right).\end{aligned}$$

Thus, (2.6) follows by subtracting them. □

2.4. **Runge approximation.** From [GLX17, Theorem 1.3], it is known that the Runge approximation holds for variable coefficients nonlocal elliptic operators. The proof of the Runge approximation is based on the unique continuation property (UCP in short) for the operator $(-\operatorname{div}(\sigma\nabla))^s$. For readers' conveniences, let us review these properties:

Proposition 2.2 (UCP, [GLX17, Theorem 1.2])**.** *Let $\sigma \in C^\infty(\mathbb{R}^n)$ satisfy (1.4), $0 < s < 1$ and $\mathcal{O} \subset \mathbb{R}^n$ a nonempty open set. Suppose that $u \in H^s(\mathbb{R}^n)$ satisfies*

$$u = (-\operatorname{div}(\sigma\nabla))^s u = 0 \text{ in } \mathcal{O}.$$

Then $u \equiv 0$ in $\mathbb{R}^n$.

Proposition 2.3 (Runge approximation)**.** *Let $\Omega, W \subset \mathbb{R}^n$ be bounded open sets with Lipschitz boundaries such that $\overline{\Omega} \cap \overline{W} = \emptyset$. Assume that $\sigma \in C^\infty(\mathbb{R}^n)$ satisfies (1.4) and $0 \le q \in L^\infty(\Omega)$. Then the following inclusions are dense*

(i) $\{u_f|_\Omega : f \in C_c^\infty(W)\} \subset L^2(\Omega)$,

(ii) $\{u_f - f : f \in C_c^\infty(W)\} \subset \widetilde{H}^s(\Omega)$,

where $u_f \in H^s(\mathbb{R}^n)$ is the solution to (1.6).

Proof. The assertion (i) has been proved in [GLX17, Lemma 5.6] and (ii) follows by combining Proposition 2.2 with [RZ23, Theorem 4.3]. □

3. The extension problem

We review some important properties of the extension problem for $(-\operatorname{div}(\sigma\nabla))^s$. We will also revisit some useful decay estimates for the solution of the extension problem. Let us start by stating the following Caffarell–Silvestre (CS) type extension:

Definition 3.1 (CS-type extension)**.** *Let $\sigma \in C^\infty(\mathbb{R}^n)$ satisfy the conditions in Section 2.2 and $0 < s < 1$. Suppose $p_t(x,z)$, $t > 0$, is the (symmetric) heat kernel related to the parabolic equation*

$$\partial_t u - \operatorname{div}(\sigma\nabla u) = 0 \text{ in } \mathbb{R}^n, \quad \lim_{t\downarrow 0} u = \delta_x.$$

For any $u \in H^s(\mathbb{R}^n)$ we define its Caffarelli–Silvestre type extension *by*

$$\mathcal{P}^s_\sigma u(x,y) := \int_{\mathbb{R}^n} P_y(x,z)u(z)\,dz, \tag{3.1}$$

where P_y is the fractional Poisson kernel given by

$$P_y(x,z) = c_s y^{2s}\int_0^\infty e^{-y^2/4t} p_t(x,z)\frac{dt}{t^{1+s}}$$

for some given constant $c_s > 0$.

Remark 3.2. *For the existence of a symmetric heat kernel p_t as in the previous definition we refer to the monograph [Gri09] and furthermore by [Dav90] we know that there are constants $\alpha_j, c_j > 0$, $j = 1, 2$, such that*

$$c_1 e^{-\alpha_1|x-z|^2/4t} t^{-n/2} \leq p_t(x,z) \leq c_2 e^{-\alpha_2|x-z|^2/4t} t^{-n/2} \tag{3.2}$$

for all $x, z \in \mathbb{R}^n$. Observe that without loss of generality we can assume that $\alpha_1 \geq 1$ and $0 < \alpha_2 \leq 1$.

Lemma 3.3. *Suppose that the assumptions of Definition 3.1 hold and denote by*

$$\widetilde{P}_y^s(x) = C_s \frac{y^{2s}}{(|x|^2 + y^2)^{\frac{n+2s}{2}}}$$

the fractional Poisson kernel related to the fractional Laplacian $(-\Delta)^s$.

(i) *There exist constants $C_1, C_2 > 0$ such that*

$$C_1 \widetilde{P}^s_{y/\sqrt{\alpha_1}}(x-z) \leq P_y(x,z) \leq C_2 \widetilde{P}^s_{y/\sqrt{\alpha_2}}(x-z). \tag{3.3}$$

(ii) *For any $u \in L^p(\mathbb{R}^n)$ with $1 \leq p \leq \infty$ and $1 + \frac{1}{r} = \frac{1}{p} + \frac{1}{q}$, there holds*

$$\|\mathcal{P}^s_\sigma u(\cdot, y)\|_{L^r(\mathbb{R}^n)} \lesssim y^{n(1-q)/q} \|u\|_{L^p(\mathbb{R}^n)}, \tag{3.4}$$

for any fixed $y > 0$. Moreover, $\mathcal{P}^s_\sigma u \in L^p_{\mathrm{loc}}(\overline{\mathbb{R}^{n+1}_+}, y^{1-2s})$ satisfies

$$\|\mathcal{P}^s_\sigma u\|_{L^p(\mathbb{R}^n \times (0,R), y^{1-2s})} \lesssim R^{2(1-s)/p} \|u\|_{L^p(\mathbb{R}^n)}, \tag{3.5}$$

for any $R > 0$.

(iii) *For any $u \in L^p(\mathbb{R}^n)$ with $1 \leq p \leq \infty$ and $1 + \frac{1}{r} = \frac{1}{p} + \frac{1}{q}$, there holds*

$$\|\nabla_{x,y} \mathcal{P}^s_\sigma u(\cdot, y)\|_{L^r(\mathbb{R}^n)} \lesssim y^{n(1-q)/q-1} \|u\|_{L^p(\mathbb{R}^n)} \tag{3.6}$$

for any fixed $y > 0$, where the exponent on the right hand side has to be interpreted as $-(n+1)$ in the case $p = 1$.

(iv) *For any $u \in L^p(\mathbb{R}^n)$ with $1 \leq p < \infty$, the function $\mathcal{P}^s_\sigma u$ solves the following extension problem*

$$\begin{cases} \mathrm{div}_{x,y}\left(y^{1-2s}\Sigma(x)\nabla_{x,y} v\right) = 0 & \text{in } \mathbb{R}^{n+1}_+, \\ v(x,0) = u(x) & \text{on } \mathbb{R}^n, \end{cases} \tag{3.7}$$

where $\Sigma(x)$ is given by (1.14).

(v) *If $u \in H^s(\mathbb{R}^n)$, then one has $\nabla_{x,y}\mathcal{P}^s_\sigma u \in L^2(\mathbb{R}^{n+1}_+, y^{1-2s})$ satisfying*

$$\|\nabla_{x,y}\mathcal{P}^s_\sigma u\|_{L^2(\mathbb{R}^{n+1}_+)} \lesssim \|u\|_{H^s(\mathbb{R}^n)} \tag{3.8}$$

and there holds

$$-\lim_{y\to 0} y^{1-2s}\partial_y \mathcal{P}^s_\sigma u = d_s(-\mathrm{div}(\sigma\nabla))^s u \tag{3.9}$$

in $H^{-s}(\mathbb{R}^n)$ for some positive constant $d_s > 0$.

Proof of Lemma 3.3. We only provide the upper bound in (i) as the lower bound works similarly. Using (3.2) and change of variables we have

$$
\begin{aligned}
P_y(x,z) &= c_s y^{2s} \int_0^\infty e^{-y^2/4t} p_t(x,z)\, \frac{dt}{t^{1+s}} \\
&\leq C y^{2s} \int_0^\infty e^{-\frac{y^2+\alpha_2|x-z|^2}{4t}} \frac{dt}{t^{n/2+1+s}} \\
&= \underbrace{C y^{2s} \int_0^\infty e^{-\frac{y^2+\alpha_2|x-z|^2}{4}\tau} \tau^{n/2+s-1}\, d\tau}_{\tau=1/t} \\
&= \underbrace{C \frac{y^{2s}}{(y^2+\alpha_2|x-z|^2)^{\frac{n+2s}{2}}} \int_0^\infty e^{-\eta}\eta^{n/2+s-1}\, d\eta}_{\eta=\frac{y^2+\alpha_2|x-z|^2}{4}\tau} \\
&= C \frac{y^{2s}}{(y^2+\alpha_2|x-z|^2)^{\frac{n+2s}{2}}}
\end{aligned} \tag{3.10}
$$

for any $y>0$. In the above computation we used that the integral converges to the Gamma function $\Gamma(n/2+s)$. Taking α_2 outside and rescaling gives the result.

Next, we prove (ii). For this we may first observe that using (3.3) we have

$$
|\mathcal{P}^s_\sigma u| \leq \int_{\mathbb{R}^n} P_y(x,z)|u(z)|\, dz \leq C \left(P^s_{y/\sqrt{\alpha_2}} * |u|\right)(x). \tag{3.11}
$$

Next, recall that by [KRZ23, Lemma 7.1] for any $s>0$ and $1\leq q<\infty$ there holds

$$
\left\|\widetilde{P}^s_y\right\|^q_{L^q(\mathbb{R}^n)} = C_s \frac{\omega_n}{2} y^{n(1-q)} B(n/2, n(q-1)/2+sq) \tag{3.12}
$$

for all $y>0$, where $\omega_n = |\partial B_1(0)|$ denotes the Lebesgue measure of $\partial B_1(0)$, and $B(x,y)$ is the Beta function[2]. Therefore, we can use Young's inequality and the previous estimate to obtain the bound (3.4). The estimate (3.5) follows from (3.4) and the same computation as in [KRZ23, Lemma 7.2].

Next, we turn to the proof of (iii). First, note that by [CJKS20, Theorem 1.2], there holds

$$
|\nabla_x p_t(x,z)| \leq c_3 t^{-\frac{n+1}{2}} e^{-\alpha_3 \frac{|x-z|^2}{4t}}
$$

for constants $c_3, \alpha_3 > 0$. Therefore, using Lebesgue's differentiation theorem and a similar computation as above we get

[2]The constant $C_s>0$ is chosen in such a way that $\left\|\widetilde{P}^s_y\right\|_{L^1(\mathbb{R}^n)} = 1$.

$$
\begin{aligned}
|\nabla_x P_y(x,z)| &= c_s y^{2s} \left| \int_0^\infty e^{-y^2/4t} \nabla_x p_t(x,z) \, \frac{dt}{t^{1+s}} \right| \\
&\leq C y^{2s} \int_0^\infty e^{-\frac{y^2+\alpha_3|x-z|^2}{4t}} \, \frac{dt}{t^{n/2+3/2+s}} \\
&= \underbrace{C y^{2s} \int_0^\infty e^{-\frac{y^2+\alpha_3|x-z|^2}{4}\tau} \tau^{n/2+s-1/2} \, d\tau}_{\tau=1/t} \\
&= \underbrace{C \frac{y^{2s}}{(y^2+\alpha_3|x-z|^2)^{\frac{n+2s+1}{2}}} \int_0^\infty e^{-\eta} \eta^{n/2+s} \, d\eta}_{\eta=\frac{y^2+\alpha_3|x-z|^2}{4}\tau} \\
&= \frac{C}{y} \frac{y^{2(s+1/2)}}{(y^2+\alpha_3|x-z|^2)^{\frac{n+2(s+1/2)}{2}}} \\
&= \frac{C}{y/\sqrt{\alpha_3}} \widetilde{P}^{s+1/2}_{y/\sqrt{\alpha_3}}(x-z).
\end{aligned}
$$

Moreover, by Lebesgue's dominated convergence theorem we may calculate

$$
\begin{aligned}
&\partial_y P_y(x,z) \\
&= c_s \partial_y \left(\int_0^\infty y^{2s} \, e^{-y^2/4t} p_t(x,z) \, \frac{dt}{t^{1+s}} \right) \\
&= c_s \left(2s y^{2s-1} \int_0^\infty e^{-y^2/4t} p_t(x,z) \, \frac{dt}{t^{1+s}} - \frac{y^{2s+1}}{2} \int_0^\infty e^{-y^2/4t} p_t(x,z) \, \frac{dt}{t^{2+s}} \right).
\end{aligned}
$$

