THE CALDERÓN PROBLEM FOR THE SCHRÖDINGER EQUATION IN TRANSVERSALLY ANISOTROPIC GEOMETRIES WITH PARTIAL DATA

YI-HSUAN LIN, GEN NAKAMURA, AND PHILIPP ZIMMERMANN

Abstract. In this work, we study the partial data Calderón problem for the anisotropic Schrödinger equation

$$(-\Delta_{\widetilde{g}} + V)u = 0 \text{ in } \Omega \times (0,\infty), \tag{0.1}$$

where $\Omega \subset \mathbb{R}^n$ is a bounded smooth domain, $\widetilde{g} = g_{ij}(x)dx^i \otimes dx^j + dy \otimes dy$ and V is translationally invariant in the y direction. Our final goal is to show that both the metric g and the potential V can be recovered from the (partial) Neumann-to-Dirichlet (ND) map on $\Gamma \times \{0\}$ with $\Gamma \Subset \Omega$. Our approach can be divided into the following steps:

Step 1. Boundary determination. We establish a novel boundary determination to identify (g, V) on Γ with the help of suitable approximate solutions for (0.1) with inhomogeneous Neumann boundary condition.

Step 2. Relation to a nonlocal elliptic inverse problem. We relate inverse problems for the Schrödinger equation with the nonlocal elliptic equation

$$(-\Delta_g + V)^{1/2} v = f \text{ in } \Omega, \tag{0.2}$$

via the Caffarelli–Silvestre type extension, where the measurements are encoded in the source-to-solution map. The nonlocality of this inverse problem allows us to recover the associated heat kernel.

Step 3. Reduction to an inverse problem for a wave equation. Combining the knowledge of the heat kernel with the Kannai type transmutation formula, we transfer the inverse problem for (0.2) to an inverse problem for the wave equation

$$(\partial_t^2 - \Delta_g + V)w = F \text{ in } \Omega \times (0,\infty), \tag{0.3}$$

where the measurement operator is also the source-to-solution map. We can finally determine (g, V) on $\Omega \setminus \Gamma$ by solving the inverse problem for (0.3).

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1. Introduction

In this paper, we investigate an inverse boundary value problem for a certain class of elliptic partial differential equations (PDEs) on the transversal domain $\Omega \times (0, \infty)$. The nowadays prototypical example of an inverse problem for an elliptic PDE was introduced by Calderón [Cal06], in which the objective is to recover the conductivity γ in the *conductivity equation*

$$\operatorname{div}(\gamma \nabla u) = 0 \text{ in } \Omega \tag{1.1}$$

from the (full) *Dirichlet-to-Neumann (DN) map* Λ_γ. From a physical point of view, this corresponds to inducing a voltage f on the boundary and measuring the resulting (normal) current $\mathsf{j} = \gamma \partial_\nu u_f|_{\partial\Omega}$ across it, where u_f is the solution of (1.1) with $u_f = f$ on $\partial\Omega$ and ν denotes the outward pointing normal vector field along $\partial\Omega$. A closely related problem is the determination of an unknown potential q in the *Schrödinger equation*

$$(-\Delta + q)\, v = 0 \text{ in } \Omega \tag{1.2}$$

from the DN map $\Lambda_q g = \partial_\nu v_g|_{\partial\Omega}$, which was resolved in [SU87] for $n \geq 3$ and [Buk08] for $n = 2$. The solutions of (1.1) and (1.2) are connected via the Liouville reduction $v = \gamma^{1/2} u$, which also gives a precise relation between Λ_γ and Λ_q only involving $\gamma|_{\partial\Omega}$ and $\partial_\nu \gamma|_{\partial\Omega}$, and by the boundary determination result of Kohn and Vogelius [KV84], the solution of the inverse problem for the Schrödinger equation directly resolves the Calderón problem under suitable regularity assumptions on γ. Let us note that the result of Kohn and Vogelius is a local boundary determination result, that is to recover γ and $\partial_\nu \gamma$ at a boundary point $x_0 \in \partial\Omega$, one only needs to know Λ_γ in a small neighborhood of x_0. For a more comprehensive account of these results, we refer the readers to the survey article [Uhl09]. Inverse problems in transversally anisotropic geometries with full data or partial data have also been considered in various models, such as [DSFKSU09, DSFKLS16, DSFKL+20, KSU07, LLLS21, FO20, FLL23, KU18].

Recently, the above type of inverse problems has been extended to nonlocal models like

$$(L + q)\, u = 0 \text{ in } \Omega, \tag{1.3}$$

where L is, for example, an elliptic nonlocal operator, and one again aims to recover the potential q and possibly some coefficients on which L may depend from the related DN map $\Lambda_{L,q}$. The first model studied in the literature [GSU20] is the case of the fractional Laplacian $L = (-\Delta)^s$, having Fourier symbol $|\xi|^{2s}$, and the resulting equation (1.3) is by now usually called *fractional Schrödinger equation*. If one assumes that the nonlocal operator L in (1.3) satisfies the *unique continuation property (UCP)*, implying the Runge approximation, then one can show that the inverse problem related to (1.3) is uniquely solvable (see, for example, [RZ23, LZ23]). Two classical examples of nonlocal operators having the UCP are the fractional Laplacian [GSU20] and the Bessel potential operator $\langle D\rangle^s$ with Fourier symbol $\left(1+|\xi|^2\right)^{s/2}$ [KPPV20]. Let us note that in both cases proving the UCP for these operators rests on the existence of a nice extension problem related to these operators, but unfortunately, up to now, there is no characterization of nonlocal operators having this property.

More precisely, Caffarelli and Silvestre [CS07] characterized the fractional Laplacian $(-\Delta)^s$ as the Dirichlet-to-Neumann operator for the associated extension problem. This point of view of the fractional Laplacian is commonly referred to as the *Caffarelli-Silvestre extension* in the literature. The UCP for this extension problem was shown in [Rül15]. For general variable coefficients nonlocal elliptic operators of order $s \in (0,1)$, Stinga and Torrea [ST10] demonstrated analogous results such that this type of nonlocal operators can be also characterized via the related extension problem. Based on this, the authors of [GLX17] solved the Calderón problem for variable coefficients nonlocal operators, whereas the analogous result for their local counterpart remains open in dimensions $n \geq 3$. Indeed, there are several uniqueness results for nonlocal inverse problems, which are still open for their local counterparts and maybe even not true, such as the drift problem [CLR20], the obstacle problem [CLL19], the inverse source problem [LL23], and the characterization via monotonicity relations [HL19, HL20]. Hence, one can regard the nonlocality as a tool that helps solve inverse problems. Recently, the works [CGRU23, LLU23, Rül23, LZ24] provide interesting connections between the nonlocal and local Calderón type inverse problems for both elliptic and parabolic equations. We also refer readers to several related articles for nonlocal operators, such as [CRZ22, KRZ23, KLZ24, CRTZ24, LZ23, LTZ24] and the references therein. Very recently, the recovery of the geometrical information (M, g) and potential V simultaneously has been investigated by [FKU24] on closed Riemannian manifolds, and we also refer readers to [Fei24, FGKU24] as the potential $V = 0$.

Based on this observation, we study in this article a class of inverse problems for (local) elliptic PDEs having a similar form as the ones emerging in the related extension problems for the aforementioned operators. In the next section, we introduce the considered model in more detail.

1.1. **Mathematical formulation.** Let Ω be a bounded smooth domain[1] in $\mathbb{R}^n$ with $n \geq 2$. Suppose that we have given on Ω a (smooth) Riemannian metric $g = (g_{ij})_{1\leq i,j\leq n}$ satisfying the uniform ellipticity condition

$$\lambda |\xi|^2 \leq g_{ij}(x)\xi^i\xi^j \leq \lambda^{-1}|\xi|^2 \text{ in } \Omega, \tag{1.4}$$

for some constant $\lambda \in (0,1)$ and for any vector $\xi = \left(\xi^1, \ldots, \xi^n\right) \in \mathbb{R}^n$. Throughout the whole article, we impose the Einstein summation convention. Let Δ_g be the Laplace-Beltrami operator given by

$$\Delta_g u := |g|^{-1/2}\partial_i\left(|g|^{1/2}g^{ij}\partial_j u\right),$$

where $|g| = \det g$, g^{ij} denotes the components of the inverse matrix g^{-1} and $\partial_j = \partial_{x^j}$.

[1] Throughout this work we say $D \subset \mathbb{R}^n$ is a domain if it is an open connected set.

To formulate the PDE problem, let us extend g to a Riemannian metric $\widetilde{g}$ on $\Omega \times \mathbb{R}_+$, where $\mathbb{R}_+ = (0, \infty)$, by setting

$$\widetilde{g} = g_{ij} dx^i \otimes dx^j + dy \otimes dy \tag{1.5}$$

or equivalently in matrix form

$$\widetilde{g}(x) = \begin{pmatrix} g(x) & 0 \\ 0 & 1 \end{pmatrix}. \tag{1.6}$$

In equation (1.5) and below, we denote the coordinates in $\Omega \times \mathbb{R}_+$ by (x, y) or $(x^1, \ldots, x^n, x^{n+1})$ and the range of the indices about we sum will always be clear from the context. Then the induced Laplace–Beltrami operator on $\Omega \times \mathbb{R}_+$ becomes

$$\Delta_{\widetilde{g}} = \Delta_g + \partial_y^2.$$

Next, let us consider the following mixed boundary value problem for *anisotropic Schrödinger equation*

$$\begin{cases} (-\Delta_{\widetilde{g}} + V)\, u = 0 & \text{in } \Omega \times \mathbb{R}_+, \\ -\partial_y u = f & \text{on } \Omega \times \{0\}, \\ u = 0 & \text{on } \partial\Omega \times \mathbb{R}_+, \end{cases} \tag{1.7}$$

where $V = V(x)$ is a given bounded nonnegative potential, which is translation invariant in the y-direction.

With the well-posedness result of equation (1.7) (see Section 2) at hand, we can define for any domain $\Gamma \subsetneq \Omega$ the related *(partial) Neumann-to-Dirichlet (ND) map*

$$\Lambda^\Gamma_{g,V} : H^{-1/2}(\overline{\Gamma}) \to H^{1/2}(\Gamma), \quad f \mapsto u_f|_\Gamma,$$

where we identify Γ with $\Gamma \times \{0\}$ and $u_f \in H^1_0(\Omega \times [0, \infty))$ is the unique solution to (1.7). The involved function spaces will be introduced in Section 2. Now, we can formulate the considered inverse problem.

- **(IP1) Inverse problem for the elliptic equation.** Can one uniquely determine the metric g and potential V in Ω by using the knowledge of the partial ND map $\Lambda^\Gamma_{g,V}$?

If $\Gamma = \Omega$, this inverse problem **(IP1)** can be viewed as the *boundary determination* problem for both g and V, since both g and V depend only on the x-variable. This can be proved by introducing suitable *approximate solutions* (see Section 3) so that both g and V can be recovered. Because of this, we assumed that $\Gamma \neq \Omega$.

Theorem 1.1 (Global recovery)**.** *Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$, be a bounded smooth domain, and $\Gamma \Subset \Omega$ be a domain with smooth boundary $\partial\Gamma$, so that Γ and $\Omega \setminus \overline{\Gamma}$ are connected. Let $g_1, g_2 \in C^\infty(\overline{\Omega}; \mathbb{R}^{n\times n})$ be two Riemannian metrics satisfying the uniform ellipticity condition (1.4) (extended to $\Omega \times \mathbb{R}_+$ via (1.5)). Assume that the two potentials $0 \leq V_1, V_2 \in C^\infty(\overline{\Omega})$ are translation invariant in the y-direction. Let $\Lambda^\Gamma_{g_j,V_j}$ be the partial ND map of*

$$\begin{cases} \left(-\Delta_{\widetilde{g}_j} + V_j\right) u_j = 0 & \textit{in } \Omega \times \mathbb{R}_+, \\ -\partial_y u_j = f & \textit{on } \Omega \times \{0\}, \\ u_j = 0 & \textit{on } \partial\Omega \times \mathbb{R}_+, \end{cases} \tag{1.8}$$

for $j = 1, 2$. Suppose that

$$\Lambda^\Gamma_{g_1,V_1} f = \Lambda^\Gamma_{g_2,V_2} f \ \textit{ on } \Gamma \textit{ for any } f \in C^\infty_c(\Gamma), \tag{1.9}$$

then there exists a diffeomorphism $\Psi : \overline{\Omega} \to \overline{\Omega}$ with $\Psi|_\Gamma = \mathrm{Id}_{\overline{\Gamma}}$ such that

$$g_1 = \Psi^* g_2 \textit{ in } \overline{\Omega} \quad \textit{and} \quad V_1 = V_2 \circ \Psi \textit{ in } \overline{\Omega},$$

where $\mathrm{Id}_{\overline{\Gamma}}$ denotes the identity map on $\overline{\Gamma}$.

Note that in the case $V \equiv 0$, we have the following uniqueness result.

Corollary 1.2. *Suppose all assumptions of Theorem 1.1 hold and let $\Lambda^{\Gamma}_{g_j,0}$ be the local ND map of*

$$\begin{cases} \Delta_{\widetilde{g}_j} u_j = 0 & \text{in } \Omega \times \mathbb{R}_+, \\ -\partial_y u_j = f & \text{on } \Omega \times \{0\}, \\ u_j = 0 & \text{on } \partial\Omega \times \mathbb{R}_+, \end{cases}$$

for $j = 1, 2$. Suppose that

$$\Lambda^{\Gamma}_{g_1,0} f = \Lambda^{\Gamma}_{g_2,0} f \text{ for any } f \in H^{-1/2}(\overline{\Gamma}),$$

holds, then there exists a diffeomorphism $\Psi \colon \overline{\Omega} \to \overline{\Omega}$ with $\Psi|_{\overline{\Gamma}} = \mathrm{Id}_{\overline{\Gamma}}$ in $\overline{\Gamma}$ such that $g_1 = \Psi^* g_2$ in Ω.

The preceding results are related to the Calderón problem on transversally anisotropic geometries. In the work [DSFKLS16], the authors investigated similar problems by using lateral boundary Cauchy data, under appropriate geometrical conditions for the manifold. However, in this work, we utilize the measurement from the bottom of the domain, which makes the problems treated in these two papers essentially different. In addition, Corollary 1.2 can be viewed as a special case of the anisotropic Calderón problem (1.1), where the scalar conductivity γ is replaced by a conductivity matrix (γ_{ij}) and incorporates the physical situation in which the medium has a directional dependent resistivity $\rho = \gamma^{-1}$. This implies that the current j does not necessarily flow into the direction of the electrical field E, as they satisfy the relation $\mathrm{j} = \rho E$, and such behavior is met in various materials. On the one hand, both the metric and the potential are y-independent, which means that g and V depend on n variables. On the other hand, the (localized) ND map $\Lambda^{\Gamma}_{g,V}$ is $2n$-dimensional, which is different from the classical Calderón type inverse problems that n-variables with $(2n-2)$ boundary measurements. Hence, we have 2-dimensional more boundary measurements that can be used in our study.

Let us point out that even if g is isotropic (i.e. $g_{ij} = \sigma\delta_{ij}$ for some scalar function σ), it is impossible to determine both g and V in general, due to the natural obstruction from the Liouville reduction. More concretely, let us use the forthcoming classical example to demonstrate why the result fails in general. Consider a positive scalar function $\sigma \in C^\infty(\overline{\Omega})$ with $\sigma = 1$ near $\partial\Omega$, and $q \in L^\infty(\Omega)$. Then the DN data of

$$-\nabla \cdot (\sigma \nabla u) + qu = 0 \text{ and } \underbrace{-\Delta w + \left(\frac{\Delta\sqrt{\sigma}}{\sqrt{\sigma}} + \frac{q}{\sigma} \right) w = 0}_{\text{Liouville's reduction: } v = \sqrt{\sigma} w}$$

are the same, that is, $(u|_{\partial\Omega},\ \partial_\nu u|_{\partial\Omega}) = (w|_{\partial\Omega},\ \partial_\nu w|_{\partial\Omega})$, where we used $\sigma = 1$ near $\partial\Omega$ and ν is the unit outer normal on $\partial\Omega$. However, it is easy to see that their coefficients could be different. This type of inverse problem is usually referred to as the *diffuse optical tomography problem* in the literature, which was investigated in [Arr99, AL98, Har09]. Therefore, one would not expect that the injectivity for the previously described Calderón problem (1.7) can be achieved.

As we mentioned before, in our model (1.7), g and V are transversally dependent, but independent of the vertical variable. A typical example is graphite, which is composed of multiple layers of graphene possessing, microscopically, a honeycomb lattice. The directionally different conducting properties of graphite rests on the fact that the layers are held together via the relatively weak Van der Waals forces, whereas one observes delocalized π-systems in each graphene layer. Based on this, the conductivity is much smaller in the transversal direction and the $\pi-$system is mostly responsible for the planar conduction. As a first approximation, one may regard the conductivity as being constant, as we do it in the problem (1.7), but the planar part of the conductivity matrix still depends on the y-coordinate as there are different forms of stacking of the graphene layers and the layers have a nonzero distance to each other. For a more detailed account of the physical properties of such materials we refer to the specialized literature (see e.g. [AM76, CFF+18]). Models having a non-trivial y-dependence will not be studied in this work.

Finally, let us mention that in the course of proving Theorem 1.1, we also establish the following unique determination result for an elliptic nonlocal inverse problem, which is a generalization of [FGKU24, Theorem 1.1].

Theorem 1.3. *Assume that Ω, Γ, (g_j, V_j) for $j = 1, 2$ are given as in Theorem 1.1 and let $(g, V) \in C^\infty(\overline{\Omega}; \mathbb{R}^{n\times n}) \times C^\infty(\overline{\Omega})$ be any pair of a uniformly elliptic Riemannian metric g and nonnegative potential V such that*

$$(g_1|_\Gamma\,, V_1|_\Gamma) = (g_2|_\Gamma\,, V_2|_\Gamma) = (g|_\Gamma, V|_\Gamma)\,. \tag{1.10}$$

Let $\mathcal{S}^\Gamma_{g_j,V_j} : C^\infty_c(\Gamma) \ni f \mapsto v^f_j|_\Gamma \in L^2(\Gamma)$ be the local source-to-solution map of

$$\begin{cases} (-\Delta_{g_j} + V_j)^{1/2} v = f & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega \end{cases}$$

for $j = 1, 2$. Suppose that

$$\mathcal{S}^\Gamma_{g_1,V_1} f = \mathcal{S}^\Gamma_{g_2,V_2} f \text{ for any } f \in C^\infty_c(\Gamma), \tag{1.11}$$

then there exists a diffeomorphism $\Psi\colon \overline{\Omega} \to \overline{\Omega}$ with $\Psi|_{\overline{\Gamma}} = \mathrm{Id}_{\overline{\Gamma}}$ such that

$$g_1 = \Psi^* g_2 \quad \text{and} \quad V_1 = V_2 \circ \Psi \text{ in } \Omega.$$

1.2. **Strategy of proof.** Next, let us explain our approach to prove Theorem 1.1 (cf. **(IP1)**).

Step 1. Boundary determination. In the first step, we establish a novel boundary determination result, which shows that the ND map on Γ, denoted by $\Lambda^\Gamma_{g,V}$, determines the metric g and the potential V on Γ. To achieve this goal, we will construct suitable approximate solutions for the anisotropic Schrödinger equation (1.7) with inhomogeneous Neumann boundary condition on the bottom $\Omega \times \{0\}$ and homogeneous Dirichlet boundary condition on the lateral boundary $\partial\Omega \times (0, \infty)$.

Step 2. Relation to a nonlocal elliptic inverse problem. In the next step, we relate via the Caffarelli–Silvestre type extension technique [CS07, ST10] (see Section 4.2) the inverse problem for the Schrödinger equation with an inverse problem for the nonlocal elliptic equation

$$(-\Delta_g + V)^{1/2}\, v = f \text{ in } \Omega, \tag{1.12}$$

where the measurements are encoded in the *source-to-solution map*. The nonlocality of this inverse problem allows us to recover the associated heat kernel of the heat operator $\partial_t - \Delta_g + V$ on $\Gamma \times (0, \infty)$. This is partially inspired by the work [FGKU24] and will be utilized in the proof of our main result (cf. **(IP2)**).

Step 3. Reduction to an inverse problem for a wave equation. In the third step, by combining the knowledge of the heat kernel with the Kannai type transmutation formula, we relate the nonlocal inverse problem for (1.12) to an inverse problem for the wave equation

$$\left(\partial_t^2 - \Delta_g + V\right) w = F \text{ in } \Omega \times (0, \infty), \tag{1.13}$$

where the measurement operator is again the source-to-solution map and the wave w vanishes on the lateral boundary $\partial\Omega$ and has zero initial conditions (cf. **(IP3)**).

By relating this measurement map with a restricted Dirichlet-to-Neumann (DN) map for the wave equation (1.13) and using existing uniqueness results for wave equations (cf. [KOP18]) we can finally determine (g, V) on $\Omega \setminus \Gamma$.

Finally, let us remark that for Calderón type inverse problems, many research articles establish unique determination results by using *complex geometrical optics* (CGO) solutions. For example in the classical Calderón problem for the Schrödinger equation $-\Delta + q$, they can be used together with a suitable integral identity to show that the Fourier transform of the difference of the potentials vanishes. The above-outlined approach does not require these special solutions, but let us emphasize that the boundary determination result also relies on oscillating approximate solutions (Lemma 3.2) and appropriate integral identities (Theorem 3.1).

1.3. **Organization of the paper.** The paper is organized as follows. In Section 2, we define the function spaces used throughout this work and prove the well-posedness of (1.7), so that the corresponding localized ND map can be defined rigorously. In Section 3, we show that the localized ND map $\Lambda^{\Gamma}_{g,V}$ determines both g and V on the open set Γ, which can be viewed as a boundary determination result. We give a characterization of the anisotropic Schrödinger equation and the associated nonlocal elliptic equation in Section 4. We also transfer our local inverse problem to a nonlocal inverse problem in this section and show that the corresponding heat kernel is determined. In Section 5, we use a Kannai-type transmutation formula together with the known heat kernels to transfer the information from the elliptic nonlocal inverse problem to an inverse problem for a wave equation. This inverse problem is eventually solved by using existing unique determination results for wave equations. Furthermore, in the Appendices A, B and C we collect some proofs of necessary background material, which we used throughout the article.

2. Preliminaries

In this section we collect some fundamental material that will be utilized throughout our work.

2.1. **Function spaces.** If U is an open subset of some Euclidean space $\mathbb{R}^m$, we denote by $L^2(U)$ and $H^1(U)$ the usual Lebesgue and Sobolev spaces concerning the Lebesgue measure. These are Hilbert spaces, carry the norms

$$\|u\|_{L^2(U)} := \left(\int_U |u|^2\,dx\right)^{1/2},$$
$$\|u\|_{H^1(U)} := \left(\|u\|^2_{L^2(U)} + \|\nabla u\|^2_{L^2(U)}\right)^{1/2},$$

and the related inner products are defined via the polarization identity. Here ∇ denotes the usual gradient concerning the Euclidean metric $h_{ij} = \delta_{ij}$. If U has a Lipschitz boundary, then clearly we have a well-defined (bounded) trace operator $H^1(U) \ni u \mapsto u|_{\partial U} \in L^2(\partial U, d\mathcal{H}^{m-1})$, where $d\mathcal{H}^{m-1}$ is the $(m-1)$-dimensional Hausdorff measure, and its image coincides with the Slobodeckij space $H^{1/2}(\partial U)$, that is the space of functions v on ∂U such that

$$\|v\|_{H^{1/2}(\partial U)} := \left(\|v\|^2_{L^2(\partial U)} + [v]^2_{H^{1/2}(\partial U)}\right)^{1/2} < \infty, \tag{2.1}$$

where $[\cdot]_{H^{1/2}(\partial U)}$ is the Gagliardo seminorm given by

$$[v]_{H^{1/2}(\partial U)} := \left(\int_{\partial U \times \partial U} \frac{|v(x) - v(y)|^2}{|x-y|^m}\, d\mathcal{H}^{m-1}(x) d\mathcal{H}^{m-1}(y)\right)^{1/2}. \tag{2.2}$$

The dual space of $H^{1/2}(\partial U)$ is denoted by $H^{-1/2}(\partial U)$. For any open set $\Gamma \subset \partial U$, the spaces $H^{1/2}(\Gamma)$ are defined exactly as in (2.1) and (2.2) up to replacing ∂U by

Γ. If $u \in H^{-1/2}(\partial U)$ is supported in $\overline{\Gamma}$, where $\Gamma \subset \partial U$ is a given open set, then we say u belongs to the space $H^{-1/2}(\overline{\Gamma})$. Next, let us observe that the trace operator is bounded as a map from $H^1(U)$ to $H^{1/2}(\partial U)$. Furthermore, for any open set $U \subset \mathbb{R}^m$ we define

$$H_0^1(U) := \text{closure of } C_c^\infty(U) \text{ in } H^1(U),$$

and if U is a Lipschitz domain, then $H_0^1(U)$ coincides with the kernel of the trace operator.

Next, we introduce some relevant notations for the Riemannian setting. If $U \subset \mathbb{R}^m$ is a given open set with coordinates $\left(x^1, \dots, x^m\right)$, Riemannian metric $h = (h_{ij})$ and inverse $h^{-1} = \left(h^{ij}\right)$, then we denote the induced Riemannian measure by

$$dV_h := |h|^{1/2} dx^1 \dots dx^m$$

with $|h| = \det(h)$ and the inner products of vector fields and 1-forms by

$$X \cdot Y := h_{ij} X^i Y^j, \quad \omega \cdot \eta := h^{ij} \omega_i \eta_j,$$

where $X = X^i \partial_i$, $Y = Y^j \partial_j$, $\omega = \omega_i dx^i$ and $\eta = \eta_j dx^j$. The latter definition is consistent with the musical isomorphism between the tangent and cotangent space, which reads in coordinates $X_i = g_{ij} X^j$. As usual we set $|X| = \sqrt{X \cdot X}$ and $|\omega| = \sqrt{\omega \cdot \omega}$, when X is a vector field and ω a 1-form. We believe that these notations will not lead to any confusion as it is always clear from the context to which we are referring. In particular, if u, v are functions on U and d denotes the exterior derivative, we have

$$du \cdot dv = h^{ij} \partial_i u \partial_j v \quad \text{and} \quad du \cdot \xi = h^{ij} \left(\partial_i u\right) \xi_j,$$

for any $\xi = (\xi_1, \dots, \xi_n) \in \mathbb{R}^n$. Furthermore, we set

$$\|u\|_{L^2(U;dV_h)} := \left(\int_U |u|^2 dV_h\right)^{1/2}$$

and

$$\|u\|_{H^1(U;dV_h)} := \left(\|u\|^2_{L^2(U;dV_h)} + \|du\|^2_{L^2(U;dV_h)}\right)^{1/2}$$

for functions u on U. Note that if the (smooth) Riemannian metric $h = (h_{ij})$ is uniformly elliptic (fulfilling the condition (1.4)), then one clearly has

$$\|u\|_{L^2(U)} \sim \|u\|_{L^2(U;dV_h)} \quad \text{and} \quad \|\nabla v\|_{L^2(U)} \sim \|dv\|_{L^2(U;dV_h)} \tag{2.3}$$

for all $u \in L^2(U)$ and $v \in H^1(U)$, where $\sim$ indicates equivalence of norms. In other words, there are positive constants c, C independent of u such that

$$c\|u\|_{L^2(U;dV_h)} \leq \|u\|_{L^2(U)} \leq C\|u\|_{L^2(U;dV_h)}.$$

Similar statements hold for the higher order spaces $H^k(U)$ and $H^k(U, dV_h)$ for $k \in \mathbb{N}$.

Finally, we introduce a function space consisting of functions with vanishing trace on part of the boundary, which is adapted to our problem (1.7). For this assume that $\Omega \subset \mathbb{R}^n$ is a Lipschitz domain carrying a uniformly elliptic Riemannian metric $g = (g_{ij})$ with canonical extension $\widetilde{g}$ to $\Omega \times \mathbb{R}_+$ (see (1.5)). Moreover, let dV_g, $dV_{\widetilde{g}}$ be the Riemannian measures on Ω and $\Omega \times \mathbb{R}_+$, respectively. Then we define

$$H_0^1(\Omega \times [0,\infty)) := \text{closure of } C_c^1(\Omega \times [0,\infty)) \text{ in } H^1(\Omega \times [0,\infty)).$$

This function space will play on the one hand the role of the solution space and on the other hand the space of test functions in the weak formulations for our mixed boundary value problems.

