A LOCAL UNIQUENESS THEOREM FOR THE FRACTIONAL SCHRÖDINGER EQUATION ON CLOSED RIEMANNIAN MANIFOLDS

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Abstract. We investigate that a potential V in the fractional Schrödinger equation $\left((-\Delta_g)^s + V\right) u = f$ can be recovered locally by using the local source-to-solution map on smooth connected closed Riemannian manifolds. To achieve this goal, we derive a related new Runge approximation property.

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1. Introduction

Let (M, g) be a smooth closed Riemannian manifold of dimension $\dim M \geq 2$, and $-\Delta_g$ be the Laplace-Beltrami operator defined on M. It is known that the domain of $-\Delta_g$ is the standard Sobolev space $H^2(M)$. Let us also denote the eigenvalue of $-\Delta_g$ on M with increasing order, by $0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \ldots$, and the first eigenfunction $\phi_0 \equiv 1$ (up to normalization). Consider $(\phi_k)_{k\in\mathbb{N}}$ to be an orthonormal basis for $\ker(-\Delta_g - \lambda_k)$ in $L^2(M)$ (with multiplicity), with respect to λ_k, for $k \in \mathbb{N}$. Let us define the L^2 inner product on M as $(f, g)_{L^2(M)} := \int_M fg \, \mathsf{dV}_g$, where dV_g denotes the natural volume element on (M, g). Given $s \in (0, 1)$, the fractional Laplace-Beltrami on M of order s as an unbounded self-adjoint operator can be defined via the spectral theorem

$$(-\Delta_g)^s u(x) = \sum_{k=0}^{\infty} \lambda_k^s \, (u, \phi_k)_{L^2(M)} \, \phi_k(x)$$

in the domain

$$\mathcal{D}\left((-\Delta_g)^s\right) := \left\{ u \in L^2(M) : \sum_{k=0}^{\infty} \lambda_k^{2s} \left| (u, \phi_k)_{L^2(M)} \right|^2 < \infty \right\} = H^{2s}(M).$$

The fractional Laplace-Beltrami operator $(-\Delta_g)^s$ can be viewed as a linear map from $H^a(M) \to H^{a-2s}(M)$, for any $a \geq 0$.

Let us characterize our mathematical model in this work. Let (M, g) be a smooth closed Riemannian manifold, and $V = V(x) \in C^\infty(M)$ be a nonnegative potential. Consider

$$\left((-\Delta_g)^s + V\right) u = f \text{ in } M. \tag{1.1}$$

Via the eigenvalue condition, it is also known that $((-\Delta_g)u, 1)_{L^2(M)} = 0$. With the above assumptions, by standard regularity theory, it is not hard to see that there exists a unique solution $u_f \in C^\infty(M)$ of the equation (1.1).

Let $\mathcal{O}\subset M$ be a nonempty open set and $f\in C_c^\infty(\mathcal{O})$. Then one can define the local *source-to-solution map* $L_{V,\mathcal{O}}$ of (1.1) via

$$L_{V,\mathcal{O}}: C_c^\infty(\mathcal{O})\ni f\mapsto u_f|_{\mathcal{O}}\in C^\infty(\mathcal{O}), \tag{1.2}$$

where $u_f\in C^\infty(M)$ is the solution to (1.1). We want to recover the potential V by using the local measurement $L_{V,\mathcal{O}}$. We can prove the next theorem:

Theorem 1.1. *Given $s\in(0,1)$, let (M,g) be smooth closed connected Riemannian manifolds of dimension $n\geq 2$, and $0\leq V_j\in C^\infty(M)$, for $j=1,2$. Let $\mathcal{O}\Subset M$ be an open set. Suppose that either $V_1=V_2$ in $\mathcal{O}$ or $V_1=V_2$ in $M\setminus\mathcal{O}$, then*

$$L_{V_1,\mathcal{O}}(f)=L_{V_2,\mathcal{O}}(f), \text{ for all } f\in C_c^\infty(\mathcal{O}). \tag{1.3}$$

implies $V_1=V_2$ in M.

The proof of Theorem 1.1 relies on suitable integral identity and approximation property. Meanwhile, a variant result of Theorem 1.1 has been addressed in [GLX17] by using exterior measurements Throughout this paper, we always denote (M,g) as a smooth, connected closed Riemannian manifold of $\dim M\geq 2$.

• **Related works.** Inverse problems for fractional/nonlocal equations have been paid attention in recent years. In the pioneering work [GSU20], the authors first proved a global uniqueness result of bounded potentials for the fractional Schrödinger equation by using the exterior Dirichlet-to-Neumann (DN) map. Afterward, there are many related results appeared in this direction, and we refer readers to [CLL19, GLX17, CLL19, CLR20, CRTZ24, FGKU24, FKU24, KLW22, RS20, RZ23, LZ24, GRSU20, KLZ24, KRZ23]). Let us point out that the essential methods in this area are *unique continuation property* and *Runge approximation*. More recently, one could also solve (local) inverse problems by using the nonlocal method (see [LNZ24]).

2. Preliminaries

We first review a simple fact.

