RESEARCH ARTICLE

On determining and breaking the gauge class in inverse problems for reaction-diffusion equations

Yavar Kian [1], Tony Liimatainen[2] and Yi-Hsuan Lin[3]

[1]Univ Rouen Normandie, CNRS, Normandie Univ, LMRS UMR 6085, F-76000 Rouen, France;
E-mail: yavar.kian@univ-rouen.fr.
[2]Department of Mathematics and Statistics, University of Helsinki, P.O 68 (Pietari Kalmin katu 5), Helsinki, Finland;
E-mail: tony.liimatainen@helsinki.fi.
[3]Department of Applied Mathematics, National Yang Ming Chiao Tung University, 1001, Ta Hsueh Road, Hsinchu, 30050,
Taiwan; E-mail: yihsuanlin3@gmail.com.

Abstract

We investigate an inverse boundary value problem of determination of a nonlinear law for reaction-diffusion processes, which are modeled by general form semilinear parabolic equations. We do not assume that any solutions to these equations are known a priori, in which case the problem has a well-known gauge symmetry. We determine, under additional assumptions, the semilinear term up to this symmetry in a time-dependent anisotropic case modeled on Riemannian manifolds, and for partial data measurements on $\mathbb{R}^n$.

Moreover, we present cases where it is possible to exploit the nonlinear interaction to break the gauge symmetry. This leads to full determination results of the nonlinear term. As an application, we show that it is possible to give a full resolution to classes of inverse source problems of determining a source term and nonlinear terms simultaneously. This is in strict contrast to inverse source problems for corresponding linear equations, which always have the gauge symmetry. We also consider a Carleman estimate with boundary terms based on intrinsic properties of parabolic equations.

Contents

1. Introduction

In this article, we address the following question: Is it possible to determine the non-linear law of a reaction-diffusion process by applying sources and measuring the corresponding flux on the boundary of the domain of diffusion? Mathematically, the question can be stated as a problem of determination of a general semilinear term appearing in a semilinear parabolic equation from boundary measurements.

Let us explain the problem precisely. Let $T > 0$ and $\Omega \subset \mathbb{R}^n$, with $n \geqslant 2$, be a bounded and connected domain with a smooth boundary. Let $a := (a_{ik})_{1 \leqslant i,k \leqslant n} \in C^\infty([0,T] \times \overline{\Omega}; \mathbb{R}^{n \times n})$ be symmetric matrix field,

$$a_{ik}(t,x) = a_{ki}(t,x), \; x \in \Omega, \; i,k = 1,\ldots,n, \; (t,x) \in [0,T] \times \overline{\Omega},$$

which fulfills the following ellipticity condition: there exists a constant $c > 0$ such that

$$\sum_{i,k=1}^n a_{ik}(t,x)\xi_i\xi_k \geqslant c|\xi|^2, \quad \text{for each } (t,x) \in [0,T] \times \overline{\Omega}, \; \xi = (\xi_1,\ldots,\xi_n) \in \mathbb{R}^n. \tag{1.1}$$

We define elliptic operators $\mathcal{A}(t)$, $t \in [0,T]$, in divergence form by

$$\mathcal{A}(t)u(t,x) := -\sum_{i,k=1}^n \partial_{x_i}\left(a_{ik}(t,x)\partial_{x_k}u(t,x)\right), \; x \in \overline{\Omega}, \; t \in [0,T].$$

Throughout the article, we set

$$Q := (0,T) \times \Omega, \quad \Sigma := (0,T) \times \partial\Omega,$$

and we refer to Σ as the lateral boundary. Let us also fix $\rho \in C^\infty([0,T] \times \overline{\Omega}; \mathbb{R}_+)$ and $b \in C^\infty([0,T] \times \overline{\Omega} \times \mathbb{R})$. Here, $\mathbb{R}_+ = (0,+\infty)$. Then, we consider the following initial boundary value problem (IBVP in short):

$$\begin{cases} \rho(t,x)\partial_t u(t,x) + \mathcal{A}(t)u(t,x) + b(t,x,u(t,x)) = 0, & (t,x) \in Q, \\ u(t,x) = f(t,x), & (t,x) \in \Sigma, \\ u(0,x) = 0, & x \in \Omega. \end{cases} \tag{1.2}$$

The parabolic *Dirichlet-to-Neumann* map (DN map in short) is formally defined by

$$\mathcal{N}_b : f \mapsto \partial_{\nu(a)}u\big|_\Sigma,$$

where u is the solution to (1.2). Here, the conormal derivative $\partial_{\nu(a)}$ associated to the coefficient a is defined by

$$\partial_{\nu(a)}v(t,x) := \sum_{i,k=1}^n a_{ik}(t,x)\partial_{x_k}v(t,x)\nu_i(x), \quad (t,x) \in \Sigma,$$

where $\nu = (\nu_1,\ldots,\nu_n)$ denotes the outward unit normal vector of $\partial\Omega$ with respect to the Euclidean $\mathbb{R}^n$ metric. The solution u used in the definition of the DN map $\mathcal{N}_b$ is unique in a specific sense so that

there is no ambiguity in the definition of $\mathcal{N}_b$. For this fact and a rigorous definition of the DN map, we refer to to Section 2.1. We write simply $\partial_\nu = \partial_{\nu(a)}$ in the case a is the $n \times n$ identity matrix $\mathrm{Id}_{\mathbb{R}^{n \times n}}$. The inverse problem we study is the following.

○ **Inverse problem (IP):** Can we recover the semilinear term b from the knowledge of the parabolic Dirichlet-to-Neumann map $\mathcal{N}_b$?

Physically, reaction diffusion equations of the form (1.2) describe several classes of diffusion processes with applications in chemistry, biology, geology, physics and ecology. This includes the spreading of biological populations [Fis37], the Rayleigh-Bénard convection [NW87] or models appearing in combustion theory [Vol14, ZFK38]. The inverse problem (IP) is equivalent to the determination of an underlying physical law of a diffusion process, described by the nonlinear expression b in (1.2), by applying different sources (e.g., heat sources) and measuring the corresponding flux at the lateral boundary Σ. The information extracted from this way is encoded into the DN map $\mathcal{N}_b$.

These last decades, problems of parameter identification in nonlinear partial differential equations have generated a large interest in the mathematical community. Among the different formulation of these inverse problems, the determination of a nonlinear law is one of the most challenging from the severe ill-posedness and nonlinearity of the problem. For diffusion equations, one of the first results in that direction can be found in [Isa93]. Later on, this result was improved by [CK18b], where the stability issue was also considered. To the best of our knowledge, the most general and complete result known so far about the determination of a semilinear term of the form $b(t, x, u)$ depending simultaneously on the time variable t, the space variable x and the solution of the equation u from knowledge of the parabolic DN map $\mathcal{N}_b$ can be found in [KU23]. Without being exhaustive, we mention the works of [Isa01, CY88, COY06] devoted to the determination of semilinear terms depending only on the solution and the determination of quasilinear terms addressed in [CK18a, EPS17, FKU22]. Finally, we mention the works of [FO20, KLU18, LLLS21, LLLS20, LLST22, KU20b, KU20a, FLL23, HL23, KKU23, CFK+21, FKU22, LL22a, Lin22, KU23, LL23, LL19] devoted to similar problems for elliptic and hyperbolic equations. Moreover, in the recent works [LLLZ22, LLL21], the authors investigated simultaneous determination problems of coefficients and initial data for both parabolic and hyperbolic equations.

Most of the above mentioned results concern the inverse problem (IP) under the assumption that the semilinear term b in (1.2) satisfies the condition

$$b(t, x, 0) = 0, \quad (t, x) \in Q. \tag{1.3}$$

This condition implies that (1.2) has at least one known solution, the trivial solution. In the same spirit, for any constant $\lambda \in \mathbb{R}$, the condition $b(t, x, \lambda) = 0$ for $(t, x) \in Q$ implies that the constant function $(t, x) \mapsto \lambda$ is a solution of the equation $\rho(t, x)\partial_t u + \mathcal{A}(t)u + b(t, x, u) = 0$ in Q. In the present article, we treat the determination of general class of semilinear terms that might not satisfy the condition (1.3). In this case, the inverse problem (IP) is even more challenging since no solutions of (1.2) may be known a priori. In fact, as observed in [Sun10] for elliptic equations, there is an obstruction to the determination of b from the knowledge of $\mathcal{N}_b$ in the form of a gauge symmetry. We demonstrate the gauge symmetry first in the form of an example.

Example 1.1. Let us consider the inverse problem (IP) for the simplest linear case

$$\begin{cases} \partial_t u - \Delta u = b_0 & \text{in } Q, \\ u = f & \text{on } \Sigma, \\ u(0, x) = 0 & \text{in } \Omega, \end{cases} \tag{1.4}$$

where the aim is to recover an unknown source term $b_0 = b_0(x, t)$. Here, Δ is the Laplacian, but it could also be replaced by a more general second-order elliptic operator whose coefficients are known. Let us

consider a function $\varphi \in C^\infty([0,T] \times \overline{\Omega})$, which satisfies $\varphi \not\equiv 0$, $\varphi(0,x) = 0$ for $x \in \Omega$ and $\varphi = \partial_\nu \varphi = 0$ on Σ, where $\partial_\nu \varphi$ denotes the Neumann derivative of φ on Σ.

Then the function $\tilde{u} := u + \varphi$ satisfies

$$
\begin{cases}
\partial_t \tilde{u} - \Delta \tilde{u} = b_0 + (\partial_t - \Delta)\varphi & \text{in } Q, \\
\tilde{u} = f & \text{on } \Sigma, \\
\tilde{u}(0,x) = 0 & \text{in } \Omega.
\end{cases}
\tag{1.5}
$$

Since u and $\tilde{u}$ have the same initial data at $t = 0$ and boundary data on Σ, we see that the DN maps of (1.4) and (1.5) are the same. However, by unique continuation properties for parabolic equations (see, for example, [SS87, Theorem 1.1]), the conditions $\varphi = \partial_\nu \varphi = 0$ on Σ and $\varphi \not\equiv 0$ imply that $(\partial_t - \Delta)\varphi \not\equiv 0$, and it follows that $b_0 + (\partial_t - \Delta)\varphi \neq b_0$. Consequently, the inverse problem (IP) cannot be uniquely solved.

In what follows, we assume that $\alpha \in (0,1)$ and refer to Section 2.1 for the definitions of various function spaces that will show up. Let us describe the gauge symmetry, or gauge invariance, of the inverse problem (IP) in detail. For this, let a function $\varphi \in C^{1+\frac{\alpha}{2},2+\alpha}([0,T] \times \overline{\Omega})$ again satisfy

$$
\varphi(0,x) = 0, \quad x \in \Omega, \quad \varphi(t,x) = \partial_{\nu(a)}\varphi(t,x) = 0, \quad (t,x) \in \Sigma,
\tag{1.6}
$$

and consider the mapping S_φ from $C^\infty(\mathbb{R}; C^{\frac{\alpha}{2},\alpha}([0,T] \times \overline{\Omega}))$ into itself defined by

$$
S_\varphi b(t,x,\mu) = b(t,x,\mu + \varphi(t,x)) + \rho(t,x)\partial_t \varphi(t,x) + \mathcal{A}(t)\varphi(t,x), \quad (t,x,\mu) \in Q \times \mathbb{R}.
\tag{1.7}
$$

As in the example above, one can easily check that $\mathcal{N}_b = \mathcal{N}_{S_\varphi b}$.

In view of this obstruction, the inverse problem (IP) should be reformulated as a problem about determining the semilinear term up to the gauge symmetry described by (1.7). We note that (1.7) implies an equivalence relation for functions in $C^\infty(\mathbb{R}; C^{\frac{\alpha}{2},\alpha}([0,T] \times \overline{\Omega}))$. Corresponding equivalence classes will be called gauge classes:

Definition 1.1 (Gauge class). We say that two nonlinearities $b_1, b_2 \in C^\infty(\mathbb{R}; C^{\frac{\alpha}{2},\alpha}([0,T] \times \overline{\Omega})$ are in the same gauge class, or equivalent up to a gauge, if there is $\varphi \in C^{1+\frac{\alpha}{2},2+\alpha}([0,T] \times \overline{\Omega})$ satisfying (1.6) such that

$$
b_1 = S_\varphi b_2.
\tag{1.8}
$$

Here, S_φ is as in (1.7). In the case we consider a partial data inverse problem, where the normal derivative of solutions is assumed to be known only on $(0,T) \times \tilde{\Gamma}$, with $\tilde{\Gamma}$ open in $\partial\Omega$, we assume $\partial_{\nu(a)}\varphi = 0$ only on $(0,T) \times \tilde{\Gamma}$ in (1.6).

Using the above definition of the gauge class and taking into account the obstruction described above, we reformulate the inverse problem (IP) as follows.

○ **Inverse problem (IP1):** Can we determine the gauge class of the semilinear term b from the full or a partial knowledge of the parabolic DN map $\mathcal{N}_b$?

There is a natural additional question raised by (IP1) – namely,

○ **Inverse problem (IP2):** When does the gauge invariance break leading to a full resolution of the inverse problem (IP)?

One can easily check that it is possible to give a positive answer to problem (IP2) when (1.3) is fulfilled. Nevertheless, as observed in the recent work [LL22b], the resolution of problem (IP2) is not restricted to such situations. The work [LL22b] provided the first examples (in an elliptic setting) how to use nonlinearity as a tool to break the gauge invariance of (IP).

Let us also remark that, following Example 1.1, when $u \mapsto b(\cdot, u)$ is affine, corresponding to the case where the equation (1.2) is linear and has a source term, there is no hope to break the gauge

invariance (1.7) and the answer to (IP2) is, in general, negative. As will be observed in this article, this is no longer the case for various classes of nonlinear terms b. We will present cases where we will be able to solve (IP) uniquely. These cases present new instances where nonlinear interaction can be helpful in inverse problems. Nonlinearity has earlier been observed to be a helpful tool by many authors in different situations such as in partial data inverse problems and in anisotropic inverse problems on manifolds; see, for example, [KLU18, FO20, LLLS21, LLLS20, LLST22, KU20a, KU20b, FLL23].

In the present article, we will address both problems (IP1) and (IP2). We will start by considering the problem (IP1) for nonlinear terms, which are quite general. Then, we will exhibit several general situations where the gauge invariance breaks and give an answer to (IP2).

We mainly restrict our analysis to semilinear terms b subjected to the condition that the map $u \mapsto b(\cdot, u)$ is analytic (the (t, x) dependence is of $b(t, x, \mu)$ will not be assumed to be analytic). The restriction to this class of nonlinear terms is motivated by the study of the challenging problems (IP1) and (IP2) in this article. Indeed, even when condition (1.3) is fulfilled, the problem (IP) is still open, in general, for semilinear terms that are not subjected to our analyticity condition (see [KU23] for the most complete results known so far for this problem). Results for semilinear elliptic equations have also been proven for cases when the nonlinear terms are, roughly speaking, globally Lipschitz (see, for example, [IN95, IS94]). For this reason, our assumptions seem reasonable for tackling problems (IP1) and (IP2) and giving the first answers to these challenging problems. Note also that the linear part of (1.2) will be associated with a general class of linear parabolic equations with variable time-dependent coefficients. Consequently, we will present results also for linear equations with full and partial data measurements.

2. Main results

In this section, we will first introduce some preliminary definitions and results required for the rigorous formulation of our problem (IP). Then, we will state our main results for problems (IP1) and (IP2).

2.1. Preliminary properties

From now on, we fix $\alpha \in (0, 1)$, and we denote by $C^{\frac{\alpha}{2}, \alpha}([0, T] \times X)$, with $X = \overline{\Omega}$ or $X = \partial\Omega$, the set of functions h lying in $C([0, T] \times X)$ satisfying

$$[h]_{\frac{\alpha}{2}, \alpha} = \sup\left\{\frac{|h(t, x) - h(s, y)|}{(|x - y|^2 + |t - s|)^{\frac{\alpha}{2}}} : (t, x), (s, y) \in [0, T] \times X, \ (t, x) \neq (s, y)\right\} < \infty.$$

Then we define the space $C^{1+\frac{\alpha}{2}, 2+\alpha}([0, T] \times X)$ as the set of functions h lying in

$$C([0, T]; C^2(X)) \cap C^1([0, T]; C(X))$$

such that

$$\partial_t h, \partial_x^\beta h \in C^{\frac{\alpha}{2}, \alpha}([0, T] \times X), \quad \beta \in (\mathbb{N} \cup \{0\})^n, \ |\beta| = 2.$$

We consider these spaces with their usual norms, and we refer to [Cho09, pp. 4] for more details. Let us also introduce the space

$$\mathcal{K}_0 := \left\{h \in C^{1+\frac{\alpha}{2}, 2+\alpha}([0, T] \times \partial\Omega) : \ h(0, \cdot) = \partial_t h(0, \cdot) = 0\right\}.$$

If $r > 0$ and $h \in \mathcal{K}_0$, we denote by

$$\mathbb{B}(h, r) := \left\{g \in \mathcal{K}_0 : \ \|g - h\|_{C^{1+\frac{\alpha}{2}, 2+\alpha}([0, T] \times \partial\Omega)} < r\right\} \tag{2.1}$$

the ball centered at h and of radius r in the space $\mathcal{K}_0$. We assume also that b fulfills the following condition:

$$b(0, x, 0) = 0, \quad x \in \partial\Omega. \tag{2.2}$$

In this article, we assume that there exists $f_0 \in \mathcal{K}_0$ such that (1.2) admits a unique solution u lying in $C^{1+\frac{\alpha}{2}, 2+\alpha}([0, T] \times \overline{\Omega})$ when $f = f_0$. We note that the existence of u requires the condition (2.2) in order to have compatibility with the initial data at time $t = 0$ and the Dirichlet data on the lateral boundary Σ (see [LSU88, pp. 319 and pp. 449] for more details).

According to [LSU88, Theorem 6.1, pp. 452], [LSU88, Theorem 2.2, pp. 429], [LSU88, Theorem 4.1, pp. 443], [LSU88, Lemma 3.1, pp. 535] and [LSU88, Theorem 5.4, pp. 448], the problem (1.2) is well-posed for any $f = f_0 \in \mathcal{K}_0$ if, for instance, there exist $c_1, c_2 \geqslant 0$ such that the semilinear term b satisfies the following sign condition

$$b(t, x, \mu)\mu \geqslant -c_1\mu^2 - c_2, \quad t \in [0, T], \ x \in \overline{\Omega}, \ \mu \in \mathbb{R}.$$

The unique existence of solutions of (1.2) for some $f \in \mathcal{K}_0$ is not restricted to such a situation. Indeed, assume that there exists $\psi \in C^{1+\frac{\alpha}{2}, 2+\alpha}([0, T] \times \overline{\Omega})$ satisfying

$$\begin{cases} \rho(t, x)\partial_t\psi(t, x) + \mathcal{A}(t)\psi(t, x) = 0 & \text{in } Q, \\ \psi(0, x) = 0 & \text{for } x \in \Omega, \end{cases}$$

such that $b(t, x, \psi(t, x)) = 0$ for $(t, x) \in [0, T] \times \overline{\Omega}$. Then, one can easily check that (1.2) admits a unique solution when $f = \psi|_\Sigma$. Moreover, applying Proposition 3.1, we deduce that (1.2) will be well-posed when $f \in \mathcal{K}_0$ is sufficiently close to $\psi|_\Sigma$ in the sense of $C^{1+\frac{\alpha}{2}, 2+\alpha}([0, T] \times \partial\Omega)$.

As will be shown in Proposition 3.1, the existence of $f_0 \in \mathcal{K}_0$ such that (1.2) admits a solution when $f = f_0$ implies that there exists $\epsilon > 0$, depending on $a, \rho, b, f_0, \Omega, T$, such that, for all $f \in \mathbb{B}(f_0, \epsilon)$, the problem (1.2) admits a solution $u_f \in C^{1+\frac{\alpha}{2}, 2+\alpha}([0, T] \times \overline{\Omega})$, which is unique in a sufficiently small neighborhood of the solution of (1.2) with boundary value $f = f_0$. Using these properties, we can define the parabolic DN map

$$\mathcal{N}_b : \mathbb{B}(f_0, \epsilon) \ni f \mapsto \partial_{\nu(a)}u_f(t, x), \quad (t, x) \in \Sigma. \tag{2.3}$$

2.2. Resolution of (IP1)

We present our first main results about recovering the gauge class of a semilinear term from the corresponding DN map.

We consider Ω_1 to be an open bounded, smooth and connected subset of $\mathbb{R}^n$ such that $\overline{\Omega} \subset \Omega_1$. We extend a and ρ into functions defined smoothly on $[0, T] \times \overline{\Omega}_1$ satisfying $\rho > 0$ and condition (1.1) with Ω replaced by Ω_1. For all $t \in [0, T]$, we set also

$$g(t) := \rho(t, \cdot)a(t, \cdot)^{-1},$$

and we consider the compact Riemannian manifold with boundary $(\overline{\Omega}_1, g(t))$.