2.2. **Well-posedness for the elliptic equation.** Let us start by defining the bilinear form related to the PDE

$$\begin{cases} (-\Delta_{\widetilde{g}} + V)\, u = 0 & \text{in } \Omega \times \mathbb{R}_+, \\ -\partial_y u = f & \text{on } \Omega \times \{0\}, \\ u = 0 & \text{on } \partial\Omega \times \mathbb{R}_+. \end{cases} \tag{2.4}$$

Proposition 2.1 (Bilinear form)**.** *Let $\Omega \subset \mathbb{R}^n$ be a Lipschitz domain endowed with a uniformly elliptic Riemannian metric $g = (g_{ij})$ and extension $\widetilde{g}$ to $\Omega \times \mathbb{R}_+$. Suppose that $V \geq 0$ is a bounded potential. Then the map $B_{g,V} \colon H^1_0(\Omega \times [0,\infty)) \times H^1_0(\Omega \times [0,\infty)) \to \mathbb{R}$ given by*

$$B_{g,V}(u,\varphi) := \int_{\Omega\times\mathbb{R}_+} (du \cdot d\varphi + V u \varphi)\, dV_{\widetilde{g}} \tag{2.5}$$

is bounded, coercive bilinear form.

Proof. The bilinearity is obvious and the boundedness is an immediate consequence of the uniform ellipticity of g, the equivalence (2.3) and Hölder's inequality. The coercivity on the other hand follows by $V \geq 0$, the Poincaré inequality (Theorem A.2) and again the uniform ellipticity of g as well as the equivalence (2.3). $\square$

Now, by the Lax–Milgram theorem we can easily establish the following well-posedness result.

Lemma 2.2 (Well-posedness)**.** *Let $\Omega \subset \mathbb{R}^n$ be a Lipschitz domain endowed with a uniformly elliptic Riemannian metric $g = (g_{ij})$ and extension $\widetilde{g}$ to $\Omega \times \mathbb{R}_+$ given by (1.6). Suppose that $V \geq 0$ is a bounded potential. Then for any $f \in H^{-1/2}(\overline{\Omega}\times\{0\})$, there exists a unique solution $u = u_f \in H^1_0(\Omega \times [0,\infty))$ of (2.4), that is there holds*

$$B_{g,V}(u,\varphi) = \left\langle f, |g|^{1/2}\varphi|_{\Omega\times\{0\}} \right\rangle \tag{2.6}$$

for all $\varphi \in H^1_0(\Omega\times[0,\infty))$, where $\langle\cdot,\cdot\rangle$ denotes the duality pairing between $H^{1/2}(\Omega\times\{0\})$ and $H^{-1/2}(\overline{\Omega}\times\{0\})$. Moreover, the unique solution u satisfies the estimate

$$\|u\|_{H^1(\Omega\times\mathbb{R}_+)} \leq C\|f\|_{H^{-1/2}(\overline{\Omega}\times\{0\})} \tag{2.7}$$

for some $C > 0$ independent of u and f.

Proof. First of all let us observe that the map $\ell_f \colon H^1_0(\Omega \times [0,\infty)) \to \mathbb{R}$ defined via

$$\ell_f(\varphi) = \left\langle f, |g|^{1/2}\varphi|_{\Omega\times\{0\}} \right\rangle$$

for $\varphi \in H^1_0(\Omega \times [0,\infty))$ is a bounded linear map. There holds

$$\begin{aligned} |\ell_f(\varphi)| &\leq C\|f\|_{H^{-1/2}(\overline{\Omega}\times\{0\})} \left\|\varphi|_{\Omega\times\{0\}}\right\|_{H^{1/2}(\Omega\times\{0\})} \\ &\leq C\|f\|_{H^{-1/2}(\overline{\Omega}\times\{0\})}\|\varphi\|_{H^1(\Omega\times\mathbb{R}_+)} \end{aligned}$$

for all $\varphi \in H^1_0(\Omega\times[0,\infty))$, where we used the trace theorem. By Proposition 2.1 we can apply the Lax–Milgram theorem and can conclude that there exists a unique $u \in H^1_0(\Omega \times [0,\infty))$ satisfying (2.6) and

$$\|u\|_{H^1(\Omega\times\mathbb{R}_+)} \leq C \left\|\ell_f\right\|_{(H^1_0(\Omega\times[0,\infty))^*} \leq C\|f\|_{H^{-1/2}(\overline{\Omega}\times\{0\})}.$$

This proves the assertion. $\square$

One has the following elliptic estimate:

Proposition 2.3 (Elliptic estimate)**.** *Let $\Omega \subset \mathbb{R}^n$ be a Lipschitz domain endowed with a uniformly elliptic Riemannian metric $g = (g_{ij})$ and extension $\widetilde{g}$ to $\Omega \times \mathbb{R}_+$*

given by (1.6)*. Suppose that* $V \geq 0$ *is a bounded potential,* $G \in L^2(\Omega \times \mathbb{R}_+)$ *and* $f \in H^{-1/2}(\Omega \times \{0\})$*. If* $v \in H_0^1(\Omega \times [0,\infty))$ *solves*

$$\begin{cases} (-\Delta_{\widetilde{g}} + V)\, v = G & \text{in } \Omega \times \mathbb{R}_+, \\ -\partial_y v = f & \text{on } \Omega \times \{0\}, \\ v = 0 & \text{on } \partial\Omega \times \mathbb{R}_+, \end{cases} \tag{2.8}$$

then there holds

$$\|v\|_{H^1(\Omega\times\mathbb{R}_+)} \leq C\big(\|G\|_{L^2(\Omega\times\mathbb{R}_+)} + \|f\|_{H^{-1/2}(\overline{\Omega}\times\{0\})}\big), \tag{2.9}$$

for some constant $C > 0$ *independent of* v*,* G *and* f*.*

Proof. Note that by assumption there holds

$$B_{g,V}(v,\varphi) = \langle G, \varphi\rangle_{L^2(\Omega\times\mathbb{R}_+, dV_{\widetilde{g}})} + \big\langle f, |g|^{1/2}\varphi|_{\Omega\times\{0\}}\big\rangle,$$

for all $\varphi \in H_0^1(\Omega \times [0,\infty))$. Using $\varphi = v$ as a test function, then the coercivity of $B_{g,V}$ (Proposition 2.5) and the trace theorem imply

$$\begin{aligned} c\|v\|^2_{H^1(\Omega\times\mathbb{R}_+)} &\leq B_{g,V}(v,v) \\ &\leq \|G\|_{L^2(\Omega\times\mathbb{R}_+, dV_{\widetilde{g}})}\|v\|_{L^2(\Omega\times\mathbb{R}_+, dV_{\widetilde{g}})} \\ &\quad + \|f\|_{H^{-1/2}(\overline{\Omega}\times\{0\})}\||g|^{1/2}v\|_{H^{1/2}(\Omega\times\{0\})} \\ &\leq C\big(\|G\|_{L^2(\Omega\times\mathbb{R}_+)} + \|f\|_{H^{-1/2}(\overline{\Omega}\times\{0\})}\big)\|v\|_{H^1(\Omega\times\mathbb{R}_+)}, \end{aligned}$$

for some $C > 0$. Hence, we can conclude the proof. $\square$

We also define the alternative bilinear form

$$\begin{aligned} \mathcal{B}_{g,V}(u,\varphi) &:= B_{g,V}(u, |g|^{-1/2}\varphi) \\ &= \int_{\Omega\times\mathbb{R}_+} \big[\widetilde{g}^{-1}\nabla_{x,y}u\cdot\nabla_{x,y}\varphi + |g|^{1/2}\, g^{-1}\nabla\, |g|^{-1/2}\cdot\nabla u\varphi + Vu\varphi\big]\,dxdy, \end{aligned} \tag{2.10}$$

where $\widetilde{g}$ is given by (1.6) and the matrix g^{-1} has coefficients g^{ij} for $1 \leq i,j \leq n$. In terms of this bilinear form, a solution $v \in H_0^1(\Omega\times[0,\infty))$ of (2.8) satisfies

$$\mathcal{B}_{g,V}(v,\varphi) = \langle G,\varphi\rangle_{L^2(\Omega\times\mathbb{R}_+)} + \big\langle f, \varphi|_{\Omega\times\{0\}}\big\rangle\,,$$

for all $\varphi \in H_0^1(\Omega\times[0,\infty))$. Here (and in the definition of $\mathcal{B}_{g,V}$) we are using that our Riemannian metric g belongs to the class $C^\infty(\overline{\Omega};\mathbb{R}^{n\times n})$.

2.3. **Neumann-to-Dirichlet map.** With the well-posedness of (2.4) and definition (2.10), we can define the partial ND map.

Proposition 2.4 (Partial ND map)**.** *Let* $\Omega \subset \mathbb{R}^n$ *be a Lipschitz domain endowed with a uniformly elliptic Riemannian metric* $g = (g_{ij})$*, and extension* $\widetilde{g}$ *to* $\Omega \times \mathbb{R}_+$ *given by* (1.6)*. Suppose that* $0 \leq V \in L^\infty(\Omega)$*, and* $\Gamma \Subset \Omega$ *is an open set with Lipschitz boundary. Then the* partial ND map $\Lambda^\Gamma_{g,V}$ *is given by*

$$\Lambda^\Gamma_{g,V} \colon H^{-1/2}(\overline{\Gamma}\times\{0\}) \to H^{1/2}(\Gamma\times\{0\}), \quad f \mapsto u_f|_{\Gamma\times\{0\}}\,,$$

where $u_f \in H_0^1(\Omega\times[0,\infty))$ *is the unique solution to*

$$\begin{cases} (-\Delta_{\widetilde{g}} + V)\, u = 0 & \text{in } \Omega \times \mathbb{R}_+, \\ -\partial_y u = f & \text{on } \Omega \times \{0\}, \\ u = 0 & \text{on } \partial\Omega \times \mathbb{R}_+ \end{cases} \tag{2.11}$$

(see Lemma 2.2), is a well-defined bounded map. Moreover, for any $F \in H^{-1/2}(\overline{\Gamma}\times\{0\})$ *there holds*

$$\big\langle F, \Lambda^\Gamma_{g,V} f\big\rangle = \mathcal{B}_{g,V}\left(u_F, u_f\right), \tag{2.12}$$

where $u_F \in H_0^1(\Omega\times[0,\infty))$ *is the unique solution to* (2.11) *with Neumann data* F*.*

Proof. First note that $\Lambda^\Gamma_{g,V}$ is a well-defined map by the inclusion $H^{-1/2}(\overline{\Gamma}\times\{0\}) \hookrightarrow H^{-1/2}(\Omega\times\{0\})$, Lemma 2.2 and the mapping properties of the trace operator. It is bounded by the trace estimates and the continuity estimate (2.7). The identity (2.12) is a direct consequence of the fact that u_F solves (2.11) and $u_f \in H^1_0(\Omega \times [0,\infty))$. This concludes the proof. $\square$

Remark 2.5. *Similar to the identity* (2.12)*, we can also derive the identity*

$$\left\langle F, |g|^{1/2}\, \Lambda^\Gamma_{g,V} f\right\rangle = B_{g,V}(u_F, u_f),$$

for any $f, F \in H^{-1/2}(\overline{\Gamma}\times\{0\})$, where u_f and $u_F \in H^1_0(\Omega\times[0,\infty))$ are the solutions to (2.11) *with Neumann data f and F, respectively, and $B_{g,V}(\cdot,\cdot)$ is defined by* (2.5).

Lemma 2.6 (Integral identity)**.** *Let $\Lambda^\Gamma_{g_j,V_j}$ be the partial ND maps of* (1.8) *for $j = 1,2$ and suppose that* (1.9) *holds. Then we have*

$$\left\langle f, |g_1|^{1/2}\, \Lambda^\Gamma_{g_1,V_1} f\right\rangle - \left\langle f, |g_2|^{1/2}\Lambda^\Gamma_{g_2,V_2} f\right\rangle = \left(B_{g_1,V_1} - B_{g_2,V_2}\right)\left(u_f^{(1)}, u_f^{(2)}\right), \tag{2.13}$$

for any $f \in C^\infty_c(\Gamma)$, where $\mathcal{B}_{g_j,V_j}$ is given by (2.10) *as $g = g_j$ and $V = V_j$ ($j = 1,2$), and $u_f^{(j)}$ is the solution to* (1.8)*, for $j = 1,2$.*

Proof. Recall that $\Lambda^\Gamma_{g_j,V_j} f = u_f^{(j)}|_{\Gamma\times\{0\}}$, for $j = 1,2$. Then by (2.6), one has

$$\begin{aligned}\left\langle f, |g_1|^{1/2}\Lambda^\Gamma_{g_1,V_1} f\right\rangle &= \left\langle f, |g_1|^{1/2}\Lambda^\Gamma_{g_2,V_2} f\right\rangle\\ &= \left\langle f, |g_1|^{1/2} u_f^{(2)}|_\Gamma\right\rangle\\ &= B_{g_1,V_1}\left(u_f^{(1)}, u_f^{(2)}\right).\end{aligned}$$

By the same argument we have

$$\left\langle f, |g_2|^{1/2}\Lambda^\Gamma_{g_2,V_2} f\right\rangle = B_{g_2,V_2}\left(u_f^{(2)}, u_f^{(1)}\right).$$

Using the symmetry of B_{g_j,V_j}, $j = 1,2$, we arrive at the formula (2.13) after subtracting the previous two identities. $\square$

3. Boundary determination

The main goal of this section is to prove that the partial ND map (1.9) implies that the Riemannian metrics and potentials coincide in Γ. Suppose the partial ND data $\left(\Gamma, g, V, \Lambda^\Gamma_{g,V}\right)$ satisfies the assumptions of Theorem 1.1, and we want to prove:

Theorem 3.1 (Local boundary determination)**.** *Let us adopt all assumptions and notations from Theorem 1.1. Suppose* (1.9) *holds, then we have*

$$g_1 = g_2 \quad \text{and} \quad V_1 = V_2 \ \text{in } \Gamma.$$

To show this, we next introduce suitable *approximate solutions* of

$$\begin{cases} (-\Delta_{\widetilde{g}} + V)\, u = 0 & \text{in } \Omega\times\mathbb{R}_+,\\ -\partial_y u = f & \text{on } \Omega\times\{0\},\\ u = 0 & \text{on } \partial\Omega\times\mathbb{R}_+.\end{cases}$$

3.1. Approximate solutions.

Let us mention that the subsequent construction is inspired by the works [KY02, LN17]. For this purpose, let us consider the sequence of Neumann data

$$\phi_N(x) = N e^{\mathrm{i}Nx\cdot\xi}\eta(x), \tag{3.1}$$

where $\eta \in C^\infty_c(\Omega)$ is an arbitrary test function, and $\mathrm{i} = \sqrt{-1}$ is the imaginary unit, and $N \geq 1$. Here $x = (x^1, \ldots, x^n)$, $\xi = (\xi_1, \ldots, \xi_n) \in \mathbb{R}^n$ is a fixed co-vector and $x \cdot \xi = x^j \xi_j$ stands for the standard inner product in the Euclidean space. Throughout this section, we will use the following notation

$$|\xi|_g = \sqrt{g^{ij}\xi_i\xi_j} \tag{3.2}$$

to distinguish it from the ususal Euclidean norm $|\xi|$ and we may notice that the uniform ellipticity of g implies that $\sqrt{\lambda}\,|\xi| \leq |\xi|_g \leq \sqrt{\lambda^{-1}}\,|\xi|$, where $\lambda > 0$ is the ellipticity constant given in (1.4). Using the above notation we have the following lemma.

Lemma 3.2 (Approximate solutions)**.** *For any $N \geq 1$, there exists a smooth approximate solution Φ_N of the form*

$$\Phi_N(x,y) = e^{N(\mathrm{i}x\cdot\xi - |\xi|_g y)}\left(\frac{\eta(x)}{|\xi|_g} + \sum_{k=1}^{2} N^{-k}\psi_k(x,Ny)\right), \tag{3.3}$$

such that

$$\begin{cases} -\partial_y \Phi_N = \phi_N & \text{on } \Omega\times\{0\} \\ \Phi_N = 0 & \text{on } \partial\Omega\times\mathbb{R}_+, \end{cases} \tag{3.4}$$

where $|\xi|_g$ is given by (3.2) and ϕ_N is given by (3.1). Here $\psi_k(x,Ny)$ is a polynomial in the variable Ny, whose coefficients are bounded in x. Moreover, we have the error estimate

$$|(-\Delta_{\widetilde{g}} + V)\,\Phi_N| \leq CN^{-1}\mathcal{P}(x,Ny)e^{-N|\xi|_g y}, \tag{3.5}$$

for $x \in \Gamma$, $y > 0$ and some constant $C > 0$ independent of $N \geq 1$, where $\mathcal{P}(x,Ny) = Q(x)P(Ny)$ with $P(Ny)$ being of polynomial growth and $Q(x)$ compactly supported in the x variable. Furthermore, if η is supported in $\Gamma \Subset \Omega$, then Φ_N is supported in Γ as the functions ψ_1, ψ_2 are.

Proof. Unless otherwise stated all differential operators in this proof act only on the x variable. The construction of approximate solutions is based on the Wentzel–Kramers–Brillouin (WKB) construction concerning the parameter $N \geq 1$. Let us first consider the function $\Phi_N(x,y)$ of the form

$$\Phi_N(x,y) = e^{\mathrm{i}Nx\cdot\xi}\Psi(x,Ny), \tag{3.6}$$

where $\xi = (\xi_1,\ldots,\xi_n)$, $x\cdot\xi = x^i\xi_i = g_{ij}x^i\xi^j$. We may calculate

$$\begin{aligned}
&\Delta_g\Phi_N \\
&= |g|^{-1/2}\partial_i\Big(|g|^{1/2}g^{ij}(\mathrm{i}N\xi_j\Psi + \partial_j\Psi)e^{\mathrm{i}Nx\cdot\xi}\Big) \\
&= \mathrm{i}N\xi_i g^{ij}\left(\mathrm{i}N\xi_j\Psi + \partial_j\Psi\right)e^{\mathrm{i}Nx\cdot\xi} \\
&\quad + g^{ij}\left[\mathrm{i}N\xi_j\partial_i\Psi + \partial_{ij}\Psi + \mathrm{div}\left(g^{-1}\right)\left(\mathrm{i}N\xi_j\Psi + \partial_j\Psi\right)\right]e^{\mathrm{i}Nx\cdot\xi} \\
&= \left[-N^2|\xi|_g^2\Psi + \mathrm{i}N(2\xi\cdot d\Psi + \mathrm{div}g^{-1}\cdot\xi\Psi) + g^{-1} : D^2\Psi + \mathrm{div}g^{-1}\cdot\nabla\Psi\right]e^{\mathrm{i}Nx\cdot\xi},
\end{aligned}$$

where we denote $\left[\mathrm{div}\left(g^{-1}\right)\right]^i = |g|^{-1/2}\partial_j\left(|g|^{1/2}g^{ij}\right)$, $D^2\Psi = (\partial_{ij}\Psi)_{1\leq i,j\leq n}$ and $A : B$ is the contraction $A^{ij}B_{ij}$. Taking into account $\Delta_{\widetilde{g}} \equiv \Delta_g + \partial_y^2$, we get

$$\begin{aligned}
(-\Delta_{\widetilde{g}} + V)\,\Phi_N = \big[&N^2\left(|\xi|_g^2\Psi - \partial_y^2\Psi\right) - \mathrm{i}N\left(2\xi\cdot d\Psi + (\mathrm{div}g^{-1}\cdot\xi)\Psi\right) \\
&-(g^{-1} : D^2\Psi + \mathrm{div}g^{-1}\cdot\nabla\Psi + V\Psi)\big]e^{\mathrm{i}Nx\cdot\xi}.
\end{aligned} \tag{3.7}$$

If we insert the ansatz

$$\Psi(x,Ny) := \sum_{k=0}^{2} N^{-k}\widetilde{\psi}_k(x,Ny). \tag{3.8}$$

into (3.7), then we obtain

$$\begin{aligned}
&(-\Delta_{\widetilde{g}}+V)\Phi_N \\
&= \big[N^2\big(|\xi|_g^2\widetilde{\psi}_0-\partial_y^2\widetilde{\psi}_0\big)+N\big(|\xi|_g^2\widetilde{\psi}_1-\partial_y^2\widetilde{\psi}_1\big)+\big(|\xi|_g^2\widetilde{\psi}_2-\partial_y^2\widetilde{\psi}_2\big)\big]e^{\mathrm{i}Nx\cdot\xi} \\
&\quad -\mathrm{i}N\big(2\xi\cdot d\widetilde{\psi}_0+(\mathrm{div}g^{-1}\cdot\xi)\widetilde{\psi}_0\big)e^{\mathrm{i}Nx\cdot\xi} \\
&\quad -\mathrm{i}\big(2\xi\cdot d\widetilde{\psi}_1+(\mathrm{div}g^{-1}\cdot\xi)\widetilde{\psi}_1\big)e^{\mathrm{i}Nx\cdot\xi} \\
&\quad -\mathrm{i}N^{-1}\big(2\xi\cdot d\widetilde{\psi}_2+(\mathrm{div}g^{-1}\cdot\xi)\widetilde{\psi}_2\big)e^{\mathrm{i}Nx\cdot\xi} \\
&\quad -\big(g^{-1}:D^2\widetilde{\psi}_0+\mathrm{div}g^{-1}\cdot\nabla\widetilde{\psi}_0+V\widetilde{\psi}_0\big)e^{\mathrm{i}Nx\cdot\xi} \\
&\quad -N^{-1}\big(g^{-1}:D^2\widetilde{\psi}_1+\mathrm{div}g^{-1}\cdot\nabla\widetilde{\psi}_1+V\widetilde{\psi}_1\big)e^{\mathrm{i}Nx\cdot\xi} \\
&\quad -N^{-2}\big(g^{-1}:D^2\widetilde{\psi}_2+\mathrm{div}g^{-1}\cdot\nabla\widetilde{\psi}_2+V\widetilde{\psi}_2\big)e^{\mathrm{i}Nx\cdot\xi} \\
&= N^2\big(|\xi|_g^2\widetilde{\psi}_0-\partial_y^2\widetilde{\psi}_0\big)e^{\mathrm{i}Nx\cdot\xi} \\
&\quad +N\big(|\xi|_g^2\widetilde{\psi}_1-\partial_y^2\widetilde{\psi}_1-\mathrm{i}(2\xi\cdot d\widetilde{\psi}_0+(\mathrm{div}g^{-1}\cdot\xi)\widetilde{\psi}_0)\big)e^{\mathrm{i}Nx\cdot\xi} \\
&\quad +\big[\big(|\xi|_g^2\widetilde{\psi}_2-\partial_y^2\widetilde{\psi}_2-\mathrm{i}(2\xi\cdot d\widetilde{\psi}_1+(\mathrm{div}g^{-1}\cdot\xi)\widetilde{\psi}_1\big) \\
&\qquad\quad -\big(g^{-1}:D^2\widetilde{\psi}_0+\mathrm{div}g^{-1}\cdot\nabla\widetilde{\psi}_0+V\widetilde{\psi}_0\big)\big]e^{\mathrm{i}Nx\cdot\xi} \\
&\quad -N^{-1}\big[-\mathrm{i}\big(2\xi\cdot d\widetilde{\psi}_2+(\mathrm{div}g^{-1}\cdot\xi)\widetilde{\psi}_2\big) \\
&\qquad\quad +g^{-1}:D^2\widetilde{\psi}_1+\mathrm{div}g^{-1}\cdot\nabla\widetilde{\psi}_1+V\widetilde{\psi}_1\big]e^{\mathrm{i}Nx\cdot\xi} \\
&\quad -N^{-2}\big(g^{-1}:D^2\widetilde{\psi}_2+\mathrm{div}g^{-1}\cdot\nabla\widetilde{\psi}_2+V\widetilde{\psi}_2\big)e^{\mathrm{i}Nx\cdot\xi}.
\end{aligned}$$

The above identity is written in terms of the orders of N.

Next, let us set

$$\begin{aligned}
L_0 &:= -\partial_y^2+|\xi|_g^2, \\
L_1 &:= 2\xi\cdot d+\mathrm{div}g^{-1}\cdot\xi, \\
L_2 &:= g^{-1}:D^2+\mathrm{div}g^{-1}\cdot\nabla+V.
\end{aligned} \tag{3.9}$$

Then the conjugate equation of Ψ_N becomes

$$\begin{aligned}
&e^{-\mathrm{i}Nx\cdot\xi}\left(-\Delta_{\widetilde{g}}+V\right)\left(e^{\mathrm{i}Nx\cdot\xi}\Psi_N\right) \\
&= N^2L_0\widetilde{\psi}_0+N\big(L_0\widetilde{\psi}_1-\mathrm{i}L_1\widetilde{\psi}_1\big)+\big(L_0\widetilde{\psi}_2-\mathrm{i}L_1\widetilde{\psi}_1-L_2\widetilde{\psi}_0\big) \\
&\quad -N^{-1}\big(-\mathrm{i}L_1\widetilde{\psi}_2+L_2\widetilde{\psi}_1\big)-N^{-2}L_2\widetilde{\psi}_2.
\end{aligned} \tag{3.10}$$

In order to prove (3.5), we aim to solve the following system of ordinary differential equations (ODEs) in the y-variable

$$\begin{cases}
L_0\widetilde{\psi}_0=0, \\
L_0\widetilde{\psi}_1=\mathrm{i}L_1\widetilde{\psi}_0, \\
L_0\widetilde{\psi}_2=\mathrm{i}L_1\widetilde{\psi}_1+L_2\widetilde{\psi}_0
\end{cases} \tag{3.11}$$

with the boundary conditions

$$\begin{cases}
-\partial_y\widetilde{\psi}_0\big|_{y=0}=\eta(x), & \widetilde{\psi}_0\to 0 \text{ as } y\to\infty, \\
-\partial_y\widetilde{\psi}_1\big|_{y=0}=0, & \widetilde{\psi}_1\to 0 \text{ as } y\to\infty, \\
-\partial_y\widetilde{\psi}_2\big|_{y=0}=0, & \widetilde{\psi}_2\to 0 \text{ as } y\to\infty.
\end{cases} \tag{3.12}$$

Notice that if (3.11) holds, then (3.10) is of order N^{-1}. Furthermore, the coefficient $V(x)$ only appears in the operator L_2 and so it enters only into $\widetilde{\psi}_2$. Similarly as in [KY02, Lemma 2.1], we can solve the system (3.11), (3.12) iteratively.

First, observe that a solution of the first ODE in (3.11) with the desired boundary conditions is

$$\widetilde{\psi}_0(x,y)=\widetilde{\eta}(x)e^{-|\xi|_g y} \quad \text{with} \quad \widetilde{\eta}(x):=\frac{\eta(x)}{|\xi|_g}. \tag{3.13}$$

By the definition of L_1 and $\widetilde{\psi}_0$ (see (3.9) and (3.13)), we may calculate

$$\begin{aligned}\mathrm{i}L_1\widetilde{\psi}_0 &= \mathrm{i}\left[2\xi\cdot d\widetilde{\eta}+(\mathrm{div}g^{-1}\cdot\xi)\widetilde{\eta}\right]e^{-|\xi|_g y}+2\mathrm{i}\widetilde{\eta}\big(\xi\cdot de^{-|\xi|_g y}\big)\\ &= \mathrm{i}\left[2\xi\cdot d\widetilde{\eta}+(\mathrm{div}g^{-1}\cdot\xi)\widetilde{\eta}\right]e^{-|\xi|_g y}+\mathrm{i}\widetilde{\eta}\frac{\left(\partial_k g^{ij}\right)\xi_k\xi_i\xi_j}{|\xi|_g}ye^{-|\xi|_g y}\\ &:= f_1(x)e^{-|\xi|_g y}+f_2(x)ye^{-|\xi|_g y}.\end{aligned} \tag{3.14}$$

Next note that for $k\in\mathbb{N}$, there holds

$$\begin{aligned}\partial_y^2\big(y^k e^{-|\xi|_g y}\big) &= \partial_y\left[\big(ky^{k-1}-|\xi|_g y^k\big)e^{-|\xi|_g y}\right]\\ &= \left[k(k-1)y^{k-2}-2k|\xi|_g y^{k-1}+|\xi|_g^2 y^k\right]e^{-|\xi|_g y}\end{aligned}$$

and thus we obtain

$$\left(-\partial_y^2+|\xi|_g^2\right)\left(y^k e^{-|\xi|_g^2 y}\right)=\left[2k|\xi|_g y^{k-1}-k(k-1)y^{k-2}\right]e^{-|\xi|_g y}. \tag{3.15}$$

Now, we make the ansatz

$$\widetilde{\psi}_1=\widetilde{\psi}_{1,0}+\widetilde{\psi}_{1,1} \quad \text{with} \quad \begin{cases}\widetilde{\psi}_{1,0}(x,y)=h_0(x)e^{-|\xi|_g y}\\ \widetilde{\psi}_{1,1}(x,y)=\left(h_1(x)y+h_2(x)y^2\right)e^{-|\xi|_g y}.\end{cases} \tag{3.16}$$

Using (3.15), we deduce that

$$\begin{aligned}L_0\widetilde{\psi}_{1,1}(x,y) &= \left[2h_1|\xi|_g+h_2\left(4|\xi|_g y-2\right)\right]e^{-|\xi|_g y}\\ &= \left[2\left(h_1|\xi|_g-h_2\right)+4h_2|\xi|_g y\right]e^{-|\xi|_g y}.\end{aligned}$$

Comparing with (3.14) infers that $\widetilde{\psi}_{1,1}$ solves the second ODE in (3.11), if we choose

$$h_1(x)=\frac{f_1(x)}{2|\xi|_g}+\frac{f_2(x)}{4|\xi|_g^2} \quad \text{and} \quad h_2(x)=\frac{f_2(x)}{4|\xi|_g}. \tag{3.17}$$

Moreover, $\widetilde{\psi}_{1,1}$ satisfies

$$-\partial_y\widetilde{\psi}_{1,1}\big|_{y=0}=-h_1 \quad \text{and} \quad \widetilde{\psi}_{1,1}\to 0 \text{ as } y\to\infty.$$

On the other hand, from (3.15) we know that

$$\widetilde{\psi}_{1,0}(x,y)=h_0(x)e^{-|\xi|_g y} \quad \text{with} \quad h_0(x)=\frac{h_1(x)}{|\xi|_g} \tag{3.18}$$

solves

$$L_0\widetilde{\psi}_{1,0}=0 \quad \text{and} \quad -\partial_y\widetilde{\psi}_{1,0}\big|_{y=0}=h_1$$

and hence $\widetilde{\psi}_1$ with h_0,h_1,h_2 as in (3.17) and (3.18) is the desired solution of the second ODEs in (3.11) with the right boundary conditions (3.12).