Lemma 2.1. *There holds*

$$\begin{aligned}\left((-\Delta_g)^s u, v\right)_{H^{-s}(M)\times H^s(M)} &= \left((-\Delta_g)^{s/2}u, (-\Delta_g)^{s/2}v\right)_{L^2(M)}\\ &= \left(u, (-\Delta_g)^s v\right)_{H^s(M)\times H^{-s}(M)},\end{aligned}$$

for all $u,v\in H^s(M)$, where $H^{-s}(M)$ denotes the dual space of $H^s(M)$.

Proof. This is followed directly by straightforward computations with eigenvalue and eigenfunction decomposition. ☐

Lemma 2.2 (Well-posedness). *Given $f\in H^{-s}(M)$, let $0\leq V\in C^\infty(M)$, then there exists a unique solution $u\in H^s(M)$ to the equation (1.1). Moreover, if $f\in C^\infty(M)$, then $u\in C^\infty(M)$.*

Proof. Consider the bilinear form associated with (1.1) to be

$$B_{g,V}(u,v):=\left((-\Delta_g)^{s/2}u, (-\Delta_g)^{s/2}v\right)_{L^2(M)}+(Vu,v)_{L^2(M)},$$

for any $u, v \in H^s(M)$. Using $V \geq 0$ in M, the standard Lax-Milgram theorem and later statement can be seen by using elliptic regularity theory. □

With the above lemma at hand, one can define the local source-to-solution map $L_{V,\mathcal{O}}$ of (1.1) rigorously, which is exactly given by (1.2). We next derive a useful integral identity.

Lemma 2.3. *Let $V, V_j \in C^\infty(M)$, for $j = 1, 2$, and $\mathcal{O} \subset M$ be a nonempty open set, then*

(i) (Self-adjointness). There holds

$$(f_1, L_{V,\mathcal{O}} f_2)_{L^2(M)} = (L_{V,\mathcal{O}} f_1, f_2)_{L^2(M)}, \tag{2.1}$$

(ii) (Integral identity). Let $V_1, V_2 \in C^\infty(M)$, then there holds that

$$((L_{V_1,\mathcal{O}} - L_{V_2,\mathcal{O}}) f_1, f_2)_{L^2(M)} = -\big((V_1 - V_2) u_1^{f_1}, u_2^{f_2}\big)_{L^2(M)}, \tag{2.2}$$

for all $f_1, f_2 \in C_c^\infty(\mathcal{O})$, where $u_j^{f_j}$ are the solutions to

$$((-\Delta_g)^s + V_j) u_j^{f_j} = f_j \text{ in } M, \text{ for } j = 1, 2. \tag{2.3}$$

Proof. We first prove the self-adjointness of the local source-to-solution map $L_{V,\mathcal{O}}$. For any $f_1, f_2 \in C_c^\infty(\mathcal{O})$, direct computations imply that

$$\begin{aligned}
(L_{V,\mathcal{O}} f_1, f_2)_{L^2(M)} = (L_{V,\mathcal{O}} f_1, f_2)_{L^2(\mathcal{O})} &= \left(u_1^{f_1}, f_2\right)_{L^2(M)} \\
&= \left(u_1^{f_1}, ((-\Delta_g)^s + V) u_2^{f_2}\right)_{L^2(M)} \\
&= \underbrace{\left(((-\Delta_g)^s + V) u_1^{f_1}, u_2^{f_2}\right)_{L^2(M)}}_{\text{By Lemma 2.1}} \\
&= (f_1, L_{V,\mathcal{O}} f_2)_{L^2(M)}.
\end{aligned}$$

By using the adjointness (2.1), the integral identity (2.2) can be seen via

$$\begin{aligned}
&((L_{V_1,\mathcal{O}} - L_{V_2,\mathcal{O}}) f_1, f_2)_{L^2(M)} \\
&= (L_{V_1,\mathcal{O}} f_1, f_2)_{L^2(M)} - (f_1, L_{V_2,\mathcal{O}} f_2)_{L^2(M)} \\
&= \underbrace{(u_1, ((-\Delta_g)^s + V_2) u_2)_{L^2(M)} - (((-\Delta_g)^s + V_1) u_1, u_2)_{L^2(M)}}_{\text{By (2.3)}} \\
&= \underbrace{-((V_1 - V_2) u_1, u_2)_{L^2(M)}}_{\text{By Lemma 2.1}},
\end{aligned}$$

which proves the assertion. □

Let us also recall a well-known unique continuation property (UCP) for the fractional Laplace-Beltrami operator.

Lemma 2.4 (UCP)**.** *Let $\emptyset \neq \mathcal{O} \subset M$ be an open subset, then $u = (-\Delta_g)^s u = 0$ in $\mathcal{O}$ implies $u \equiv 0$.*

The proof can be found in [GLX17, FGKU24] with different approaches.

3. Proof of main results

To prove Theorem 1.1, we first show a useful approximation result.

Lemma 3.1 (Runge approximation). *Consider the set*

$$\mathcal{R} := \left\{ u_f|_{\mathcal{O}} : \textit{for any } f \in C_c^\infty(\mathcal{O}) \right\},$$

where u_f is the solution to (1.1). Then $\mathcal{R}$ is dense in $H^s(\mathcal{O})$.