Assumption 2.1. *Throughout this article, we assume that $(\overline{\Omega}_1, g(t))$ is a simple Riemannian manifold for all $t \in [0, T]$. That is, we assume that for any point $x \in \overline{\Omega}_1$, the exponential map $\exp_x$ is a diffeomorphism from some closed neighborhood of 0 in $T_x\overline{\Omega}_1$ onto $\overline{\Omega}_1$ and $\partial\Omega_1$ is strictly convex.*

From now on, for any Banach space X, we denote by $\mathbb{A}(\mathbb{R}; X)$ the set of analytic functions on $\mathbb{R}$ as maps taking values in X. That is, for any $b \in \mathbb{A}(\mathbb{R}; X)$ and $\mu \in \mathbb{R}$, b has convergent X-valued Taylor series on a neighborhood of μ.

Theorem 2.1. *Let $a := (a_{ik})_{1 \leqslant i,k \leqslant n} \in C^{\infty}([0,T] \times \overline{\Omega}; \mathbb{R}^{n \times n})$ satisfy (1.1) and $\rho \in C^{\infty}([0,T] \times \overline{\Omega}; \mathbb{R}_{+})$. Let $b_j \in \mathbb{A}(\mathbb{R}; C^{\frac{\alpha}{2},\alpha}([0,T] \times \overline{\Omega}))$, which satisfies (2.2) as $b = b_j$, for $j = 1,2$. We also assume that there exists $f_0 \in \mathcal{K}_0$ such that problem (1.2), with $f = f_0$ and $b = b_j$, admits a unique solution for $j = 1,2$. Then, the condition*

$$\mathcal{N}_{b_1}(f) = \mathcal{N}_{b_2}(f), \quad f \in \mathbb{B}(f_0, \epsilon) \tag{2.4}$$

implies that there exists $\varphi \in C^{1+\frac{\alpha}{2},2+\alpha}([0,T] \times \overline{\Omega})$ satisfying (1.6) such that

$$b_1 = S_{\varphi} b_2 \tag{2.5}$$

with S_{φ} the map defined by (1.7).

Remark 2.1. Let us list the data that can be recovered by our methods without the assumption of analyticity of b in the μ-variable. In this case, we can only recover the Taylor series of b in μ-variable at shifted points. Indeed, assume as in Theorem 2.1 with the exception that the nonlinearities b_1 and b_2 are not analytic in the μ-variable, and fix (t, x). In this case, by inspecting the proof of Theorem 2.1 (see Section 4), we can show that

$$\partial_{\mu}^k b_1(t, x, u_{1,0}(t,x)) = \partial_{\mu}^k b_2(t, x, u_{2,0}(t,x)), \quad \text{for any } k \in \mathbb{N},$$

where $u_{j,0}(t,x)$, for $j = 1,2$, is the solution to (1.2) with coefficient $b = b_j$ and boundary value $f = f_0 \in \mathcal{K}_0$. Thus, we see that the formal Taylor series of $b_1(t, x, \cdot)$ at $u_{1,0}(t,x)$ is that of $b_2(t, x, \cdot)$ shifted by $u_{2,0}(t,x) - u_{1,0}(t,x)$, which is typically nonzero as we do not assume that we know any solutions to (1.2) a priori.

By assuming analyticity in Theorem 2.1, we are able to connect the Taylor series of b_1 and b_2 in the μ-variable at different points, which leads to (2.5) in the end. This is one motivation for the analyticity assumption. Note that we do not assume analyticity in the other variables.

For our second result, let us consider a *partial data result* when $\mathcal{A}(t) = -\Delta$ (that is, $a = \text{Id}_{\mathbb{R}^{n \times n}}$ is an $n \times n$ identity matrix) and $\rho \equiv 1$. More precisely, consider the front and back sets of $\partial\Omega$

$$\Gamma_{\pm}(x_0) := \{x \in \partial\Omega : \pm(x - x_0) \cdot \nu(x) \geqslant 0\}$$

with respect to a source $x_0 \in \mathbb{R}^n \setminus \overline{\Omega}$. Then, our second main result is stated as follows:

Theorem 2.2. *For $n \geqslant 3$ and Ω simply connected, let $a = (a_{ik})_{1 \leqslant i,k \leqslant n} = \text{Id}_{\mathbb{R}^{n \times n}}$ and $\rho \equiv 1$. Let $b_j \in \mathbb{A}(\mathbb{R}; C^{\frac{\alpha}{2},\alpha}([0,T] \times \overline{\Omega}))$, which satisfies (2.2) as $b = b_j$, for $j = 1,2$. We also assume that there exists $f_0 \in \mathcal{K}_0$ such that problem (1.2), with $f = f_0$ and $b = b_j$, admits a unique solution, for $j = 1,2$. Fix $x_0 \in \mathbb{R}^n \setminus \overline{\Omega}$ and consider $\tilde{\Gamma}$ a neighborhood of $\Gamma_{-}(x_0)$ on $\partial\Omega$. Then, the condition*

$$\mathcal{N}_{b_1} f(t,x) = \mathcal{N}_{b_2} f(t,x), \quad (t,x) \in (0,T) \times \tilde{\Gamma}, \text{ for any } f \in \mathbb{B}(f_0, \epsilon) \tag{2.6}$$

implies that there exists $\varphi \in C^{1+\frac{\alpha}{2},2+\alpha}([0,T] \times \overline{\Omega})$ satisfying

$$\varphi(0,x) = 0, \quad x \in \Omega, \quad \varphi(t,x) = 0, \quad (t,x) \in \Sigma, \tag{2.7}$$

$$\partial_{\nu}\varphi(t,x) = 0, \quad (t,x) \in (0,T) \times \tilde{\Gamma} \tag{2.8}$$

such that

$$b_1 = S_{\varphi} b_2. \tag{2.9}$$

We will be able to break the gauge condition $b_1 = S_{\varphi} b_2$ in Theorems 2.1 and 2.2 in various cases. We present these results separately in the next section.

2.3. Breaking the gauge in the sense of (IP2)

In several situations, the gauge class (2.9) could be broken, and one can fully determine the semilinear term b in (1.2) from its parabolic DN map. We will present below classes of nonlinearities when such phenomenon occurs. We start by considering general elements of $b \in \mathbb{A}(\mathbb{R}; C^{\frac{\alpha}{2}, \alpha}([0,T] \times \overline{\Omega}))$ for which the gauge invariance (2.9) breaks.

Corollary 2.1. *Let the conditions of Theorem 2.1 be fulfilled and assume that there exists $\kappa \in C^{\frac{\alpha}{2}, \alpha}([0,T] \times \overline{\Omega})$ such that*

$$b_1(t, x, \kappa(t, x)) = b_2(t, x, \kappa(t, x)), \quad (t, x) \in [0, T] \times \overline{\Omega}. \tag{2.10}$$

Then, the condition (2.4) implies that $b_1 = b_2$. In the same way, assuming that the conditions of Theorem 2.2 are fulfilled, the condition (2.6) implies that $b_1 = b_2$.

Corollary 2.2. *Let the conditions of Theorem 2.1 be fulfilled, and assume that there exists $h \in C^{\alpha}(\overline{\Omega})$, $G \in C^{\frac{\alpha}{2}, \alpha}([0,T] \times \overline{\Omega})$ and $\theta \in (0, T]$ satisfying the condition*

$$\inf_{x \in \Omega} |G(\theta, x)| > 0, \tag{2.11}$$

such that

$$b_1(t, x, 0) - b_2(t, x, 0) = h(x)G(t, x), \quad (t, x) \in [0, T] \times \overline{\Omega}. \tag{2.12}$$

Assume also that the solutions $u_{j,0}$ of (1.2), $j = 1, 2$, with $f = f_0$ and $b = b_j$, satisfy the condition

$$u_{1,0}(\theta, x) = u_{2,0}(\theta, x), \quad x \in \Omega. \tag{2.13}$$

Then, the condition (2.4) implies that $b_1 = b_2$. Moreover, by assuming that the conditions of Theorem 2.2 are fulfilled, the condition (2.6) implies that $b_1 = b_2$.

Now let us consider elements $b \in \mathbb{A}(\mathbb{R}; C^{\frac{\alpha}{2}, \alpha}([0,T] \times \overline{\Omega}))$ which are polynomials of the form

$$b(t, x, \mu) = \sum_{k=0}^{N} b_k(t, x)\mu^k, \quad (t, x, \mu) \in [0, T] \times \overline{\Omega} \times \mathbb{R}. \tag{2.14}$$

For this class of nonlinear terms, we can prove the following.

Theorem 2.3. *Let the condition of Theorem 2.1 be fulfilled, and assume that, for $j = 1, 2$, there exists $N_j \geqslant 2$ such that*

$$b_j(t, x, \mu) = \sum_{k=0}^{N_j} b_{j,k}(t, x)\mu^k, \quad (t, x, \mu) \in [0, T] \times \overline{\Omega} \times \mathbb{R}. \tag{2.15}$$

Let ω be an open subset of $\mathbb{R}^n$ such that $\omega \subset \Omega$ and J is a dense subset of $(0, T) \times \omega$. We assume also that, for $N = \min(N_1, N_2)$, the conditions

$$\min\left(\left| (b_{1,N-1} - b_{2,N-1})(t, x) \right|, \sum_{j=1}^{2} \left| (b_{j,N} - b_{j,N-1})(t, x) \right| \right) = 0, \ (t, x) \in J, \tag{2.16}$$

$$\left| b_{1,N}(t, x) \right| > 0, \quad (t, x) \in J \tag{2.17}$$

$$b_{1,0}(t, x) = b_{2,0}(t, x), \quad (t, x) \in (0, T) \times (\Omega \setminus \overline{\omega}) \tag{2.18}$$

hold true. Then the condition (2.4) *implies that* $b_1 = b_2$. *In addition, assuming that the conditions of Theorem* 2.2 *are fulfilled, condition* (2.6) *implies that* $b_1 = b_2$.

We make the following remark.

Remark 2.2.

(i) The preceding theorem, in particular, says that the inverse source problem of recovering a source function F from the DN map of

$$\partial_t u - \Delta u + u^2 = F$$

is uniquely solvable. This is in strict contrast to the inverse source problem for the linear equation $\partial_t u - \Delta u = F$ that always has a gauge invariance explained in Example 1.1: the sources F and $\widetilde{F} := F + \partial_t \varphi - \Delta \varphi$ in Q have the same DN map. Here, the only restrictions for φ are given in (1.6) and thus typically $F \neq \widetilde{F}$.

(ii) Inverse source problems for semilinear elliptic equations were studied in [LL22b]. There it was shown that if in the notation of the corollary $b_{1,N-1} = b_{2,N-1}$ and $b_{1,N} \neq 0$ in Ω (so that (2.16) and (2.17) hold), then the gauge breaks. With natural replacements, the corollary generalizes [LL22b, Corollary 1.3] in the elliptic setting. This can be seen by inspecting its proof.

Let us then consider nonlinear terms $b \in \mathbb{A}(\mathbb{R}; C^{\frac{\alpha}{2}, \alpha}([0,T] \times \overline{\Omega}))$ of the form

$$b(t,x,\mu) = b_1(t,x)h(t,b_2(t,x)\mu) + b_0(t,x), \quad (t,x,\mu) \in [0,T] \times \overline{\Omega} \times \mathbb{R}. \tag{2.19}$$

We start by considering nonlinear terms of the form (2.19) with $b_2 \equiv 1$.

Theorem 2.4. *Let the conditions of Theorem* 2.1 *be fulfilled, and assume that, for* $j = 1, 2$, *there exists* $h_j \in \mathbb{A}(\mathbb{R}; C^{\frac{\alpha}{2}}([0,T]))$ *such that*

$$b_j(t,x,\mu) = b_{j,1}(t,x)h_j(t,\mu) + b_{j,0}(t,x), \quad (t,x,\mu) \in [0,T] \times \overline{\Omega} \times \mathbb{R}. \tag{2.20}$$

Assume also that, for all $t \in (0,T)$, *there exist* $\mu_t \in \mathbb{R}$ *and* $n_t \in \mathbb{N}$ *such that*

$$\partial_\mu^{n_t} h_1(t, \cdot) \not\equiv 0 \text{ and } \partial_\mu^{n_t} h_1(t, \mu_t) = 0, \quad t \in (0,T). \tag{2.21}$$

Moreover, we assume that

$$b_{1,1}(t,x) \neq 0, \quad (t,x) \in Q, \tag{2.22}$$

and that for all $t \in (0,T)$, *there exists* $x_t \in \partial \Omega$ *such that*

$$b_{1,1}(t,x_t) = b_{2,1}(t,x_t) \neq 0, \quad t \in (0,T). \tag{2.23}$$

Then, the condition (2.4) *implies that* $b_1 = b_2$. *Moreover, assuming that the conditions of Theorem* 2.2 *are fulfilled, the condition* (2.6) *implies that* $b_1 = b_2$.

Under a stronger assumption imposed on the expression h, we can also consider nonlinear terms of the form (2.19) with $b_2 \not\equiv 1$.

Theorem 2.5. *Let the conditions of Theorem* 2.1 *be fulfilled, and assume that, for* $j = 1, 2$, *there exists* $h_j \in \mathbb{A}(\mathbb{R}; C^{\frac{\alpha}{2}}([0,T]))$ *such that*

$$b_j(t,x,\mu) = b_{j,1}(t,x)h_j(t,b_{j,2}(t,x)\mu) + b_{j,0}(t,x), \quad (t,x,\mu) \in [0,T] \times \overline{\Omega} \times \mathbb{R}. \tag{2.24}$$

Assume also that, for all $t \in (0,T)$, *there exists* $n_t \in \mathbb{N}$ *such that*

$$\partial_\mu^{n_t} h_1(t, \cdot) \not\equiv 0 \text{ and } \partial_\mu^{n_t} h_1(t,0) = 0, \quad t \in (0,T). \tag{2.25}$$

Moreover, we assume that

$$b_{1,1}(t,x) \neq 0 \text{ and } b_{1,2}(t,x) \neq 0, \quad (t,x) \in Q, \tag{2.26}$$

and that for all $t \in (0,T)$, there exists $x_t \in \partial\Omega$ such that

$$b_{1,1}(t,x_t) = b_{2,1}(t,x_t) \neq 0 \text{ and } b_{1,2}(t,x_t) = b_{2,2}(t,x_t) \neq 0 \quad t \in (0,T). \tag{2.27}$$

Then, the condition (2.4) implies that $b_1 = b_2$. Moreover, assuming that the conditions of Theorem 2.2 are fulfilled, the condition (2.6) implies that $b_1 = b_2$.

Finally, we consider nonlinear terms $b \in \mathbb{A}(\mathbb{R}; C^{\frac{\alpha}{2},\alpha}([0,T] \times \overline{\Omega}))$ satisfying

$$b(t,x,\mu) = b_1(t,x)G(x,b_2(t,x)\mu) + b_0(t,x), \quad (t,x,\mu) \in [0,T] \times \Omega \times \mathbb{R}. \tag{2.28}$$

Corollary 2.3. *Let the conditions of Theorem 2.1 be fulfilled, and assume that, for $j = 1,2$, there exists $G \in \mathbb{A}(\mathbb{R}; C^\alpha(\overline{\Omega}))$ such that*

$$b_j(t,x,\mu) = b_{j,1}(t,x)G(x,b_{j,2}(t,x)\mu) + b_{j,0}(t,x), \quad (t,x,\mu) \in [0,T] \times \Omega \times \mathbb{R}. \tag{2.29}$$

Assume also that one of the following conditions is fulfilled:

(i) *We have $b_{1,2} = b_{2,2}$, and for all $x \in \Omega$, there exists $n_x \in \mathbb{N}$ and $\mu_x \in \mathbb{R}$ such that*

$$\partial_\mu^{n_x} G(x,\cdot) \not\equiv 0 \text{ and } \partial_\mu^{n_x} G(x,\mu_x) = 0, \quad x \in \Omega. \tag{2.30}$$

(ii) *For all $x \in \Omega$, there exists $n_x \in \mathbb{N}$ such that*

$$\partial_\mu^{n_x} G(x,\cdot) \not\equiv 0 \text{ and } \partial_\mu^{n_x} G(x,0) = 0, \quad x \in \Omega. \tag{2.31}$$

Moreover, we assume that condition (2.26) is fulfilled. Then, the condition (2.4) implies that $b_1 = b_2$. In addition, by assuming that the conditions of Theorem 2.2 are fulfilled, condition (2.6) implies that $b_1 = b_2$.

Remark 2.3. Let us observe that the results of Theorems 2.4 and 2.5 and Corollary 2.3 are mostly based on the conditions (2.21) and (2.26) imposed to nonlinear terms of the form (2.19), and on the conditions (2.30) and (2.31) imposed to nonlinear terms of the form (2.28). These conditions are rather general, and they will be fulfilled in various situations for different class of functions. For instance, assuming that the function h_1 takes the form

$$h_1(t,\mu) = P(t,\mu)\exp(Q(t,\mu)), \quad (t,\mu) \in [0,T] \times \mathbb{R}$$

with $P, Q \in \mathbb{A}(\mathbb{R}; C^{\frac{\alpha}{2}}([0,T]))$, condition (2.21) will be fulfilled if we assume that there exists $\sigma \in C^{\alpha/2}([0,T])$ such that

$$\left(\partial_\mu P(t,\mu) + P(t,\mu)\partial_\mu Q(t,\mu)\right)\big|_{\mu=\sigma(t)} = 0, \quad t \in [0,T].$$

Such a condition will, of course, be fulfilled when $h_1(t,\mu) = \mu e^\mu$, $(t,\mu) \in [0,T] \times \mathbb{R}$.

More generally, let $\sigma \in C^{\frac{\alpha}{2}}([0,T])$ be arbitrary chosen, and for each $t \in [0,T]$, consider $N_t \in \mathbb{N}$. Assuming that the function h_1 satisfies the following property

$$h_1(t,\mu) = \sum_{k=0}^{N_t-1} a_k(t)(\mu - \sigma(t))^k + \underset{\mu \to \sigma(t)}{\mathcal{O}}\left((\mu - \sigma(t))^{N_t+1}\right), \quad t \in [0,T], \tag{2.32}$$

with $a_k \in C^{\frac{\alpha}{2}}([0,T])$, $k \in \mathbb{N} \cup \{0\}$, one can easily check that condition (2.21) will be fulfilled since we have

$$\partial_\mu^{N_t} h_1(t, \sigma(t)) = 0, \quad t \in [0,T].$$

Condition (2.26) will be fulfilled under the same condition provided that h_1 satisfies (2.32) with $\sigma \equiv 0$. Moreover, let $g \in C^\alpha(\overline{\Omega})$, and for each $x \in \Omega$, consider $N_x \in \mathbb{N}$. Assuming that the function G in (2.30) satisfies the property

$$G(x, \mu) = \sum_{k=0}^{N_x - 1} \beta_k(x)(\mu - g(x))^k + \underset{\mu \to g(x)}{\mathcal{O}}\left((\mu - g(x))^{N_x+1}\right), \quad x \in \Omega, \tag{2.33}$$

with functions $\beta_k \in C^\alpha(\overline{\Omega})$, $k \in \mathbb{N} \cup \{0\}$, it is clear that condition (2.30) will be fulfilled since we have

$$\partial_\mu^{N_x} G(x, g(x)) = 0, \quad x \in \Omega.$$

The same is true for condition (2.31) when G satisfies (2.33) with $g \equiv 0$.

Finally, via previous observations, we can also determine the order coefficients for *linear* parabolic equations.

Corollary 2.4 (Global uniqueness with partial data). *Adopting all notations in Theorem* 2.2, *let* $q_j = q_j(t,x) \in C^\infty([0,T] \times \overline{\Omega})$ *and* $b_j(t,x,\mu) = q_j(t,x)\mu$ *for* $j = 1, 2$. *Then* (2.4) *implies* $q_1 = q_2$ *in* Q.

We mention that we could also prove that the assumptions of Theorem 2.1 and $b_j(t,x,\mu) = q_j(t,x)\mu$, $j = 1, 2$, imply $q_1 = q_2$.

2.4. Comments about our results

To the best of our knowledge, Theorems 2.1 and 2.2 give the first positive answer to the inverse problem (IP1) for semilinear parabolic equations. In addition, the results of Theorem 2.1 and 2.2 extend the analysis of [LL22b] that considered a problem similar to (IP1) for elliptic equations, but which did not fully answer the question raised by (IP1). In that sense, Theorems 2.1 and 2.2 give the first positive answer to (IP1) for a class of elliptic PDEs as well. While Theorem 2.1 is stated for general class of parabolic equations, Theorem 2.2 gives a result with measurements restricted to a neighborhood of the back set with respect to a source $x_0 \in \mathbb{R}^n \setminus \overline{\Omega}$ in the spirit of the most precise partial data results stated for linear elliptic equations such as [KSU07]. Note that in contrast to [KSU07], the source x_0 is not necessary outside the convex hull of $\overline{\Omega}$ and, as observed in [KSU07], when Ω is convex, the measurements of Theorem 2.2 can be restricted to any open set of $\partial\Omega$. Even for linear equations, Theorems 2.1 and 2.2 improve in precision and generality the earlier works of [CK18b, Isa91] dealing with determination of time dependent coefficients appearing in linear parabolic equations.

We gave a positive answer to the problem (IP2) and show that the gauge breaks for three different classes of semilinear terms:

1) Semilinear terms with prescribed information in Corollaries 2.1 and 2.2,
2) Polynomial semilinear terms in Theorem 2.3,
3) Semilinear terms with separated variables of the form (2.19) or (2.28) in Theorems 2.4 and 2.5 and in Corollary 2.3.