Next, let us compute $L_1\widetilde{\psi}_1$ and $L_2\widetilde{\psi}_0$. The first one is easily seen via

$$\begin{aligned}L_1\widetilde{\psi}_1 &= 2\xi\cdot d\widetilde{\psi}_1+\left(\mathrm{div}g^{-1}\cdot\xi\right)\widetilde{\psi}_1\\ &= \sum_{k=0}^{2}y^k\left[2\xi\cdot dh_k+\left(\mathrm{div}g^{-1}\cdot\xi\right)h_k\right]e^{-|\xi|_g y}-\sum_{k=0}^{2}y^{k+1}h_k\left(2\xi\cdot d|\xi|_g\right)e^{-|\xi|_g y}.\end{aligned}$$

For the second one, let us observe that

$$\begin{aligned}
&\partial_k\left(\widetilde{\eta}e^{-|\xi|_g y}\right) = \left(\partial_k\widetilde{\eta} - y\widetilde{\eta}\partial_k|\xi|_g\right)e^{-|\xi|_g y}\\
&\partial^2_{k\ell}\left(\widetilde{\eta}e^{-|\xi|_g y}\right) = \left\{\partial^2_{k\ell}\widetilde{\eta} - y\left[\left(\partial^2_{k\ell}|\xi|_g\right)\widetilde{\eta} + \left(\partial_k\widetilde{\eta}\right)\left(\partial_\ell|\xi|_g\right) + \left(\partial_\ell\widetilde{\eta}\right)\left(\partial_k|\xi|_g\right)\right]\right.\\
&\qquad\qquad\left. + y^2\left(\partial_k|\xi|_g\right)\left(\partial_\ell|\xi|_g\right)\widetilde{\eta}\right\}e^{-|\xi|_g y}
\end{aligned} \tag{3.19}$$

for $1 \leq \ell, k \leq n$. This implies

$$\begin{aligned}
L_2\widetilde{\psi}_0 &= g^{-1} : D^2\widetilde{\psi}_0 + \mathrm{div}g^{-1}\cdot\nabla\widetilde{\psi}_0 + V\widetilde{\psi}_0\\
&= \left(g^{-1} : D^2\widetilde{\eta} + \mathrm{div}g^{-1}\cdot\nabla\widetilde{\eta} + V\widetilde{\eta}\right)e^{-|\xi|_g y}\\
&\quad - y\left[g^{-1} : D^2|\xi|_g + 2d\widetilde{\eta}\cdot d|\xi|_g + \widetilde{\eta}\,\mathrm{div}g^{-1}\cdot\nabla|\xi|_g\right]e^{-|\xi|_g y}\\
&\quad + y^2|d|\xi|_g|^2\widetilde{\eta}e^{-|\xi|_g y}.
\end{aligned}$$

Therefore, we can write

$$\mathrm{i}L_1\widetilde{\psi}_1 + L_2\widetilde{\psi}_0 = \left(F_1 + yF_2 + y^2F_3 + y^3F_4\right)e^{-|\xi|_g y},$$

for appropriate functions F_1, F_2, F_3 and F_4. As we want to find $\widetilde{\psi}_2$ solving

$$L_0\widetilde{\psi}_2 = \left(F_1 + yF_2 + y^2F_3 + y^3F_4\right)e^{-|\xi|_g y}, \tag{3.20}$$

the identity (3.15) suggests the ansatz

$$\widetilde{\psi}_2 = \widetilde{\psi}_{2,0} + \widetilde{\psi}_{2,1} \tag{3.21}$$

with

$$\begin{cases}
\widetilde{\psi}_{2,0}(x,y) = H_0(x)e^{-|\xi|_g y},\\
\widetilde{\psi}_{2,1}(x,y) = \left(H_1(x)y + H_2(x)y^2 + H_3(x)y^3 + H_4(x)y^4\right)e^{-|\xi|_g y},
\end{cases}$$

where again we use the zeroth order term to correct the Neumann data. Using (3.15) we can write

$$\begin{aligned}
L_0\widetilde{\psi}_{2,1} &= \left[2|\xi|_g H_1 + H_2\left(4|\xi|_g y - 2\right) + H_3\left(6|\xi|_g y^2 - 6y\right)\right.\\
&\qquad\left. + H_4\left(8|\xi|_g y^3 - 12y^2\right)\right]e^{-|\xi|_g y}\\
&= \left[2\left(|\xi|_g H_1 - H_2\right) + \left(4|\xi|_g H_2 - 6H_3\right)y\right.\\
&\qquad\left. + \left(6|\xi|_g H_3 - 12H_4\right)y^2 + 8|\xi|_g H_4 y^3\right]e^{-|\xi|_g y}.
\end{aligned}$$

By comparing this expression to (3.20) in terms of the order of the y-variable, we see that if the algebraic system

$$\begin{cases}
F_1 = 2(|\xi|_g H_1 - H_2),\\
F_2 = 4|\xi|_g H_2 - 6H_3,\\
F_3 = 6|\xi|_g H_3 - 12H_4,\\
F_4 = 8|\xi|_g H_4,
\end{cases}$$

holds true, then $\widetilde{\psi}_{2,1}$ solves the ODE (3.20). Thus, the coefficients are given by

$$\begin{cases}
H_1 = \frac{4|\xi|_g^3F_1 + 2|\xi|_g^2F_2 + 2|\xi|_gF_3 + 3F_4}{8|\xi|_g^4},\\
H_2 = \frac{2|\xi|_g^2F_2 + 2|\xi|_gF_3 + 3F_4}{8|\xi|_g^3},\\
H_3 = \frac{2|\xi|_gF_3 + 3F_4}{12|\xi|_g^2},\\
H_4 = \frac{F_4}{8|\xi|_g}.
\end{cases} \tag{3.22}$$

The function $\widetilde{\psi}_{2,1}$ has Neumann data $-H_1$ and hence as above we choose

$$\widetilde{\psi}_{2,0}(x,y) = H_0(x)e^{-|\xi|_g y} \quad \text{with} \quad H_0(x) = \frac{H_1(x)}{|\xi|_g}. \tag{3.23}$$

Then $\widetilde{\psi}_2$ given by (3.21), (3.22) and (3.23) solves the last equation in (3.11) with the correct boundary conditions (3.12).

Now, since $\widetilde{\psi}_j$, $j = 0, 1, 2$ solve (3.11), the identity (3.10) implies

$$(-\Delta_{\widetilde{g}} + V)\,\Phi_N = -e^{\mathrm{i}Nx\cdot\xi}\big[N^{-1}(-L_1\widetilde{\psi}_2 + L_2\widetilde{\psi}_1) + N^{-2}L_2\widetilde{\psi}_2\big].$$

To further simplify this identity, we next calculate the operators. Using the expansions (3.16), (3.21) and the identity (3.19), we get

$$\begin{aligned}
L_1\widetilde{\psi}_2 &= \sum_{k=0}^{4} y^k\,(L_1 H_k)\,e^{-|\xi|_g y} - \sum_{k=0}^{4} y^{k+1}\,(2\xi\cdot d|\xi|_g)\,H_k e^{-|\xi|_g y},\\
L_2\widetilde{\psi}_1 &= \sum_{k=0}^{2} y^k\,(L_2 h_k)\,e^{-|\xi|_g y} + \sum_{k=0}^{2} y^k h_k\big(g^{-1} : D^2 e^{-|\xi|_g y} + \mathrm{div} g^{-1}\cdot\nabla e^{-|\xi|_g y}\big)\\
&\quad + \sum_{k=0}^{2} y^k dh_k\cdot de^{-|\xi|_g y}\\
&= \sum_{k=0}^{2} y^k (L_2 h_k) e^{-|\xi|_g y}\\
&\quad - \sum_{k=0}^{2} y^{k+1}\big[h_k\,\big(g^{-1} : D^2|\xi|_g + \mathrm{div} g^{-1}\cdot\nabla|\xi|_g\big) + 2dh_k\cdot d|\xi|_g\big]e^{-|\xi|_g y}\\
&\quad + \sum_{k=0}^{2} y^{k+2} h_k |d|\xi|_g|^2 e^{-|\xi|_g y},
\end{aligned}$$

and

$$\begin{aligned}
L_2\widetilde{\psi}_2 &= \sum_{k=0}^{4} y^k (L_2 H_k) e^{-|\xi|_g y}\\
&\quad - \sum_{k=0}^{4} y^{k+1}\left[H_k\,\big(g^{-1} : D^2|\xi|_g + \mathrm{div} g^{-1}\cdot\nabla|\xi|_g\big) + 2dH_k\cdot d|\xi|_g\right] e^{-|\xi|_g y}\\
&\quad + \sum_{k=0}^{4} y^{k+2} H_k |d|\xi|_g|^2 e^{-|\xi|_g y}.
\end{aligned}$$

Therefore, we can write

$$(-\Delta_{\widetilde{g}} + V)\,\Phi_N = -e^{N(\mathrm{i}x\cdot\xi - |\xi|_g y)}\left[N^{-1}\alpha(x)P_5(Ny) + N^{-2}\beta(x)P_6(Ny)\right], \tag{3.24}$$

where $\alpha, \beta \in C_c^\infty(\Omega)$ and P_j is a polynomial of degree at most j. The representation (3.24) immediately implies the estimate (3.5). Next, observe that the constructed function $\Phi_N(x,y) = e^{\mathrm{i}Nx\cdot\xi}\sum_{k=0}^{2} N^{-k}\widetilde{\psi}_k(x, Ny)$ has by (3.6), (3.8) and (3.12), the Neumann data

$$-\partial_y\Phi_N(x,y)|_{y=0} = Ne^{\mathrm{i}Nx\cdot\xi}\sum_{k=0}^{2} N^{-k}(-\partial_y\widetilde{\psi}_k(x,0)) = Ne^{\mathrm{i}Nx\cdot\xi}\eta(x) = \phi_N(x).$$

Finally, the error estimate (3.5) and the assertion on the support are direct consequences of the above construction. Therefore, we have constructed approximate solutions, if we define ψ_j for $j = 0, 1, 2$ via $\widetilde{\psi}_j = \psi_j e^{-|\xi|_g y}$ (see (3.13), (3.16) and (3.21)), and from the above considerations we can conclude the proof. $\square$

3.2. **Proof of Theorem 3.1.** With the approximate solutions (3.3) at hand, we can prove Theorem 3.1.

Proof of Theorem 3.1. For the ease of notation, let us set $g = g_j$ and $V = V_j$ for either $j = 1$ or $j = 2$. Let u_N be the solution of

$$\begin{cases} (-\Delta_{\widetilde{g}} + V)\, u_N = 0 & \text{in } \Omega \times \mathbb{R}_+, \\ -\partial_y u_N = \phi_N & \text{on } \Omega \times \{0\}, \\ u_N = 0 & \text{on } \partial\Omega \times \mathbb{R}_+, \end{cases}$$

where ϕ_N is given by (3.1). Clearly, the same reasoning as in Section 2 works, if the Neumann data and related functions are complex-valued. Let Φ_N be the approximate solution of u_N with $-\partial_y \Phi_N|_{y=0} = \phi_N$. Note that we have

$$\partial_y u_N = \partial_y \Phi_N \text{ and } \partial_y r_N = 0 \text{ in } \Omega \times \{0\}, \tag{3.25}$$

where r_N is the remainder term given by $r_N = u_N - \Phi_N$. Via (2.12), one has

$$\left\langle \phi_N, \Lambda^{\Gamma}_{g,V} \overline{\phi_N} \right\rangle = \mathcal{B}_{g,V}(u_N, \overline{u_N}),$$

where $\mathcal{B}_{g,V}$ is the bilinear form given by (2.10) and $\overline{\phi_N}$ denotes the complex conjugate of ϕ_N. Thus, using the decomposition $u_N = \Phi_N + r_N$, we get

$$\begin{aligned} &\left\langle \phi_N, \Lambda^{\Gamma}_{g,V} \overline{\phi_N} \right\rangle \\ &= \int_{\Omega\times\mathbb{R}_+} \left[\widetilde{g}^{-1} \nabla_{x,y} u_N \cdot \nabla_{x,y} \overline{u_N} + |g|^{1/2} g^{-1} \nabla |g|^{-1/2} \cdot \nabla u_N \, \overline{u_N} + V |u_N|^2 \right] dxdy \\ &:= I_N + II_N, \end{aligned}$$

where we set

$$\begin{aligned} I_N &:= \int_{\Omega\times\mathbb{R}_+} \widetilde{g}^{-1} \nabla_{x,y} \Phi_N \cdot \nabla_{x,y} \overline{\Phi_N} \, dxdy, \\ II_N &:= \int_{\Omega\times\mathbb{R}_+} \Big[2\widetilde{g}^{-1} \mathrm{Re}\left(\nabla_{x,y} \Phi_N \cdot \nabla_{x,y} \overline{r_N} \right) + \widetilde{g}^{-1} \nabla_{x,y} r_N \cdot \nabla_{x,y} \overline{r_N} \\ &\qquad + |g|^{1/2} g^{-1} \nabla |g|^{-1/2} \cdot \nabla u_N \, \overline{u_N} + V |u_N|^2 \Big] dxdy. \end{aligned}$$

Here $\mathrm{Re}(f)$ stands for the real part of the complex-valued function f. Let us next estimate I_N and II_N separately.

Step 1. Estimate of I_N.

Let us first compute the L^2-norm of Φ_N. By (3.3) and the change of variables $z = Ny$ one easily obtains the bound

$$\begin{aligned} \|\Phi_N\|_{L^2(\Omega\times\mathbb{R}_+)} &\le \left\| e^{-N|\xi|_g y} \frac{\eta}{|\xi|_g} \right\|_{L^2(\Omega\times\mathbb{R}_+)} + \sum_{k=1}^{2} N^{-k} \left\| e^{-N|\xi|_g y} \psi_k(\cdot, Ny) \right\|_{L^2(\Omega\times\mathbb{R}_+)} \\ &\lesssim N^{-1/2} \end{aligned} \tag{3.26}$$

for $N \ge 1$. Again using the representation formula (3.3), a direct computation yields that

$$\nabla_{x,y} \Phi_N = \left[N \begin{pmatrix} \mathrm{i}\xi - y \nabla |\xi|_g \\ -|\xi|_g \end{pmatrix} \frac{\eta}{|\xi|_g} + q(x, Ny) \right] e^{N(\mathrm{i}x\cdot\xi - |\xi|_g y)}, \tag{3.27}$$

where $q(x, Ny)$ is of polynomial growth in Ny and a bounded function x. Similarly as for the L^2 norm of Φ_N, the identity (3.27) and the change of variables $z = Ny$ imply the following gradient estimate

$$
\begin{aligned}
\|\nabla_{x,y}\Phi_N\|_{L^2(\Omega\times\mathbb{R}_+)} &\leq N \left\| \begin{pmatrix} \mathrm{i}\xi \\ -|\xi|_g \end{pmatrix} \frac{\eta}{|\xi|_g} e^{-N|\xi|_g y} \right\|_{L^2(\Omega\times\mathbb{R}_+)} \\
&\quad + \|\widetilde{q}(x,Ny)e^{-N|\xi|_g y}\|_{L^2(\Omega\times\mathbb{R}_+)} \\
&\lesssim N^{1/2}
\end{aligned} \tag{3.28}
$$

for $N \geq 1$, where $\widetilde{q}$ is of polynomial growth in Ny and bounded in x.

On the other hand, with the representation formula (3.27) at hand, a direct computation ensures that

$$
\begin{aligned}
I_N &= \int_{\Omega\times\mathbb{R}_+} \widetilde{g}^{-1}\nabla_{x,y}\Phi_N \cdot \nabla_{x,y}\overline{\Phi_N}\, dxdy \\
&= N^2 \int_{\Gamma\times\mathbb{R}_+} \widetilde{g}^{-1} \begin{pmatrix} \mathrm{i}\xi - y\nabla|\xi|_g \\ -|\xi|_g \end{pmatrix} \cdot \begin{pmatrix} -\mathrm{i}\xi - y\nabla|\xi|_g \\ -|\xi|_g \end{pmatrix} \frac{\eta^2}{|\xi|_g^2} e^{-2N|\xi|_g y}\, dxdy \\
&\quad + N\int_{\Gamma\times\mathbb{R}_+} p(x,Ny)e^{-2N|\xi|_g y}\, dxdy \\
&= N^2 \int_{\Gamma\times\mathbb{R}_+} (2|\xi|_g^2 + |d|\xi|_g|^2 y^2)\frac{\eta^2}{|\xi|_g^2} e^{-2N|\xi|_g y}\, dxdy + \mathcal{O}(1) \\
&= 2N^2 \int_{\Gamma\times\mathbb{R}_+} e^{-2N|\xi|_g y}\eta^2\, dxdy + N^{-1}\int_{\Gamma\times\mathbb{R}_+} \frac{|d|\xi|_g|^2}{|\xi|_g^2} y^2\eta^2 e^{-2|\xi|_g y}\, dxdy + \mathcal{O}(1) \\
&= N\int_\Gamma |\xi|_g^{-1}\eta^2\, dx + \mathcal{O}(1)
\end{aligned} \tag{3.29}
$$

for $\xi \neq 0$, where $p(x, Ny)$ is of polynomial growth in Ny and a bounded function in x. Moreover, in the last equality we used the fundamental theorem of calculus and the notation $\mathcal{O}(1)$ or more generally $\mathcal{O}(N^\alpha)$ for some $\alpha \in \mathbb{R}$ means that the term has growth N^α as $N \to \infty$. Multiplying (3.29) by N^{-1}, one can see that

$$
\lim_{N\to\infty} N^{-1} I_N = \lim_{N\to\infty} \left\{ \int_\Gamma |\xi|_g^{-1}\eta^2\, dx + \mathcal{O}\left(N^{-1}\right) \right\} = \int_\Gamma |\xi|_g^{-1}\eta^2\, dx. \tag{3.30}
$$

Step 2. Estimate of II_N.

Notice that

$$
\begin{aligned}
II_N := \int_{\Omega\times\mathbb{R}_+} \Big[& 2\widetilde{g}^{-1}\mathrm{Re}\left(\nabla_{x,y}\Phi_N \cdot \nabla_{x,y}\overline{r_N}\right) + \widetilde{g}^{-1}\nabla_{x,y}r_N \cdot \nabla_{x,y}\overline{r_N} \\
& + |g|^{1/2} g^{-1}\nabla |g|^{-1/2} \cdot \nabla\left(\Phi_N + r_N\right)\left(\overline{\Phi_N + r_N}\right) \\
& + V\left|\Phi_N + r_N\right|^2 \Big] dx,
\end{aligned} \tag{3.31}
$$

where we used $u_N = \Phi_N + r_N$ again. By the construction of Φ_N, (3.4) and (3.25), we see that r_N is a solution to

$$
\begin{cases}
\left(-\Delta_g + V\right) r_N = -\left(-\Delta_g + V\right)\Phi_N & \text{in } \Omega\times\mathbb{R}_+, \\
-\partial_y r_N = 0 & \text{on } \Omega\times\{0\}, \\
r_N = 0 & \text{on } \partial\Omega\times\mathbb{R}_+.
\end{cases}
$$

By the elliptic estimate (2.9) and the change of variables $z = Ny$, there holds

$$\begin{aligned}\|r_N\|_{H^1(\Omega\times\mathbb{R}_+)} &\lesssim \|(-\Delta_g + V)\,\Phi_N\|_{L^2(\Omega\times\mathbb{R}_+)}\\ &\lesssim N^{-1}\big\|\mathcal{P}(x,Ny)e^{-N|\xi|_g y}\big\|_{L^2(\Omega\times\mathbb{R}_+)}\\ &\lesssim N^{-3/2}\end{aligned}\tag{3.32}$$

for $N \geq 1$. Using Hölder's inequality, (3.26), (3.28) and (3.32), we can estimate (3.31) as

$$\begin{aligned}|II_N| \lesssim\ & \|\nabla_{x,y}\Phi_N\|_{L^2(\Omega\times\mathbb{R}_+)}\|\nabla_{x,y}r_N\|_{L^2(\Omega\times\mathbb{R}_+)} + \|r_N\|^2_{H^1(\Omega\times\mathbb{R}_+)}\\ &+ \|\nabla_{x,y}\Phi_N\|_{L^2(\Omega\times\mathbb{R}_+)}(\|\Phi_N\|_{L^2(\Omega\times\mathbb{R}_+)} + \|r_N\|_{L^2(\Omega\times\mathbb{R}_+)})\\ &+ \|\nabla_{x,y}r_N\|_{L^2(\Omega\times\mathbb{R}_+)}\|\Phi_N\|_{L^2(\Omega\times\mathbb{R}_+)} + \|\Phi_N\|^2_{L^2(\Omega\times\mathbb{R}_+)}\\ \lesssim\ & 1\end{aligned}\tag{3.33}$$

for all $\xi \neq 0$ and $N \geq 1$. Multiplying (3.33) by N^{-1} and passing to the limit $N \to \infty$, we get

$$\lim_{N\to\infty} N^{-1} II_N = 0.\tag{3.34}$$

Combining (3.30) and (3.34), we get

$$\lim_{N\to\infty} N^{-1}\left\langle \phi_N, \Lambda^{\Gamma}_{g,V}\overline{\phi_N}\right\rangle = \lim_{N\to\infty} N^{-1}\left(I_N + II_N\right) = \int_\Gamma |\xi|_g^{-1}\,\eta^2\,dx.\tag{3.35}$$

Step 3. Recovery the metric g on Γ.

Now, suppose the condition (1.9) holds, then one can determine the metric $(g_{ij}(x))$ on Γ by varying $0 \neq \xi = (\xi_1,\dots,\xi_n) \in \mathbb{R}^n$. More precisely, let $u_N^{(j)} = \Phi_N^{(j)} + r_N^{(j)}$ be the solutions to

$$\begin{cases}\left(-\Delta_{g_j} - \partial_y^2\right)u_N^{(j)} + V_j u_N^{(j)} = 0 & \text{in } \Omega\times\mathbb{R}_+,\\ -\partial_y u_N^{(j)}(x,0) = \phi_N(x) & \text{on } \Omega\times\{0\},\\ u_N^{(j)} = 0 & \text{on } \partial\Omega\times\mathbb{R}_+,\end{cases}$$

where ϕ_N is given by (3.1), $\Phi_N^{(j)}$ stands for the approximate solution constructed by Lemma 3.2, and $r_N^{(j)}$ is the remainder term, for $j = 1,2$ and $N \geq 1$. With these approximate solutions at hand, by using (3.35), we have

$$\begin{aligned}\int_\Gamma |\xi|_{g_1}^{-1}\,\eta^2\,dx &= \lim_{N\to\infty} N^{-1}\left\langle \phi_N, \Lambda^{\Gamma}_{g_1,V_1}\overline{\phi_N}\right\rangle\\ &= \lim_{N\to\infty} N^{-1}\left\langle \phi_N, \Lambda^{\Gamma}_{g_1,V_2}\overline{\phi_N}\right\rangle\\ &= \int_\Gamma |\xi|_{g_2}^{-1}\,\eta^2\,dx,\end{aligned}$$

for any test function $\eta \in C_c^\infty(\Gamma)$, and for any $0 \neq \xi = (\xi_1,\dots,\xi_n) \in \mathbb{R}^n$. Thus, after polarization of test functions, we deduce $|\xi|_{g_1} = |\xi|_{g_2}$ on Γ, for any $\xi = (\xi_1,\dots,\xi_n) \in \mathbb{R}^n$. Therefore, we deduce that $g_1^{k\ell}\zeta_k\eta_\ell = g_2^{k\ell}\zeta_k\eta_\ell$ on Γ, for all $\zeta,\eta \in \mathbb{R}^n$ and hence

$$g_1 = g_2 \text{ on } \Gamma.$$

Step 4. Recovery the potential V on Γ.