Using (3.2), (3.10) and the same change of variables as above we get

$$
\begin{aligned}
&|\partial_y P_y(x,z)| \\
&\leq C \left(y^{-1} P_y(x,z) + y^{2s+1} \int_0^\infty e^{-y^2/4t} p_t(x,z) \, \frac{dt}{t^{2+s}} \right) \\
&\leq C \left(\frac{y^{2s-1}}{(y^2+\alpha_2|x-z|^2)^{\frac{n+2s}{2}}} + y^{2s+1} \int_0^\infty e^{-\frac{y^2+\alpha_2|x-z|^2}{4t}} \, \frac{dt}{t^{n/2+2+s}} \right) \\
&= C \left(\frac{y^{2s-1}}{(y^2+\alpha_2|x-z|^2)^{\frac{n+2s}{2}}} + y^{2s+1} \int_0^\infty e^{-\frac{y^2+\alpha_2|x-z|^2}{4}\tau} \tau^{n/2+s} \, d\tau \right) \\
&\leq C \left(\frac{y^{2s-1}}{(y^2+\alpha_2|x-z|^2)^{\frac{n+2s}{2}}} + \frac{y^{2s+1}}{(y^2+\alpha_2|x-z|^2)^{\frac{n+2(s+1)}{2}}} \right) \\
&= \frac{C}{y} \left(\frac{y^{2s}}{(y^2+\alpha_2|x-z|^2)^{\frac{n+2s}{2}}} + \frac{y^{2(s+1)}}{(y^2+\alpha_2|x-z|^2)^{\frac{n+2(s+1)}{2}}} \right) \\
&= \frac{C}{y/\sqrt{\alpha_2}} \left(\widetilde{P}^s_{y/\sqrt{\alpha_2}}(x-z) + \widetilde{P}^{s+1}_{y/\sqrt{\alpha_2}}(x-z) \right).
\end{aligned}
$$

As in (3.11) this implies

$$
|\nabla_x \mathcal{P}^s_\sigma u| \leq \frac{C}{y/\sqrt{\alpha_3}} \left(\widetilde{P}^{s+1/2}_{y/\sqrt{\alpha_3}} * |u| \right)
$$

and

$$
|\partial_y \mathcal{P}^s_\sigma u| \leq \frac{C}{y/\sqrt{\alpha_2}} \left(\widetilde{P}^s_{y/\sqrt{\alpha_2}} + \widetilde{P}^{s+1}_{y/\sqrt{\alpha_2}} \right) * |u|. \tag{3.13}
$$

Therefore, by Young's inequality and (3.12) we may estimate

$$
\begin{aligned}
&\|\nabla_{x,y}\mathcal{P}^s_\sigma u(\cdot,y)\|_{L^r(\mathbb{R}^n)} \\
&\leq \frac{C}{y}\left(\left\|\widetilde{P}^{s+1/2}_{y/\sqrt{\alpha_3}}*|u|\right\|_{L^r(\mathbb{R}^n)}+\left\|\widetilde{P}^{s}_{y/\sqrt{\alpha_2}}*|u|\right\|_{L^r(\mathbb{R}^n)}+\left\|\widetilde{P}^{s+1}_{y/\sqrt{\alpha_2}}*|u|\right\|_{L^r(\mathbb{R}^n)}\right) \\
&\leq Cy^{n(1-q)/q-1}\|u\|_{L^p(\mathbb{R}^n)},
\end{aligned}
$$

which proves (iii).

Now, we come to the proof of (iv). That $P^s_y(\cdot,z)\in L^2(\mathbb{R}^n)$ solves the PDE in (3.7) has been proven in [ST10, Theorem 2.2]. By standard result it then follows that $\mathcal{P}^s_\sigma(u)$ is a solution to the desired PDE. Furthermore, again by [ST10, Theorem 2.2] the boundary value is attained in the L^p sense.

Finally, we prove the assertion (v). If $u\in C^\infty_c(\mathbb{R}^n)$, then we may conclude from [ST10, Theorem 1.1] that formula (3.9) holds in the L^2 sense. For the general case choose $u_k\in C^\infty_c(\mathbb{R}^n)$, $k\in\mathbb{N}$, such that $u_k\to u$ in $H^s(\mathbb{R}^n)$ as $k\to\infty$. Since $\mathcal{P}^s_\sigma u_k$ solves the extension problem (3.7), using the previous L^2 convergence we get

$$
\begin{aligned}
&\int_{\mathbb{R}^{n+1}_+} y^{1-2s}\Sigma\nabla_{x,y}\mathcal{P}^s_\sigma u_k\cdot\nabla_{x,y}\mathcal{P}^s_\sigma u_k\,dxdy \\
&= -\int_{\mathbb{R}^n\times\{y=0\}}\left(y^{1-2s}\partial_y\mathcal{P}^s_\sigma u_k\right)\mathcal{P}^s_\sigma u_k\,dx \\
&= d_s\int_{\mathbb{R}^n}\left((-\operatorname{div}(\sigma\nabla))^s u_k\right)u_k\,dx \\
&= d_s\left\langle(-\operatorname{div}(\sigma\nabla))^s u_k,u_k\right\rangle_{H^{-s}(\mathbb{R}^n)\times H^s(\mathbb{R}^n)}.
\end{aligned}
$$

Now, using the linearity of $u\mapsto\mathcal{P}^s_\sigma u$, (3.6) with $r=p=2$ and Fatou's lemma we may estimate

$$
\begin{aligned}
&\int_{\mathbb{R}^{n+1}_+} y^{1-2s}\Sigma\nabla_{x,y}\mathcal{P}^s_\sigma u\cdot\nabla_{x,y}\mathcal{P}^s_\sigma u\,dxdy \\
&\leq \liminf_{k\to\infty}\int_{\mathbb{R}^{n+1}_+} y^{1-2s}\Sigma\nabla_{x,y}\mathcal{P}^s_\sigma u_k\cdot\nabla_{x,y}\mathcal{P}^s_\sigma u_k\,dxdy \\
&= d_s\liminf_{k\to\infty}\left\langle(-\operatorname{div}(\sigma\nabla))^s u_k,u_k\right\rangle_{H^{-s}(\mathbb{R}^n)\times H^s(\mathbb{R}^n)} \\
&= d_s\left\langle(-\operatorname{div}(\sigma\nabla))^s u,u\right\rangle_{H^{-s}(\mathbb{R}^n)\times H^s(\mathbb{R}^n)}.
\end{aligned}
$$

By the uniform ellipticity of σ, we directly get the desired estimate (3.8). Finally, we can argue as in [GLX17, Proposition 4.3] to conclude that (3.9) holds for general $u\in H^s(\mathbb{R}^n)$. This concludes the proof. □

Remark 3.4. *To make the formal computations* (1.16) *rigorous, we only need to check*

$$
\lim_{y\to\infty} y^{1-2s}\partial_y v=0 \text{ in } \mathbb{R}^n, \tag{3.14}
$$

where v is the solution to the extension problem (1.12)*. To this end, we can choose $r=\infty$, $p=q=2$ in* (3.6)*, then the limit* (3.14) *holds automatically.*

4. From nonlocal to local

In this section, we derive key ingredients in order to prove our main results. In Section 4.1 we prove a regularity result for H^1_{loc}-solutions of

$$
-\operatorname{div}(\sigma\nabla v)=d_s(-\operatorname{div}(\sigma\nabla))^s u \text{ in } \mathbb{R}^n
$$

for $u \in H^s(\mathbb{R}^n)$ and review the local PDE solved by the functions U_f, which are defined via the equation (1.15) with $u = u_f$. Then in Section 4.2 we provide via the geometric Hahn–Banach theorem our new Runge approximation, which allows to approximation solutions of $\mathrm{div}(\sigma\nabla v) = 0$ by the functions U_f.

4.1. **Some auxiliary lemmas.** We begin with the aforementioned basic regularity result.

Lemma 4.1 (Regularity result)**.** *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain, $0 < s < 1$, $\sigma \in C^\infty(\mathbb{R}^n)$ a uniformly elliptic diffusion coefficient with $\sigma|_{\Omega_e} = 1$ and $u \in H^s(\mathbb{R}^n)$. Suppose that $v \in H^1_{\mathrm{loc}}(\mathbb{R}^n)$ is a solution to*

$$-\mathrm{div}(\sigma\nabla v) = d_s(-\mathrm{div}(\sigma\nabla))^s u \text{ in } \mathbb{R}^n, \tag{4.1}$$

where d_s is some constant depending only on $s \in (0,1)$. Then the Hessian $\nabla^2 v \in H^{-s}(\mathbb{R}^n;\mathbb{R}^{n\times n})$ satisfies

$$\left\|\nabla^2 v\right\|_{H^{-s}(\mathbb{R}^n)} \leq C\|-\Delta v\|_{H^{-s}(\mathbb{R}^n)} \leq C\left(\|\nabla v\|_{L^2(\Omega)} + \|u\|_{H^s(\mathbb{R}^n)}\right), \tag{4.2}$$

for some constant $C > 0$ independent of u and v, but depending on $\|\sigma\|_{C^{0,1}(\mathbb{R}^n)}$. Moreover, we have $-\mathrm{div}(\sigma\nabla v) \in H^{-s}(\mathbb{R}^n)$ and equation (4.1) holds in $H^{-s}(\mathbb{R}^n)$.

Proof. By (4.1), let us first note that there holds

$$-\sigma\Delta v = \nabla\sigma\cdot\nabla v + d_s(-\mathrm{div}(\sigma\nabla))^s u$$

in $\mathscr{D}'(\mathbb{R}^n)$. Using that $(-\mathrm{div}(\sigma\nabla))^s$ is a bounded linear operator from $H^s(\mathbb{R}^n)$ to $H^{-s}(\mathbb{R}^n)$ and $\sigma|_{\Omega_e} = 1$, we deduce that there holds

$$\begin{aligned}\left|\int_{\mathbb{R}^n}\nabla v\cdot\nabla(\sigma\varphi)\,dx\right| &\leq C\left(\|\nabla\sigma\|_{L^\infty(\Omega)}\|\nabla v\|_{L^2(\Omega)}\|\varphi\|_{L^2(\Omega)} + \|u\|_{H^s(\mathbb{R}^n)}\|\varphi\|_{H^s(\mathbb{R}^n)}\right)\\ &\leq C\left(\|\nabla v\|_{L^2(\Omega)}\|\varphi\|_{L^2(\Omega)} + \|u\|_{H^s(\mathbb{R}^n)}\|\varphi\|_{H^s(\mathbb{R}^n)}\right)\end{aligned}$$

for all $\varphi \in C_c^\infty(\mathbb{R}^n)$. As $\sigma \in C^\infty(\mathbb{R}^n)$ is uniformly elliptic, we can replace $\varphi \in C_c^\infty(\mathbb{R}^n)$ by $\psi/\sigma \in C_c^\infty(\mathbb{R}^n)$ with $\psi \in C_c^\infty(\mathbb{R}^n)$ to get

$$\left|\int_{\mathbb{R}^n}\nabla v\cdot\nabla\psi\,dx\right| \leq C\left(\|\nabla v\|_{L^2(\Omega)}\|\psi\|_{L^2(\Omega)} + \|u\|_{H^s(\mathbb{R}^n)}\|\psi\|_{H^s(\mathbb{R}^n)}\right),$$

for all $\psi \in C_c^\infty(\mathbb{R}^n)$. Here, we used the uniform ellipticity (1.4) of σ, $\sigma|_{\Omega_e} = 1$ and the resulting Lipschitz continuity of σ on $\mathbb{R}^n$, which implies that the multiplication map $H^s(\mathbb{R}^n) \ni v \mapsto \sigma^{-1}v \in H^s(\mathbb{R}^n)$ is bounded (see [DNPV12, Lemma 5.3]). More precisely, we used that the Lipschitz continuity and uniform ellipticity of σ ensure that σ^{-1} is Lipschitz continuous. This in turn follows from the simple estimate

$$|\sigma^{-1}(x) - \sigma^{-1}(y)| = \left|\frac{\sigma(x)-\sigma(y)}{\sigma(x)\sigma(y)}\right| \leq C|\sigma(x)-\sigma(y)| \leq C[\sigma]_{C^{0,1}(\mathbb{R}^n)}|x-y|,$$

where in the first inequality we used the uniform ellipticity of σ and $[\sigma]_{C^{0,1}(\mathbb{R}^n)}$ denotes the usual Lipschitz seminorm.