This seems to be the most complete overview of situations where one can give a positive answer to problem (IP2). While [LL22b] considered also such phenomena for polynomial nonlinear terms and

some specific examples, the conditions of Corollary 2.1 and 2.2, Theorems 2.4 and 2.5 and Corollary 2.3, leading to a positive answer for (IP2), seem to be new. In Remark 2.3, we gave several concrete and general examples of semilinear terms satisfying the conditions of Theorems 2.4, 2.5 and Corollary 2.3.

The proof of our results is based on a combination of the higher-order linearization technique, application of suitable class of geometric optics solutions for parabolic equations, Carleman estimates, properties of holomorphic functions and different properties of parabolic equations. Theorem 2.1 is deduced from the linearized result of Proposition 4.1 that we prove by using geometric optics solutions for parabolic equations. These solutions are built by using the energy estimate approach introduced in the recent work of [Fei23].

We mention that Assumption 2.1 is used in this work mainly for two purposes. Our construction of geometric optics solutions is based on the use of global polar normal coordinates on the manifolds $(\overline{\Omega}_1, g(t))$, $t \in [0,T]$. In addition, Assumption 2.1 guarantees the injectivity of the geodesic ray transform on the manifolds $(\overline{\Omega}_1, g(t))$, $t \in [0,T]$, which is required in our proofs of Theorems 2.1 and 2.2. It is not clear at the moment how Assumption 2.1 can be relaxed in the construction of geometric optics solutions for parabolic equations of the form considered in this paper.

This allows us to consider problem (IP1) for general class of semilinear parabolic equations with variable coefficients. In Theorem 2.2, we combine this class of geometric optics solutions with a Carleman estimate with boundary terms stated in Lemma 6.1 in order to restrict the boundary measurements to a part of the boundary. Note that the weight under consideration in Lemma 6.1 is not a limiting Carleman weight for parabolic equations. This is one reason why we cannot apply such Carleman estimates for making also a restriction on the support of the Dirichlet input in Theorem 2.2.

It is worth mentioning that our results for problem (IP1) and (IP2) can be applied to *inverse source problems* for nonlinear parabolic equations. This important application is discussed in Section 8. There we show how the nonlinear interaction allows to solve this problem for general classes of source terms, depending simultaneously on the time and space variables. Corresponding problems for linear equations cannot be solved uniquely (see Example 1.1 or, for example, [KSXY22, Appendix A]). In that sense, our analysis exhibits a new consequence of the nonlinear interaction, already considered for examples in [FO20, KLU18, LLLS21, LLLS20, LLST22, KU20b, KU20a, FLL23]), by showing how nonlinearity can help for the resolution of inverse source problems for parabolic equations.

We remark that while Theorem 2.1 is true for $n \geqslant 2$, we can only prove Theorem 2.2 for dimension $n \geqslant 3$. The fact that we cannot prove Theorem 2.2 for $n = 2$ is related to the Carleman estimate of Lemma 6.1 that we can only derive for $n \geqslant 3$. Since this Carleman estimate is a key ingredient in the proof of Theorem 2.2, we need to exclude the case $n = 2$ in the statement of this result.

Finally, let us observed that, under the suitable assumption of simplicity stated in Assumption 2.1, the result of Theorem 2.1 can be applied to the determination of a semilinear term for reaction diffusion equations on a Riemannian manifold with boundary equipped with a time-dependent metric.

2.5. *Outline of the paper*

This article is organized as follows. In Section 3, we consider the forward problem by proving the well-posedness of (1.2) under suitable conditions, and we recall some properties of the higher-order linearization method for parabolic equations. Section 4 is devoted to the proof of Theorem 2.1, while in Section 5, we prove Proposition 4.1. In Section 6, we prove Theorem 2.2, and in Section 7, we consider our results related to problem (IP2). Finally, in Section 8, we discuss the applications of our results to inverse source problems for parabolic equations. In the Appendix A, the outline of the proof of the Carleman estimates of Lemma 6.1 is presented.

3. The forward problem and higher-order linearization

Recall that in this article we assume that there is a solution u_0 to (1.2) corresponding to a lateral boundary data f_0.

3.1. Well-posedness for Dirichlet data close to f_0

In this subsection, we consider the well-posedness for the problem (1.2), whenever the boundary datum f is sufficiently close to f_0 with respect to $C^{1+\frac{\alpha}{2},2+\alpha}([0,T] \times \partial\Omega)$. For this purpose, we consider the Banach space $\mathcal{K}_0$ with the norm of the space $C^{1+\frac{\alpha}{2},2+\alpha}([0,T] \times \partial\Omega)$. Our local well-posedness result is stated as follows.

Proposition 3.1. *Let $a := (a_{ik})_{1 \leqslant i,k \leqslant n} \in C^\infty([0,T] \times \overline{\Omega}; \mathbb{R}^{n \times n})$ satisfy (1.1), $\rho \in C^\infty([0,T] \times \overline{\Omega}; \mathbb{R}_+)$ and $b \in C^\infty(\mathbb{R}; C^{\frac{\alpha}{2},\alpha}([0,T] \times \overline{\Omega}))$. We assume also that there exists a boundary value $f_0 \in \mathcal{K}_0$ such that the problem (1.2) with $f = f_0$ admits a unique solution $u_0 \in C^{1+\frac{\alpha}{2},2+\alpha}([0,T] \times \overline{\Omega})$. Then there exists $\epsilon > 0$ depending on a, ρ, b, f_0, Ω, T, such that, for all $f \in \mathbb{B}(f_0, \epsilon)$, the problem (1.2) admits a unique solution $u_f \in C^{1+\frac{\alpha}{2},2+\alpha}([0,T] \times \overline{\Omega})$ satisfying*

$$\left\| u_f - u_0 \right\|_{C^{1+\frac{\alpha}{2},2+\alpha}([0,T] \times \overline{\Omega})} \leqslant C \| f - f_0 \|_{C^{1+\frac{\alpha}{2},2+\alpha}([0,T] \times \partial\Omega)}. \tag{3.1}$$

Moreover, the map $\mathbb{B}(f_0, \epsilon) \ni f \mapsto u_f \in C^{1+\frac{\alpha}{2},2+\alpha}([0,T] \times \overline{\Omega})$ is C^∞ in the Fréchet sense.

Proof. Let us first observe that we may look for a solution u_f by splitting it into two terms by $u_f = u_0 + v$, where v solves

$$\begin{cases} \rho\partial_t v + \mathcal{A}(t)v + b(t,x,v+u_0) - b(t,x,u_0) = 0 & \text{in } (0,T) \times \Omega, \\ v = h & \text{on } (0,T) \times \partial\Omega, \\ v(0,x) = 0 & \text{for } x \in \Omega, \end{cases} \tag{3.2}$$

with $h := f - f_0$. Therefore, it is enough for our purpose to show that there exists $\epsilon > 0$ depending on a, ρ, b, f_0, Ω, T, such that for $h \in \mathbb{B}(0, \epsilon)$, the problem (3.2) admits a unique solution $v_h \in C^{1+\frac{\alpha}{2},2+\alpha}([0,T] \times \overline{\Omega})$ satisfying

$$\| v_h \|_{C^{1+\frac{\alpha}{2},2+\alpha}([0,T] \times \overline{\Omega})} \leqslant C \| h \|_{C^{1+\frac{\alpha}{2},2+\alpha}([0,T] \times \partial\Omega)}. \tag{3.3}$$

We introduce the spaces

$$\mathcal{H}_0 := \left\{ u \in C^{1+\frac{\alpha}{2},2+\alpha}([0,T] \times \overline{\Omega}) : u|_{\{0\}\times\overline{\Omega}} \equiv 0, \ \partial_t u|_{\{0\}\times\partial\Omega} \equiv 0 \right\},$$

$$\mathcal{L}_0 := \left\{ F \in C^{\frac{\alpha}{2},\alpha}([0,T] \times \overline{\Omega}) : F|_{\{0\}\times\partial\Omega} \equiv 0 \right\}.$$

Then, let us introduce the map $\mathcal{G}$ defined by

$$\mathcal{G} : \mathcal{K}_0 \times \mathcal{H}_0 \to \mathcal{L}_0 \times \mathcal{K}_0,$$
$$(h,v) \mapsto (\rho\partial_t v + \mathcal{A}(t)v + b(t,x,v+u_0) - b(t,x,u_0), v|_\Sigma - h).$$

We will find a solution to (1.2) by applying the implicit function theorem to the map $\mathcal{G}$. Using the fact that $b \in C^\infty(\mathbb{R}; C^{\frac{\alpha}{2},\alpha}([0,T] \times \overline{\Omega}))$, it follows that the map $\mathcal{G}$ is C^∞ on $\mathcal{K}_0 \times \mathcal{H}_0$ in the Fréchet sense. Moreover, we have $\mathcal{G}(0,0) = (0,0)$ and

$$\partial_v \mathcal{G}(0,0)w = \left(\rho\partial_t w + \mathcal{A}(t)w + \partial_\mu b(t,x,u_0)w, w|_\Sigma \right).$$

In order to apply the implicit function theorem, we will prove that the map $\partial_v \mathcal{G}(0,0)$ is an isomorphism from $\mathcal{H}_0$ to $\mathcal{L}_0 \times \mathcal{K}_0$. For this purpose, let us fix $(F, h) \in \mathcal{L}_0 \times \mathcal{K}_0$, and let us consider the linear parabolic problem

$$\begin{cases} \rho \partial_t w + \mathcal{A}(t)w + \partial_\mu b(t, x, u_0)w = F(t, x) & \text{in } Q, \\ w = h & \text{on } \Sigma, \\ w(0, x) = 0 & x \in \Omega. \end{cases} \tag{3.4}$$

Applying [LSU88, Theorem 5.2, Chapter IV, page 320], we deduce that problem (3.4) admits a unique solution $w \in \mathcal{H}_0$ satisfying

$$\|w\|_{C^{1+\frac{\alpha}{2}, 2+\alpha}([0,T] \times \overline{\Omega})} \leqslant C \Big(\|F\|_{C^{\frac{\alpha}{2}, \alpha}([0,T] \times \overline{\Omega})} + \|h\|_{C^{1+\frac{\alpha}{2}, 2+\alpha}([0,T] \times \partial\Omega)} \Big),$$

for some constant $C > 0$ independent of w, F and h. From this result, we deduce that $\partial_v \mathcal{G}(0,0)$ is an isomorphism from $\mathcal{H}_0$ to $\mathcal{L}_0 \times \mathcal{K}_0$.

Therefore, applying the implicit function theorem (see, for example, [RR06, Theorem 10.6]), we deduce that there exists $\epsilon > 0$ depending on a, b, ρ, f_0, Ω, T, and a smooth map ψ from $\mathbb{B}(0, \epsilon)$ to $\mathcal{H}_0$, such that, for all $h \in \mathbb{B}(0, \epsilon)$, we have $\mathcal{G}(h, \psi(h)) = (0, 0)$. This proves that $v = \psi(h)$ is a solution of (3.2) for all $h \in \mathbb{B}(0, \epsilon)$.

For the uniqueness of the solution of (3.2), let us consider $v_1 \in C^{1+\frac{\alpha}{2}, 2+\alpha}([0, T] \times \overline{\Omega})$ to be a solution of (3.2), and let us show that $v_1 = v$. For this purpose, we fix $w = v_1 - v$ and notice that w solves the following initial boundary value problem

$$\begin{cases} \rho \partial_t w + \mathcal{A}(t)w + q(t, x)w = 0 & \text{in } Q, \\ w = 0 & \text{on } \Sigma, \\ w(0, x) = 0 & \text{for } x \in \Omega, \end{cases} \tag{3.5}$$

with

$$q(t, x) = \int_0^1 \partial_\mu b(t, x, s v_1(t, x) + (1 - s)v(t, x) + u_0(t, x))\, ds, \quad (t, x) \in Q.$$

Then the uniqueness of the solutions of (3.5) implies that $w \equiv 0$, and by the same way that $v = v_1$. Therefore, $v = \psi(h)$ is the unique solution of (3.2). Combining this with the fact that ψ is smooth from $\mathbb{B}(0, \epsilon)$ to $\mathcal{H}_0$ and $\psi(0) = 0$, we obtain (3.1). Finally, recalling that, for all $f \in \mathbb{B}(f_0, \epsilon)$, $u_f = u_0 + \psi(f - f_0)$ with ψ a C^∞ map from $\mathbb{B}(0, \epsilon)$ to $C^{1+\frac{\alpha}{2}, 2+\alpha}([0, T] \times \overline{\Omega})$, we deduce that the map $\mathbb{B}(f_0, \epsilon) \ni f \mapsto u_f \in C^{1+\frac{\alpha}{2}, 2+\alpha}([0, T] \times \overline{\Omega})$ is C^∞. This completes the proof of the proposition. $\quad\square$

3.2. Linearizations of the problem

In this subsection, we assume that the conditions of Proposition 3.1 are fulfilled. Let us introduce $m \in \mathbb{N} \cup \{0\}$ and consider the parameter $s = (s_1, \ldots, s_{m+1}) \in (-1, 1)^{m+1}$. Fixing $h_1, \ldots, h_{m+1} \in \mathbb{B}(0, \frac{\epsilon}{m+1})$, we consider $u = u_s$ the solution of

$$\begin{cases} \rho(t, x)\partial_t u(t, x) + \mathcal{A}(t)u(t, x) + b(t, x, u(t, x)) = 0 & \text{in } Q, \\ u = f_0 + \displaystyle\sum_{i=1}^{m+1} s_i h_i & \text{on } \Sigma, \\ u(0, x) = 0 & \text{for } x \in \Omega. \end{cases} \tag{3.6}$$

Following the proof of Proposition 3.1, we know that the map $s \mapsto u_s$ is lying in

$$C^\infty\left((-1,1)^{m+1}; C^{1+\frac{\alpha}{2},2+\alpha}([0,T] \times \overline{\Omega})\right).$$

Then we are able to differentiate (3.6) with respect to the s parameter.

Let us introduce the solution of the first linearized problem

$$\begin{cases} \rho(t,x)\partial_t v + \mathcal{A}(t)v + \partial_\mu b(t,x,u_0)v = 0 & \text{in } Q, \\ v = h & \text{on } \Sigma, \\ v(0,x) = 0 & \text{for } x \in \Omega. \end{cases} \tag{3.7}$$

Using the facts that $u_s|_{s=0} = u_0$ and that the map $s \mapsto u_s$ is smooth, we see that if v_ℓ is the solution of (3.7) with $h = h_\ell$, $\ell = 1, \ldots, m+1$, then we have

$$\partial_{s_\ell} u_s\big|_{s=0} = v_\ell, \quad \ell = 1, \ldots, m+1, \tag{3.8}$$

in the sense of functions taking values in $C^{1+\frac{\alpha}{2},2+\alpha}([0,T] \times \overline{\Omega})$.

Now let us turn to the expression $\partial_{s_{\ell_1}} \partial_{s_{\ell_2}} u_s\big|_{s=0}$, $\ell_1, \ell_2 = 1, \ldots, m+1$. For this purpose, we introduce the function $w_{\ell_1,\ell_2} \in C^{1+\frac{\alpha}{2},2+\alpha}([0,T] \times \overline{\Omega})$ solving the second linearized problem

$$\begin{cases} \rho(t,x)\partial_t w_{\ell_1,\ell_2} + \mathcal{A}(t)w_{\ell_1,\ell_2} + \partial_\mu b(t,x,u_0)w_{\ell_1,\ell_2} = -\partial_\mu^2 b(t,x,u_0)v_{\ell_1}v_{\ell_2} & \text{in } Q, \\ w_{\ell_1,\ell_2} = 0 & \text{on } \Sigma, \\ w_{\ell_1,\ell_2}(0,x) = 0 & \text{for } x \in \Omega. \end{cases} \tag{3.9}$$

Repeating the above arguments, we obtain that

$$\partial_{s_{\ell_1}} \partial_{s_{\ell_2}} u_s\big|_{s=0} = w_{\ell_1,\ell_2} \tag{3.10}$$

is the solution to (3.9). Then, by iterating the above arguments, one has the following result.

Lemma 3.1 (Higher-order linearizations). *Let $m \in \mathbb{N}$. The function*

$$w^{(m+1)} = \partial_{s_1} \partial_{s_2} \cdots \partial_{s_{m+1}} u_s\big|_{s=0} \tag{3.11}$$

is well defined in the sense of functions taking values in $C^{1+\frac{\alpha}{2},2+\alpha}([0,T] \times \overline{\Omega})$. Moreover, $w^{(m+1)}$ solves the $(m+1)$-th order linearized problem

$$\begin{cases} \rho(t,x)\partial_t w^{(m+1)} + \mathcal{A}(t)w^{(m+1)} + \partial_\mu b(t,x,u_0)w^{(m+1)} = H^{(m+1)} & \text{in } Q, \\ w^{(m+1)} = 0 & \text{on } \Sigma, \\ w^{(m+1)}(0,x) = 0 & \text{for } x \in \Omega. \end{cases} \tag{3.12}$$

Here, we have

$$H^{(m+1)} = -\partial_\mu^{m+1} b(t,x,u_0)v_1 \cdots v_{m+1} + K^{(m+1)}, \tag{3.13}$$

where all the functions are evaluated at the point (t,x) and $K^{(m+1)}(t,x)$ depends only on a, ρ, Ω, T, $\partial_\mu^k b(t,x,u_0)$, $k = 0, \ldots, m$, $v_1, \ldots, v_{m+1}$, and $w^{(k+1)}$, for $k = 1, \ldots, m-1$. Here, $v_1, \ldots, v_{m+1}$ are the solutions of (3.7) with $h = h_\ell$, $\ell = 1, \ldots, m+1$.

4. Proof of Theorem 2.1

In this section, we will prove Theorem 2.1 by admitting the proof of the following denseness result.

Proposition 4.1. *Adopting all the conditions of Theorem 2.1, consider $F \in C([0,T] \times \overline{\Omega})$, $m \in \mathbb{N}$ and $q_0, \ldots, q_m \in C^{\frac{\alpha}{2}, \alpha}([0,T] \times \overline{\Omega})$. Assume that the identity*

$$\int_0^T \int_\Omega F v_1 \cdots v_m w \, dx dt = 0 \tag{4.1}$$

holds true for all $v_j \in C^{1+\frac{\alpha}{2}, 2+\alpha}([0,T] \times \overline{\Omega}))$ and $w \in C^{1+\frac{\alpha}{2}, 2+\alpha}([0,T] \times \overline{\Omega}))$ solving the following equations

$$\begin{cases} \rho(t,x)\partial_t v_j + \mathcal{A}(t)v_j + q_j v_j = 0, & \text{in } Q, \\ v_j(0,x) = 0 & \text{for } x \in \Omega, \end{cases} \tag{4.2}$$

for $j = 1, \ldots, m$, and

$$\begin{cases} -\partial_t(\rho(t,x)w) + \mathcal{A}(t)w + q_0 w = 0, & \text{in } Q, \\ w(T,x) = 0 & \text{for } x \in \Omega, \end{cases} \tag{4.3}$$

respectively. Then $F \equiv 0$.

We next use the above proposition to show Theorem 2.1. The proof of the proposition will be postponed to Section 5.

Proof of Theorem 2.1. We will prove this theorem in two steps. We will start by proving that the assumption

$$\mathcal{N}_{b_1}(f) = \mathcal{N}_{b_2}(f), \text{ for all } f \in \mathbb{B}(f_0, \epsilon)$$

implies that

$$\partial_\mu^k b_1(t, x, u_{1,0}(t,x)) = \partial_\mu^k b_2(t, x, u_{2,0}(t,x)), \quad (t,x) \in [0,T] \times \overline{\Omega} \tag{4.4}$$

holds true for all $k \in \mathbb{N}$, with $u_{j,0}$ being the solution of (1.2) with $b = b_j$ and $f = f_0$, for $j = 1, 2$. Then we will complete the proof by showing that (4.4) implies the claim

$$b_1 = S_\varphi b_2$$

for some $\varphi \in C^{1+\frac{\alpha}{2}, 2+\alpha}([0,T] \times \overline{\Omega})$ satisfying the condition (1.6).

