

ENTANGLEMENT PRINCIPLE AND FRACTIONAL CALDERÓN PROBLEM FOR NONLOCAL PARABOLIC OPERATORS

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ABSTRACT. We examine inverse problems for the variable-coefficient nonlocal parabolic operator $(\partial_t - \Delta_g)^s$, where $0 < s < 1$. This article makes two primary contributions. First, we introduce a novel entanglement principle for these operators under suitable smoothness conditions. Second, we prove that lower-order perturbations can be uniquely determined from the associated Dirichlet-to-Neumann map using this principle. However, due to insufficient solution regularity, direct application of the entanglement principle to the inverse problem is not feasible. To address this, we derive a modified entanglement principle, enabling the effective resolution of related inverse problems.

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1. INTRODUCTION

Inverse problems for space-fractional equations have garnered considerable attention in recent years, not only due to their distinctive mathematical features, but also because of their wide range of applications in physics, biology, finance, and related fields.

A pioneering breakthrough in this direction is the resolution of the Calderón problem for the fractional Schrödinger equation (see [GSU20]), which concerns the recovery of an unknown bounded potential from exterior measurements. One of the key contributions in [GSU20] is the establishment of the *unique continuation property* (UCP) for the fractional Laplacian operator $(-\Delta)^s$, $0 < s < 1$, which states that

$$u = (-\Delta)^s u = 0 \quad \text{in a nonempty open subset of } \mathbb{R}^n \implies u \equiv 0 \text{ in } \mathbb{R}^n.$$

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This fundamental property leads to the *Runge approximation property*, which asserts that any L^2 -function on a given open set can be approximated by solutions of the fractional Schrödinger equation $((-\Delta)^s + q)u = 0$. Either UCP or Runge approximation can then be used to show that a lower-order perturbation q can be uniquely determined from exterior data, under suitable regularity conditions on q .

Following this seminal work [GSU20], a substantial body of research has emerged on inverse problems for various space-fractional models. For example, simultaneous recovery results for multiple parameters were obtained in [CLL19, CLR20], while the determination of bounded potentials for anisotropic nonlocal Schrödinger equations was investigated in [GLX17]. These problems remain open in the local case $s = 1$ for dimensions $n \geq 3$, suggesting that nonlocality appears to provide significant advantages in addressing such inverse problems. For further developments on both linear and nonlinear nonlocal inverse problems in various settings, we refer the reader to the following articles [HL19, HL20, LL22, GRSU20, CMRU22, RS20, RS18, GRSU20, LLR20, LZ23, GU21] and the references therein.

In particular, owing to the close connection between nonlocal and local settings, interior coefficients can be recovered either via reductions based on the Caffarelli–Silvestre extension (see, e.g., [CGRU23, Rül25, LLU23, LZ24]) or by employing heat semigroup methods on closed Riemannian manifolds (see, e.g., [FGKU25, Fei24, FKU24, Lin24]). Both approaches share a similar philosophy: they transfer certain nonlocal information to its local counterpart, or to the associated heat equation. In contrast, a local-to-nonlocal reduction was studied in [LNZ24] for the classical Schrödinger equation in transversal anisotropic geometry.

More recently, the study of perturbation by nonlocal operators has gathered interest because of their intrinsic mathematical properties and potential applications. The corresponding unique continuation property, referred to as the *entanglement principle*, was established in [FKU24] for the fractional Laplace–Beltrami operator and in [FL24] for the fractional Laplacian. Its remarkable capacity to disentangle contributions from each fractional power in a nonlocal operator is expected to inspire further developments in nonlocal inverse problems. For a comprehensive survey of this rapidly evolving field, we refer the reader to the recent monograph [LL25].

1.1. Mathematical formulations and main results. In this work, we focus on inverse problems associated with fractional nonlocal parabolic operators. Let's begin by defining the parabolic operator

$$\mathcal{H}_g := \partial_t - \Delta_g,$$

where the Laplace–Beltrami operator Δ_g is defined as follows:

$$\Delta_g = \frac{1}{\sqrt{|g|}} \sum_{j,k=1}^n \frac{\partial}{\partial x_j} \left(\sqrt{|g|} g^{jk} \frac{\partial}{\partial x_k} \right).$$

Here the metric $g = (g_{jk}(x))_{1 \leq j,k \leq n} \in C^\infty(\mathbb{R}^n; \mathbb{R}^{n \times n})$ satisfies the ellipticity condition, i.e., there exists a constant $\lambda \in (0, 1)$ such that

$$(1.1) \quad \lambda |\xi|^2 \leq \sum_{j,k=1}^n g_{jk}(x) \xi_j \xi_k \leq \lambda^{-1} |\xi|^2,$$

for any $x \in \mathbb{R}^n$ and for any $\xi = (\xi_1, \dots, \xi_n) \in \mathbb{R}^n$. Also, $|g|$ stands for the absolute value of the determinant of g , and g^{jk} are the components of the inverse of $g = (g_{jk}(x))_{1 \leq j,k \leq n}$.

The fractional powers of the parabolic operator $\mathcal{H}_g$ is defined by

$$\mathcal{H}_g^s u := (\partial_t - \Delta_g)^s u, \quad s > 0,$$

of a function $u = u(x, t) : \mathbb{R}^{n+1} \rightarrow \mathbb{R}$, $n \in \mathbb{N}$. These space-time nonlocal operators are found in various applications, including continuous-time random walks and mathematical biology. In contrast with the operators like $\partial_t u + (-\Delta_g)^s$, the space and time variables in $\mathcal{H}_g^s$ are coupled together with order s in time t and order $2s$ in space x .

To define such operators, when $g = I_n$ (the $n \times n$ identity matrix), we simply write $\mathcal{H} := \partial_t - \Delta$ as a heat operator, where Δ denotes the classical Laplace operator. In this case, since the coefficients of $\mathcal{H}$ are constants, the operator $\mathcal{H}^s$ can be defined via the Fourier transform as follows:

$$\widehat{(\mathcal{H}^s u)}(\xi, \rho) = (i\rho + |\xi|^2)^s \widehat{u}(\xi, \rho), \quad \text{for } (\xi, \rho) \in \mathbb{R}^n \times \mathbb{R},$$

for $u \in \mathcal{S}(\mathbb{R}^{n+1})$, which is the Schwartz space of smooth, rapidly decreasing functions in $\mathbb{R}^n \times \mathbb{R}$. Here $\widehat{f}$ denotes the Fourier transform of f in both space and time variables (x, t) . For the latter purpose, we also use $\mathcal{F}_x(f)$ and $\mathcal{F}_t(f)$ to denote the Fourier transform of f with respect to x and t , respectively. However, using the method of the Fourier transform to define the operator $\mathcal{H}_g^s$ with variable coefficients is not directly applicable. Fortunately, another more flexible approach through the parabolic language of semigroups is available to handle $\mathcal{H}_g^s$, see for instance, [ST17] and also the discussion in Section 2. In particular, it can also provide the explicit formula for $\mathcal{H}_g^s$ at the space-time point (x, t) .

1.1.1. Previous work on nonlocal parabolic inverse problems. Let us revisit inverse problems related to nonlocal parabolic operators. We consider the Calderón-type problem for the equation $(\mathcal{H}_g^s + V)u = 0$ with a *time-dependent* potential $V = V(x, t)$. The goal is to recover V from exterior measurements Λ_V defined below.

Let u_f be the solution to the problem

$$(1.2) \quad \begin{cases} (\mathcal{H}_g^s + V)u = 0 & \text{in } \Omega_T, \\ u = f & \text{in } (\Omega_e)_T, \\ u = 0 & \text{in } \mathbb{R}^n \times \{t \leq -T\}, \end{cases}$$

where we use the standing notations

$$\Omega_e := \mathbb{R}^n \setminus \overline{\Omega},$$

and

$$A_T := A \times (-T, T),$$

for any subset $A \subset \mathbb{R}^n$ and a fixed real number $T > 0$ throughout this work. Notice that our initial condition in (1.2) is needed due to the natural nonlocality of the operator $\mathcal{H}_g^s$.

To study inverse problems for (1.2), we require an additional eigenvalue condition: Suppose that $\{0\}$ is not a Dirichlet eigenvalue of (1.2) in the sense that

$$(1.3) \quad \begin{cases} \text{If } u \in \mathbf{H}^s(\mathbb{R}^{n+1}) \text{ solves } \begin{cases} (\mathcal{H}_g^s + V)u = 0 & \text{in } \Omega_T, \\ u = 0 & \text{in } (\Omega_e)_T, \\ u = 0 & \text{in } \mathbb{R}^n \times \{t \leq -T\}, \end{cases} \\ \text{then } u \equiv 0 \text{ in } \mathbb{R}_T^n. \end{cases}$$

Here the space $\mathbf{H}^s(\mathbb{R}^{n+1})$ is defined in Section 2. It is well-known that for all bounded potentials $V \geq 0$, (1.3) is satisfied automatically. Since the condition (1.3) ensures unique solvability of the forward problem for (1.2) (see Section 2), we can formally define the *Dirichlet-to-Neumann* (DN) map Λ_V of (1.2) by

$$\Lambda_V : \widetilde{\mathbf{H}}^s((\Omega_e)_T) \rightarrow \mathbf{H}^{-s}((\Omega_e)_T), \quad f \mapsto \mathcal{H}_g^s u_f|_{(\Omega_e)_T}.$$

By using the DN map Λ_V , it has been shown in [LLR20, BS24] that the time-dependent potential V can be recovered uniquely.

We would also like to point out that in the works [LLU22, LLU23], the authors show the nonlocal-to-local reduction with respect to both the heat semigroup and the Caffarelli-Silvestre extension approaches. More precisely, they demonstrated that the nonlocal DN map for nonlocal parabolic equations can determine their local DN map for local parabolic equations.

1.1.2. Main results. In this article, we extend our study to the models with the fractional parabolic operators $\sum_{k=1}^N b_k \mathcal{H}_g^{\alpha_k}$ for constants b_k . As mentioned above, several previous works have been devoted to studying inverse problems for the operator $\mathcal{H}_g^s$ with local perturbation. However, when nonlocal operators perturb it, the complicated nonlocal interactions contributed from different terms make the problem challenging to solve. To decouple their entangled effect, we introduce a novel approach in Theorem 1.2 and use it to study the related inverse problems in Theorem 1.4.

Now we state the setting of the problem to be investigated and the key approach. Given an integer $N \geq 2$, let $\mathcal{O} \subset \mathbb{R}^n$ be a nonempty open set. Suppose that $u_k \in C^\infty(\mathbb{R}^{n+1})$, for $k = 1, \dots, N$. Inspired by the papers [FKU24, FL24] which address the entanglement issue for fractional elliptic operators, we are interested in the following question:

(IP-1) *Given $N \geq 2$. Suppose that $\{u_k\}_{k=1}^N \subset C^\infty(\mathbb{R}^{n+1})$ satisfies*

$$u_1|_{\mathcal{O}_T} = \dots = u_N|_{\mathcal{O}_T} = \left(\sum_{k=1}^N b_k \mathcal{H}_g^{\alpha_k} u_k \right) \Big|_{\mathcal{O}_T} = 0,$$

where $\{b_k\} \subset \mathbb{C} \setminus \{0\}$, and $\{\alpha_k\} \subset (0, \infty) \setminus \mathbb{N}$ are given real numbers. Does there hold $u_k \equiv 0$ in $\mathbb{R}_T^n$ for all $k = 1, \dots, N$?

The notation $\mathbb{N}$ denotes the set of all positive integers, and $\mathbb{Z}$ denotes the set of all integers. Note that when $N = 1$, for a single operator $\mathcal{H}_g^s$, this property is referred to as the *unique continuation property* (UCP) and has been studied in [LLR20, BS24].

Remark 1.1. *Restrictions on the exponents $\{\alpha_k\}_{k=1}^N$ are necessary, since for general choices of $\{\alpha_k\}_{k=1}^N$, Question (IP-1) does not always hold. To illustrate this, consider the case $N = 2$. Let $\alpha \in (0, \infty) \setminus \mathbb{N}$ and set $\alpha_1 = \alpha + m$, $\alpha_2 = \alpha$ for some $m \in \mathbb{N}$. Given a nonempty open subset $\mathcal{O} \subset \mathbb{R}^n$, let $u_1 \in C^\infty(\mathbb{R}_T^n)$ be a nontrivial function such that $u_1 = 0$ in $\mathcal{O}_T$. Since $\mathcal{H}_g^m$ is a local operator for $m \in \mathbb{N}$, by defining $u_2 := -\mathcal{H}_g^m u_1$, it follows that $u_2 = 0$ in $\mathcal{O}_T$ and, moreover, the equation*

$$\mathcal{H}_g^{\alpha_1} u_1 + \mathcal{H}_g^{\alpha_2} u_2 = 0 \quad \text{in } \mathcal{O}_T.$$

However, u_1 and u_2 are not trivial functions.

The above counterexample shows that no such principle exists for local operators, and it leads to the following optimal condition for our entanglement principle:

Assumption 1. *We assume $\{\alpha_k\}_{k=1}^N \subset (0, \infty) \setminus \mathbb{N}$ with $\alpha_1 < \alpha_2 < \dots < \alpha_N$, and that they satisfy*

$$\alpha_k - \alpha_j \notin \mathbb{Z} \quad \text{for all } j \neq k,$$

which is required to ensure a positive answer to Question (IP-1). This nonresonance condition guarantees that the fractional powers remain genuinely distinct and cannot be reduced to local operators by integer shifts.