Thus, the distribution $-\Delta v$ satisfies

$$|\langle -\Delta v, \varphi\rangle| \leq C\left(\|\nabla v\|_{L^2(\Omega)} + \|u\|_{H^s(\mathbb{R}^n)}\right)\|\varphi\|_{H^s(\mathbb{R}^n)}$$

for all $\varphi \in C_c^\infty(\mathbb{R}^n)$ and hence continuously extends to an element in $H^{-s}(\mathbb{R}^n)$ with

$$\|-\Delta v\|_{H^{-s}(\mathbb{R}^n)} \leq C\left(\|\nabla v\|_{L^2(\Omega)} + \|u\|_{H^s(\mathbb{R}^n)}\right).$$

Note that

$$\widehat{\partial_{jk}v} = \frac{\xi_j\xi_k}{|\xi|^2}\widehat{-\Delta v}$$

for all $1 \leq j,k \leq n$ and hence by Plancherel's theorem there holds

$$\begin{aligned}\|\partial_{jk}v\|_{H^{-s}(\mathbb{R}^n)} &= C\left\|\langle\xi\rangle^{-s}\widehat{\partial_{jk}v}\right\|_{L^2(\mathbb{R}^n)} = C\left\|\langle\xi\rangle^{-s}\frac{\xi_j\xi_k}{|\xi|^2}\widehat{-\Delta v}\right\|_{L^2(\mathbb{R}^n)} \\ &\leq C\left\|\langle\xi\rangle^{-s}\widehat{-\Delta v}\right\|_{L^2(\mathbb{R}^n)} = C\left\|-\Delta v\right\|_{H^{-s}(\mathbb{R}^n)}\end{aligned}$$

for all $1 \leq j,k \leq n$. Hence, we can conclude the proof of (4.2). Next, recall that we have

$$-\operatorname{div}(\sigma\nabla v) = -\sigma\Delta v - \chi_\Omega\nabla\sigma\cdot\nabla v$$

in $\mathscr{D}'(\mathbb{R}^n)$, where we denote by χ_A its characteristic function for a set $A \subset \mathbb{R}^n$. Hence, we have

$$\begin{aligned}\|-\operatorname{div}(\sigma\nabla v)\|_{H^{-s}(\mathbb{R}^n)} &\leq \|-\sigma\Delta v\|_{H^{-s}(\mathbb{R}^n)} + \|\chi_\Omega\nabla\sigma\cdot\nabla v\|_{H^{-s}(\mathbb{R}^n)} \\ &\leq C\|-\Delta v\|_{H^{-s}(\mathbb{R}^n)} + \|\nabla\sigma\cdot\nabla v\|_{L^2(\Omega)} \\ &\leq C\left(\|\nabla v\|_{L^2(\Omega)} + \|u\|_{H^s(\mathbb{R}^n)}\right).\end{aligned}$$

Here we used again the Lipschitz continuity of σ and [DNPV12, Lemma 5.3]. This shows that $-\operatorname{div}(\sigma\nabla v) \in H^{-s}(\mathbb{R}^n)$. Finally, the fact that (4.1) holds in $H^{-s}(\mathbb{R}^n)$ follows from the density of $C_c^\infty(\mathbb{R}^n)$ in $H^s(\mathbb{R}^n)$.

Indeed, let $\varphi \in H^s(\mathbb{R}^n)$ and choose $\varphi_k \in C_c^\infty(\mathbb{R}^n)$ such that $\varphi_k \to \varphi$ in $H^s(\mathbb{R}^n)$. Then using $-\operatorname{div}(\sigma\nabla v), (-\operatorname{div}(\sigma\nabla))^s u \in H^{-s}(\mathbb{R}^n)$, we obtain

$$\begin{aligned}\langle -\operatorname{div}(\sigma\nabla v), \varphi\rangle_{H^{-s}(\mathbb{R}^n)\times H^s(\mathbb{R}^n)} &= \lim_{k\to\infty}\langle -\operatorname{div}(\sigma\nabla v), \varphi_k\rangle_{H^{-s}(\mathbb{R}^n)\times H^s(\mathbb{R}^n)} \\ &= \lim_{k\to\infty}\int_{\mathbb{R}^n}\sigma\nabla v\cdot\nabla\varphi_k\,dx \\ &= d_s\lim_{k\to\infty}\langle(-\operatorname{div}(\sigma\nabla))^s u, \varphi_k\rangle_{H^{-s}(\mathbb{R}^n)\times H^s(\mathbb{R}^n)} \\ &= d_s\langle(-\operatorname{div}(\sigma\nabla))^s u, \varphi\rangle_{H^{-s}(\mathbb{R}^n)\times H^s(\mathbb{R}^n)}.\end{aligned}$$

This completes the proof. □

Lemma 4.2. *Let $\Omega \subset \mathbb{R}^n$ be a smoothly bounded domain and $0 < s < 1$. Assume that σ is a smooth uniformly elliptic function with $\sigma|_{\Omega_e} = 1$ and $0 \leq q \in L^\infty(\Omega)$. Let $u_f \in H^s(\mathbb{R}^n)$ be the unique solution to (1.6) with $f \in H^s(\mathbb{R}^n)$ and let $\mathcal{P}^s_\sigma u_f$ denote the Caffarelli-Silvestre type extension of u_f. Then*

$$U_f(x) := \int_0^\infty y^{1-2s}\mathcal{P}^s_\sigma u_f(x,y)\,dy$$

belongs to $H^1_{\mathrm{loc}}(\mathbb{R}^n)$. Moreover, the function U_f solves

$$-\operatorname{div}(\sigma\nabla v) = d_s(-\operatorname{div}(\sigma\nabla))^s u_f \text{ in } \mathbb{R}^n \tag{4.3}$$

and

$$\begin{cases}\operatorname{div}(\sigma\nabla v) = d_s q u_f & \text{in } \Omega, \\ v = U_f|_{\partial\Omega} & \text{on } \partial\Omega.\end{cases} \tag{4.4}$$

Proof. By [CGRU23, Proposition 6.1], it is known that $U_f \in H^1_{\mathrm{loc}}(\mathbb{R}^n)$ and by [CGRU23, Theorem 3] that it then solves (4.3). Now, by using the nonlocal equation (1.6), the equations (4.4) holds true as we want. □

Remark 4.3. *Let $f_1, f_2 \in C_c^\infty(W)$ be arbitrary exterior data, $u^{(j)}_{f_\ell} \in H^s(\mathbb{R}^n)$ be the solution to (1.10), for $j = 1,2$. Consider the Caffarelli-Silvestre type extension $\mathcal{P}^s_{\sigma_j}(u^{(j)}_{f_\ell})$ for $j,\ell \in \{1,2\}$. By Lemma 4.2, then the function*

$$U^{(j)}_{f_\ell}(x) := \int_0^\infty y^{1-2s}\mathcal{P}^s_{\sigma_j}u^{(j)}_{f_\ell}(x,y)\,dy$$

solves the equations

$$-\operatorname{div}\left(\sigma_j \nabla U_{f_\ell}^{(j)}\right) = d_s \left(-\operatorname{div}(\sigma_j \nabla)\right)^s u_{f_\ell}^{(j)} \text{ in } \mathbb{R}^n,$$

for $j, \ell \in \{1,2\}$*. Let us point out that the superscript* (j) *corresponds to have conductivity* σ_j *and potential* q_j *in the extension problem and the nonlocal PDE, respectively, the subscript* f_ℓ *to the exterior data in the nonlocal PDE. In particular, as* $j = \ell \in \{1,2\}$, $U_{f_j}^{(j)}$ *satisfies*

$$\begin{cases} \operatorname{div}(\sigma_j \nabla v) = d_s q_j u_{f_j}^{(j)} & \text{in } \Omega, \\ v = U_{f_j}^{(j)}\Big|_{\partial\Omega} & \text{on } \partial\Omega \end{cases}$$

in the $H^{-s}(\mathbb{R}^n)$ *sense (see Lemma 4.1).*

4.2. **A new Runge approximation.** With the preceding analysis, we want to prove the following novel Runge type approximation result:

Proposition 4.4 (Runge approximation)**.** *Let*

$$\begin{aligned} \mathcal{D} &:= \left\{ U_f(x) = \int_0^\infty y^{1-2s} \mathcal{P}_\sigma u_f(x,y)\, dy \,:\, f \in C_c^\infty(W) \right\}, \\ \mathcal{D}' &:= \left\{ U_f|_{\partial\Omega} : U_f \in \mathcal{D} \right\}, \end{aligned}$$

and

$$S := \left\{ v \in H^1(\Omega) : \operatorname{div}(\sigma \nabla v) = 0 \text{ in } \Omega \right\}.$$

Given $v \in S$*, for any* $\epsilon > 0$*, there exists* $U_f \in \mathcal{D}$ *such that*

$$\|U_f - v\|_{H^1(\Omega)} < \epsilon. \tag{4.5}$$

Furthermore, given $g \in H^{1/2}(\partial\Omega)$*, for any* $\epsilon > 0$*, one can also find* $U_f|_{\partial\Omega} \in \mathcal{D}'$ *such that*

$$\left\| U_f|_{\partial\Omega} - g \right\|_{H^{1/2}(\partial\Omega)} < \epsilon. \tag{4.6}$$

Note that (4.5) implies that any $H^{1/2}(\partial\Omega)$ function can be also approximated by a sequence of functions in $\mathcal{D}'$ and hence yielding the assertion (4.6).

One can expect the approximation result holds since the CS-type extension only brings the coefficient inside the nonlocal operator $(-\operatorname{div}(\sigma\nabla))^s$ for the equation $((-\operatorname{div}(\sigma\nabla))^s + q)\, u = 0$ in Ω into the higher dimensional space. This can be regarded as a Robin-type boundary condition for the extension problem (3.7) on $\Omega \times \{0\}$. We first give a formal proof for the case $s = 1/2$ and $\sigma = 1$.

A formal proof of Proposition 4.4. To demonstrate our idea, let us consider the case $s = 1/2$ and $\sigma = 1$. Suppose (4.5) does not hold, i.e., there is a $v \in S$ and $\epsilon > 0$ such that

$$\|U_f - v\|_{H^1(\Omega)} \geq \epsilon > 0, \text{ for all } f \in C_c^\infty(W). \tag{4.7}$$

By the above inequality, $v \notin \mathcal{D}$ and in particular $v \in S$ cannot be zero, otherwise it leads to a contradiction. Let

$$A := \overline{\mathcal{D}}^{H^1(\Omega)} \text{ and } B := \{v\},$$

by the above condition, we have $A \cap B = \emptyset$, where A and B are closed convex sets in $H^1(\Omega)$ with B being compact. In fact, for any $U \in A$, there exists $f \in C_c^\infty(W)$ such that

$$\|U - U_f\|_{H^1(\Omega)} < \epsilon/2.$$

But then (4.7) implies

$$\|U - v\|_{H^1(\Omega)} \geq \|U_f - v\|_{H^1(\Omega)} - \|U - U_f\|_{H^1(\Omega)} \geq \epsilon/2 > 0.$$

As this holds for any $U \in A$, we can deduce that $v \notin A$ and hence $A \cap B = \emptyset$.