Step 1. Determination of the Taylor coefficients

We will show (4.4) holds true, for all $k \in \mathbb{N}$, by recursion. Note first that, for $j = 1, 2$, the function $(t,x) \mapsto \partial_\mu b_j(t, x, u_{j,0}(t,x))$ belongs to $C^{\frac{\alpha}{2}, \alpha}([0,T] \times \overline{\Omega})$. Thus, for $j = 1, 2$, we can consider $v_{j,1} \in C^{1+\frac{\alpha}{2}, 2+\alpha}([0,T] \times \overline{\Omega}))$ satisfying (4.2) with $q_j(t,x) = \partial_\mu b_j(t, x, u_{j,0}(t,x))$ and $w \in C^{1+\frac{\alpha}{2}, 2+\alpha}([0,T] \times \overline{\Omega}))$ satisfying (4.3) with $q_0(t,x) = \partial_\mu b_1(t, x, u_{1,0}(t,x))$. We assume here that $v_{1,1}|_\Sigma = h = v_{2,1}|_\Sigma$ for some $h \in \mathbb{B}(0,1)$. Applying the first-order linearization, we find

$$\partial_{\nu(a)} v_{1,1}|_\Sigma = \partial_s \mathcal{N}_{b_1}(f_0 + sh) = \partial_s \mathcal{N}_{b_2}(f_0 + sh) = \partial_{\nu(a)} v_{2,1}|_\Sigma.$$

Thus, fixing $v_1 = v_{1,1} - v_{2,1}$, we deduce that v_1 satisfies the conditions

$$
\begin{cases}
\rho(t,x)\partial_t v_1 + \mathcal{A}(t)v_1 + \partial_\mu b_1(t,x,u_{1,0}(t,x))v_1 = G(t,x) & \text{in } Q, \\
v_1 = \partial_{v_a} v_1 = 0 & \text{on } \Sigma, \\
v(0,x) = 0 & \text{for } x \in \Omega,
\end{cases}
\tag{4.5}
$$

where

$$
G(t,x) = \big(\partial_\mu b_2(t,x,u_{2,0}(t,x)) - \partial_\mu b_1(t,x,u_{1,0}(t,x))\big)v_{2,1}(t,x), \quad (t,x) \in Q.
$$

Multiplying the equation (4.5) by w and integrating by parts, we obtain the identity

$$
\int_0^T \int_\Omega \big(\partial_\mu b_2(t,x,u_{2,0}) - \partial_\mu b_1(t,x,u_{1,0})\big)v_{2,1}w \, dxdt
$$

$$
= \int_0^T \int_\Omega \big(\rho(t,x)\partial_t v_1 + \mathcal{A}(t)v_1 + \partial_\mu b_1(t,x,u_{1,0}(t,x))v_1\big)w \, dxdt
$$

$$
= \underbrace{\int_0^T \int_\Omega \big(-\partial_t(\rho w) + \mathcal{A}(t)w + \partial_\mu b_1(t,x,u_{1,0}(t,x))w\big)v_1 \, dxdt}_{\text{Since } v_1 = \partial_{v(a)} v_1 = 0 \text{ on } \Sigma,\ v_1(0,x) = 0 \text{ and } w(T,x) = 0 \text{ in } \Omega.}
$$

$$
= 0.
$$

Using the the fact that $v_{2,1}$ can be seen as an arbitrary chosen element of $C^{1+\frac{\alpha}{2},2+\alpha}([0,T]\times\overline{\Omega}))$ satisfying (4.2) and applying Proposition 4.1, we deduce that (4.4) holds true for $k = 1$. Moreover, by the unique solvability of (3.7), we deduce that $v_{1,1} = v_{2,1}$ in Q.

Now, let us fix $m \in \mathbb{N}$ and assume that, for $k = 1, \ldots, m$, (4.4) holds true and

$$
w_1^{(k)} = w_2^{(k)} \text{ in } Q.
\tag{4.6}
$$

Consider $v_j \in C^{1+\frac{\alpha}{2},2+\alpha}([0,T]\times\overline{\Omega}))$ satisfying (4.2) with $q_j(t,x) = \partial_\mu b_1(t,x,u_{1,0}(t,x))$ for $j = 1,\ldots,m+1$, and $w \in C^{1+\frac{\alpha}{2},2+\alpha}([0,T]\times\overline{\Omega}))$ satisfying (4.2) with $q_0(t,x) = \partial_\mu b_1(t,x,u_{1,0}(t,x))$. We fix $h_j = v_j|_\Sigma$, $j = 1,\ldots,m+1$, and proceeding to the higher-order linearization described in Lemma 3.1, we obtain

$$
\partial_{v(a)} w_j^{(m+1)}\Big|_\Sigma = \partial_{s_1} \ldots \partial_{s_{m+1}} \mathcal{N}_{b_j}(f_0 + s_1 h_1 + \ldots + s_{m+1}h_{m+1})\big|_{s=0},
$$

with $s = (s_1,\ldots,s_{m+1})$ and $w_j^{(m+1)}$ solving (3.12) as $b = b_j$, for $j = 1,2$. Then the condition (2.4) implies

$$
\partial_{v(a)} w_1^{(m+1)}\Big|_\Sigma = \partial_{v(a)} w_2^{(m+1)}\Big|_\Sigma.
$$

Fixing $w^{(m+1)} = w_1^{(m+1)} - w_2^{(m+1)}$ and applying Lemma 3.1, we deduce that $w^{(m+1)}$ satisfies the condition

$$
\begin{cases}
\rho(t,x)\partial_t w^{(m+1)} + \mathcal{A}(t)w^{(m+1)} + \partial_\mu b_1(t,x,u_{1,0}(t,x))w^{(m+1)} = \mathcal{K} & \text{in } Q, \\
w^{(m+1)} = \partial_{v_a} w^{(m+1)} = 0 & \text{on } \Sigma, \\
w^{(m+1)}(0,x) = 0 & \text{for } x \in \Omega,
\end{cases}
\tag{4.7}
$$

where $\mathcal{K} = (\partial_\mu^{m+1} b_2(t,x,u_{2,0}) - \partial_\mu^{m+1} b_1(t,x,u_{1,0}))v_1 \cdots v_{m+1}$. Here, we used the assumption for this recursion argument that (4.4) and (4.6) hold true for $k = 1, \ldots, m$. Multiplying the equation (4.7) by w

and integrating by parts, we obtain

$$\int_0^T \int_\Omega \left(\partial_\mu^{m+1} b_2(t,x,u_{2,0}) - \partial_\mu^{m+1} b_1(t,x,u_{1,0}) \right) v_1 \cdots v_{m+1} w \, dx dt = 0.$$

Applying again Proposition 4.1, we find that (4.4) holds true for $k = 1, \ldots, m+1$. By unique solvability of (3.7), we also have $w_1^{(m+1)} = w_2^{(m+1)}$ in Q. It follows that (4.4) holds true for all $k \in \mathbb{N}$.

Step 2. Gauge invariance.

In this step, we will show that (2.5) holds with some $\varphi \in C^{1+\frac{\alpha}{2}, 2+\alpha}([0,T] \times \overline{\Omega})$ satisfying (1.6). We will choose here $\varphi = u_{2,0} - u_{1,0}$ and, thanks to (2.4), we know that φ fulfills condition (1.6). We fix $(t,x) \in [0,T] \times \overline{\Omega}$ and consider the map

$$G_j : \mathbb{R} \ni \mu \mapsto b_j(t,x,u_{j,0}(t,x) + \mu) - b_j(t,x,u_{j,0}(t,x)), \quad j = 1,2.$$

For $j = 1, 2$, using the fact that $b_j \in \mathbb{A}(\mathbb{R}; C^{\frac{\alpha}{2}, \alpha}([0,T] \times \overline{\Omega}))$, we deduce that the map $G = G_1 - G_2$ is analytic with respect to $\mu \in \mathbb{R}$. It is clear that

$$G(0) = G_1(0) - G_2(0) = 0 - 0 = 0.$$

Moreover, (4.4) implies that

$$G^{(k)}(0) = \partial_\mu^k b_1(t,x,u_{1,0}(t,x)) - \partial_\mu^k b_2(t,x,u_{2,0}(t,x)) = 0, \quad k \in \mathbb{N}.$$

Combining this with the fact that G is analytic with respect to $\mu \in \mathbb{R}$, we deduce that there must exist $\delta > 0$ such that

$$G(\mu) = 0, \quad \mu \in (-\delta, \delta).$$

Then, the unique continuation of analytic functions implies that $G \equiv 0$. It follows that, for all $\mu \in \mathbb{R}$, we have

$$b_1(t,x,u_{1,0}(t,x) + \mu) - b_1(t,x,u_{1,0}(t,x)) = b_2(t,x,u_{2,0}(t,x) + \mu) - b_2(t,x,u_{2,0}(t,x)).$$

Recalling that

$$-b_j(t,x,u_{j,0}(t,x)) = \rho(t,x)\partial_t u_{j,0}(t,x) + \mathcal{A}(t)u_{j,0}(t,x), \quad j = 1,2,$$

we deduce that

$$
\begin{aligned}
& b_1(t,x,u_{1,0}(t,x) + \mu) \\
& = b_2(t,x,u_{2,0}(t,x) + \mu) + b_1(t,x,u_{1,0}(t,x)) - b_2(t,x,u_{2,0}(t,x)) \\
& = b_2(t,x,u_{2,0}(t,x) + \mu) + \rho(t,x)\partial_t(u_{2,0} - u_{1,0})(t,x) + \mathcal{A}(t)(u_{2,0} - u_{1,0})(t,x) \\
& = b_2(t,x,u_{2,0}(t,x) + \mu) + \rho(t,x)\partial_t\varphi(t,x) + \mathcal{A}(t)\varphi(t,x), \quad \mu \in \mathbb{R}.
\end{aligned}
\tag{4.8}
$$

Considering (4.8) with $\mu_1 = u_{1,0}(t,x) + \mu$, we obtain

$$
\begin{aligned}
b_1(t,x,\mu_1) & = b_2(t,x,u_{2,0}(t,x) - u_{1,0}(t,x) + \mu_1) + \rho(t,x)\partial_t\varphi(t,x) + \mathcal{A}(t)\varphi(t,x) \\
& = b_2(t,x,\varphi(t,x) + \mu_1) + \rho(t,x)\partial_t\varphi(t,x) + \mathcal{A}(t)\varphi(t,x), \quad \mu_1 \in \mathbb{R}.
\end{aligned}
$$

Using the fact that here $(t,x) \in [0,T] \times \overline{\Omega}$ is arbitrary chosen, we deduce that (2.5) holds true with $\varphi = u_{2,0} - u_{1,0}$. This completes the proof of the theorem. $\qquad \square$

5. Proof of Proposition 4.1

In order to prove Proposition 4.1, we need to construct special solutions, which helps us to prove the completeness of products of solutions.

5.1. *Constructions of geometric optics solutions*

For the proof of Proposition 4.1, we will need to consider the construction of *geometrical optics* (GO in short) solutions. More precisely, fixing $q \in C^{\frac{\alpha}{2}, \alpha}([0, T] \times \overline{\Omega})$, we will consider GO solutions to the equation

$$\begin{cases} \rho(t, x)\partial_t v + \mathcal{A}(t)v + qv = 0 & \text{in } Q, \\ v(0, x) = 0 & \text{for } x \in \Omega, \end{cases} \tag{5.1}$$

as well as GO solutions for the formal adjoint equation

$$\begin{cases} -\rho(t, x)\partial_t w + \mathcal{A}(t)w - \partial_t \rho(t, x)w + qw = 0 & \text{in } Q, \\ w(T, x) = 0 & \text{for } x \in \Omega, \end{cases} \tag{5.2}$$

belonging to the space $C^{1+\frac{\alpha}{2}, 2+\alpha}([0, T] \times \overline{\Omega})$. Following the recent construction of [Fei23], in this section, we give a construction of these GO solutions that will depend on a large asymptotic positive parameter τ with $\tau \gg 1$ and concentrate on geodesics with respect to the metric $g(t) = \rho(t, \cdot)a(t, \cdot)^{-1}$ in $\overline{\Omega_1}$ that passes through a point $x_0 \in \partial\Omega_1$, whenever Assumption 2.1 holds. Here, Ω_1 is a domain satisfying Assumption 2.1 containing $\overline{\Omega}$. Let us construct the GO solution as follows.

First, we consider solutions of the form

$$v(t, x) = e^{\tau^2 t + \tau \psi(t, x)} \big[c_+(t, x) + R_{+, \tau}(t, x) \big], \quad (t, x) \in Q, \tag{5.3}$$

and

$$w(t, x) = e^{-\tau^2 t - \tau \psi(t, x)} \big[c_-(t, x) + R_{-, \tau}(t, x) \big], \quad (t, x) \in Q, \tag{5.4}$$

to equations (5.1) and (5.2), respectively. The phase functions and principal terms $c_{\pm}$ of the GOs will be constructed by using polar normal coordinate on the manifold $(\overline{\Omega_1}, g(t))$, for $t \in [0, T]$.

Let us define the differential operators $L_{\pm}$, $P_{\tau, \pm}$ on Ω_1 by

$$\begin{aligned} L_+ &= \rho(t, x)\partial_t + \mathcal{A}(t) + q(t, x), \\ L_- &= -\rho(t, x)\partial_t + \mathcal{A}(t) - \partial_t \rho(t, x) + q(t, x), \end{aligned} \tag{5.5}$$

and

$$P_{\tau, \pm} = e^{\mp(\tau^2 t + \tau \psi(t, x))} L_{\pm} \Big(e^{\pm(\tau^2 t + \tau \psi(t, x))} \Big). \tag{5.6}$$

Via a straightforward computation, we can write

$$P_{\tau, \pm} v = \tau^2 \mathcal{I} v + \tau \mathcal{J}_{\pm} v + L_{\pm} v,$$

where

$$\mathcal{I} = \rho(t, x) - \sum_{i,k=1}^{n} a_{ik}(t, x)\partial_{x_i}\psi\partial_{x_k}\psi, \tag{5.7}$$

and

$$\mathcal{J}_{\pm}v := \mp 2 \sum_{i,k=1}^{n} a_{ik}(t,x)\partial_{x_i}\psi\partial_{x_k}v + [\rho(t,x)\partial_t\psi \pm \mathcal{A}(t)\psi]v. \tag{5.8}$$

Next, we want to choose ψ in such a way that the eikonal equation $\mathcal{I} = 0$ is satisfied in Q. Hence, after choosing ψ, we seek for $c_{\pm}$ solving the transport equations

$$\mathcal{J}_+ c_+ = 0 \quad \text{and} \quad \mathcal{J}_- c_- = 0. \tag{5.9}$$

Since for all $t \in [0,T]$, the Riemannian manifold $(\overline{\Omega_1}, g(t))$ is assumed to be simple, the eikonal equation $\mathcal{I} = 0$ can be solved globally on $\overline{Q}$. This is known, but let us show how it is done. For this, let us fix $x_0 \in \partial\Omega_1$ and consider the polar normal coordinates (r, θ) on $(\overline{\Omega_1}, g(t))$ given by $x = \exp_{x_0}(r\theta)$, where $r > 0$ and

$$\theta \in S_{x_0,t}(\overline{\Omega_1}) := \left\{ v \in \mathbb{R}^n : |v|_{g(t)[x_0]} = 1 \right\}.$$

According to the Gauss lemma, in these coordinates, the metric takes the form

$$dr^2 + g_0(t, r, \theta),$$

where $g_0(t, r, \theta)$ is a metric defined on $S_{x_0,t}(\overline{\Omega_1})$, which depends smoothly on t and r. In fact, we choose

$$\psi(t,x) = \operatorname{dist}_{g(t)}(x_0, x), \quad (t,x) \in [0,T] \times \overline{\Omega}, \tag{5.10}$$

where $\operatorname{dist}_{g(t)}(\cdot, \cdot)$ is the Riemannian distance function associated with the metric $g(t)$, for $t \in [0,T]$. As ψ is given by r in the polar normal coordinates, one can easily check that ψ solves $\mathcal{I} = 0$.

Let us now turn to the transport equations (5.9). We write $c_{\pm}(t, r, \theta) = c_{\pm}(t, \exp_{x_0}(r\theta))$ and use this notation to indicate the representation in the polar normal coordinates also for other functions. Then, using this notation and following [Fei23, Section 5.1.2], we deduce that, in these polar normal coordinates with respect to $x_0 \in \partial\Omega_1$, the equations in (5.9) become

$$\partial_r c_{\pm} + \left(\frac{\partial_r \beta}{4\beta}\right) c_{\pm} \mp \frac{\partial_t \psi(t, r, \theta)}{2} c_{\pm} = 0 \tag{5.11}$$

with $\beta(t, r, \theta) = \det(g_0(t, r, \theta))$. Note that in this equation, there is no differentiation in the θ-variable. This fact will allow us to localize GO solutions near geodesics. We fix

$$r_0 = \inf_{t \in [0,T]} \operatorname{dist}_{g(t)}(\partial\Omega_1, \overline{\Omega})$$

and recall that $r_0 > 0$. For any $h \in C^{\infty}(S_{x_0,t}(\overline{\Omega_1}))$ and $\chi \in C_0^{\infty}(0,T)$, the functions

$$c_+(t, r, \theta) = \chi(t)h(\theta)\beta(t, r, \theta)^{-1/4} \exp\left(\int_{r_0}^{r} \frac{\partial_t \psi(t, s, \theta)}{2} ds\right), \tag{5.12}$$

$$c_-(t, r, \theta) = \chi(t)\beta(t, r, \theta)^{-1/4} \exp\left(-\int_{r_0}^{r} \frac{\partial_t \psi(t, s, \theta)}{2} ds\right) \tag{5.13}$$

are respectively solutions of the transport equations (5.9). Moreover, the regularity of the coefficients ρ, a and the simplicity of the manifold $(\overline{\Omega_1}, g(t))$ implies that solutions of the transport equation (5.11) $c_{\pm} \in C^{\infty}([0,T] \times \overline{\Omega})$.

In order to complete our construction of GO solutions, we need to show that it is possible to construct the remainder terms $R_{\pm,\tau} \in C^{1+\frac{\alpha}{2},2+\alpha}([0,T] \times \overline{\Omega})$ satisfying the decay property

$$\left\| R_{\pm,\tau} \right\|_{L^2(Q)} \leqslant C\, |\tau|^{-1}, \tag{5.14}$$

for some constant $C > 0$ independent of τ, positive and large enough, as well as the initial and final condition

$$R_{+,\tau}(0,x) = R_{-,\tau}(T,x) = 0, \quad x \in \Omega.$$

For this purpose, we recall that for ψ given by (5.10), we have $P_{\tau,\pm} = L_\pm + \tau \mathcal{J}_\pm$ with $L_\pm$ and $\mathcal{J}_\pm$ defined by (5.5)–(5.8). Then, according to (5.9), we have

$$L_\pm \left[e^{\tau^2 t + \tau \psi(t,x)} c_\pm(t,x) \right] = e^{\tau^2 t + \tau \psi(t,x)} P_{\tau,\pm} c_\pm(t,x)$$
$$= e^{\tau^2 t + \tau \psi(t,x)} L_\pm c_\pm.$$

Therefore, the conditions $L_+ v = 0$ and $L_- w = 0$ are fulfilled if and only if $R_{\pm,\tau}$ solves

$$P_{\tau,\pm} R_{\pm,\tau}(t,x) = -L_\pm c_\pm(t,x), \quad (t,x) \in (0,T) \times \Omega.$$

We will choose $R_{\pm,\tau}$ to be the solution of the IBVP

$$\begin{cases} P_{\tau,+} R_{+,\tau}(t,x) = -L_+ c_+(t,x) & \text{in } Q, \\ R_{+,\tau}(t,x) = 0 & \text{on } \Sigma, \\ R_{+,\tau}(0,x) = 0 & \text{for } x \in \Omega \end{cases} \tag{5.15}$$

and

$$\begin{cases} P_{\tau,-} R_{-,\tau}(t,x) = -L_- c_-(t,x) & \text{in } Q, \\ R_{-,\tau}(t,x) = 0 & \text{on } \Sigma, \\ R_{-,\tau}(T,x) = 0 & \text{for } x \in \Omega. \end{cases} \tag{5.16}$$

Note that $L_\pm c_\pm$ is independent of τ. We give the following extension of the energy estimate approach under consideration [Fei23] for problem (5.15)–(5.16).

Proposition 5.1. *There exists $\tau_0 > 0$, depending only on Ω, T, a, ρ, q, such that problem (5.15)–(5.16) admits a unique solution $R_{\pm,\tau} \in C^{1+\frac{\alpha}{2},2+\alpha}([0,T] \times \overline{\Omega})$ satisfying the estimate*

$$\tau \left\| R_{\pm,\tau} \right\|_{L^2(Q)} + \tau^{\frac{1}{2}} \left\| R_{\pm,\tau} \right\|_{L^2(0,T;H^1(\Omega))} \leqslant C \| L_\pm c_\pm \|_{L^2(Q)}, \quad \tau > \tau_0. \tag{5.17}$$

Proof. The proof of this proposition is based on arguments similar to [Fei23, Proposition 4.1] that we adapt to problem (5.15)–(5.16) whose equations are more general than the ones under consideration in [Fei23]. For this reason and for sake of completeness, we give the full proof of this proposition. We only show the result for $R_{+,\tau}$, the same property for $R_{-,\tau}$ can be deduced by applying similar arguments. Let us first observe that

$$P_{\tau,+} = \rho(t,x)\partial_t + \mathcal{A}(t) + B(t,x) \cdot \nabla_x + V(t,x)$$

with

$$B(t,x) \cdot \nabla_x v(t,x) = -2\tau \sum_{i,k=1}^n a_{ik}(t,x) \partial_{x_i} \psi \partial_{x_k} v(t,x),$$

$$V(t,x) = q(t,x) + \tau \rho(t,x) \partial_t \psi + \tau \mathcal{A}(t)\psi.$$

Thus, observing that $B \in C^\infty([0,T] \times \overline{\Omega})^n$, $V \in C^{\frac{\alpha}{2}, \alpha}([0,T] \times \overline{\Omega})$, $L_+ c_+ \in C_0^\infty(0,T; C^\infty(\overline{\Omega}))$ and applying [LSU88, Theorem 5.2, Chapter IV, page 320], we deduce that (5.15) admits a unique solution $R_{+,\tau} \in C^{1+\frac{\alpha}{2}, 2+\alpha}([0,T] \times \overline{\Omega})$, and we only need to check estimate (5.17).