Our first main result, which decouples the entangled effects, is stated as follows.

Theorem 1.2 (Entanglement principle). *Let $\mathcal{O} \subset \mathbb{R}^n$ be a nonempty open set for $n \geq 2$. Let $N \in \mathbb{N}$, $T > 0$, and $\{\alpha_k\}_{k=1}^N \subset (0, \infty) \setminus \mathbb{N}$ satisfy **Assumption 1**. Suppose $g \in C^\infty(\mathbb{R}^n; \mathbb{R}^{n \times n})$*

satisfy (1.1). Assume that $\{u_k\}_{k=1}^N \subset C^\infty((-\infty, T); \mathcal{S}(\mathbb{R}^n))$ satisfy the following estimates: given any $\beta = (\beta_0, \beta_1, \dots, \beta_n) \in (\mathbb{N} \cup \{0\})^{n+1}$, there exist positive constants C_β and δ such that

$$(1.4) \quad |D_{x,t}^\beta u_k(x, t)| \leq C_\beta |\varphi_\beta(x)| e^{\delta t}, \quad |\beta| \geq 0 \quad \text{for } (x, t) \in \mathbb{R}^n \times \{t \leq -T\},$$

for $k = 1, \dots, N$, where $D_{x,t}^\beta = \frac{\partial^{|\beta|}}{\partial_t^{\beta_0} \partial x_1^{\beta_1} \dots \partial x_n^{\beta_n}}$ and $\varphi_\beta \in \mathcal{S}(\mathbb{R}^n)$. If

$$(1.5) \quad u_1|_{\mathcal{O}_T} = \dots = u_N|_{\mathcal{O}_T} = 0 \quad \text{and} \quad \left(\sum_{k=1}^N b_k \mathcal{H}_g^{\alpha_k} u_k \right) \Big|_{\mathcal{O}_T} = 0$$

hold for some $\{b_k\}_{k=1}^N \subset \mathbb{C} \setminus \{0\}$, then $u_k \equiv 0$ in $\mathbb{R}_T^n$ for all $k = 1, \dots, N$.

Theorem 1.2 extends the entanglement principle established in [FKU24, FL24] for elliptic operators to the parabolic setting. Specifically, [FKU24] proved the validity of the entanglement principle for the Laplace–Beltrami operator in the compact case, while [FL24] addressed the non-compact case for the classical Laplace operator.

We also would like to emphasize that the decay condition (1.4) will not impose any additional assumption in the study of inverse problem in Theorem 1.4. Moreover, when $N = 1$, (1.4) can be removed in Theorem 1.2, see Remark 3.4 for more detailed discussions.

Remark 1.3. In our case, owing to the extra time variable in the fractional parabolic equation, the exponents $\{\alpha_k\}_{k=1}^N$ only need to satisfy **Assumption 1**. This is different from the fractional Laplacian considered in [FL24]. Indeed, [FL24] requires additional assumptions on the odd dimensions to remove the resonance effect, which is only a technical reason.

As an application of the above entanglement principle, we study the unique determination of a time-dependent potential V in fractional poly-parabolic operators defined by

$$(1.6) \quad P_V := \sum_{k=1}^N b_k \mathcal{H}_g^{s_k} + V,$$

where $V = V(x, t) \in L^\infty(\Omega_T)$. Here $0 < s_1 < \dots < s_N < 1$ and $b_k > 0$ for all $1 \leq k \leq N$ (the positivity of b_k is needed for the forward problem). We consider the initial exterior value problem

$$(1.7) \quad \begin{cases} P_V u = 0 & \text{in } \Omega_T, \\ u = f & \text{in } (\Omega_e)_T, \\ u = 0 & \text{in } \mathbb{R}^n \times \{t \leq -T\}. \end{cases}$$

Assume that

$$\{0\} \text{ is not a Dirichlet eigenvalue of } P_V.$$

Theorem 2.4 guarantees the well-posedness of the problem (1.7) and allows us to define the corresponding exterior DN map

$$\Lambda_V : \mathbf{H}^{s_N}((\Omega_e)_T) \rightarrow \mathbf{H}^{-s_N}((\Omega_e)_T), \quad f \mapsto \sum_{k=1}^N b_k \mathcal{H}_g^{s_k} u_f \Big|_{(\Omega_e)_T},$$

where u_f is the unique solution of (1.7). A rigorous definition of the DN map can be found in Section 2.4.

We are interested in the following question.

(IP-2) Can one determine the potential $V \in L^\infty(\Omega_T)$ using the exterior DN map Λ_V of (1.7)?

The second main result of the paper answers this question.

Theorem 1.4 (Global uniqueness). *Given $N \in \mathbb{N}$, $\{b_k\}_{k=1}^N \subset (0, \infty)$, and $0 < s_1 < \dots < s_N < 1$, let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain for $n \geq 2$, and $W_1, W_2 \Subset \Omega_e$ be nonempty open subsets. Suppose $g \in C^\infty(\mathbb{R}^n; \mathbb{R}^{n \times n})$ satisfy (1.1). Let $V_j = V_j(x, t) \in L^\infty(\Omega_T)$, and Λ_{V_j} be the DN map of*

$$\begin{cases} (\sum_{k=1}^N b_k \mathcal{H}_g^{s_k} + V_j)u = 0 & \text{in } \Omega_T, \\ u = f & \text{in } (\Omega_e)_T, \\ u = 0 & \text{in } \mathbb{R}^n \times \{t \leq -T\}, \end{cases}$$

for $j = 1, 2$. Then the relation

$$\Lambda_{V_1} f|_{(W_2)_T} = \Lambda_{V_2} f|_{(W_2)_T}, \quad \text{for any } f \in C_c^\infty((W_1)_T)$$

implies that $V_1 = V_2$ in Ω_T .

Note that Theorem 1.4 extends the earlier results in [LLR20, BS24] (the works studied a global uniqueness for the case $N = 1$) to multiple terms of nonlocal parabolic operators.

1.2. Organization of the article. In Section 2, we recall several functional spaces and introduce nonlocal parabolic operators through the semigroup theory, together with well-posedness results for initial exterior value problems and rigorous definitions of the DN maps. Section 3 is devoted to establishing the entanglement principle for nonlocal parabolic operators. Finally, in Section 4, we prove the remaining main result regarding the global uniqueness of the potential in Theorem 1.4.

2. PRELIMINARIES

In this section, we introduce the function spaces used in this paper and recall several useful properties of the nonlocal parabolic operator $\mathcal{H}_g^s$.

2.1. Function spaces. We start by recalling the (fractional) Sobolev spaces. Given $a \in \mathbb{R}$, $H^a(\mathbb{R}^n) = W^{a,2}(\mathbb{R}^n)$ is the L^2 -based fractional Sobolev space (see [DNPV12] for example) with the norm

$$\|u\|_{H^a(\mathbb{R}^n)} := \|\mathcal{F}_x^{-1} \{ \langle \xi \rangle^a \mathcal{F}_x u \} \|_{L^2(\mathbb{R}^n)},$$

where $\langle \xi \rangle = (1 + |\xi|^2)^{\frac{1}{2}}$. Let $\mathcal{O} \subset \mathbb{R}^n$ be an open set. We define

$$H^a(\mathcal{O}) := \{u|_{\mathcal{O}} : u \in H^a(\mathbb{R}^n)\},$$

$$\tilde{H}^a(\mathcal{O}) := \text{closure of } C_c^\infty(\mathcal{O}) \text{ in } H^a(\mathbb{R}^n).$$

The space $H^a(\mathcal{O})$ is complete under the norm

$$\|u\|_{H^a(\mathcal{O})} := \inf \{ \|v\|_{H^a(\mathbb{R}^n)} : v \in H^a(\mathbb{R}^n) \text{ and } v|_{\mathcal{O}} = u \}.$$

Given an open set $B \subset \mathbb{R}^{n+1}$, if $f = f(x, t)$ and $g = g(x, t)$ are L^2 functions in B , we denote the L^2 inner product by

$$(f, g)_B := \int_B f \bar{g} \, dx dt.$$

For the nonlocal space-time operator $\mathcal{H}^s = (\partial_t - \Delta)^s$, we will work on the following Lions-Magenes Sobolev spaces $H^{s,s/2}(\mathbb{R}^n \times \mathbb{R})$ (see [LM72, Chapter 4.2] and in particular equation (2.3) there). To simplify this notation and to emphasize the coupling between time and space variables, hereinafter we abbreviate it by $\mathbf{H}^s(\mathbb{R}^{n+1})$. More precisely, for $a \in \mathbb{R}$, we consider

$$\mathbf{H}^a(\mathbb{R}^{n+1}) := \{u \in L^2(\mathbb{R}^{n+1}) : \|u\|_{\mathbf{H}^a(\mathbb{R}^{n+1})} < \infty\} = H^{a,a/2}(\mathbb{R}^n \times \mathbb{R}),$$

where

$$(2.1) \quad \|u\|_{\mathbf{H}^a(\mathbb{R}^{n+1})}^2 = \int_{\mathbb{R}^{n+1}} (1 + |i\rho + |\xi|^2|^a) |\hat{u}(\xi, \rho)|^2 d\rho d\xi < \infty.$$

Note that $|\rho + |\xi|^2| = (|\rho|^2 + |\xi|^4)^{1/2}$ and $2^{-1/2}(|\rho| + |\xi|^2) \leq (|\rho|^2 + |\xi|^4)^{1/2} \leq |\rho| + |\xi|^2$. As the “classical” fractional Sobolev spaces (see [McL00]), the following notations follow naturally by treating spacetime together. For an open set O and a closed set F in $\mathbb{R}^{n+1}$, $n \geq 1$, we define

$$\begin{aligned}\mathbf{H}^a(O) &:= \{u|_O : u \in \mathbf{H}^a(\mathbb{R}^{n+1})\}, \\ \tilde{\mathbf{H}}^a(O) &:= \text{closure of } C_c^\infty(O) \text{ in } \mathbf{H}^a(\mathbb{R}^{n+1}), \\ \mathbf{H}_F^a &= \mathbf{H}_F^a(\mathbb{R}^{n+1}) := \{u \in \mathbf{H}^a(\mathbb{R}^{n+1}) : \text{supp}(u) \subset F\}.\end{aligned}$$

Also, $C_c^\infty(\mathbb{R}^{n+1})$ is dense in $\mathbf{H}^a(\mathbb{R}^{n+1})$ under the norm $\|\cdot\|_{\mathbf{H}^a(\mathbb{R}^{n+1})}$. Moreover,

$$(\mathbf{H}^a(O))^* = \tilde{\mathbf{H}}^{-a}(O), \quad (\tilde{\mathbf{H}}^a(O))^* = \mathbf{H}^{-a}(O), \quad \text{for } a \in \mathbb{R}.$$

2.2. The nonlocal parabolic operator. The definition for the nonlocal parabolic operator $\mathcal{H}_g^s$ was given in [Bal60] (also see [BDLCRS21, BS24]) for $0 < s < 1$. It is known that heat operator $\partial_t - \Delta_g$ in $\mathbb{R}^n \times \mathbb{R}$ possesses a globally defined fundamental solution $p_\tau(x, y)$, which satisfies

$$e^{\tau \Delta_g} 1(x) = \int_{\mathbb{R}^n} p_\tau(x, y) dV_g(y) = 1, \quad \text{for every } x \in \mathbb{R}^n \text{ and } \tau > 0,$$

where $e^{\tau \Delta_g}$ stands for the heat semigroup associated to the operator Δ_g . For $u \in \mathcal{S}(\mathbb{R}^n)$, we have

$$e^{\tau \Delta_g} u(x) = \int_{\mathbb{R}^n} p_\tau(x, y) u(y) dV_g(y), \quad \text{for every } x \in \mathbb{R}^n \text{ and } \tau > 0.$$

Here the Riemannian volume form dV_g is given by $dV_g(y) = \sqrt{|g|} dy$. For the purpose of simplifying the notation, in what follows, we will only use dy to represent $dV_g(y)$. Meanwhile, the heat kernels $p_\tau(x, y)$ satisfies

$$(2.2) \quad C_1 \left(\frac{1}{4\pi\tau} \right)^{n/2} e^{-\frac{c_1|x-y|^2}{4\tau}} \leq p_\tau(x, y) \leq C_2 \left(\frac{1}{4\pi\tau} \right)^{n/2} e^{-\frac{c_2|x-y|^2}{4\tau}},$$

for some positive constants c_1, c_2, C_1 and C_2 and for all $x, y \in \mathbb{R}^n$, $\tau > 0$.

Since ∂_t and $-\Delta_g$ are commutable, we have $e^{-\tau \mathcal{H}_g} = e^{\tau \Delta_g} \circ e^{-\tau \partial_t}$, and the evolution semigroup is given by

$$(2.3) \quad e^{-\tau \mathcal{H}_g} u(x, t) := e^{\tau \Delta_g} u(x, t - \tau) = \int_{\mathbb{R}^n} p_\tau(x, y) u(y, t - \tau) dy, \quad \tau > 0, \quad \text{for } u \in \mathcal{S}(\mathbb{R}^{n+1}),$$

where $p_\tau(x, y)$ is the heat kernel given as before, and $\mathcal{S}(\mathbb{R}^{n+1})$ stands for the Schwartz space. Meanwhile, it is held that

$$e^{-\tau \mathcal{H}_g} 1(x, t) = \int_{\mathbb{R}^n} p_\tau(x, y) dy = 1, \quad \text{for every } (x, t) \in \mathbb{R}^{n+1}, \text{ and } \tau > 0.$$

Note that $\{e^{-\tau \mathcal{H}_g}\}_{\tau \geq 0}$ is a strongly continuous contractive semigroup such that¹

$$\|e^{-\tau \mathcal{H}_g} u - u\|_{L^2(\mathbb{R}^{n+1})} = \mathcal{O}(\tau), \quad \text{as } \tau \rightarrow 0.$$

Let us first give the explicit formula of $\mathcal{H}_g^s$ via the heat semigroup.