Now, by the geometric form of the Hahn-Banach theorem (for example, see [Bre11, Theorem 1.7]), there exists a (continuous) linear functional $\varphi \in (H^1(\Omega))^* = \widetilde{H}^{-1}(\Omega)$ (dual space of $H^1(\Omega)$) and $\alpha \in \mathbb{R}$ such that

$$\varphi(U) < \alpha < \varphi(v), \text{ for all } U \in A. \tag{4.8}$$

Note that we necessarily have $\alpha > 0$ as $f = 0$ corresponds to $U_f = 0$. The condition (4.8) especially implies that $\varphi(U_f) = 0$ for all $f \in C_c^\infty(W)$. To see this assume that $\varphi(U_f) \neq 0$ for some $f \in C_c^\infty(W)$. As the mappings $f \mapsto U_f$ and φ are linear, one can replace f by $\mu f \in C_c^\infty(W)$ to obtain $\mu\varphi(U_f) \neq 0$ for all $0 \neq \mu \in \mathbb{R}$. Hence, by choosing a suitable $\mu \in \mathbb{R}$ one gets a contradiction to (4.8).

Next, we aim to show that

$$\varphi(U_f) = 0 \text{ for all } f \in C_c^\infty(W) \implies \varphi(v) = 0.$$

To this end, let us consider the adjoint problem

$$\begin{cases} \Delta_{x,y} w(x,y) = \varphi & \text{in } \mathbb{R}^{n+1}_+ \\ w(x,0) = 0 & \text{on } \Omega_e \times \{0\}, \\ -\lim_{y\to 0} \partial_y w(x,y) + q(x) w(x,0) = 0 & \text{on } \Omega \times \{0\}. \end{cases} \tag{4.9}$$

For simplicity, let us set $\widetilde{u}_f(x,y) := \mathcal{P}_1^{1/2} u_f(x,y)$ in the following derivation. By the equation (4.9), direct computations show that

$$\begin{aligned} 0 = \varphi(U_f) &= \left\langle \varphi, \int_0^\infty \widetilde{u}_f \, dy \right\rangle_{\widetilde{H}^{-1}(\Omega)\times H^1(\Omega)} \\ &= \int_{\mathbb{R}^{n+1}_+} (\Delta_{x,y} w) \widetilde{u}_f \, dydx \\ &= -\int_{\mathbb{R}^n} \widetilde{u}_f \lim_{y\to 0} \partial_y w \, dx - \int_{\mathbb{R}^{n+1}_+} \nabla_{x,y} w \cdot \nabla_{x,y} \widetilde{u}_f \, dydx \\ &= \underbrace{-\int_{\Omega} \widetilde{u}_f \lim_{y\to 0} \partial_y w \, dx}_{-\partial_y w + qw = 0 \text{ on } \Omega\times\{0\}} - \int_{\Omega_e} \widetilde{u}_f \lim_{y\to 0} \partial_y w \, dx + \int_{\mathbb{R}^n} w \lim_{y\to 0} \partial_y \widetilde{u}_f \, dx \\ &= -\int_{\Omega} u_f q w \, dx - \int_W f \lim_{y\to 0} \partial_y w \, dx - \int_{\Omega} w(-\Delta)^{1/2} u_f \, dx \\ &= -\int_W f \lim_{y\to 0} \partial_y w \, dx, \end{aligned}$$

where we used $\left((-\Delta)^{1/2} + q\right) u_f = 0$ in Ω in the last equality, and recall that $\widetilde{u}_f$ satisfies

$$\begin{cases} \Delta_{x,y} \widetilde{u}_f = 0 & \text{in } \mathbb{R}^{n+1}_+, \\ \widetilde{u}_f(x,0) = u_f(x) & \text{on } \mathbb{R}^n \end{cases} \quad \text{with} \quad -\lim_{y\to 0} \partial_y \widetilde{u}_f = (-\Delta)^{1/2} u_f \text{ in } \mathbb{R}^n$$

Here we used the fact that $d_s = 1$ as $s = 1/2$. Due to the arbitrariness of $f \in C_c^\infty(W)$, there must hold $\partial_y w(x,0) = 0$ in W. Moreover, the weak UCP for $\Delta_{x,y}$ and $\mathrm{supp}(\varphi) \subset \overline{\Omega}$ imply that $w = 0$ in $\Omega_e \times (0,\infty)$. This infers particularly that

$$w = \partial_\nu w = 0 \text{ on } \partial\Omega \times (0,\infty),$$

where ν denotes the unit outer normal of $\partial\Omega$ pointing towards Ω_e.

In the next step, we want to show that $\varphi(v) = 0$. To get rid of boundary contributions on the set $\Omega \times \{0\}$, let us choose $\beta_1 \in C_c^\infty((0,\infty))$ with $\mathrm{supp}(\beta_1) \subset$

$(1,2)$, $\beta_1 \geq 0$ and $\int_0^\infty \beta_1 \, dy = 1$. Now, consider

$$\beta_k(y) := \frac{1}{k}\beta_1(y/k) \text{ for } k \in \mathbb{N}.$$

By a change of variables one can easily see that $\int_0^\infty \beta_k \, dy = 1$ for all $k \in \mathbb{N}$. Similarly as in the previous computations, one has

$$\begin{aligned}
-\varphi(v) &= -\varphi\left(v \int_0^\infty \beta_k \, dy\right) \\
&= -\int_{\Omega\times(0,\infty)} (\Delta_{x,y} w)\beta_k v \, dydx \\
&= -\underbrace{\int_{\partial\Omega\times(0,\infty)} \partial_\nu w v \beta_k \, dydS}_{=0, \text{ since } \partial_\nu w=0 \text{ on } \partial\Omega\times(0,\infty)} + \underbrace{\int_{\Omega\times\{0\}} \partial_y w v \beta_k \, dx}_{=0, \text{ since } \beta_k(0)=0} \\
&\quad + \int_{\Omega\times(0,\infty)} \nabla_{x,y} w \cdot \nabla_{x,y} (v\beta_k) \, dydx \\
&= \int_\Omega \nabla v \cdot \nabla \left(\int_0^\infty w\beta_k \, dy\right) dx + \int_{\Omega\times(0,\infty)} v\partial_y w \partial_y \beta_k \, dydx \\
&= \underbrace{\int_{\partial\Omega} \partial_\nu v \left(\int_0^\infty w\beta_k \, dy\right) dS}_{=0, \text{ since } w=0 \text{ on } \partial\Omega\times(0,\infty)} - \underbrace{\int_\Omega \Delta v \left(\int_0^\infty w\beta_k \, dy\right) dx}_{=0, \text{ since } \Delta v=0 \text{ in } \Omega} \\
&\quad + \int_{\Omega\times(0,\infty)} v\partial_y w \partial_y \beta_k \, dydx,
\end{aligned}$$

for all $k \in \mathbb{N}$. Hence, take $k \to \infty$, we have

$$-\varphi(v) = \lim_{k\to\infty} \int_{\Omega\times(0,\infty)} v\partial_y w \partial_y \beta_k \, dydx.$$

Our final aim is to prove that the above limit is zero. As claimed in [CGRU23, Section 3.1], we can obtain

$$\lim_{k\to\infty} \int_{\Omega\times(0,\infty)} v\partial_y w \partial_y \beta_k \, dydx = \lim_{k\to\infty} \left(k^{-2} \int_{\Omega\times(k,2k)} v\partial_y w \partial_y \beta_1 \, dydx\right) = 0,$$

which shows $\varphi(v) = 0$ as desired. However, this contradicts the condition (4.8) so that $v \in S$ as asserted cannot exist. Finally, by the trace theorem, it is easy to see that any $H^{1/2}(\partial\Omega)$ function can be approximated by a sequence of functions in $\mathcal{D}'$ as well. □

The above derivation demonstrates the idea to show our new Runge approximation. Next, based on this approach, we want to prove the general case rigorously, that is in which σ may not be 1 in Ω and $s \neq 1/2$. One can see from the formal proof that the adjoint equation (4.9) plays an essential role in this argument. Therefore, let us first study the existence of a solution to this problem. To this end, let us introduce the following subspaces of the homogeneous and non-homogeneous weighted Sobolev spaces:

$$\begin{aligned}
\dot{H}^1_0(\mathbb{R}^{n+1}_+, y^{1-2s}) &:= \left\{ g \in \dot{H}^1(\mathbb{R}^{n+1}_+, y^{1-2s}) : g = 0 \text{ on } \Omega_e \times \{0\} \right\}, \\
\dot{H}^1_{\mathrm{loc},0}(\overline{\mathbb{R}^{n+1}_+}, y^{1-2s}) &:= \left\{ g \in \dot{H}^1_{\mathrm{loc}}(\overline{\mathbb{R}^{n+1}_+}, y^{1-2s}) : g = 0 \text{ on } \Omega_e \times \{0\} \right\}, \\
H^1_c(\overline{\mathbb{R}^{n+1}_+}, y^{1-2s}) &:= \left\{ g \in H^1(\mathbb{R}^{n+1}_+, y^{1-2s}) : g \text{ has compact support in } \overline{\mathbb{R}^{n+1}_+} \right\}, \\
H^1_{c,0}(\overline{\mathbb{R}^{n+1}_+}, y^{1-2s}) &:= \left\{ g \in H^1_c(\overline{\mathbb{R}^{n+1}_+}, y^{1-2s}) : g = 0 \text{ on } \Omega_e \times \{0\} \right\}.
\end{aligned}$$

Lemma 4.5 (Solvability of the adjoint problem)**.** *Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain and $0 < s < 1$. Assume that Σ is given by (1.14) such that $\sigma \in C^\infty(\mathbb{R}^n)$ is uniformly elliptic with $\sigma = 1$ in Ω_e. Let $0 \leq q \in L^\infty(\Omega)$, and $\varphi \in \widetilde{H}^{-1}(\Omega)$. Consider*

$$\begin{cases}
\mathrm{div}_{x,y}\left(y^{1-2s}\Sigma\nabla_{x,y} w\right) = y^{1-2s}\varphi & \text{in } \mathbb{R}^{n+1}_+, \\
w = 0 & \text{on } \Omega_e \times \{0\}, \\
-\lim_{y\to 0} y^{1-2s}\partial_y w + d_s q w = 0 & \text{on } \Omega \times \{0\}.
\end{cases} \tag{4.10}$$

Then the problem (4.10) is solvable in $\dot{H}^1_{\mathrm{loc},0}(\overline{\mathbb{R}^{n+1}_+}, y^{1-2s})$ in the sense that

$$\begin{aligned}
&\int_{\mathbb{R}^{n+1}_+} y^{1-2s}\Sigma\nabla_{x,y} w \cdot \nabla_{x,y}\psi \, dxdy + d_s \int_{\Omega\times\{0\}} q w \psi \, dx \\
&= -\left\langle \varphi, \int_0^\infty y^{1-2s}\psi(\cdot, y)\, dy \right\rangle_{\widetilde{H}^{-1}(\Omega)\times H^1(\Omega)},
\end{aligned} \tag{4.11}$$

for any $\psi \in H^1_{c,0}(\overline{\mathbb{R}^{n+1}_+}, y^{1-2s})$.

Remark 4.6. *Let us give a short explanation to require the identity (4.11). To this end regard φ as a function supported in Ω. Then multiplying (4.10) by $\psi \in H^1_{c,0}(\overline{\mathbb{R}^{n+1}_+}, y^{1-2s})$ and integrating the resulting equation over $\mathbb{R}^{n+1}_+$ gives*

$$\begin{aligned}
&\int_{\Omega\times(0,\infty)} y^{1-2s}\varphi\psi \, dxdy \\
&= \int_{\mathbb{R}^{n+1}_+} \mathrm{div}_{x,y}\left(y^{1-2s}\Sigma\nabla_{x,y} w\right)\psi \, dydx \\
&= -\int_{\mathbb{R}^n\times\{0\}} y^{1-2s}\partial_y w\psi \, dx - \int_{\mathbb{R}^{n+1}_+} y^{1-2s}\Sigma\nabla_{x,y} w \cdot \nabla_{x,y}\psi \, dydx \\
&= -d_s \int_{\Omega\times\{0\}} q w\psi \, dx - \int_{\Omega_e\times\{0\}} y^{1-2s}\partial_y w\psi \, dx - \int_{\mathbb{R}^{n+1}_+} y^{1-2s}\Sigma\nabla_{x,y} w \cdot \nabla_{x,y}\psi \, dydx.
\end{aligned}$$

In the second equality we performed the usual integration by parts and in the third equality we used the boundary conditions. Now, be the assumption $\psi \in H^1_{c,0}(\overline{\mathbb{R}^{n+1}_+}, y^{1-2s})$ the middle term in the last line vanishes and we arrive at the identity (4.11).