First, let us note that by (2.12) and the fact that all coefficients (g_j, V_j) for $j = 1,2$ are real-valued one has $\langle\overline{\phi_N}, |g_j|^{1/2}\Lambda^{\Gamma}_{g_j,V_j}\phi_N\rangle \in \mathbb{R}$, for $j = 1,2$. Next, observe that in the complex-valued case formula (2.13) in Lemma 2.6 becomes

$$\left\langle \overline{f}, |g_1|^{1/2}\,\Lambda^{\Gamma}_{g_1,V_1}f\right\rangle - \left\langle \overline{f}, |g_2|^{1/2}\Lambda^{\Gamma}_{g_2,V_2}f\right\rangle = (B_{g_1,V_1} - B_{g_2,V_2})\big(u_f^{(1)}, \overline{u_f^{(2)}}\big).$$

Thus, by $g_1 = g_2$ in Γ and (1.9), there holds that

$$
\begin{aligned}
0 &= \langle \overline{\phi_N}, |g_1|^{1/2} \Lambda^{\Gamma}_{g_1,V_1}\phi_N\rangle - \langle \overline{\phi_N}, |g_2|^{1/2}\Lambda^{\Gamma}_{g_2,V_2}\phi_N\rangle \\
&= (B_{g_1,V_1} - B_{g_2,V_2})\left(u_N^{(1)}, \overline{u_N^{(2)}}\right) \\
&= \int_{\Omega\times\mathbb{R}_+} \left(|g_1|^{1/2} g_1^{-1} - |g_2|^{1/2} g_2^{-1}\right)\nabla u_N^{(1)}\cdot\nabla\overline{u_N^{(2)}}\, dxdy \\
&\quad + \int_{\Omega\times\mathbb{R}_+} \left(|g_1|^{1/2} V_1 - |g_2|^{1/2} V_2\right) u_N^{(1)}\overline{u_N^{(2)}}\, dxdy.
\end{aligned} \tag{3.36}
$$

for $j = 1, 2$. As in Step 3, we expand

$$
|g_k|^{1/2} g_k^{-1}\nabla u_N^{(1)}\cdot\nabla\overline{u_N^{(2)}} = |g_k|^{1/2} g_k^{-1}\left(\nabla\Phi_N^{(1)} + \nabla r_N^{(1)}\right)\cdot\left(\overline{\nabla\Phi_N^{(2)} + \nabla r_N^{(2)}}\right) \tag{3.37}
$$

for $k = 1, 2$. Next, inserting (3.37) into (3.36) and using $g_1 = g_2$ in Γ as well as $\operatorname{supp}\Phi_N^{(j)} \subset \Gamma$, we get

$$
\begin{aligned}
0 = &\int_{\Omega\times\mathbb{R}_+} \left(|g_1|^{1/2} g_1^{-1} - |g_2|^{1/2} g_2^{-1}\right)\nabla r_N^{(1)}\cdot\nabla\overline{r_N^{(2)}}\, dxdy \\
&+ \int_{\Omega\times\mathbb{R}_+} \left(|g_1|^{1/2} V_1 - |g_2|^{1/2} V_2\right)\left(\Phi_N^{(1)} + r_N^{(1)}\right)\left(\overline{\Phi_N^{(2)} + r_N^{(2)}}\right) dxdy.
\end{aligned} \tag{3.38}
$$

Applying the error estimate (3.32) of $r_N^{(j)}$ for $j = 1, 2$ and Hölder's inequality, we have

$$
\left|\int_{\Omega\times\mathbb{R}_+} \left(|g_1|^{1/2} g_1^{-1} - |g_2|^{1/2} g_2^{-1}\right)\nabla r_N^{(1)}\cdot\nabla\overline{r_N^{(2)}}\, dxdy\right| \lesssim N^{-3}. \tag{3.39}
$$

On the other hand, for the second term in (3.38), one can see that

$$
\begin{aligned}
&\int_{\Omega\times\mathbb{R}_+} \left(|g_1|^{1/2} V_1 - |g_2|^{1/2} V_2\right)\left(\Phi_N^{(1)} + r_N^{(1)}\right)\left(\overline{\Phi_N^{(2)} + r_N^{(2)}}\right) dxdy \\
= &\int_{\Omega\times\mathbb{R}_+} \left(|g_1|^{1/2} V_1 - |g_2|^{1/2} V_2\right)\Phi_N^{(1)}\overline{\Phi_N^{(2)}}\, dxdy + \mathcal{O}\left(N^{-2}\right),
\end{aligned} \tag{3.40}
$$

where we used (3.26) and (3.32). Hence, (3.38), (3.39) and (3.40) imply

$$
N\int_{\Omega\times\mathbb{R}_+} \left(|g_1|^{1/2} V_1 - |g_2|^{1/2} V_2\right)\Phi_N^{(1)}\overline{\Phi_N^{(2)}}\, dxdy = \mathcal{O}\left(N^{-1}\right),
$$

which gives

$$
\lim_{N\to\infty} N\int_{\Omega\times\mathbb{R}_+} \left(|g_1|^{1/2} V_1 - |g_2|^{1/2} V_2\right)\Phi_N^{(1)}\overline{\Phi_N^{(2)}}\, dxdy = 0. \tag{3.41}
$$

Thus, from the representation formula (3.3) and the change of variables $z = Ny$ we can conclude that

$$
\begin{aligned}
&\int_{\Omega\times\mathbb{R}_+} \left(|g_1|^{1/2} V_1 - |g_2|^{1/2} V_2\right)\Phi_N^{(1)}\overline{\Phi_N^{(2)}} dxdy \\
= &\int_{\Omega\times\mathbb{R}_+} \left(|g_1|^{1/2} V_1 - |g_2|^{1/2} V_2\right) e^{-N(|\xi|_{g_1} + |\xi|_{g_2})y} \frac{\eta^2}{|\xi|_{g_1}|\xi|_{g_2}}\, dxdy + \mathcal{O}\left(N^{-2}\right) \\
= &\, N^{-1}\int_{\Gamma\times\mathbb{R}_+} |g_1|^{1/2}(V_1 - V_2)\, e^{-2|\xi|_{g_1} y}\frac{\eta^2}{|\xi|_{g_1}^2}\, dxdy + \mathcal{O}\left(N^{-2}\right),
\end{aligned}
$$

where we used in the last equality that $g_1 = g_2$ on Γ and $\eta \in C_c^\infty(\Gamma)$. Inserting this into (3.41) and using the fundamental theorem of calculus, we deduce that

$$\begin{aligned}0 &= \int_{\Gamma\times\mathbb{R}_+} |g_1|^{1/2}\,(V_1 - V_2)\, e^{-2|\xi|_{g_1} y}\, \frac{\eta^2}{|\xi|_{g_1}^2}\, dxdy \\ &= \int_{\Gamma} |g_1|^{1/2}\,(V_1 - V_2)\, \frac{\eta^2}{2|\xi|_{g_1}^3}\, dx\end{aligned}$$

for any $\eta \in C_c^\infty(\Gamma)$. This shows by the usual polarization argument that

$$V_1 = V_2 \text{ on } \Gamma.$$

This concludes the proof. □

4. Inverse problem for nonlocal equations

We start by reviewing in Section 4.1 the definition of fractional powers of elliptic operators. In Section 4.2 we recall the extension property of elliptic variable coefficient nonlocal operators. This helps us in Section 4.3 to relate the Neumann derivative with the square root of an elliptic operator. Finally, in Section 4.4 we show that the ND map uniquely determines the heat kernel of the operator $-\Delta_g + V$.

4.1. Fractional powers of $-\Delta_g + V$.

As usual, let $\Omega \subset \mathbb{R}^n$ denote a bounded smooth domain. For any uniformly elliptic Riemannian metric $g \in C^\infty(\overline{\Omega}; \mathbb{R}^{n\times n})$ and potential $0 \le V \in C^\infty(\overline{\Omega})$, we introduce the operator

$$\mathsf{P}_{g,V} := -\Delta_g + V \tag{4.1}$$

on $L^2(\Omega, dV_g)$ with homogeneous Dirichlet condition on $\partial\Omega$, that is, it has domain

$$\mathsf{Dom}\,(\mathsf{P}_{g,V}) = \left\{u \in H_0^1(\Omega, dV_g)\,;\, \mathsf{P}_{g,V} u \in L^2(\Omega; dV_g)\right\}, \tag{4.2}$$

where $\mathsf{P}_{g,V} u \in L^2(\Omega; dV_g)$ has to be understood in the weak sense. Below, we will show that[2] $\mathsf{Dom}(\mathsf{P}_{g,V}) = H_0^1(\Omega) \cap H^2(\Omega)$. Arguing as in [Bre11, Theorem 8.22, Theorem 9.31], one deduces that there exists a Hilbert basis $(\phi_k)_{k\in\mathbb{N}} \subset H_0^1(\Omega)$ of $L^2(\Omega, dV_g)$ and a sequence $(\lambda_k)_{k\in\mathbb{N}} \subset \mathbb{R}_+$ with $\lambda_k \to \infty$ as $k \to \infty$ such that

$$\begin{cases} \mathsf{P}_{g,V}\phi_k = \lambda_k\phi_k & \text{in } \Omega, \\ \phi_k = 0 & \text{on } \partial\Omega, \end{cases}$$

for all $k \in \mathbb{N}$. Moreover, by [Bre11, Theorem 9.25, Remark 24] it follows that $\phi_k \in C^\infty(\overline{\Omega})$.

Next observe that any $u \in \mathsf{Dom}(\mathsf{P}_{g,V})$ with spectral decomposition $u = \sum_{k\ge1} u_k\phi_k$ satisfies

$$\|P_{g,V}u\|_{L^2(\Omega,dV_g)}^2 = \sum_{k\ge1} \lambda_k^2\, |u_k|^2 < \infty \tag{4.3}$$

and there holds

$$\mathsf{P}_{g,V} u = \sum_{k\ge1} \lambda_k u_k \phi_k. \tag{4.4}$$

The identities (4.4) and (4.3) suggest a natural definition for the fractional powers $\mathsf{P}_{g,V}^s$, $0 < s < 1$ (more details are given in Appendix B). To define it, let us introduce the spaces $\widetilde{H}_{g,V}^{2s}(\Omega)$ consisting of all $u \in L^2(\Omega, dV_g)$ such that

$$\sum_{k\ge1} \lambda_k^{2s}\, |u_k|^2 < \infty,$$

[2]Here and in the following, we make repeatedly use of the fact that $H^k(\Omega) = H^k(\Omega, dV_g)$ with equivalent norms by the ellipticity (1.4) of g (see Section 2.1).

where u has the spectral decomposition $u=\sum_{k\geq 1}u_k\phi_k$ in $L^2(\Omega,dV_g)$ (i.e. $u_k=\langle u,\phi_k\rangle_{L^2(\Omega,dV_g)}$ for $k\in\mathbb{N}$). Note that $\widetilde{H}^{2s}_{g,V}(\Omega)$ equipped with the inner product

$$\langle u,v\rangle_{\widetilde{H}^{2s}_{g,V}(\Omega)}=\sum_{k\geq 1}\lambda_k^{2s}u_kv_k \tag{4.5}$$

for $u,v\in\widetilde{H}^{2s}_{g,V}(\Omega)$ becomes a Hilbert space. This is true for any $s\geq 0$. Similarly, for $s<0$, we denote by $H^{-2s}_{g,V}(\Omega)$ the set of all $u=\sum_{k\geq 1}u_k\phi_k$ satisfying

$$\|u\|_{H^{-2s}_{g,V}(\Omega)}=\sum_{k\geq 1}\lambda_k^{-2s}\left|u_k\right|^2<\infty.$$

If one defines the inner product similarly as in (4.5), then $H^{-2s}_{g,V}(\Omega)$ becomes a Hilbert space and one can identify the dual space $(\widetilde{H}^{2s}_{g,V}(\Omega))^*$ and $H^{-2s}_{g,V}(\Omega)$ with equivalent norms.

Hence, for $u\in\widetilde{H}^{2s}_{g,V}(\Omega)$, we can define

$$\mathsf{P}^s_{g,V}u=\sum_{k\geq 1}\lambda_k^s u_k\phi_k\in L^2(\Omega,dV_g) \tag{4.6}$$

and $\mathsf{Dom}\left(\mathsf{P}^s_{g,V}\right)=\widetilde{H}^{2s}_{g,V}(\Omega)$. By construction we have

$$\left\|\mathsf{P}^s_{g,V}u\right\|_{L^2(\Omega;dV_g)}=\|u\|_{\widetilde{H}^{2s}_{g,V}(\Omega)},\text{ for all } u\in\widetilde{H}^{2s}_{g,V}(\Omega).$$

Lemma 4.1. *We have*

$$\mathsf{Dom}\left(\mathsf{P}_{g,V}\right)\hookrightarrow\mathsf{Dom}\left(\mathsf{P}^s_{g,V}\right) \tag{4.7}$$

and

$$\widetilde{H}^s_{g,V}(\Omega)\hookrightarrow\widetilde{H}^t_{g,V}(\Omega). \tag{4.8}$$

for all $0\leq t<s<\infty$.

Proof. To see (4.7), let $k_0\in\mathbb{N}$ be the smallest natural number such that $\lambda_{k_0}\geq 1$. Then for $u=\sum_{k\geq 1}u_k\phi_k$ we have

$$\begin{aligned}\sum_{k\geq 1}\lambda_k^{2s}|u_k|^2&\leq\sum_{1\leq k\leq k_0-1}\lambda_k^{2s}|u_k|^2+\sum_{k\geq k_0}\lambda_k^{2s}|u_k|^2\\&\leq\sum_{k\geq 1}|u_k|^2+\sum_{k\geq 1}\lambda_k^2|u_k|^2\\&\leq\|u\|^2_{L^2(\Omega,dV_g)}+\|\mathsf{P}_{g,V}u\|^2_{L^2(\Omega,dV_g)}<\infty.\end{aligned}$$

We only prove (4.8) for $t=0$, that is

$$\widetilde{H}^{2s}_{g,V}(\Omega)\hookrightarrow L^2(\Omega,dV_g), \tag{4.9}$$

and the general result is followed by a simple modification. A direct calculation shows

$$\begin{aligned}\|u\|^2_{L^2(\Omega,dV_g)}&\leq\sum_{1\leq k\leq k_0-1}|u_k|^2+\sum_{k\geq k_0}|u_k|^2\\&\leq\sum_{1\leq k\leq k_0-1}\frac{\lambda_k^{2s}}{\lambda_k^{2s}}|u_k|^2+\sum_{k\geq k_0}\lambda_k^{2s}|u_k|^2\\&\leq\sum_{k\geq 1}\lambda_k^{2s}|u_k|^2\\&=\|u\|_{\widetilde{H}^{2s}_{g,V}(\Omega)},\end{aligned}$$

where we used the first eigenvalue $\lambda_1>0$. ☐

Remark 4.2. *Let us remark that in the special case $g_{ij} = \delta_{ij}$ and $V = 0$, the operator defined via* (4.6) *is called the* spectral fractional Laplacian, *and for more details, in particular an alternative characterization of $\widetilde{H}^{2s}_{(\delta_{ij}),0}(\Omega)$, we refer the interested reader to* [CT10] *and* [BSV14, Section 3.1.3].

The following integration by parts formula holds.

Lemma 4.3. *For all $u, v \in \widetilde{H}^{2s}_{g,V}(\Omega)$, we have*

$$\left\langle \mathsf{P}^s_{g,V} u, v \right\rangle_{L^2(\Omega,dV_g)} = \left\langle u, \mathsf{P}^s_{g,V} v \right\rangle_{L^2(\Omega,dV_g)} = \left\langle \mathsf{P}^{s/2}_{g,V} u, \mathsf{P}^{s/2}_{g,V} v \right\rangle_{L^2(\Omega,dV_g)}. \tag{4.10}$$

Proof. The proof can be easily seen by using (4.6) and straightforward computations. □

It is easily seen that the last expression in the integration by parts formula (4.10) is precisely the inner product in $\widetilde{H}^s_{g,V}(\Omega)$. Because of this, for given $f \in (\widetilde{H}^s_{g,V}(\Omega))^*$, we say $u\colon \Omega \to \mathbb{R}$ (weakly) solves

$$\begin{cases} \mathsf{P}^s_{g,V} u = f & \text{in } \Omega, \\ u = 0 & \text{in } \partial\Omega, \end{cases}$$

if $u \in \widetilde{H}^s_{g,V}(\Omega)$ and

$$\langle u, v \rangle_{\widetilde{H}^s_{g,V}(\Omega)} = \langle f, v \rangle, \tag{4.11}$$

where $\langle \cdot, \cdot \rangle$ denotes the duality pairing between $\widetilde{H}^s_{g,V}(\Omega)$ and $(\widetilde{H}^s_{g,V}(\Omega))^*$. In fact, it is not hard to see that $(\widetilde{H}^s_{g,V}(\Omega))^* = H^{-s}_{g,V}(\Omega)$ and

$$\langle f, v \rangle = \sum_{k \geq 1} f_k v_k$$

for $f \in (\widetilde{H}^s_{g,V}(\Omega))^*$ and $v \in \widetilde{H}^s_{g,V}(\Omega)$ (see [BSV14, Section 7.9]).

Next, we want to relate the fractional powers $\mathsf{P}^s_{g,V}$ given by (4.6) with the associated heat semigroup $e^{-t\mathsf{P}_{g,V}}$, $t \geq 0$. First, we have the next lemma.

Lemma 4.4. *There holds*

$$e^{-t\mathsf{P}_{g,V}} u = \sum_{k \geq 1} e^{-t\lambda_k} u_k \phi_k \tag{4.12}$$

for any $u \in L^2(\Omega, dV_g)$.

The proof of the above lemma is in Appendix B. Furthermore, note that the uniform ellipticity (1.4) of g, $V \geq 0$ and the Poincaré inequality imply

$$\left\| e^{-t\mathsf{P}_{g,V}} \right\|_{L(L^2(\Omega,dV_g))} \leq e^{-\gamma t} \leq 1, \tag{4.13}$$

for all $t \geq 0$ and some $\gamma > 0$ (cf. e.g. [Are06, Theorem 3.4.3]). Additionally, by [Are06, Example 9.2.2] we have $e^{-t\mathsf{P}_{g,V}} \geq 0$ for $t \geq 0$.

Recall that the Gamma function is defined by

$$\Gamma(s) := \int_0^\infty e^{-t} t^{s-1}\, dt,$$

and one has

$$\lambda^s = \frac{1}{\Gamma(-s)} \int_0^\infty \left(e^{-t\lambda} - 1 \right) \frac{dt}{t^{1+s}} \tag{4.14}$$

for $\lambda > 0$ and $0 < s < 1$. Then using fundamental properties of $\mathsf{P}_{g,V}$ (see Lemma B.1) and (4.14), one can show that there holds

$$\mathsf{P}^s_{g,V} u = \frac{1}{\Gamma(-s)} \int_0^\infty \left(e^{-t\mathsf{P}_{g,V}} u - u \right) \frac{dt}{t^{1+s}}. \tag{4.15}$$

for $u \in \widetilde{H}^{2s}_{g,V}(\Omega)$, which is called semigroup formula for $\mathsf{P}^s_{g,V}$. It is well-known that this holds in a very general setting, but in our case, the argument is more elementary.

In fact, first of all taking in (4.14) $\lambda = \lambda_k$, multiplying by $u_k\phi_k$ and summing k over $\{1, \dots, m\}$ we get

$$\mathsf{P}^s_{g,V} \sum_{k=1}^m u_k\phi_k = \frac{1}{\Gamma(-s)} \int_0^\infty \sum_{k=1}^m \left(e^{-t\lambda_k} - 1\right) u_k\phi_k \frac{dt}{t^{1+s}}$$

for all $m \in \mathbb{N}$. Here, we used $\phi_k \in \mathsf{Dom}(\mathsf{P}_{g,V})$, (4.7), (4.6) and (4.12). By construction the left hand side converges to $\mathsf{P}^s_{g,V}u$ in $L^2(\Omega, dV_g)$ and hence passing to the limit $m \to \infty$ gives

$$\mathsf{P}^s_{g,V}u = \lim_{m\to\infty} \frac{1}{\Gamma(-s)} \int_0^\infty \sum_{k=1}^m \left(e^{-t\lambda_k} - 1\right) u_k\phi_k \frac{dt}{t^{1+s}}$$

in $L^2(\Omega, dV_g)$. Hence, for all $v \in L^2(\Omega, dV_g)$ there holds

$$\begin{aligned}
&\left\langle \mathsf{P}^s_{g,V}u, v\right\rangle_{L^2(\Omega,dV_g)} \\
&= \lim_{m\to\infty} \frac{1}{\Gamma(-s)} \left\langle \int_0^\infty \sum_{k=1}^m \left(e^{-t\lambda_k} - 1\right) u_k\phi_k \frac{dt}{t^{1+s}}, v\right\rangle_{L^2(\Omega,dV_g)} \\
&= \lim_{m\to\infty} \frac{1}{\Gamma(-s)} \int_0^\infty \left\langle \sum_{k=1}^m \left(e^{-t\lambda_k} - 1\right) u_k\phi_k, v\right\rangle_{L^2(\Omega,dV_g)} \frac{dt}{t^{1+s}} \\
&= \frac{1}{\Gamma(-s)} \sum_{k=1}^\infty \int_0^\infty \left(e^{-t\lambda_k} - 1\right) u_k v_k \frac{dt}{t^{1+s}}
\end{aligned} \tag{4.16}$$

where we set $v_k = \langle v, \phi_k\rangle_{L^2(\Omega,dV_g)}$. Next, note that

$$\begin{aligned}
&\sum_{k=1}^\infty \int_0^\infty \left(1 - e^{-t\lambda_k}\right) |u_k||v_k| \frac{dt}{t^{1+s}} \\
\leq &\sum_{k=1}^\infty \left(\int_0^{1/\lambda_k} \left(1 - e^{-t\lambda_k}\right) \frac{dt}{t^{1+s}} + \int_{1/\lambda_k}^\infty \left(1 - e^{-t\lambda_k}\right) \frac{dt}{t^{1+s}} \right) |u_k||v_k|.
\end{aligned} \tag{4.17}$$

Now, the second integral in the right-hand side of (4.17) can be bounded as

$$\int_{1/\lambda_k}^\infty \left(1 - e^{-t\lambda_k}\right) \frac{dt}{t^{1+s}} \leq \int_{1/\lambda_k}^\infty \frac{dt}{t^{1+s}} \lesssim \lambda_k^s,$$

whereas the change of variables $\tau = t\lambda_k$ in the first integral yields

$$\int_0^{1/\lambda_k} \left(1 - e^{-t\lambda_k}\right) \frac{dt}{t^{1+s}} = \lambda_k^s \int_0^1 \left(1 - e^{-\tau}\right) \frac{d\tau}{\tau^{1+s}} \lesssim \lambda_k^s.$$

Inserting these estimates into (4.17) gives

$$\begin{aligned}
\sum_{k=1}^\infty \int_0^\infty \left(1 - e^{-t\lambda_k}\right) |u_k||v_k| \frac{dt}{t^{1+s}} &\lesssim \sum_{k=1}^\infty \lambda_k^s |u_k||v_k| \\
&\lesssim \sum_{k=1}^\infty \lambda_k^{2s}|u_k|^2 + \sum_{k=1}^\infty |v_k|^2 \\
&\lesssim \|u\|^2_{\widetilde{H}^{2s}_{g,V}(\Omega)} + \|v\|^2_{L^2(\Omega,dV_g)} < \infty.
\end{aligned}$$

Therefore, we can invoke Fubini's theorem in (4.16) to get

$$\begin{aligned}\left\langle \mathsf{P}^s_{g,V}u,v\right\rangle_{L^2(\Omega,dV_g)} &= \frac{1}{\Gamma(-s)}\int_0^\infty \sum_{k=1}^\infty \left(e^{-t\lambda_k}-1\right)u_k v_k\,\frac{dt}{t^{1+s}}\\ &= \frac{1}{\Gamma(-s)}\int_0^\infty \left\langle \left(e^{-t\mathsf{P}_{g,V}}-1\right)u,v\right\rangle_{L^2(\Omega,dV_g)}\frac{dt}{t^{1+s}},\end{aligned}$$

where we used (4.6) and (4.12). Hence, we have established (4.15).

Next, let us introduce the negative powers of $\mathsf{P}_{g,V}$. For fixed $0<s<1$, we set

$$\mathsf{P}^{-s}_{g,V}u=\sum_{k\geq 1}\lambda_k^{-s}u_k\phi_k,$$

which is well-defined for $u\in L^2(\Omega,dV_g)$ as $\lambda_k>0$. One easily verifies by a direct calculation that $\mathsf{P}^s_{g,V}$ is an isomorphism as a map from $\widetilde{H}^{2s}_{g,V}(\Omega)$ to $L^2(\Omega)$ and there holds

$$\mathsf{P}^{-s}_{g,V}\mathsf{P}^s_{g,V}=\mathrm{Id}_{\widetilde{H}^{2s}_{g,V}(\Omega)}\quad\text{and}\quad \mathsf{P}^s_{g,V}\mathsf{P}^{-s}_{g,V}=\mathrm{Id}_{L^2(\Omega,dV_g)}. \tag{4.18}$$

Let us remark here that through the integration by parts formula (4.10), the operator $\mathsf{P}^s_{g,V}$ can be extended to a continuous map from $\widetilde{H}^s_{g,V}(\Omega)$ to $H^{-s}_{g,V}(\Omega)$ and its again an isomorphism with inverse $\mathsf{P}^{-s}_{g,V}$. Furthermore, if one uses the identity

$$\lambda^{-s}=\frac{1}{\Gamma(s)}\int_0^\infty e^{-t\lambda}\frac{dt}{t^{1-s}},\quad \lambda>0,$$

then there holds

$$\mathsf{P}^{-s}_{g,V}u=\frac{1}{\Gamma(s)}\int_0^\infty e^{-t\mathsf{P}_{g,V}}u\frac{dt}{t^{1-s}} \tag{4.19}$$

for $u\in L^2(\Omega,dV_g)$. Note that the right-hand side of (4.19) converges in $L^2(\Omega,dV_g)$ due to (4.13). In fact, (4.13) implies

$$\begin{aligned}\int_0^\infty \left\|e^{-t\mathsf{P}_{g,V}}u\right\|_{L^2(\Omega,dV_g)}\frac{dt}{t^{1-s}} &\leq \left(\int_0^\infty e^{-\gamma t}\frac{dt}{t^{1-s}}\right)\|u\|_{L^2(\Omega,dV_g)}\\ &\leq \frac{\Gamma(s)}{\gamma^s}\|u\|_{L^2(\Omega,dV_g)}.\end{aligned} \tag{4.20}$$

Again to see the identity (4.19) one can rely on the abstract theory or argue similarly as for (4.15) via an expansion in eigenfunctions and using the identity (4.12).

4.2. **The Neumann derivative and the nonlocal equation.** We start by recalling that, in a similar vein as the fractional Laplacian $(-\Delta)^s$ [CS07], a wide class of nonlocal operators can be recovered as Neumann derivatives of solutions to suitable extension problems. For example in [ST10, Theorem 1.1] it is shown that if μ is a (σ-finite) nonnegative measure on $\Omega\subset\mathbb{R}^n$, $\mathcal{L}$ is a nonnegative, densely defined, self-adjoint operator on $L^2(\Omega,d\mu)$ with domain $\mathsf{Dom}(\mathcal{L})$ and $w\in\mathsf{Dom}\,(\mathcal{L}^s)$ for some $0<s<1$, then

$$W(x,y)=\frac{1}{\Gamma(s)}\int_0^\infty e^{-t\mathcal{L}}\left(\mathcal{L}^s w\right)(x)e^{-y^2/4t}\frac{dt}{t^{1-s}} \tag{4.21}$$

solves

$$\begin{cases}\mathcal{L}W-\frac{1-2s}{y}\partial_y W-\partial_y^2 W=0 & \text{in } \Omega\times\mathbb{R}_+,\\ W=w & \text{on } \Omega\times\{0\}\end{cases} \tag{4.22}$$

and one has

$$-\lim_{y\to 0^+}y^{1-2s}\partial_y W=c_s\mathcal{L}^s w \text{ on } \Omega\times\{0\}, \tag{4.23}$$

where $c_s > 0$ is a constant depending only on s. The previous limit has to be understood in the $L^2(\Omega, d\mu)$ sense.

In particular, as $s = 1/2$, we can connect (1.7) to a nonlocal equation. Let us make a few remarks.

(a) We observe that actually in our special case $s = 1/2$ and $\mathcal{L} = \mathsf{P}_{g,V} = -\Delta_g + V$, the above result follows by a more elementary argument for smooth functions. More precisely, let $g \in C^\infty(\overline{\Omega}; \mathbb{R}^{n\times n})$, $V \in C^\infty(\overline{\Omega})$ be independent of y, and Ω has smooth boundary, then one can easily see, arguing as in [CS07], that the operator

$$Tf = -\partial_y u|_{y=0},$$

where $u \in H_0^1(\Omega \times \mathbb{R}_+)$ uniquely solves

$$\begin{cases} (-\Delta_g + V)\, u - \partial_y^2 u = 0 & \text{in } \Omega \times \mathbb{R}_+, \\ u = 0 & \text{on } \partial\Omega \times \mathbb{R}_+, \\ u = f & \text{on } \Omega \times \{0\}, \end{cases}$$

which is a positive operator with $T^2 f = (-\Delta_g + V) f$. Therefore, we have

$$Tf = (-\Delta_g + V)^{1/2} f$$

for $f \in C^\infty(\overline{\Omega})$ vanishing on $\partial\Omega$. The identity $T^2 f = (-\Delta_g + V)\, f$, for $f \in C^\infty(\overline{\Omega})$ with $f = 0$ on $\partial\Omega$, which can be seen as follows

$$\begin{aligned} T^2 f &= T\left(-\partial_y u|_{y=0}\right) = \partial_y^2 u|_{y=0} \\ &= (-\Delta_g + V)\, u|_{y=0} = (-\Delta_g + V)\, f, \end{aligned}$$

since both g and V are y-independent.

(b) Furthermore, in our case $\mathcal{L} = \mathsf{P}_{g,V}$, we get from [Sti10, Section 3] that under the additional boundary condition at infinity

$$\lim_{y\to\infty} W(x,y) = 0 \text{ weakly in } L^2(\Omega, dV_g)$$

that W given by (4.21) is the unique solution of (4.22) and in particular coincides with the one obtained via the Fourier method, i.e. making the ansatz $W(x,y) = \sum_{k\geq 1} c_k(y)\phi_k$.

4.3. **ND map and source-to-solution map.** We next transfer the ND map of (1.7) to the source-to-solution map for the nonlocal elliptic equation

$$\mathsf{P}_{g,V}^{1/2} v = f \text{ in } \Omega. \tag{4.24}$$

By (4.7) and our notion of weak solutions to (4.24) (see in particular (4.11)), we know that for any $f \in C_c^\infty(\Omega)$ there exists a unique solution $v \in \widetilde{H}_{g,V}^{1/2}(\Omega)$ of (4.24).

Hence, taking into account (4.9), for a given open subset $\Gamma \subsetneq \Omega$ we can define the local *source-to-solution* map corresponding to (4.24) by

$$\mathcal{S}_{g,V}^\Gamma \colon C_c^\infty(\Gamma) \to L^2(\Gamma), \quad f \mapsto v^f\big|_\Gamma,$$

for any $f \in C_c^\infty(\Gamma)$, where $v^f \in \widetilde{H}_{g,V}^{1/2}(\Omega)$ is the solution to (4.24). Clearly, the source-to-solution map could be defined on a larger space like $H_{g,V}^{-1/2}(\Omega)$ by our notion of weak solutions, but $C_c^\infty(\Gamma)$ is for our purposes enough. This naturally leads to the following inverse problem:

(IP2) Inverse problem for the nonlocal elliptic equation. Can one determine the metric g and potential V in Ω by using the knowledge of the local source-to-solution map $\mathcal{S}_{g,V}^\Gamma$?

Recalling that with the boundary determination at hand, we know the information of both g and V on the measured open subset $\Gamma \Subset \Omega$. We next assert that measurements in the inverse problem **(IP1)** determine the measurements in **(IP2)**.