Definition 2.1 (Balakrishnan formula). *Given $s \in (0, 1)$ and $u \in \mathcal{S}(\mathbb{R}^{n+1})$, the nonlocal parabolic operator $\mathcal{H}_g^s$ can be defined as (see [BS24, Section 2])*

$$(2.4) \quad \mathcal{H}_g^s u(x, t) := \frac{1}{\Gamma(-s)} \int_0^\infty (e^{-\tau \mathcal{H}_g} u(x, t) - u(x, t)) \frac{d\tau}{\tau^{1+s}}.$$

¹The notation $\mathcal{O}(\tau)$ is the Bachmann–Landau notation.

Note that $\mathcal{H}_g^s u \in L^2(\mathbb{R}^{n+1})$ for $u \in \mathcal{S}(\mathbb{R}^{n+1})$ by the functional calculus. Using (2.3), we can rewrite (2.4) as follows:

$$\mathcal{H}_g^s u(x, t) = \int_0^\infty \int_{\mathbb{R}^n} (u(y, t - \tau) - u(x, t)) \mathcal{K}_s(x, y, \tau) dy d\tau,$$

where

$$\mathcal{K}_s(x, y, \tau) := \frac{1}{\Gamma(-s)} \frac{p_\tau(x, y)}{\tau^{1+s}}.$$

In particular, as $g = I_n$, the kernel $\mathcal{K}_s(x, y, \tau)$ has an explicit representation formula

$$\mathcal{K}_s(x, y, \tau) = \frac{1}{(4\pi)^{n/2} \Gamma(-s)} \frac{e^{-\frac{|x-y|^2}{4\tau}}}{\tau^{n/2+1+s}},$$

where we used the heat kernel for the heat operator $\partial_\tau - \Delta$ precisely.

Remark 2.2. Recall that the fractional Laplace–Beltrami operator can also be defined in a similar way, which is

$$\begin{aligned} (-\Delta_g)^s v(x) &:= \frac{1}{\Gamma(-s)} \int_0^\infty (e^{\tau \Delta_g} v(x) - v(x)) \frac{d\tau}{\tau^{1+s}} \\ &= \int_0^\infty \int_{\mathbb{R}^n} (v(y) - v(x)) \mathcal{K}_s(x, y, \tau) dy d\tau. \end{aligned}$$

Using the Fourier transform with respect to the time variable $t \in \mathbb{R}$, one can express $\mathcal{H}_g^s u$ in terms of the Fourier transform. In doing so, we first denote by E_λ the spectral measure associated to $\mathcal{H}_g$, i.e.,

$$\mathcal{H}_g = - \int_0^\infty \lambda dE_\lambda.$$

We then observe that the heat semigroup $\{e^{\tau \Delta_g}\}_{\tau \geq 0}$ can be written by spectral measures as an identity of gamma functions [BS24, Section 2]:

$$(2.5) \quad e^{\tau \Delta_g} = \int_0^\infty e^{-\lambda \tau} dE_\lambda \quad \text{and} \quad \frac{1}{\Gamma(-s)} \int_0^\infty \frac{e^{-(\lambda + i\sigma)\tau} - 1}{\tau^{1+s}} d\tau = (\lambda + i\sigma)^s,$$

for $\lambda > 0$ and $\sigma \in \mathbb{R}$, where $i = \sqrt{-1}$. Taking the Fourier transform in the time variable on (2.3) yields

$$\mathcal{F}_t(e^{-\tau \mathcal{H}_g} u)(x, \sigma) = e^{-i\sigma \tau} e^{\tau \Delta_g} (\mathcal{F}_t u(\cdot, \sigma))(x).$$

Together with (2.5), this gives the Fourier analogue of the definition (2.4) as follows:

$$\mathcal{F}_t(\mathcal{H}_g^s u)(\cdot, \sigma) = \int_0^\infty (\lambda + i\sigma)^s dE_\lambda(\mathcal{F}_t u(\cdot, \sigma)).$$

Moreover, we define the adjoint operator $\mathcal{H}_{g,*}^s$ of $\mathcal{H}_g^s$ in terms of the spectral resolution in the following manner

$$\mathcal{F}_t(\mathcal{H}_{g,*}^s u)(\cdot, \sigma) = \int_0^\infty (\lambda - i\sigma)^s dE_\lambda(\mathcal{F}_t u(\cdot, \sigma)), \quad \text{for } u \in \mathcal{S}(\mathbb{R}^{n+1}).$$

We also recall the following property from [BS24, Section 2]

$$\begin{aligned} \langle \mathcal{H}_g^s f, h \rangle &= \langle \mathcal{H}_g^{s/2} f, \mathcal{H}_{g,*}^{s/2} h \rangle = \langle f, \mathcal{H}_{g,*}^s h \rangle \\ &= \int_{\mathbb{R}} \int_0^\infty (\lambda + i\sigma)^s d\langle E_\lambda \mathcal{F}_t f, \overline{\mathcal{F}_t h} \rangle(\cdot, \sigma) d\sigma, \quad \text{for } f, h \in \mathcal{S}(\mathbb{R}^{n+1}). \end{aligned}$$

From the resolution of the parabolic version of the Kato square root problem in [AEN20] and interpolation type argument, $\mathbf{H}^s(\mathbb{R}^{n+1})$ is the completion of $\mathcal{S}(\mathbb{R}^{n+1})$ with respect to the following norm:

$$(2.6) \quad \left(\int_{\mathbb{R}} \int_0^\infty \left((1 + |\lambda + i\sigma|^2)^{s/2} d\|E_\lambda(\mathcal{F}_t u(\cdot, \sigma))\|^2 \right) d\sigma \right)^{1/2}, \quad s \in (0, 1), \quad \text{for } u \in \mathcal{S}(\mathbb{R}^{n+1}).$$

Therefore, we get from the Cauchy-Schwarz inequality that

$$(2.7) \quad \langle \mathcal{H}_g^s f, h \rangle = \langle \mathcal{H}_g^{s/2} f, \mathcal{H}_{g,*}^{s/2} h \rangle \leq C \|f\|_{\mathbf{H}^s(\mathbb{R}^{n+1})} \|h\|_{\mathbf{H}^s(\mathbb{R}^{n+1})}$$

for some constant $C > 0$ independent of f, h . This leads to the mapping properties

$$\mathcal{H}_g^s : \mathbf{H}^s(\mathbb{R}^{n+1}) \rightarrow \mathbf{H}^{-s}(\mathbb{R}^{n+1}) \quad \text{and} \quad \mathcal{H}_{g,*}^s : \mathbf{H}^s(\mathbb{R}^{n+1}) \rightarrow \mathbf{H}^{-s}(\mathbb{R}^{n+1}).$$

We refer to [BS24, Section 2] for related discussions.

For general $\alpha \in (0, \infty) \setminus \mathbb{N}$, we write $\alpha = m + s$, where m is the integer part of α and $s \in (0, 1)$. Based on [MCSA01, Chapter 5], we can write

$$\mathcal{H}_g^\alpha = \mathcal{H}_g^{m+s} = \mathcal{H}_g^s(\mathcal{H}_g^m) = \mathcal{H}_g^m(\mathcal{H}_g^s),$$

where $\mathcal{H}_g^m = (\partial_t - \Delta_g)^m$ is a local differential operator.

2.3. The well-posedness. In this section, we will show the unique existence of solutions to the initial exterior value problems (1.7) by adapting the arguments developed in [LLR20, BS24], in which the well-posed problems (1.2) were proved for a constant and a variable coefficient, respectively.

To show the problem (1.7) has a unique solution, we consider the following initial exterior value problem instead:

$$(2.8) \quad \begin{cases} P_V u = F & \text{in } \Omega_T, \\ u = f & \text{in } (\Omega_e)_T, \\ u = 0 & \text{in } \mathbb{R}^n \times \{t \leq -T\}, \end{cases}$$

where $F \in (\mathbf{H}_{\Omega_T}^{s_N})^*$, $f \in \mathbf{H}^{s_N}((\Omega_e)_T)$, and P_V is defined in (1.6).

Let $0 < s_1 < \dots < s_N < 1$, and consider the sesquilinear form $B_V(\cdot, \cdot)$ on $\mathbf{H}^{s_N}(\mathbb{R}^{n+1}) \times \mathbf{H}^{s_N}(\mathbb{R}^{n+1})$ defined by

$$B_V(u, w) := \sum_{k=1}^N b_k(\mathcal{H}_g^{s_k/2} u, \mathcal{H}_{g,*}^{s_k/2} w)_{L^2(\mathbb{R}^{n+1})} + (Vu, w)_{L^2(\Omega_T)}.$$

According to [LLR20, CLR20], we need to study a time-localized problem. We denote the cut-off of a function $u(x, t)$ on the time variable t by

$$u_T(t, x) := u(t, x) \chi_{[-T, T]}(t),$$

where $\chi_{[-T, T]}(t)$ is a characteristic function for $t \in \mathbb{R}$. Since the characteristic function is a multiplier in the Sobolev space $H^\gamma(\mathbb{R})$ for $|\gamma| < \frac{1}{2}$, we have $u_T \in \mathbf{H}^s(\mathbb{R}^{n+1})$ when $u \in \mathbf{H}^s(\mathbb{R}^{n+1})$ for $0 < s < 1$, see [LLR20, Section 2]. We are ready to define the weak solution for (2.8).

Definition 2.3. Let Ω be a bounded open set in $\mathbb{R}^n$ and $T > 0$. Assume $0 < s_1 < \dots < s_N < 1$, and $V \in L^\infty(\Omega_T)$ such that 0 is not a Dirichlet eigenvalue of the problem (2.8). Given $F \in (\mathbf{H}_{\Omega_T}^{s_N})^*$ and $f \in \tilde{\mathbf{H}}^{s_N}((\Omega_e)_T)$, we say that $u \in \mathbf{H}^{s_N}(\mathbb{R}^{n+1})$ is a weak solution of (2.8) if $v := (u - f)_T \in \mathbf{H}_{\Omega_T}^{s_N}$ and

$$B_V(u, w) = \langle F, w \rangle_{(\mathbf{H}_{\Omega_T}^{s_N})^* \times \mathbf{H}_{\Omega_T}^{s_N}}, \quad \text{for any } w \in \mathbf{H}_{\Omega_T}^{s_N}.$$

Theorem 2.4 (Well-posedness). *Let Ω be a bounded open set in $\mathbb{R}^n$ and $T > 0$. Let $N \in \mathbb{N}$ and $\{b_k\}_{k=1}^N \subset (0, \infty)$. Suppose $g \in C^\infty(\mathbb{R}^n; \mathbb{R}^{n \times n})$ satisfies (1.1). Assume $0 < s_1 < \dots < s_N < 1$ and $V \in L^\infty(\Omega_T)$ such that 0 is not a Dirichlet eigenvalue of the problem (2.8). Given $F \in (\mathbf{H}_{\Omega_T}^{s_N})^*$ and $f \in \tilde{\mathbf{H}}^{s_N}((\Omega_e)_T)$, there exists a unique solution $u_T \in \mathbf{H}^{s_N}(\mathbb{R}^{n+1})$ to the problem (2.8) satisfying*

$$\|u_T\|_{\mathbf{H}^{s_N}(\mathbb{R}^{n+1})} \leq C(\|F\|_{(\mathbf{H}_{\Omega_T}^{s_N})^*} + \|f\|_{\mathbf{H}^{s_N}((\Omega_e)_T)})$$

for some constant $C > 0$ independent of F , f , and u .

Proof. Let $v := (u - f)_T$ and $\tilde{F} := F - P_V f$, then $v \in \mathbf{H}_{\Omega_T}^{s_N}$ and $v_T = v$. It suffices to show that for $\tilde{F} \in (\mathbf{H}_{\Omega_T}^{s_N})^*$, there exists a unique solution $v \in \mathbf{H}_{\Omega_T}^{s_N}$ such that

$$B_V(v, w) = \langle \tilde{F}, w \rangle_{(\mathbf{H}_{\Omega_T}^{s_N})^* \times \mathbf{H}_{\Omega_T}^{s_N}}, \quad \text{for any } w \in \mathbf{H}_{\Omega_T}^{s_N}.$$

Consider the bilinear form

$$B_V(v, w) + \mu(v, w)_{L^2(\Omega_T)},$$

for $\mu \geq \|\min\{V, 0\}\|_{L^\infty(\Omega_T)}$ in $\mathbf{H}_{\Omega_T}^{s_N}$. The boundedness of this bilinear form

$$\begin{aligned} B_V(v, w) + \mu(v, w)_{L^2(\Omega_T)} &\leq C \sum_{k=1}^N \|v\|_{\mathbf{H}^{s_k}(\mathbb{R}^{n+1})} \|w\|_{\mathbf{H}^{s_k}(\mathbb{R}^{n+1})} + C \|v\|_{L^2(\mathbb{R}^{n+1})} \|w\|_{L^2(\mathbb{R}^{n+1})} \\ &\leq C \|v\|_{\mathbf{H}^{s_N}(\mathbb{R}^{n+1})} \|w\|_{\mathbf{H}^{s_N}(\mathbb{R}^{n+1})} \quad \text{for any } v, w \in \mathbf{H}_{\Omega_T}^{s_N}, \end{aligned}$$

follows directly from (2.7).