In this proof, we set

$$v = R_{+,\tau}, \quad K = -L_+ c_+,$$

and without loss of generality, we assume that both v and K are real valued. We fix $\lambda > 0$, and we multiply (5.15) by $ve^{\lambda\psi}$ in order to get

$$\int_Q (-L_+ c_+)e^{\lambda\psi} \, dxdt = \int_Q Kve^{\lambda\psi} \, dxdt = \int_Q (\rho \partial_t v + \mathcal{A}(t)v + \tau \, \mathcal{J}_+ v + qv)ve^{\lambda\psi} \, dxdt$$

$$:= I + II + III,$$

where

$$I = \int_Q (\rho \partial_t v + qv + \tau \rho(t,x) \partial_t \psi v) v e^{\lambda\psi} \, dxdt,$$

$$II = \int_Q (\mathcal{A}(t)v) v e^{\lambda\psi} \, dxdt,$$

$$III = \tau \int_Q \left(-2 \sum_{i,k=1}^n a_{ij}(t,x) \partial_{x_i} \psi \partial_{x_k} v + \mathcal{A}(t)\psi v \right) v e^{\lambda\psi} \, dxdt.$$

For I, using the fact that $v_{|t=0} = 0$ and integrating by parts, we get

$$I = \frac{1}{2} \int_\Omega \rho(T,x)v(T,x)^2 e^{\lambda\psi(T,x)} \, dx - \frac{1}{2} \int_Q \partial_t(\rho e^{\lambda\psi})v^2 \, dxdt + \int_Q (qv + \tau \rho(t,x) \partial_t \psi v)ve^{\lambda\psi} \, dxdt$$

$$\geq -\frac{1}{2} \int_Q \partial_t \left(\rho e^{\lambda\psi} \right)v^2 \, dxdt + \int_Q (qv + \tau \rho(t,x) \partial_t \psi v)ve^{\lambda\psi} \, dxdt$$

$$\geq -(C_1 + C_2\tau + C_3\lambda) \int_Q v^2 e^{\lambda\psi} \, dxdt,$$

with C_1, C_2, C_3 three positive constants independent of λ and τ.

For II, using the fact that $v|_\Sigma = 0$, applying (1.1) and integrating by parts, we find

$$II = \int_Q \left(\sum_{i,k=1}^n a_{ik}(t,x) \partial_{x_i} v \partial_{x_k} v \right) e^{\lambda\psi} \, dxdt + \frac{1}{2} \int_Q \sum_{i,k=1}^n a_{ik}(t,x) \partial_{x_i}(v^2) \partial_{x_k} \left(e^{\lambda\psi} \right) dxdt$$

$$\geq c \int_Q |\nabla_x v|^2 e^{\lambda\psi} \, dxdt - \frac{1}{2} \int_Q v^2 \mathcal{A}(t) \left(e^{\lambda\psi} \right) dxdt$$

$$\geq c \int_Q |\nabla_x v|^2 e^{\lambda\psi} \, dxdt - \left(C_4\lambda + C_5\lambda^2 \right) \int_Q v^2 e^{\lambda\psi} \, dxdt,$$

with constants $c, C_4, C_5 > 0$ independent of λ and τ.

Finally, for *III*, using the that $\mathcal{I} = 0$, with $\mathcal{I}$ defined by (5.7), we find

$$
\begin{aligned}
III &= -\tau \int_Q \sum_{i,k=1}^n a_{ik}(t,x)\partial_{x_i}\psi \partial_{x_k}(v^2)e^{\lambda\psi}\,dxdt + \tau \int_Q (\mathcal{A}(t)\psi)v^2 e^{\lambda\psi}\,dxdt \\
&= \tau \int_Q \sum_{i,k=1}^n a_{ik}(t,x)\partial_{x_i}\psi \partial_{x_k}\left(e^{\lambda\psi}\right)v^2\,dxdt \\
&= \tau\lambda \int_Q \left(\sum_{i,k=1}^n a_{ik}(t,x)\partial_{x_i}\psi \partial_{x_k}\psi\right)v^2 e^{\lambda\psi}\,dxdt \\
&= \tau\lambda \int_Q \rho(t,x)v^2 e^{\lambda\psi}\,dxdt \\
&\geqslant \left(\inf_{(t,x)\in Q}\rho(t,x)\right)\tau\lambda \int_Q v^2 e^{\lambda\psi}\,dxdt \\
&:= C_6\tau\lambda \int_Q v^2 e^{\lambda\psi}\,dxdt.
\end{aligned}
$$

Combining these estimates of *I*, *II* and *III*, we find

$$
\int_Q Kve^{\lambda\psi}\,dxdt \geqslant c\int_Q |\nabla_x v|^2 e^{\lambda\psi}\,dxdt + \left(-C_1 - C_2\tau - C_3\lambda - C_4\lambda - C_5\lambda^2 + C_6\tau\lambda\right)\int_Q v^2 e^{\lambda\psi}\,dxdt.
$$

Choosing $\lambda \geqslant \frac{3C_2}{C_6}$ and

$$
\tau_0 = \frac{3\left(\frac{C_1}{\lambda} + C_3 + C_4 + C_5\lambda\right)}{C_6},
$$

we deduce that

$$
\int_Q Kve^{\lambda\psi}\,dxdt \geqslant c\int_Q |\nabla_x v|^2 e^{\lambda\psi}\,dxdt + \frac{C_6}{3}\tau\lambda \int_Q v^2 e^{\lambda\psi}\,dxdt, \quad \tau > \tau_0.
$$

Applying Cauchy-Schwarz inequality, for $\tau > \tau_0$, we get

$$
\begin{aligned}
c\int_Q |\nabla_x v|^2 e^{\lambda\psi}\,dxdt + \frac{C_6}{3}\tau\lambda \int_Q v^2 e^{\lambda\psi}\,dxdt &\leqslant \left(\tau^{-1}\int_Q K^2 e^{\lambda\psi}\,dxdt\right)^{\frac{1}{2}}\left(\tau\int_Q v^2 e^{\lambda\psi}\,dxdt\right)^{\frac{1}{2}} \\
&\leqslant \frac{\tau^{-1}}{2}\int_Q K^2 e^{\lambda\psi}\,dxdt + \frac{\tau}{2}\int_Q v^2 e^{\lambda\psi}\,dxdt,
\end{aligned}
$$

which implies that

$$
c\int_Q |\nabla_x v|^2 e^{\lambda\psi}\,dxdt + \tau\left(\frac{C_6}{3}\lambda - \frac{1}{2}\right)\int_Q v^2 e^{\lambda\psi}\,dxdt \leqslant \frac{\tau^{-1}}{2}\int_Q K^2 e^{\lambda\psi}\,dxdt.
$$

Fixing $\lambda = \frac{3(C_2+1)}{C_6}$, we obtain

$$
\tau\int_Q |\nabla_x v|^2\,dxdt + \tau^2 \int_Q v^2\,dxdt \leqslant C\int_Q K^2\,dxdt, \quad \tau > \tau_0,
$$

where $C > 0$ is a constant independent of τ. From this last estimate, we deduce (5.17). □

Note that the the energy estimate (5.17) is only subjected to the requirement that ψ solves the eikonal equation $\mathcal{I} = 0$ in Q. For this result, the simplicity assumption is not required.

Applying Proposition 5.1, we deduce the existence of $R_{\pm,\tau}$ fulfilling condition (5.15)–(5.16) and the decay estimate (5.14). Armed with these class of GO solutions, we are now in position to complete the proof of Proposition 4.1.

5.2. *Completion of the proof of Proposition 4.1*

We will show Proposition 4.1 by iteration.

Proof of Proposition 4.1. We start by showing that the claim of Proposition 4.1 holds true for $m = 1$. We fix $x_0 \in \partial\Omega_1$, $t_0 \in (0, T)$ and for $\chi_* \in C_0^\infty(-1, 1)$ satisfying

$$\int_{\mathbb{R}} \chi_*(t)^2 \, dt = 1,$$

and we set

$$\chi_\delta(t) = \delta^{-\frac{1}{2}} \chi_* \left(\delta^{-1}(t - t_0) \right), \quad \text{for } \delta \in (0, \min(T - t_0, t_0)).$$

We consider $v_1 \in C^{1+\frac{\alpha}{2},2+\alpha}([0,T] \times \overline{\Omega})$ (resp. $w \in C^{1+\frac{\alpha}{2},2+\alpha}([0,T] \times \overline{\Omega})$) of the form (5.3) (resp. (5.4)) satisfying (5.1) (resp. (5.2)) with $q = q_1$ (resp. $q = q_0$) and $R_{+,\tau}$ (resp. $R_{-,\tau}$) satisfying the decay property (5.14). Here, we choose $\chi = \chi_\delta$ in the expression of the function v_1 and w. Then, condition (4.1) implies that

$$\int_0^T \int_\Omega F(t,x)c_{+,0}c_{-,0} \, dxdt = \lim_{\tau \to +\infty} \int_0^T \int_\Omega F(t,x)v_1 w \, dxdt = 0. \tag{5.18}$$

From now on, for $t \in [0, T]$, we denote by $\partial_+ S_t(\overline{\Omega_1})$ the unit sphere bundle

$$\partial_+ S_t(\overline{\Omega_1}) := \left\{ (x, \theta) \in S_t(\overline{\Omega_1}) : x \in \partial\Omega_1, \ \langle \theta, \nu_t(x) \rangle_{g(t)} < 0 \right\},$$

where ν_t denotes the outward unit normal vector of $\partial\Omega_1$ with respect to the metric $g(t)$. We also denote for any $(y, \theta) \in \partial_+ S_t(\overline{\Omega_1})$ by $\ell_{t,+}(y, \theta)$ the time of existence in $\overline{\Omega_1}$ of the maximal geodesic $\gamma_{y,\theta}$, with respect to the metric $g(t)$, satisfying $\gamma_{y,\theta}(0) = y$ and $\gamma'_{y,\theta}(0) = \theta$.

Consider $\tilde{F} \in L^\infty((0, T) \times \Omega_1)$ defined by

$$\tilde{F}(t, x) = \begin{cases} (\det(g(t)))^{-\frac{1}{2}} F(t, x), & \text{for } (t, x) \in Q \\ 0, & \text{for } (t, x) \in (0, T) \times (\Omega_1 \setminus \Omega) \end{cases}.$$

Then we have

$$\int_0^T \int_{\overline{\Omega_1}} \tilde{F}(t, x)c_{+,0}c_{-,0} \, dV_{g(t)}(x)dt = \int_0^T \int_{\Omega_1} \tilde{F}(t, x)c_{+,0}c_{-,0} \sqrt{\det(g(t))} \, dxdt = 0,$$

where $dV_{g(t)}$ is the Riemannian volume of $(\overline{\Omega}_1, g(t))$. Passing to polar normal coordinates, we obtain

$$\int_0^T \chi_\delta(t)^2 \int_0^{\ell_{t,+}(y,\theta)} \int_{S_{x_0,t}(\overline{\Omega}_1)} h(\theta) \tilde{F}(t,r,\theta) \, d\theta dr dt = 0.$$

Using the fact that $F \in C([0,T] \times \overline{\Omega})$, we deduce that $\tilde{F} \in C([0,T]; L^\infty(\Omega_1))$, and letting $\delta \to 0$, we obtain

$$\int_0^{\ell_{t_0,+}(x_0,\theta)} \int_{S_{x_0,t_0}(\overline{\Omega}_1)} h(\theta) \tilde{F}(t_0,r,\theta) \, d\theta dr = 0.$$

Applying the fact that in this identity $h \in C^\infty(S_{x_0,t_0}(\overline{\Omega}_1))$ is arbitrary chosen, we deduce that

$$\int_0^{\ell_{t_0,+}(x_0,\theta)} \tilde{F}(t_0, \gamma_{x_0,\theta}(s)) \, ds = \int_0^{\ell_{t_0,+}(x_0,\theta)} \tilde{F}(t_0,r,\theta) \, dr = 0, \quad (x_0,\theta) \in \partial_+ S_{t_0}(\overline{\Omega}_1).$$

Combining this with the facts that in this identity, $x_0 \in \partial\Omega_1$ was arbitrary chosen, that the manifold $(\overline{\Omega}_1, g(t_0))$ is assumed to be simple and that the geodesic ray transform is injective on simple manifolds, we deduce that $\tilde{F}(t_0, \cdot) \equiv 0$ on Ω_1. Thus, $F(t_0, \cdot) \equiv 0$. Combining this with the fact that here $t_0 \in (0,T)$ is arbitrary chosen and $F \in C([0,T] \times \overline{\Omega})$, we deduce that $F \equiv 0$.

Now let us fix $m \geqslant 1$, and assume that (4.1) for this m implies that $F \equiv 0$. Fix $G \in C([0,T] \times \overline{\Omega})$, and assume that

$$\int_0^T \int_\Omega G v_1 \cdots v_{m+1} w \, dx dt = 0$$

for all $v_1, \ldots, v_{m+1} \in C^{1+\frac{\alpha}{2},2+\alpha}([0,T] \times \overline{\Omega}))$ satisfying (4.2) and all $w \in C^{1+\frac{\alpha}{2},2+\alpha}([0,T] \times \overline{\Omega}))$ satisfying (4.3). Fixing $F = G v_1$, we deduce that $F \equiv 0$, and multiplying F by an arbitrary chosen $w \in C^{1+\frac{\alpha}{2},2+\alpha}([0,T] \times \overline{\Omega}))$ satisfying (4.3), and integrating, we deduce that

$$\int_0^T \int_\Omega G v_1 w \, dx dt = 0$$

for all $v_1 \in C^{1+\frac{\alpha}{2},2+\alpha}([0,T] \times \overline{\Omega}))$ satisfying (4.2) and all $w \in C^{1+\frac{\alpha}{2},2+\alpha}([0,T] \times \overline{\Omega}))$ satisfying (4.3). Then, the above argumentation implies that $G \equiv 0$. This proves the assertion. $\qquad \square$

6. Proof of Theorem 2.2

In this section, we assume that $n \geqslant 3$ and $\psi(x) = |x - x_0|$, $x \in \Omega$, for $x_0 \in \mathbb{R}^n \setminus \overline{\Omega}$. Note that the function ψ satisfies the eikonal equation

$$|\nabla_x \psi(x)| = 1, \text{ for } x \in \Omega. \tag{6.1}$$

We start by considering the following new Carleman estimate whose proof is postponed to Appendix A.

Lemma 6.1. *Let $q \in L^\infty(Q)$ and $v \in H^1(Q) \cap L^2(0,T; H^2(\Omega))$ satisfy the condition*

$$v|_\Sigma = 0, \quad v|_{t=0} = 0. \tag{6.2}$$

Then, there exists $\tau_0 > 0$ depending on T, Ω and $\|q\|_{L^\infty(Q)}$ such that for all $\tau > \tau_0$, the following estimate

$$\tau \int_0^T \int_{\Gamma_+(x_0)} e^{-2(\tau^2 t + \tau\psi(x))} |\partial_\nu v|^2 |\partial_\nu \psi(x)| d\sigma(x) dt + \tau^2 \int_Q e^{-2(\tau^2 t + \tau\psi(x))} |v|^2 dxdt$$

$$\leqslant C \Bigg(\int_Q e^{-2(\tau^2 t + \tau\psi(x))} |(\partial_t - \Delta_x + q)v|^2 \, dxdt \tag{6.3}$$

$$+ \tau \int_0^T \int_{\Gamma_-(x_0)} e^{-2(\tau^2 t + \tau\psi(x))} |\partial_\nu v|^2 |\partial_\nu \psi(x)| d\sigma(x) \, dt \Bigg)$$

holds true.

Remark 6.1. The weight function ψ under consideration in the Carleman estimates (6.3) is chosen in accordance with the construction of geometric optics solutions of the form (5.3)–(5.4) introduced in Section 5.1. Namely, we need to consider weight functions ψ that can be the phase of such geometric optics solutions. For this reason, we need to consider weight functions that satisfy the eikonal equation $\mathcal{I} = 0$ which takes the form (6.1) in the context of Theorem 2.2.

Armed with these results, we are now in a position to complete the proof of Theorem 2.2.

Proof of Theorem 2.2. Following the proof of Theorem 2.1, we only need to prove that (4.4) holds true. We will prove this by a recursion argument. Let us first observe that since $\tilde{\Gamma}$ is a neighborhood of $\Gamma_-(x_0)$, there exists $\epsilon > 0$ such that $B(x_0, \epsilon) \cap \overline{\Omega} = \emptyset$, and for all $y \in B(x_0, \epsilon)$, we have

$$\Gamma_-(y, \epsilon) := \{x \in \partial\Omega : \ (x - y) \cdot \nu(x) \leqslant \epsilon\} \subset \tilde{\Gamma}.$$

We start by considering (4.4) for $k = 1$. For $j = 1, 2$, consider $v_{j,1} \in C^{1+\frac{\alpha}{2}, 2+\alpha}([0,T] \times \overline{\Omega})$ satisfying (4.2) with $b = b_j$ and $w \in C^{1+\frac{\alpha}{2}, 2+\alpha}([0,T] \times \overline{\Omega})$ satisfying (4.3) with $b = b_1$. We assume here that $v_{1,1}|_\Sigma = h = v_{2,1}|_\Sigma$ for some $h \in \mathcal{K}_0$. Fixing $v_1 = v_{1,1} - v_{2,1}$, we deduce that v_1 satisfies the conditions

$$\begin{cases} \partial_t v_1 - \Delta v_1 + \partial_\mu b_1(t, x, u_{1,0}) v_1 = F(t, x) & \text{in } Q, \\ v_1 = 0 & \text{on } \Sigma, \\ \partial_\nu v_1 = 0 & \text{on } (0, T) \times \tilde{\Gamma}, \\ v(0, x) = 0 & \text{for } x \in \Omega, \end{cases}$$

with

$$F(t, x) = \big(\partial_\mu b_2(t, x, u_{2,0}(t,x)) - \partial_\mu b_1(t, x, u_{1,0}(t,x))\big) v_{2,1}(t,x), \quad (t,x) \in Q.$$

Multiplying the above equation by w and integrating by parts, we obtain

$$\int_0^T \int_\Omega \big(\partial_\mu b_2(t, x, u_{2,0}) - \partial_\mu b_1(t, x, u_{1,0})\big) v_{2,1} w \, dxdt - \int_\Sigma \partial_\nu v_1(t, x) w(t, x) \, d\sigma(x) dt = 0.$$

Moreover, applying the first-order linearization, we find

$$\partial_\nu v_{1,1}\big|_{(0,T)\times\Gamma_-(y,\epsilon)} = \partial_s \mathcal{N}_{b_1}(f_0 + sh)\big|_{(0,T)\times\Gamma_-(y,\epsilon)}$$

$$= \partial_s \mathcal{N}_{b_2}(f_0 + sh)\big|_{(0,T)\times\Gamma_-(y,\epsilon)}$$

$$= \partial_\nu v_{2,1}\big|_{(0,T)\times\Gamma_-(y,\epsilon)},$$

and it follows that

$$\int_0^T \int_\Omega (\partial_\mu b_2(t,x,u_{2,0}) - \partial_\mu b_1(t,x,u_{1,0})) v_{2,1} w \, dxdt$$
$$= \int_0^T \int_{\partial\Omega \backslash \Gamma_-(y,\epsilon)} \partial_\nu v_1(t,x) w(t,x) \, d\sigma(x)dt, \tag{6.4}$$

with $v_{2,1}$ (resp. w) an arbitrary chosen element of $C^{1+\frac{\alpha}{2},2+\alpha}([0,T] \times \overline{\Omega})$ satisfying (4.2) (resp. (4.3)). In particular, we can apply this identity to $v_{2,1}$ and w two GO solutions of the form (5.3) and (5.4) with $\psi(x) = |x - y|$ and $c_\pm$ given by

$$c_+(t,x) = \chi_\delta(t) h\left(\frac{x-y}{|x-y|}\right) |x-y|^{-(n-1)/2},$$
$$c_-(t,x) = \chi_\delta(t)|x-y|^{-(n-1)/2},$$

for $(t,x) \in [0,T] \times (\mathbb{R}^n \backslash \{y\})$ with $h \in C^\infty(\mathbb{S}^{n-1})$ and

$$\chi_\delta(t) = \delta^{-\frac{1}{2}} \chi_*(\delta^{-1}(t-t_0)), \text{ for } \delta \in (0, \min(T - t_0, t_0)),$$

where $\chi_* \in C_0^\infty(-1, 1)$ satisfies

$$\int_{\mathbb{R}} \chi_*(t)^2 \, dt = 1.$$

Note that the construction of such GO solutions is a consequence of the fact that we can find Ω_2 an open neighborhood of $\overline{\Omega}$ such that $\psi \in C^\infty(\overline{\Omega_2})$ solves the eikonal equation

$$|\nabla_x \psi(x)|^2 = 1, \quad x \in \Omega_2,$$

as well as an application of Proposition 5.1. In addition, we built this class of GO solutions by following the arguments used in Section 5.1 where the polar normal coordinates will be replaced by polar coordinates centered at y. Note that in such coordinates, $\psi = r$ and the transport equations (5.11) are just

$$\partial_r c_\pm + \left(\frac{\partial_r \beta}{4\beta}\right) c_\pm = 0,$$

where β is an angle dependent multiple of $r^{2(n-1)}$.