Lemma 4.5. *Let Ω, Γ, (g_1, V_1) and (g_2, V_2) be given as in Theorem 1.1. Suppose (1.9) holds, then one has*

$$\mathcal{S}^{\Gamma}_{g_1,V_1} f = \mathcal{S}^{\Gamma}_{g_2,V_2} f \text{ for any } f \in C_c^\infty(\Gamma), \tag{4.25}$$

where $\mathcal{S}_{g_j,V_j} : C_c^\infty(\Gamma) \ni f \mapsto v_j^f\big|_\Gamma \in L^2(\Gamma)$ is the local source-to-solution map of

$$\mathsf{P}^{1/2}_{g_j,V_j} v_j^f = f \text{ in } \Omega.$$

for $j = 1, 2$.

Proof. Let us start by recalling that the boundary determination result established in Section 3 ensures that

$$g_1 = g_2 \text{ and } V_1 = V_2 \text{ on } \Gamma.$$

Next, we show (4.25). For a given $f \in C_c^\infty(\Gamma)$, we denote by $u_j^f \in H_0^1(\Omega \times [0,\infty))$ the unique solutions of (1.8) for $j = 1, 2$ (see Lemma 2.2).

Claim 4.6. *For $j = 1, 2$, we have $u_j^f \in H^3(\Omega \times [0, R))$ for any $R > 0$.*

Let us offer the proof of Claim 4.6 in Appendix A. By using the previous claim and suitable trace theorems, we know that $u_j^f|_{y=0} \in H_0^1(\Omega) \cap H^2(\Omega)$ and hence $u_j^f|_{y=0} \in \mathsf{Dom}(\mathsf{P}^s_{g_j,V_j})$ by (4.7). In fact, Claim 4.6 ensures that $u_j^f \in H^1(0, R; H^2(\Omega))$ for fixed $R > 0$ and so by the trace theorem we have $u_j^f \in C([0, R]; H^2(\Omega))$, which gives $u_j^f|_{y=0} \in H^2(\Omega)$. Next, let us note that $u_j^f \in H_0^1(\Omega \times [0,\infty)) \hookrightarrow L^2(0,\infty; H_0^1(\Omega))$ ensures $u_j^f(\cdot, y) \in H_0^1(\Omega)$ for a.e. $y > 0$. Thus, we can deduce from $u_j^f \in C([0, R]; H^1(\Omega))$ that $u_j^f|_{y=0} \in H_0^1(\Omega)$ as $H_0^1(\Omega)$ is a closed subspace of $H^1(\Omega)$. Let $\varphi \in L^2(\Omega, dV_g)$ be fixed and consider the function $U_j : \mathbb{R}_+ \to \mathbb{R}$ defined by

$$U_j^f(y) := \int_\Omega u_j^f(x, y)\varphi(x)\, dV_g(x).$$

Using $u_j^f \in H_0^1(\Omega \times [0,\infty)) \hookrightarrow H^1(\mathbb{R}_+; L^2(\Omega; dV_g))$ we know that $U_j^f \in H^1(\mathbb{R}_+)$ and hence the Sobolev embedding ensures the uniform continuity of U_j^f on $[0,\infty)$. But then we get

$$U_j^f \to 0 \text{ as } y \to \infty.$$

Thus, we can invoke the uniqueness statement (b) of Section 4.2 to see that u_j^f is the unique solution of the extension problem

$$\begin{cases} (-\Delta_g + V)\, u - \partial_y^2 u = 0 & \text{in } \Omega \times \mathbb{R}_+, \\ u = 0 & \text{on } \partial\Omega \times \mathbb{R}_+, \\ u = u_j^f\Big|_{y=0} & \text{on } \Omega \times \{0\}, \end{cases}$$

for $j = 1, 2$. Thus, from (4.23) with $\mathcal{L} = \mathsf{P}_{g_j,V_j}$ and $s = 1/2$, we get that $v_f^j = u_f^j|_{y=0}$ satisfies

$$\begin{cases} \mathsf{P}^{1/2}_{g_j,V_j} v = f & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega. \end{cases} \tag{4.26}$$

Now, v_j^f is indeed the unique solution to this problem by $v_j^f \in H_0^1(\Omega)\cap H^2(\Omega)$ for $j=1,2$, (4.7) and the discussion at the beginning of the section. Combining (4.26) with the condition (1.9), we have

$$v_1^f = v_2^f \text{ in } \Gamma, \text{ for any } f \in C_c^\infty(\Gamma),$$

or stated alternatively (4.25). This proves the assertion. □

4.4. **Determination of heat kernel.** The purpose of this section is to show that if (g_j, V_j) is is prescribed on the measurement set Γ and the source-to-solution maps $\mathcal{S}^\Gamma_{g_j,V_j}$ coincide on Γ, then the Schwartz kernels $e^{-t\mathsf{P}_{g_j,V_j}}(\cdot,\cdot)$ of the semigroup $e^{-t\mathsf{P}_{g_j,V_j}}$ related to $\partial_t + \mathsf{P}_{g_j,V_j}$ in $\Omega\times(0,\infty)$ (see Appendix B for more details) agree on Γ. More precisely, we have the following lemma.

Lemma 4.7. *Assume that Ω, Γ, (g_j,V_j) for $j=1,2$ are given as in Theorem 1.1 and let $(g,V)\in C^\infty(\overline{\Omega};\mathbb{R}^{n\times n})\times C^\infty(\overline{\Omega})$ be any pair of a uniformly elliptic Riemannian metric g and nonnegative potential V such that (1.10) holds. Let $\mathcal{S}^\Gamma_{g_j,V_j}\colon C_c^\infty(\Gamma)\ni f\mapsto v_j^f|_\Gamma \in L^2(\Gamma)$ be the local source-to-solution map of (4.26) for $j=1,2$. Suppose that (1.11) holds, then we have*

$$e^{-t\mathsf{P}_{g_1,V_1}}(x,z) = e^{-t\mathsf{P}_{g_2,V_2}}(x,z) \text{ for } x,z\in\Gamma \text{ and } t>0. \tag{4.27}$$

Notice that the conditions (1.10) and (1.11) are the conclusions of Theorem 3.1 and Lemma 4.5.

Proof of Lemma 4.7. Fix any nonempty open subset $\mathcal{O}_1 \Subset \Gamma$ and let $f\in C_c^\infty(\mathcal{O}_1)\subset \mathsf{Dom}(\mathsf{P}^k_{g,V})$ for all $k\in\mathbb{N}_0=\mathbb{N}\cup\{0\}$ (see Lemma B.2). Using (1.10), we deduce the identity

$$(-\Delta_{g_1}+V_1)^k f = (-\Delta_{g_2}+V_2)^k f = (-\Delta_g + V)^k f \in C_c^\infty(\mathcal{O}_1)$$

for any $k\in\mathbb{N}_0$. Therefore, using the preceding identity, (4.18) and (1.11), we can deduce that

$$\mathsf{P}^{-1/2}_{g_1,V_1}(-\Delta_g+V)^k f = \mathsf{P}^{-1/2}_{g_2,V_2}(-\Delta_g+V)^k f \text{ on } \Gamma.$$

Then (4.19) ensures that

$$\int_0^\infty \left(e^{-t\mathsf{P}_{g_j,V_j}} - e^{-t\mathsf{P}_{g_j,V_j}}\right)(-\Delta_g+V)^k f\,\frac{dt}{t^{1/2}} = 0 \text{ on } \Gamma \tag{4.28}$$

for $k\in\mathbb{N}_0$ (see (4.20)). We also recall that by Lemma B.3, (c) we have

$$e^{-t\mathsf{P}_{g_j,V_j}}(-\Delta_g+V)^k f \in C^\infty(\overline{\Omega}\times[0,\infty))$$

for $j=1,2$.

Next, we follow arguments from [FGKU24, Section 2] (see [GU21, Proposition 3.1] for nonlocal elliptic operators and [LLU22, Section 4] for nonlocal parabolic operators). We first note that by the semigroup property of $e^{-t\mathsf{P}_{g,V}}$, $t\geq 0$, we have the commutativity

$$e^{-t\mathsf{P}_{g_j,V_j}}\left(-\Delta_{g_j}+V_j\right)^k g = \left(-\Delta_{g_j}+V_j\right)^k e^{-t\mathsf{P}_{g_j,V_j}} g \tag{4.29}$$

and

$$\partial_t^k\left(e^{-t\mathsf{P}_{g_j,V_j}} g\right) = (-1)^k\left(-\Delta_{g_j}+V_j\right)^k e^{-t\mathsf{P}_{g_j,V_j}} g \tag{4.30}$$

for all $g\in\mathsf{Dom}(\mathsf{P}^k_{g,V})$ (see (B.12) and Lemma B.3, (a)). Thus, inserting (4.29) and (4.30) into (4.28), we have

$$\int_0^\infty \partial_t^k\left(e^{-t\mathsf{P}_{g_1,V_1}} - e^{-t\mathsf{P}_{g_2,V_2}}\right) f\,\frac{dt}{t^{1/2}} = 0 \text{ on } \Gamma, \text{ for all } k\in\mathbb{N}_0. \tag{4.31}$$

We claim that there are no boundary contributions when performing in (4.31) an integration by parts. Using the above relations, Lemma B.3, (b), (B.13), (4.13) and (B.14), we get

$$\begin{aligned}
&\left\|\partial_t^k\left(e^{-t\mathsf{P}_{g_1,V_1}}-e^{-t\mathsf{P}_{g_2,V_2}}\right)f\right\|_{L^\infty(\Omega)}\\
&\lesssim\sum_{i=1}^{2}\left\|\partial_t^k e^{-t\mathsf{P}_{g_i,V_i}}f\right\|_{\mathsf{Dom}(\mathsf{P}_{g_i,V_i}^{m-k})}\\
&\lesssim\sum_{i=1}^{2}\sum_{\ell=0}^{m-k}\left\|\mathsf{P}_{g_i,V_i}^{\ell}\partial_t^k e^{-t\mathsf{P}_{g_i,V_i}}f\right\|_{L^2(\Omega,dV_g)}\\
&\lesssim\sum_{i=1}^{2}\sum_{\ell=0}^{m-k}\left\|e^{-t\mathsf{P}_{g_i,V_i}}\mathsf{P}_{g_i,V_i}^{\ell+k}f\right\|_{L^2(\Omega,dV_g)}\\
&\lesssim\sum_{i=1}^{2}e^{-\gamma_i t}\sum_{\ell=0}^{m-k}\left\|\mathsf{P}_{g_i,V_i}^{\ell+k}f\right\|_{L^2(\Omega,dV_g)}\\
&\lesssim\|f\|_{H^{2m}(\Omega,dV_g)}e^{-\gamma t},
\end{aligned}\tag{4.32}$$

where $\gamma=\min(\gamma_1,\gamma_2)>0$. In the calculation above m is chosen such that $m-k>n/4$. Note that formula (4.32) shows that for $t\to\infty$ there are no boundary contributions.

To proceed, we want to estimate the left-hand side of (4.32) for $t>0$ and $x\in\mathcal{O}_2$, where $\mathcal{O}_2$ is a nonempty open subset of Γ such that $\overline{\mathcal{O}}_1\cap\overline{\mathcal{O}}_2=\emptyset$. Indeed, by (B.10) and $f\in C_c^\infty(\mathcal{O}_1)$ we may write

$$\begin{aligned}
&\partial_t^k\left[\left(e^{-t\mathsf{P}_{g_1,V_1}}-e^{-t\mathsf{P}_{g_2,V_2}}\right)f\right](x)\\
&=(-1)^k\left[\left(e^{-t\mathsf{P}_{g_1,V_1}}-e^{-t\mathsf{P}_{g_2,V_2}}\right)(-\Delta_g+V)^k f\right](x)\\
&=(-1)^k\int_{\mathcal{O}_1}\left[\left(e^{-t\mathsf{P}_{g_1,V_1}}(x,z)-e^{-t\mathsf{P}_{g_2,V_2}}(x,z)\right)(-\Delta_g+V)^k f(z)\right]dV_g(z),
\end{aligned}\tag{4.33}$$

for $x\in\mathcal{O}_2$, where $e^{-t\mathsf{P}_{g_j,V_j}}(x,z)\geq 0$ is the (bounded) Schwartz kernel of $e^{-t\mathsf{P}_{g_j,V_j}}$ for $j=1,2$. Via (4.33), we have

$$\begin{aligned}
&\left|\partial_t^k\left[\left(e^{-t\mathsf{P}_{g_1,V_1}}-e^{-t\mathsf{P}_{g_2,V_2}}\right)f\right](x)\right|\\
&\leq\int_{\mathcal{O}_1}\left|\left(e^{-t\mathsf{P}_{g_1,V_1}}(x,z)-e^{-t\mathsf{P}_{g_2,V_2}}(x,z)\right)(-\Delta_g+V)^k f(z)\right|dV_g(z)\\
&\leq\left\|e^{-t\mathsf{P}_{g_1,V_1}}(\cdot,\cdot)-e^{-t\mathsf{P}_{g_2,V_2}}(\cdot,\cdot)\right\|_{L^\infty(\mathcal{O}_2\times\mathcal{O}_1)}\left\|(-\Delta_g+V)^k f\right\|_{L^1(\mathcal{O}_1,dV_g)},
\end{aligned}$$

for $x\in\mathcal{O}_2$ and any $k\in\mathbb{N}_0$. Moreover, we can use a Gaussian upper bound for the kernel $e^{-t\mathsf{P}_{g_j,V_j}}(\cdot,\cdot)$ (see (B.11)) to obtain

$$\begin{aligned}
&\left|\partial_t^k\left[\left(e^{-t\mathsf{P}_{g_1,V_1}}-e^{-t\mathsf{P}_{g_2,V_2}}\right)f\right](x)\right|\\
&\leq ct^{-n/2}e^{-b\,(\mathrm{dist}(\mathcal{O}_1,\mathcal{O}_2))^2/t}e^{\omega t}\left\|(-\Delta_g+V)^k f\right\|_{L^1(\mathcal{O}_1,dV_g)},
\end{aligned}\tag{4.34}$$

for $x\in\mathcal{O}_2$, any $k\in\mathbb{N}_0$ and $t>0$, where $b,c>0$, $\omega\in\mathbb{R}$ only depend on the heat kernel $e^{-t\mathsf{P}_{g_j,V_j}}$ (cf. (B.11)) and $\mathrm{dist}(\mathcal{O}_1,\mathcal{O}_2):=\inf\{|x_1-x_2|\,;\,x_1\in\mathcal{O}_1,\ x_2\in\mathcal{O}_2\}$. This shows that we also do not have a boundary contribution at $t=0$ as by assumption $\mathrm{dist}(\mathcal{O}_1,\mathcal{O}_2)>0$.

Therefore, using $e^{-t\mathsf{P}_{g_j,V_j}}f\in C^\infty(\overline{\Omega}\times[0,\infty))$ for $j=1,2$, (4.32) and (4.34), an integration by parts (k times) with respect to the t-variable in (4.31) yields that

$$\int_0^\infty\left[\left(e^{-t\mathsf{P}_{g_1,V_1}}-e^{-t\mathsf{P}_{g_2,V_2}}\right)f\right](x)\,\frac{dt}{t^{k+1/2}}=0,$$

for $x \in \mathcal{O}_2$ and any $k \in \mathbb{N}_0$. In particular, by using the change of variables $\zeta = \frac{1}{t}$, we obtain

$$\int_0^\infty \phi_x(\zeta)\zeta^k \, d\zeta = 0, \tag{4.35}$$

for $x \in \mathcal{O}_2$ and any $k \in \mathbb{N}_0$, where introduced for fixed $x \in \mathcal{O}_2$ the function $\phi_x \colon (0,\infty) \to \mathbb{R}$ by

$$\phi_x(\zeta) := \frac{\left(e^{-\frac{1}{\zeta}\mathsf{P}_{g_1,V_1}} - e^{-\frac{1}{\zeta}\mathsf{P}_{g_2,V_2}}\right) f(x)}{\zeta^{1/2}}.$$

Claim 4.8. *The functions $(\phi_x)_{x\in\mathcal{O}_2}$ have the following properties*

(a) $\phi_x \in C^\infty((0,\infty)) \cap L^2((0,\infty))$

(b) and for some $\alpha > 0$ we have $\mathcal{L}(\phi_x)(s) = 0$ for $0 < s < \alpha$, where $\mathcal{L}\colon L^2((0,\infty)) \to L^2((0,\infty))$ is the Laplace transform defined by

$$\mathcal{L}f(s) = \int_0^\infty f(t)e^{-st} \, dt$$

for $f \in L^2((0,\infty))$ and $s > 0$.

Proof of Claim 4.8. The smoothness assertion follows immediately from Lemma B.3, (c). To see $\phi_x \in L^2((0,\infty))$, we use the change of variables $\zeta = 1/t$ to write

$$\begin{aligned}
\|\phi_x\|^2_{L^2((0,\infty))} &= \int_0^\infty \frac{\left|\left(e^{-\frac{1}{\zeta}\mathsf{P}_{g_1,V_1}} - e^{-\frac{1}{\zeta}\mathsf{P}_{g_2,V_2}}\right) f(x)\right|^2}{\zeta} \, d\zeta \\
&= \int_0^1 \left|\left(e^{-t\mathsf{P}_{g_1,V_1}} - e^{-t\mathsf{P}_{g_2,V_2}}\right) f(x)\right|^2 \frac{dt}{t} \\
&\quad + \int_1^\infty \left|\left(e^{-t\mathsf{P}_{g_1,V_1}} - e^{-t\mathsf{P}_{g_2,V_2}}\right) f(x)\right|^2 \frac{dt}{t}.
\end{aligned}$$

The second integral is finite as $e^{-t\mathsf{P}_{g_j,V_j}} f \in L^2(0,\infty; H^1_0(\Omega))$ for $j = 1,2$ (see (B.6)) and using (4.34) the first integral can be estimated as

$$\begin{aligned}
&\int_0^1 \left|\left(e^{-t\mathsf{P}_{g_1,V_1}} - e^{-t\mathsf{P}_{g_2,V_2}}\right) f(x)\right|^2 \frac{dt}{t} \\
&\lesssim \|f\|_{L^1(\mathcal{O}_1,dV_g)} \int_0^1 t^{-n/2} e^{-d/t} \frac{dt}{t} \\
&= d^{-n/2} \|f\|_{L^1(\mathcal{O}_1,dV_g)} \int_d^\infty e^{-\tau}\tau^{n/2-1} \, d\tau \\
&\lesssim d^{-n/2} \|f\|_{L^1(\mathcal{O}_1,dV_g)} \Gamma(n/2) < \infty
\end{aligned}$$

for some constant $d > 0$. This establishes $\phi_x \in L^2((0,\infty))$ and hence completes the proof of assertion (a).

First, recall that we have

$$\left| e^{-s\zeta} - \sum_{k=0}^N \frac{(-s\zeta)^k}{(N+1)!} \right| = \frac{e^{-s\xi}}{(N+1)!}(-s\zeta)^{N+1},$$

for any $N \in \mathbb{N}$, $\zeta > 0$, $s > 0$ and some fixed $\xi \in (0,\zeta)$. As the Laplace transform is a bounded operator from $L^2((0,\infty))$ to itself, we know that $\mathcal{L}(\phi_x)(s)$ makes sense for $s > 0$ (up to a set of measure zero). By (4.35), we have

$$\int_0^\infty \phi_x(\zeta)e^{-s\zeta} \, d\zeta = \int_0^\infty \phi_x(\zeta) \left(e^{-s\zeta} - \sum_{k=0}^N \frac{(-s\zeta)^k}{(N+1)!} \right) d\zeta,$$

for any $N \in \mathbb{N}$ and $s > 0$. Therefore, we may estimate

$$\begin{aligned}
&\left|\int_0^\infty \phi_x(\zeta)\left(e^{-s\zeta} - \sum_{k=0}^{N}\frac{(-s\zeta)^k}{(N+1)!}\right)d\zeta\right| \\
&\lesssim \frac{s^{N+1}}{(N+1)!}\int_0^\infty |\phi_x(\zeta)|\,\zeta^{N+1}\,d\zeta \\
&= \frac{s^{N+1}}{(N+1)!}\left(\int_0^1 |\phi_x(\zeta)|\,\zeta^{N+1}\,d\zeta + \int_1^\infty |\phi_x(\zeta)|\,\zeta^{N+1}\,d\zeta\right) \\
&\lesssim \frac{s^{N+1}}{(N+1)!}\left(\|\phi_x\|_{L^2((0,\infty))} + \int_1^\infty |\phi_x(\zeta)|\,\zeta^{N+1}\,d\zeta\right).
\end{aligned} \tag{4.36}$$

The last integral can be controlled by using the Gaussian bound (4.34) as

$$\begin{aligned}
\int_1^\infty |\phi_x(\zeta)|\,\zeta^{N+1}\,d\zeta &\lesssim \int_1^\infty e^{-\alpha\zeta}e^{\omega/\zeta}\zeta^{N+n/2+1/2}\,d\zeta \\
&\lesssim \int_1^\infty e^{-\alpha\zeta}\zeta^{N+n/2+1/2}\,d\zeta \\
&= \alpha^{-(N+n/2+3/2)}\int_d^\infty e^{-\rho}\rho^{N+n/2+1/2}\,d\rho \\
&\lesssim \alpha^{-(N+n/2+3/2)}\Gamma(N+n/2+3/2).
\end{aligned}$$

for some $\alpha > 0$. Next, let us recall that for any $\beta \in \mathbb{C}$ we have the asymptotics

$$\Gamma(x+\beta) \sim \Gamma(x)x^\beta \text{ as } x \to \infty. \tag{4.37}$$

Inserting this into (4.36) and using (4.37), we arrive at the estimate

$$\begin{aligned}
&\left|\int_0^\infty \phi_x(\zeta)\left(e^{-s\zeta} - \sum_{k=0}^{N}\frac{(-s\zeta)^k}{(N+1)!}\right)d\zeta\right| \\
&\lesssim \frac{s^{N+1}}{(N+1)!}\left(\|\phi_x\|_{L^2((0,\infty))} + \alpha^{-(N+n/2+3/2)}\Gamma(N+n/2+3/2)\right) \\
&\lesssim \frac{s^{N+1}}{(N+1)!} + \frac{s^{N+1}}{\alpha^{N+n/2+3/2}}\frac{\Gamma(N+n/2+3/2)}{\Gamma(N+2)} \\
&\sim \frac{s^{N+1}}{(N+1)!} + \left(\frac{s}{\alpha}\right)^N (N+2)^{(n-1)/2} \\
&\lesssim \frac{s^{N+1}}{(N+1)!} + \left(\frac{s}{\alpha}\right)^N N^{(n-1)/2} \\
&= 0
\end{aligned}$$

as $N \to \infty$. Here, we used that as $N \to \infty$ the first term goes to zero for all $s > 0$ and the second term as long as $0 < s < \alpha$. Hence, we deduce that

$$\mathcal{L}(\phi_x)(s) = 0 \text{ for } 0 < s < \alpha$$

and this concludes the proof of (b). Hence, Claim 4.8 is proved. ☐

Since $\phi_x \in L^2((0,\infty))$ its Laplace transform can be extended analytically to the right half plane of $\mathbb{C}$ and thus (b) of Claim 4.8 together with the identity theorem for analytic functions guarantee that $\mathcal{L}\phi_x = 0$ for $s > 0$. Now, we can invoke the inversion formula to deduce $\phi_x(\zeta) = 0$ for $\zeta > 0$. This in turn implies

$$\left[\left(e^{-t\mathsf{P}_{g_1,V_1}} - e^{-t\mathsf{P}_{g_2,V_2}}\right)f\right](x) = 0, \text{ for } t > 0 \text{ and } x \in \mathcal{O}_2.$$

On the other hand, via the condition (1.10), the function

$$v = \left(e^{-t\mathsf{P}_{g_1,V_1}} - e^{-t\mathsf{P}_{g_2,V_2}}\right)f$$

is a solution to

$$
\begin{cases} (\partial_t + \mathsf{P}_{g,V})\, v = 0 & \text{in } \Gamma \times (0,\infty), \\ v = 0 & \text{in } \mathcal{O}_2 \times (0,\infty), \end{cases} \tag{4.38}
$$

where we utilized the notation $g = g_1 = g_2$ and $V = V_1 = V_2$ on the open subset $\Gamma \Subset \Omega$. We may deduce from the fact that v solves (4.38), Γ is connected and the unique continuation property of solutions to heat equations (see, for example, [Lin90, Sections 1 and 4]) that

$$
\left[\left(e^{-t\mathsf{P}_{g_1,V_1}} - e^{-t\mathsf{P}_{g_2,V_2}}\right) f\right](x) = 0, \text{ for } t > 0 \text{ and } x \in \Gamma. \tag{4.39}
$$

Let us also note that for any given $f \in C_c^\infty(\Gamma)$ we can always choose open sets $\mathcal{O}_1, \mathcal{O}_2 \subset \Gamma$ such that $\operatorname{supp} f \subset \mathcal{O}_1 \Subset \Gamma$ and $\overline{\mathcal{O}}_2 \cap \overline{\mathcal{O}}_1 = \emptyset$. Hence, by (4.39) there holds

$$
e^{-t\mathsf{P}_{g_1,V_1}} f\big|_{\Gamma} = e^{-t\mathsf{P}_{g_2,V_2}} f\big|_{\Gamma}, \text{ for } t > 0, \tag{4.40}
$$

for any $f \in C_c^\infty(\Gamma)$. Finally, (4.40) and (B.10) yield that

$$
e^{-t\mathsf{P}_{g_1,V_1}}(x,z) = e^{-t\mathsf{P}_{g_2,V_2}}(x,z) \text{ for } t > 0 \text{ and } x, z \in \Gamma,
$$

which implies that the condition (4.27) holds. This completes the proof. □

5. Inverse problem for wave equations

In this section, we introduce another key tool – the *Kannai type transmutation formula* (see [Kan77]). This will transfer solutions of wave equations to solutions of heat equations, via time integration against suitable kernel functions (see eq. (5.9)). Using Lemma 4.7 allows us to relate the inverse problem **(IP2)** to an inverse source problem for the associated wave equation

$$
\begin{cases} \left(\partial_t^2 + \mathsf{P}_{g,V}\right) w = F & \text{in } \Omega \times [0,\infty), \\ w(0) = w_0, \quad \partial_t w(0) = w_1 & \text{in } \Omega. \end{cases} \tag{5.1}
$$

By establishing a unique determination for this inverse problem, we will prove in Section 5.3 our main result, Theorem 1.1.

Before proceeding, let us collect some relevant well-posedness and regularity results for the Cauchy problem (5.1), whose proof is presented in Appendix C for completeness.

Theorem 5.1. *Let $\Omega \subset \mathbb{R}^n$ be a smoothly bounded domain, $g \in C^\infty(\overline{\Omega}; \mathbb{R}^{n\times n})$ a uniformly elliptic Riemannian metric, $V \in C^\infty(\overline{\Omega})$ be a nonnegative potential and let $\mathsf{P}_{g,V}$ be the unbounded operator introduced in (4.1)-(4.2).*

(a) Suppose that $w_0 \in H^2(\Omega, dV_g) \cap H_0^1(\Omega, dV_g)$, $w_1 \in H_0^1(\Omega, dV_g)$ and $F \in C^1([0,\infty); L^2(\Omega, dV_g))$. Then there exists a unique function w satisfying

$$
\begin{cases} w \in C([0,\infty); \mathsf{Dom}(\mathsf{P}_{g,V})), \\ \partial_t w \in C([0,\infty); H_0^1(\Omega; dV_g)), \\ \partial_t^2 w \in C([0,\infty); L^2(\Omega, dV_g)) \end{cases} \tag{5.2}
$$

and solving the Cauchy problem (5.1).

(b) If $w_j \in \bigcap_{k\in\mathbb{N}} H^k(\Omega, dV_g)$ satisfy $\mathsf{P}_{g,V}^k w_j \in H_0^1(\Omega, dV_g)$ for $k \in \mathbb{N}_0$, $j = 0, 1$ and $F \in C_c^\infty(\Omega \times (0,\infty))$, then the unique solution w of (5.1) belongs to $C^\infty(\overline{\Omega} \times [0,\infty))$.

(c) *Under the assumptions of assertion (a), the unique solution w of (5.1) has the representation formula*

(5.3)
$$\begin{aligned} w(t) &= \sum_{k\geq 1}\left[\cos(t\lambda_k^{1/2})w_0^k + \frac{\sin(t\lambda_k^{1/2})}{\lambda_k^{1/2}}w_1^k + \int_0^t \frac{\sin((t-\tau)\lambda_k^{1/2})}{\lambda_k^{1/2}}F_k(\tau)\,d\tau\right]\phi_k \\ &= \cos(t\mathsf{P}_{g,V}^{1/2})w_0 + \frac{\sin(t\mathsf{P}_{g,V}^{1/2})}{\mathsf{P}_{g,V}^{1/2}}w_1 + \int_0^t \frac{\sin((t-\tau)\mathsf{P}_{g,V}^{1/2})}{\mathsf{P}_{g,V}^{1/2}}F(\tau)\,d\tau, \end{aligned}$$

where $F_k(t) = \langle F(t), \phi_k\rangle_{L^2(\Omega,dV_g)}$ and $w_j^k = \langle w_j, \phi_k\rangle_{L^2(\Omega,dV_g)}$ for $k \in \mathbb{N}$, $t \geq 0$, $j = 0,1$.