We now prove the coercity in the space $\mathbf{H}_{\Omega_T}^{s_N}$. Note that for $k = 1, \dots, N$, we have for $v \in H_{\Omega_T}^{s_N}$ that

$$\begin{aligned} \langle \mathcal{H}_g^{s_k/2} v, \mathcal{H}_{g,*}^{s_k/2} v \rangle &= \int_{\mathbb{R}} \int_0^\infty (\lambda + i\sigma)^{s_k} d\|E_\lambda(\mathcal{F}_t v)(\cdot, \sigma)\|^2 d\sigma \\ (2.9) \quad &= \int_{\mathbb{R}} \int_0^\infty |\lambda + i\sigma|^{s_k} (\cos(s_k \theta) + i \sin(s_k \theta)) d\|E_\lambda(\mathcal{F}_t v)(\cdot, \sigma)\|^2 d\sigma \\ &= \int_{\mathbb{R}} \int_0^\infty |\lambda + i\sigma|^{s_k} \cos(s_k \theta) d\|E_\lambda(\mathcal{F}_t v)(\cdot, \sigma)\|^2 d\sigma, \end{aligned}$$

where $\tan \theta = \sigma/\lambda$ and we utilized the fact that $\sin(s_k \theta)$ is an odd function in the last step. Since $\lambda > 0$ implies $\theta \in (-\frac{\pi}{2}, \frac{\pi}{2})$, we have for all $s_k \in (0, 1)$ that

$$\cos(s_k \theta) \geq \cos\left(\frac{s_k \pi}{2}\right) \geq \min_{1 \leq k \leq N} \cos\left(\frac{s_k \pi}{2}\right) =: c_s > 0.$$

Therefore, using the fact $\mu \geq \|\min\{V, 0\}\|_{L^\infty(\Omega_T)}$, equation (2.9) and the equivalent norm between (2.6) and (2.1), we obtain

(2.10)

$$B_V(v, v) + \mu(v, v)_{L^2(\Omega_T)} \geq \sum_{k=1}^N b_k (\mathcal{H}_g^{s_k/2} v, \mathcal{H}_{g,*}^{s_k/2} v)_{\mathbb{R}^{n+1}} \geq C \sum_{k=1}^N \int_{\mathbb{R}^{n+1}} |i\rho + |\xi|^2|^{s_k} |\hat{v}(\xi, \rho)|^2 d\xi d\rho.$$

Applying the Hardy-Littlewood-Sobolev inequality for the x -variable and the fact that v is compactly supported in x -variable, it yields for any $s_k \in (0, 1)$,

$$(2.11) \quad \int_{\mathbb{R}^{n+1}} |i\rho + |\xi|^2|^{s_k} |\hat{v}(\xi, \rho)|^2 d\xi d\rho \geq \int_{\mathbb{R}} \|(-\Delta_x)^{s_k/2} \mathcal{F}_t v(\cdot, \rho)\|_{L^2(\mathbb{R}^n)}^2 d\rho \geq C \|v\|_{L^2(\mathbb{R}^{n+1})}^2,$$

see [LLR20, BGMN21] for a detailed explanation. Coercivity then follows from (2.10) and (2.11), that is, $B_V(v, v) + \mu(v, v)_{L^2(\Omega_T)} \geq C \|v\|_{\mathbf{H}^{s_N}(\mathbb{R}^{n+1})}^2$.

By Lax–Milgram theorem, there exists a unique solution $v = G_\mu \tilde{F} \in H_{\Omega_T}^{s_N}$ such that

$$B_V(v, w) + \mu(v, w)_{L^2(\Omega_T)} = \langle \tilde{F}, w \rangle_{(\mathbf{H}_{\Omega_T}^{s_N})^* \times \mathbf{H}_{\Omega_T}^{s_N}}, \quad \text{for any } w \in \mathbf{H}_{\Omega_T}^{s_N},$$

along with

$$\|v\|_{H_{\Omega_T}^{s_N}} \leq C \|\tilde{F}\|_{(H_{\Omega_T}^{s_N})^*}.$$

In particular, $G_\mu : (\mathcal{H}_{\Omega_T}^{s_N})^* \rightarrow \mathcal{H}_{\Omega_T}^{s_N}$ is bounded and by the compact Sobolev embedding, the operator $G_\mu : L^2(\Omega_T) \rightarrow L^2(\Omega_T)$ is compact. Then the spectral theorem implies that the eigenvalues of G_μ are $\frac{1}{\lambda_j + \mu}$ with $\lambda_j \rightarrow +\infty$. Fredholm alternative and the assumption 0 is not a Dirichlet eigenvalue of P_V ensure the existence and uniqueness of the problem under consideration. $\square$

Remark 2.5. *The assumption that $\{0\}$ is not a Dirichlet eigenvalue of P_V implies $\{0\}$ is not a Dirichlet eigenvalue of the adjoint of P_V . Similarly, we can establish the well-posedness result for the adjoint problem to (2.8). Under the hypothesis of Theorem 2.4, there exists a unique solution $u \in \mathbf{H}^{s_N}(\mathbb{R}^{n+1})$ to the future exterior problem*

$$\begin{cases} (\sum_{k=1}^N b_k \mathcal{H}_{g,*}^{s_k} + V)u = F & \text{in } \Omega_T, \\ u = f & \text{in } (\Omega_e)_T, \\ u = 0 & \text{in } \mathbb{R}^n \times \{t \geq T\}. \end{cases}$$

2.4. The Dirichlet-to-Neumann map. Based on the well-posedness results of initial exterior problems (1.7), let us define the corresponding DN maps Λ_V by means of the bilinear form B_V .

We first introduce the following quotient spaces for our exterior data by

$$\mathbb{X} := \mathbf{H}^{s_N}(\mathbb{R}^n \times [-T, T]) / \mathbf{H}_{\Omega_T}^{s_N},$$

equipped with the norm

$$\|[f]\|_{\mathbb{X}} := \inf_{\phi \in \mathbf{H}_{\Omega_T}^{s_N}} \|f + \phi\|_{H^s(\mathbb{R}^n)}, \quad \text{for } f \in \mathbf{H}^{s_N}(\mathbb{R}^n \times [-T, T]).$$

Denote $\mathbb{X}^*$ as the dual of $\mathbb{X}$. We now define the DN maps as follows:

$$\langle \Lambda_V[f], [\zeta] \rangle_{\mathbb{X}^* \times \mathbb{X}} := B_V(u_f, \zeta), \quad \text{for } [f], [\zeta] \in \mathbb{X},$$

where $u_f \in \mathbf{H}^{s_N}(\mathbb{R}^{n+1})$ is the solution of (1.7) with the Dirichlet data f .

Analogously, one can also define the adjoint DN maps by utilizing the following natural pairing property

$$\langle [f], \Lambda_V^*[\zeta] \rangle_{\mathbb{X} \times \mathbb{X}^*} := \langle \Lambda_V[f], [\zeta] \rangle_{\mathbb{X}^* \times \mathbb{X}}, \quad \text{for } [f], [\zeta] \in \mathbb{X},$$

Also, the adjoint DN maps can be represented as

$$\langle [f], \Lambda_V^*[\zeta] \rangle_{\mathbb{X}_1 \times \mathbb{X}_1^*} = B_V(f, u_\zeta), \quad \langle [h],$$

where $u_\zeta \in \mathbf{H}^{s_N}(\mathbb{R}^{n+1})$ is the solution of the adjoint equation $(\sum_{k=1}^N b_k \mathcal{H}_{g,*}^{s_k} + V)u_\zeta = 0$ with the Dirichlet data ζ in $(\Omega_e)_T$ and $u_\zeta = 0$ for $t \geq T$. To simplify the notations, we use f to denote $[f]$.

Proposition 2.6. *Let Ω be a bounded open set in $\mathbb{R}^n$ and $T > 0$. Let $N \in \mathbb{N}$. Assume that $\{b_k\}_{k=1}^N \subset (0, \infty)$, $0 < s_1 < \dots < s_N < 1$, and $V \in L^\infty(\Omega_T)$ such that 0 is not a Dirichlet eigenvalue of the problem (2.8). Then the DN map Λ_V defined above is well-defined and bounded.*

Proof. We first show that Λ_V only depends on the equivalence classes. For $f, \zeta \in \mathbf{H}^{s_N}(\mathbb{R}^n \times [-T, T])$. Let $\phi, \psi \in \mathbf{H}_{\Omega_T}^{s_N}$. Since u_f and $u_{f+\phi}$ both solve the equation (1.7) with the same exterior data, Theorem 2.4 implies $u_f = u_{f+\phi}$. By the linearity of B_V in the second component, it yields

$$B_V(u_{f+\phi}, \zeta + \psi) = B_V(u_f, \zeta + \psi) = B_V(u_f, \zeta) + B_V(u_f, \psi).$$

Using the fact that $\text{supp}(\psi) \subset \overline{\Omega_T}$ and u_f solves (1.7), we get $B_V(u_f, \psi) = 0$. This proves that $\langle \Lambda_V(f + \phi), (\zeta + \psi) \rangle = \langle \Lambda_V f, \zeta \rangle$ and thus Λ_V is well-defined.

The boundedness of Λ_V follows from

$$|\langle \Lambda_V f, \zeta \rangle| \leq |B_V(u_{f+\phi}, \zeta + \psi)| \leq C \|u_{f+\phi}\|_{H^{s_N}(\mathbb{R}^{n+1})} \|\zeta + \psi\|_{H^{s_N}(\mathbb{R}^{n+1})}$$

and taking the infimum with respect to $\phi, \psi \in \mathbf{H}_{\Omega_T}^{s_N}$. $\square$

3. THE ENTANGLEMENT PRINCIPLE

The aim of this section is to show the entanglement principles for the fractional parabolic operators on the Euclidean domain. To this end, we first recall the result demonstrated in [FKU24, Proposition 3.1], which will play a crucial role of decoupling the mixed fractional parabolic operators later.

Proposition 3.1 ([FKU24, Proposition 3.1]). *Let $N \in \mathbb{N}$ and $\{\alpha_k\}_{k=1}^N \subset (0, \infty) \setminus \mathbb{N}$ satisfy **Assumption 1**. Given $a > 0$, suppose that $\{f_k\}_{k=1}^N \subset C^\infty((0, \infty))$, there exist positive constants c and δ such that the function $f = f_k$ ($k = 1, \dots, N$) fulfills*

$$(3.1) \quad |f(\tau)| \leq ce^{-\delta\tau}, \quad \tau \in (a, \infty), \quad \text{and} \quad |f(\tau)| \leq ce^{-\frac{\delta}{\tau}}, \quad \tau \in (0, a].$$

Additionally, if there exists $\ell \in \mathbb{N} \cup \{0\}$ such that

$$\sum_{k=1}^N \Gamma(m+1+\alpha_k) \int_0^\infty f_k(\tau) \tau^{-m} d\tau = 0, \quad \text{for all } m = \ell, \ell+1, \ell+2, \dots,$$

then $f_k(\tau) = 0$ for all $\tau \in (0, \infty)$, and for all $k = 1, \dots, N$.

Note that [FKU24, Proposition 3.1] shows the case when $a = 1$. The same result still holds for any given constant $a > 0$ by following the same arguments there.

Remark 3.2. *Let us emphasize the essential difference of the entanglement principle between [FKU24, FL24] and this work.*

- (i) *The works [FKU24, FL24] investigate the entanglement principle for nonlocal elliptic operators, and we study an analogous tenet for the nonlocal parabolic operator. Particularly, in [FKU24], the authors investigated the entanglement principle for fractional Laplace–Beltrami operators on closed Riemannian manifolds. Thanks to the compactness, the first inequality in (3.1) can be achieved naturally by its heat kernel estimate.*
- (ii) *In [FL24], the authors considered the same problem on $\mathbb{R}^n$, for the fractional Laplace operator. Due to the lack of compactness, the first inequality in (3.1) can not be satisfied. Hence, the authors introduced the super-exponential decay condition, allowing them to transfer the problem to the spherical mean vanishing property. In this work, we are in the non-compact setting as well as [FL24]. However, thanks to the representation formulas (2.3) and (2.4), one can introduce a suitable decay condition of $u(x, t)$ with respect to the time variable, so that the first inequality in (3.1) still holds. Given this, one may expect the entanglement principle to hold for the nonlocal parabolic operator.*
- (iii) *Let us point out that the entanglement principle for fractional Laplace–Beltrami operators $(-\Delta_g)^s$ remains open in the non-compact Euclidean space $\mathbb{R}^n$, which seems to be a challenging problem to resolve.*

The following theorem lays the foundation of the proof of Theorem 1.2 for $\{\alpha_k\}_{k=1}^N \subset (0, \infty) \setminus \mathbb{N}$.