Remark 4.7. *For the function w in Lemma 4.5, by the regularity for w, we could define* $\lim_{y\to 0} y^{1-2s}\partial_y w \in \dot{H}^{-s}_{\mathrm{loc}}(\mathbb{R}^n)$ *by*

$$\begin{aligned}-\int_{\mathbb{R}^n}\psi(\cdot,0)\lim_{y\to 0} y^{1-2s}\partial_y w\,dx := \int_{\mathbb{R}^{n+1}_+} y^{1-2s}\Sigma\nabla_{x,y}w\cdot\nabla_{x,y}\psi\,dxdy \\ + \left\langle \varphi, \int_0^\infty y^{1-2s}\psi(\cdot,y)\,dy\right\rangle_{\widetilde{H}^{-1}(\Omega)\times H^1(\Omega)}\end{aligned} \tag{4.12}$$

for any $\psi \in H^1_c(\overline{\mathbb{R}^{n+1}_+}, y^{1-2s})$ *(cf. Remark 4.6).*

Proof of Lemma 4.5. The construction of solutions to (4.10) is similar to the proof of [CGRU23, Lemma 3.2]. As $\varphi \in \widetilde{H}^{-1}(\Omega)$ has compact support, there exists a function $u_1 \in \dot{H}^1(\mathbb{R}^n)$ solving

$$\mathrm{div}\,(\sigma\nabla u_1) = \varphi \text{ in } \mathbb{R}^n.$$

In fact, this solution can be obtained as a minimizer of the weakly lower semicontinuous, convex, coercive energy functional $E\colon \dot{H}^1(\mathbb{R}^n)\to\mathbb{R}$ defined by

$$E(\psi) := \frac{1}{2}\int_{\mathbb{R}^n}\sigma|\nabla\psi|^2\,dx + \langle\varphi,\psi\rangle_{\widetilde{H}^{-1}(\Omega)\times H^1(\Omega)} \tag{4.13}$$

for $\psi\in\dot{H}^1(\mathbb{R}^n)$ (see [Str08, Chapter I, Theorem 1.2]). The fact that E is convex is immediate from the definition and hence let us shortly argue how one gets the other two properties.

(i) (Coercivity). Note that by the Sobolev embedding, we have $\dot{H}^1(\mathbb{R}^n)\hookrightarrow L^{\frac{2n}{n-2}}(\mathbb{R}^n)$ and thus, by boundedness of Ω, this ensures the continuity of the restriction $\dot{H}^1(\mathbb{R}^n)\ni\psi\mapsto\psi|_\Omega\in H^1(\Omega)$. This in turn guarantees the coercivity estimate

$$E(\psi)\geq C_1\|\psi\|^2_{\dot{H}^1(\mathbb{R}^n)} - C_2\|\varphi\|^2_{\widetilde{H}^{-1}(\Omega)} \geq -C_2\|\varphi\|^2_{\widetilde{H}^{-1}(\Omega)}$$

for some constants $C_1, C_2>0$.

(ii) (Weak lower semicontinuity). Note that weak lower semicontinuity follows from the fact that

$$\|\psi\|_\sigma := \left(\int_{\mathbb{R}^n}\sigma|\nabla\psi|^2\,dx\right)^{1/2}$$

is an equivalent norm on $\dot{H}^1(\mathbb{R}^n)$, as σ is uniformly elliptic, and

$$\dot{H}^1(\mathbb{R}^n)\ni\psi\mapsto\langle\varphi,\psi|_\Omega\rangle_{\widetilde{H}^{-1}(\Omega)\times H^1(\Omega)}$$

is a continuous linear functional (see (i)).

By $\widetilde{u}_1(x,y)\equiv u_1(x)$, we denote the trivial extension of u_1 into the y-direction.

Now, let us consider the problem

$$\begin{cases}\mathrm{div}_{x,y}\left(y^{1-2s}\Sigma\nabla_{x,y}\widetilde{u}_2\right)=0 & \text{in } \mathbb{R}^{n+1}_+,\\ \widetilde{u}_2=0 & \text{on } \Omega_e\times\{0\},\\ -\lim_{y\to0}y^{1-2s}\partial_y\widetilde{u}_2 + d_sq\widetilde{u}_2 = -\lim_{y\to0}y^{1-2s}\partial_y\mathcal{P}^s_\sigma u_1 & \text{on } \Omega\times\{0\},\end{cases} \tag{4.14}$$

where $\mathcal{P}^s_\sigma$ is the Caffarelli-Silvestre extension operator given by (3.1). Performing a similar analysis as in the proof of [CGRU23, Lemma 3.2], one sees that

$$\lim_{y\to0}y^{1-2s}\partial_y\mathcal{P}^s_\sigma u_1\in\dot{H}^{-s}(\mathbb{R}^n)+L^2(\Omega)\hookrightarrow H^{-s}(\Omega). \tag{4.15}$$

The relation (4.15) follows in fact by decomposing u_1 as

$$u_1 = \eta_R u_1 + (1-\eta_R)m_h(D)u_1 + (1-\eta_R)m_\ell(D)u_1 = v_R + v_h + v_\ell,$$

where $R>0$ is chosen such that $\overline{\Omega}\subset B_R$, $\eta_R\in C_c^\infty(B_{2R})$ satisfies $\eta_R|_{\overline{B}_R}=1$ and the Fourier multipliers $m_\ell(D)$, $m_h(D)$ have symbols ρ and $1-\rho$ with $\rho\in C_c^\infty(B_2)$ satisfying $\rho|_{\overline{B}_1}=1$. As $v_R\in H^1(\mathbb{R}^n)\hookrightarrow H^s(\mathbb{R}^n)$, Lemma 3.3, (v) implies

$$-\lim_{y\to 0} y^{1-2s}\partial_y\mathcal{P}_\sigma^s v_R = d_s(-\operatorname{div}(\sigma\nabla))^s v_R\in \dot{H}^{-s}(\mathbb{R}^n),$$

where we used the property (2.1) for the kernel of $(-\operatorname{div}(\sigma\nabla))^s$ as well as the Gagliardo–Slobodeckij characterization of $\dot{H}^s(\mathbb{R}^n)$. Next, we assert that the same holds for the function v_h. To see this it is enough to show that $v_h\in H^1(\mathbb{R}^n)$. Note that there holds

$$\int_{\mathbb{R}^n}|(1-\rho)\widehat{u}_1|^2\,d\xi\leq C\int_{\mathbb{R}^n\setminus\overline{B}_1}|\xi|^{-2}\left|\xi\widehat{u}_1\right|^2\,d\xi\leq C\left\|\nabla u_1\right\|_{L^2(\mathbb{R}^n)}^2,$$

and hence we have $v_h\in L^2(\mathbb{R}^n)$. By a simple calculation one also gets $\nabla v_h\in L^2(\mathbb{R}^n)$ and hence $v_h\in H^1(\mathbb{R}^n)$.

Note that there holds

$$m_\ell(D)u_1=\mathcal{F}^{-1}\left(\rho\widehat{u}_1\right)=\check{\rho}*u_1$$

with $\check{\rho}\in\mathscr{S}(\mathbb{R}^n)$. It is immediate from Young's inequality that $m_h(D)u_1\in\dot{H}^1(\mathbb{R}^n)$. Hence, the same holds for v_h and moreover $v_h=0$ on $\overline{B}_R$. Finally, we can rely on [CGRU23, Lemma 6.4], which is indeed a direct consequence of (3.13), to see that

$$-\lim_{y\to 0}y^{1-2s}\partial_y\mathcal{P}_\sigma^s v_\ell\in L^2(\Omega).$$

Let us discuss the solvability of (4.14). By the trace theorem $\dot{H}^1(\mathbb{R}^{n+1}_+,y^{1-2s})\hookrightarrow\dot{H}^s(\mathbb{R}^n)$, the linear functional

$$\dot{H}^1(\mathbb{R}^{n+1}_+,y^{1-2s})\ni\phi\mapsto\int_\Omega\phi(\cdot,0)\lim_{y\to 0}y^{1-2s}\mathcal{P}_\sigma^s u_1\,dx \tag{4.16}$$

is bounded, if it is interpreted accordingly (see (4.15)). As above, via the direct method in the calculus of variations, the problem (4.14) admits a solution $\widetilde{u}_2\in\dot{H}^1(\mathbb{R}^{n+1}_+,y^{1-2s})$.

More precisely, one can introduce the lower semicontinuous, coercive, convex energy functional

$$\begin{aligned}\mathcal{E}\left(\phi\right):=&\frac{1}{2}\int_{\mathbb{R}^{n+1}_+}y^{1-2s}\Sigma\nabla_{x,y}\phi\cdot\nabla_{x,y}\phi\,dxdy+\frac{d_s}{2}\int_{\Omega\times\{0\}}q\left|\phi\right|^2\,dx\\&+\int_{\Omega\times\{0\}}\phi\,\left(y^{1-2s}\mathcal{P}_\sigma^s u_1\right)\,dx,\end{aligned}\tag{4.17}$$

for $\phi\in\dot{H}^1_0(\mathbb{R}^{n+1}_+,y^{1-2s})$, and deduce that there exists a minimizer of (4.17), which is denoted by $\widetilde{u}_2\in\dot{H}^1(\mathbb{R}^{n+1}_+,y^{1-2s})$. The aforementioned properties of $\mathcal{E}$ can be seen similarly as for the simpler functional E given by (4.13). In fact, the convexity is again clear and the first integral represents an equivalent norm on $\dot{H}^1(\mathbb{R}^{n+1}_+,y^{1-2s})$ and $0\leq q\in L^\infty(\Omega)$. Next, we assert that the embedding $\dot{H}^1(\mathbb{R}^{n+1},y^{1-2s})\hookrightarrow L^2(\Omega)$ is compact. Indeed, this follows by the chain of embeddings

$$\dot{H}^1(\mathbb{R}^{n+1},y^{1-2s})\hookrightarrow\dot{H}^s(\mathbb{R}^n)\hookrightarrow H^s(\Omega)\hookrightarrow L^2(\Omega),$$

where the first continuous embedding is the trace theorem for the weighted space $\dot{H}^1(\mathbb{R}^{n+1},y^{1-2s})$, the second the restriction to Ω and the last the compact Rellich–Kondrachov theorem. Thus, by using the equivalent norm induced from the first term in the definition of $\mathcal{E}$, (4.16), and this compactness assertion, we see that $\mathcal{E}$ is weakly lower semicontinuous. It is also easy to see by the above estimates and $q\geq 0$ that $\mathcal{E}$ is coercive and hence as $H^1_0(\mathbb{R}^{n+1}_+,y^{1-2s})$ is weakly closed, we can again apply [Str08, Theorem 1.2] to conclude the existence of a minimizer $\widetilde{u}_2 \in \dot{H}^1_0(\mathbb{R}^{n+1}_+, y^{1-2s})$ of $\mathcal{E}$.

As (4.14) are the Euler–Lagrange equations of this minimization problem, we have obtained a solution of it, which is $\widetilde{u}_2$. Moreover, by non-negativity of q, (4.15) and (4.16), $\widetilde{u}_2$ satisfies the energy estimate

$$\|\widetilde{u}_2\|_{\dot{H}^1(\mathbb{R}^{n+1}_+, y^{1-2s})} \leq C \left\| \lim_{y\to 0} y^{1-2s}\partial_y \mathcal{P}^s_\sigma u_1 \right\|_{H^{-s}(\Omega)} < \infty,$$

for some constant $C > 0$ independent of $\widetilde{u}_2$.