From now on, for any source $F \in C^1([0,\infty); L^2(\Omega, dV_g))$, we denote by $w^F \in C([0,\infty); \mathsf{Dom}(\mathsf{P}_{g,V}))$ the unique solution to the Cauchy problem for the wave equation with zero initial data

$$\begin{cases} \left(\partial_t^2 + \mathsf{P}_{g,V}\right) w = F & \text{in } \Omega\times(0,\infty), \\ w(0) = \partial_t w(0) = 0 & \text{in } \Omega. \end{cases}$$

Next, using this notation, we introduce the (local) *source-to-solution map* by

$$\begin{aligned} \mathscr{J}_{g,V}^{\Gamma} \colon C^1([0,\infty); L^2(\overline{\Gamma}, dV_g)) &\to C([0,\infty); H^2(\Gamma)), \\ F &\mapsto w^F|_{\Gamma\times[0,\infty)}, \end{aligned} \tag{5.4}$$

where $\Gamma \Subset \Omega$ and $L^2(\overline{\Gamma}, dV_g)$ denotes the collection of functions $G \in L^2(\Omega, dV_g)$ with $\operatorname{supp} G \subset \overline{\Gamma}$. Observe that by Theorem 5.1, (b) we know that $\mathscr{J}_{g,V}^{\Gamma}F \in C^\infty(\Gamma\times[0,\infty))$, whenever $F \in C_c^\infty(\Omega\times(0,\infty))$. The above considerations lead naturally to the following inverse problem.

(IP3) Inverse problem for the wave equation. Can one uniquely determine the metric g and potential V from the local source-to-solution map $\mathscr{J}_{g,V}^{\Gamma}$?

The rest of this section is structured as follows. In Section 5.1 we show that the inverse problem **(IP2)** can be related to **(IP3)**, and in Section 5.2 we establish an affirmative answer to the question **(IP3)** and finally in Section 5.3 we proof our main result, Theorem 1.1.

5.1. Kannai type transmutation and relation between (IP2) and (IP3).

The following lemma is similar to the one in [FGKU24, Section 3] and we offer the proof for the sake of completeness.

Lemma 5.2. *Let Ω, Γ, (g_1, V_1) and (g_2, V_2) be given as in Theorem 1.1. Consider the local source-to-solution map $\mathscr{J}_{g_j,V_j}^{\Gamma}$ of*

$$\begin{cases} \left(\partial_t^2 + \mathsf{P}_{g_j,V_j}\right) w_j = F & \text{in } \Omega\times(0,\infty), \\ w_j(0) = \partial_t w_j(0) = 0 & \text{in } \Omega, \end{cases} \tag{5.5}$$

for $j = 1,2$ and suppose that the conditions (1.10) and (4.27) hold for some pair $(g,V) \in C^\infty(\overline{\Omega}; \mathbb{R}^{n\times n}) \times C^\infty(\overline{\Omega})$ consisting of a uniformly elliptic Riemannian metric g and nonnegative potential V. Then there holds

$$\mathscr{J}_{g_1,V_1}^{\Gamma}F = \mathscr{J}_{g_2,V_2}^{\Gamma}F, \text{ for any } F \in C_c^\infty(\Gamma\times(0,\infty)). \tag{5.6}$$

Proof of Lemma 5.2. First note that via the Fourier inversion formula, we have

$$e^{-t\lambda^2} = \frac{1}{\sqrt{4\pi t}}\int_{-\infty}^{\infty} e^{-\frac{\zeta^2}{4t}} e^{\mathrm{i}\zeta\lambda}\,d\zeta = \frac{1}{\sqrt{4\pi t}}\int_{-\infty}^{\infty} e^{-\frac{\zeta^2}{4t}}\cos(\zeta\lambda)\,d\zeta, \quad t > 0. \tag{5.7}$$

For $\lambda \neq 0$, an integration by parts in (5.7) yields that

$$
\begin{aligned}
e^{-t\lambda^2} &= \frac{2}{4\sqrt{\pi}t^{3/2}} \int_0^\infty \zeta e^{-\frac{\zeta^2}{4t}} \frac{\sin(\zeta\lambda)}{\lambda}\, d\zeta \\
&= \frac{1}{4\sqrt{\pi}t^{3/2}} \int_0^\infty e^{-\frac{\tau}{4t}} \frac{\sin\left(\tau^{1/2}\lambda\right)}{\lambda}\, d\tau.
\end{aligned} \tag{5.8}
$$

Here we used that the sine function is odd and the change of variables $\tau = \zeta^2$. For any $f \in L^2(\Omega, dV_g)$, Lemma 4.4, (5.8) and Fubini's theorem ensure that the Kannai type transmutation formula holds, that is

$$
e^{-t\mathsf{P}_{g_j,V_j}} f = \frac{1}{4\sqrt{\pi}t^{3/2}} \int_0^\infty e^{-\frac{\tau}{4t}} \frac{\sin(\tau^{1/2}\mathsf{P}_{g_j,V_j}^{1/2})}{\mathsf{P}_{g_j,V_j}^{1/2}} f\, d\tau \text{ in } L^2(\Omega, dV_g) \tag{5.9}
$$

for $j = 1, 2$. If $f \in C_c^\infty(\Gamma)$, then (B.10), (4.27) and (5.9) imply

$$
\int_0^\infty e^{-\frac{\tau}{4t}} \left(\frac{\sin\left(\tau^{1/2}\mathsf{P}_{g_1,V_1}^{1/2}\right)}{\mathsf{P}_{g_1,V_1}^{1/2}} f \right)(x)\, d\tau = \int_0^\infty e^{-\frac{\tau}{4t}} \left(\frac{\sin\left(\tau^{1/2}\mathsf{P}_{g_2,V_2}^{1/2}\right)}{\mathsf{P}_{g_2,V_2}^{1/2}} f \right)(x)\, d\tau, \tag{5.10}
$$

for $t > 0$ and $x \in \Gamma$. This holds in the sense that we test in the L^2-sense the expression under the integral against any $h \in C_c^\infty(\Gamma)$. Applying the inverse Laplace transform in (5.10), we obtain

$$
\left(\frac{\sin\left(\tau^{1/2}\mathsf{P}_{g_1,V_1}^{1/2}\right)}{\mathsf{P}_{g_1,V_1}^{1/2}} f \right)(x) = \left(\frac{\sin\left(\tau^{1/2}\mathsf{P}_{g_2,V_2}^{1/2}\right)}{\mathsf{P}_{g_2,V_2}^{1/2}} f \right)(x), \tag{5.11}
$$

for $\tau > 0$ and $x \in \Gamma$. Therefore, with the representation formula (5.3) and $F \in C_c^\infty(\Gamma \times (0,\infty))$, by using (5.11), one can conclude

$$
w_1^F(x,t) = w_2^F(x,t) \text{ in } \Gamma \times [0,\infty),
$$

which proves (5.6). This proves the assertion. □

5.2. **Simultaneous determination for wave equations.** The goal of this section is to prove the following affirmative answer to the inverse problem **(IP3)**.

Theorem 5.3. *Let Ω, Γ, (g_1, V_1) and (g_2, V_2) be given as in Theorem 1.1. Let $\mathcal{J}_{g_j,V_j}^\Gamma$ be the local source-to-solution map of (5.5). Suppose that the conditions (1.10) and (5.6) hold for some pair $(g, V) \in C^\infty(\overline{\Omega}; \mathbb{R}^{n\times n}) \times C^\infty(\overline{\Omega})$ consisting of a uniformly elliptic Riemannian metric g and nonnegative potential V, then there exists a diffeomorphism $\Psi\colon \overline{\Omega} \to \overline{\Omega}$ with $\Psi|_{\overline{\Gamma}} = \mathrm{Id}_{\overline{\Gamma}}$ on $\overline{\Gamma}$ such that $g_1 = \Psi_* g_2$ and $V_1 = V_2 \circ \Psi$ in Ω.*

We will reduce the proof of Theorem 5.3 to a unique determination problem in [KOP18].

Proof of Theorem 5.3. Let us start by recalling that by assumption $(\overline{\Omega}, g)$ is a compact, connected, smooth manifold with smooth boundary $\partial\Omega$ and we have given the following data

$$
\left(\Gamma,\, g|_\Gamma, V|_\Gamma,\, \mathcal{J}_{g,V}^\Gamma\right), \tag{5.12}
$$

where $\mathcal{J}_{g,V}^\Gamma$ denotes the local source-to-solution map for the wave equation with zero initial data (see (5.4)). Now, we aim to recover (g, V) in the connected set $\Omega' := \Omega \setminus \overline{\Gamma}$ up to a diffeomorphism. We divide the proof of Theorem 5.3 into two steps:

Step 1. Source-to-solution data (5.12) determines the restricted DN map.

Let us consider the wave equation in the domain $\Omega' \times (0,\infty)$:

$$\begin{cases} \left(\partial_t^2 + \mathsf{P}_{g,V}\right)\widetilde{w} = 0 & \text{in } \Omega' \times (0,\infty), \\ \widetilde{w} = f & \text{on } \partial\Gamma \times (0,\infty), \\ \widetilde{w} = 0 & \text{on } \partial\Omega \times (0,\infty), \\ \widetilde{w}(0) = \partial_t\widetilde{w}(0) = 0 & \text{in } \Omega'. \end{cases} \tag{5.13}$$

It is known that the restricted DN map for the wave equation (5.13) can be defined by

$$\Lambda^{\mathrm{w},\partial\Gamma,T}_{g,V} \colon f|_{\partial\Gamma\times(0,T)} \mapsto \partial_{\nu_g}\widetilde{w}_f\big|_{\partial\Gamma\times(0,T)}, \tag{5.14}$$

for any $T>0$, where $f \in C^\infty_c(\partial\Gamma \times (0,T))$ and w_f is the unique solution to (5.13). The well-posedness of (5.13) can be obtained as follows: First extend the boundary condition f to a function $\bar{f} \in C^\infty_c(\overline{\Omega'} \times [0,\infty))$ with $\bar{f}|_{t=0} = 0$, $\bar{f}|_{\partial\Omega\times[0,\infty)} = 0$ and set $\widetilde{w} = \widetilde{v} + \bar{f}$. Then $\widetilde{v}$ solves a wave equation of the form (5.1), which uniquely exists by Theorem 5.1 and hence showing the well-posedness of (5.13). By [KOP18, Lemma 4.2], it is known that the data (5.12) determines the map $\Lambda^{\mathrm{w},\partial\Gamma,T}_{g,V}$.

Step 2. Determination of the metric and potential from the restricted DN map. By Step 1, (1.10) and (5.6) we have

$$\Lambda^{\mathrm{w},\partial\Gamma,T}_{g_1,V_1} f = \Lambda^{\mathrm{w},\partial\Gamma,T}_{g_2,V_2} f, \text{ for any } f \in C^\infty_c(\partial\Gamma \times (0,T)),$$

for any $T>0$, where $\Lambda^{\mathrm{w},\partial\Gamma,T}_{g_j,V_j}$ stands for the restricted DN map given by (5.14), for $j=1,2$. We may apply [KOP18, Theorem 1.1] (with $E_j = \overline{\Omega'} \times \mathbb{C}$, $\mathcal{S}_j = \partial\Gamma$ and $\phi = \mathrm{Id}_{\partial\Gamma\times\mathbb{C}}$) to conclude that there exists a hermitian vector bundle isomorphism $\Phi \colon \overline{\Omega'} \times \mathbb{C} \to \overline{\Omega'} \times \mathbb{C}$ such that $\Phi|_{\partial\Gamma\times\mathbb{C}} = \phi$ and there holds

$$\Psi^* g_2 = g_1 \text{ and } \Phi^* V_2 = V_1,$$

where $\Psi \colon \overline{\Omega'} \to \overline{\Omega'}$ is the induced diffeomorphism of Φ. Thus, we have

$$\Phi(x,v) = (\Psi(x), c(x)v),$$

where $c\colon \overline{\Omega'} \to \mathbb{C}$ is smooth scalar function with $c(x) = 1$ on $\partial\Gamma$. Observe that $V_1 = \Phi^* V_2$ means nothing else in the scalar case than $V_1 = V_2 \circ \Psi$. ☐

Remark 5.4. *Notice that the results in [KOP18] hold in the more general vector-valued setting, where the potential V is not longer scalar-valued as in our case. Moreover, in [KOP18], the authors even allowed the leading order operator to have a drift term, which emerges from an additional vector potential A. Thus, in the present article we do not invoke the full strength of the results in [KOP18].*

5.3. **Proof of Theorem 1.1.** Last but not least, we can show Theorem 1.1.

Proof of Theorem 1.1. First as the ND data agree (see (1.9)), the boundary determination (Theorem 3.1) shows that

$$g_1 = g_2 \text{ and } V_1 = V_2 \text{ on } \Gamma. \tag{5.15}$$

Furthermore, Lemma 4.5 guarantees that

$$\mathcal{S}^\Gamma_{g_1,V_1} f = \mathcal{S}^\Gamma_{g_2,V_2} f \text{ for all } f \in C^\infty_c(\Gamma), \tag{5.16}$$

where $\mathcal{S}^\Gamma_{g_j,V_j}$ is the source-to-solution map for the nonlocal equation

$$\mathsf{P}^{1/2}_{g_j,V_j} v = f \text{ in } \Omega,$$

for $j=1,2$. Next, let us fix any extension (g,V), consisting of a uniformly elliptic Riemannian metric g and nonnegative potential V, of $(g_1|_\Gamma, V_1|_\Gamma)$ to the whole domain $\overline{\Omega}$. By Lemma 4.7 we know from (5.15) and (5.16) that the Schwartz kernels of the corresponding heat semigroups agree on Γ, that is

$$e^{-t\mathsf{P}_{g_1,V_1}}(x,z)=e^{-t\mathsf{P}_{g_2,V_2}}(x,z) \text{ for } t>0 \text{ and } x,z\in\Gamma. \tag{5.17}$$

By Lemma 5.2 the conditions (5.15), (5.16) and (5.17) ensure that

$$\mathcal{J}^{\Gamma}_{g_1,V_1}F=\mathcal{J}^{\Gamma}_{g_2,V_2}F \text{ for any } F\in C_c^\infty(\Gamma\times(0,\infty)), \tag{5.18}$$

where $\mathcal{J}^{\Gamma}_{g_j,V_j}$ denotes the source-to-solution map for the wave equation

$$\begin{cases}\left(\partial_t^2+\mathsf{P}_{g_j,V_j}\right)w=F & \text{in } \Omega\times[0,\infty),\\ w(0)=w_0,\quad \partial_t w(0)=w_1 & \text{in } \Omega,\end{cases}$$

for $j=1,2$. Finally, using (5.15) and (5.18) we can apply Theorem 5.3 to establish the assertion of Theorem 1.1. $\square$

Remark 5.5. *Let us note that the above proof of Theorem 1.1 also establishes Theorem 1.3 and the methods in this work can be used to study more general versions of it. For example, one can consider the problem*

$$\begin{cases}\left(-\Delta_g+V\right)^s u=f & \text{in } \Omega,\\ u=0 & \text{on } \partial\Omega,\end{cases}$$

where $f\in C_c^\infty(\Gamma)$, $\Gamma\Subset\Omega$ is a given smooth domain and $0<s<1$. If $(g|_\Gamma,V|_\Gamma)$ and the local source-to-solution map $\mathcal{S}^{s,\Gamma}_{g,V}\colon f|_\Gamma\mapsto u_f|_\Gamma$ are prescribed, for any $f\in C_c^\infty(\Gamma)$, then one could apply similar methods as in this work to determine simultaneously (g,V) in Ω up to a diffeomorphism.

Appendix A. Reflection and Poincaré inequality

To derive a suitable Poincaré inequality, we will make use of the following simple lemma on first-order reflections.

Lemma A.1 (First order reflection)**.** *Let $\Omega\subset\mathbb{R}^n$ be an open set. Then for any function $u\colon\Omega\times[0,\infty)\to\mathbb{R}$, we define its* first order reflection *$\widetilde{u}\colon\Omega\times\mathbb{R}\to\mathbb{R}$ by*

$$\widetilde{u}(x,y):=\begin{cases}u(x,y), & \text{if } \ y\geq 0\\ -3u(x,-y)+4u(x,-y/2), & \text{if } \ y\leq 0\end{cases}$$

and set $u_+:=\overline{u}|_{\Omega\times[0,\infty)}$, $u_-:=\overline{u}|_{\Omega\times(-\infty,0]}$. If $u\in C_c^1(\Omega\times[0,\infty))$, then there holds

- *(a) $\widetilde{u}\in C_c^1(\Omega\times\mathbb{R})$,*
- *(b) $u_+|_{y=0}=u_-|_{y=0}$,*
- *(c) $\partial_y u_+|_{y=0}=\partial_y u_-|_{y=0}$,*
- *(d) $\partial_{x^i}u_+|_{y=0}=\partial_{x^i}u_-|_{y=0}$ for $1\leq i\leq n$*
- *(e) and there exists $C>0$ independent of u such that*

$$\|\widetilde{u}\|_{L^2(\Omega\times\mathbb{R})}\leq C\|u\|_{L^2(\Omega\times\mathbb{R}_+)} \quad \text{and} \quad \|\nabla\widetilde{u}\|_{L^2(\Omega\times\mathbb{R})}\leq C\|\nabla u\|_{L^2(\Omega\times\mathbb{R}_+)}. \tag{A.1}$$

If $u\in H_0^1(\Omega\times[0,\infty))$, then $\widetilde{u}\in H_0^1(\Omega\times\mathbb{R})$ and the estimate (A.1) still holds.

Proof. For the first part of Lemma A.1 dealing with functions in $C_c^1(\Omega\times[0,\infty))$, we refer to [Eva10, Section 5.4]. Now, suppose that $u\in H_0^1(\Omega\times[0,\infty))$ and choose $(\varphi_k)_{k\in\mathbb{N}}\subset C_c^1(\Omega\times[0,\infty))$ such that $\varphi_k\to u$ in $H^1(\Omega\times\mathbb{R}_+)$ as $k\to\infty$. By (a) we know that $\widetilde{\varphi}_k\in C_c^1(\Omega\times\mathbb{R})$. Using (A.1) we deduce that $(\widetilde{\varphi}_k)_{k\in\mathbb{N}}$ is a Cauchy sequence in $H^1(\Omega\times\mathbb{R})$ and hence there exists $v\in H^1(\Omega\times\mathbb{R})$ such that $\widetilde{\varphi}_k\to v$ in $H^1(\Omega\times\mathbb{R})$ as $k\to\infty$. On the other hand, up to extracting a subsequence we have

$\widetilde{\varphi}_k \to \widetilde{u}$ a.e. in $\Omega\times\mathbb{R}$ as $k\to\infty$ and hence $v=\widetilde{u}$ in $\Omega\times\mathbb{R}$. Thus, $\widetilde{u}\in H_0^1(\Omega\times\mathbb{R})$ as v belongs to this space. Now, by (A.1) we have

$$\|\widetilde{\varphi}_k\|_{L^2(\Omega\times\mathbb{R})}\leq C\,\|\varphi_k\|_{L^2(\Omega\times\mathbb{R}_+)} \quad\text{and}\quad \|\nabla\widetilde{\varphi}_k\|_{L^2(\Omega\times\mathbb{R})}\leq C\,\|\nabla\varphi_k\|_{L^2(\Omega\times\mathbb{R}_+)}$$

for all $k\in\mathbb{N}$ and hence passing to the limit $k\to\infty$ gives

$$\|\widetilde{u}\|_{L^2(\Omega\times\mathbb{R})}\leq C\|u\|_{L^2(\Omega\times\mathbb{R}_+)} \quad\text{and}\quad \|\nabla\widetilde{u}\|_{L^2(\Omega\times\mathbb{R})}\leq C\|\nabla u\|_{L^2(\Omega\times\mathbb{R}_+)},$$

which concludes the proof. ☐

This lemma allows us to establish the following Poincaré inequality.

Theorem A.2 (Poincaré inequality)**.** *Let $\Omega\subset\mathbb{R}^n$ be a bounded domain endowed with a uniformly elliptic Riemannian metric $g=(g_{ij})$ and extension $\widetilde{g}$ to $\Omega\times\mathbb{R}_+$. Then there exists $C>0$ such that there holds*

$$\|u\|_{L^2(\Omega\times\mathbb{R}_+,dV_{\widetilde{g}})}\leq C\|du\|_{L^2(\Omega\times\mathbb{R}_+;dV_{\widetilde{g}})} \tag{A.2}$$

for all $u\in H_0^1(\Omega\times[0,\infty))$.

Proof. Let $u\in H_0^1(\Omega\times[0,\infty))$ and denote by $\widetilde{u}\in H_0^1(\Omega\times\mathbb{R})$ the corresponding first order reflection of u from Lemma A.1. Then by the classical Poincaré inequality, we know that there holds

$$\|\widetilde{u}\|_{L^2(\Omega\times\mathbb{R})}\leq C\,\|\nabla\widetilde{u}\|_{L^2(\Omega\times\mathbb{R})}$$

for some $C>0$ independent of $\overline{u}$. Using (A.1), we deduce

$$\|u\|_{L^2(\Omega\times\mathbb{R}_+)}\leq\|\widetilde{u}\|_{L^2(\Omega\times\mathbb{R})}\leq C\,\|\nabla\widetilde{u}\|_{L^2(\Omega\times\mathbb{R})}\leq C\|\nabla u\|_{L^2(\Omega\times\mathbb{R}_+)}.$$

The uniform ellipticity of g ensures the equivalences (2.3) and thus the estimate (A.2) follows. ☐

At the end of this section, let us prove Claim 4.6.

Proof of Claim 4.6. For $j=1,2$, let us set $u=u_j^f$, $g=g_j$ and $V=V_j$. By construction $u\in H_0^1(\Omega\times[0,\infty))$ solves

$$\begin{cases}(-\Delta_{\widetilde{g}}+V)\,u=0 & \text{in }\Omega\times\mathbb{R}_+,\\ u=0 & \text{on }\partial\Omega\times\mathbb{R}_+,\\ -\partial_y u=f & \text{on }\Omega\times\{0\}.\end{cases}$$

Since $f\in C_c^\infty(\Gamma)$, we can find $F\in C_c^\infty(\Gamma\times\mathbb{R})$ such that $\partial_y F|_{y=0}=f$. For example one can take $F(x,y)=yf(x)\rho(y)$, where $\rho\in C_c^\infty(\mathbb{R})$ is a cutoff function with $\rho=1$ in a neighborhood of $y=0$. Now, we may observe that $v=u-F\in H_0^1(\Omega\times[0,\infty))$ solves

$$\begin{cases}(-\Delta_{\widetilde{g}}+V)\,v=G & \text{in }\Omega\times\mathbb{R}_+,\\ v=0 & \text{on }\partial\Omega\times\mathbb{R}_+,\\ -\partial_y v=0 & \text{on }\Omega\times\{0\}\end{cases}$$

with $G=-(-\Delta_{\widetilde{g}}+V)\,F$.

Next, with $\partial_y v|_{\Omega\times\{0\}}=0$, let us define the *even reflection* of v by

$$v^*(x,y)=\begin{cases}v(x,y), & \text{for }(x,y)\in\Omega\times[0,\infty)\\ v(x,-y), & \text{for }(x,y)\in\Omega\times(-\infty,0)\end{cases}.$$

It is well-known that $v^*\in H_0^1(\Omega\times\mathbb{R})$ with

$$\partial_y v^*(x,y)=\begin{cases}\partial_y v(x,y), & \text{in }\Omega\times[0,\infty)\\ -\partial_y v(x,-y), & \text{in }\Omega\times(-\infty,0).\end{cases}$$

Then a simple calculation shows that v^* solves

$$\begin{cases}(-\Delta_{\widetilde{g}} + V)\, v = G^* & \text{in } \Omega\times\mathbb{R},\\ v = 0 & \text{on } \partial\Omega\times\mathbb{R},\end{cases}$$

where G^* denotes the even reflection of G. Let $\eta\in C_c^\infty(\Omega\times\mathbb{R})$, then $w=\eta v^*\in H^1(\mathbb{R}^{n+1})$ (extended by zero outside of Ω) solves

$$(-\Delta_{\widetilde{g}} + V)\, w = H^* \text{ in } \mathbb{R}^{n+1}, \tag{A.3}$$

where $H^* := \eta G^* - v^*\Delta_{\widetilde{g}}\eta - 2dv^*\cdot d\eta - 2\partial_y v^*\partial_y\eta \in L^2(\mathbb{R}^{n+1})$. Hence, elliptic regularity theory implies $w\in H^2_{\mathrm{loc}}(\mathbb{R}^{n+1})$ and thus $v\in H^2(\omega\times[0,R))$ for all $\omega\Subset\Omega$ and $R>0$. Although in general G^* is not regular for regular functions G, in our case close to $y=0$ we have $G^*=|y|(-\Delta_g+V)f$ and therefore $G^*\in H^1(\mathbb{R}^{n+1})$. This in turn implies $H^*\in H^1(\mathbb{R}^{n+1})$. Thus, by differentiating (A.3) and arguing as before we get $v\in H^3(\omega\times[0,R))$ for any $\omega\Subset\Omega$ and $R>0$. Boundary regularity can be obtained precisely as in [GT83, Theorem 8.12] by using the method of difference quotients. One obtains that for any $x\in\partial\Omega$, there exists $r>0$ such that $v\in H^2(B_r(x)\times[0,R))$ for any $R>0$ and hence by a covering $\partial\Omega$ with such balls and taking into account the interior regularity result, we get $v\in H^3(\Omega\times[0,R))$ for any $R>0$. Going back from v to our original solution u, the Claim 4.6 is followed. □

Appendix B. Heat semigroup and powers of $-\Delta_g+V$

B.1. **Functional analytic properties of $-\Delta_g+V$ and heat semigroup.** Let us make the following observations, which were used repeatedly throughout this article.

(a) There holds

$$\partial_k|g|^{\pm 1/2} = \pm\frac{|g|^{\pm1/2}}{2}g^{ij}\partial_k g_{ji} \tag{B.1}$$

for all $1\leq k\leq n$. Hence, iteratively we get $|g|^{\pm1/2}\in C^\infty(\overline{\Omega})$.

(b) We have $\varphi\in H_0^1(\Omega)$ if and only if $|g|^{1/2}\varphi\in H_0^1(\Omega)$.

(c) Suppose that $u\in H^1(\Omega)$ (weakly) solves

$$(-\Delta_g+q) = f \text{ in } \Omega \tag{B.2}$$

for some $f\in L^2(\Omega,dV_g)$ and $q\in L^\infty(\Omega)$, that is

$$\int_\Omega (du\cdot d\varphi + qu\varphi)\, dV_g = \int_\Omega f\varphi\, dV_g$$

for all $\varphi\in H_0^1(\Omega)$. Then by (B.1) we deduce that

$$\begin{aligned}&\int_\Omega f\varphi|g|^{1/2}\,dx\\ =&\int_\Omega\left(g^{ij}\partial_i u\partial_j(|g|^{1/2}\varphi) - (g^{ij}\partial_i u\partial_j|g|^{1/2})\varphi + qu|g|^{1/2}\varphi\right)dx\\ =&\int_\Omega\left(g^{ij}\partial_i u\partial_j(|g|^{1/2}\varphi) - \frac{1}{2}(g^{ij}g^{k\ell}\partial_j g_{\ell k}\partial_i u)|g|^{1/2}\varphi + qu|g|^{1/2}\varphi\right)dx.\end{aligned}$$

Thus, by (b) we can replace φ in the previous formula by $|g|^{-1/2}\psi$ with $\psi\in H_0^1(\Omega)$ and obtain that $u\in H^1(\Omega)$ solves (B.2) if and only if $u\in H^1(\Omega)$ solves

$$-\mathrm{div}(g\nabla u) + b\cdot\nabla u + qu = f \text{ in } \Omega, \tag{B.3}$$

where $b = \left(b^1, \ldots, b^n\right) \in C^\infty(\overline{\Omega}, \mathbb{R}^n)$ is given by

$$b^i = -\frac{1}{2} g^{ij} g^{k\ell} \partial_j g_{\ell k}, \text{ for } i = 1, \ldots, n. \tag{B.4}$$

Lemma B.1. *The operator $\mathsf{P}_{g,V}$ has the following properties:*

(a) $\mathsf{P}_{g,V}$ *is symmetric meaning that*

$$\langle \mathsf{P}_{g,V} u, v \rangle_{L^2(\Omega, dV_g)} = \langle u, \mathsf{P}_{g,V} v \rangle_{L^2(\Omega, dV_g)} \text{ for all } u, v \in \mathsf{Dom}\left(\mathsf{P}_{g,V}\right).$$

(b) $\mathsf{P}_{g,V}$ *is maximal monotone, that is there holds*

(i) Monotonicity: For all $u \in \mathsf{Dom}\left(\mathsf{P}_{g,V}\right)$ *one has*

$$\langle \mathsf{P}_{g,V} u, u \rangle_{L^2(\Omega, dV_g)} \geq 0,$$

(ii) Maximality: $\mathrm{Ran}\left(1 + \mathsf{P}_{g,V}\right) = L^2(\Omega, dV_g)$.