Theorem 3.3. *Let $\mathcal{O} \subset \mathbb{R}^n$ be a nonempty open set for $n \geq 2$. Let $N \in \mathbb{N}$, $T > 0$, and $0 < s_1 < \dots < s_N < 1$. Suppose that $\{v_k\}_{k=1}^N \subset C^\infty((-\infty, T); \mathcal{S}(\mathbb{R}^n))$ satisfies the following estimates: given*

any multi-index $\beta = (\beta_0, \beta_1, \dots, \beta_n) \in (\mathbb{N} \cup \{0\})^{n+1}$, there exist positive constants C_β and δ such that

$$(3.2) \quad |D_{x,t}^\beta v_k(x, t)| \leq \begin{cases} C_0 |\varphi_0(x)| e^{\delta t}, & |\beta| = 0 \\ C_\beta |\varphi_\beta(x)|, & |\beta| \geq 1 \end{cases}, \quad \text{for } (x, t) \in \mathbb{R}^n \times \{t \leq -T\},$$

and $k = 1, \dots, N$, where $\varphi_\beta \in \mathcal{S}(\mathbb{R}^n)$. If

$$(3.3) \quad v_1|_{\mathcal{O}_T} = \dots = v_N|_{\mathcal{O}_T} = 0 \quad \text{and} \quad \left(\sum_{k=1}^N \mathcal{H}_g^{s_k} v_k \right) \Big|_{\mathcal{O}_T} = 0$$

hold, then $v_k \equiv 0$ in $\mathbb{R}_T^n$ for all $k = 1, \dots, N$.

Proof. Similar to the arguments as in [FKU24, FL24, LLU22], via the condition (3.3), the iteration arguments yield for $m = 1, 2, \dots$ that

$$\mathcal{H}_g^m v_1|_{\mathcal{O}_T} = \dots = \mathcal{H}_g^m v_N|_{\mathcal{O}_T} = 0, \quad \text{and} \quad \left(\sum_{k=1}^N \mathcal{H}_g^{s_k} \mathcal{H}_g^m v_k \right) \Big|_{\mathcal{O}_T} = 0.$$

Let $\omega \Subset \mathcal{O}$ be an open nonempty subset such that

$$(3.4) \quad \text{dist}(\omega, \mathbb{R}^n \setminus \overline{\mathcal{O}}) \geq 2\kappa,$$

for some constant $\kappa > 0$. For $(x, t) \in \omega_T$, we have by (2.4) that

$$(3.5) \quad \begin{aligned} 0 &= \sum_{k=1}^N \mathcal{H}_g^{s_k} \mathcal{H}_g^m v_k(x, t) = \sum_{k=1}^N \frac{1}{\Gamma(-s_k)} \int_0^\infty (e^{-\tau \mathcal{H}_g} \mathcal{H}_g^m v_k)(x, t) \frac{d\tau}{\tau^{1+s_k}} \\ &= \underbrace{\sum_{k=1}^N \frac{1}{\Gamma(-s_k)} \int_0^\infty (-1)^m \partial_\tau^m (e^{-\tau \mathcal{H}_g} v_k)(x, t) \frac{d\tau}{\tau^{1+s_k}}}_{(e^{-\tau \mathcal{H}_g} \mathcal{H}_g^m v_k)(x, t) = (-1)^m \partial_\tau^m (e^{-\tau \mathcal{H}_g} v_k)(x, t)}. \end{aligned}$$

Next, fix $t_0 \in (-T, T)$, we shall show no contribution arises at the endpoints when conducting integration by parts in τ . That is, for $\ell = 0, 1, \dots, m-1$, the following terms

$$(3.6) \quad \partial_\tau^\ell (e^{-\tau \mathcal{H}_g} v_k)(x, t_0) \frac{1}{\tau^{s_k+m-\ell}}$$

vanish at $\tau \rightarrow 0^+$ and $\tau \rightarrow +\infty$. Since $t_0 - (-T) > 0$, let us denote

$$a := t_0 - (-T) = t_0 + T > 0.$$

Under this assumption, we have

$$(3.7) \quad t_0 - \tau \in (-\infty, -T) \quad \text{if } \tau \in (a, \infty),$$

and

$$(3.8) \quad t_0 - \tau \in [-T, T) \quad \text{if } \tau \in (0, a].$$

To show the boundary terms (3.6) vanish, it suffices to show the following two estimates. For $\tau \in (a, \infty)$, we have

$$\begin{aligned}
(3.9) \quad |\partial_\tau^\ell (e^{-\tau \mathcal{H}_g} v_k)(x, t_0)| &= | (e^{-\tau \mathcal{H}_g} \mathcal{H}_g^\ell v_k)(x, t_0) | \\
&\leq \underbrace{C_2 \int_{\mathbb{R}^n} \left(\frac{1}{4\pi\tau} \right)^{n/2} e^{-\frac{c_2|x-y|^2}{4\tau}} |\mathcal{H}_g^\ell v_k(y, t_0 - \tau)| dy}_{\text{By (2.2)}} \\
&\leq \underbrace{\frac{C}{\tau^{n/2}} \int_{\mathbb{R}^n} e^{-\frac{c_2|x-y|^2}{4\tau}} |\varphi_\ell(y)| dy}_{\text{By (3.7) and (3.2) and } \mathcal{H}_g^\ell = (\partial_t - \Delta_g)^\ell} \\
&\leq \underbrace{C \|\varphi_\ell\|_{L^\infty(\mathbb{R}^n)}}_{\int_{\mathbb{R}^n} e^{-\frac{|x|^2}{4\tau}} dx = (4\pi\tau)^{\frac{n}{2}}} ,
\end{aligned}$$

for some $\varphi_\ell \in \mathcal{S}(\mathbb{R}^n)$, and thus

$$|\partial_\tau^\ell (e^{-\tau \mathcal{H}_g} v_k)(x, t_0) \frac{1}{\tau^{s_k+m-\ell}}| \leq C \|\varphi_\ell\|_{L^\infty(\mathbb{R}^n)} \frac{1}{\tau^{s_k+m-\ell}} \rightarrow 0, \quad \text{as } \tau \rightarrow \infty.$$

Similarly, for $\tau \in (0, a]$, we have

$$\begin{aligned}
(3.10) \quad |\partial_\tau^\ell (e^{-\tau \mathcal{H}_g} v_k)(x, t_0)| &\leq C_2 \int_{\mathbb{R}^n} \left(\frac{1}{4\pi\tau} \right)^{n/2} e^{-\frac{c_2|x-y|^2}{4\tau}} |\mathcal{H}_g^\ell v_k(y, t_0 - \tau)| dy \\
&\leq \underbrace{\frac{C}{\tau^{n/2}} \int_{\mathbb{R}^n \setminus \mathcal{O}} e^{-\frac{c_2|x-y|^2}{4\tau}} |\mathcal{H}_g^\ell v_k(y, t_0 - \tau)| dy}_{\text{By (3.8) so that } \mathcal{H}_g^\ell v_k(y, t_0 - \tau) = 0 \text{ for } y \in \mathcal{O}} \\
&\leq \frac{C}{\tau^{n/2}} e^{-\frac{c_2\kappa^2}{\tau}} \sup_{t \in (-T, T)} \left\| \mathcal{H}_g^\ell v_k(\cdot, t) \right\|_{L^1(\mathbb{R}^n \setminus \overline{\mathcal{O}})} \\
&\leq C e^{-\frac{c}{\tau}}, \quad \text{for } x \in \omega \Subset \mathcal{O},
\end{aligned}$$

which leads to

$$|\partial_\tau^\ell (e^{-\tau \mathcal{H}_g} v_k)(x, t_0) \frac{1}{\tau^{s_k+m-\ell}}| \leq C e^{-\frac{c}{\tau}} \frac{1}{\tau^{s_k+m-\ell}} \rightarrow 0, \quad \text{as } \tau \rightarrow 0^+.$$

Since we have shown that the boundary values vanish, by applying m -times integration by parts in τ to (3.5), we get

$$\begin{aligned}
0 &= \sum_{k=1}^N \frac{\gamma_k}{\Gamma(-s_k)} \int_0^\infty (e^{-\tau \mathcal{H}_g} v_k)(x, t_0) \tau^{-(m+1+s_k)} d\tau \\
&= \sum_{k=1}^N \Gamma(m+1+s_k) \int_0^\infty f_k(\tau) \tau^{-m} d\tau, \quad \text{for any } m \in \mathbb{N} \cup \{0\},
\end{aligned}$$

where the function

$$(3.11) \quad f_k(\tau) := \frac{1}{\Gamma(-s_k)\Gamma(1+s_k)} (e^{-\tau \mathcal{H}_g} v_k)(x, t_0) \tau^{-(1+s_k)},$$

and the constant γ_k is defined as $\gamma_k := (1+s_k)(2+s_k)\dots(m+s_k) = \frac{\Gamma(m+1+s_k)}{\Gamma(1+s_k)}$. Note that the smoothness of $v_k(x, t)$ and $p_\tau(x, y)$ yield $f_k(\tau) \in C^\infty((0, \infty))$.

In addition, we will show that $f_k(\tau)$ satisfies the bound (3.1) so that we will be able to apply Proposition 3.1. As (3.10) for $\ell = 0$ implies the case $\tau \in (0, a]$, it remains to show the exponential decay in the interval $\tau \in (a, \infty)$. To this end, by following a similar argument as in (3.9) and utilizing (3.2) with $|\beta| = 0$, it gives rise to the succeeding estimate for $f_k(\tau)$, $\tau \in (a, \infty)$:

$$\begin{aligned}
 |f_k(\tau)| &\leq |(e^{-\tau \mathcal{H}_g} v_k)(x, t_0) \tau^{-(1+s_k)}| \\
 &\leq \underbrace{C_2 \tau^{-(1+s_k)} \int_{\mathbb{R}^n} \left(\frac{1}{4\pi\tau}\right)^{n/2} e^{-\frac{c_2|x-y|^2}{4\tau}} |v_k(y, t_0 - \tau)| dy}_{\text{By (2.2)}} \\
 &\leq \underbrace{\frac{C \tau^{-(1+s_k)}}{\tau^{n/2}} e^{\delta(t_0 - \tau)} \int_{\mathbb{R}^n} e^{-\frac{c_2|x-y|^2}{4\tau}} |\varphi_0(y)| dy}_{\text{By (3.7) and (3.2)}} \\
 &\leq \underbrace{C \tau^{-(1+s_k)} e^{\delta(t_0 - \tau)} \|\varphi_0\|_{L^\infty(\mathbb{R}^n)}}_{\int_{\mathbb{R}^n} e^{-\frac{|x|^2}{4\tau}} dx = (4\pi\tau)^{\frac{n}{2}}} \\
 &\leq \underbrace{C e^{\delta T} e^{-\delta\tau} \|\varphi_0\|_{L^\infty(\mathbb{R}^n)}}_{\text{since } t_0 \in (-T, T)},
 \end{aligned}$$

for some $\varphi_0 \in \mathcal{S}(\mathbb{R}^n)$. With this estimate, we can now apply Proposition 3.1 to obtain that $f_k(\tau)$ is identically zero, for $\tau \in (0, \infty)$. Indeed the definition of $f_k(\tau)$ in (3.11) implies

$$(e^{-\tau \mathcal{H}_g} v_k)(x, t_0) = 0 \quad \text{for } x \in \omega, \tau > 0 \text{ and } k = 1, \dots, N.$$

Since $t_0 \in (-T, T)$ can be arbitrary, we further deduce

$$(3.12) \quad (e^{-\tau \mathcal{H}_g} v_k)(x, t) = 0 \quad \text{for } (x, t) \in \omega_T, \tau > 0 \text{ and } k = 1, \dots, N.$$

Now, since $\omega \Subset \mathcal{O}$ and $\kappa > 0$ in (3.4) are arbitrarily chosen, as a result, substituting (3.12) and (3.3) into (2.4), we have

$$v_k = \mathcal{H}_g^{s_k} v_k = 0, \text{ in } \mathcal{O}_T, \text{ for } k = 1, \dots, N.$$

Finally, applying the (weak) UCP for nonlocal parabolic operators $\mathcal{H}_g^s$, $s \in (0, 1)$ (see [BS24, LLR20]), we can ensure $v_k \equiv 0$ in $\mathbb{R}_T^n$, for $k = 1, \dots, N$. This concludes the proof. $\square$

With Theorem 3.3, we can prove Theorem 1.2.

Proof of Theorem 1.2. For $\alpha_k = m_k + s_k \in \mathbb{R}_+ \setminus \mathbb{N}$, where m_k is the integer part of α_k and $s_k \in (0, 1)$ is the fractional part of α_k . In particular, we have $\mathcal{H}_g^{\alpha_k} = \mathcal{H}_g^{m_k + s_k} = \mathcal{H}_g^{s_k}(\mathcal{H}_g^{m_k})$. Note that $\{u_k\}_{k=1}^N \subset C^\infty((-\infty, T); \mathcal{S}(\mathbb{R}^n))$, and we let

$$v_k := b_k \mathcal{H}_g^{m_k} u_k, \quad b_k \in \mathbb{C} \setminus \{0\}, \quad \text{for } k = 1, \dots, N,$$

which is also in $C^\infty((-\infty, T); \mathcal{S}(\mathbb{R}^n))$. Since $\mathcal{H}_g^{m_k}$ is a local operator, the condition (1.5) implies (3.3), and the condition (1.4) leads to (3.2). By Theorem 3.3, we deduce $v_k = 0$ in $\mathbb{R}_T^n$, and thus $\mathcal{H}_g^{m_k} u_k = 0$ in $\mathbb{R}_T^n$ with $u_k|_{\mathcal{O}_T} = 0$. Lastly, the UCP of the classical parabolic operators $\mathcal{H}_g^{m_k}$ in $\mathbb{R}_T^n$ leads to the desired result $u_k = 0$ in $\mathbb{R}_T^n$. This proves the assertion. $\square$

We conclude this section with some remarks regarding the exponential decay conditions.

Remark 3.4.