Proof. By Lemma 2.2, the solution u_f of (1.1) belongs to $H^s(M)$ for any $f \in H^{-s}(M)$, then we have $\mathcal{R} \subset H^s(\mathcal{O})$. Using the Hahn-Banach theorem, consider $\varphi \in \widetilde{H}^{-s}(\mathcal{O}) = (H^s(\mathcal{O}))^*$ (dual space of $H^s(\mathcal{O})$), which satisfies

$$\varphi\left(u_f\right) = 0, \text{ for all } f \in C_c^\infty(\mathcal{O}),$$

then we aim to show prove $\varphi \equiv 0$. Let $w \in H^s(M)$ be the solution to

$$\left((-\Delta_g)^s + V\right) w = \varphi \text{ in } M, \tag{3.1}$$

then we have

$$\begin{aligned} 0 &= \langle \varphi, u_f \rangle_{\widetilde{H}^{-s}(\mathcal{O}) \times H^s(\mathcal{O})} = \langle \left((-\Delta_g)^s + V\right) w, u_f \rangle_{H^{-s}(M) \times H^s(M)} \\ &= \underbrace{\left(\left((-\Delta_g)^s + V\right) u_f, w\right)_{L^2(M)}}_{\text{By Lemma 2.1}} = (f, w)_{L^2(M)} = (f, w)_{L^2(\mathcal{O})}, \end{aligned}$$

for any $f \in C_c^\infty(\mathcal{O})$. This implies $w = 0$ in $\mathcal{O}$.

On the other hand, multiplying (3.1) by the function w, one has

$$\begin{aligned} \langle \varphi, w \rangle_{H^{-s}(M) \times H^s(M)} &= \int_M w \left((-\Delta_g)^s + V\right) w \, \mathsf{dV}_g \\ &= \underbrace{\int_M \left(\left| (-\Delta_g)^{s/2} w \right|^2 + V|w|^2\right) \mathsf{dV}_g \geq 0}_{\text{By using } V \geq 0 \text{ in } M}, \end{aligned}$$

which implies

$$\underbrace{\langle \varphi, w \rangle_{H^{-s}(M) \times H^s(M)} = \langle \varphi, w \rangle_{\widetilde{H}^{-s}(\mathcal{O}) \times H^s(\mathcal{O})} = 0}_{\text{By using } \varphi \in \widetilde{H}^{-s}(\mathcal{O}) \text{ and } w=0 \text{ in } \mathcal{O}}.$$

Hence, we have $0 \leq \int_M \left(|(-\Delta_g)^s w|^2 + V|w|^2 \right) \mathsf{dV}_g = 0$, such that w must be zero in M, which yields that $0 = \varphi \in \widetilde{H}^{-s}(\mathcal{O})$ by using the adjoint equation (3.1). ∎

Proof of Theorem 1.1. For any $f_1, f_2 \in C_c^\infty(\mathcal{O})$, let u_j be the solution of (2.3), for $j = 1, 2$. Let us first consider $V_1 = V_2$ in $M \setminus \mathcal{O}$, combining with (1.3) and (2.2), then we can obtain

$$\int_{\mathcal{O}} (V_1 - V_2)\, u_1^{f_1} u_2^{f_2} \, \mathsf{dV}_g = \int_M (V_1 - V_2)\, u_1^{f_1} u_2^{f_2} \, \mathsf{dV}_g = 0. \tag{3.2}$$

By Lemma 3.1, given any $h \in H^s(\mathcal{O})$, one can find sequences of functions $\left(f_1^{f_1^{(k)}}\right)_{k \in \mathbb{N}}$, $\left(f_2^{(\ell)}\right)_{\ell \in \mathbb{N}}$ such that the corresponding solutions satisfy $u_1^{(k)} \to h$

and $u_2^{f_2^{(\ell)}} \to 1$ in $H^s(\mathcal{O})$, as $k,\ell\to\infty$. Now, inserting these sequences of solutions into (3.2) and taking limits, there holds that

$$\int_{\mathcal{O}} (V_1 - V_2)\, h \, \mathrm{dV}_g = 0.$$

Due to arbitrariness of $h \in H^s(\mathcal{O})$, there holds $V_1 = V_2$ in $\mathcal{O}$ so that $V_1 = V_2$ in M.

On the other hand, if $V_1 = V_2$ in $\mathcal{O}$, then the condition (1.3) implies that $(-\Delta_g)^s\big(u_1^f - u_2^f\big) = u_1^f - u_2^f = 0$ in $\mathcal{O}$. The UCP implies $u^f := u_1^f = u_2^f$ in M for any $f \in C_c^\infty(\mathcal{O})$. This implies $(V_1 - V_2)\, u^f = 0$ in $M \setminus \mathcal{O}$. Similar to the arguments as in [GRSU20, LZ24], by the smoothness of u^f and UCP for $(-\Delta_g)^s$, it is not hard to show $V_1 = V_2$ in $M \setminus \mathcal{O}$. This implies $V_1 = V_2$ in M as well. □

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