With this choice of the functions $v_{2,1}$ and w, we obtain by Cauchy-Schwarz inequality that

$$\left| \int_0^T \int_{\partial\Omega \backslash \Gamma_-(y,\epsilon)} \partial_\nu v_1(t,x) w(t,x) \, d\sigma(x)dt \right|$$
$$\leqslant C \left(\int_0^T \int_{\partial\Omega \backslash \Gamma_-(y,\epsilon)} |\partial_\nu v_1(t,x)|^2 e^{-2(\tau^2 t + \tau\psi(x))} \, d\sigma(x)dt \right)^{\frac{1}{2}}. \tag{6.5}$$

In addition, the Carleman estimate (6.3) and the fact that $\partial_\nu v_1|_{(0,T)\times\Gamma_-(y)} = 0$ imply that, for $\tau > 0$ sufficiently large, we have

$$\int_0^T \int_{\partial\Omega\backslash\Gamma_-(y,\epsilon)} |\partial_\nu v_1(t,x)|^2 e^{-2(\tau^2 t + \tau\psi(x))} \, d\sigma(x)dt$$

$$\leqslant C\epsilon^{-1} \int_0^T \int_{\partial\Omega\backslash\Gamma_-(y,\epsilon)} |\partial_\nu v_1(t,x)|^2 e^{-2(\tau^2 t + \tau\psi(x))} \partial_\nu\psi(x) \, d\sigma(x)dt$$

$$\leqslant C\epsilon^{-1} \int_0^T \int_{\Gamma_+(y)} |\partial_\nu v_1(t,x)|^2 e^{-2(\tau^2 t + \tau\psi(x))} \partial_\nu\psi(x) \, d\sigma(x)dt$$

$$\leqslant C\tau^{-1} \int_Q |\partial_t v_1 - \Delta v_1 + \partial_\mu b_1(t,x,u_0)v_1|^2 e^{-2(\tau^2 t + \tau\psi(x))} \, dxdt$$

$$\underbrace{\hspace{8cm}}$$

Here we use the Carleman estimate (6.3) with $\partial_\nu v_1|_{(0,T)\times\Gamma_-(y)}=0$.

$$\leqslant C\tau^{-1} \int_Q |(\partial_\mu b_2(t,x,u_0) - \partial_\mu b_1(t,x,u_0))v_{2,1}|^2 e^{-2(\tau^2 t + \tau\psi(x))} \, dxdt$$

$$\leqslant C\tau^{-1} \int_Q |\partial_\mu b_2(t,x,u_0) - \partial_\mu b_1(t,x,u_0)|^2 |c_+|^2 \, dxdt$$

$$\leqslant C\tau^{-1},$$

where $C > 0$ is a constant independent of τ. Therefore, for $\tau > 0$ sufficiently large, we obtain

$$\left| \int_0^T \int_{\partial\Omega\backslash\Gamma_-(y,\epsilon)} \partial_\nu v_1(t,x)w(t,x) \, d\sigma(x)dt \right| \leqslant C\tau^{-\frac{1}{2}},$$

and in a similar way to Proposition 4.1, sending $\tau \to +\infty$, we find

$$\int_Q (\partial_\mu b_2(t,x,u_0) - \partial_\mu b_1(t,x,u_0))c_+c_- \, dxdt = 0.$$

By using the polar coordinates, sending $\delta \to 0$ and repeating the arguments of Proposition 4.1, one can get

$$\int_0^{+\infty} \int_{\mathbb{S}^{n-1}} G(t,y+r\theta)h(\theta) \, d\theta dr = 0, \quad t \in (0,T),$$

where $G := \partial_\mu b_2(t,x,u_{2,0}) - \partial_\mu b_1(t,x,u_{1,0})$ in Q extended to $(0,T) \times \mathbb{R}^n$ by zero.

Using the fact that $h \in C^\infty(\mathbb{S}^{n-1})$ is arbitrary chosen, we get

$$\int_0^{+\infty} G(t,y+r\theta) \, dr = 0, \quad t \in (0,T), \ \theta \in \mathbb{S}^{n-1},$$

and the condition on the support of G implies that

$$\int_{\mathbb{R}} G(t,y+s\theta) \, ds = 0, \quad t \in (0,T), \ \theta \in \mathbb{S}^{n-1}.$$

Since this last identity holds true for all $y \in B(x_0,\epsilon)$, we obtain

$$\int_{\mathbb{R}} G(t,y+s\theta) \, ds = 0, \quad t \in (0,T), \ \theta \in \mathbb{S}^{n-1}, \ y \in B(x_0,\epsilon). \tag{6.6}$$

In addition, since $G = 0$ on $(0,T) \times \mathbb{R}^n \setminus Q$, we know that

$$G(t,x) = 0, \quad t \in (0,T), \ x \in B(x_0,\epsilon)$$

and, combining this with (6.6), we are in position to apply [IM20, Theorem 1.2] in order to deduce that, for all $t \in (0, T)$, $G(t, \cdot) \equiv 0$. It follows that $G \equiv 0$ and (4.4) holds true for $k = 1$.

Now, let us fix $m \in \mathbb{N}$ and assume that (4.4) holds true for $k = 1, \ldots, m$. Consider $v_1, \ldots, v_{m+1} \in C^{1 + \frac{\alpha}{2}, 2 + \alpha}([0, T] \times \overline{\Omega})$ satisfying (4.2) with $b = b_1$ and $w \in C^{1 + \frac{\alpha}{2}, 2 + \alpha}([0, T] \times \overline{\Omega})$ satisfying (4.2) with $b = b_1$. We fix $h_j = v_j|_{\Sigma}$, $j = 1, \ldots, m + 1$, and proceeding to the higher-order linearization described in Lemma 3.1, we obtain

$$\partial_{\nu(a)} w_j^{(m+1)}\Big|_{\Sigma} = \partial_{s_1} \ldots \partial_{s_{m+1}} \mathcal{N}_{b_j}(f_0 + s_1 h_1 + \ldots + s_{m+1} h_{m+1})\big|_{s=0},$$

with $s = (s_1, \ldots, s_{m+1})$ and $w_j^{(m+1)}$ solving (3.9) with $b = b_j$ for $j = 1, 2$. Then, (2.6) implies

$$\partial_{\nu(a)} w_1^{(m+1)}\Big|_{(0,T) \times \tilde{\Gamma}} = \partial_{\nu(a)} w_2^{(m+1)}\Big|_{(0,T) \times \tilde{\Gamma}}$$

and, fixing $w^{(m+1)} = w_1^{(m+1)} - w_2^{(m+1)}$ in Q, and applying Lemma 3.1, we deduce that $w^{(m+1)}$ satisfies the condition

$$\begin{cases} \partial_t w^{(m+1)} - \Delta w^{(m+1)} + \partial_\mu b_1(t, x, u_0) w^{(m+1)} = \mathcal{K} & \text{in } Q, \\ w^{(m+1)} = 0 & \text{on } \Sigma, \\ \partial_\nu w^{(m+1)} = 0 & \text{on } (0, T) \times \tilde{\Gamma}, \\ w^{(m+1)}(0, x) = 0 & \text{for } x \in \Omega, \end{cases}$$

where $\mathcal{K} = (\partial_\mu^{m+1} b_2(t, x, u_{2,0}) - \partial_\mu^{m+1} b_1(t, x, u_{1,0})) v_1 \cdot \ldots \cdot v_{m+1}$. Multiplying this equation by w and integrating by parts, we obtain

$$\int_0^T \int_\Omega (\partial_\mu^{m+1} b_2(t, x, u_{2,0}) - \partial_\mu^{m+1} b_1(t, x, u_{1,0})) v_1 \cdot \ldots \cdot v_{m+1} \cdot w \, dx dt$$
$$- \int_0^T \int_{\partial\Omega \backslash \tilde{\Gamma}} \partial_\nu w^{(m+1)} w(t, x) d\sigma(x) \, dt = 0.$$

We choose v_{m+1} and w two GO solutions of the form (5.3) and (5.4) with $\psi(x) = |x - y|$, $y \in B(x_0, \epsilon)$, and we fix

$$H(t, x) = \left(\partial_\mu^{m+1} b_2(t, x, u_{2,0}) - \partial_\mu^{m+1} b_1(t, x, u_{1,0})\right) v_1 \cdot \ldots \cdot v_m(t, x), \quad (t, x) \in Q$$

that we extend by zero to $(0, T) \times \mathbb{R}^n$. Applying Cauchy-Schwarz inequality and the fact that $\Gamma_-(y, \epsilon) \subset \tilde{\Gamma}$, we find

$$\left| \int_0^T \int_{\partial\Omega \backslash \tilde{\Gamma}} \partial_\nu w^{(m+1)}(t, x) w(t, x) \, d\sigma(x) dt \right|$$
$$\leqslant C \left(\int_0^T \int_{\partial\Omega \backslash \Gamma_-(y, \epsilon)} |\partial_\nu w^{(m+1)}(t, x)|^2 e^{-2(\tau^2 t + \tau \psi(x))} d\sigma(x) dt \right)^{\frac{1}{2}},$$

and, applying the Carleman estimate (6.3) and repeating the above argumentation, we have

$$\left| \int_0^T \int_{\partial\Omega\setminus\Gamma_-(y,\epsilon)} \partial_\nu w^{(m+1)}(t,x)w(t,x)\, d\sigma(x)dt \right|^2$$
$$\leqslant C\tau^{-1}\int_Q |H(t,x)|^2 |v_{m+1}(t,x)|^2 e^{-2(\tau^2 t + \tau\psi(x))}\, dxdt$$
$$\leqslant C\tau^{-1}.$$

Thus, we obtain

$$\lim_{\tau\to+\infty}\int_Q H(t,x)v_{m+1}(t,x)w(t,x)\, dxdt = 0,$$

and repeating the above argumentation, we get

$$\left(\partial_\mu^{m+1} b_2(t,x,u_{2,0}) - \partial_\mu^{m+1} b_1(t,x,u_{1,0})\right)v_1\cdot\ldots\cdot v_m(t,x) = H(t,x) = 0, \quad (t,x)\in Q.$$

Multiplying this expression by any $w\in C^{1+\frac{\alpha}{2},2+\alpha}([0,T]\times\overline{\Omega})$ satisfying (4.2) with $b=b_1$, we obtain

$$\int_Q \left(\partial_\mu^{m+1} b_2(t,x,u_{2,0}) - \partial_\mu^{m+1} b_1(t,x,u_{1,0})\right)v_1\cdot\ldots\cdot v_m w\, dxdt = 0,$$

and applying Proposition 4.1, we can conclude that (4.4) holds true for $k = 1,\ldots, m+1$. It follows that (4.4) holds true for all $k\in\mathbb{N}$, and repeating the arguments used in the second step of the proof of Theorem 2.1, we can conclude that (2.6) implies (2.5) with the function $\varphi = u_{2,0} - u_{1,0}$ satisfying (2.7)–(2.8). □

7. Breaking the gauge class

This section is devoted to the proof of the positive answers that we give to problem (IP2) in the theorems and corollaries of Section 2.2.

Proof of Corollary 2.1. We start by assuming that the conditions of Theorem 2.1 are fulfilled and by proving that (2.4) implies $b_1 = b_2$. By Theorem 2.1, condition (2.4) implies that there exists $\varphi\in C^{1+\frac{\alpha}{2},2+\alpha}([0,T]\times\overline{\Omega})$ satisfying (1.6) such that

$$b_1(t,x,\mu) = b_2(t,x,\mu+\varphi(t,x)) + \rho(t,x)\partial_t\varphi(t,x) + \mathcal{A}(t)\varphi(t,x), \quad (t,x,\mu)\in Q\times\mathbb{R}. \tag{7.1}$$

We will prove that $\varphi\equiv 0$, which implies that $b_1 = b_2$. Choosing $\mu = \kappa(t,x)$ and applying (2.10), we obtain

$$\rho(t,x)\partial_t\varphi(t,x) + \mathcal{A}(t)\varphi(t,x) + b_2(t,x,\kappa(t,x)+\varphi(t,x)) - b_2(t,x,\kappa(t,x))$$
$$= b_1(t,x,\kappa(t,x)) - b_2(t,x,\kappa(t,x)) = 0, \quad (t,x)\in Q.$$

Moreover, we have

$$b_2(t,x,\kappa(t,x)+\varphi(t,x)) - b_2(t,x,\kappa(t,x)) = \left(\int_0^1 \partial_\mu b_2(t,x,\kappa(t,x)+s\varphi(t,x))ds\right)\varphi(t,x)$$
$$:= q(t,x)\varphi(t,x), \quad \text{for } (t,x)\in Q.$$

Therefore, φ fulfills the condition

$$\begin{cases} \rho(t,x)\partial_t\varphi(t,x) + \mathcal{A}(t)\varphi(t,x) + q(t,x)\varphi(t,x) = 0 & \text{in } Q, \\ \varphi(t,x) = 0 & \text{on } \Sigma, \\ \varphi(0,x) = 0 & \text{for } x \in \Omega, \end{cases} \tag{7.2}$$

and the uniqueness of solutions for this problem implies that $\varphi \equiv 0$. Thus, (7.1) implies $b_1 = b_2$.

Using similar arguments, one can check that, by assuming the conditions of Theorem 2.2, (2.6) implies also that $b_1 = b_2$. $\qquad\square$

Proof of Corollary 2.2. Let us assume that the conditions of Theorem 2.1 and (2.4) are fulfilled. By Theorem 2.1 there exists $\varphi \in C^{1+\frac{\alpha}{2},2+\alpha}([0,T] \times \overline{\Omega})$ satisfying (1.6) such that (7.1) is fulfilled. In particular, by choosing $\mu = 0$, we have

$$b_1(t,x,0) - b_2(t,x,0) = b_2(t,x,\varphi(t,x)) - b_2(t,x,0) + \rho(t,x)\partial_t\varphi(t,x) + \mathcal{A}(t)\varphi(t,x) \text{ in } Q.$$

Combining this with (2.12) and (1.6), we deduce that φ fulfills the condition

$$\begin{cases} \rho(t,x)\partial_t\varphi(t,x) + \mathcal{A}(t)\varphi(t,x) + q(t,x)\varphi(t,x) = h(x)G(t,x) & \text{in } Q, \\ \varphi(t,x) = \partial_{\nu(a)}\varphi(t,x) = 0 & \text{on } \Sigma, \\ \varphi(0,x) = 0 & \text{for } x \in \Omega, \end{cases} \tag{7.3}$$

with

$$q(t,x) = \int_0^1 \partial_\mu b_2(t,x,s\varphi(t,x))\, ds, \quad (t,x) \in Q.$$

Moreover, following the proof of Theorem 2.1, we know that $\varphi = u_{2,0} - u_{1,0}$ and the additional assumption (2.13)

$$u_{1,0}(\theta,x) = u_{2,0}(\theta,x)$$

implies that

$$\varphi(\theta,x) = 0, \quad x \in \Omega.$$

Combining this with (2.11),

$$\inf_{x\in\Omega}|G(\theta,x)| > 0,$$

(7.3) and applying [IY98, Theorem 3.4], we obtain $h \equiv 0$. Then the source term in (7.3) is zero, and uniqueness of solutions implies that $\varphi \equiv 0$. Thus, (7.1) implies $b_1 = b_2$. The last statement of the corollary can be deduced from similar arguments. $\qquad\square$

We are ready to prove Theorem 2.3.

Proof of Theorem 2.3. We assume first that the conditions of Theorem 2.1 and (2.4) are fulfilled. Let us first prove that we can assume that $N_1 = N_2$. Indeed, assuming that $N_1 \neq N_2$, we may assume without loss of generality that $N_1 > N_2$. From (7.1), we deduce that

$$b_{1,N_1}(t,x) = \lim_{\lambda\to+\infty} \frac{b_1(t,x,\lambda)}{\lambda^{N_1}}$$

$$= \lim_{\lambda\to+\infty} \frac{b_2(t,x,\lambda + \varphi(t,x)) + \rho(t,x)\partial_t\varphi(t,x) + \mathcal{A}(t)\varphi(t,x)}{\lambda^{N_1}} = 0,$$

for $(t, x) \in Q$. In the same way, we can prove by iteration that

$$b_{1,N_1} = b_{1,N_1-1} = \cdots = b_{1,N_2+1} \equiv 0.$$

Therefore, from now on, we assume that $N_1 = N_2 = N$. In view of (7.1), for all $(t, x, \mu) \in Q \times \mathbb{R}$, we get by renumbering the sums

$$\sum_{k=0}^{N} b_{1,k}(t,x)\mu^k = \sum_{k=1}^{N} b_{2,k}(t,x)\left(\sum_{j=1}^{k}\binom{k}{j}\varphi(t,x)^{k-j}\mu^j\right)$$
$$+ \rho(t,x)\partial_t\varphi(t,x) + \mathcal{A}(t)\varphi(t,x) + b_2(t,x,\varphi(t,x))$$
$$= \sum_{j=1}^{N}\left(\sum_{k=j}^{N} b_{2,k}(t,x)\binom{k}{j}\varphi(t,x)^{k-j}\right)\mu^j$$
$$+ \rho(t,x)\partial_t\varphi(t,x) + \mathcal{A}(t)\varphi(t,x) + b_2(t,x,\varphi(t,x)).$$

It follows that

$$b_{1,j}(t,x) = \sum_{k=j}^{N} b_{2,k}(t,x)\binom{k}{j}\varphi(t,x)^{k-j}, \quad (t,x) \in Q, \ j = 1, \ldots, N, \tag{7.4}$$

and

$$b_{1,0}(t,x) - b_{2,0}(t,x) = \rho(t,x)\partial_t\varphi(t,x) + \mathcal{A}(t)\varphi(t,x) + b_2(t,x,\varphi(t,x)) - b_2(t,x,0), \tag{7.5}$$

for $(t, x) \in Q$.

Applying (7.4) with $j = N$ and $j = N - 1$, we obtain

$$b_{1,N} = b_{2,N}, \quad b_{1,N-1} = b_{2,N}\varphi + b_{2,N-1}. \tag{7.6}$$

Moreover, the fact that the condition (2.16) holds true on the dense set J combined with the fact that $b_{j,k} \in C([0,T] \times \overline{\Omega})$, $j = 1, 2$ and $k = N - 1, N$ implies that

$$\min\left(\left|(b_{1,N-1} - b_{2,N-1})(t,x)\right|, \sum_{j=1}^{2}\left|(b_{j,N} - b_{j,N-1})(t,x)\right|\right) = 0, \ (t,x) \in (0,T) \times \omega.$$

This condition implies that for all $(t, x) \in (0, T) \times \omega$, we have either $b_{2,N-1}(t,x) = b_{1,N-1}(t,x)$ or $b_{j,N}(t,x) = b_{j,N-1}(t,x)$, $j = 1, 2$. Combining this with (7.6) and the assumption that $|b_{1,N}(t,x)| > 0$ for $(t, x) \in J$, we deduce that $\varphi = 0$ on $(0, T) \times \omega$. Thus, (7.5) implies

$$(b_{1,0} - b_{2,0})(t,x) = \rho\partial_t\varphi(t,x) + \mathcal{A}(t)\varphi(t,x) + b_2(t,x,\varphi(t,x)) - b_2(t,x,0) = 0,$$

for $(t, x) \in (0, T) \times \omega$. Finally, the assumption (2.18) implies that $b_{1,0} = b_{2,0}$ everywhere on Q. Thus, φ satisfies (7.2) with

$$q(t,x) := \int_0^1 \partial_\mu b_2(t,x,s\varphi(t,x)) \, ds, \quad (t,x) \in Q.$$

Therefore, the uniqueness of solutions of (7.2) implies that $\varphi \equiv 0$, and it follows that $b_1 = b_2$. The last statement of the theorem can be proved with similar arguments. $\square$

Proof of Theorem 2.4. We assume that the conditions of Theorem 2.1 and (2.4) are fulfilled. We start by showing that $\partial_\mu h_1 = \partial_\mu h_2$. For this purpose, combining (1.6) and (7.1) with (2.20), we obtain

$$b_{1,1}(t,x)h_1(t,\mu) + b_{1,0}(t,x) = b_{2,1}(t,x)h_2(t,\mu) + b_{2,0}(t,x) + \mathcal{A}(t)\varphi(t,x), \quad (t,x,\mu) \in \Sigma \times \mathbb{R}.$$

Differentiating both sides of this identity with respect to μ, we get

$$b_{1,1}(t,x)\partial_\mu h_1(t,\mu) = b_{2,1}(t,x)\partial_\mu h_2(t,\mu), \quad (t,x,\mu) \in \Sigma \times \mathbb{R},$$

and (2.23) implies that

$$b_{1,1}(t,x_t)(\partial_\mu h_1(t,\mu) - \partial_\mu h_2(t,\mu)) = 0, \quad (t,\mu) \in (0,T) \times \mathbb{R},$$

with $b_{1,1}(t,x_t) \neq 0, t \in (0,T)$. It follows that $\partial_\mu h_1 = \partial_\mu h_2$.