- (i) The exponential decay condition (3.2) is needed only for $N \geq 2$ (entangled nonlocal parabolic) in Theorem 3.3 in order to break the nonlocal effect arising from every fractional operator $\mathcal{H}_g^{s_k}$. When $N = 1$ (single $\mathcal{H}_g^s$), however, the UCP holds without such decay condition and it has been shown in the works [BS24, LLR20, LLU22]. As a result, (1.4) in Theorem 1.2 can be removed when one considers $N = 1$.
- (ii) Although the decay conditions (1.4) and (3.2) seem strong, for the study of inverse problems, both (1.4) and (3.2) hold automatically provided that the solution of the initial exterior value problem has zero initial data, namely, $u = 0$ in $\mathbb{R}^n \times \{t \leq -T\}$, see Section 4 for detailed discussions.

4. INVERSE PROBLEMS AND PROOF OF MAIN RESULTS

4.1. Global uniqueness for the fractional poly-parabolic operators. It is known that the proof of uniqueness can be established by employing the UCP together with the Runge approximation property. In what follows, we present an alternative formulation of the entanglement principle by additionally imposing an initial value vanishing condition (4.1), which helps to shorten the arguments for the UCP in [LLR20, BS24]. It is worth mentioning that when we deal with the inverse problem, (4.1) is fulfilled naturally due to the initial condition in the problem under consideration.

Proposition 4.1 (Modified entanglement principle). *Let $\mathcal{O} \subset \mathbb{R}^n$ be a nonempty open set for $n \geq 2$. Let $N \in \mathbb{N}$, $T > 0$, $\{b_k\}_{k=1}^N \subset (0, \infty)$, and $0 < s_1 < \dots < s_N < 1$. Let $u_k \in \mathbf{H}^{s_k}(\mathbb{R}^{n+1})$, for $k = 1, \dots, N$. If*

$$(4.1) \quad u_1 = \dots = u_N = 0 \text{ in } \mathbb{R}^n \times \{t \leq -T\},$$

and

$$(4.2) \quad u_1|_{\mathcal{O}_T} = \dots = u_N|_{\mathcal{O}_T} = \left(\sum_{k=1}^N b_k \mathcal{H}_g^{s_k} u_k \right) \Big|_{\mathcal{O}_T} = 0,$$

hold, then $u_1 = \dots = u_N = 0$ in $\mathbb{R}_T^n$ for all $k = 1, \dots, N$.

Proof. The proof can be reduced to Proposition 3.1 after an appropriate deduction. Note that the functions $u_k \in \mathbf{H}^{s_k}(\mathbb{R}^{n+1})$ are not necessarily smooth, thus Theorem 3.3 cannot be applied directly to the current setting. To address this lack of smoothness, we utilize certain properties of the heat kernel, combined with a smooth mollifier, which enables us to approximate the functions u_k and thereby overcome the regularity issue.

To this end, for $\varepsilon > 0$, we denote

$$T_\varepsilon := T - \varepsilon.$$

Consider the one-dimensional standard mollifier $\varphi \in C_0^\infty(\mathbb{R})$ with compact support $\text{supp } \varphi \subset (-1, 1)$, and satisfy $0 \leq \varphi$ and $\|\varphi\|_{L^2(\mathbb{R})} = 1$. For each $\varepsilon > 0$, we define $\varphi_\varepsilon(t) := \varepsilon^{-1} \varphi(t/\varepsilon)$ and thus $\varphi_\varepsilon \in C^\infty(\mathbb{R})$ with $\text{supp } \varphi_\varepsilon \subset (-\varepsilon, \varepsilon)$. For each $x \in \mathbb{R}^n$, since $u_k(x, \cdot)$ is locally integrable in t variable, the function

$$u_{k,\varepsilon}(x, t) := (u_k * \varphi_\varepsilon)(x, t) = \int_{-\varepsilon}^{\varepsilon} u_k(x, t - \eta) \varphi_\varepsilon(\eta) d\eta, \quad t \in (-\infty, T_\varepsilon),$$

and $u_{k,\varepsilon}(x, \cdot) \in C^\infty(\mathbb{R})$. Also, $u_{k,\varepsilon}(x, \cdot) \rightarrow u_k(x, \cdot)$ almost everywhere as $\varepsilon \rightarrow 0$ for $k = 1, \dots, N$.

Next, for $s \in (0, 1)$, recalling the definition (2.4) and using a direct computation give

$$(\mathcal{H}_g^s u_{k,\varepsilon})(x, t) = (\mathcal{H}_g^s (u_k * \varphi_\varepsilon))(x, t) = ((\mathcal{H}_g^s u_k) * \varphi_\varepsilon)(x, t), \quad (x, t) \in \mathcal{O}_{T_\varepsilon},$$

which can be seen since $\mathcal{H}_g^s u(x, t)$ is defined via a convolution in t . This implies

$$\sum_{k=1}^N ((\mathcal{H}_g^{s_k} u_k) * \varphi_\varepsilon)(x, t) = \sum_{k=1}^N (\mathcal{H}_g^{s_k} u_{k,\varepsilon})(x, t), \quad (x, t) \in \mathcal{O}_{T_\varepsilon}.$$

Together with (4.2) and (4.1), we get

$$(4.3) \quad \begin{aligned} u_{1,\varepsilon} = \dots = u_{N,\varepsilon} = 0, \text{ in } (\mathcal{O} \times (-T_\varepsilon - 2\varepsilon, T_\varepsilon)) \cup (\mathbb{R}^n \times \{t \leq -T_\varepsilon - 2\varepsilon\}), \\ \sum_{k=1}^N \mathcal{H}_g^{s_k} u_{k,\varepsilon} = 0, \quad \text{ in } \mathcal{O}_{T_\varepsilon}. \end{aligned}$$

Applying $\mathcal{H}_g^m$, $m = 1, 2, \dots$, to (4.3) leads to

$$\mathcal{H}_g^m u_{1,\varepsilon}|_{\mathcal{O}_{T_\varepsilon}} = \dots = \mathcal{H}_g^m u_{N,\varepsilon}|_{\mathcal{O}_{T_\varepsilon}} = \left(\sum_{k=1}^N b_k \mathcal{H}_g^{m+s_k} u_{k,\varepsilon} \right) \Big|_{\mathcal{O}_{T_\varepsilon}} = 0,$$

Let $\omega \Subset \mathcal{O}$ such that

$$\text{dist}(\omega, \mathbb{R}^n \setminus \overline{\mathcal{O}}) \geq 2\kappa,$$

for some constant $\kappa > 0$. For $(x, t) \in \omega_{T_\varepsilon}$, we have by (2.4) that

$$(4.4) \quad \begin{aligned} 0 &= \sum_{k=1}^N b_k \mathcal{H}_g^{s_k} \mathcal{H}_g^m u_{k,\varepsilon}(x, t) = \sum_{k=1}^N \frac{b_k}{\Gamma(-s_k)} \int_0^\infty (e^{-\tau \mathcal{H}_g} \mathcal{H}_g^m u_{k,\varepsilon})(x, t) \frac{d\tau}{\tau^{1+s_k}} \\ &= \sum_{k=1}^N \frac{b_k}{\Gamma(-s_k)} \int_0^\infty (-1)^m \partial_\tau^m (e^{-\tau \mathcal{H}_g} u_{k,\varepsilon})(x, t) \frac{d\tau}{\tau^{1+s_k}}. \end{aligned}$$

Next, fix $t_0 \in (-T_\varepsilon, T_\varepsilon)$, we shall show no contribution arises at the endpoints when conducting integration by parts in τ . That is, for $\ell = 0, 1, \dots, m-1$, the following terms

$$(4.5) \quad \partial_\tau^\ell (e^{-\tau \mathcal{H}_g} u_{k,\varepsilon})(x, t_0) \frac{1}{\tau^{s_k+m-\ell}}$$

vanish at $\tau \rightarrow 0^+$ and $\tau \rightarrow +\infty$.

Let us denote

$$a := t_0 - (-T) = t_0 + T > 0,$$

and split τ into the following regions:

$$t_0 - \tau \in (-\infty, -T_\varepsilon - 2\varepsilon) \quad \text{if } \tau \in (a + \varepsilon, \infty),$$

and

$$t_0 - \tau \in [-T_\varepsilon - 2\varepsilon, T_\varepsilon) \quad \text{if } \tau \in (0, a + \varepsilon].$$

We first show the boundary terms (4.5) vanish when $\tau \rightarrow \infty$. For $\tau \in (a + \varepsilon, \infty)$, we have $t_0 - \tau \in (-\infty, -T_\varepsilon - 2\varepsilon)$, and

$$u_{1,\varepsilon} = \dots = u_{N,\varepsilon} = 0 \text{ in } \mathbb{R}^n \times \{t \leq -T_\varepsilon - 2\varepsilon\}.$$

Therefore

$$(4.6) \quad \begin{aligned} |\partial_\tau^\ell (e^{-\tau \mathcal{H}_g} u_{k,\varepsilon})(x, t_0)| &= | (e^{-\tau \mathcal{H}_g} \mathcal{H}_g^\ell u_{k,\varepsilon})(x, t_0) | \\ &\leq C_2 \int_{\mathbb{R}^n} \left(\frac{1}{4\pi\tau} \right)^{n/2} e^{-\frac{c_2|x-y|^2}{4\tau}} |\mathcal{H}_g^\ell u_{k,\varepsilon}(y, t_0 - \tau)| dy \\ &= 0. \end{aligned}$$

Now, notice the function $(e^{-\tau \mathcal{H}_g} u_{k,\varepsilon})(x, t)$ is C^∞ -smooth for $(x, t, \tau) \in \mathbb{R}^{n+1} \times (0, \infty)$. This can be seen via the integral formula (2.3) and the heat kernel $p_\tau(x, y)$ is C^∞ -smooth for $(x, y) \in \mathbb{R}^n \times \mathbb{R}^n$ and $\tau > 0$ (see [Dav90], Chapter 5). Moreover, it is also known that the function $(e^{-\tau \mathcal{H}_g} u_{k,\varepsilon})(x, t)$ satisfies

$$(4.7) \quad \begin{cases} (\partial_\tau + \mathcal{H}_g)(e^{-\tau \mathcal{H}_g} u_{k,\varepsilon})(x, t) = 0 & \text{for } (x, t, \tau) \in \mathbb{R}^{n+1} \times \mathbb{R}_+, \\ \lim_{\tau \rightarrow 0^+} (e^{-\tau \mathcal{H}_g} u_{k,\varepsilon})(x, t) = u_{k,\varepsilon}(x, t) & \text{for } (x, t) \in \mathbb{R}^{n+1}, \end{cases}$$

where the above limit holds in the L^2 -sense, for all $k = 1, \dots, N$.

Next, we shall show no contribution arises at $\tau \rightarrow 0^+$. For $\tau \in (0, a + \varepsilon]$, we have $t_0 - \tau \in [-T_\varepsilon - 2\varepsilon, T_\varepsilon]$ and

$$u_{1,\varepsilon} = \dots = u_{N,\varepsilon} = 0 \text{ in } \mathcal{O} \times (-T_\varepsilon - 2\varepsilon, T_\varepsilon).$$

For $x \in \omega \Subset \mathcal{O}$, using (4.7) and binomial expansion, the Lebesgue dominated convergence theorem infers that

$$(4.8) \quad \begin{aligned} |\partial_\tau^\ell (e^{-\tau \mathcal{H}_g} u_{k,\varepsilon})(x, t_0)| &= |(\partial_t - \Delta_g)^\ell (e^{-\tau \mathcal{H}_g} u_{k,\varepsilon})(x, t_0)| \\ &= \left| \int_{\mathbb{R}^n} \sum_{i=1}^{\ell} \binom{\ell}{i} \partial_t^i (-\Delta_g)^{\ell-i} (p_\tau(x, y) u_{k,\varepsilon})(y, t_0 - \tau) dy \right| \\ &= \left| \sum_{i=1}^{\ell} \binom{\ell}{i} \int_{\mathbb{R}^n} (-\Delta_g)^{\ell-i} p_\tau(x, y) \partial_t^i u_{k,\varepsilon}(y, t_0 - \tau) dy \right| \\ &= \left| \sum_{i=1}^{\ell} \binom{\ell}{i} \int_{\mathbb{R}^n \setminus \mathcal{O}} (-\Delta_g)^{\ell-i} p_\tau(x, y) \partial_t^i u_{k,\varepsilon}(y, t_0 - \tau) dy \right| \\ &= \left| \sum_{i=1}^{\ell} \binom{\ell}{i} \int_{\mathbb{R}^n \setminus \mathcal{O}} (-\partial_\tau)^{\ell-i} p_\tau(x, y) \partial_t^i u_{k,\varepsilon}(y, t_0 - \tau) dy \right|, \end{aligned}$$

where we used $p_\tau(x, y)$ is the heat kernel solving the heat equation

$$(\partial_\tau - \Delta_g) p_\tau(x, y) = 0,$$

for $x \neq y$, $x, y \in \mathbb{R}^n$ and $\tau > 0$ (since $x \in \omega$ and $y \in \mathbb{R}^n \setminus \mathcal{O}$). Utilizing [Gri95, Theorem 3.1], it is known that the heat kernel satisfies the following time-derivative estimate

$$(4.9) \quad |\partial_\tau^{\ell-i} p_\tau(x, y)| \leq C \frac{(1 + |x - y|^2 / \tau)^{N'}}{\tau^{\ell-i} \min(\tau, R^2)^l} e^{-\frac{|x-y|^2}{4\tau}},$$

for any $\ell \in \mathbb{N}$, $R > 0$, $x, y \in \mathbb{R}^n$ and for some constants $C, N', l > 0$ with $N' = \ell - i + l + 1$. Thus, inserting (4.9) into (4.8), we have

$$(4.10) \quad |\partial_\tau^\ell (e^{-\tau \mathcal{H}_g} u_{k,\varepsilon})(x, t_0)| \leq C \sum_{i=1}^{\ell} \binom{\ell}{i} \int_{\mathbb{R}^n \setminus \mathcal{O}} \frac{(1 + |x - y|^2 / \tau)^{N'}}{\tau^{\ell-i} \min(\tau, R^2)^l} e^{-\frac{|x-y|^2}{4\tau}} |\partial_t^i u_{k,\varepsilon}(y, t_0 - \tau)| dy.$$