Now, consider the function

$$w := \widetilde{u}_1 - \mathcal{P}^s_\sigma u_1 + \widetilde{u}_2,$$

then $w \in \dot{H}^1_{\mathrm{loc},0}(\overline{\mathbb{R}^{n+1}_+}, y^{1-2s})$ and w solves (4.10). Furthermore, there holds

$$w(x,0) = u_1(x) - \mathcal{P}^s_\sigma u_1(x,0) + \widetilde{u}_2(x,0) = \widetilde{u}_2(x,0) \text{ for } x \in \mathbb{R}^n,$$

which shows that indeed w vanishes on $\Omega_e \times \{0\}$. Using that $\widetilde{u}_1 = u_1$ is independent of y, this ensures that

$$\begin{aligned} -\lim_{y\to 0} y^{1-2s}\partial_y w &= \lim_{y\to 0} y^{1-2s}\partial_y \mathcal{P}^s_\sigma u_1 - \lim_{y\to 0} y^{1-2s}\partial_y \widetilde{u}_2 \\ &= -d_s q \widetilde{u}_2(x,0) = -d_s q w(x,0) \text{ for } x \in \Omega. \end{aligned} \tag{4.18}$$

More concretely, for any $\psi \in H^1_c(\overline{\mathbb{R}^{n+1}_+}, y^{1-2s})$, there holds that

$$\int_{\mathbb{R}^{n+1}_+} y^{1-2s}\Sigma\nabla_{x,y}\mathcal{P}^s_\sigma u_1 \cdot \nabla_{x,y}\psi \, dxdy = -\int_{\mathbb{R}^n} \psi(x,0) \lim_{y\to 0} y^{1-2s}\partial_y \mathcal{P}^s_\sigma u_1 \, dx,$$

$$\begin{aligned} \int_{\mathbb{R}^{n+1}_+} y^{1-2s}\Sigma\nabla_{x,y}\widetilde{u}_2 \cdot \nabla_{x,y}\psi \, dxdy &= -\int_{\Omega} \psi(x,0) \lim_{y\to 0} y^{1-2s}\partial_y \mathcal{P}^s_\sigma u_1 \, dx \\ &\quad - \int_\Omega \psi(x,0) d_s q \widetilde{u}_2 \, dx \\ &\quad - \int_{\Omega_e} \psi(x,0) \lim_{y\to 0} y^{1-2s}\partial_y \widetilde{u}_2 \, dx. \end{aligned}$$

In addition, one has

$$\begin{aligned} \int_{\mathbb{R}^{n+1}_+} y^{1-2s}\Sigma\nabla_{x,y}\widetilde{u}_1 \cdot \nabla_{x,y}\psi \, dxdy &= \int_{\mathbb{R}^{n+1}_+} y^{1-2s}\sigma\nabla u_1 \cdot \nabla\psi \, dxdy \\ &= \int_{\mathbb{R}^n} \sigma\nabla u_1 \cdot \nabla\left(\int_0^\infty y^{1-2s}\psi \, dy\right) dx \\ &= -\left\langle \varphi, \int_0^\infty y^{1-2s}\psi \, dy \right\rangle_{H^{-1}(\Omega)\times H^1(\Omega)}, \end{aligned}$$

where we used that Minkowski's inequality implies $\int_0^\infty y^{1-2s}\psi(x,y)\,dy \in H^1(\Omega)$. Using the weak definition of the Neumann derivative (see (4.12)), we deduce from the above computations that there holds

$$\begin{aligned} &-\int_{\mathbb{R}^n} \psi(x,0) \lim_{y\to 0} y^{1-2s}\partial_y w \, dx \\ &= \int_{\mathbb{R}^{n+1}_+} y^{1-2s}\Sigma\nabla_{x,y} w \cdot \nabla_{x,y}\psi \, dxdy + \left\langle \varphi, \int_0^\infty y^{1-2s}\psi(\cdot,y)\,dy \right\rangle_{\widetilde{H}^{-1}(\Omega)\times H^1(\Omega)} \\ &= -\int_\Omega \psi(x,0) d_s q(x)\widetilde{u}_2(x,0)\,dx - \int_{\Omega_e} \psi(x,0) \lim_{y\to 0} y^{1-2s}\partial_y \widetilde{u}_2 \, dx \end{aligned}$$

for any $\psi \in H^1_c(\overline{\mathbb{R}^{n+1}_+}, y^{1-2s})$. Hence, in particular this establishes (4.18). This completes the proof. $\square$

Now, we are ready to prove the asserted Runge approximation.

Proof of Proposition 4.4. Let us begin to show that the Runge approximation (4.5) holds. Suppose (4.5) does not hold, i.e., there exists $v \in S$ and $\epsilon > 0$ such that

$$\|U_f - v\|_{H^1(\Omega)} \geq \epsilon > 0, \text{ for all } f \in C_c^\infty(W). \tag{4.19}$$

Arguing as in the formal proof above, it is enough to show the following implication:

$$\varphi(U_f) = 0 \text{ for all } f \in C_c^\infty(W) \implies \varphi(v) = 0. \tag{4.20}$$

Let us point out that the rigorous version of the above formal proof, and its generalization to $s \in (0,1)$ as well as variable σ, is almost the same as the proof of [CGRU23, Proposition 3.1], where one needs to consider the same vertical and transversal cutoff functions and use suitable decay properties of $\mathcal{P}^s_\sigma u_f$ as $y \to \infty$ (see Lemma 3.3). To this end, consider $\eta_k(y) := \eta_1(y/k)$, where $\eta_1 \in C_c^\infty([0,2])$ is a smooth cutoff function with $\eta_1 = 1$ near $y = 0$ and $\int_0^\infty y^{1-2s}\eta_1\, dy = 1$. It is not hard to check that $k^{2s-2}\int_0^\infty y^{1-2s}\eta_k(y)\, dy = \int_0^\infty y^{1-2s}\eta_1\, dy = 1$, for all $k \in \mathbb{N}$. Without loss of generality, we may assume that there exists a large $R > 0$ such that $\overline{\Omega} \cup \overline{W} \subset B_R$. In addition, let us introduce the sequence $\zeta_k(x) := \zeta_1(x/k)$, where $\zeta_1 \in C_c^\infty(B_{2R})$ is a smooth radial cutoff function with $\zeta_1 \equiv 1$ in $\overline{B}_R$. Let

$$\left(\mathcal{P}^s_\sigma u_f\right)_k(x,y) := \zeta_k(x)\eta_k(y)\mathcal{P}^s_\sigma u_f(x,y)$$

for $k \in \mathbb{N}$, which belongs to $H^1_c(\overline{\mathbb{R}^{n+1}_+}, y^{1-2s})$. By (4.20) and Remark 4.7, we get

$$\begin{aligned}
0 = \varphi(U_f) &= \lim_{k\to\infty}\left\langle \varphi, \int_0^\infty y^{1-2s}\left(\mathcal{P}^s_\sigma u_f\right)_k dy \right\rangle_{\widetilde{H}^{-1}(\Omega)\times H^1(\Omega)} \\
&= \lim_{k\to\infty}\left(-\int_{\mathbb{R}^n}\left(\mathcal{P}^s_\sigma u_f\right)_k(\cdot,0)\lim_{y\to 0} y^{1-2s}\partial_y w\, dx\right. \\
&\qquad \left. -\int_{\mathbb{R}^{n+1}_+} y^{1-2s}\Sigma\nabla_{x,y}w\cdot\nabla_{x,y}\left(\mathcal{P}^s_\sigma u_f\right)_k dxdy\right) \\
&= -\int_W f\left(\lim_{y\to 0}y^{1-2s}\partial_y w\right)dx - \int_\Omega u_f\left(\lim_{y\to 0}y^{1-2s}\partial_y w\right)dx + \lim_{k\to\infty} I_k,
\end{aligned} \tag{4.21}$$

where we used $\mathcal{P}^s_\sigma u_f(x,0) = u_f(x)$ for $x \in \mathbb{R}^n$, $\eta_k(0) = 1$ and $u_f|_{\Omega_e} = f \in C_c^\infty(W)$. Furthermore, we set

$$\begin{aligned}
I_k := &-\int_{\mathbb{R}^{n+1}_+} y^{1-2s}\Sigma\left(\eta_k\zeta_k\nabla_{x,y}w\right)\cdot\nabla_{x,y}\mathcal{P}^s_\sigma u_f\, dxdy \\
&-\int_{\mathbb{R}^{n+1}_+} y^{1-2s}\Sigma\mathcal{P}^s_\sigma u_f\nabla_{x,y}\left(\eta_k\zeta_k\right)\cdot\nabla_{x,y}w\, dxdy.
\end{aligned} \tag{4.22}$$

Moreover, in (4.21), we used that one has

$$\lim_{k\to\infty}\int_0^\infty y^{1-2s}\zeta_k(x)\eta_k(y)\mathcal{P}^s_\sigma u_f(x,y)\, dy = \int_0^\infty y^{1-2s}\mathcal{P}^s_\sigma u_f(x,y)\, dy \text{ in } H^1(\Omega).$$

We next claim that there holds

$$-\int_\Omega u_f\left(\lim_{y\to 0}y^{1-2s}\partial_y w\right)dx + \lim_{k\to\infty} I_k = 0.$$

Using the product rule for the first term in (4.22) and for the second term an integration by parts, we deduce

$$\begin{aligned}
I_k &= -\int_{\mathbb{R}^{n+1}_+} y^{1-2s}\Sigma\nabla_{x,y}\left(\eta_k\zeta_k w\right)\cdot\nabla_{x,y}\mathcal{P}^s_\sigma u_f\,dxdy \\
&\quad + \int_{\mathbb{R}^{n+1}_+} y^{1-2s} w\Sigma\nabla_{x,y}\left(\eta_k\zeta_k\right)\cdot\nabla_{x,y}\mathcal{P}^s_\sigma u_f\,dxdy \\
&\quad - \int_{\mathbb{R}^{n+1}_+} y^{1-2s}\Sigma\mathcal{P}^s_\sigma u_f\nabla_{x,y}\left(\eta_k\zeta_k\right)\cdot\nabla_{x,y} w\,dxdy \\
&= -\int_{\mathbb{R}^{n+1}_+} y^{1-2s}\Sigma\nabla_{x,y}\left(\eta_k\zeta_k w\right)\cdot\nabla_{x,y}\mathcal{P}^s_\sigma u_f\,dxdy \\
&\quad + \int_{\mathbb{R}^{n+1}_+} y^{1-2s} w\Sigma\nabla_{x,y}\left(\eta_k\zeta_k\right)\cdot\nabla_{x,y}\mathcal{P}^s_\sigma u_f\,dxdy \\
&\quad + \int_{\mathbb{R}^{n+1}_+} w\,\mathrm{div}_{x,y}\left(y^{1-2s}\Sigma\mathcal{P}^s_\sigma u_f\nabla_{x,y}\left(\eta_k\zeta_k\right)\right)dxdy \\
&= \int_{\mathbb{R}^n} \zeta_k w(x,0)\lim_{y\to 0} y^{1-2s}\partial_y\mathcal{P}^s_\sigma u_f\,dx \\
&\quad + 2\int_{\mathbb{R}^{n+1}_+} y^{1-2s} w\Sigma\nabla_{x,y}\left(\eta_k\zeta_k\right)\cdot\nabla_{x,y}\mathcal{P}^s_\sigma u_f\,dxdy \\
&\quad + \int_{\mathbb{R}^{n+1}_+} w y^{1-2s}\mathcal{P}^s_\sigma u_f\,\mathrm{div}_{x,y}\left(\Sigma\nabla_{x,y}\left(\eta_k\zeta_k\right)\right)dxdy \\
&\quad + (1-2s)\int_{\mathbb{R}^{n+1}_+} w y^{-2s}\mathcal{P}^s_\sigma u_f\zeta_k\partial_y\eta_k\,dxdy.
\end{aligned}$$

In the second equality, we used and integration by parts and $\eta_k(y) = 1$ for small y and all $k \in \mathbb{N}$. In the last equality, we used the product rule for the term in the sixth line and for the term in the fourth line again an integration by parts together with $\eta_k(0) = 1$ for $k \in \mathbb{N}$ as well as the PDE for $\mathcal{P}^s_\sigma u_f$.