Now let us show that the function φ of (7.1) is identically zero. Fixing $t \in (0,T)$ and applying the derivative at order n_t with respect to μ on both sides of (7.1), we get

$$b_{1,1}(t,x)\partial_\mu^{n_t} h_1(t,\mu) = b_{2,1}(t,x)\partial_\mu^{n_t} h_2(t,\mu + \varphi(t,x)) = b_{2,1}(t,x)\partial_\mu^{n_t} h_1(t,\mu + \varphi(t,x)),$$

for $(x,\mu) \in \Omega \times \mathbb{R}$. Fixing $\mu = \mu_t - \varphi(t,x)$ and applying (2.21), we get

$$b_{1,1}(t,x)\partial_\mu^{n_t} h_1(t,\mu_t - \varphi(t,x)) = b_{2,1}(t,x)\partial_\mu^{n_t} h_1(t,\mu_t) = 0, \quad x \in \Omega,$$

and (2.22) implies

$$\partial_\mu^{n_t} h_1(t,\mu_t - \varphi(t,x)) = 0, \quad x \in \Omega.$$

However, since $\mathbb{R} \ni \mu \mapsto \partial_\mu^{n_t} h_1(t,\mu)$ is analytic, either it is uniformly vanishing or its zeros are isolated. By (2.21), we have that $\partial_\mu^{n_t} h_1(t,\cdot) \not\equiv 0$ for $t \in (0,T)$. Thus, the zeros of $\mathbb{R} \ni \mu \mapsto \partial_\mu^{n_t} h_1(t,\mu)$ are isolated. Using the fact that $\overline{\Omega} \ni x \mapsto \varphi(t,x)$ is continuous, we deduce that the map $\overline{\Omega} \ni x \mapsto \varphi(t,x)$ is constant. Then, recalling that $\varphi(t,x) = 0$ for $x \in \partial\Omega$, we deduce that $\varphi(t,\cdot) \equiv 0$. Since here $t \in (0,T)$ is arbitrary chosen, we deduce that $\varphi \equiv 0$, and it follows that $b_1 = b_2$. The last statement of the theorem can be proved with similar arguments. □

Proof of Theorem 2.5. We will only consider the other statement of the theorem since the last statement can be deduced from similar arguments. Namely, we will prove that (2.4) and the conditions of Theorem 2.1 imply that $b_1 = b_2$. We start by showing that $\partial_\mu h_1 = \partial_\mu h_2$. For this purpose, combining (1.6) and (7.1) with (2.24), we obtain

$$b_{1,1}(t,x)h_1(t,b_{1,2}(t,x)\mu) + b_{1,0}(t,x) = b_{2,1}(t,x)h_2(t,b_{2,2}(t,x)\mu) + b_{2,0}(t,x) + \mathcal{A}(t)\varphi(t,x),$$

for $(t,x,\mu) \in \Sigma \times \mathbb{R}$. Differentiating both sides of this identity with respect to μ, we get

$$b_{1,2}(t,x)b_{1,1}(t,x)\partial_\mu h_1(t,b_{1,2}(t,x)\mu) = b_{2,2}(t,x)b_{2,1}(t,x)\partial_\mu h_2(t,b_{2,2}(t,x)\mu), \quad (t,x,\mu) \in \Sigma \times \mathbb{R},$$

and (2.27) implies that

$$b_{1,1}(t,x_t)b_{1,2}(t,x_t)(\partial_\mu h_1(t,b_{1,2}(t,x_t)\mu) - \partial_\mu h_2(t,b_{1,2}(t,x_t)\mu)) = 0, \quad (t,\mu) \in (0,T) \times \mathbb{R},$$

with $b_{1,1}(t,x_t) \neq 0$ and $b_{1,2}(t,x_t) \neq 0, t \in (0,T)$. It follows that $\partial_\mu h_1 = \partial_\mu h_2$.

Now let us show that the function φ of (7.1) is identically zero. Fixing $t \in (0, T)$ and applying the derivative at order n_t with respect to μ on both sides of (7.1), we get

$$b_{1,1}(t, x)(b_{1,2}(t, x))^{n_t} \partial_\mu^{n_t} h_1(t, b_{1,2}(t, x)\mu)$$
$$= b_{2,1}(t, x))(b_{2,2}(t, x))^{n_t} \partial_\mu^{n_t} h_2(t, b_{2,2}(t, x)(\mu + \varphi(t, x)))$$
$$= b_{2,1}(t, x)(b_{2,2}(t, x))^{n_t} \partial_\mu^{n_t} h_1(t, b_{2,2}(t, x)(\mu + \varphi(t, x))), \quad (x, \mu) \in \Omega \times \mathbb{R}.$$

Fixing $\mu = -\varphi(t, x)$ and applying (2.27), we get

$$b_{1,1}(t, x)(b_{1,2}(t, x))^{n_t} \partial_\mu^{n_t} h_1(t, -b_{1,2}(t, x)\varphi(t, x)) = b_{2,1}(t, x)(b_{2,2}(t, x))^{n_t} \partial_\mu^{n_t} h_1(t, 0) = 0, \quad x \in \Omega,$$

and (2.26) implies

$$\partial_\mu^{n_t} h_1(t, -b_{1,2}(t, x)\varphi(t, x))) = 0, \quad x \in \Omega.$$

However, in view of (2.25), since $\mathbb{R} \ni \mu \mapsto \partial_\mu^{n_t} h_1(t, \mu)$ is analytic, $\partial_\mu^{n_t} h_1(t, \cdot) \not\equiv 0$ and $\overline{\Omega} \ni x \mapsto b_{1,2}(t, x)\varphi(t, x)$ is continuous, we deduce that the map $\overline{\Omega} \ni x \mapsto b_{1,2}(t, x)\varphi(t, x)$ is constant. Then, recalling that $\varphi(t, x) = 0$ for $x \in \partial\Omega$ and applying (2.26), we deduce that $\varphi(t, \cdot) \equiv 0$. Since here $t \in (0, T)$ is arbitrary chosen, we deduce that $\varphi \equiv 0$, and it follows that $b_1 = b_2$. $\qquad \square$

Proof of Corollary 2.3. Again, we will only consider the first statement of the corollary as the other statement follows similarly. Namely, we will prove that (2.4) and the conditions of Theorem 2.1 imply that $b_1 = b_2$. For this purpose, we only need to prove that the function φ of (7.1) is identically zero. We start by assuming that condition (i) is fulfilled. Fixing $x \in \Omega$ and applying the derivative at order n_x with respect to μ on both sides of (7.1), we get

$$b_{1,1}(t, x)(b_{1,2}(t, x))^{n_x} \partial_\mu^{n_x} G(x, b_{1,2}(t, x)\mu)$$
$$= b_{2,1}(t, x)(b_{2,2}(t, x))^{n_x} \partial_\mu^{n_x} G(x, b_{2,2}(t, x)(\mu + \varphi(t, x)))$$
$$= b_{2,1}(t, x)(b_{1,2}(t, x))^{n_x} \partial_\mu^{n_x} G(x, b_{1,2}(t, x)(\mu + \varphi(t, x))), \quad (t, \mu) \in (0, T) \times \mathbb{R}.$$

Applying (2.26), fixing $\mu = \frac{\mu_x}{b_{1,2}(t, x)} - \varphi(t, x)$ and using (2.30), we get

$$b_{1,1}(t, x)(b_{1,2}(t, x))^{n_x} \partial_\mu^{n_x} G(x, \mu_x - b_{1,2}(t, x)\varphi(t, x)) = b_{2,1}(t, x)(b_{1,2}(t, x))^{n_x} \partial_\mu^{n_x} G(x, \mu_x) = 0,$$

for $t \in (0, T)$. Then, (2.26) implies

$$\partial_\mu^{n_x} G(x, \mu_x - b_{1,2}(t, x)\varphi(t, x)) = 0, \quad t \in (0, T).$$

However, since $\mathbb{R} \ni \mu \mapsto \partial_\mu^{n_x} G(x, \mu)$ is analytic and $[0, T] \ni t \mapsto b_{1,2}(t, x)\varphi(t, x)$ is continuous, we deduce that either $\partial_\mu^{n_x} G(x, \cdot) \equiv 0$ or that the map $[0, T] \ni t \mapsto b_{1,2}(t, x)\varphi(t, x)$ is constant. Since $\partial_\mu^{n_x} G(x, \cdot) \not\equiv 0$, we deduce that the map $[0, T] \ni t \mapsto b_{1,2}(t, x)\varphi(t, x)$ is constant. Then, recalling that $\varphi(0, \cdot) = 0$ and applying (2.26), we deduce that, for all $t \in [0, T]$, $\varphi(t, x) = 0$. Since here $x \in \Omega$ is arbitrary chosen, we deduce that $\varphi \equiv 0$, and it follows that $b_1 = b_2$.

Combining the above argumentation and the arguments used for the proof of Theorem 2.5, one can easily check that (2.4) implies also that $b_1 = b_2$ when condition (ii) is fulfilled. This completes the proof of the corollary. $\qquad \square$

Proofs of Corollary 2.4. With the conclusion of Theorem 2.2 at hand, combined with $b_j(t, x, \mu) = q_j(t, x)\mu$ for $(t, x, \mu) \in Q \times \mathbb{R}$ and $j = 1, 2$, both (2.5) and (2.9) yield that

$$q_1(x, t)\mu = S_\varphi(q_2(x, t)\mu) = q_2(t, x)(\mu + \varphi(t, x)) + \rho(t, x)\partial_t \varphi(t, x) + \mathcal{A}(t)\varphi(t, x), \qquad (7.7)$$

for any $(t, x, \mu) \in Q \times \mathbb{R}$. In particular, plugging $\mu = 0$ into (7.7), the function φ satisfies the IBVP

$$\begin{cases} \rho(t,x)\partial_t\varphi(t,x) + \mathcal{A}(t)\varphi(t,x) + q_2(t,x)\varphi(t,x) = 0 & \text{in } Q, \\ \varphi(t,x) = 0 & \text{on } \Sigma, \\ \varphi(0,x) = 0 & \text{in } \Omega. \end{cases}$$

By the uniqueness of the above IBVP, we must have $\varphi \equiv 0$ in Q. Now, by using (7.7) again, we have $q_1(t,x)\mu = q_2(t,x)\mu$, for all $\mu \in \mathbb{R}$, which implies $q_1 = q_2$ as desired. This completes the proof. $\qquad \square$

8. Application to the simultaneous determination of nonlinear and source terms

One of the important application of our results is to inverse source problems, where the aim is to recover both the source and nonlinear terms simultaneously. In this section, we consider the IBVP

$$\begin{cases} \rho(t,x)\partial_t u(t,x) + \mathcal{A}(t)u(t,x) + d(t,x,u(t,x)) = F(t,x) & \text{in } Q, \\ u(t,x) = f(t,x) & \text{on } \Sigma, \\ u(0,x) = 0 & \text{for } x \in \Omega, \end{cases} \qquad (8.1)$$

with $d \in \mathbb{A}(\mathbb{R}; C^{\frac{\alpha}{2},\alpha}([0,T] \times \overline{\Omega}))$ and $F \in C^{\frac{\alpha}{2},\alpha}([0,T] \times \overline{\Omega})$ satisfying the conditions

$$F(0,x) = 0, \quad x \in \partial\Omega, \qquad (8.2)$$

$$d(t,x,0) = 0, \quad (t,x) \in Q. \qquad (8.3)$$

The latter condition is just for presentational purposes and can be removed by redefining F. In a similar way to the problem studied above, we assume here that there exists $f = f_0 \in \mathcal{K}_0$ such that (8.1) admits a unique solution for $f = f_0$. Then, applying Proposition 3.1, we can prove that there exists $\epsilon > 0$, depending on $a, \rho, d, F, f_0, \Omega, T$, such that, for all $f \in \mathbb{B}(f_0, \epsilon)$, (8.1) admits a unique solution $u_f \in C^{1+\frac{\alpha}{2},2+\alpha}([0,T] \times \overline{\Omega})$ that lies in a sufficiently small neighborhood of the solution u_0 of (8.1) when $f = f_0$. Using these properties, we can define the parabolic DN map

$$\mathcal{M}_{(d,F)} : \mathbb{B}(f_0, \epsilon) \ni f \mapsto \partial_{\nu(a)} u(t,x)\big|_\Sigma,$$

where u solves (8.1).

We consider in this section the inverse problem of determining simultaneously the nonlinear term d and the source term F appearing in (8.1). Similarly to the problem (IP), there will be a gauge invariance for this inverse problem. Indeed, fix $\varphi \in C^{1+\frac{\alpha}{2},2+\alpha}([0,T] \times \overline{\Omega})$ satisfying (1.6) and consider the map U_φ mapping $C^\infty(\mathbb{R}; C^{\frac{\alpha}{2},\alpha}([0,T] \times \overline{\Omega})) \times C^{\frac{\alpha}{2},\alpha}([0,T] \times \overline{\Omega})$ into itself and defined by $U_\varphi(d,F) = (d_\varphi, F_\varphi)$ with

$$\begin{aligned} d_\varphi(t,x,\mu) &= d(t,x,\mu + \varphi(t,x)) - d(t,x,\varphi(t,x)), \\ F_\varphi(t,x) &= F(t,x) - \rho(t,x)\partial_t\varphi(t,x) - \mathcal{A}(t)\varphi(t,x) - d(t,x,\varphi(t,x)), \end{aligned} \qquad (8.4)$$

for $(t,x,\mu) \in Q \times \mathbb{R}$. Then, one can easily check that $\mathcal{M}_{(d,F)} = \mathcal{M}_{U_\varphi(d,F)}$. Note that (8.4) is equivalent to (1.7) for

$$b(t,x,\mu) = d(t,x,\mu) - F(t,x), \quad (t,x,\mu) \in Q \times \mathbb{R}.$$

Following, this property, in general, the best one can expect for our inverse problem is the determination of $U_\varphi(d,F)$ from $\mathcal{M}_{(d,F)}$ for some $\varphi \in C^{1+\frac{\alpha}{2},2+\alpha}([0,T] \times \overline{\Omega})$ satisfying (1.6). Our first result will be stated in that sense.

Proposition 8.1. *Let $a := (a_{ik})_{1 \leqslant i,k \leqslant n} \in C^\infty([0,T] \times \overline{\Omega}; \mathbb{R}^{n \times n})$ satisfy (1.1), $\rho \in C^\infty([0,T] \times \overline{\Omega}; \mathbb{R}_+)$ and, for $j = 1, 2$, let $d_j \in \mathbb{A}(\mathbb{R}; C^{\frac{\alpha}{2}, \alpha}([0,T] \times \overline{\Omega})) \cap C^\infty([0,T] \times \overline{\Omega} \times \mathbb{R})$ and $F_j \in C^\infty([0,T] \times \overline{\Omega})$ satisfy (8.2)–(8.3) with $d = d_j$ and $F = F_j$. We assume also that there exists $f_0 \in \mathcal{K}_0$ such that problem (8.1), with $f = f_0$ and $d = d_j$, $F = F_j$, $j = 1, 2$, is well-posed. Then, the condition*

$$\mathcal{M}_{(d_1, F_1)} = \mathcal{M}_{(d_2, F_2)} \tag{8.5}$$

implies that there exists $\varphi \in C^{1+\frac{\alpha}{2}, 2+\alpha}([0,T] \times \overline{\Omega})$ satisfying (1.6) such that

$$(d_1, F_1) = U_\varphi(d_2, F_2), \tag{8.6}$$

where U_φ is the map defined by (8.4).

Proof. Fixing

$$b_j(t, x, \mu) = d_j(t, x, \mu) - F_j(t, x), \quad (t, x, \mu) \in Q \times \mathbb{R},$$

one can check that $\mathcal{N}_{b_j} = \mathcal{M}_{(d_j, F_j)}$ and (8.5) implies (2.4). Then, applying Theorem 2.1, we deduce that (2.5) holds true which clearly implies (8.6). □

It is well known that when $\mu \mapsto d(t, x, \mu)$ is linear, $(t, x) \in Q$, there is no hope to determine general class of source terms $F \in C^\infty([0,T] \times \overline{\Omega})$ satisfying the condition of Proposition 8.1 from the knowledge of the map $\mathcal{M}_{(d,F)}$; see Example 1.1 or, for example, [KSXY22, Appendix A]. Nevertheless, this invariance breaks for several classes of nonlinear terms d for which we can prove the simultaneous determination of d and F from $\mathcal{M}_{(d,F)}$. More precisely, applying Corollary 2.1 and Theorems 2.3, 2.4, 2.5, we can show the following.

Corollary 8.1. *Let the condition of Proposition 8.1 be fulfilled, and assume that, for $j = 1, 2$, the nonlinear term d_j satisfies one of the following conditions:*

(i) *There exists $\kappa \in C^{\frac{\alpha}{2}, \alpha}([0,T] \times \overline{\Omega})$ such that*

$$d_1(t, x, \kappa(t, x)) - d_2(t, x, \kappa(t, x)) = F_1(t, x) - F_2(t, x), \quad (t, x) \in [0,T] \times \overline{\Omega}.$$

(ii) *There exists $N_j \geqslant 2$ such that*

$$d_j(t, x, \mu) = \sum_{k=1}^{N_j} d_{j,k}(t, x) \mu^k, \quad (t, x, \mu) \in [0,T] \times \overline{\Omega} \times \mathbb{R}, \ j = 1, 2.$$

Moreover, for $N = \min(N_1, N_2)$ and J a dense subset of Q, we have

$$\min\left(|(d_{1,N-1} - d_{2,N-1})(t, x)|, \sum_{j=1}^{2} |(d_{j,N} - d_{j,N-1})(t, x)| \right) = 0, \ (t, x) \in J,$$

$$d_{1,N}(t, x) \neq 0, \quad (t, x) \in J.$$

(iii) *There exists $h_j \in \mathbb{A}(\mathbb{R}; C^{\frac{\alpha}{2}}([0,T]))$ such that*

$$d_j(t, x, \mu) = q_j(t, x) h_j(t, \mu), \quad (t, x, \mu) \in [0,T] \times \overline{\Omega} \times \mathbb{R}, \ j = 1, 2.$$

Assume also that, for all $t \in (0, T)$, there exist $\mu_t \in \mathbb{R}$ and $n_t \in \mathbb{N}$ such that

$$\partial_\mu^{n_t} h_1(t, \cdot) \not\equiv 0, \quad \partial_\mu^{n_t} h_1(t, \mu_t) = 0, \quad t \in (0, T).$$

Moreover, we assume that

$$q_1(t,x) \neq 0, \quad (t,x) \in Q,$$

and that for all $t \in (0,T)$, there exists $x_t \in \partial\Omega$ such that

$$q_1(t,x_t) = q_2(t,x_t) \neq 0, \quad t \in (0,T).$$

(iv) *There exists $h_j(t,\cdot) \in \mathbb{A}(\mathbb{R}; C^{\frac{\alpha}{2}}([0,T]))$ such that*

$$d_j(t,x,\mu) = q_{j,1}(t,x)h_j(t,q_{j,2}(t,x)\mu), \quad (t,x,\mu) \in [0,T] \times \overline{\Omega} \times \mathbb{R}, \ j = 1,2.$$

Assume also that, for all $t \in (0,T)$, there exists $n_t \in \mathbb{N}$ such that

$$\partial_\mu^{n_t} h_1(t,\cdot) \not\equiv 0, \quad \partial_\mu^{n_t} h_1(t,0) = 0, \quad t \in (0,T).$$

Moreover, we assume that

$$q_{1,1}(t,x) \neq 0, \quad q_{1,2}(t,x) \neq 0 \quad (t,x) \in Q,$$

and that for all $t \in (0,T)$, there exists $x_t \in \partial\Omega$ such that

$$q_{1,1}(t,x_t) = q_{2,1}(t,x_t) \neq 0, \quad q_{1,2}(t,x_t) = q_{2,2}(t,x_t) \neq 0, \quad t \in (0,T).$$

Then (8.5) implies that $d_1 = d_2$ and $F_1 = F_2$.

Proof. Fixing

$$b_j(t,x,\mu) = d_j(t,x,\mu) - F_j(t,x), \quad (t,x,\mu) \in Q \times \mathbb{R},$$

one can check that $\mathcal{N}_{b_j} = \mathcal{M}_{(d_j,F_j)}$ and (8.5) implies (2.4). Then, applying Corollary 2.1 and Theorem 2.3, 2.4, 2.5, we deduce that for semilinear terms d_j satisfying one of the conditions (i), (ii), (iii), (iv), we have

$$d_1(t,x,\mu) - F_1(t,x) = b_1(t,x,\mu) = b_2(t,x,\mu) = d_2(t,x,\mu) - F_2(t,x), \quad (t,x,\mu) \in Q \times \mathbb{R}.$$

Choosing $\mu = 0$ in the above identity and applying (8.3), we deduce that $F_1 = F_2$ and then $d_1 = d_2$. □

A. Carleman estimates

In the end of this paper, we prove the Carleman estimate in Section 6. For the sake of convenience, we also state the result as follows.