Thanks to the bound (4.10), for $\tau \in (0, a + \varepsilon]$, by the Hölder inequality, we have

$$\begin{aligned}
 (4.11) \quad & \int_{\mathbb{R}^n \setminus \mathcal{O}} \frac{(1 + |x - y|^2 / \tau)^{N'}}{\tau^{\ell-i} \min(\tau, R^2)^l} e^{-\frac{|x-y|^2}{4\tau}} |\partial_t^i u_{k,\varepsilon}(y, t_0 - \tau)| dy \\
 & \leq C e^{-\frac{c_0}{\tau}} \sup_{t_0 - \tau \in (-T_\varepsilon - 2\varepsilon, T_\varepsilon)} \|\partial_t^i u_{k,\varepsilon}(\cdot, t_0 - \tau)\|_{L^2(\mathbb{R}^n)} \underbrace{\left(\int_{\kappa}^{\infty} \frac{(1 + |\rho|^2 / \tau)^{2N'}}{\tau^{2\ell-2i} \min(\tau, R^2)^{2l}} e^{-\frac{c_1|\rho|^2}{\tau}} \rho^{n-1} d\rho \right)^{1/2}}_{\text{change of variable } \rho/\sqrt{\tau} \mapsto \rho} \\
 & \leq C e^{-\frac{c_0}{\tau}} \sup_{t_0 - \tau \in (-T_\varepsilon - 2\varepsilon, T_\varepsilon)} \|\partial_t^i u_{k,\varepsilon}(\cdot, t_0 - \tau)\|_{L^2(\mathbb{R}^n)} \underbrace{\left(\frac{\tau^{n/2}}{\tau^{2\ell-2i} \min(\tau, R^2)^{2l}} \int_0^{\infty} (1 + \rho^2)^{2N'} \rho^{n-1} e^{-c_1 \rho^2} d\rho \right)^{1/2}}_{\text{finite}},
 \end{aligned}$$

and, moreover, the Minkowski inequality and Young's inequality give the following bound

$$\|\partial_t^i u_{k,\varepsilon}(\cdot, t_0 - \tau)\|_{L^2(\mathbb{R}^n)} \leq \|u\|_{L^2(\mathbb{R}^{n+1})} \|\partial_t^i \varphi_\varepsilon\|_{L^2(\mathbb{R})} \leq \|u\|_{\mathbf{H}^{s_k}(\mathbb{R}^{n+1})} \|\partial_t^i \varphi_\varepsilon\|_{L^2(\mathbb{R})}$$

for some constants $c_0, c_1, C > 0$. Recalling that $u_{k,\varepsilon}(x, t)$ is smooth in t , using the estimate (4.11), one can ensure

$$\left| \partial_\tau^\ell (e^{-\tau \mathcal{H}_g} u_{k,\varepsilon})(x, t_0) \frac{1}{\tau^{s_k + m - \ell}} \right| \rightarrow 0, \quad \text{as } \tau \rightarrow 0^+.$$

Since we have shown that the boundary values vanish, by applying m -times integration by parts in τ to (4.4), we get

$$\begin{aligned}
 0 &= \sum_{k=1}^N \frac{\gamma_k}{\Gamma(-s_k)} \int_0^\infty (e^{-\tau \mathcal{H}_g} u_{k,\varepsilon})(x, t_0) \tau^{-(m+1+s_k)} d\tau \\
 &= \sum_{k=1}^N \Gamma(m+1+s_k) \int_0^\infty f_{k,\varepsilon}(\tau) \tau^{-m} d\tau, \quad \text{for any } m \in \mathbb{N} \cup \{0\},
 \end{aligned}$$

where for each fixed $\varepsilon > 0$, the function $f_{k,\varepsilon}(\tau)$ is defined as

$$(4.12) \quad f_{k,\varepsilon}(\tau) := \frac{b_k}{\Gamma(-s_k)\Gamma(1+s_k)} (e^{-\tau \mathcal{H}_g} u_{k,\varepsilon})(x, t_0) \tau^{-(1+s_k)},$$

and the constant γ_k is defined as

$$\gamma_k := (1+s_k)(2+s_k)\dots(m+s_k) = \frac{\Gamma(m+1+s_k)}{\Gamma(1+s_k)}.$$

Note that the smoothness of $u_{k,\varepsilon}(x, t)$ in time variable brings out $f_{k,\varepsilon}(\tau) \in C^\infty((0, \infty))$.

In addition, we will show that $f_{k,\varepsilon}(\tau)$ satisfies the bound (3.1) so that we will be able to apply Proposition 3.1.

- For $\tau \in (a + \varepsilon, \infty)$, $f_{k,\varepsilon}(\tau) = 0$ is implied by (4.6) by taking $\ell = 0$.
- For $\tau \in (0, a + \varepsilon]$, we can derive similarly as in (4.11) by taking $\ell = i = 0$ and get

$$|f_{k,\varepsilon}(\tau)| \leq C e^{-\frac{c_2}{\tau}},$$

for some constant $c_2 > 0$.

With this estimate, we can now apply Proposition 3.1 to obtain that $f_{k,\varepsilon}(\tau)$ is identically zero, for $\tau \in (0, \infty)$ and for all $k = 1, \dots, N$. Indeed the definition of $f_{k,\varepsilon}(\tau)$ in (4.12) implies

$$(e^{-\tau \mathcal{H}_g} u_{k,\varepsilon})(x, t_0) = 0 \quad \text{for } x \in \omega, \tau > 0 \text{ and } k = 1, \dots, N.$$

Since $t_0 \in (-T_\varepsilon, T_\varepsilon)$ can be arbitrary, we further deduce

$$(e^{-\tau \mathcal{H}_g} u_{k,\varepsilon})(x, t) = 0 \quad \text{for } (x, t) \in \omega_{T_\varepsilon}, \quad \tau > 0 \text{ and } k = 1, \dots, N.$$

Now, since $\omega \Subset \mathcal{O}$ and $\kappa > 0$ are arbitrarily chosen, we further have

$$u_{k,\varepsilon} = \mathcal{H}_g^{s_k} u_{k,\varepsilon} = 0, \quad \text{in } \mathcal{O}_{T_\varepsilon}, \quad \text{for } k = 1, \dots, N.$$

Applying the (weak) UCP for nonlocal parabolic operators $\mathcal{H}_g^{s_k}$, $s_k \in (0, 1)$ (see [BS24, LLR20]), we can ensure $u_{k,\varepsilon} \equiv 0$ in $\mathbb{R}_{T_\varepsilon}^n$, for $k = 1, \dots, N$. Then $u_k \equiv 0$ in $\mathbb{R}_T^n$ follows from the fact that $u_{k,\varepsilon}$ converges to u_k almost everywhere as $\varepsilon \rightarrow 0$. This proves the assertion. $\square$

Remark 4.2. *Let us emphasize that the nonlocal operator $\mathcal{H}_g^s$ has constant coefficients in the time variable. This allows the use of a convolution argument to relax the regularity assumptions for certain functions. Consequently, one may weaken the regularity hypotheses in Theorem 1.2.*

Remark 4.3. *The entanglement principle also applies to the adjoint fractional poly-parabolic operator $\sum_{k=1}^N b_k \mathcal{H}_{g,*}^{s_k}$. More precisely, if*

$$u_1 = \dots = u_N = 0 \quad \text{in } \mathbb{R}^n \times \{t \geq T\},$$

and

$$u_1|_{\mathcal{O}_T} = \dots = u_N|_{\mathcal{O}_T} = \left(\sum_{k=1}^N b_k \mathcal{H}_{g,*}^{s_k} u_k \right) \Big|_{\mathcal{O}_T} = 0,$$

then $u_k \equiv 0$ in $\mathbb{R}_T^n$ for all $k = 1, \dots, N$. The proof proceeds in the same way as that of Proposition 4.1, except that one reverses the sign in the t -variable.

To study the inverse problems, we only need one single function in the entanglement principle to prove our result, i.e., $u := u_1 = \dots = u_N$ in $\mathbb{R}^{n+1}$. Below, we will apply Proposition 4.1 to prove the Runge approximation for fractional poly-parabolic operators.

For $0 < s_1 < \dots < s_N < 1$, $\{b_k\}_{k=1}^N \subset (0, \infty)$, and $T > 0$, we recall the notation $\Omega_T = \Omega \times (-T, T) \subset \mathbb{R}^{n+1}$. Let $V \in L^\infty(\Omega_T)$ satisfy the eigenvalue condition (1.3). For $f \in \tilde{\mathbf{H}}^{s_N}((\Omega_e)_T)$, let $u_f \in \mathbf{H}^{s_N}(\mathbb{R}^{n+1})$ solve the problem

$$(4.13) \quad \begin{cases} \left(\sum_{k=1}^N b_k \mathcal{H}_g^{s_k} + V \right) u_f = 0 & \text{in } \Omega_T, \\ u_f = f & \text{in } (\Omega_e)_T, \\ u_f = 0 & \text{in } \mathbb{R}^n \times \{t \leq -T\}. \end{cases}$$

It is known that $\chi_{(-\infty, T]}(t) u_f(t, x)$ is the unique solution of (4.13).

Lemma 4.4 (Runge approximation). *For $n \geq 1$, let $W \subset \Omega_e$ be a nonempty open subset and $T > 0$ be a real number. Then the set*

$$\mathcal{R} = \{u_f|_{\Omega_T} : u_f \text{ is the solution to (4.13), } f \in C_c^\infty(W_T)\}$$

is dense in $L^2(\Omega_T)$.

Proof. The proof is standard and relies on the Hahn-Banach theorem. It suffices to show that if $(v, w)_{L^2(\Omega_T)} = 0$ for all $v \in \mathcal{R}$, then necessarily $w \equiv 0$. To proceed, let $w \in L^2(\Omega_T)$. Assume that

$$(\chi_{(-\infty, T]} u_f, w)_{L^2(\Omega_T)} = (u_f, w)_{L^2(\Omega_T)} = 0, \quad \text{for all } f \in C_c^\infty(W_T),$$

where $\chi_{(-\infty, T]} u_f$ denotes the unique solution of (4.13) in Ω_T . Here we have used the fact that, as before, the future data does not influence the solution in Ω_T .

Next, let $\phi \in \mathbf{H}^s(\mathbb{R}^{n+1})$ be the solution of

$$(4.14) \quad \begin{cases} \left(\sum_{k=1}^N b_k \mathcal{H}_{g,*}^{s_k} + V \right) \phi = w & \text{in } \Omega_T, \\ \phi = 0 & \text{in } (\Omega_e)_T \cup (\mathbb{R}^n \times (\mathbb{R} \setminus (-T, T))), \end{cases}$$

where the well-posedness of (4.14) is guaranteed by Remark 2.5. Then,

$$(4.15) \quad 0 = (u_f, w)_{L^2(\Omega_T)} = \left(u_f - f, \left(\sum_{k=1}^N b_k \mathcal{H}_{g,*}^{s_k} + V \right) \phi \right)_{L^2(\mathbb{R}_T^n)} = - \left(f, \sum_{k=1}^N b_k \mathcal{H}_{g,*}^{s_k} \phi \right)_{L^2(W_T)},$$

for all $f \in C_c^\infty(W_T)$, where in the last identity we used the fact that f is supported in W_T and u_f solves (4.13). As (4.15) holds for all $f \in C_c^\infty(W_T)$, it yields

$$\sum_{k=1}^N b_k \mathcal{H}_{g,*}^{s_k} \phi = 0 \text{ in } W_T.$$

Combining it with $\phi = 0$ in W_T (from (4.14)), we apply the entanglement principle (see Remark 4.3) to deduce

$$\phi = 0 \text{ in } \mathbb{R}_T^n.$$

Moreover, from (4.14) again, the exterior condition of ϕ in the past and future time vanish, which implies $\phi \equiv 0$ in $\mathbb{R}^{n+1}$. Hence we infer that $\mathcal{H}_{g,*}^{s_k} \phi = 0$ in $\mathbb{R}^{n+1}$, for all $k = 1, \dots, N$. Finally, by substituting this ϕ back into (4.14), we can conclude $w \equiv 0$, which proves the Runge approximation. $\square$

Before proving Theorem 1.4, we also need the following integral identity.