Furthermore, $\mathsf{P}_{g,V}$ *is a self-adjoint operator on* $L^2(\Omega, dV_g)$.

Proof. Note that (a) and (i) of (b) follow by a simple integration by parts as $\mathsf{Dom}(\mathsf{P}_{g,V}) = H_0^1(\Omega) \cap H^2(\Omega)$ (see Lemma B.2 below). Hence, we only need to show (ii). To see this, fix some $f \in L^2(\Omega, dV_g)$, then one observes that

$$\ell_f \colon H_0^1(\Omega) \to \mathbb{R}, \quad \langle \ell_f, \varphi \rangle = \int_\Omega f \varphi \, dV_g$$

is a continuous linear form on $H_0^1(\Omega)$ and as g is uniformly elliptic as well as $0 \leq V \in L^\infty(\Omega)$ an equivalent inner product on $H_0^1(\Omega)$ is given by

$$\langle u, v \rangle_{g,V} = \int_\Omega \left(g^{ij} \partial_i u \partial_j v + (V+1) uv\right) dV_g.$$

Therefore, by the Riesz representation theorem, there exists a unique $u \in H_0^1(\Omega)$ satisfying

$$\langle u, \varphi \rangle_{g,V} = \int_\Omega f \varphi \, dV_g \text{ for all } \varphi \in H_0^1(\Omega).$$

As explained above, using $\varphi = |g|^{-1/2} \psi$ with $\psi \in H_0^1(\Omega)$ as a test function, we get an equation of the type (B.3) and then we can invoke the usual elliptic regularity theory to deduce $u \in H^2(\Omega)$ (see [GT83, Theorem 8.12]). Hence, an integration by parts guarantees that $u \in \mathsf{Dom}\left(\mathsf{P}_{g,V}\right)$ satisfies

$$\left(\mathrm{Id} + \mathsf{P}_{g,V}\right) u = f \text{ in } L^2(\Omega, dV_g)$$

and this establishes (ii). Now, we can apply [Bre11, Proposition 7.6] to infer that $\mathsf{P}_{g,V}$ is in fact a self-adjoint operator on $L^2(\Omega, dV_g)$. □

Now, we explain the reason for the validity of the identities (4.3) and (4.4). The first identity follows by using the orthonormality of $(\phi_k)_{k \in \mathbb{N}}$ and $\mathsf{P}_{g,V} \phi_k = \lambda_k \phi_k$. If $u \in \mathsf{Dom}(\mathsf{P}_{g,V})$, then the identity (4.3) guarantees the $(\lambda_k u_k)_{k \in \mathbb{N}} \subset \ell^2(\mathbb{N})$. Let

$$U_m = \sum_{k=1}^m u_k \phi_k \text{ and } V_m = \sum_{k=1}^m \lambda_k u_k \phi_k.$$

By construction, we have $U_m \in \mathsf{Dom}(\mathsf{P}_{g,V})$ and $\mathsf{P}_{g,V} U_m = V_m$, for $m \in \mathbb{N}$. Then clearly $U_m \to u$ in $L^2(\Omega, dV_g)$ and as V_m is a Cauchy sequence in $L^2(\Omega, dV_g)$ it converges to some limit in $L^2(\Omega, dV_g)$. By [Bre11, Proposition 7.1] maximal monotone operators are closed and hence we may conclude that $\mathsf{P}_{g,V} u = \sum_{k=1}^\infty \lambda_k u_k \phi_k$.

Since $\mathsf{P}_{g,V}$ is a symmetric, maximal monotone operator, [Are06, Theorem 2.3.1] implies that $-\mathsf{P}_{g,V}$ generates a C_0-semigroup of contractive, self-adjoint operators on $L^2(\Omega, dV_g)$, which we denote as usual by $\left(e^{-t\mathsf{P}_{g,V}}\right)_{t \geq 0}$ (the heat kernel of $\partial_t + \mathsf{P}_{g,V}$). Here, contractive means nothing else than $\left\|e^{-t\mathsf{P}_{g,V}}\right\|_{L(L^2(\Omega; dV_g))} \leq 1$, where $L(L^2(\Omega, dV_g))$ denotes the operator norm from $L^2(\Omega, dV_g)$ to $L^2(\Omega, dV_g)$. More precisely, for a given function $f \in L^2(\Omega, dV_g)$, the function $U(t) := e^{-t\mathsf{P}_{g,V}} f$ is the unique solution of

$$\begin{cases} u \in C([0,\infty); L^2(\Omega, dV_g)) \cap C^1((0,\infty); L^2(\Omega, dV_g)), \\ u \in C((0,\infty); \mathsf{Dom}(\mathsf{P}_{g,V})), \\ (\partial_t + \mathsf{P}_{g,V})\, u = 0 \text{ in } (0,\infty), \\ u(0) = f \end{cases} \tag{B.5}$$

(see [Bre11, Theorem 7.7]). In (B.5) the space $\mathsf{Dom}(\mathsf{P}_{g,V})$ is regarded as a Hilbert space with an inner product given by

$$\langle u, v\rangle_{\mathsf{Dom}(\mathsf{P}_{g,V})} = \langle u, v\rangle_{L^2(\Omega; dV_g)} + \langle \mathsf{P}_{g,V} u, \mathsf{P}_{g,V} v\rangle_{L^2(\Omega; dV_g)}.$$

Furthermore, we have

$$U \in L^2(0,\infty; H^1_0(\Omega, dV_g)) \tag{B.6}$$

with

$$\begin{split} &\frac{\|U(t)\|^2_{L^2(\Omega;dV_g)}}{2} + \int_0^t \left(\|dU(\tau)\|^2_{L^2(\Omega,dV_g)} + \left\|V^{1/2}U(\tau)\right\|^2_{L^2(\Omega,dV_g)}\right) d\tau \\ &= \frac{\|f\|^2_{L^2(\Omega,dV_g)}}{2} \end{split} \tag{B.7}$$

for all $t > 0$. To see this let us consider the function $\varphi \in C^1((0,\infty))$ given by

$$\varphi(t) = \frac{\|U(t)\|^2_{L^2(\Omega,dV_g)}}{2}.$$

Using $U \in C^1((0,\infty); L^2(\Omega, dV_g)) \cap C((0,\infty); \mathsf{Dom}(\mathsf{P}_{g,V}))$, we get

$$\begin{aligned} \varphi'(\tau) &= \langle U(\tau), \partial_t U(\tau)\rangle_{L^2(\Omega,dV_g)} \\ &= -\langle U(\tau), \mathsf{P}_{g,V} U(\tau)\rangle_{L^2(\Omega,dV_g)} \\ &= -\|dU(\tau)\|^2_{L^2(\Omega,dV_g)} - \left\|V^{1/2}U(\tau)\right\|^2_{L^2(\Omega;dV_g)} \end{aligned}$$

for any $\tau > 0$. Therefore, the fundamental theorem of calculus implies

$$\begin{aligned} \varphi(t) - \varphi(\epsilon) &= \int_\epsilon^t \varphi'(\tau)\, d\tau \\ &= -\int_\epsilon^t \left(\|dU(\tau)\|^2_{L^2(\Omega,dV_g)} + \left\|V^{1/2}U(\tau)\right\|^2_{L^2(\Omega,dV_g)}\right) d\tau \end{aligned}$$

for all $0 < \epsilon < t < \infty$. As $U \in C([0,\infty); L^2(\Omega, dV_g))$ and $U(0) = f$, we obtain in the limit $\epsilon \to 0$ the energy identity (B.7). But now the energy inequality shows

$$\int_0^t \|dU(\tau)\|^2_{L^2(\Omega,dV_g)}\, d\tau \le \frac{\|f\|^2_{L^2(\Omega,dV_g)}}{2} < \infty$$

for all $t \ge 0$, which in turn implies $U \in L^2(0,\infty; H^1_0(\Omega, dV_g))$ and hence establishes (B.6). Finally, from the fact that $U \in C((0,\infty); \mathsf{Dom}(\mathsf{P}_{g,V})) \cap L^2(0,\infty; H^1_0(\Omega, dV_g))$ it also follows that

$$\partial_t U \in L^2(0,\infty; H^{-1}(\Omega, dV_g)). \tag{B.8}$$

Proof of Lemma 4.4. First note that by the Galerkin method, using the finite-dimensional subspaces spanned by $(\phi_k)_{k\in\mathbb{N}} \subset H^1_0(\Omega, dV_g)$ as the Galerkin approximation of $L^2(\Omega, dV_g)$, the problem

$$\begin{cases} (\partial_t + \mathsf{P}_{g,V})\, u = 0 & \text{in } \Omega \times (0,T), \\ u = 0 & \text{on } \partial\Omega, \\ u(0) = f & \text{in } \Omega \end{cases} \tag{B.9}$$

has a unique (weak) solution

$$u \in L^2(0,T; H_0^1(\Omega, dV_g)) \text{ with } \partial_t u \in L^2(0,T; H^{-1}(\Omega, dV_g)),$$

for any $T > 0$ (see [Eva10, Chapter 7] or [DL92, Chapter XVIII]). By construction, the approximate solutions u_m are given by

$$u_m(t) = \sum_{k=1}^{m} e^{-\lambda_k t} f_k \phi_k \ \text{ with } \ f_k = \langle f, \phi_k \rangle_{L^2(\Omega, dV_g)},$$

which converge in $L^2(\Omega, dV_g)$ to the solution u, that is

$$u(t) = \sum_{k=1}^{\infty} e^{-\lambda_k t} f_k \phi_k$$

(see [DL92, Chapter XVIII, Section 3.5, Remark 4]). By uniqueness of the problem (B.9), (B.6) and (B.8), we deduce that

$$U(t) = \sum_{k=1}^{\infty} e^{-\lambda_k t} f_k \phi_k.$$

This concludes the proof. □

Next, let us recall that by [AE97, Theorem 3.1] (see also [Aro68, Section 7]) there exists $b, c > 0$, $\omega \in \mathbb{R}$ and $K_t = K_t(x,z) \in L^\infty(\Omega \times \Omega)$ such that

$$e^{-t\mathsf{P}_{g,V}} \varphi(x) = \int_\Omega K_t(x,z) \varphi(z) \, dV_g(z) \text{ for a.e. } x \in \Omega \tag{B.10}$$

for all $t > 0$, $\varphi \in L^2(\Omega, dV_g)$ and

$$|K_t(x,z)| \le c t^{-n/2} e^{-b|x-z|^2/t} e^{\omega t} \text{ for a.e. } x, z \in \Omega. \tag{B.11}$$

By the above discussion, we have $K_t \ge 0$. In the following, we set

$$e^{-t\mathsf{P}_{g,V}}(\cdot,\cdot) := K_t(\cdot,\cdot).$$

B.2. **Integer powers of $-\Delta_g + V$.** Next, let us introduce integer powers of our operator $\mathsf{P}_{g,V} = -\Delta_g + V$, and recall some regularity results for solutions of (B.5) for regular initial conditions. For $2 \le k \in \mathbb{N}$, we set

$$\mathsf{Dom}\left(\mathsf{P}_{g,V}^k\right) = \left\{ v \in \mathsf{Dom}(\mathsf{P}_{g,V}^{k-1}) \, ; \, \mathsf{P}_{g,V} v \in \mathsf{Dom}(\mathsf{P}_{g,V}^{k-1}) \right\}, \tag{B.12}$$

and define

$$\mathsf{P}_{g,V}^k = \underbrace{\mathsf{P}_{g,V} \cdots \mathsf{P}_{g,V}}_{\text{k times}}.$$

It is easily seen that the space $\mathsf{Dom}(\mathsf{P}_{g,V}^k)$, $k \ge 1$, is a Hilbert space if we endow it with the inner product

$$\langle u, v \rangle_{\mathsf{Dom}(\mathsf{P}_{g,V}^k)} = \sum_{j=0}^{k} \left\langle \mathsf{P}_{g,V}^j u, \mathsf{P}_{g,V}^j v \right\rangle_{L^2(\Omega, dV_g)}. \tag{B.13}$$

Lemma B.2. *For all $k \ge 1$, we have*

$$\mathsf{Dom}(\mathsf{P}_{g,V}^k) \hookrightarrow H^{2k}(\Omega) \tag{B.14}$$

and there holds

$$\mathsf{Dom}(\mathsf{P}_{g,V}^k) = \left\{ u \in H^{2k}(\Omega) \, ; \, u, P_{g,V} u, \dots, P_{g,V}^{k-1} u \in H_0^1(\Omega) \right\}. \tag{B.15}$$

Proof. Let us prove it via mathematical induction.

Case 1. $k=1$:

Let $u \in \mathsf{Dom}(\mathsf{P}_{g,V})$. By (4.2), the function $u \in H_0^1(\Omega)$ solves

$$\mathsf{P}_{g,V} u = f \text{ in } \Omega$$

for some $f \in L^2(\Omega)$. Therefore $u \in H_0^1(\Omega)$ satisfies

$$-\mathrm{div}(g\nabla u) + b\cdot\nabla u + Vu = f \text{ in } \Omega, \tag{B.16}$$

where $b \in C^\infty(\overline{\Omega};\mathbb{R}^n)$ is defined as in (B.4), but then [GT83, Theorem 8.12] implies $u \in H^2(\Omega)$ with

$$\|u\|_{H^2(\Omega)} \lesssim \|u\|_{L^2(\Omega)} + \|f\|_{L^2(\Omega)}.$$

This in turn shows that

$$\|u\|_{H^2(\Omega)} \lesssim \|u\|_{L^2(\Omega;dV_g)} + \|\mathsf{P}_{g,V} u\|_{L^2(\Omega;dV_g)} \lesssim \|u\|_{\mathsf{Dom}(\mathsf{P}_{g,V})}$$

(see (B.13)). This establishes (B.14) in the case $k=1$. On the other hand by definition of $\mathsf{Dom}(\mathsf{P}_{g,V})$ we know $u \in H_0^1(\Omega)$. Hence, we have

$$\mathsf{Dom}(\mathsf{P}_{g,V}) \subset \{u \in H^2(\Omega)\,;\, u \in H_0^1(\Omega)\}.$$

The reversed inclusion is also true. Thus, (B.15) holds for $k=1$.

Case 2. $k-1 \mapsto k$:

Let $u \in \mathsf{Dom}(\mathsf{P}_{g,V}^k)$. As $\mathsf{Dom}(\mathsf{P}_{g,V}^{k-1}) \hookrightarrow H^{2k-2}(\Omega)$ and $u \in H_0^1(\Omega)$ solves (B.16) with $f \in H^{2k-2}(\Omega)$, elliptic regularity theory [GT83, Theorem 8.13] implies that $u \in H^k(\Omega)$ and

$$\begin{aligned}\|u\|_{H^k(\Omega)} &\lesssim \|u\|_{L^2(\Omega)} + \|\mathsf{P}_{g,V} u\|_{H^{2k-2}(\Omega)} \\ &\lesssim \|u\|_{L^2(\Omega,dV_g)} + \|\mathsf{P}_{g,V} u\|_{\mathsf{Dom}(\mathsf{P}_{g,V}^{k-1})} \\ &\lesssim \|u\|_{\mathsf{Dom}(\mathsf{P}_{g,V}^k)}.\end{aligned}$$

In the second estimate, we used $\mathsf{P}_{g,V} u \in \mathsf{Dom}(\mathsf{P}_{g,V}^{k-1})$ and $\mathsf{Dom}(\mathsf{P}_{g,V}^{k-1}) \hookrightarrow H^{2k-2}(\Omega)$. Therefore, we get (B.14). As $u \in \mathsf{Dom}(\mathsf{P}_{g,V}^{k-1})$, we know already

$$u, \mathsf{P}_{g,V} u, \ldots, \mathsf{P}_{g,V}^{k-2} u \in H_0^1(\Omega).$$

As $\mathsf{P}_{g,V} u \in \mathsf{Dom}(\mathsf{P}_{g,V}^{k-1})$, we also have $\mathsf{P}_{g,V}^{k-1} u \in H_0^1(\Omega)$ and thus

$$\mathsf{Dom}(\mathsf{P}_{g,V}^k) \subset \left\{u \in H^{2k}(\Omega)\,;\, u, P_{g,V} u, \ldots, P_{g,V}^{k-1} u \in H_0^1(\Omega)\right\}.$$

The other inclusion $\supset$ is again easily seen by the induction hypothesis. Hence, we have established (B.15) and can conclude the proof. □

Lemma B.3 (Regularity of heat semigroup)**.** *Let the notation be as above and in particular for given $f \in L^2(\Omega, dV_g)$ denote by $u = e^{-t\mathsf{P}_{g,V}} f$ the unique solution to* (B.5)*.*

(a) *If $f \in \mathsf{Dom}(\mathsf{P}_{g,V}^k)$ for some $k \in \mathbb{N}$, then there holds*

$$u \in C^{k-j}([0,\infty); \mathsf{Dom}(\mathsf{P}_{g,V}^j)) \text{ for all } j = 0,1,\ldots,k.$$

(b) *If $f \in \mathsf{Dom}(\mathsf{P}_{g,V}^k)$ for some $k > n/4$ and $0 \le j \le k$ satisfies $j > n/4$, then there holds $u \in C^{k-j}([0,\infty); C^{\ell_j,\alpha_j}(\overline{\Omega}))$ and*

$$\left\|\partial_t^{k-j} u(t)\right\|_{C^{\ell_j,\alpha_j}(\overline{\Omega})} \lesssim \left\|\partial_t^{k-j} u(t)\right\|_{\mathsf{Dom}(\mathsf{P}_{g,V}^j)} \tag{B.17}$$

for any $t \geq 0$. *The exponents* $\ell_j \in \mathbb{N}_0$, $\alpha_j \in (0,1]$ *are given by*

$$\ell = \begin{cases} [2j - n/2], & \text{if } 2j - n/2 \notin \mathbb{N}, \\ 2j - n/2 - 1, & \text{if } 2j - n/2 \in \mathbb{N} \end{cases} \quad \text{and}$$

$$\alpha \in \begin{cases} [0, 2j - [2j - n/2] - n/2], & \text{if } n/2 \notin \mathbb{N}, \\ [0,1), & \text{if } n/2 \in \mathbb{N}, \end{cases}$$

where $[x] = \max\{k \in \mathbb{Z}\,;/, x \geq k\}$ *for* $x \in \mathbb{R}$.

(c) *If* $f \in \mathsf{Dom}(\mathsf{P}_{g,V}^k)$ *for all* $k \in \mathbb{N}$, *then there holds* $u \in C^\infty(\overline{\Omega} \times [0,\infty))$.

Proof. The statement (a) is an immediate consequence of [Bre11, Theorem 7.4-7.5]. Next, let us prove the assertion (b). First, by (a) we know that $\partial_t^{k-j} u(t) \in \mathsf{Dom}(\mathsf{P}_{g,V}^j)$ for all $t \geq 0$ and thus Lemma B.2 implies

$$\left\|\partial_t^{k-j} u(t)\right\|_{H^{2j}(\Omega)} \lesssim \left\|\partial_t^{k-j} u(t)\right\|_{\mathsf{Dom}(\mathsf{P}_{g,V}^j)}.$$

Therefore, by the Sobolev embedding, we arrive at the estimate (B.17). The statement (c) is a direct consequence of (b). ☐

Appendix C. Well-posedness of the wave equation

We next give the proof of the Theorem 5.1.

Proof of Theorem 5.1. For (a), let us start by rewriting (5.1) as a system of first-order equations

$$\begin{cases} \partial_t w - w' = 0 & \text{in } \Omega \times (0,\infty), \\ \partial_t w' + \mathsf{P}_{g,V} w = F & \text{in } \Omega \times (0,\infty) \end{cases}$$

or in operator notation as

$$\partial_t W + \mathcal{P}_{g,V} W = \widetilde{F}, \tag{C.1}$$

where we put $W = (w, w')$, $\widetilde{F} = (0, F)$ and

$$\mathcal{P}_{g,V} W := \begin{pmatrix} 0 & -\mathrm{Id} \\ \mathsf{P}_{g,V} & 0 \end{pmatrix} \begin{pmatrix} w \\ w' \end{pmatrix} = \begin{pmatrix} -w' \\ \mathsf{P}_{g,V} w \end{pmatrix}. \tag{C.2}$$

We aim to apply the Hille–Yosida theory for the evolution problem (C.1). For this purpose, let us introduce the Hilbert space

$$\mathcal{H} := H_0^1(\Omega, dV_g) \times L^2(\Omega, dV_g)$$

endowed with the inner product

$$\langle W_1, W_2 \rangle_{\mathcal{H}} = \langle dw_1, dw_2 \rangle_{L^2(\Omega, dV_g)} + \langle w_1', w_2' \rangle_{L^2(\Omega, dV_g)},$$

for $W_j = \left(w_j, w_j'\right) \in \mathcal{H}$ $(j = 1,2)$, and interpret $\mathcal{P}_{g,V}$ as an unbounded operator on $\mathcal{H}$ with domain

$$\mathsf{Dom}(\mathcal{P}_{g,V}) = \mathsf{Dom}(\mathsf{P}_{g,V}) \times H_0^1(\Omega, dV_g). \tag{C.3}$$

Let $C_0 > 0$ be the Poincaré constant on Ω, that is there holds

$$\|u\|_{L^2(\Omega, dV_g)}^2 \leq C_0 \|du\|_{L^2(\Omega, dV_g)}^2 \tag{C.4}$$

for all $u \in H_0^1(\Omega, dV_g)$, and let $\lambda \geq 0$ satisfy

$$\lambda \geq \frac{\|V\|_{L^\infty(\Omega)}}{2\min\left(C_0^{-1}, 1\right)}. \tag{C.5}$$

This constant will be fixed throughout the whole proof.

Claim C.1. $\mathcal{P}_{g,V} + \lambda$ *is a maximal monotone operator on* $\mathcal{H}$.

Proof of Claim C.1. Let us show the following facts:

(a) *Monotonicity:* For $W=(w,w')\in \mathsf{Dom}(\mathcal{P}_{g,V})$, we may calculate

$$\begin{aligned}\langle \mathcal{P}_{g,V}W,W\rangle_{\mathcal{H}} &= \langle (-w',\mathsf{P}_{g,V}w),(w,w')\rangle_{\mathcal{H}} \\ &= -\langle dw',dw\rangle_{L^2(\Omega,dV_g)}+\langle \mathsf{P}_{g,V}w,w'\rangle_{L^2(\Omega,dV_g)} \\ &= \langle Vw,w'\rangle_{L^2(\Omega,dV_g)}.\end{aligned}$$

Next, observe that by (C.4) we have

$$\begin{aligned}\|W\|^2_{\mathcal{H}} &= \|dw\|^2_{L^2(\Omega,dV_g)}+\|w'\|^2_{L^2(\Omega,dV_g)} \\ &\geq \min\left(C_0^{-1},1\right)\left(\|w\|^2_{L^2(\Omega,dV_g)}+\|w'\|^2_{L^2(\Omega,dV_g)}\right).\end{aligned}$$

This implies

$$\begin{aligned}&\langle (\mathcal{P}_{g,V}+\lambda)W,W\rangle_{\mathcal{H}} \\ &\geq \langle Vw,w'\rangle_{L^2(\Omega,dV_g)}+\lambda\min\left(C_0^{-1},1\right)\left(\|w\|^2_{L^2(\Omega,dV_g)}+\|w'\|^2_{L^2(\Omega,dV_g)}\right) \\ &\geq \left(\lambda\min\left(C_0^{-1},1\right)-\frac{\|V\|_{L^\infty(\Omega)}}{2}\right)\left(\|w\|^2_{L^2(\Omega,dV_g)}+\|w'\|^2_{L^2(\Omega,dV_g)}\right) \\ &\geq 0.\end{aligned}$$

(b) *Maximality:* Let $H=(h,h')\in\mathcal{H}$. We want to show that there exists $W=(w,w')\in\mathsf{Dom}(\mathcal{P}_{g,V})$ such that

$$(\mathcal{P}_{g,V}+(\lambda+1))W=H. \tag{C.6}$$

Note that this is equivalent to the condition that W solves

$$\begin{cases}(\lambda+1)w-w'=h & \text{in } H^1_0(\Omega,dV_g),\\ \mathsf{P}_{g,V}w+(\lambda+1)w'=h' & \text{in } L^2(\Omega,dV_g).\end{cases}$$

Inserting the first equation into the second one, we arrive at the following equation for w:

$$\mathsf{P}_{g,V}w+(\lambda+1)^2w=(\lambda+1)h+h' \text{ in } L^2(\Omega,dV_g).$$

It follows from Lax–Milgram theorem and elliptic regularity theory [GT83, Theorem 8.12] that this problem has a unique solution $w\in H^2(\Omega,dV_g)\cap H^1_0(\Omega,dV_g)$. Hence, by defining

$$w'=-(\lambda+1)w+h\in H^1_0(\Omega,dV_g)$$

we arrive at a solution $W=(w,w')$ of the original problem (C.6).

This proves the Claim C.1. □

Hence, we have shown that $\mathcal{P}_{g,V}+\lambda$, for $\lambda\geq 0$ satisfying (C.5) is a maximal monotone operator. Therefore, we can use [Bre11, Theorem 7.4] to see that for any $W_0=(w_0,w_1)$ with $w_0\in H^2(\Omega,dV_g)\cap H^1_0(\Omega,dV_g)$ and $w_1\in H^1_0(\Omega,dV_g)$, there exists a unique function

$$W_\lambda\in C([0,\infty);\mathsf{Dom}(\mathcal{P}_{g,V}))\cap C^1([0,\infty);\mathcal{H})$$

satisfying

$$\begin{cases}\partial_t W_\lambda+(\mathcal{P}_{g,V}+\lambda)W_\lambda=0 \text{ for } t\geq 0,\\ W_\lambda(0)=W_0.\end{cases}$$

Moreover, we have

$$\|W_\lambda\|_{\mathcal{H}}\leq\|W_0\|_{\mathcal{H}}. \tag{C.7}$$

But then the function $W = e^{\lambda t} W_\lambda$ satisfies

$$W \in C([0,\infty); \mathsf{Dom}(\mathcal{P}_{g,V})) \cap C^1([0,\infty); \mathcal{H}) \tag{C.8}$$

and

$$\begin{cases} (\partial_t + \mathcal{P}_{g,V})\, W = 0 \text{ for } t \geq 0, \\ W(0) = W_0. \end{cases} \tag{C.9}$$

It is immediate to see that this solution W is again unique. Now, for each $t \geq 0$ let us define the linear map

$$T_t \colon \mathsf{Dom}(\mathcal{P}_{g,V}) \to \mathsf{Dom}(\mathcal{P}_{g,V}), \quad W_0 \mapsto W(t),$$

where $W = W(t)$ is the unique solution constructed above with the initial condition $W(0) = W_0$. Note that by (C.7) the linear operators T_t satisfy the continuity estimate

$$\|T_t\|_{L(\mathcal{H})} \leq e^{\lambda t}, \text{ for } t \geq 0. \tag{C.10}$$

As $\mathcal{P}_{g,V} + \lambda$ is maximal monotone, we can deduce that $\mathsf{Dom}(\mathcal{P}_{g,V})$ is dense in $\mathcal{H}$ and this in turn allows us to extend the family T_t to maps in $L(\mathcal{H})$ such that the bound (C.10) still holds. This extension will still be denoted by T_t. It is not difficult to see that this extension T_t satisfies the semigroup property, and is strongly continuous. We may estimate

$$\begin{aligned} \|T_t W_0 - W_0\|_{\mathcal{H}} &\leq \left\|T_t(W_0 - W_0^k)\right\|_{\mathcal{H}} + \left\|T_t W_0^k - W_0^k\right\|_{\mathcal{H}} + \left\|W_0^k - W_0\right\|_{\mathcal{H}} \\ &\leq \left(\|T_t\|_{L(\mathcal{H})} + 1 \right) \left\|W_0^k - W_0\right\|_{\mathcal{H}} + \left\|T_t W_0^k - W_0^k\right\|_{\mathcal{H}}, \end{aligned}$$

for any $W_0 \in \mathcal{H}$, where $W_0^k \in \mathsf{Dom}(\mathcal{P}_{g,V})$ is any sequence satisfying $W_0^k \to W$ in $\mathcal{H}$ as $k \to \infty$. Passing to the limit $t \to 0$ shows the strong continuity of $(T_t)_{t\geq 0}$ on $\mathcal{H}$. Hence, $(T_t)_{t\geq 0}$ is a C_0-semigroup. Furthermore, since the solution of (C.9) satisfies (C.8), we have by construction

$$\partial_t W(0) = -\lim_{t\to 0} \mathcal{P}_{g,V} W = -\mathcal{P}_{g,V} W_0$$

for any $W_0 \in \mathsf{Dom}(\mathcal{P}_{g,V})$. Therefore, $-\mathcal{P}_{g,V}$ is the generator of the C_0-semigroup $(T_t)_{t\geq 0}$ (cf. [BS18, Lemma 7.1.17] and [Are06, Exercise 2.6.2]).