Lemma A.1. *Let $n \geqslant 3$, $q \in L^\infty(Q)$ and $v \in H^1(Q) \cap L^2(0,T;H^2(\Omega))$ satisfy the condition*

$$v|_\Sigma = 0, \quad v|_{t=0} = 0. \tag{A.1}$$

Then, there exists $\tau_0 > 0$ depending on T, Ω and $\|q\|_{L^\infty(Q)}$ such that for all $\tau > \tau_0$, the estimate

$$\tau \int_0^T \int_{\Gamma_+(x_0)} e^{-2(\tau^2 t + \tau\psi(x))} |\partial_\nu v|^2 |\partial_\nu \psi(x)| d\sigma(x) dt + \tau^2 \int_Q e^{-2(\tau^2 t + \tau\psi(x))} |v|^2 dxdt$$

$$\leqslant C \left(\int_Q e^{-2(\tau^2 t + \tau\psi(x))} |(\partial_t - \Delta_x + q)v|^2 \, dxdt \right. \tag{A.2}$$

$$\left. + \tau \int_0^T \int_{\Gamma_-(x_0)} e^{-2(\tau^2 t + \tau\psi(x))} |\partial_\nu v|^2 |\partial_\nu \psi(x)| d\sigma(x) \, dt \right)$$

holds true.

Proof. Recall that $\psi(x) = |x - x_0|$, $x \in \Omega$, for $x_0 \in \mathbb{R}^n \setminus \overline{\Omega}$, then ψ satisfies the eikonal equation $|\nabla_x \psi(x)| = 1$ for $x \in \Omega$. Without loss of generality, we assume that u is real valued and $q = 0$. In order to prove the estimate (A.2), we fix $v \in C^2(\overline{Q})$ satisfying (A.1), $s > 0$, and we set

$$w = e^{-(\tau^2 t + \tau\psi(x) - s\frac{\psi(x)^2}{2})} v$$

in such a way that

$$e^{-(\tau^2 t + \tau\psi(x) - s\frac{\psi(x)^2}{2})} (\partial_t - \Delta_x) v = P_{\tau,s} w, \tag{A.3}$$

where $P_{\tau,s}$ is given by

$$P_{\tau,s} = \partial_t - \Delta + 2s\tau\psi - \tau\Delta\psi + s\psi\Delta\psi - s^2\psi^2 + s - 2\tau\nabla\psi \cdot \nabla + 2s\psi\nabla\psi \cdot \nabla.$$

Here, we used the fact that the function ψ satisfies the eikonal equation.

We next decompose $P_{\tau,s}$ into two parts $P_{\tau,s} = P_{\tau,s,+} + P_{\tau,s,-}$ with

$$P_{\tau,s,+} := -\Delta + 2s\tau\psi - \tau\Delta\psi + s\psi\Delta\psi - s^2\psi^2 + s,$$
$$P_{\tau,s,-} := \partial_t - 2\tau\nabla\psi \cdot \nabla + 2s\psi\nabla\psi \cdot \nabla.$$

Then, it follows that

$$\|P_{\tau,s}w\|_{L^2(Q)}^2 \geqslant 2 \int_Q (P_{\tau,s,+}w)(P_{\tau,s,-}w) dxdt \tag{A.4}$$

$$:= I + II + III + IV + V + VI + VII,$$

where

$$I = -2 \int_Q \partial_t w \Delta w \, dxdt, \quad II = 4\tau \int_Q \Delta w \nabla\psi \cdot \nabla w \, dxdt,$$

$$III = 4s\tau \int_Q \psi w \partial_t w \, dxdt, \quad IV = -8s\tau^2 \int_Q \psi w \nabla\psi \cdot \nabla w \, dxdt,$$

$$V = -4s \int_Q \Delta w \psi \nabla\psi \cdot \nabla w \, dxdt, \quad VI = 8s^2\tau \int_Q \psi^2 w \nabla\psi \cdot \nabla w \, dxdt,$$

and

$$VII = 2 \int_Q \left[-\tau\Delta\psi + s\psi\Delta\psi - s^2\psi^2 + s \right] w (P_{\tau,s,-}w) dxdt.$$

Recalling that $w|_\Sigma = 0$ and $w|_{t=0} = 0$, fixing

$$c_* = \inf_{x \in \Omega} \psi(x) > 0$$

and integrating by parts, we find

$$I = 2 \int_Q \partial_t \nabla w \cdot \nabla w \, dx dt = \int_Q \partial_t |\nabla w|^2 \, dx dt = \int_\Omega |\nabla w(T, x)|^2 \, dx \geqslant 0,$$

$$III = 2s\tau \int_Q \psi \partial_t(w^2) \, dx dt = 2s\tau \int_\Omega \psi(x) w(T, x)^2 \, dx \geqslant 2c_* s\tau \int_\Omega w(T, x)^2 \, dx \geqslant 0,$$

$$IV = -4s\tau^2 \int_Q (x - x_0) \cdot \nabla(w^2) \, dx dt = 4s\tau^2 \int_Q \mathrm{div}(x - x_0) w^2 \, dx dt$$

$$= 4ns\tau^2 \int_Q w^2 \, dx dt \geqslant 0,$$

and similarly,

$$VI = -4s^2\tau \int_Q \mathrm{div}(\psi^2 \nabla \psi) w^2 \, dx dt = -4(n+1)s^2\tau \int_Q |x - x_0| w^2 \, dx dt,$$

where we utilized the fact that $\psi^2 \nabla \psi = |x - x_0|(x - x_0)$.

Now, let us consider II. Integrating by parts and using the fact that $w|_\Sigma = 0$, we get

$$II = 4\tau \int_\Sigma \partial_\nu w \nabla \psi \cdot \nabla w \, d\sigma(x) dt - 4\tau \int_Q \nabla w \cdot \nabla(\nabla \psi \cdot \nabla w) \, dx dt$$

$$= 4\tau \int_\Sigma (\partial_\nu w)^2 \partial_\nu \psi \, d\sigma(x) dt - 4\tau \int_Q D^2\psi(\nabla w, \nabla w) \, dx dt - 2\tau \int_Q \nabla \psi \cdot \nabla(|\nabla w|^2) \, dx dt$$

$$= 2\tau \int_\Sigma (\partial_\nu w)^2 \partial_\nu \psi \, d\sigma(x) dt - 4\tau \int_Q D^2\psi(\nabla w, \nabla w) \, dx dt + 2\tau \int_Q \Delta \psi |\nabla w|^2 \, dx dt$$

$$= 2\tau \int_\Sigma (\partial_\nu w)^2 \partial_\nu \psi \, d\sigma(x) dt - 4\tau \int_Q D^2\psi(\nabla w, \nabla w) \, dx dt + 2\tau \int_Q \frac{n-1}{|x - x_0|} |\nabla w|^2 \, dx dt.$$

However, one can check that

$$D^2\psi(x) = |x - x_0|^{-3} \left(|x - x_0|^2 \mathrm{Id}_{\mathbb{R}^{n \times n}} - N(x) \right), \quad x \in \Omega$$

with $N(x) = ((x_i - x_0^i)(x_j - x_0^j))_{1 \leqslant i, j \leqslant n}$ where $x = (x_1, \ldots, x_n)$ and $x_0 = (x_0^1, \ldots, x_0^n)$. Moreover, it can be proved that $N(x)$ is a symmetric matrix whose eigenvalues are either 0 or $|x - x_0|^2$. Thus, we get

$$0 \leqslant D^2\psi(x)(\nabla w(t, x), \nabla w(t, x)) \leqslant \frac{|\nabla w(t, x)|^2}{|x - x_0|}, \quad \text{for } (t, x) \in Q,$$

and it follows that

$$\begin{aligned} II &\geqslant 2\tau \int_\Sigma (\partial_\nu w)^2 \partial_\nu \psi \, d\sigma(x) dt + \tau \int_Q \frac{2(n-3)}{|x - x_0|} |\nabla w|^2 \, dx dt \\ &\geqslant 2\tau \int_\Sigma (\partial_\nu w)^2 \partial_\nu \psi \, d\sigma(x) dt + c_1(n-3)\tau \int_Q |\nabla w|^2 \, dx dt, \end{aligned} \tag{A.5}$$

where $c_1 = \inf_{x \in \Omega} \frac{2}{|x - x_0|}$.

Applying similar computations for V, one has that

$$
\begin{aligned}
V &= -2s \int_Q \Delta w \nabla \psi^2 \cdot \nabla w \, dxdt \\
&= -2s \int_\Sigma \partial_\nu w \nabla \psi^2 \cdot \nabla w \, d\sigma(x) dt + 2s \int_Q \nabla w \cdot \nabla(\nabla \psi^2 \cdot \nabla w) \, dxdt \\
&= -2s \int_\Sigma (\partial_\nu w)^2 \partial_\nu \psi^2 \, d\sigma(x) dt + 2s \int_Q D^2 \psi^2 (\nabla w, \nabla w) \, dxdt + s \int_Q \nabla \psi^2 \cdot \nabla(|\nabla w|^2) \, dxdt \\
&= -s \int_\Sigma (\partial_\nu w)^2 \partial_\nu \psi^2 \, d\sigma(x) dt + 2s \int_Q D^2 \psi^2 (\nabla w, \nabla w) \, dxdt - s \int_Q \Delta \psi^2 |\nabla w|^2 \, dxdt \quad\quad \text{(A.6)} \\
&= -s \int_\Sigma (\partial_\nu w)^2 \partial_\nu \psi^2 \, d\sigma(x) dt + 2s \int_Q D^2 \psi^2 (\nabla w, \nabla w) \, dxdt - sn \int_Q |\nabla w|^2 \, dxdt, \\
&\geqslant \underbrace{-s \int_0^T \int_{\Gamma_+(x_0)} 2\psi (\partial_\nu w)^2 \partial_\nu \psi \, d\sigma(x) dt + s(4-n) \int_Q |\nabla w|^2 \, dxdt}_{} .
\end{aligned}
$$

Here we use $\Delta \psi^2 = \Delta |x - x_0|^2 = n$ and $D^2 \psi^2 (\nabla w, \nabla w) = 2|\nabla w|^2$.

In addition, the sum of the last two terms on the right-hand side of (A.5) and (A.6) is

$$
\begin{cases}
s \int_Q |\nabla w|^2 \, dxdt & \text{for } n = 3, \\
c_1(n-3)\tau \int_Q |\nabla w|^2 \, dxdt + s(4-n) \int_Q |\nabla w|^2 \, dxdt & \text{for } n \geqslant 4.
\end{cases}
\quad\quad \text{(A.7)}
$$

By choosing $\tau \geqslant \frac{s(4-n)}{c_1(n-3)}$ for $n \geqslant 4$, both cases appearing in (A.7) are nonnegative. Now fixing $c_2 = 2 \sup_{x \in \partial\Omega} \psi(x)$, and choosing

$$
\tau > \tau_1(s) := \begin{cases}
c_2 s & \text{for } n = 3, \\
\max\left(c_2 s, \frac{s(4-n)}{c_1(n-3)}\right) & \text{for } n \geqslant 4,
\end{cases}
$$

we can deduce that

$$
II + V \geqslant \tau \int_0^T \int_{\Gamma_+(x_0)} (\partial_\nu w)^2 |\partial_\nu \psi| \, d\sigma(x) dt - 2\tau \int_0^T \int_{\Gamma_-(x_0)} (\partial_\nu w)^2 |\partial_\nu \psi| \, d\sigma(x) dt.
$$

In addition, repeating the above argumentation, it is not hard to check that there exists a constant $c_0 > 0$ independent of s and τ such that

$$
VI + VII \geqslant -c_0 \left((\tau^2 + s^2\tau + s^2) \int_Q w^2 \, dxdt + (\tau + s^2) \int_\Omega w(T,x)^2 \, dx \right).
$$

Thus, choosing $s = c_0(1 + c_*^{-1}) + 1$, $\tau_0 = \max(\tau_1(s), \frac{sn}{c_1}, 3s^2 + 1, \frac{sc_0}{c_*})$ and applying the above estimates, for all $\tau > \tau_0$, we obtain

$$
\begin{aligned}
\|P_{\tau,s} w\|_{L^2(Q)}^2 &\geqslant 2 \int_Q P_{\tau,s,+} w P_{\tau,s,-} w \, dxdt \\
&\geqslant c_1 \tau^2 \int_Q w^2 \, dxdt + \tau \int_0^T \int_{\Gamma_+(x_0)} (\partial_\nu w)^2 |\partial_\nu \psi| \, d\sigma(x) dt \\
&\quad - 2\tau \int_0^T \int_{\Gamma_-(x_0)} (\partial_\nu w)^2 |\partial_\nu \psi| \, d\sigma(x) dt,
\end{aligned}
$$

with $c_1 > 0$ depending only on Ω. From this last estimate and the fact that

$$\partial_\nu \psi(x) = \frac{(x - x_0) \cdot \nu}{|x - x_0|}, \quad x \in \partial\Omega,$$

we deduce easily (A.2). □

References

[CFK+21] C. I. Cârstea, A. Feizmohammadi, Y. Kian, K. Krupchyk and G. Uhlmann, 'The calderón inverse problem for isotropic quasilinear conductivities', *Adv. Math.* **391** (2021), 107956.

[Cho09] M. Choulli, *Une introduction aux problèmes inverses elliptiques et paraboliques* vol. **65** (Springer Science & Business Media, 2009).

[CK18a] P. Caro and Y. Kian, 'Determination of convection terms and quasi-linearities appearing in diffusion equations', Preprint, 2018, arXiv:1812.08495.

[CK18b] M. Choulli and Y. Kian, 'Logarithmic stability in determining the time-dependent zero order coefficient in a parabolic equation from a partial dirichlet-to-neumann map. application to the determination of a nonlinear term', *J. Math. Pures Appl.* **114** (2018), 235–261.

[COY06] M. Choulli, E. M. Ouhabaz and M. Yamamoto, 'Stable determination of a semilinear term in a parabolic equation', *Commun. Pure and Appl. Anal.* **5**(3) (2006), 447.

[CY88] J. R. Cannon and H. Yin, 'A uniqueness theorem for a class of nonlinear parabolic inverse problems', *Inverse Problems* **4**(2) (1988), 411.

[EPS17] H. Egger, J.-F. Pietschmann and M. Schlottbom, 'On the uniqueness of nonlinear diffusion coefficients in the presence of lower order terms', *Inverse Problems* **33**(11) (2017), 16. Id/No 115005.

[Fei23] A. Feizmohammadi, 'An inverse boundary value problem for isotropic nonautonomous heat flows', *Math. Ann.* (2023), **1–39**.

[Fis37] R. A. Fisher, 'The wave of advance of advantageous genes', *Ann. Eugen.* **7** (1937), 355–369.

[FKU22] A. Feizmohammadi, Y. Kian and G. Uhlmann, 'An inverse problem for a quasilinear convection-diffusion equation', *Nonlinear Anal.* **222** (2022), 30. Id/No 112921.

[FLL23] A. Feizmohammadi, T. Liimatainen and Y.-H. Lin, 'An inverse problem for a semilinear elliptic equation on conformally transversally anisotropic manifolds', *Ann. PDE* **9**(2) (2023), Paper No. 12, 54.

[FO20] A. Feizmohammadi and L. Oksanen. 'An inverse problem for a semi-linear elliptic equation in Riemannian geometries', *J. Differential Equations* **269**(6) (2020), 4683–4719.

[HL23] B. Harrach and Y.-H. Lin, 'Simultaneous recovery of piecewise analytic coefficients in a semilinear elliptic equation', *Nonlinear Anal.* **228** (2023), 113188.

[IM20] J. Ilmavirta and K. Mönkkönen, 'Unique continuation of the normal operator of the x-ray transform and applications in geophysics', *Inverse Problems* **36**(4) (2020), 23. Id/No 045014.

[IN95] V. Isakov and A. I. Nachman, 'Global uniqueness for a two-dimensional semilinear elliptic inverse problem', *Trans. Amer. Math. Soc.* **347**(9) (1995), 3375–3390.

[IS94] V. Isakov and J. Sylvester, 'Global uniqueness for a semilinear elliptic inverse problem', *Comm. Pure Appl. Math.* **47**(10) (1994), 1403–1410.

[Isa91] V. Isakov, 'Completeness of products of solutions and some inverse problems for PDE', *J. Differential Equations* **92**(2) (1991), 305–316.

[Isa93] V. Isakov, 'On uniqueness in inverse problems for semilinear parabolic equations', *Arch. Ration. Mech. Anal.* **124**(1) (1993), 1–12.

[Isa01] V. Isakov, 'Uniqueness of recovery of some systems of semilinear partial differential equations', *Inverse Problems* **17**(4) (2001), 607.

[IY98] O. Y. Imanuvilov and M. Yamamoto, 'Lipschitz stability in inverse parabolic problems by the Carleman estimate', *Inverse Problems* **14**(5) (1998), 1229.

[KKU23] Y. Kian, K. Krupchyk and G. Uhlmann, 'Partial data inverse problems for quasilinear conductivity equations', *Math Ann.* **385** (2023), 1611–1638.

[KLU18] Y. Kurylev, M. Lassas and G. Uhlmann, 'Inverse problems for Lorentzian manifolds and non-linear hyperbolic equations', *Invent. Math.* **212**(3) (2018), 781–857.

[KSU07] C. Kenig, J. Sjöstrand and G. Uhlmann, 'The Calderón problem with partial data', *Ann. Math.* **165**(2) (2007), 567–591.

[KSXY22] Y. Kian, É. Soccorsi, Q. Xue and M. Yamamoto, 'Identification of time-varying source term in time-fractional diffusion equations', *Commun. Math. Sci.* **20**(1) (2022), 53–84.

[KU20a] K. Krupchyk and G. Uhlmann, 'Partial data inverse problems for semilinear elliptic equations with gradient nonlinearities' *Math Res. Lett.* **27**(6) (2020), 1801–1824.

[KU20b] K. Krupchyk and G. Uhlmann, 'A remark on partial data inverse problems for semilinear elliptic equations', *Proc. Amer. Math. Soc.* **148** (2020), 681–685.

[KU23] Y. Kian and G. Uhlmann, 'Recovery of nonlinear terms for reaction diffusion equations from boundary measurements', *Arch. Ration. Mech. Anal.* **247**(1) (2023), 6.

[Lin22] Y.-H. Lin, 'Monotonicity-based inversion of fractional semilinear elliptic equations with power type nonlinearities', *Calc. Var. Partial Differential Equations* **61**(5) (2022), 1–30.

[LL19] R.-Y. Lai and Y.-H. Lin, 'Global uniqueness for the fractional semilinear Schrödinger equation', *Proc. Amer. Math. Soc.* **147**(3) (2019), 1189–1199.

[LL22a] R.-Y. Lai and Y.-H. Lin, 'Inverse problems for fractional semilinear elliptic equations', *Nonlinear Anal.* **216** (2022), 112699.

[LL22b] T. Liimatainen and Y.-H. Lin, 'Uniqueness results and gauge breaking for inverse source problems of semilinear elliptic equations, Preprint, 2022, arXiv:2204.11774.

[LL23] Y.-H. Lin and H. Liu, 'Inverse problems for fractional equations with a minimal number of measurements', *Commun. Anal. Comput.* **1**(1) (2023), 72–93.

[LLL21] Y.-H. Lin, H. Liu and X. Liu, 'Determining a nonlinear hyperbolic system with unknown sources and nonlinearity', Preprint, 2021, arXiv:2107.10219.

[LLLS20] M. Lassas, T. Liimatainen, Y.-H. Lin and M. Salo, 'Partial data inverse problems and simultaneous recovery of boundary and coefficients for semilinear elliptic equations', *Rev. Mat. Iberoam.* **37**(4) (2020), 1553–1580.

[LLLS21] M. Lassas, T. Liimatainen, Y.-H. Lin and M. Salo, 'Inverse problems for elliptic equations with power type nonlinearities', *J. Math. Pures Appl.* **145** (2021), 44–82.

[LLLZ22] Y.-H. Lin, H. Liu, X. Liu and S. Zhang, 'Simultaneous recoveries for semilinear parabolic systems' *Inverse Problems* **38**(11) (2022), 115006.

[LLST22] T. Liimatainen, Y.-H. Lin, M. Salo and T. Tyni, 'Inverse problems for elliptic equations with fractional power type nonlinearities', *J. Differential Equations* **306** (2022), 189–219.

[LSU88] O. A. Ladyženskaja, V. A. Solonnikov and N. N. Ural'ceva, *Linear and Quasi-Linear Equations of Parabolic Type* vol. 23 (American Mathematical Soc., 1988).

[NW87] A. C. Newell and J. A Whitehead, 'Finite bandwidth, finite amplitude convection', *J. Fluid Mech.* **38** (1987), 118–139.

[RR06] M. Renardy and R. C. Rogers, *An Introduction to Partial Differential Equations* vol. 13 (Springer Science & Business Media, 2006).

[SS87] J.-C. Saut and B. Scheurer, 'Unique continuation for some evolution equations', *J. Differential Equations* **66** (1987), 118–139.

[Sun10] Z. Sun, 'An inverse boundary-value problem for semilinear elliptic equations', *Electron. J. Differential Equations [electronic only]* **2010** (2010), 1–5.

[Vol14] V. Volpert, *Elliptic Partial Differential Equations: Volume 2: Reaction-Diffusion Equations* vol. **104** (Springer, 2014).

[ZFK38] Y. B. Zeldovich and D. A. Frank-Kamenetsky, 'A theory of thermal propagation of flame', *Acta Physicochim URSS* **9** (1938), 341–350.