Lemma 4.5 (Integral identity). *Let $\Omega_T \subset \mathbb{R}^{n+1}$ be the bounded open set and let $V_1, V_2 \in L^\infty(\Omega_T)$ satisfy the eigenvalue condition (1.3). Then, for any exterior Dirichlet data $f_1, f_2 \in \tilde{\mathbf{H}}^{s_N}((\Omega_e)_T)$, we have*

$$(4.16) \quad \langle (\Lambda_{V_1} - \Lambda_{V_2}) f_1, f_2 \rangle_{\mathbf{H}^s((\Omega_e)_T)^* \times \mathbf{H}^s((\Omega_e)_T)} = ((V_1 - V_2) u_1, u_2)_{\Omega_T},$$

where $u_1 \in \mathbf{H}^{s_N}(\mathbb{R}^{n+1})$ is the weak solution of

$$\begin{cases} \left(\sum_{k=1}^N b_k \mathcal{H}_g^{s_k} + V_1 \right) u_1 = 0 & \text{in } \Omega_T, \\ u_1 = f_1 & \text{in } (\Omega_e)_T, \\ u_1 = 0 & \text{in } \mathbb{R}^n \times \{t \leq -T\}, \end{cases}$$

and $u_2 \in \mathbf{H}^{s_N}(\mathbb{R}^{n+1})$ is the weak solution of

$$\begin{cases} \left(\sum_{k=1}^N b_k \mathcal{H}_{g,*}^{s_k} + V_2 \right) u_2 = 0 & \text{in } \Omega_T, \\ u_2 = f_2 & \text{in } (\Omega_e)_T, \\ u_2 = 0 & \text{in } \mathbb{R}^n \times \{t \geq T\}. \end{cases}$$

Proof. By the adjoint property, the DN map, one has

$$\begin{aligned} & \langle (\Lambda_{V_1} - \Lambda_{V_2}) f_1, f_2 \rangle_{\mathbf{H}^s((\Omega_e)_T)^* \times \mathbf{H}^s((\Omega_e)_T)} \\ &= \langle \Lambda_{V_1} f_1, f_2 \rangle_{\mathbf{H}^s((\Omega_e)_T)^* \times \mathbf{H}^s((\Omega_e)_T)} - \langle f_1, \Lambda_{V_2}^* f_2 \rangle_{\mathbf{H}^s((\Omega_e)_T) \times \mathbf{H}^s((\Omega_e)_T)^*} \\ &= B_{V_1}(u_1, u_2) - B_{V_2}(u_1, u_2) \\ &= ((V_1 - V_2) u_1|_{\Omega_T}, u_2|_{\Omega_T})_{\Omega_T}. \end{aligned}$$

This completes the proof. $\square$

Now, we can prove Theorem 1.4.

Proof of Theorem 1.4. We follow the same argument as the proof of [LLR20, Theorem 1.1]. If $\Lambda_{V_1} f|_{(W_2)_T} = \Lambda_{V_2} f|_{(W_2)_T}$ for any $f \in C_c^\infty((W_1)_T)$, where W_1 and W_2 are nonempty open subsets of Ω_e . By the integral identity (4.16), we have

$$\int_{\Omega_T} (V_1 - V_2) u_1 \overline{u_2} dx dt = 0,$$

where $u_1, u_2 \in \mathbf{H}^{s_N}(\mathbb{R}^n)$ solve $(\sum_{k=1}^N b_k \mathcal{H}_g^{s_k} + V_1) u_1 = 0$ and $(\sum_{k=1}^N b_k \mathcal{H}_{g,*}^{s_k} + V_2) u_2 = 0$ with $u_1 = 0$ for $\{t \leq -T\}$ and $u_2 = 0$ for $\{t \geq T\}$. Also, u_1, u_2 have the same exterior value $f \in C_c^\infty((W_1)_T)$.

Given an arbitrary $\phi \in L^2(\Omega_T)$ and by using the Runge approximation of Lemma 4.4, there exists two sequences of functions $\{u_\ell^1\}_{\ell \in \mathbb{N}}, \{u_\ell^2\}_{\ell \in \mathbb{N}} \subset \mathbf{H}^{s_N}(\mathbb{R}^{n+1})$ that fulfill

$$\begin{aligned} \left(\sum_{k=1}^N b_k \mathcal{H}_g^{s_k} + V_1 \right) u_\ell^1 &= \left(\sum_{k=1}^N b_k \mathcal{H}_{g,*}^{s_k} + V_2 \right) u_\ell^2 = 0 \text{ in } \Omega_T, \\ \text{supp}(u_\ell^1) &\subseteq \overline{(\Omega_1)_T}, \quad \text{supp}(u_\ell^2) \subseteq \overline{(\Omega_2)_T}, \\ u_\ell^1|_{\Omega_T} &= \phi + r_\ell^1, \quad u_\ell^2|_{\Omega_T} = 1 + r_\ell^2, \end{aligned}$$

where $\Omega_1, \Omega_2 \subset \mathbb{R}^n$ are two open sets containing Ω , and $r_\ell^1, r_\ell^2 \rightarrow 0$ in $L^2(\Omega_T)$ as $\ell \rightarrow \infty$. By substituting the solutions u_ℓ^j into the integral identity and passing to the limit as $\ell \rightarrow \infty$, we infer that

$$\int_{\Omega_T} (V_1 - V_2) \phi dx dt = 0.$$

As $\phi \in L^2(\Omega_T)$ is arbitrary, we can conclude that $V_1 = V_2$ in Ω_T . This completes the proof. $\square$

STATEMENTS AND DECLARATIONS

Data availability statement. No datasets were generated or analyzed during the current study.

Conflict of Interests. Hereby, we declare there are no conflicts of interest.

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REFERENCES

- [AEN20] Pascal Auscher, Moritz Egert, and Kaj Nyström. L^2 well-posedness of boundary value problems for parabolic systems with measurable coefficients. *J. Eur. Math. Soc. (JEMS)*, 22(9):2943–3058, 2020.
- [Bal60] Alampallam V. Balakrishnan. Fractional powers of closed operators and the semigroups generated by them. *Pacific J. Math.*, 10:419–437, 1960.
- [BDLCRS21] Animesh Biswas, Marta De León-Contreras, and Pablo Raúl Stinga. Harnack inequalities and Hölder estimates for master equations. *SIAM J. Math. Anal.*, 53(2):2319–2348, 2021.
- [BGMN21] Agnid Banerjee, Nicola Garofalo, Isidro H. Munive, and Duy-Minh Nhieu. The Harnack inequality for a class of nonlocal parabolic equations. *Commun. Contemp. Math.*, 23(6):Paper No. 2050050, 23, 2021.
- [BS24] Agnid Banerjee and Soumen Senapati. The Calderón problem for space-time fractional parabolic operators with variable coefficients. *SIAM J. Math. Anal.*, 56(4):4759–4810, 2024.
- [CGRU23] Giovanni Covi, Tuhin Ghosh, Angkana Rüland, and Gunther Uhlmann. A reduction of the fractional Calderón problem to the local Calderón problem by means of the Caffarelli-Silvestre extension. *arXiv preprint arXiv:2305.04227*, 2023.
- [CLL19] Xinlin Cao, Yi-Hsuan Lin, and Hongyu Liu. Simultaneously recovering potentials and embedded obstacles for anisotropic fractional Schrödinger operators. *Inverse Probl. Imaging*, 13(1):197–210, 2019.

- [CLR20] Mihajlo Cekic, Yi-Hsuan Lin, and Angkana Rüland. The Calderón problem for the fractional Schrödinger equation with drift. *Cal. Var. Partial Differential Equations*, 59(91), 2020.
- [CMRU22] Giovanni Covi, Keijo Mönkkönen, Jesse Railo, and Gunther Uhlmann. The higher order fractional Calderón problem for linear local operators: Uniqueness. *Adv. Math.*, 399:Paper No. 108246, 2022.
- [Dav90] Edward Brian Davies. *Heat kernels and spectral theory*, volume 92 of *Cambridge Tracts in Mathematics*. Cambridge University Press, Cambridge, 1990.
- [DNPV12] Eleonora Di Nezza, Giampiero Palatucci, and Enrico Valdinoci. Hitchhiker’s guide to the fractional Sobolev spaces. *Bull. Sci. Math.*, 136(5):521–573, 2012.
- [Fei24] Ali Feizmohammadi. Fractional Calderón problem on a closed Riemannian manifold. *Trans. Amer. Math. Soc.*, 377(4):2991–3013, 2024.
- [FGKU25] Ali Feizmohammadi, Tuhin Ghosh, Katya Krupchyk, and Gunther Uhlmann. Fractional anisotropic Calderón problem on closed Riemannian manifolds. *J. Differential Geom.*, 131(2):–, 2025.
- [FKU24] Ali Feizmohammadi, Katya Krupchyk, and Gunther Uhlmann. Calderón problem for fractional Schrödinger operators on closed Riemannian manifolds. *arXiv preprint arXiv:2407.16866*, 2024.
- [FL24] Ali Feizmohammadi and Yi-Hsuan Lin. Entanglement principle for the fractional Laplacian with applications to inverse problems. *arXiv preprint arXiv:2412.13118*, 2024.
- [GLX17] Tuhin Ghosh, Yi-Hsuan Lin, and Jingni Xiao. The Calderón problem for variable coefficients nonlocal elliptic operators. *Comm. Partial Differential Equations*, 42(12):1923–1961, 2017.
- [Gri95] Alexander Grigoryan. Upper bounds of derivatives of the heat kernel on an arbitrary complete manifold. *J. Funct. Anal.*, 127(2):363–389, 1995.
- [GRSU20] Tuhin Ghosh, Angkana Rüland, Mikko Salo, and Gunther Uhlmann. Uniqueness and reconstruction for the fractional Calderón problem with a single measurement. *J. Funct. Anal.*, 279(1):108505, 42, 2020.
- [GSU20] Tuhin Ghosh, Mikko Salo, and Gunther Uhlmann. The Calderón problem for the fractional Schrödinger equation. *Anal. PDE*, 13(2):455–475, 2020.
- [GU21] Tuhin Ghosh and Gunther Uhlmann. The Calderón problem for nonlocal operators. *arXiv:2110.09265*, 2021.
- [HL19] Bastian Harrach and Yi-Hsuan Lin. Monotonicity-based inversion of the fractional Schrödinger equation I. Positive potentials. *SIAM J. Math. Anal.*, 51(4):3092–3111, 2019.
- [HL20] Bastian Harrach and Yi-Hsuan Lin. Monotonicity-based inversion of the fractional Schrödinger equation II. General potentials and stability. *SIAM J. Math. Anal.*, 52(1):402–436, 2020.
- [Lin24] Yi-Hsuan Lin. The fractional anisotropic Calderón problem for a nonlocal parabolic equation on closed Riemannian manifolds. *arXiv preprint arXiv:2410.17750*, 2024.
- [LL22] Ru-Yu Lai and Yi-Hsuan Lin. Inverse problems for fractional semilinear elliptic equations. *Nonlinear Anal.*, 216:Paper No. 112699, 21, 2022.
- [LL25] Yi-Hsuan Lin and Hongyu Liu. *Inverse Problems for Integro-differential Operators*, volume 222 of *Applied Mathematical Sciences*. Springer, Cham, 2025.
- [LLR20] Ru-Yu Lai, Yi-Hsuan Lin, and Angkana Rüland. The Calderón problem for a space-time fractional parabolic equation. *SIAM J. Math. Anal.*, 52(3):2655–2688, 2020.
- [LLU22] Ching-Lung Lin, Yi-Hsuan Lin, and Gunther Uhlmann. The Calderón problem for nonlocal parabolic operators. *arXiv preprint arXiv:2209.11157*, 2022.
- [LLU23] Ching-Lung Lin, Yi-Hsuan Lin, and Gunther Uhlmann. The Calderón problem for nonlocal parabolic operators: A new reduction from the nonlocal to the local. *arXiv preprint arXiv:2308.09654*, 2023.
- [LM72] Jacques-Louis Lions and Enrico Magenes. *Non-homogeneous boundary value problems and applications. Vol. II*, volume Band 182 of *Die Grundlehren der mathematischen Wissenschaften*. Springer-Verlag, New York-Heidelberg, 1972. Translated from the French by P. Kenneth.
- [LNZ24] Yi-Hsuan Lin, Gen Nakamura, and Philipp Zimmermann. The Calderón problem for the Schrödinger equation in transversally anisotropic geometries with partial data. *arXiv preprint arXiv:2408.08298*, 2024.
- [LZ23] Yi-Hsuan Lin and Philipp Zimmermann. Unique determination of coefficients and kernel in nonlocal porous medium equations with absorption term. *arXiv preprint arXiv:2305.16282*, 2023.
- [LZ24] Yi-Hsuan Lin and Philipp Zimmermann. Approximation and uniqueness results for the nonlocal diffuse optical tomography problem. *arXiv preprint arXiv:2406.06226*, 2024.
- [McL00] William McLean. *Strongly elliptic systems and boundary integral equations*. Cambridge University Press, Cambridge, 2000.
- [MCSA01] Celso Martínez Carracedo and Miguel Sanz Alix. *The theory of fractional powers of operators*, volume 187 of *North-Holland Mathematics Studies*. North-Holland Publishing Co., Amsterdam, 2001.

- [RS18] Angkana Rüland and Mikko Salo. Exponential instability in the fractional Calderón problem. *Inverse Problems*, 34(4):045003, 21, 2018.
- [RS20] Angkana Rüland and Mikko Salo. The fractional Calderón problem: low regularity and stability. *Non-linear Anal.*, 193:111529, 56, 2020.
- [Rül25] Angkana Rüland. Revisiting the anisotropic fractional Calderón problem. *Int. Math. Res. Not. IMRN*, (5):Paper No. rnaf036, 28, 2025.
- [ST17] Pablo Raúl Stinga and José L. Torrea. Regularity theory and extension problem for fractional nonlocal parabolic equations and the master equation. *SIAM J. Math. Anal.*, 49(5):3893–3924, 2017.

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