Therefore, by the support conditions for the cutoff functions η_k, ζ_k, we can write

(4.23)

$$\begin{aligned}
&-\int_\Omega u_f\left(\lim_{y\to 0} y^{1-2s}\partial_y w\right)dx + \lim_{k\to\infty} I_k \\
&= \lim_{k\to\infty}\int_{B_{2Rk}}\int_0^{2k} w y^{1-2s}\Big\{2\Sigma\nabla_{x,y}\left(\zeta_k\eta_k\right)\cdot\nabla_{x,y}\mathcal{P}^s_\sigma u_f \\
&\qquad + \left(\mathrm{div}_{x,y}\left(\Sigma\nabla_{x,y}\left(\eta_k\zeta_k\right)\right) + \frac{1-2s}{y}\zeta_k\partial_y\eta_k\right)\mathcal{P}^s_\sigma u_f\Big\}\,dydx.
\end{aligned}$$

More concretely, we have utilized that the boundary terms on $\mathbb{R}^n \times \{0\}$ are well-defined, and the equation for the adjoint equation (4.10) implies that

$$\begin{aligned}
&-\int_\Omega u_f\left(\lim_{y\to 0} y^{1-2s}\partial_y w\right)dx + \underbrace{\lim_{k\to\infty}\int_{\mathbb{R}^n}\zeta_k w(x,0)\lim_{y\to 0} y^{1-2s}\partial_y\mathcal{P}^s_\sigma u_f\,dx}_{w=0 \text{ on } \Omega_e\times\{0\} \text{ and } \zeta_k\to 1 \text{ as } k\to\infty} \\
&= -\int_\Omega u_f\underbrace{\left(\lim_{y\to 0} y^{1-2s}\partial_y w\right)}_{=d_s q w}dx + \int_{\Omega\times\{0\}} w\underbrace{\lim_{y\to 0} y^{1-2s}\partial_y\mathcal{P}^s_\sigma u_f}_{=-d_s(-\mathrm{div}(\sigma\nabla))^s u_f = d_s q u_f}\,dx \\
&= 0,
\end{aligned}$$

which implies the identity (4.23).

In addition, the rest of the proof is to ensure the right hand side of (4.23) equals to zero. One can follow the same arguments as in the rigorous proof of [CGRU23, Proposition 3.1] to obtain

$$|I_k| \leq C\left(I_{1,k}(w) + I_{2,k}(w)\right),$$

for some constant $C>0$ independent of $\mathcal{P}^s_\sigma u_f$, w and k, where

$$\begin{aligned} I_{1,k}(w) &:= k^{-1}\int_{\mathbb{R}^n}\int_k^{2k} y^{1-2s}|w|\left(|\nabla_{x,y}\mathcal{P}^s_\sigma u_f| + \frac{|\mathcal{P}^s_\sigma u_f|}{k}\right)dydx,\\ I_{2,k}(w) &:= k^{-1}\int_{B_{2Rk}\setminus B_{Rk}}\int_0^{2k} y^{1-2s}|w|\left(|\nabla_{x,y}\mathcal{P}^s_\sigma u_f| + \frac{|\mathcal{P}^s_\sigma u_f|}{k}\right)dydx, \end{aligned}$$

for $k\in\mathbb{N}$. Recalling that the function w is constructed via $w=\widetilde{u}_1-\mathcal{P}^s_\sigma u_1+\widetilde{u}_2$, and they have the same regularity properties as the ones in [CGRU23]. Therefore, the rest of the argument is the same as in the rigorous proof of [CGRU23, Proposition 3.1], so let us skip this lengthy analysis and for more details we refer the reader to the aforementioned article. As a result, there must hold $\lim_{k\to\infty} I_{1,k} = \lim_{k\to\infty} I_{2,k} = 0$. Hence, we deduce that

$$\int_W f\left(\lim_{y\to 0} y^{1-2s}\partial_y w\right)dx = 0, \text{ for all } f\in C^\infty_c(W).$$

Since $f\in C^\infty_c(W)$ is arbitrary, we infer $\lim_{y\to 0} y^{1-2s}\partial_y w = 0$ in $W\times\{0\}$. Then the UCP for (4.10) in the exterior domain implies that $w\equiv 0$ in $\Omega_e\times(0,\infty)$ as desired.

Now, for getting the implication (4.20), let us utilize the same cutoff functions $\beta_k(y)$ with $\beta_k(0)=0$ for $k\in\mathbb{N}$ (as constructed in Step 2 of the proof of [CGRU23, Proposition 3.1] or the proof of [LLU23, Proposition 6.1]) to avoid boundary contributions on $\Omega\times\{0\}$. We can conclude by repeating the same argument as in the rest of the proof of [CGRU23, Proposition 3.1] and so the implication (4.20) holds. For an outline of this argument, we refer the reader to the formal proof with $\sigma=1$ and $s=1/2$.

Now, as explained in the formal proof, the conclusion $\varphi(v)=0$ contradicts the existence of a real number α satisfying (4.8) and hence we may deduce that such a function $v\in S$ having the property (4.19) cannot exist. Therefore, the Runge approximation holds eventually. □

Remark 4.8. *Notice that all above arguments hold even when σ is an anisotropic, uniformly elliptic coefficient. We only need the isotropy of σ to derive the global uniqueness result. Let us emphasize that the condition $\sigma=1$ in Ω_e is needed in order to derive $\lim_{k\to\infty} I_k = 0$ in the rigorous proof of Proposition 4.4, which is completely the same situation as in* [CGRU23].

5. Proof of main results

In this section, we show our main uniqueness results.

Proof of Theorem 1.1. By using the (weak) UCP for the extension problem[3], one can easily conclude that the nonlocal Cauchy data $\left(f|_W, (-\operatorname{div}(\sigma\nabla))^s u_f|_W\right)$ determines

$$\left(y^{1-2s}\mathcal{P}^s_\sigma u_f\big|_{\partial\Omega\times(0,\infty)}, y^{1-2s}\partial_\nu\mathcal{P}^s_\sigma u_f\big|_{\partial\Omega\times(0,\infty)}\right)$$

and also $\left(U_f|_{\partial\Omega}, \partial_\nu U_f|_{\partial\Omega}\right)$, since as usual the functions U_f are given by

$$U_f = \int_0^\infty y^{1-2s}\mathcal{P}^s_\sigma u_f\, dy.$$

[3]See also the argument in the proof of Theorem 1.2.

Now, using Proposition 4.4 we can deduce that the nonlocal DN map $\Lambda^s_{\sigma,q}$ for (1.6) determines the (full data) DN map Λ_σ for the conductivity equation $\operatorname{div}(\sigma\nabla v)=0$ in Ω. Therefore, by the classical uniqueness result for the Calderón problem in dimensions $n\geq 3$ (see [SU87]), the diffusion coefficient σ is uniquely determined in Ω as by assumption we already know that $\sigma=1$ on $\partial\Omega$. Thus, it remains to determine $q\in L^\infty(\Omega)$ from the nonlocal DN map $\Lambda^s_{\sigma,q}$ with fixed σ, but this is an immediate consequence of [GLX17, Theorem 1.1]. This completes the proof. □

Remark 5.1. *One can see that the condition $q\geq 0$ in Ω is needed by our construction of solutions to the adjoint equation (4.10). However, In the proof of Theorem 1.1, we do not really need this sign condition. Hence, one might be able to generalize our result to general potential $q\in L^\infty(\Omega)$ if one can find an alternative construction of solutions to (4.10), so that the corresponding new Runge approximation still holds.*

Moreover, without using Proposition 4.4, we are able to prove a local uniqueness result.

Proof of Theorem 1.2. Let $u^{(j)}_{f_\ell}\in H^s(\mathbb{R}^n)$ be the solution to

$$\begin{cases}\left(\left(-\operatorname{div}\left(\sigma_j\nabla\right)\right)^s+q_j\right)u^{(j)}_{f_\ell}=0 & \text{in } \Omega,\\ u^{(j)}_{f_\ell}=f_\ell & \text{in } \Omega_e,\end{cases}$$

for $j,\ell\in\{1,2\}$. By Lemma 4.2, we know that the functions

$$U^{(j)}_{f_\ell}=\int_0^\infty y^{1-2s}\mathcal{P}^s_{\sigma_j}u^{(j)}_{f_\ell}\,dy\in H^1_{\mathrm{loc}}(\mathbb{R}^n)$$

solve

$$-\operatorname{div}\left(\sigma_j\nabla U^{(j)}_{f_\ell}\right)=d_s\left(-\operatorname{div}\left(\sigma_j\nabla\right)\right)^s u^{(j)}_{f_\ell}\ \text{in } \mathbb{R}^n \tag{5.1}$$

for $j,\ell\in\{1,2\}$, and by Lemma 4.1 the equation (5.1) holds in $H^{-s}(\mathbb{R}^n)$. Furthermore, note that $\mathcal{P}^s_{\sigma_j}u^{(j)}_{f_\ell}$ solves the degenerate elliptic equation

$$\begin{cases}\operatorname{div}_{x,y}\left(y^{1-2s}\Sigma_j\nabla_{x,y}\left(\mathcal{P}^s_{\sigma_j}u^{(j)}_{f_\ell}\right)\right)=0 & \text{in } \mathbb{R}^{n+1}_+,\\ \left(\mathcal{P}^s_{\sigma_j}u^{(j)}_{f_\ell}\right)(x,0)=u^{(j)}_{f_\ell}(x) \text{ on } \mathbb{R}^n,\end{cases}$$

for $j,\ell\in\{1,2\}$. The above equation is derived from the Caffarelli-Silvestre type extension, and it has nothing to do with the nonlocal equation for $u^{(j)}_{f_\ell}\in H^s(\mathbb{R}^n)$, for $\ell=1,2$.

In particular, the condition (1.11) implies

$$\begin{aligned}-\lim_{y\to 0}y^{1-2s}\partial_y\mathcal{P}^s_{\sigma_1}u^{(1)}_{f_\ell}&=d_s\left(-\operatorname{div}\left(\sigma_1\nabla\right)\right)^s u^{(1)}_{f_\ell}\\ &=d_s\left(-\operatorname{div}\left(\sigma_2\nabla\right)\right)^s u^{(2)}_{f_\ell}=-\lim_{y\to 0}y^{1-2s}\partial_y\mathcal{P}^s_{\sigma_2}u^{(2)}_{f_\ell}\ \text{in } W,\end{aligned}$$

whenever $f_\ell\in C^\infty_c(W)$. By the assumption $\sigma_1=\sigma_2$ in the open neighborhood $\mathcal{N}\subset\overline{\Omega}$ of $\partial\Omega$, one knows $\sigma:=\sigma_1=\sigma_2$ in $\mathcal{N}\cup\Omega_e$. In particular, for any $f_\ell\in C^\infty_c(W)$ the difference $V=\mathcal{P}^s_{\sigma_1}u^{(1)}_{f_\ell}-\mathcal{P}^s_{\sigma_2}u^{(2)}_{f_\ell}$ satisfies

$$\begin{cases}\operatorname{div}_{x,y}\left(y^{1-2s}\Sigma\nabla_{x,y}V\right)=0 & \text{in } (\mathcal{N}\cup\Omega_e)\times(0,\infty),\\ V=\lim_{y\to 0}y^{1-2s}\partial_y V=0 & \text{on } W\times\{0\}.\end{cases} \tag{5.2}$$

Then by the UCP for the PDE in (5.2), one can conclude that

$$\mathcal{P}^s_{\sigma_1}u^{(1)}_{f_\ell}=\mathcal{P}^s_{\sigma_2}u^{(2)}_{f_\ell}\ \text{in } (\mathcal{N}\cup\Omega_e)\times(0,\infty), \tag{5.3}$$

for any $f_\ell \in C_c^\infty(W)$ and $\ell = 1, 2$. To obtain this one can directly invoke the results in [GLX17, Section 5] or argue as follows. First of all, by the condition $\sigma|_{\Omega_e} = 1$, there holds

$$\begin{cases} \mathrm{div}_{x,y}\left(y^{1-2s}\nabla_{x,y}V\right) = 0 & \text{in } (\mathcal{N}\cup\Omega_e)\times(0,\infty), \\ V = \lim_{y\to 0} y^{1-2s}\partial_y V = 0 & \text{on } W\times\{0\}. \end{cases}$$

Secondly, after an even reflection of V which requires that the normal derivative of V vanishes on $W\times\{0\}$, we deduce from [FKS82, Theorem 2.3.12] that V is locally Hölder continuous on $W\times[0,\infty)$. Therefore, we can apply [Rül15, Proposition 2.2] to see that $V = 0$ on $B_r((x,0))\cap\overline{\mathbb{R}^{n+1}_+}$ for some $x\in W$ and $r>0$, where $B_r((x,0))$ is the ball in $\mathbb{R}^{n+1}$ with radius $r>0$ and center $(x,0)$ such that $B_r((x,0))\cap\{y = 0\}\subset W$. But now we can apply the usual UCP for the differential operator in (5.2) on the sets of the form $(\mathcal{N}\cup\Omega_e)\times(y_0,y_1)$ for any sufficiently small $y_0>0$ and $y_1>y_0$ and may conclude that $V=0$ on these sets. Thus, in the end we get $V=0$ in $(\mathcal{N}\cup\Omega_e)\times(0,\infty)$ as claimed.