Now, we turn our attention to the inhomogeneous problem (C.1). Note that by the assumptions on F, we have $\widetilde{F} \in C^1([0,\infty); \mathcal{H})$ and deduce from [BS18, Lemma 7.1.14] that

$$W_F(t) = \int_0^t T_{t-\tau} \widetilde{F}(\tau)\, d\tau \tag{C.11}$$

is continuously differentiable as a map from $[0,\infty)$ to $\mathcal{H}$, $W_F(t) \in \mathsf{Dom}(\mathcal{P}_{g,V})$ for all $t \geq 0$. Meanwhile, via the equation (C.1), differentiate (C.11) with respect to t, then there holds

$$\mathcal{P}_{g,V} W_F(t) + \widetilde{F}(t) = \partial_t W_F(t) = T_t \widetilde{F}(0) + \int_0^t T(t-\tau) \partial_\tau \widetilde{F}(\tau)\, d\tau$$

for all $t \geq 0$. Thus, [DL92, Chapter XVII, B, §1, Theorem 1] guarantees that for any $W_0 = (w_0, w_1) \in \mathsf{Dom}(\mathcal{P}_{g,V})$ there exists a function

$$W \in C([0,\infty); \mathcal{H}) \cap C^1((0,\infty); \mathcal{H}) \cap C((0,\infty); \mathsf{Dom}(\mathcal{P}_{g,V}))$$

satisfying the initial-value problem

$$\begin{cases} (\partial_t + \mathcal{P}_{g,V})\, W = \widetilde{F} \text{ for } t > 0 \\ W(0) = W_0. \end{cases} \tag{C.12}$$

Indeed, the solution W is given by Duhamel's formula

$$W(t) = T_t W_0 + \int_0^t T_{t-\tau} \widetilde{F}(\tau)\, d\tau$$

and so it is unique. Note that $W_0 \in \mathsf{Dom}(\mathcal{P}_{g,V})$, (C.8) and $W_F \in C^1([0,\infty);\mathcal{H})$ implies $W \in C^1([0,\infty);\mathcal{H})$. Thus, (C.12) shows

$$\mathcal{P}_{g,V} W = \widetilde{F} - \partial_t W \in C([0,\infty);\mathcal{H}).$$

Hence, we can deduce that

$$W \in C^1([0,\infty);\mathcal{H}) \cap C([0,\infty);\mathsf{Dom}(\mathcal{P}_{g,V})) \tag{C.13}$$

and the PDE (C.12) holds for $t \geq 0$.

Next, let us write $W(t) = (w(t), w'(t))$ for $t \geq 0$. By (C.13), (C.12), (C.2), $W_0 = (w_0, w_1)$ and $\widetilde{F} = (0, F)$, we deduce that $w'(t) = \partial_t(w)$ for $t \geq 0$,

$$C([0,\infty); H^2(\Omega, dV_g) \cap H_0^1(\Omega, dV_g)) \text{ with } \begin{cases} \partial_t w \in C([0,\infty); H_0^1(\Omega; dV_g)), \\ \partial_t^2 w \in C([0,\infty); L^2(\Omega, dV_g)), \end{cases} \tag{C.14}$$

and w solves

$$\left(\partial_t^2 + \mathsf{P}_{g,V}\right) w = F \text{ on } [0,\infty).$$

Observe that this solution w is unique as if $\widetilde{w}$ is another solution, then $v = w - \widetilde{w}$ solves the homogeneous problem

$$\begin{cases} \left(\partial_t^2 + \mathsf{P}_{g,V}\right) v = 0 \text{ for } t \geq 0, \\ v(0) = \partial_t v(0) = 0. \end{cases}$$

As $\partial_t v \in C([0,\infty); H_0^1(\Omega))$ we get

$$\left\langle \left(\partial_t^2 + \mathsf{P}_{g,V}\right) v, \partial_t v \right\rangle_{L^2(\Omega, dV_g)} = 0$$

for any $t \geq 0$. By (C.14), we may calculate

$$\begin{aligned} \left\langle \partial_t^2 v, \partial_t v \right\rangle_{L^2(\Omega, dV_g)} &= \frac{1}{2} \partial_t \|\partial_t v\|^2_{L^2(\Omega, dV_g)}, \\ \langle V v, \partial_t v \rangle_{L^2(\Omega, dV_g)} &= \frac{1}{2} \partial_t \left\| V^{1/2} v \right\|^2_{L^2(\Omega, dV_g)}, \\ \langle -\Delta_g v, \partial_t v \rangle_{L^2(\Omega, dV_g)} &= \langle dv, d\partial_t v \rangle_{L^2(\Omega, dV_g)} = \frac{1}{2} \partial_t \|dv\|^2_{L^2(\Omega, dV_g)}. \end{aligned}$$

Hence, we deduce

$$\partial_t \left(\|\partial_t v\|^2_{L^2(\Omega, dV_g)} + \|dv\|^2_{L^2(\Omega, dV_g)} + \left\| V^{1/2} v \right\|^2_{L^2(\Omega, dV_g)} \right) = 0.$$

Therefore, we may conclude that $v = 0$ as $v(0) = \partial_t v(0) = 0$. This demonstrates (a).

For (b), let w_0, w_1 and F be given as in the assumption. Recall from Section B.2 that for any $k \in \mathbb{N}$ the powers $\mathcal{P}_{g,V}^k$ are the unbounded operators

$$\mathcal{P}_{g,V}^k = \underbrace{\mathcal{P}_{g,V} \cdots \mathcal{P}_{g,V}}_{k\text{-times}} \text{ on } \mathcal{H}$$

with domain

$$\mathsf{Dom}(\mathcal{P}_{g,V}^k) = \left\{ U \in \mathsf{Dom}(\mathcal{P}_{g,V}^{k-1})\,;\, \mathcal{P}_{g,V} U \in \mathsf{Dom}(\mathcal{P}_{g,V}^{k-1}) \right\},$$

which becomes a Hilbert space under the inner product

$$\langle W_1, W_2 \rangle_{\mathsf{Dom}(\mathcal{P}_{g,V}^k)} = \sum_{j=0}^k \left\langle \mathcal{P}_{g,V}^j W_1, \mathcal{P}_{g,V}^j W_2 \right\rangle_{\mathcal{H}},$$

for all $W_j \in \mathsf{Dom}(\mathcal{P}^k_{g,V})$ and for $j = 1, 2$.

Claim C.2. *For any $k \in \mathbb{N}$ the following assertions hold:*

(a) We have

$$\mathsf{Dom}(\mathcal{P}^k_{g,V}) = \mathcal{Q}^k_{g,V}, \tag{C.15}$$

where $\mathcal{Q}^k_{g,V}$ denotes the set

$$\left\{ \begin{pmatrix} w \\ w' \end{pmatrix} ; \begin{array}{c} w \in H^{k+1}(\Omega, dV_g) \text{ s.t. } w, \ldots, \mathsf{P}^{[k/2]}_{g,V} w \in H^1_0(\Omega, dV_g) \\ w' \in H^k(\Omega, dV_g) \text{ s.t. } w', \ldots, \mathsf{P}^{[(k+1)/2]-1}_{g,V} w' \in H^1_0(\Omega, dV_g) \end{array} \right\}$$

and there holds

$$\mathsf{Dom}(\mathcal{P}^k_{g,V}) \hookrightarrow H^{k+1}(\Omega, dV_g) \times H^k(\Omega, dV_g). \tag{C.16}$$

(b) Let $\mathcal{P}^{(k)}_{g,V}$ be defined by

$$\mathcal{P}^{(k)}_{g,V} \colon \mathsf{Dom}(\mathcal{P}^k_{g,V}) \subset \mathsf{Dom}(\mathcal{P}^{k-1}_{g,V}) \to \mathsf{Dom}(\mathcal{P}^{k-1}_{g,V}), \quad U \mapsto \mathcal{P}_{g,V} U,$$

then $\mathcal{P}^{(k)}_{g,V} + \lambda$ is maximal monotone on $\mathsf{Dom}(\mathcal{P}^{k-1}_{g,V})$ for $k \in \mathbb{N}$, where we use the convention $\mathsf{Dom}(\mathcal{P}^0_{g,V}) = \mathcal{H}$.

Proof of Claim C.2. For (a), note that in the case $k = 1$ the identity (C.15) holds by (C.3) and (B.15). Moreover, the embedding (C.16) follows from (B.14). So let us suppose that the assertions in (a) hold for $k - 1$ and choose any $W = (w, w') \in \mathsf{Dom}(\mathcal{P}^k_{g,V})$. In particular, this implies that

$$\mathcal{P}_{g,V} W = \begin{pmatrix} -w' \\ \mathsf{P}_{g,V} w \end{pmatrix} \in \mathsf{Dom}(\mathcal{P}^{k-1}_{g,V}) = \mathcal{Q}^{k-1}_{g,V}. \tag{C.17}$$

Therefore, we have $\mathsf{P}_{g,V} w \in H^{k-1}(\Omega, dV_g)$ and $w' \in H^k(\Omega, dV_g)$. By elliptic regularity theory, this ensures $w \in H^{k+1}(\Omega, dV_g)$ with

$$\begin{aligned} \|w\|_{H^{k+1}(\Omega, dV_g)} &\lesssim \|w\|_{L^2(\Omega, dV_g)} + \|\mathsf{P}_{g,V} w\|_{H^{k-1}(\Omega, dV_g)} \\ &\lesssim \|w\|_{L^2(\Omega, dV_g)} + \left\| \begin{pmatrix} -w' \\ \mathsf{P}_{g,V} w \end{pmatrix} \right\|_{H^k(\Omega, dV_g) \times H^{k-1}(\Omega, dV_g)} \\ &\lesssim \|w\|_{L^2(\Omega, dV_g)} + \left\| \begin{pmatrix} -w' \\ \mathsf{P}_{g,V} w \end{pmatrix} \right\|_{\mathsf{Dom}(\mathcal{P}^{k-1}_{g,V})} \\ &\lesssim \|dw\|_{L^2(\Omega, dV_g)} + \sum_{j=0}^{k-1} \left\| \mathcal{P}^{j+1}_{g,V} W \right\|_{\mathcal{H}} \\ &\lesssim \|W\|_{\mathcal{H}} + \sum_{j=0}^{k-1} \left\| \mathcal{P}^{j+1}_{g,V} W \right\|_{\mathcal{H}} \\ &\lesssim \|W\|_{\mathsf{Dom}(\mathcal{P}^k_{g,V})}. \end{aligned}$$

In the above calculation, we used the Poincaré inequality, the uniform ellipticity, and the induction hypothesis. On the other hand (C.17) together with

$$\mathsf{Dom}(\mathcal{P}^{k-1}_{g,V}) \hookrightarrow H^k(\Omega, dV_g) \times H^{k-1}(\Omega, dV_g)$$

shows

$$\|w'\|_{H^k(\Omega, dV_g)} \lesssim \left\| \begin{pmatrix} -w' \\ \mathsf{P}_{g,V} w \end{pmatrix} \right\|_{\mathsf{Dom}(\mathcal{P}^{k-1}_{g,V})} \lesssim \|\mathcal{P}_{g,V} W\|_{\mathsf{Dom}(\mathcal{P}^{k-1}_{g,V})} \lesssim \|W\|_{\mathsf{Dom}(\mathcal{P}^k_{g,V})}.$$

Hence, we have established the embedding (C.16). Furthermore, by (C.17) and the induction hypothesis, we know that

$$\begin{aligned} w', \dots, \mathsf{P}_{g,V}^{[(k-1)/2]} w' &\in H_0^1(\Omega, dV_g) \\ w, \mathsf{P}_{g,V} w, \dots, \mathsf{P}_{g,V}^{[k/2]} w &\in H_0^1(\Omega, dV_g). \end{aligned}$$

Noting that $[(k+1)/2] - 1 = [(k-1)/2]$ gives $W \in \mathcal{Q}_{g,V}^k$ and hence $\mathsf{Dom}(\mathcal{P}_{g,V}^k) \subset \mathcal{Q}_{g,V}^k$. Let us next prove the reverse inclusion. If $W \in \mathcal{Q}_{g,V}^k$, then by monotonicity and induction hypothesis $W \in \mathcal{Q}_{g,V}^{k-1} = \mathsf{Dom}(\mathcal{P}_{g,V}^{k-1})$. Moreover, $W \in \mathcal{Q}_{g,V}^k$ implies

$$\begin{aligned} w' &\in H^k(\Omega, dV_g), \\ w', \dots, \mathsf{P}_{g,V}^{[(k-1)/2]} w' &\in H_0^1(\Omega, dV_g), \end{aligned}$$

and

$$\begin{aligned} \mathsf{P}_{g,V} w &\in H^{k-1}(\Omega, dV_g), \\ \mathsf{P}_{g,V} w, \dots, \mathsf{P}_{g,V}^{[k/2]-1}(\mathsf{P}_{g,V} w) &\in H_0^1(\Omega). \end{aligned}$$

Note that $\mathsf{P}_{g,V} w \in H^{k-1}(\Omega, dV_g)$ followed from $w \in H^{k+1}(\Omega)$. Thus, we get

$$\mathcal{P}_{g,V} W = \begin{pmatrix} -w' \\ \mathsf{P}_{g,V} w \end{pmatrix} \in \mathcal{Q}_{g,V}^{k-1} = \mathsf{Dom}(\mathcal{P}_{g,V}^{k-1})$$

and so we can conclude the proof of the inclusion $\mathcal{Q}_{g,V}^k \subset \mathsf{Dom}(\mathcal{P}_{g,V}^k)$.

For (b), we already know that it holds in the case $k = 1$ (see Claim C.1), so let us suppose that it holds for $k - 1$. Then we may calculation

$$\begin{aligned} \left\langle (\mathcal{P}_{g,V}^{(k)} + \lambda) U, U \right\rangle_{\mathsf{Dom}(\mathcal{P}_{g,V}^{k-1})} &= \sum_{j=1}^{k-1} \left\langle \mathcal{P}_{g,V}^j \left(\mathcal{P}_{g,V} + \lambda \right) U, \mathcal{P}_{g,V}^j U \right\rangle_{\mathcal{H}} \\ &= \sum_{j=1}^{k-1} \left\langle \left(\mathcal{P}_{g,V} + \lambda \right) \mathcal{P}_{g,V}^j U, \mathcal{P}_{g,V}^j U \right\rangle_{\mathcal{H}} \geq 0 \end{aligned}$$

for any $U \in \mathsf{Dom}(\mathcal{P}_{g,V}^k)$. Above we used the case $k = 1$ and that $U \in \mathsf{Dom}(\mathcal{P}_{g,V}^k)$ implies $\mathcal{P}_{g,V}^j U \in \mathsf{Dom}(\mathcal{P}_{g,V}^{k-j}) \subset \mathsf{Dom}(\mathcal{P}_{g,V})$ for $j = 0, 1, \dots, k-1$.

Next, let us prove the range condition. For this purpose suppose that $H = (h, h') \in \mathsf{Dom}(\mathcal{P}_{g,V}^{k-1})$. Then we wish to solve

$$\left(\mathcal{P}_{g,V}^{(k)} + (\lambda + 1) \right) U = H$$

in $\mathsf{Dom}(\mathcal{P}_{g,V}^k)$. By induction hypothesis there exists $U \in \mathsf{Dom}(\mathcal{P}_{g,V}^{k-1})$ such that

$$\left(\mathcal{P}_{g,V} + (\lambda + 1) \right) U = H.$$

In particular, $u, u' \in H_0^1(\Omega, dV_g)$ satisfy

$$\left(\mathsf{P}_{g,V} + (\lambda + 1)^2 \right) u = (\lambda + 1) h + h' \text{ and } u' = -(\lambda + 1) u + h.$$

Since $(\lambda + 1) h + h' \in H^{k-1}(\Omega, dV_g)$, elliptic regularity theory guarantees that $u \in H^{k+1}(\Omega, dV_g)$. However, as $h \in H^k(\Omega, dV_g)$, we know $u' \in H^k(\Omega, dV_g)$. Moreover, by $U \in \mathsf{Dom}(\mathcal{P}_{g,V}^{k-1})$ and $H \in \mathsf{Dom}(\mathcal{P}_{g,V}^k)$ we know that

$$\begin{aligned} \mathsf{P}_{g,V} u = & \underbrace{-(\lambda + 1)^2 u}_{\in H^{k+1}(\Omega, dV_g)} + \underbrace{(\lambda + 1) h}_{\in H^k(\Omega)} + \underbrace{h'}_{\in H^{k-1}(\Omega)} \\ & \in \left\{ v \in H^{k-1}(\Omega, dV_g) \, ; \, v, \dots, \mathsf{P}_{g,V}^{[(k-2)/2]} v \in H_0^1(\Omega, dV_g) \right\} \end{aligned}$$

and

$$u' = -(\lambda+1)u + h \in \big\{v \in H^k(\Omega, dV_g)\,;\, v, \dots, \mathsf{P}_{g,V}^{[(k-1)/2]} v \in H_0^1(\Omega, dV_g)\big\}$$
$$= \big\{v \in H^k(\Omega, dV_g)\,;\, v, \dots, \mathsf{P}_{g,V}^{[(k+1)/2]-1} v \in H_0^1(\Omega, dV_g)\big\}.$$

The first identity implies

$$\mathsf{P}_{g,V} u, \dots, \mathsf{P}_{g,V}^{[k/2]} u \in H_0^1(\Omega, dV_g)$$

and thus

$$U \in \mathcal{Q}_{g,V}^k = \mathsf{Dom}(\mathsf{P}_{g,V}^k).$$

So, we have shown the range condition and hence $\mathsf{P}_{g,V}^{(k)} + \lambda$ is maximal monotone. This shows Claim C.2. □

Next, we aim to show:

Claim C.3. *If $W_0 = (w_0, w_1) \in \mathsf{Dom}(\mathsf{P}_{g,V}^k)$, $\widetilde{F} = (0, F)$ with $F \in C_c^\infty(\Omega \times (0,\infty))$, then the above constructed unique solution $W \in C^1([0,\infty); \mathcal{H}) \cap C([0,\infty); \mathsf{Dom}(\mathsf{P}_{g,V}))$ of (C.12) satisfies*

$$W \in C^{k-j}([0,\infty); \mathsf{Dom}(\mathsf{P}_{g,V}^j)) \text{ for } j = 0, 1, \dots, k. \tag{C.18}$$

Proof of Claim C.3. For $k = 1$ there is nothing to prove and hence let us consider the case $k = 2$. Arguing as in the case $k = 1$, we can conclude from Claim C.2 that $-\mathcal{P}_{g,V}^{(2)}$ generates a C_0-semigroup $(T_t^{(2)})_{t\geq 0}$ on $\mathsf{Dom}(\mathcal{P}_{g,V})$. As $F \in C_c^\infty(\Omega \times (0,\infty))$, we have $\widetilde{F} = (0, F) \in C^1([0,\infty); \mathsf{Dom}(\mathsf{P}_{g,V}))$. Relying on the same arguments as for $k = 1$, we obtain a unique solution

$$W \in C^1([0,\infty); \mathsf{Dom}(\mathsf{P}_{g,V})) \cap C([0,\infty); \mathsf{Dom}(\mathsf{P}_{g,V}^2))$$

of

$$\begin{cases} (\partial_t + \mathsf{P}_{g,V})\, W = \widetilde{F} \text{ for } t \geq 0, \\ W(0) = W_0. \end{cases}$$

In particular, we see that this solution coincides with the unique solution constructed for $k = 1$. We next assert that $W \in C^2([0,\infty); \mathcal{H})$. By definition of the norm $\|\cdot\|_{\mathsf{Dom}(\mathsf{P}_{g,V})}$, we see that $\mathsf{P}_{g,V} \in L(\mathsf{Dom}(\mathsf{P}_{g,V}))$. Hence, $W \in C^1([0,\infty); \mathsf{Dom}(\mathsf{P}_{g,V}))$ implies

$$\mathsf{P}_{g,V} W \in C^1([0,\infty); \mathcal{H}) \text{ with } \partial_t\,(\mathsf{P}_{g,V} W) = \mathsf{P}_{g,V}\,(\partial_t W) \text{ for } t \geq 0.$$

Therefore, we get $\partial_t W = \widetilde{F} - \mathcal{P}_{g,V} \in C^1([0,\infty); \mathcal{H})$ and thus $W \in C^2([0,\infty); \mathcal{H})$ as asserted. Moreover, this implies that $\partial_t W$ solves

$$\begin{cases} \partial_t \widetilde{W} + \mathcal{P}_{g,V} \widetilde{W} = \partial_t \widetilde{F} \text{ for } t \geq 0, \\ \widetilde{W}(0) = \widetilde{F}(0) - \mathcal{P}_{g,V} W_0 = -\mathcal{P}_{g,V} W_0. \end{cases} \tag{C.19}$$

Next, we prove the general case $k \geq 3$ by induction. Suppose that the result holds for $k-1$. By the case $k = 2$, we know that the unique solution W satisfies (C.18) for $k = 2$ and

$$\partial_t W \in C([0,\infty); \mathsf{Dom}(\mathcal{P}_{g,V})) \cap C^1([0,\infty), \mathcal{H})$$

solves (C.19) with $\widetilde{W}_0 := -\mathcal{P}_{g,V} W_0 \in \mathsf{Dom}(\mathcal{P}_{g,V}^{k-1})$. By induction hypothesis this implies

$$\partial_t W \in C^{k-1-j}([0,\infty); \mathsf{Dom}(\mathcal{P}_{g,V}^j)) \text{ for } j = 0, 1, \dots, k-1$$

and hence

$$W \in C^{k-j}([0,\infty); \mathsf{Dom}(\mathcal{P}_{g,V}^j)) \text{ for } j = 0, 1, \dots, k-1.$$

Therefore, it remains to prove that $W \in C([0,\infty); \mathsf{Dom}(\mathcal{P}_{g,V}^k))$. As

$$\partial_t W \in C([0,\infty); \mathsf{Dom}(\mathcal{P}_{g,V}^{k-1})),$$

the PDE for W shows

$$\mathcal{P}_{g,V}W = \widetilde{F} - \partial_t W \in C([0,\infty); \mathsf{Dom}(\mathcal{P}_{g,V}^{k-1})).$$

Thus, we get $W \in C([0,\infty); \mathsf{Dom}(\mathcal{P}_{g,V}^{k}))$ as we want. This proves Claim C.3. □

Hence, if the assumptions of (b) hold, then the Claims C.2 and C.3 guarantee that for all $k \in \mathbb{N}$ we have

$$W \in C^{k-j}([0,\infty); H^{j+1}(\Omega, dV_g) \times H^j(\Omega, dV_g)) \text{ for } j = 0, 1, \ldots, k.$$

But then the corresponding solution w of the wave equation (5.1) belongs to $C^\infty(\overline{\Omega} \times [0,\infty))$ and we can conclude the proof of (b).

Finally, for (c), suppose that the conditions of (a) hold, let $w \in C^2([0,\infty); L^2(\Omega, dV_g))$ be the unique solution of (5.2) and set

$$w_i^k = \langle w_i, \phi_k \rangle_{L^2(\Omega, dV_g)}, \quad w_k = \langle w, \phi_k \rangle_{L^2(\Omega, dV_g)}, \quad \text{and } F_k = \langle F, \phi_k \rangle_{L^2(\Omega, dV_g)}$$

for $i = 0, 1$ and $k \in \mathbb{N}$, where w_0 and w_1 are the initial data in the wave equation (5.1). By the C^2-regularity of w in time, we see from (5.2) that there holds

$$\begin{cases} \partial_t^2 w_k + \lambda_k w_k = F_k \text{ for } t \geq 0 \\ w_k(0) = w_0^k, \quad \partial_t w_k(0) = w_1^k \end{cases} \tag{C.20}$$

for all $k \in \mathbb{N}$. Note that $\omega_k \in C^2([0,\infty))$ given by

$$\omega_k(t) = \cos(t\lambda_k^{1/2}) w_0^k + \frac{\sin(t\lambda_k^{1/2})}{\lambda_k^{1/2}} w_1^k + \int_0^t \frac{\sin((t-\tau)\lambda_k^{1/2})}{\lambda_k^{1/2}} F_k(\tau)\, d\tau$$

solves (C.20).

On the other hand, if $u_j \in C^2([0,\infty))$, $j = 1, 2$, solve (C.20), then $v = u_1 - u_2 \in C^2([0,\infty))$ satisfies

$$\begin{cases} \partial_t^2 v + \lambda_k v = 0 \text{ for } t \geq 0 \\ w_k(0) = \partial_t w_k(0) = 0. \end{cases}$$

Observe that

$$\eta(t) = |\partial_t v(t)|^2 + |v(t)|^2 \in C^1([0,\infty))$$

satisfies

$$\partial_t \eta(t) \leq (1 + \lambda_k)\, \eta(t)$$

and hence $\eta(0) = 0$ as well as Gronwall's inequality guarantees that $\eta(t) = 0$ for all $t \geq 0$. This in turn implies that $v(t) = 0$ for all $t \geq 0$. Therefore, we may conclude that $w_k = \omega_k$ and we get the first formula in (5.3). The second formula in (5.3) follows by Fubini's theorem. In fact, for any $h = \sum_{k \geq 1} h_k \phi_k \in L^2(\Omega, dV_g)$, we obtain by the first formula

$$\begin{aligned}
\langle w(t), h \rangle_{L^2(\Omega, dV_g)} &= \sum_{k \geq 1} \left(\cos\left(t\lambda_k^{1/2}\right) w_0^k h_k + \frac{\sin\left(t\lambda_k^{1/2}\right)}{\lambda_k^{1/2}} w_1^k h_k \right) \\
&\quad + \sum_{k \geq 1} \int_0^t \frac{\sin\left((t-\tau)\lambda_k^{1/2}\right)}{\lambda_k^{1/2}} F_k(\tau) h_k\, d\tau \\
&= \left\langle \cos\left(t\mathsf{P}_{g,V}^{1/2}\right) w_0, h \right\rangle_{L^2(\Omega, dV_g)} + \left\langle \frac{\sin\left(t\mathsf{P}_{g,V}^{1/2}\right)}{\mathsf{P}_{g,V}^{1/2}} w_1, h \right\rangle_{L^2(\Omega, dV_g)} \\
&\quad + \sum_{k \geq 1} \int_0^t \frac{\sin\left((t-\tau)\lambda_k^{1/2}\right)}{\lambda_k^{1/2}} F_k(\tau) h_k\, d\tau.
\end{aligned}$$

As the quotient under the integral is uniformly bounded in k, we can invoke Fubini's theorem to get

$$\begin{aligned}\langle w(t),h\rangle_{L^2(\Omega,dV_g)} &= \left\langle \cos\left(t\mathsf{P}_{g,V}^{1/2}\right)w_0,h\right\rangle_{L^2(\Omega,dV_g)} + \left\langle \frac{\sin\left(t\mathsf{P}_{g,V}^{1/2}\right)}{\mathsf{P}_{g,V}^{1/2}}w_1,h\right\rangle_{L^2(\Omega,dV_g)} \\ &\quad + \int_0^t \sum_{k\geq 1} \frac{\sin\left((t-\tau)\lambda_k^{1/2}\right)}{\lambda_k^{1/2}} F_k(\tau)h_k\,d\tau \\ &= \left\langle \cos\left(t\mathsf{P}_{g,V}^{1/2}\right)w_0,h\right\rangle_{L^2(\Omega,dV_g)} + \left\langle \frac{\sin\left(t\mathsf{P}_{g,V}^{1/2}\right)}{\mathsf{P}_{g,V}^{1/2}}w_1,h\right\rangle_{L^2(\Omega,dV_g)} \\ &\quad + \int_0^t \left\langle \frac{\sin\left((t-\tau)\mathsf{P}_{g,V}^{1/2}\right)}{\mathsf{P}_{g,V}^{1/2}}F(\tau),h\right\rangle_{L^2(\Omega,dV_g)} d\tau\end{aligned}$$

and hence the second formula in (5.3) holds. This concludes the proof of Theorem 5.1. □

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Department of Applied Mathematics, National Yang Ming Chiao Tung University, Hsinchu, Taiwan
Email address: yihsuanlin3@gmail.com

Department of Mathematics, Hokkaido University, Japan & Research Center of Mathematics for Social Creativity, Research Institute for Electronic Science, Hokkaido University, Japan
Email address: nakamuragenn@gmail.com

Departament de Matemàtiques i Informàtica, Universitat de Barcelona, Barcelona,Spain
Email address: philipp.zimmermann@ub.edu