Thus, via the definition of the function $U^{(j)}_{f_\ell}$ for $j,\ell\in\{1,2\}$, then there holds that

$$U^{(1)}_{f_\ell} = \int_0^\infty y^{1-2s}\mathcal{P}^s_{\sigma_1}u^{(1)}_{f_\ell}(\cdot,y)\,dy = \int_0^\infty y^{1-2s}\mathcal{P}^s_{\sigma_2}u^{(2)}_{f_\ell}(\cdot,y)\,dy = U^{(2)}_{f_\ell} \tag{5.4}$$

in $\mathcal{N}\cup\Omega_e$, which will be used in the forthcoming proof.

We next show

$$q_1 = q_2 \text{ in } \mathcal{N}\cap\Omega.$$

Combining the condition (1.11) and (2.6) in Lemma 2.1, we obtain

$$B_{\sigma_1,q_1}\left(u^{(1)}_{f_1}, u^{(2)}_{f_2}\right) - B_{\sigma_2,q_2}\left(u^{(2)}_{f_1}, u^{(2)}_{f_2}\right) = 0,$$

which by the definition of the bilinear forms (see (2.3)) is equivalent to

$$\begin{aligned} &\left\langle (-\mathrm{div}(\sigma_1\nabla))^s\, u^{(1)}_{f_1} - (-\mathrm{div}(\sigma_2\nabla))^s\, u^{(2)}_{f_1}, u^{(2)}_{f_2}\right\rangle_{H^{-s}(\mathbb{R}^n)\times H^s(\mathbb{R}^n)} \\ &\quad + \int_\Omega \left(q_1 u^{(1)}_{f_1} - q_2 u^{(2)}_{f_1}\right) u^{(2)}_{f_2}\,dx = 0. \end{aligned} \tag{5.5}$$

First inserting (5.1) into (5.5) and then decomposing $u^{(2)}_{f_2} = \left(u^{(2)}_{f_2} - f_2\right) + f_2$, we obtain by (5.4) the identity

$$\begin{aligned} &-\frac{1}{d_s}\left\langle \mathrm{div}\left(\sigma_1\nabla U^{(1)}_{f_1}\right) - \mathrm{div}\left(\sigma_2\nabla U^{(2)}_{f_1}\right), u^{(2)}_{f_2} - f_2\right\rangle_{H^{-s}(\mathbb{R}^n)\times H^s(\mathbb{R}^n)} \\ &\quad + \int_\Omega \left(q_1 u^{(1)}_{f_1} - q_2 u^{(2)}_{f_1}\right)\left(u^{(2)}_{f_2} - f_2\right) dx = 0. \end{aligned}$$

Recall that by Lemma 4.1, we have

$$\begin{aligned} &\left\langle \mathrm{div}\left(\sigma_1\nabla U^{(1)}_{f_1}\right) - \mathrm{div}\left(\sigma_2\nabla U^{(2)}_{f_1}\right), \varphi\right\rangle_{H^{-s}(\mathbb{R}^n)\times H^s(\mathbb{R}^n)} \\ &= \left\langle \mathrm{div}\left(\sigma_1\nabla U^{(1)}_{f_1}\right) - \mathrm{div}\left(\sigma_2\nabla U^{(2)}_{f_1}\right), \varphi\right\rangle_{H^{-s}(\Omega)\times \widetilde{H}^s(\Omega)} \\ &\le \left(\left\|\mathrm{div}\left(\sigma_1\nabla U^{(1)}_{f_1}\right)\right\|_{H^{-s}(\Omega)} + \left\|\mathrm{div}\left(\sigma_2\nabla U^{(2)}_{f_1}\right)\right\|_{H^{-s}(\Omega)}\right)\|\varphi\|_{\widetilde{H}^s(\Omega)} \end{aligned}$$

for all $\varphi\in\widetilde{H}^s(\Omega)$. We next want to derive useful integral identities in order to show the uniqueness of the potentials.

Let us assume that $\varphi \in C_c^\infty(\mathcal{N}\cap\Omega)$, via the Runge approximation (Proposition 2.3), then there exists a sequence of exterior data $\{f_{2,m}\}_{m\in\mathbb{N}} \subset C_c^\infty(W)$ such that

$$u^{(2)}_{f_{2,m}} - f_{2,m} \to \varphi \text{ in } \widetilde{H}^s(\Omega) \text{ as } m\to\infty.$$

With this sequence of functions at hand, we obtain

$$\begin{aligned}
&\lim_{m\to\infty} \left\langle \operatorname{div}\left(\sigma_1 \nabla U^{(1)}_{f_1}\right) - \operatorname{div}\left(\sigma_2 \nabla U^{(2)}_{f_1}\right), u^{(2)}_{f_{2,m}} - f_{2,m} \right\rangle_{H^{-s}(\mathbb{R}^n)\times H^s(\mathbb{R}^n)} \\
&= \left\langle \operatorname{div}\left(\sigma_1 \nabla U^{(1)}_{f_1}\right) - \operatorname{div}\left(\sigma_2 \nabla U^{(2)}_{f_1}\right), \varphi \right\rangle_{H^{-s}(\mathbb{R}^n)\times H^s(\mathbb{R}^n)} \\
&= \left\langle \operatorname{div}\left(\sigma_1 \nabla U^{(1)}_{f_1}\right) - \operatorname{div}\left(\sigma_2 \nabla U^{(2)}_{f_1}\right), \varphi \right\rangle_{\mathscr{D}'(\mathbb{R}^n)\times \mathscr{D}(\mathbb{R}^n)} \\
&= \int_{\mathbb{R}^n} \left[U^{(1)}_{f_1} \operatorname{div}(\sigma_1\nabla\varphi) - U^{(2)}_{f_1} \operatorname{div}(\sigma_2\nabla\varphi)\right] dx \\
&= \int_{\mathcal{N}\cap\Omega} \left[U^{(1)}_{f_1} \operatorname{div}(\sigma_1\nabla\varphi) - U^{(2)}_{f_1} \operatorname{div}(\sigma_2\nabla\varphi)\right] dx \\
&= 0,
\end{aligned} \tag{5.6}$$

where we used that $\operatorname{supp}(\varphi)\subset \mathcal{N}\cap\Omega$ and the last integral vanishes by the fact that $\sigma_1=\sigma_2$ in $\mathcal{N}$ and (5.4). Therefore, by passing to the limit $m\to\infty$ in (5.5) (with $f_2 = f_{2,m}$) and using (5.6) we get

$$\int_{\mathcal{N}\cap\Omega} \left(q_1 u^{(1)}_{f_1} - q_2 u^{(2)}_{f_1}\right)\varphi\, dx = 0$$

for all $\varphi\in C_c^\infty(\mathcal{N}\cap\Omega)$. Hence, we can conclude that

$$q_1 u^{(1)}_{f_1} = q_2 u^{(2)}_{f_1} \text{ in } \mathcal{N}\cap\Omega.$$

Moreover, by using (5.3), we also have

$$u^{(1)}_{f_1}(x) = \left(\mathcal{P}^s_{\sigma_1} u^{(1)}_{f_1}\right)(x,0) = \left(\mathcal{P}^s_{\sigma_2} u^{(2)}_{f_1}\right)(x,0) = u^{(2)}_{f_1}(x), \text{ for } x\in\mathcal{N}.$$

This implies

$$(q_1 - q_2)\, u^{(1)}_{f_1} = 0 \text{ in } \mathcal{N}\cap\Omega,$$

for any $f_1 \in C_c^\infty(W)$. Fix any $x\in\mathcal{N}\cap\Omega$ and a nonzero $f_1\in C_c^\infty(W)$. If $u^{(1)}_{f_1}$ would vanish in a neighborhood of x, then $u^{(1)}_{f_1}$ would solve $(-\operatorname{div}(\sigma_1\nabla))^s v = 0$ in that neighborhood. The UCP for $(-\operatorname{div}(\sigma_j\nabla))^s$ for $j=1,2$ (Proposition 2.2) would then imply $u^{(1)}_{f_1} = 0$ in $\mathbb{R}^n$, which contradicts the assumption $f_1\neq 0$. Hence, we can select a sequence $(x_k)_{k\in\mathbb{N}}$ such that $u^{(1)}_{f_1}(x_k)\neq 0$ for all $k\in\mathbb{N}$ and $x_k\to x$ as $k\to\infty$. This guarantees

$$q_1(x_k) = q_2(x_k)$$

for all $k\in\mathbb{N}$. By the condition $q_1, q_2\in C^0(\Omega)$, passing to the limit $k\to\infty$, we may conclude that $q_1(x) = q_2(x)$. As $x\in\mathcal{N}\cap\Omega$ was arbitrary, we find that $q_1 = q_2$ in $\mathcal{N}\cap\Omega$ as we wish. This proves the assertion. □

6. Conclusion remark and further discussion

With our novel methods at hand, it seems that one can split the nonlocal part and the local lower order terms via the CS-type extension and determine them separately. Let us revisit the nonlocal elliptic equation with drift. Given $s\in(1/2,1)$, consider the nonlocal elliptic equation with drift

$$\begin{cases} \left((-\operatorname{div}(\sigma\nabla))^s + b\cdot\nabla + c\right) u = 0 & \text{in } \Omega, \\ u = f & \text{in } \Omega_e. \end{cases} \tag{6.1}$$

A proof of the well-posedness of the Dirichlet problem (6.1) can be found in [CLR20] (the argument is similar to the case $\sigma = 1$). Let $\Lambda_{\sigma,b,c}$ be the DN map of (6.1), then one could also determine the positive scalar function σ, drift term b and potential c in Ω. The method is similar to our present work, namely that one can use the geometric form of Hahn-Banach theorem combined with a suitable adjoint problem, so that any solution $v \in H^1(\Omega)$ to the conductivity equation $\mathrm{div}(\sigma\nabla v) = 0$ can be approximated by a sequence of CS extension solutions as in Section 4. Once the leading coefficient σ is determined, one can directly apply the same trick as demonstrated in [CLR20] to recover b and c, via suitable Runge approximation for the nonlocal equation $\left((-\mathrm{div}(\sigma\nabla))^s + b\cdot\nabla + c\right)u = 0$ in Ω.

Remark 6.1. *From our novel methods introduced in this work, one may not expect to recover the matrix-valued leading coefficient σ, since the reduced local equation involves a long-standing open problem. That is, can one determine an anisotropic leading coefficient σ (up to isometry) for the conductivity equation $\mathrm{div}\,(\sigma\nabla v) = 0$ by using its boundary Cauchy data? So far, to our best knowledge, there is no affirmative answer for the dimension $n \geq 3$.*

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